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SUBHARMONIC RESONANCE

IN NONLINEAR SYSTEMS

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INTRODUCTION

Forced oscillations whose frequency is a fraction 1/2, 1/3, ... 1/n of that of the external applied force are called <u>subharmonic oscillations</u>. These vibrations have been observed in nonlinear mechanical systems and in nonlinear electrical systems.[1,2] Even in the auditory system of the human ear the occurrence of subharmonics has been detected when the hearing system is exposed to intense sound waves.[3] The ear under such a condition receives sounds which are not contained in the emitted acoustic radiation. Sounds of lower frequencies than that of the sound waves impinging upon the eardrum are called <u>undertones</u>. These undertones or subharmonics are attributed to the nonlinearities in the hearing mechanisms.

Although some theoretical and experimental investigations on subharmonics in nonlinear systems have been reported[4, 5, 6, 7], their behavior and the conditions under which they may exist are imperfectly understood. The existing theory has been developed with the assumptions that the nonlinearity and the damping in the system is small. From experimental observations it seems that the subharmonics are more likely to occur in a system with large nonlinearity. In this respect, the small nonlinearity assumption is a serious limitation to the existing theory. In order to provide theoretical consideration for the predominant nonlinear cases we should become more concerned with the problem of formulating theoretical conditions without assuming the small nonlinearity restriction.

STATEMENT OF THE PROBLEM

Some experimental data with regard to the behavior of subharmonic oscillations are available.[1,7] It has been observed that the occurrence of these oscillations in systems having nonlinear restoring forces depends upon the amplitude of excitation, the nonlinearity and the amount of damping in the system. The theoretical problem is to formulate some conditions which will indicate their various behaviors from the equation of motion with large nonlinearity.

Another important experimental observation has been that of transition of subharmonics in a system with the presence of several types of symmetric nonlinear restoring forces. There exists a problem of finding how the existence of certain subharmonic is related to the form of nonlinear restoring force. This phase of study will be both theoretical and experimental. The fact that the phenomena were observed in nearly-predominant nonlinear systems requires a method of development which does not have to take place in the neighborhood of linear forced vibration.

LINE OF APPROACH TO THE THEORETICAL PROBLEM

The theoretical treatment of subharmonics in nonlinear systems according to the iteration method of Stoker[4] and the method of Mandelstam and Papalexi[6] has been developed on the assumptions of small nonlinearity and small damping. The results of these methods can be used as a guide in formulating more general conditions. From the analysis of the above methods, one obvious advantage of the assumptions is in simplifying the results. The assumption of small damping is in agreement with experiment, but the small nonlinearity restriction must be removed in order to consider our problems. Since the oscillations are predominantly subharmonic over a wide range of frequency, some simplified results may be obtained by assuming the amplitude of the subharmonic is large compared to that of the harmonic component. This leads to the useful idea that the wave form of the oscillation can be represented essentially by components of two different frequencies. With these assumptions, one may develop a variational procedure according to the idea of Ritz[8] to study the subharmonic solutions to a nonlinear differential equation. It seems likely that the formulations will be more general since no restriction on the nonlinearity is imposed.

1. THE VARIATIONAL METHOD

Let us consider the energy integral

$$I = \int_{t_o}^{t_1} (T - V + R)dt \tag{1.1}$$

where T is the kinetic energy, V, the potential energy of the system and R denotes the energy dissipated by frictional forces. We will define R by the equation

$$\delta R = f \, \delta q \tag{1.2}$$

in which f represents the frictional forces and δq is a small arbitrary displacement. We shall also specify that δq is a function of time and it must vanish at the limits of a chosen time interval. Under these specifications we want to find the condition which would make the integral I a minimum or a maximum.

According to the usual variational method the variation of I is set equal to zero.(Generalized Hamilton's Principle)

$$\delta I = \delta\int_{t_o}^{t_1}(T - V)dt + \int_{t_o}^{t_1} f\,\delta q dt = 0 \tag{1.3}$$

It can easily be shown that the variation of the integral of (1.3) can be expressed as

$$\delta\int_{t_o}^{t_1}(T - V)dt = \int_{t_o}^{t_1}\frac{\partial}{\partial q}(T - V)\,\delta q dt - \int_{t_o}^{t_1}\frac{d}{dt}\frac{\partial(T - V)}{\partial \dot{q}}\,\delta q dt \text{--} \tag{1.4}$$

Hence we obtain

$$\int_{t_o}^{t_1}\left(\frac{d}{dt}\;\frac{\partial(T - V)}{\partial \dot{q}} - \frac{\partial(T - V)}{\partial q} - f\;\right)\delta q dt = 0 \tag{1.5}$$

Since δq is arbitrary we see that the condition that would make the integral I a minimum or a maximum is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = f \tag{1.6}$$

where $L = T - V$. Eq. 1.6 is the equation of motion for the system. The interesting point which we want to stress in this discussion is this: if Eq. 1.6 is given, Eq. 1.5 can be used to find the necessary conditions for which an <u>assumed</u>

$q(t)$ can satisfy Eq. 1.3 in the best possible way. Since we are dealing with equations which have periodic solutions the integral (1.5) will be evaluated over a complete cycle of the periodic occurrence. Thus we write

$$\int_0^{2\pi/\omega} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - f\right) \delta q\, dt = 0 \tag{1.7}$$

We shall attempt to present a solution of the problem of subharmonic oscillations by using Eq. 1.7 to determine the coefficients of a Fourier series solution.

2. SUBHARMONIC OSCILLATIONS IN SYSTEMS WITH NONLINEAR SYMMETRIC RESTORING FORCE.

In nonlinear systems with viscous damping and having a periodic forcing function of frequency $n\omega$, it is possible to have forced oscillations of frequency ω in addition to those of $n\omega$. These lower frequency oscillations are called subharmonic of order 1/n. From experiment we know that subharmonics do not occur if the damping is too great or the nonlinearity in the system is too small. An important feature of a subharmonic is that its amplitude is usually large compared to that of the harmonic component and in this respect the term <u>subharmonic resonance</u> might be very appropriate in denoting its occurrence. Fig. 1 shows the ordinary harmonic oscillations and the subharmonic oscillations of order 1/3. At <u>A</u> the nonlinear system was subjected to a pulse disturbance which gave rise to the subharmonic oscillations.

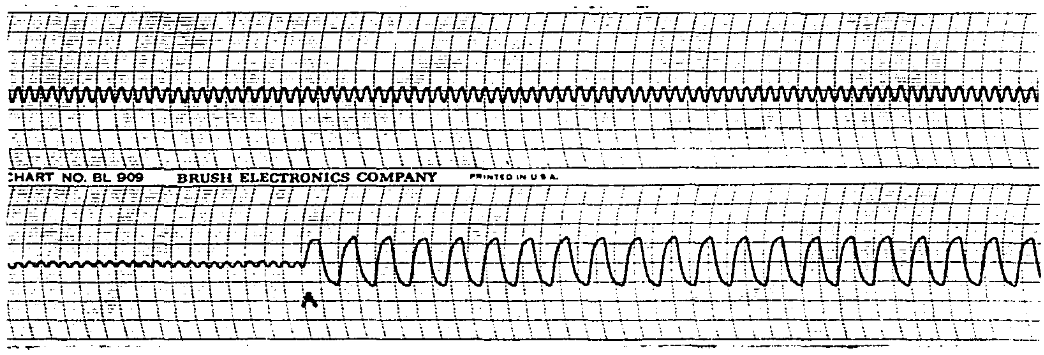


Fig. 1

Harmonic analysis has shown that the waveform of the oscillations over a relatively large range of frequency can be represented essentially by two components of a Fourier series. These experimental facts are some of the advance insight about the character of the subharmonic solution.

Let us consider a nonlinear system represented by the following equation

$$\ddot{\theta} + c\dot{\theta} + \alpha\theta + \beta g(\theta) = F \cos (n\omega t + \phi) \qquad (2.1)$$

It is clear that this equation can be expressed in the form of Eq. 1.8. Since we are interested only in the steady state oscillations the solution to Eq. 2.1 may be represented by a Fourier series

$$\theta = \sum_{k=0}^{\infty} A_k \cos k\omega t + \sum_{k=0}^{\infty} B_k \sin k\omega t$$

where $B_n = 0, \quad k = 0, 1, 2, 3, \text{---} n, \text{----}$

However, from what has been said about the waveform of the subharmonic we will assume that a two-term approximation is sufficient. In general the approximate solution will be

$$\theta = a_o + A \cos \omega t + B \sin \omega t + a_n \cos n\omega t \qquad (2.2)$$

The a_o is introduced to account for any asymmetry due to asymmetric restoring force. If we limit our analysis to systems with symetric forces the approximate solution for a subharmonic of order $1/n$ is

$$\theta = A \cos \omega t + B \sin \omega t + a_n \cos n\omega t \qquad (2.3)$$

The arbitrary variation of the displacement $\delta\theta$ is expressed as

$$\delta\theta = \delta A \cos \omega t + \delta B \sin \omega t + \delta a_n \cos n\omega t -$$

$$- (A \sin \omega t - B \cos \omega t + a_n n \sin n\omega t)\, \delta(\omega t) \qquad (2.4)$$

in which δA, δB, δa_n and $\delta\omega$ are arbitrary variations. Letting $\delta q = \delta\theta$ and applying Eq. 1.7, we find

$$(\omega^2 - \alpha)A - \frac{\beta}{\pi}\int_0^{2\pi} g(\theta) \cos\tau \, d\tau + c\omega B = 0 \qquad (2.5)$$

$$(\omega^2 - \alpha)B - \frac{\beta}{\pi}\int_0^{2\pi} g(\theta) \sin\tau \, d\tau - c\omega A = 0 \qquad (2.6)$$

$$\left[(n\omega)^2 - \alpha\right]a_n - \frac{\beta}{\pi}\int_0^{2\pi} g(\theta) \cos n\tau d\tau + F \cos \phi = 0 \qquad (2.7)$$

$$c\omega (a_1^2 + n^2 a_n^2) = a_n n F \sin \phi \qquad (2.8)$$

where $a_1^2 = A^2 + B^2$, and $\tau = \omega t$. The four conditions are necessary for the existence of the subharmonic oscillation of order $1/n$ appearing in the form given by Eq. 2.3. It is of interest to note that (2.7) and (2.8) reduce to the familiar conditions for linear forced vibrations if $\beta = 0$ and

$a_1 = 0$. The expression (2.8) is a relation between the energy supplied by the exciting force and the energy dissipated through viscous damping over a cycle of oscillation. From this relation we note that no pure subharmonic can exist in a damped system.

If we solve for a_1 (the amplitude of the subharmonic) in terms of a_n, we get

$$a_1 = \left\{ \frac{n\, a_n}{c\omega} \left(F \sin \phi - n a_n c\omega \right) \right\}^{1/2} \tag{2.9}$$

For real value of a_1 we must have

$$c \leq \left| \frac{F \sin \phi}{n \omega a_n} \right| \tag{2.10}$$

This inequality shows that the subharmonic of any order 1/n cannot occur unless the damping coefficient c is smaller than a quantity which depends upon certain parameters of the system. We note that c must be smaller for smaller amplitude F of the excitation and for higher order n of subharmonic. We shall see later that the nonlinear coefficient β is essential in determining a more specific restriction on the damping.

