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I hereby recommend that the thesis prepared under my supervision by Herman Russell Branson

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be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy

Approved by:

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Part I

THE DIFFERENTIAL ACTION OF SOFT X-RAY ON TUBIFEX TUBIFEX

A dissertation submitted to the

Graduate School
of the University of Cincinnati

in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

1939

by

Herman Russell Branson

B.S. Virginia State College 1936

TABLE OF CONTENTS:

INTRODUCTION:

This work was undertaken to ascertain the reaction of selected regions of the small chaetopod, Tubifex tubifex, to soft X-radiation. These animals were selected for several reasons. Preliminary study revealed the abundance of these animals in Burnet Woods lake, thus solving the important problem of the acquisition of specimens. After getting a supply of the animals keeping them is no trouble. They require practically no attention. The animals breed and replenish the store. One container holding approximately a quart of mud and water supplied approximately fifty animals a week for over six months.

The simple morphology of Tubifex tubifex is possibly the happiest feature of these animals for irradiation work. The photomicrograph on the following page shows them to be practically transparent, making the detection and identification of injured sections immediate and striking. The other photographs in the main sections of this dissertation support this contention.

Fig. a. A photomicrograph of a normal animal. The magnification is approximately 30 times, the animal was about 2 cm. in length. This length, 2 cm., was the average length of animals used in these experiments. In this picture the animal is curled with its head in.

Zoologically these animals are cousins of the ordinary earthworm. The complete classification runs:

Phylum:	Annulata
Class:	Chaetopoda
Sub-class:	Oligochaeta
Order:	Microdrili
Family:	Tubificidae
Genus:	Tubifex
Species:	tubifex.

These animals live in the mud along the shore line of lakes and ponds. The name suggests their habit of life: tube-dwellers. They manufacture tubes of mud held together by a tenacious secretion from the epidermal unicellular glands. The animals live in these tubes head down, with their posterior segments out and waving briskly to keep up the circulation. The animals are extremely sensitive to noise and light. A tap on the container or a shadow across them will cause the waving to stop as though questioning. If there is no further disturbance, they resume their waving but the slightest foreign sound will cause them to withdraw into their tubes. Of course any sort of rude jar will cause immediate withdrawal.

For the work on the sectional irradiation the X-ray tube shown in fig. c was used. It is housed in the center unit on the floor of the soft X-ray laboratory. The power unit for this tube supplied half-wave rectified current. Otherwise the diagram of the connections is the same as that of fig. b which represents that of the tube used for over-all irradiation. In this over-all irradiation, the animals wereplaced in small petri dishes on the brass table. This table is atop a screw which enabled the experimenter to maintain the selected region at a fixed distance from the focal spot throughout all of the individual and group irradiations. All of this apparatus is housed in the large lead house in the soft X-ray laboratory. In order to protect the animals from stray X-radiation coming from kenotrons, the dishes were placed in lead receptacles. In all of these experiments there was no appreciable rise in the temperature of the container in which the animals were placed during irradiation.

A preliminary report of this work covering some of the qualitative conclusions was published in "Nature", March 26, 1938, under the title "Effects of Soft X-rays upon Chaetopods".