It is also worth noting that for $\beta = 0$, we can eliminate A and B from equations (2.5) and (2.6) and obtain

$$(\omega^2 - \alpha)^2 = -(c\omega)^2 \tag{2.11}$$

Since $(\omega^2 - \alpha)^2$ must be real, we may conclude that equations (2.5) and (2.6) are incompatible for $\beta = 0$. It follows then that subharmonics are not expected in a linear system with damping. This conclusion may also be obtained from (2.9) and (2.10) since the equality sign must hold for the case in which $\beta = 0$.

So far we have obtained some interesting general information without knowing what $g(\theta)$ is. In order to investigate the characteristics of subharmonic resonance in a system having a nonlinear symmetric restoring force $g(\theta)$, we will fix our attention to the problem of obtaining the relationship between the frequency and the amplitude of the subharmonic oscillation.

3. SUBHARMONIC OF ORDER 1/3 RD FOR $g(\theta) = \theta^3$

When the nonlinear restoring force $g(\theta)$ is given by $g(\theta) = \theta^3$, it is possible to excite subharmonic response in a system represented by Eq. 2.1. The type of subharmonic always observed in this system is of the order 1/3 i.e. the frequency is one-third that of the applied force. This system has been particularly investigated by Stoker[4] and Hayashi[5] but most of the reported results are not applicable in the absence of <u>linear</u> restoring force or when the nonlinear coefficient β becomes large. We shall first consider the effect of damping by analyzing the existence conditions described in section 2.

If we set n = 3 in the approximate solution (2.3) and $g(\theta) = \theta^3$ in (2.5), (2.6) and (2.7) we obtain

$$(\omega^2 - \alpha - (3/4)\beta a_1^2 - 3/2\beta a_3^2)A - Bc\omega = 3/4\beta a_3 (A^2 - B^2) \tag{3.1}$$

$$(\omega^2 - \alpha - (3/4)\beta a_1^2 - 3/2\beta a_3^2)B + Ac\omega = -3/2\beta a_3 AB \tag{3.2}$$

$$(9\omega^2 - \alpha)a_3 - \beta/4(A^3 + 3a_3^3 + 6A^2 a_3) -$$

$$- 3/2\beta(B^2 a_3 + 3/2AB^2) + F\cos\phi = 0 \tag{3.3}$$

$$c\omega(a_1^2 + 9a_3^2) = 3a_3 F\sin\phi \tag{3.4}$$

If we square both sides of (3.1) and (3.2) and add, we obtain

$$\omega^2 = \alpha + 3/4\beta(a_1^2 + 2a_3^2) \pm \left[(3/4\beta a_1 a_3)^2 - (c\omega)^2\right]^{1/2} \tag{3.5}$$

It is apparent that for real value of ω^2 we must have

$$c \leq \left| 3/4 \quad \frac{\beta\, a_1\, a_3}{\omega} \right| \tag{3.6}$$

This inequality indicates an upper limit on the value of c. It is interesting to note that if we now assume that β is small and the damping is also small in (3.1), (3.2), (3.3) and (3.4), we can approximate a_3 by

$$a_3 \simeq -F/8\alpha \tag{3.7}$$

Replacing a_3 in the inequality (3.6) we have

$$c \leq \left| \frac{3}{32} \quad \frac{\beta a_1 F}{\omega\alpha} \right| \tag{3.8}$$

which is the result obtained by Stoker[4] through an iteration procedure. This is a major result on the effect of viscous damping according to the small nonlinearity theory. It is encouraging to find our conditions reducible to the results of another method of investigation.

It is obvious that inequality (3.8) has no meaning if $\alpha = 0$. In order to consider the cases in which β is large or α is small or zero, we must avoid the approximation (3.7) Suppose we let c increase to the maximum value in (3.5). When this condition is reached we have

$$c = \frac{3}{4} \quad \frac{\beta a_1 a_3}{\omega} \tag{3.9}$$

and $$\omega^2 = \alpha + 3/4\ \beta(a_1^2 + 2a_3^2) \tag{3.10}$$

From these two relations we can solve for a_1

$$a_1^2 = 2/3\ \left(\frac{\omega^2 - \alpha}{\beta}\right) \pm 1/2\left[\frac{16}{9}\left(\frac{\omega^2 - \alpha}{\beta}\right)^2 - 8\left(\frac{4c\omega}{3\beta}\right)^2\right]^{1/2} \tag{3.11}$$

Here we see that the maximum c is

$$c = \frac{(\omega^2 - \alpha)}{2\sqrt{2}\,\omega} \tag{3.12}$$

for which Eq. (3.11) becomes

$$a_1^2 = 2/3\ \left(\frac{\omega^2 - \alpha}{\beta}\right) \tag{3.13}$$

Knowing a_1^2 we can find a_3 from Eq. (3.10).

Thus

$$a_3^2 = 1/3\left(\frac{\omega^2 - \alpha}{\beta}\right) \tag{3.14}$$

In view of (3.13) and (3.14) the ratio of a_1 to a_3 when the damping increases to the quantity expressed by (3.12), is $\pm\sqrt{2}$. Hence Eq. 3.9 becomes

$$c = \frac{3\ \ \beta a_1^2}{4\sqrt{2}\ \ \omega} \tag{3.15}$$

We can now eliminate a_1^2 from (3.15) by using Eq. (3.4) for

$a_1^2 = 2a_3^2$. Thus we have

$$c^3 = \frac{27}{242\sqrt{2}} \frac{\beta F^2}{\omega^3} \sin^2 \phi \tag{3.16}$$

from which we can write

$$c^3 \leq \left| \frac{27 \quad \beta F^2}{242 \sqrt{2} \quad \omega^3} \right| \tag{3.17}$$

This inequality shows that the subharmonic of order 1/3 cannot exist unless the damping is kept below a certain value which depends essentially on F, β and ω. The result does not incorporate the quantity α, but it should be remembered that the upper limit on c in Eq. 3.12 depends on α and ω. That is to say, for larger α we must have a larger ω for a given C. Eq.(3.12) may be interpreted as a lower limit of ω for given α and c. In this sense we write Eq. 3.12 as

$$\frac{(\omega^2 - \alpha)}{2\omega\sqrt{2}} \geq c \tag{3.18}$$

EXPERIMENTAL

We shall compare the theoretical restrictions (3.18) and (3.17) with some experiments performed with the aid of an electromechanical analog.[9] Fig. 2 illustrates the lower

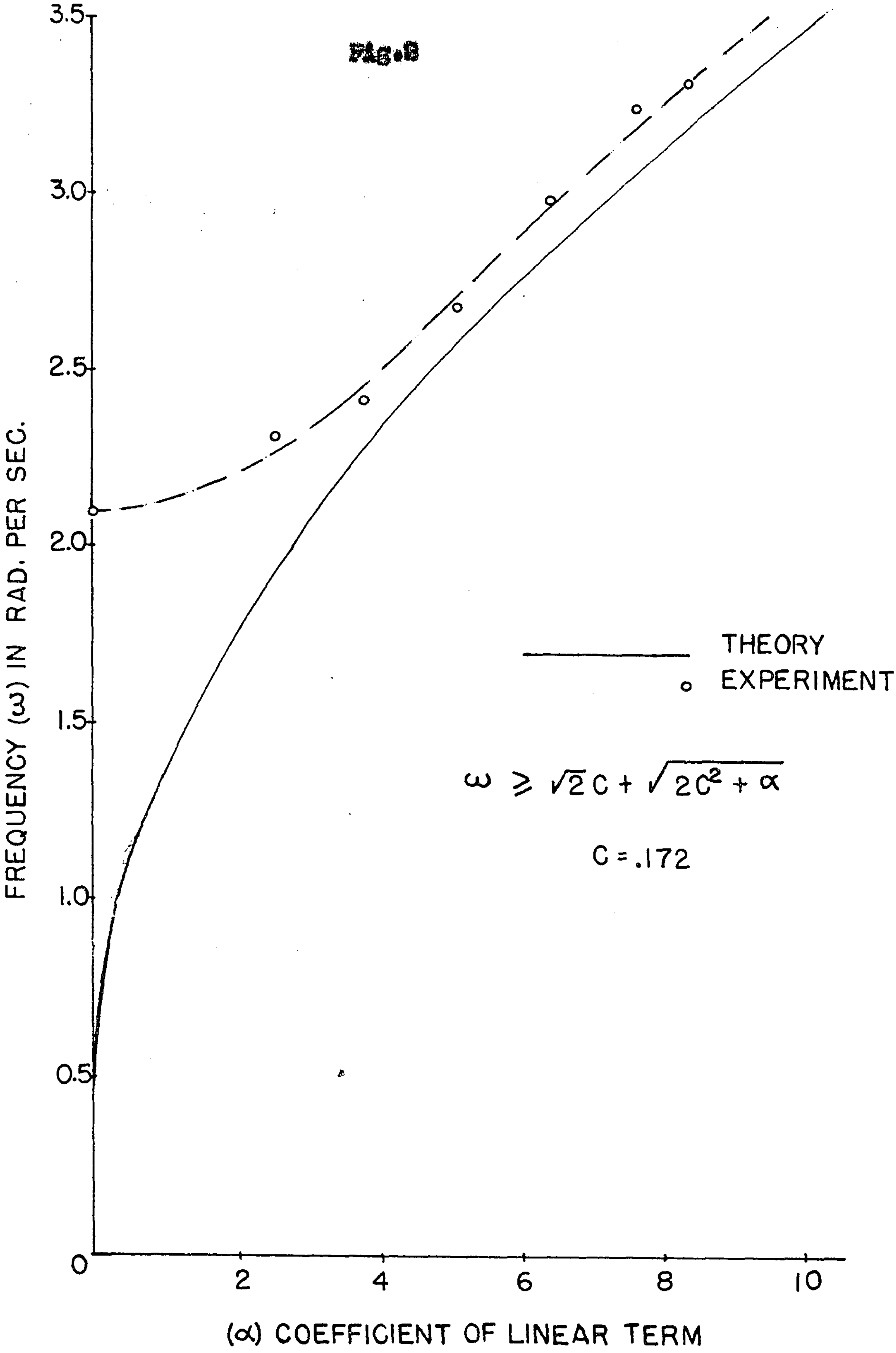
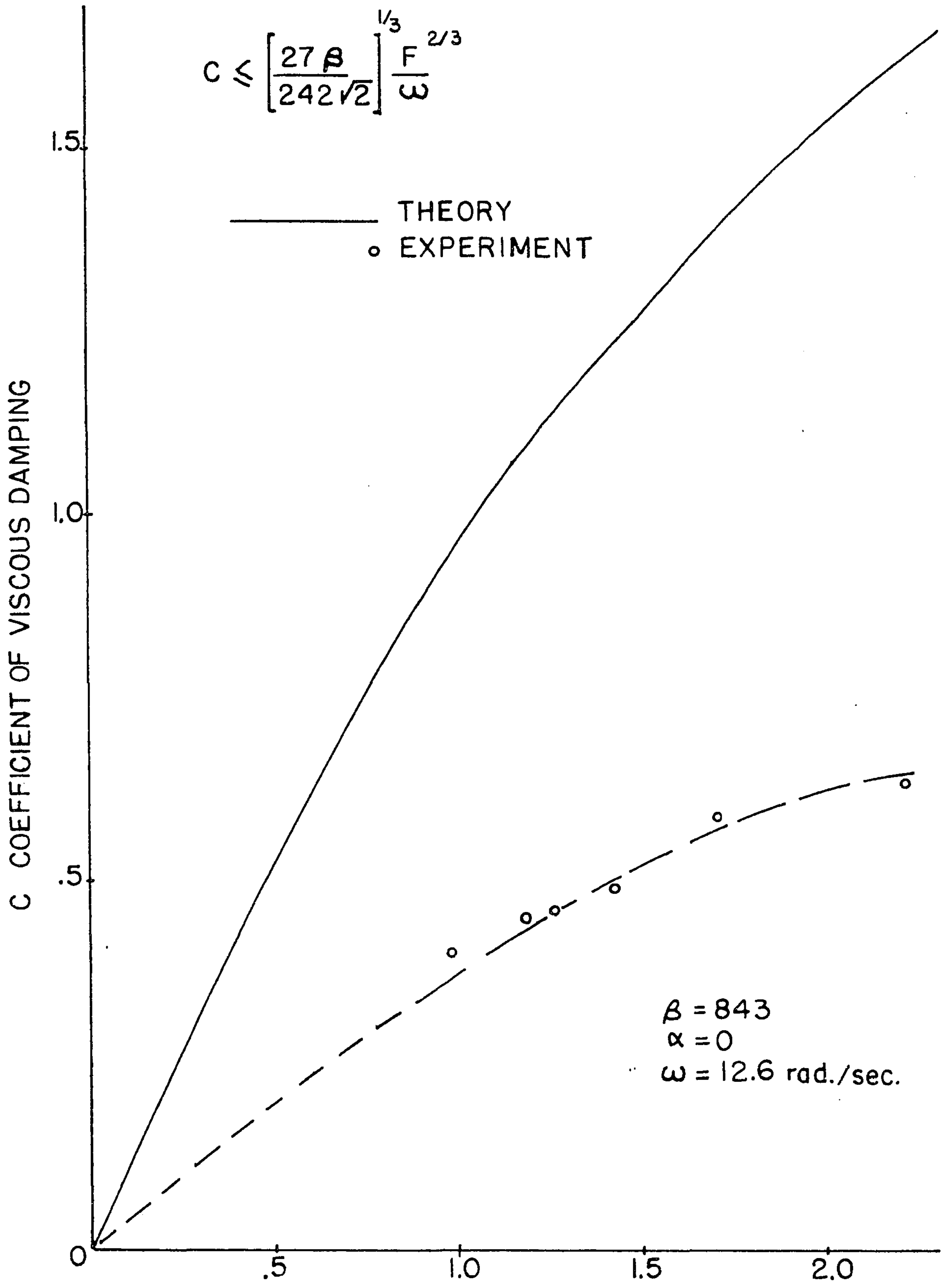


FIG. 8
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
FREQUENCY (ω) IN RAD. PER SEC.
2
4
6
8
10
(α) COEFFICIENT OF LINEAR TERM
THEORY
EXPERIMENT
$\omega \geq \sqrt{2}C + \sqrt{2C^2 + \alpha}$
C = .172

Fig. 3



$C \leq \left[\frac{27\beta}{242\sqrt{2}}\right]^{1/3} \frac{F}{\omega}^{2/3}$
THEORY
EXPERIMENT
$\beta = 843$
$\alpha = 0$
$\omega = 12.6$ rad./sec.
C COEFFICIENT OF VISCOUS DAMPING
AMPLITUDE (F) OF THE EXCITATION
0
.5
1.0
1.5
2.0

limit on the frequency ω for a given c as a function of α. For very small value of α, a significant deviation from the theoretical limit appears. This may be due to the effect of small amount of dry friction in the experimental system since the viscous damping becomes less effective as the velocity decreases. Fig. 3 shows that for a particular value of β, ω, and $\alpha = 0$, there is a limiting relationship between c and F. A comparison of the results with the restriction given by (3.17) indicates that the theoretical limit is a little over two times as high as indicated by experiment. The restriction (3.17) seems to show the right trend. The essential point of these comparisons is that the limiting conditions are applicable to cases with the absence of linear restoring force which the small nonlinearity theory failed to explain.

FREQUENCY - SUBHARMONIC AMPLITUDE RELATION

We now turn to the problem of obtaining a frequency-amplitude relationship for subharmonic resonance of order 1/3rd. It should be mentioned that the problem of eliminating a_3 from Eq. 3.5 presents great difficulties of algebraic nature. For simplification we shall neglect the damping in the system. For $c = 0$ we need to consider only Eq. 3.2

and Eq. 3.5 in the following form

$$\omega^2 - \alpha - 3/4\beta\,(a_1^2 + a_1 a_3 + a_3^2) = 0 \tag{3.19}$$

and

$$(9\omega^2 - \alpha)a_3 - \beta/4(a_1^3 + 3a_3^3 + 6a_1^2 a_3) + F = 0 \tag{3.20}$$

Since we are interested in the condition in which a_1 is large compared to a_3, we shall neglect the a_3 and a_3^2 terms in (3.19). and obtain as a first order approximation for a_1

$$\omega^2 \simeq \alpha + (3/4)\beta a_1^2 \tag{3.21}$$

Now neglecting the a_3^3 term in (3.20) and using (3.21) we get

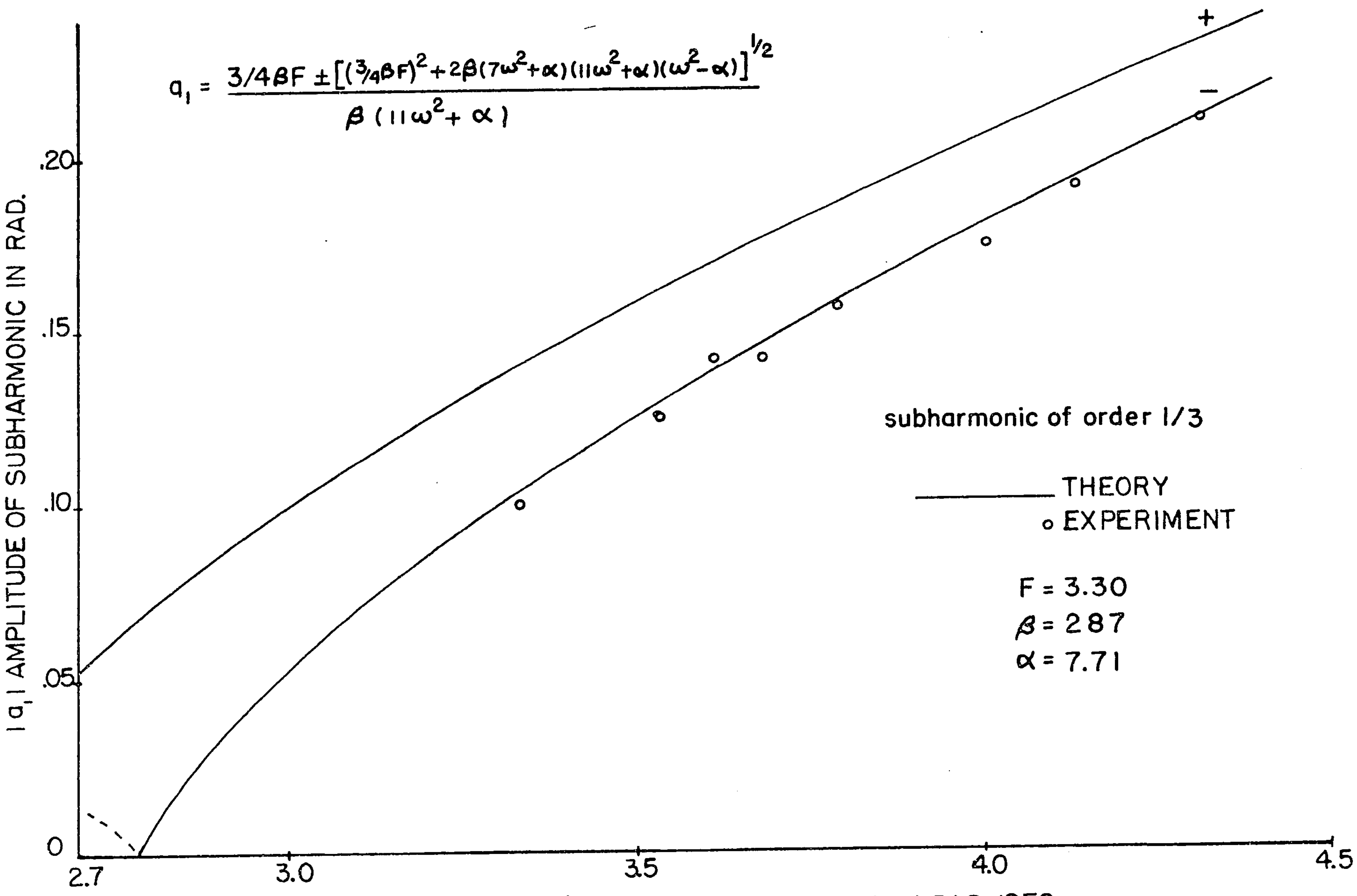
$$a_3 \simeq \frac{(\omega^2 - \alpha)\, a_1 - 3F}{3\,(7\omega^2 + \alpha)} \tag{3.22}$$

For the next order approximation we neglect only the a_3^2 term in (3.19) and using (3.22) we obtain

$$\tfrac{1}{2}\beta(11\omega^2 + \alpha)a_1^2 - \tfrac{3}{4}\beta F a_1 - (\omega^2 - \alpha)(7\omega^2 + \alpha) = 0 \tag{3.23}$$

Solving for a_1, we find

$$a_1 = \frac{\frac{3}{4}\beta F \pm \left[(\frac{3}{4}\beta F)^2 + 2\beta(\omega^2 - \alpha)(7\omega^2 + \alpha)(11\omega^2 + \alpha)\right]^{1/2}}{\beta(11\omega^2 + \alpha)} \tag{3.24}$$



$$q_1 = \frac{3/4\beta F \pm \left[(3/4\beta F)^2 + 2\beta(7\omega^2+\alpha)(11\omega^2+\alpha)(\omega^2-\alpha)\right]^{1/2}}{\beta(11\omega^2+\alpha)}$$
subharmonic of order 1/3
THEORY
EXPERIMENT
F = 3.30
β = 287
α = 7.71
+
−
|q_1| AMPLITUDE OF SUBHARMONIC IN RAD.
.20
.15
.10
.05
0
2.7
3.0
3.5
4.0
4.5
FREQUENCY (ω) OF THE SUBHARMONIC IN RAD./SEC.