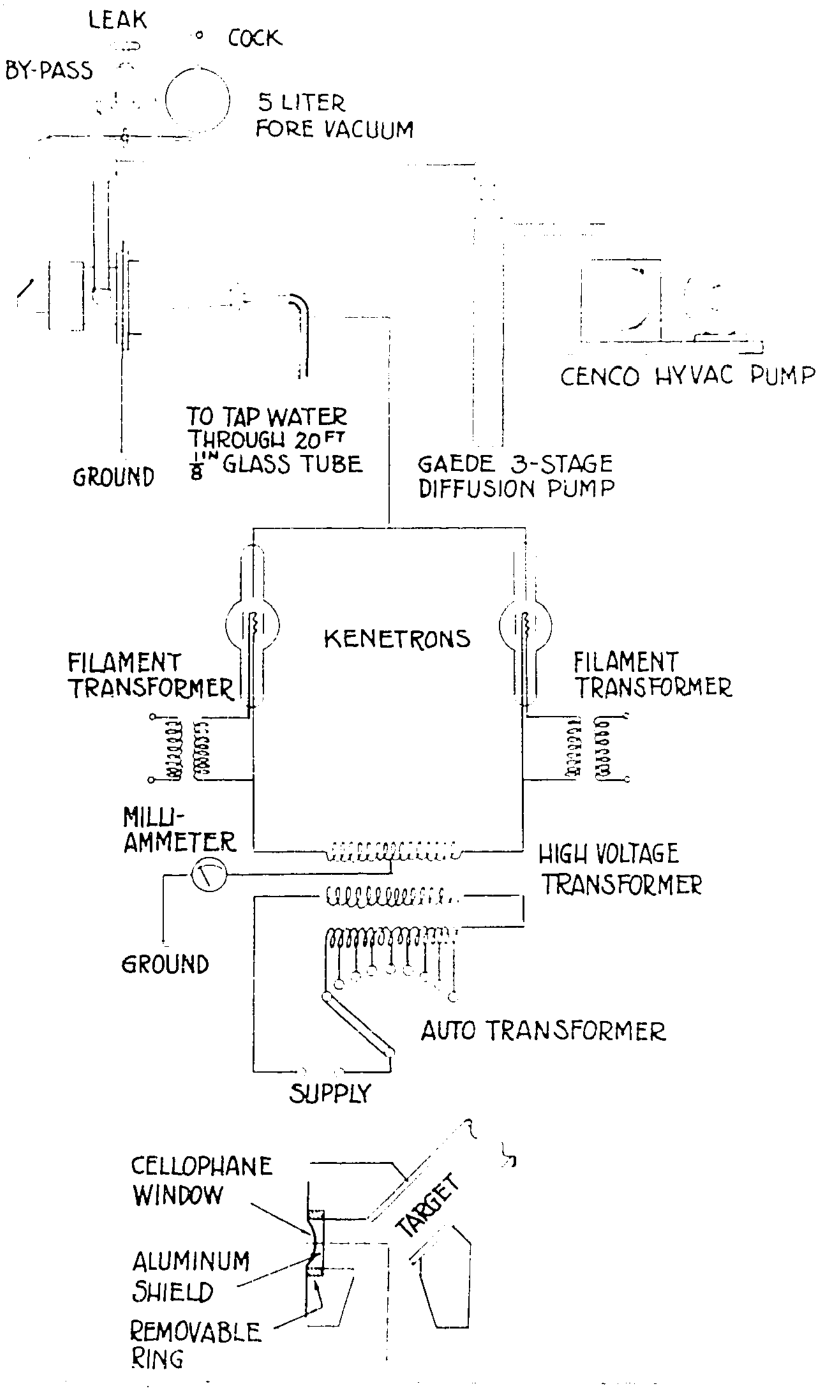


Fig. b. The diagram of the X-ray tube circuit. A discussion of the characteristics of this circuit was given by Kersten in "Radiology", XXIII, 60-63 (1934).

Fig. c. This photograph shows the prepared glass slide fixed to the brass carrier and the unit mounted on the X-ray tube. The animals are placed in the groove. The hook on the right is attached to the lead shutter. The line is brought out through a hole in lead covering around the unit, making it possible to give correctly timed exposures without chance of a burn to the experimenter.