Fig. 4

Here we see there are two possible values of a_1. The interpretation of this result is that there may be two possible waveforms for the subharmonic, one in which a_1 and a_3 have the same sign and the other case in which a_1 and a_3 have the opposite sign. It is not apparent from our development which sign is to be taken. If the negative sign is assumed, we see that the magnitude of F might not have too much effect on the size of a_1.

EXPERIMENTAL

Fig. 4 shows that a_1 increases in magnitude with increasing frequency ω.(It should be remembered that the frequency of the exciting force is 3ω in this case). The experimental values agree with the lower curve which corresponds to (3.24) with the negative sign. The experimental points denote the peak amplitude of the oscillations which are predominantly subharmonic. The inherent amount of viscous damping in the experimental system is given by c = .172 which corresponds to the damping factor b = .024. About twelve complete oscillations were observed before the motion was damped out completely.

Although the system contains some damping the experimental data is in good agreement with the theory. Since damping must be small in order to maintain the subharmonic over a wide range of frequency, Eq. 3.24 might be sufficiently accurate to describe the frequency - amplitude response for the subharmonic of order 1/3rd. in this system with or without damping.

POSSIBLE EXISTENCE OF OTHER ORDER SUBHARMONICS.

As mentioned at the beginning of our discussion concerning the subharmonics in the system having the nonlinear characteristic $g(\theta) = \theta^3$, the subharmonic of order 1/3rd. is the only type observed for θ^3 restoring force. The question now arises as to the possibility of exciting the subharmonic of order 1/2 or other orders in this system. The higher order type might not be expected in view of what has been said about the damping for higher n. We shall consider the possible existence of the order 1/2. If we set n = 2 in the approximate solution (2.3) and apply it to (2.5) and (2.6) for $g(\theta) = \theta^3$, we find

$$(\omega^2 - \alpha - \tfrac{3}{4}\beta a_1^2 - \tfrac{3}{2}\beta a_2^2)A + Bc\omega = 0 \qquad (3.25)$$

$$(\omega^2 - \alpha - \tfrac{3}{4}\beta a_1^2 - \tfrac{3}{2}\beta a_2^2)B - Ac\omega = 0 \qquad (3.26)$$

Eliminating A and B from these two equations we have

$$(\omega^2 - \alpha - \tfrac{3}{4}\beta a_1^2 - \tfrac{3}{2}\beta a_2^2)^2 = -(c\omega)^2 \qquad (3.27)$$

Since the quantity in parenthesis on the left side must be real, Eq. 3.27 cannot exist unless the damping is absent. In view of this result, we do not expect a subharmonic of order 1/2 to be excited in the experimental system. It is concluded that the subharmonic of order 1/3rd would be the type most likely to occur if $g(\theta) = \theta^3$. The latter remark is in line with experimental observation.

4. SUBHARMONICS IN A SYSTEM WITH SYMMETRIC RESTORING FORCE PROPORTIONAL TO θ^2

Since we are interested in how the order of subharmonics is related to the form of the nonlinear characteristics, we choose to study the system in which $g(\theta) = |\theta|\,\theta$. From a reported experiment[6] in which transition of subharmonics was observed, it is suspected that the symmetric restoring force proportional to θ^2 is essential in sustaining a subharmonic of order 1/2.

In order to apply our method of analysis to a system having a nonlinear characteristic $g(\theta) = |\theta|\,\theta$, we must have some advance insight about the waveform of $\theta(t)$ because the $g(\theta) = \theta^2$ changes sign with $\theta(t)$. Since we do not know exactly where $\theta(t)$ changes sign over a whole cycle, the evaluation of the integral in (2.5), (2.6) and (2.7) seems to be a very difficult problem. However, if we assume that the damping is small and the oscillations will be predominantly subharmonic, it might be satisfactory if we evaluate the integral involving $g(\theta) = |\theta|\theta$ as follows

$$\int_0^{2\pi} |\theta|\theta\, K(\tau)d\tau = \int_0^{\pi/2} |\theta|\theta\, K(\tau)d\tau + \int_{3\pi/2}^{2\pi} |\theta|\theta\, K(\tau)d\tau - \int_{\pi/2}^{3\pi/2} |\theta|\theta\, K(\tau)d\tau \qquad (4.1)$$

where $K(\tau)$ is $\cos\tau$, or $\sin\tau$, or $\cos n\tau$.

Let us set $n = 2$ and determine the conditions for the existence of subharmonic of order 1/2 in the following system

$$\ddot{\theta} + c\dot{\theta} + \beta|\theta|\theta = F\cos(2\omega t + \phi) \qquad (4.2)$$

From (2.5), (2.6), (2.7), and (2.8), we obtain

$$(\omega^2 - \alpha - \frac{8\beta}{3\pi}A)A - \frac{4\beta}{3\pi}B^2 - \frac{28}{15\pi}\beta a_2^2 - c\omega B = 0 \qquad (4.3)$$

$$(\omega^2 - \alpha - \frac{8\beta}{3\pi}A)B - c\omega A = 0 \qquad (4.4)$$

$$(4\omega^2 - \alpha - \frac{56\beta}{15\pi}A)a_2 + F\cos\phi = 0 \qquad (4.5)$$

$$c\omega(a_1^2 + 4a_2^2) - 2a_2 F\sin\phi = 0 \qquad (4.6)$$

From these conditions we shall attempt to formulate a relation between the frequency and the subharmonic amplitude for the case in which damping is neglected. For $c = 0$, we need to consider only (4.3) and (4.5) which reduce to

$$(\omega^2 - \alpha - \frac{8\beta}{3\pi}a_1)a_1 - \frac{28\beta}{15\pi}a_2^2 = 0 \qquad (4.7)$$

$$(4\omega^2 - \alpha - \frac{56\beta}{15\pi}a_1)a_2 - F = 0 \qquad (4.8)$$

Assuming a_2 is small compared to a_1 we may write

$$\omega^2 - \alpha \simeq \frac{8\beta}{3\pi}a_1 \qquad (4.9)$$

Replacing a_1 in (4.8) with the help of (4.9) we have

$$a_2^2 = \frac{25F^2}{(13\omega^2 + 2\alpha)^2} \qquad (4.10)$$

Substituting the above expression for a_2^2 in (4.7) we get

$$a_1^2 - \frac{3\pi}{8\beta}(\omega^2 - \alpha)a_1 + \frac{35F^2}{2(13\omega^2 + 2\alpha)^2} = 0 \qquad (4.11)$$

The solution of (4.11) is

$$a_1 = \frac{3\pi}{16\beta}(\omega^2 - \alpha) \pm$$

$$\pm \frac{1}{2}\left[\left(\frac{3\pi}{8\beta}(\omega^2 - \alpha)\right)^2 - \frac{70F^2}{(13\omega^2 + 2\alpha)^2}\right]^{1/2} \qquad (4.12)$$

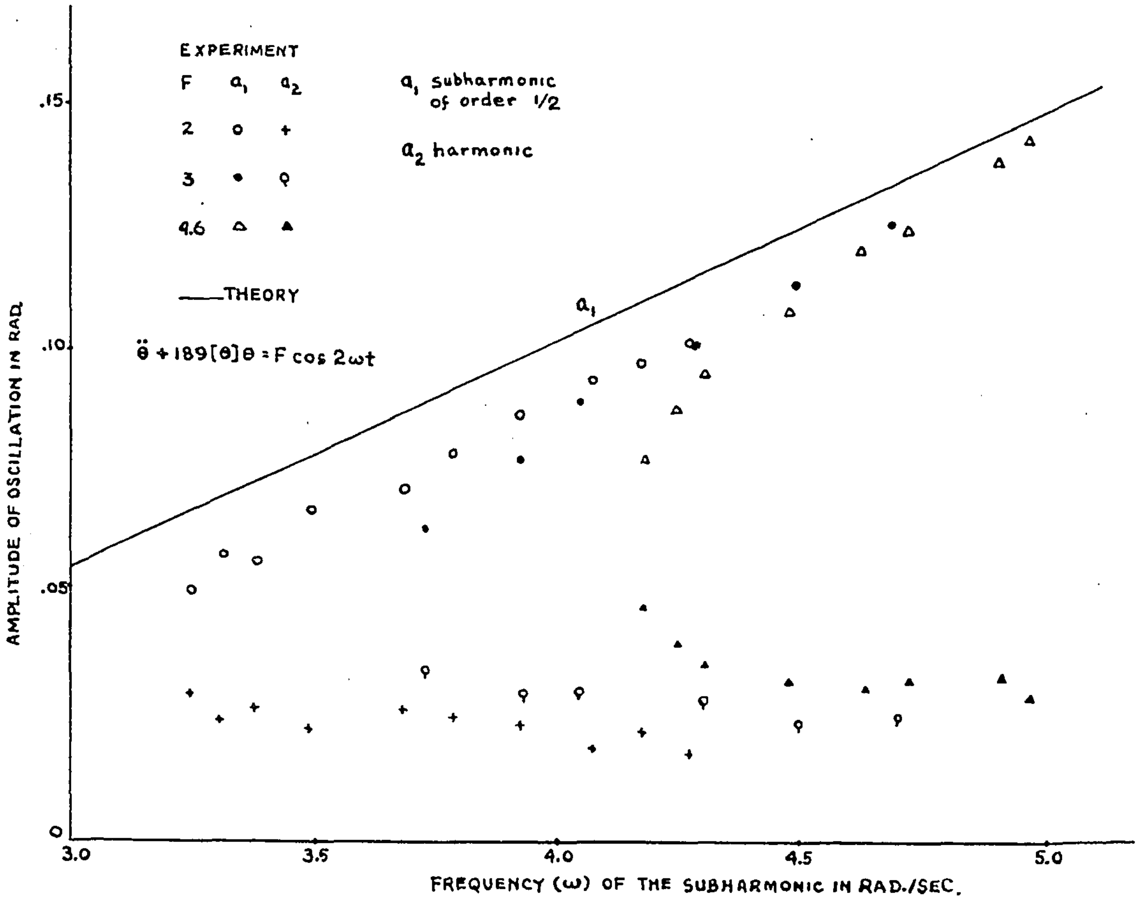
Since a_1 must be real, we must specify

$$F \leq \frac{3\pi}{13\beta}\left(\frac{4}{70}\right)^{1/2}(\omega^2 - \alpha)(13\omega^2 + 2\alpha) \tag{4.13}$$

Thus for larger amplitude F of the excitation, the frequency ω of the subharmonic of order 1/2 would have to be higher. In view of the approximation (4.9) we should take the plus sign in (4.12).

EXPERIMENTAL

If the parameters of the system are set at certain values the subharmonic of order 1/2 can be excited within a certain range of the frequency which depends upon the amplitude F of the excitation. The amplitude of oscillations increases with the frequency. The agreement of Eq. 4.9 with the experimental results is shown by Fig. 5. The lower curves correspond to the harmonic component given by Eq. 4.10. For larger F the range of frequency in which the subharmonic of order 1/2 can be maintained is shifted to the region of higher frequency as implied by (4.13). These results were obtained for $\alpha = 0$ in which case the subharmonic was easier to initiate than in the case when $\alpha > 0$.



EXPERIMENT
F a1 a2
2 o +
3 • ♀
4.6 Δ ▲
——THEORY
$\ddot{\theta} + 189[\theta]\theta = F \cos 2\omega t$
a1 subharmonic of order 1/2
a2 harmonic
a1
AMPLITUDE OF OSCILLATION IN RAD.
.15
.10
.05
0
3.0
3.5
4.0
4.5
5.0
FREQUENCY (ω) OF THE SUBHARMONIC IN RAD./SEC.

Fig. 5

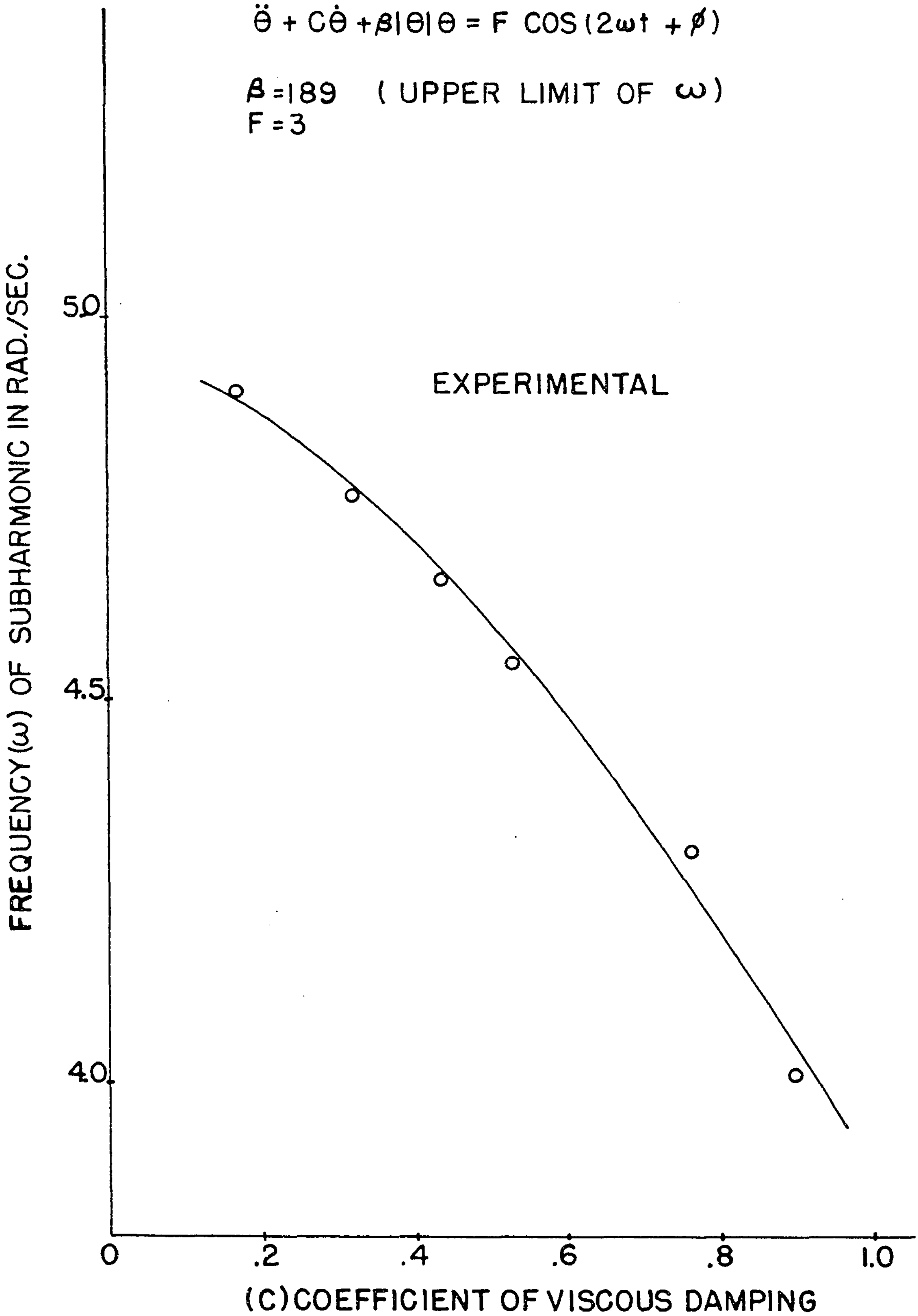
In this experiment a rather interesting phenomenon was observed; as the frequency increased, the amplitude of the subharmonic of order 1/2 increased up to a certain value at which a transition from a subharmonic of 1/2 to that of order 1/3 occurred. This result shows that subharmonic oscillation of order 1/n does occur when the highest degree of power of the nonlinear terms in $g(\theta)$ is less than n. According to previous investigation(5) this occurrence was not to be expected. The fact that transition takes place when there is only one type of nonlinear characteristic present may be helpful in understanding this interesting behavior which so far has received no explanation.

So far we have the following experimental data about the transition. If the damping is too great or the linearity (α) is too large, the transition does not occur. Table I indicates for a given F the approximate amplitudes at which the subharmonic of order 1/2 begins to decay into other forms of oscillations for various values of α.

TABLE I

$F = 3.00$, $c = 0.17$	(α)	(ω) rad/sec.	approx. a_1 (rad)
	0	4.91	.140
	3.22	4.90	.170
	7.23	4.93	.105
	9.67	4.92	.081
	12.90	4.94	.068
	16.10	4.90	.035

Fig. 6


$\ddot{\theta} + C\dot{\theta} + \beta|\theta|\theta = F \cos(2\omega t + \phi)$
$\beta = 189$ (UPPER LIMIT OF ω)
$F = 3$
EXPERIMENTAL
FREQUENCY (ω) OF SUBHARMONIC IN RAD./SEC.
5.0
4.5
4.0
0
.2
.4
.6
.8
1.0
(C) COEFFICIENT OF VISCOUS DAMPING

It is interesting to note that the occurrence starts at approximately the same frequency independent of the values of α. The effect of increasing α is in decreasing the amplitude. For $\alpha = 0$ to about $\alpha = 3.22$, the transition from the order of 1/2 to that of 1/3 was observed provided that the damping is kept to a minimum. It is supposed that the subharmonic of order 1/2 is interrupted by the viscous damping which increases with the frequency. Fig. 6 shows the relation between the damping (c) and the approximate frequency at which the subharmonic of order 1/2 decays into harmonic oscillations. The transition to a subharmonic of order 1/3 seemed possible only for small c. So far we have not been successful in formulating any specific theoretical condition to predict the transition.

It should be mentioned that if the damping is sufficiently low, the subharmonic of order 1/3 can be excited very easily in this system for $\alpha = 0$. With larger α its occurrence seemed less likely. The forcing frequency has to be higher in order to generate this subharmonic for larger α.

THEORETICAL CONSIDERATION OF SUBHARMONIC OF ORDER 1/3 IN $g(\theta) = |\theta|\,\theta$ SYSTEM

In order to provide some theoretical consideration for the subharmonic of order 1/3 in the system represented by

Eq. 4.2, we set n = 3 in (2.5), (2.6) (2.7) and (2.8) and obtain

$$(\omega^2-\alpha-\frac{8\beta}{3\pi}A-\frac{16\beta}{15\pi}a_3)A-c\omega B+\frac{4\beta}{5\pi}a_3^2-\frac{4\beta}{3\pi}B^2=0 \qquad (4.14)$$

$$(\omega^2-\alpha-\frac{8\beta A}{3\pi}-\frac{56}{15\pi}\beta a_3)B+c\omega A=0 \qquad (4.15)$$

$$(9\omega^2-\alpha)a_3+\frac{32\beta}{15\pi}A^2-\frac{28\beta}{15\pi}B^2+\frac{8\beta}{9\pi}a_3^2+\frac{8\beta A}{5\pi}a_3$$

$$-F\cos\phi=0 \qquad (4.16)$$

$$(a_1^2+9a_3^2)c\omega-3a_3F\sin\phi=0 \qquad (4.17)$$

For simplification, we shall assume as before that a_3 is small compared to a_1 and the damping can be neglected. From (4.14) and (4.16) we get approximately

$$\omega^2-\alpha-\frac{8\beta a_1}{3\pi}-\frac{56}{15\pi}\beta a_3=0 \qquad (4.18)$$

and

$$(9\omega^2-\alpha)a_3-\frac{32\beta}{15\pi}a_1^2+\frac{8\beta a_1a_3}{5\pi}+F=0 \qquad (4.19)$$

If $\frac{7}{5}\frac{a_3}{a_1}\ll 1$, we further simplify (4.18) by writing

$$\frac{8\beta a_1}{3\pi}\simeq(\omega^2-\alpha) \qquad (4.20)$$

In view of (4.20), Eq. 4.19 becomes

$$a_3 = 5\left[\frac{9\pi^2\,(\omega^2-\alpha)^2 - 64\beta^2 F}{64\beta^2\,(48\,\omega^2 - 2\,\alpha\,)}\right] \tag{4.21}$$

Substituting this result in (4.18), we get the second order approximation for a_1 as

$$a_1 = \frac{3\pi\,(\omega^2-\alpha)}{8\beta} - \left[\frac{9\pi^2(\omega^2-\alpha)^2 - 64\beta^2 F}{32\beta^2\,(48\,\omega^2 - 2\alpha)}\right] \tag{4.22}$$

EXPERIMENTAL

As shown in Fig. 7, Eq. 4.22 is in fairly good agreement with experiment. The approximation seems to be better for higher frequency. This should be expected since a_3 decreases with ω while a_1 increases with ω as shown in (4.21) and (4.22). The experimental results shown are for $\alpha = 0$.