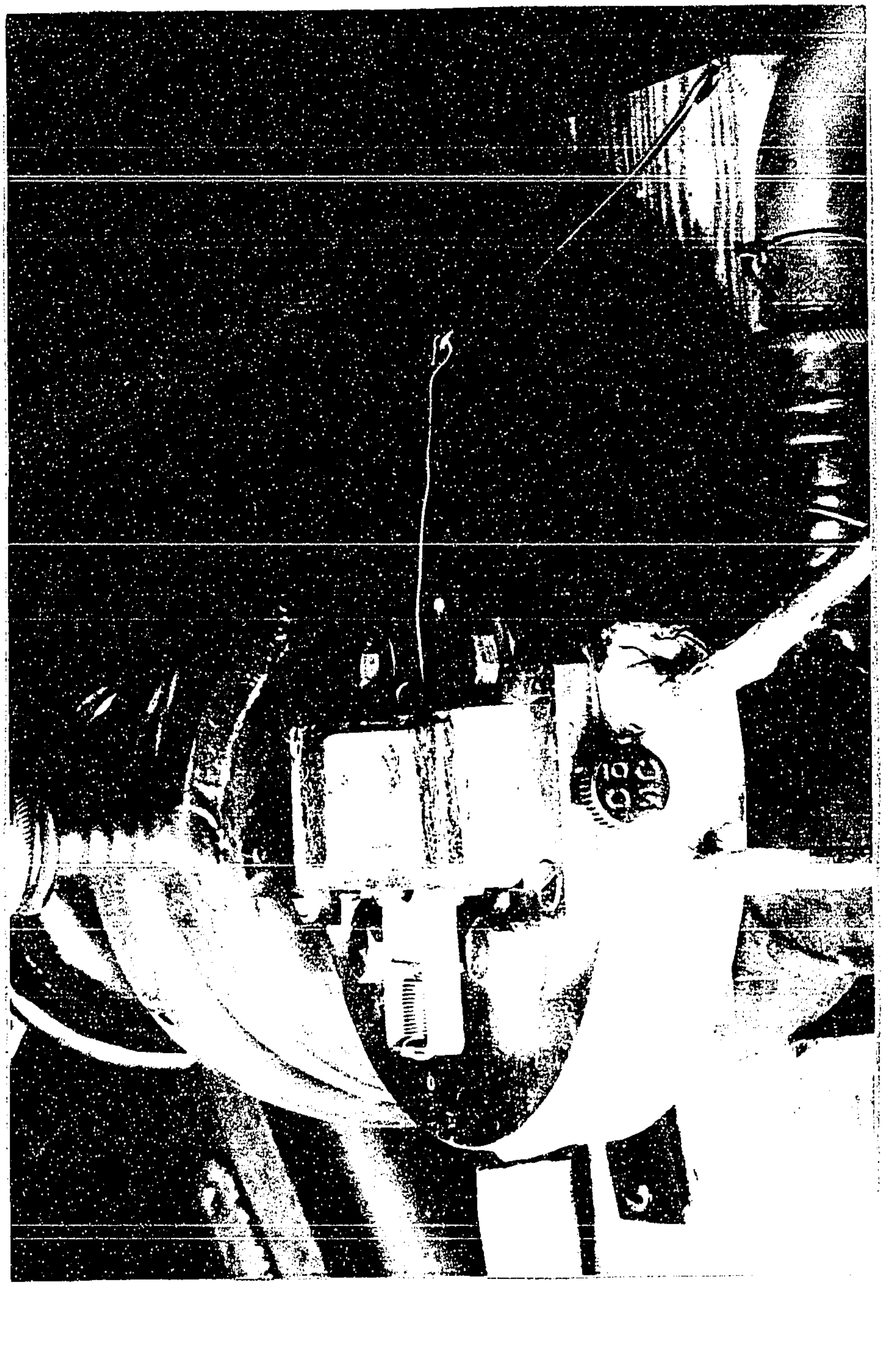


Fig. d. The apparatus used for over-all irradiation. The petri dishes carrying the specimens were placed on the brass shelf which was kept at 1" from the focal spot.

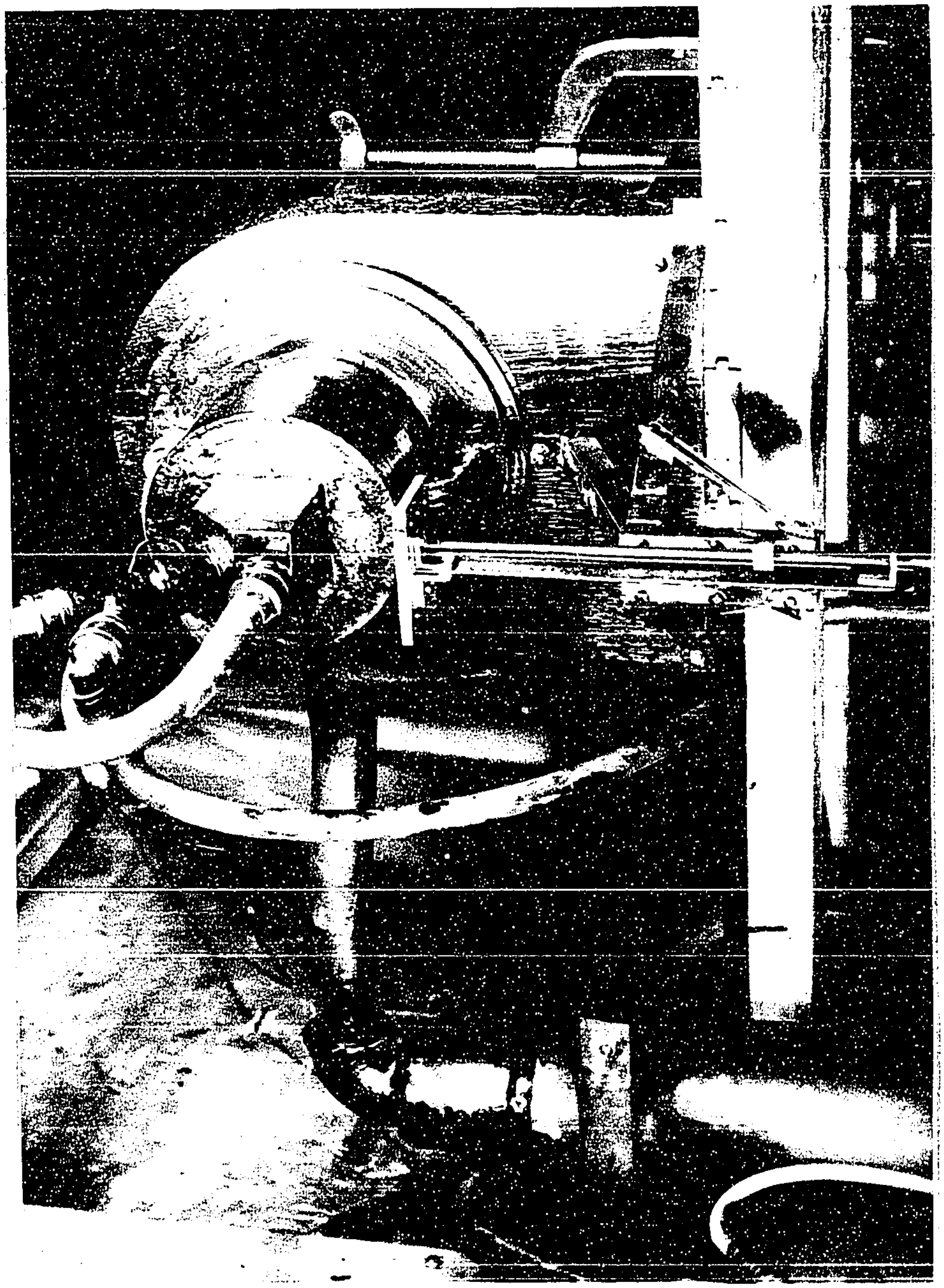
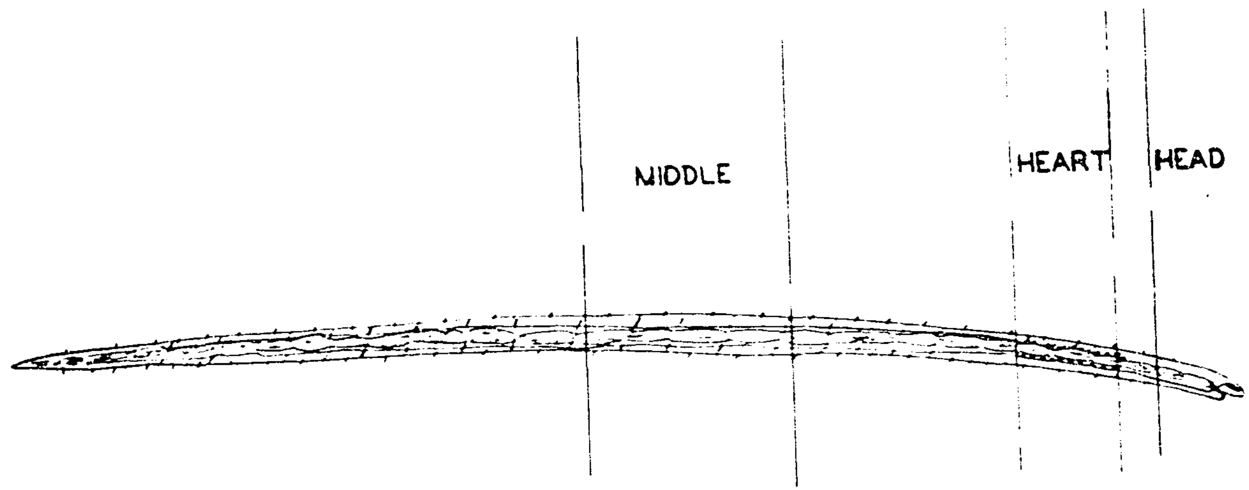


Fig. e. This schematic drawing shows the regions selected for irradiation in the first part of these experiments.

POSITIONS OF IRRADIATION



It is well known that posterior regeneration in Tubifex tubifex occurs when it is seperated with a knife at any level behind the tenth somite; and that following X-ray irradiation, regeneration is inhibited (Stone, 1932). In this paper experiments are described in which the seperation was accomplished by irradiating several somites with an intense collimated beam of soft X-rays, and also some in which irradiation of the entire animal caused the death of only the posterior.