Fig. 8, illustrates the characteristics of the subharmonics as a function of the frequency of the applied force The vertical dash line shows how the subharmonic of order 1/2 reaches a certain frequency and jumps to the lower curve representing a subharmonic of order 1/3. As the frequency increases, the oscillations follow this characteristic with

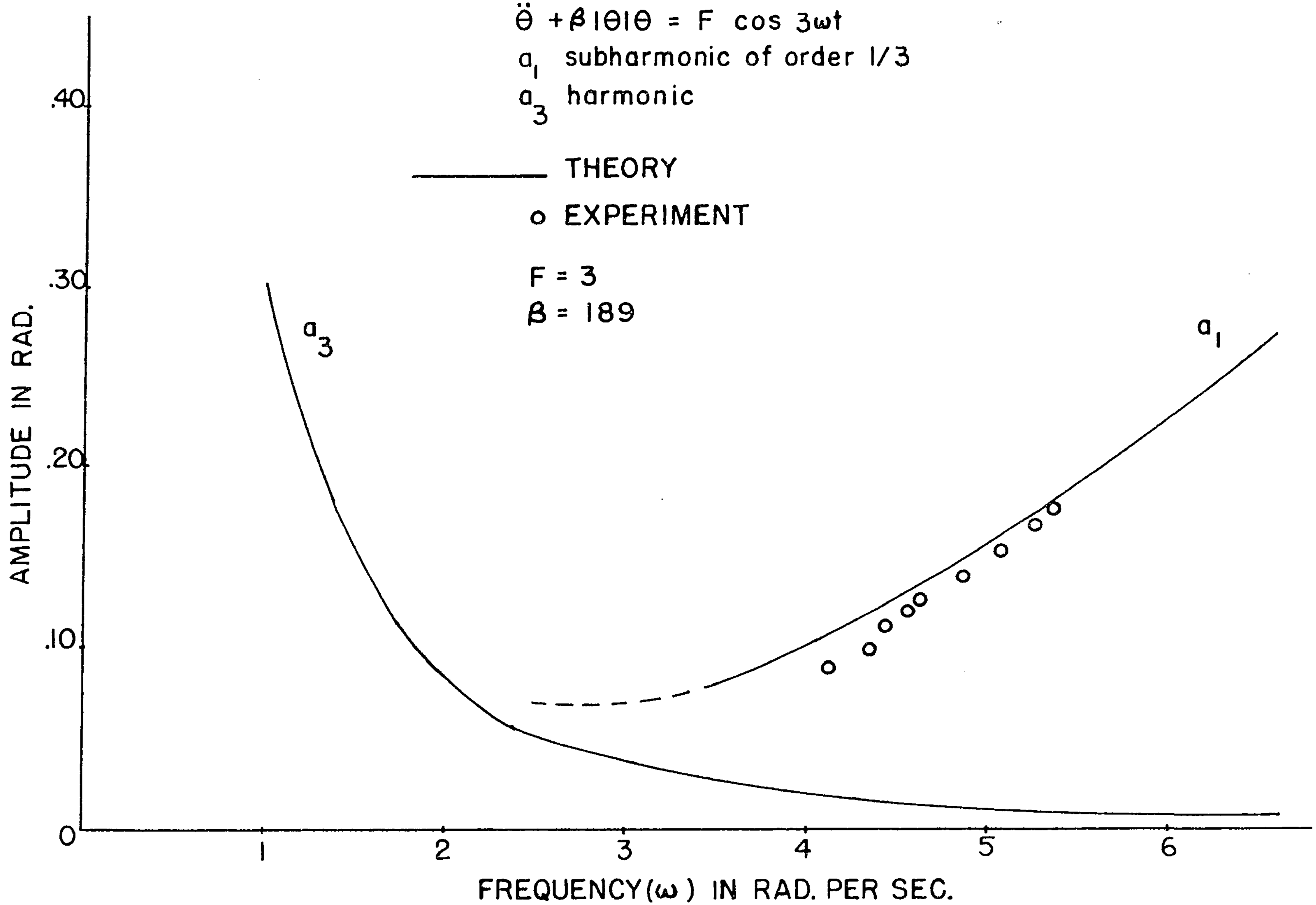
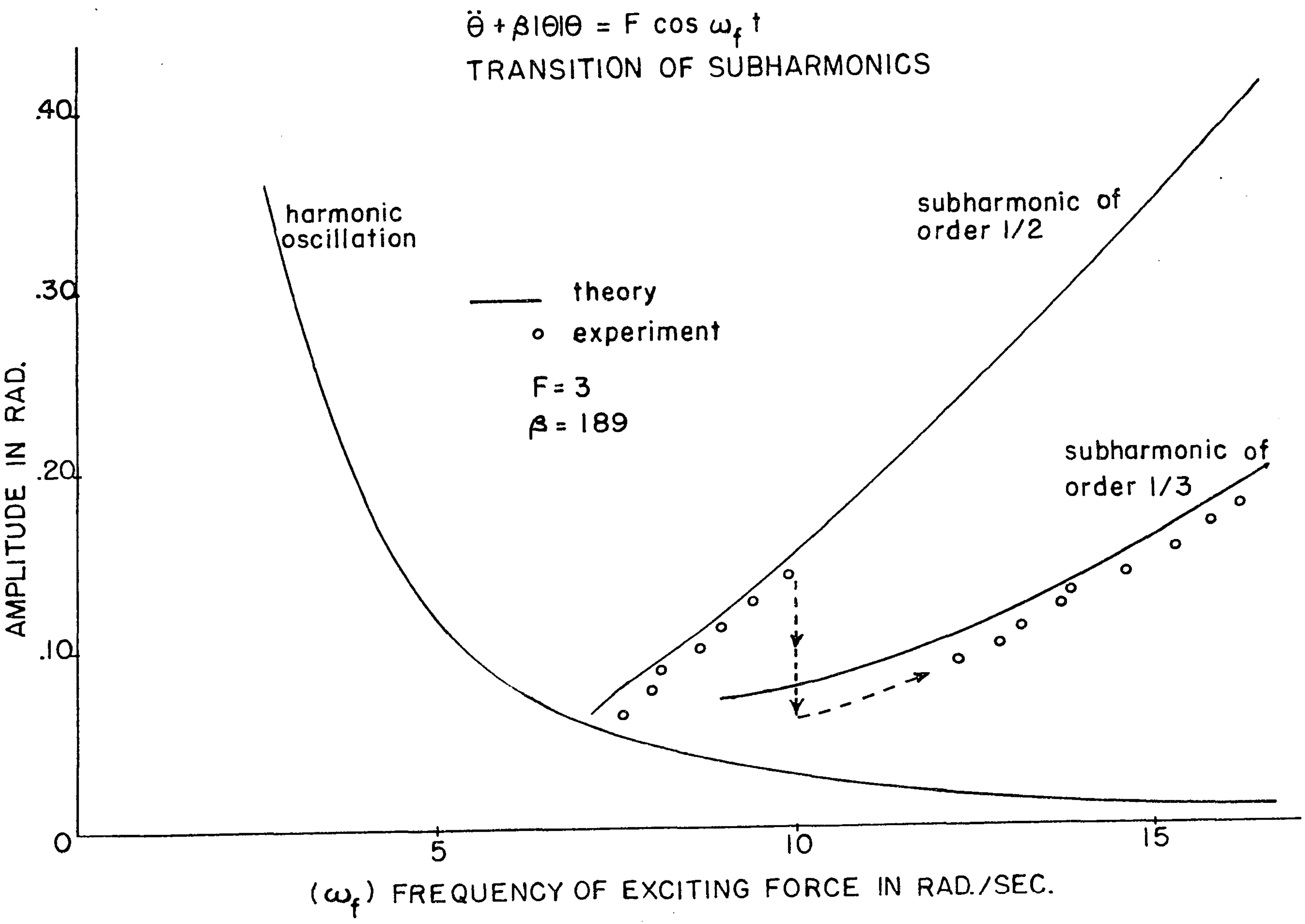

$\ddot{\theta} + \beta|\theta|\theta = F \cos 3\omega t$
a_1 subharmonic of order 1/3
a_3 harmonic
THEORY
o EXPERIMENT
F = 3
β = 189
a_1
a_3
AMPLITUDE IN RAD.
.40
.30
.20
.10
0
1
2
3
4
5
6
FREQUENCY(ω) IN RAD. PER SEC.

Fig. 7



$\ddot{\theta} + \beta|\theta|\theta = F \cos \omega_f t$
TRANSITION OF SUBHARMONICS
harmonic oscillation
subharmonic of order 1/2
subharmonic of order 1/3
theory
experiment
F = 3
β = 189
AMPLITUDE IN RAD.
.40
.30
.20
.10
0
5
10
15
(ω_f) FREQUENCY OF EXCITING FORCE IN RAD./SEC.

Fig. 8

a frequency 1/3 that of the exciting force.

5. SUBHARMONICS IN SYSTEMS WITH NONLINEAR CHARACTERISTICS $g(\theta) = |\theta^3|\theta$

The next logical step in this investigation was to study the subharmonic oscillations in connection with the form of nonlinearity expressed by $|\theta^3|\theta$. At this point there was a problem of procedure in getting some information in the simplest way. In view of the fact that the theoretical treatment becomes more complicated for higher degree nonlinear term, it was decided that the problem should be considered experimentally first.

Although the term θ^2 and θ^3 were absent from the non-linear characteristic $g(\theta)$ in this case, the subharmonic of order 1/3 and that of 1/2 were observed. The higher order such as 1/4 could not be detected. Perhaps one of the reasons for its absence is that the damping in the experimental system was too great in view of what has been said in section 2.

It was interesting to note that if the amplitude F of the excitation is sufficiently high, the damping is kept to a minimum and α is small compared to β, the subharmonic of order 1/3 was very dominant in this system. The subharmonic of order 1/2 could not be excited unless α was increased to suppress the one of order 1/3. In this respect the range of occurrence for the subharmonic of order 1/2 was very much constricted for smaller α.

SOME THEORETICAL CONSIDERATION

For simplicity we shall consider the following system with very little damping in it

$$\ddot{\theta} + \alpha\theta + \beta\,|\theta^3|\,\theta = F\cos 3\omega t \tag{5.1}$$

If we set $n = 3$ and $g(\theta) = |\theta^3|\,\theta$ in (2.5) and (2.7), we obtain after neglecting the higher order in a_3

$$a_1(\omega^2 - \alpha) - \frac{32}{15\pi}\beta a_1^4 - \frac{128}{35\pi}\beta a_1^3 a_3 \simeq 0 \tag{5.2}$$

and

$$a_3(9\omega^2 - \alpha) - \frac{32}{35\pi}\beta a_1^4 - \frac{1408}{105\pi} a_1^3 a_3 - F \simeq 0 \tag{5.3}$$

We can write (5.2) as

$$\omega^2 - \alpha = \frac{32\beta}{15\pi} a_1^3 \left(1 + \frac{12a_3}{7a_1}\right) \tag{5.4}$$

Assuming $a_3/a_1 \ll 1$, we have

$$\omega^2 - \alpha \simeq \frac{32}{15\pi}\beta a_1^3 \tag{5.5}$$

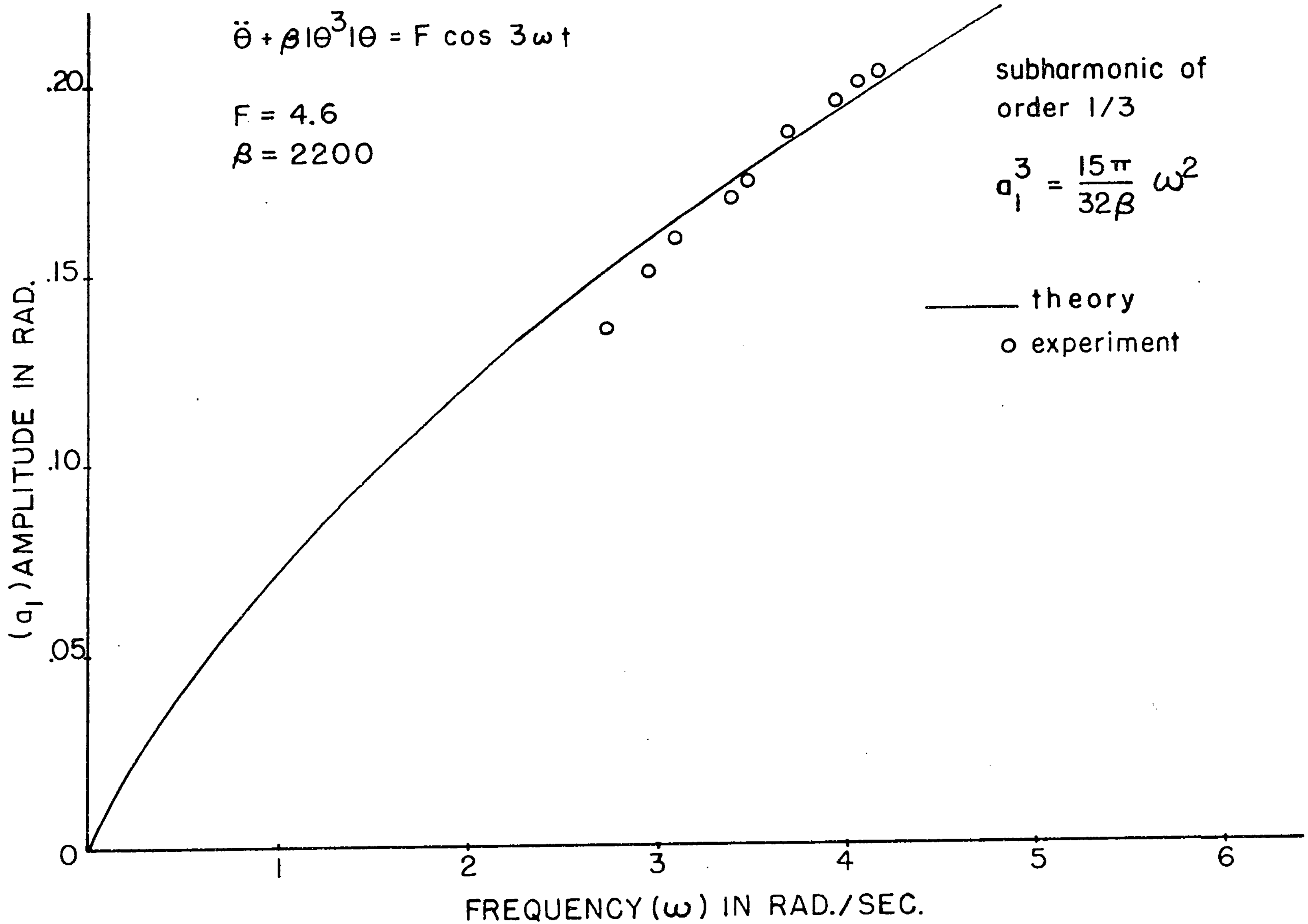

$\ddot{\theta} + \beta|\theta^3|\theta = F \cos 3\omega t$
$F = 4.6$
$\beta = 2200$
subharmonic of order 1/3
$a_1^3 = \frac{15\pi}{32\beta} \omega^2$
theory
o experiment
(a_1) AMPLITUDE IN RAD.
.20
.15
.10
.05
0
1
2
3
4
5
6
FREQUENCY (ω) IN RAD./SEC.

Fig. 9

and therefore we may express a_3 as

$$a_3 \simeq \frac{\frac{32\beta}{5\pi} a_1^4 - 7F}{51\omega^2 + 5\alpha} \tag{5.6}$$

It is evident that the next order approximation will be too complicated for analysis. It is supposed that Eq. 5.5 will be sufficiently accurate to indicate the frequency-amplitude response for the subharmonic of order 1/3 in this system. Fig. 9 indicates the agreement of (5.5) with the experimental results. Again as we would expect the approximation is better in the higher frequency range in which the a_1 is large compared to a_3.

The theoretical consideration for the subharmonic of order 1/2 can be carried out in the same way as in the above analysis. The results can be stated as follows for $n = 2, \quad c \simeq 0, \quad a_2/a_1 \ll 1$

$$(\omega^2 - \alpha) \simeq \frac{32\beta}{15\pi} a_1^3 \tag{5.7}$$

and

$$a_2 \simeq \frac{7F}{26\,\omega^2 - 47\alpha} \tag{5.8}$$

where $2\omega = \omega_f =$ frequency of the forcing function.

The difference between (5.5) and (5.7) is that in the former, $3\omega = \omega_f$ and in the latter, $2\omega = \omega_f$. The interesting point to be raised in view of all the first order approximations we have been making is that all these first approximations are the frequency formulae for the free non-linear oscillation of the system being disturbed by a periodic force n times this frequency.

CONCLUSIONS

I) The principal aim of the theoretical investigation was to formulate more general conditions for the existence of subharmonic oscillations in nonlinear systems. Special attention was given to systems with symmetric nonlinear restoring forces mainly because some experimental information about the subharmonic in these systems has been reported and was in need of theoretical consideration.

A variational procedure of studying the subharmonic solution to the equation of motion was developed according to the idea of Ritz[8]. The application of this variational method led to four general conditions for an assumed approximate solution. The method seems to be applicable with any type of restoring force without restriction on the magnitude of the nonlinearity. This means that the development does not have to be in the neighborhood of the linear forced vibration as it is in the treatment by other methods(perturbation or iteration).

Some interesting general information can readily be extracted from the general conditions. It is shown that subharmonics cannot exist in a linear system with damping. The inequality (2.10) indicates that subharmonic oscillations of any order 1/n cannot occur unless the damping coefficient c is smaller than a quantity which depends upon the amplitude F of the excitation and the order ($1/n$) of the subharmonic. Since the harmonic component(a_n) decreases slowly with the frequency, it seems likely that the damping coefficient c would have to be taken smaller in order to sustain the subharmonic of order $1/n$ as n becomes larger.

2) From the analysis of the subharmonic of order 1/3 in the system having (θ^3) type of restoring force, it is shown that certain important results obtained by the iteration procedure of Stoker[4] are contained in the results of our variational method. Further consideration leads to restrictions on the damping and frequency for predominantly non-linear cases which other methods failed to explain. The restrictions are in agreement with experimental evidence. It is also shown that a relatively simple frequency - amplitude relation can be obtained for the predominantly subharmonic oscillation in the absence of damping. Since the damping must be small in order to maintain the subharmonic over a wide range of frequency, it is reasonable to conclude that this relation(3.24) is sufficiently accurate to describe the frequency amplitude response even though some damping is present. Further analysis of this system shows that subharmonic of order 1/2 is not expected to occur. In view of this result and the damping restriction

(2.10), it is concluded that the subharmonic of order 1/3 would be the one most likely to occur. The latter remark is in line with experimental observation.

5) In order to make further study of the connection between the order of the subharmonic and the form of the nonlinear restoring force, the systems having nonlinear characteristics $|\theta|\,\theta$ and $|\theta^3|\,\theta$ were investigated separately. In the $|\theta|\,\theta$ system the subharmonic of order 1/2 and that of order 1/3 can be excited experimentally. In particular, a spontaneous transition of subharmonics can be observed when the system is predominantly nonlinear and the damping is sufficiently small. The above observations are rather interesting and important in certain respects: they show that the subharmonic of order 1/n does occur when the highest degree of the power of the nonlinear terms in the $g(\theta)$ expression is less than n. This was not to be expected according to a previous investigation[5]. It indicates that the subharmonic of order 1/2 can be generated in a system given by (2.1) without having an unsymmetrical nonlinear restoring force. Finally, the transition from the order 1/2 to that of 1/3 shows that a single type of nonlinearity may contribute to the support of more than one type of subharmonic oscillation.

So far, it is not apparent from our theoretical development how the transition of subharmonics can be predicted . It has only been shown that the frequency-amplitude relations can be obtained for these subharmonics in the absence of damping. This is in fairly good agreement with experiment. It is supposed that in the transition, the subharmonic of order 1/2 is interrupted by the viscous damping which increases with the frequency. The amplitude then decays to that of a lower frequency oscillation i.e. the order 1/3 of lower amplitude. This may happen provided that the damping is sufficiently low for the existence of subharmonic of order 1/3.

4) In the $|\theta^3|\ \theta$ system the subharmonic of order 1/2 and that of order 1/3 were the only types detected. The higher order subharmonic could <u>not</u> be excited. It seems that this is what we should expect in view of the restriction (2.10) on the coefficient of damping for the higher order subharmonics. In this case for low damping the subharmonic of order 1/3 is the dominant type when the system is predominantly nonlinear. The subharmonic of order 1/2 can be initiated only if α is increased to suppress the one of order 1/3. The transition of subharmonics was not observed in this system. It is suspected that the subharmonic of order 1/3 may occur in all symmetric nonlinear systems.

The theoretical consideration for the $|\theta^3|\ \theta$ system becomes rather complicated. It is concluded that the analysis of the system with higher exponent nonlinearity will present great difficulties of algebraic nature. The assumption of predominantly subharmonic oscillations leads to important simplifications in formulating the frequency-amplitude relations which are in fairly good agreement with experiment.

5) In general we see that the assumption of predominant subharmonic oscillations is valid in the upper range of the frequency in which the amplitude of oscillations is large. It is worth noting that the first order approximation used in getting the frequency-amplitude relations for the subharmonics is equivalent to the expression describing the frequency of the free nonlinear oscillation of the system excited by a periodic force with n times this frequency. In effect we begin the approximation with the free nonlinear vibration by assuming that the forced vibrations are predominantly subharmonic. The physical implication of this result is that the free nonlinear oscillations can be sustained by a disturbing force with approximately n times the frequency of the free vibration. The integer n depends upon the type of nonlinear restoring force. This phenomenon may occur with damping in the system provided that the damping is not too great and the amplitude of the excitation is sufficiently high.

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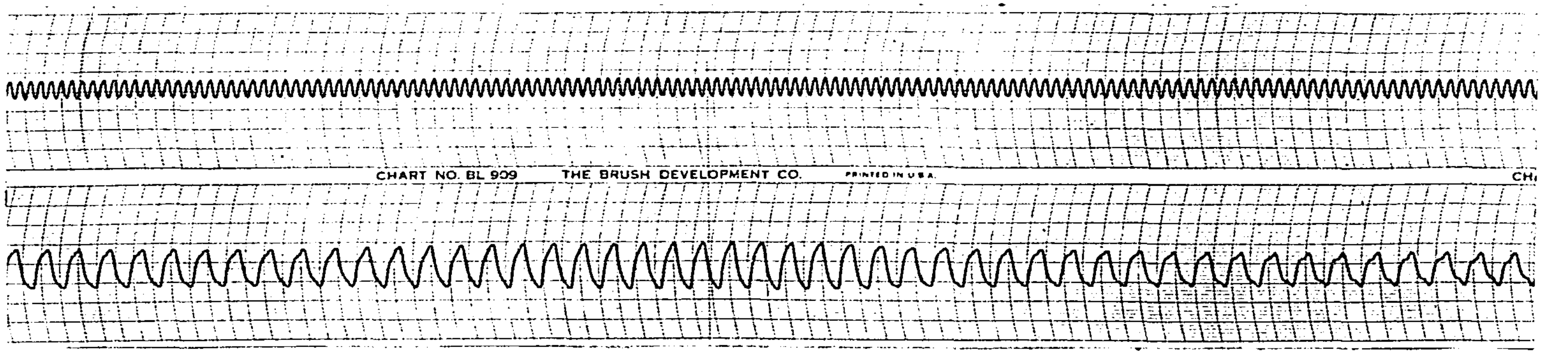
Oscillogram (1)

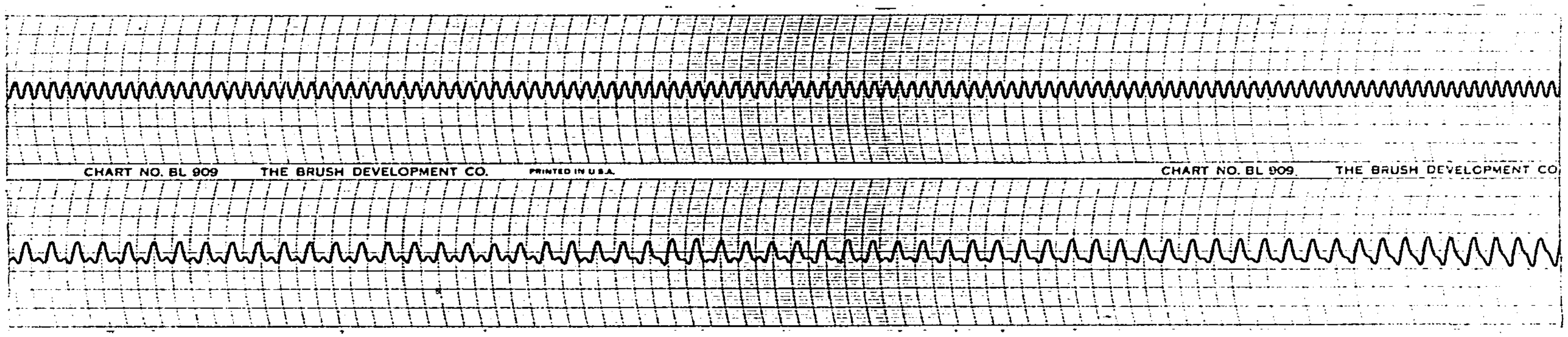


Subharmonic of Order 1/3

$$\ddot{\theta} + \alpha\theta + \beta\theta^3 = F \cos 3\omega t$$

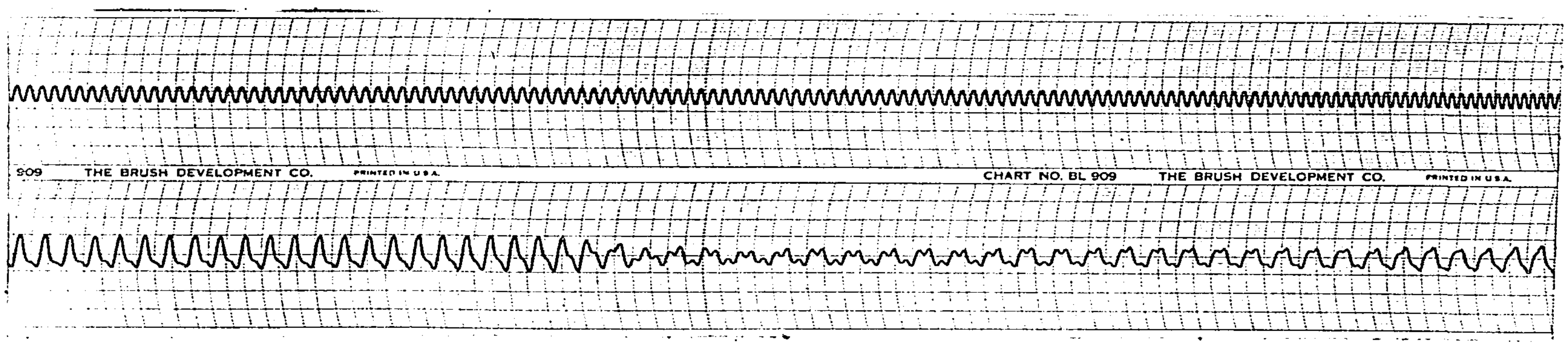
The amplitude of the subharmonic varies as a function of the frequency of the forcing function.

Oscillogram (2)



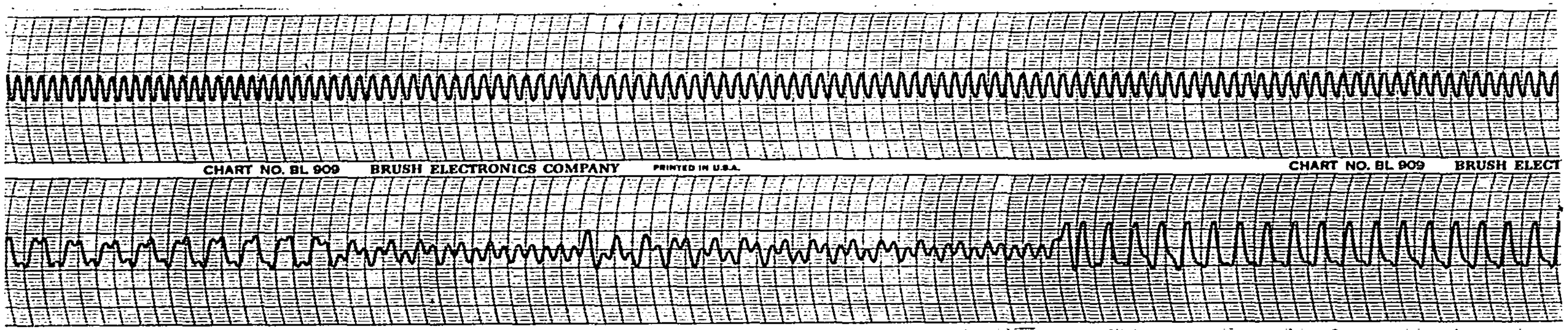
Subharmonic of order 1/2 With Increasing Frequency

Oscillogram (3)



Transition of Subharmonics
With Increasing Frequency

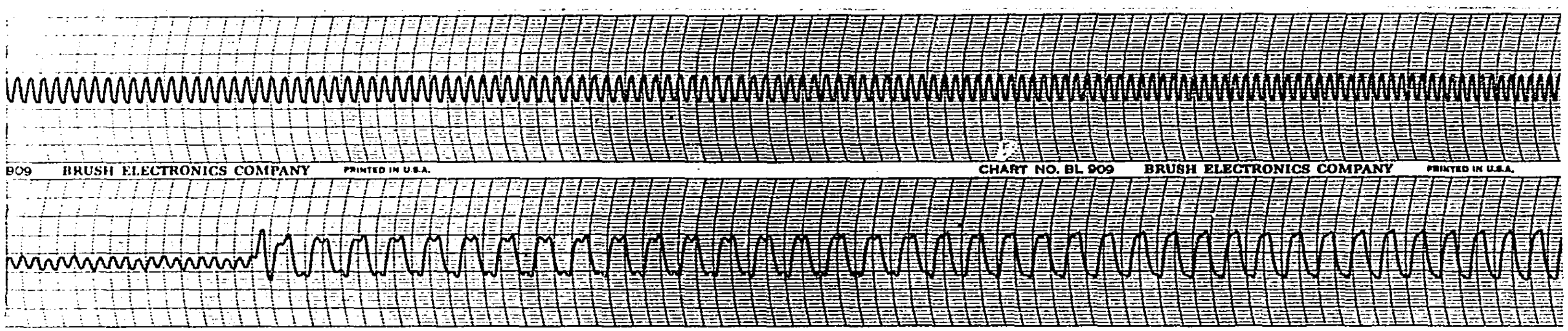
Oscillogram (4)



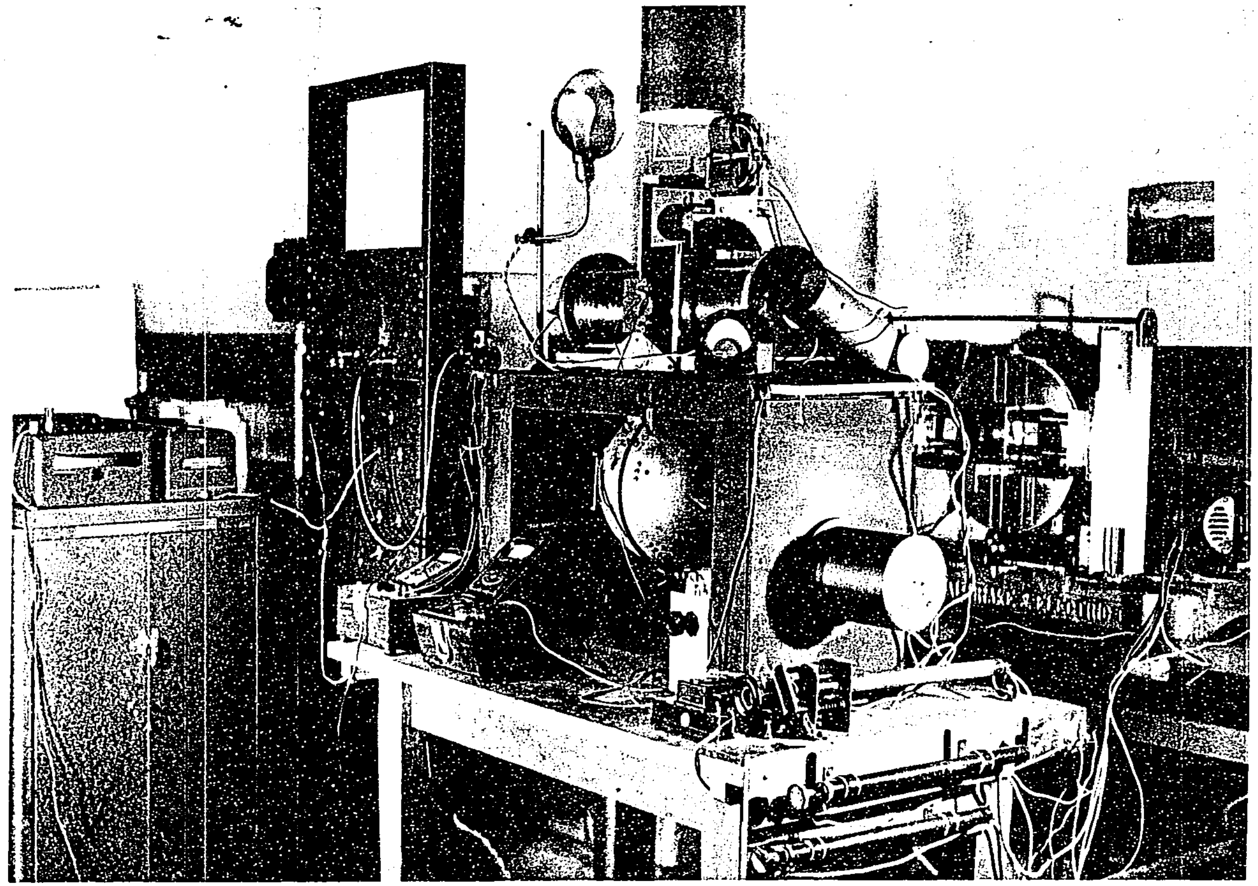
Subharmonic
of order 1/5

Initiation of Subharmonic of
order 1/2

Oscillogram (5)



Initiation of Subharmonic of Order 1/3



Electromechanical Analog