METHODS

Mud containing the animals was collected at frequent intervals for a period of over a year along the shore of Burnet Woods lake, near the University, and kept in porcelain pans at room temperature in the laboratory. The animals were removed from the mud just before they were irradiated. After irradiation they were kept in clear water in small Petri dishes or in the depressions of well dishes.

The work may be conveniently divided into two parts of which the first is concerned with the irradiation of a very small part of the head, the heart, or the middle regions of the animals: while the second is concerned with the irradiation of the animals. The first part required that a relatively intense beam of X-rays strike a samll portion of the animal, while a less intense beam spread over a large was needed in the second part, so that different X-ray tubes and different techniques for holding the animals in place during irradiation were employed. The X-ray tubes were of the gas type, made of metal with porcelain insulators. Each had a copper target and windows thin enough so that the most intense part of the transmitted radiation had a wave-length of 1.54 Å Angstroms. The tube used for the first part of the work (Figure 1,A) was operated at 35 peak kv., and 2- ma., and the one used for the second part (Figure 1,B) at 25 peak kv., and 10 ma. In order to hold the animals still in the first part of the work they were narcotized by immersing them in a 0.2% solution of chloretone for a few seconds just before the period of irradiation. Preliminary experiments had shown that the animals could be left in this solution for over 20 minutes without injury or debility, and that narcotizing did not prevent both halves from living after being cut.

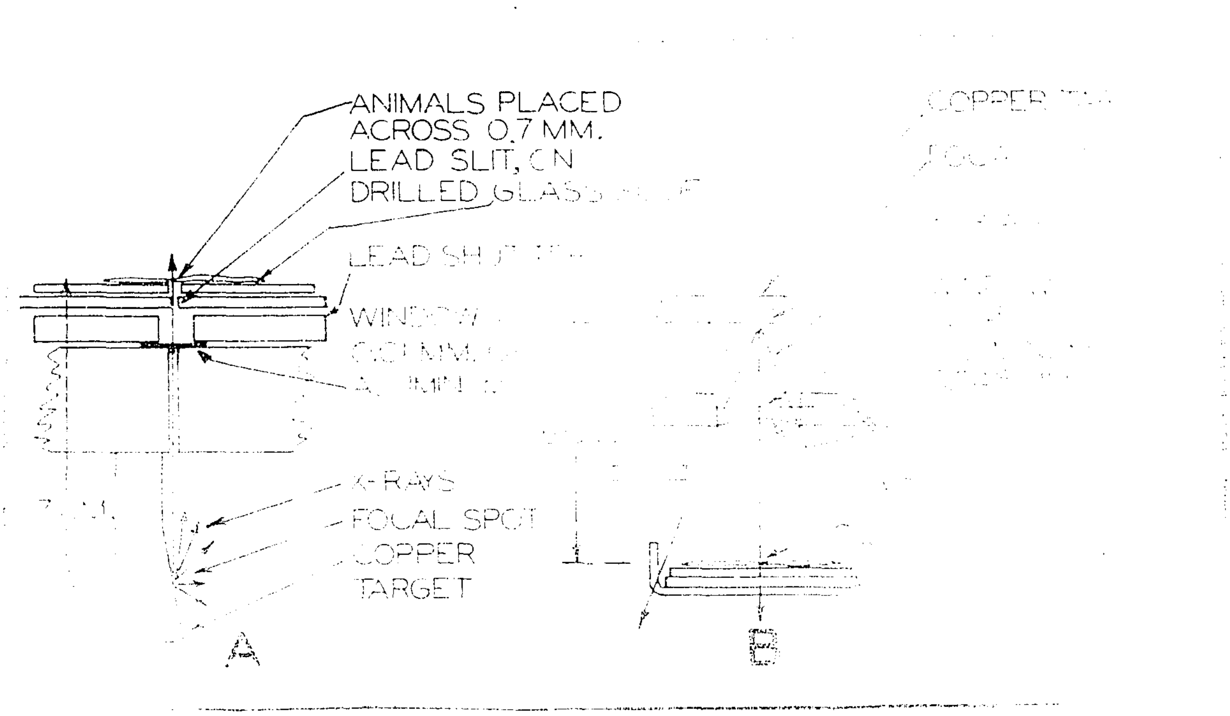


Fig. 1.-Schematic drawing showing how the animals were placed during irradiation.

Part I. Differential action of X-rays on selected regions.-

The authors expected that when several somites were irradiated using the arrangement illustrated in Figure 1-A, these would die after several hours, the worm would sever, and the two parts would survive. This happened in some cases but it was evident after a few trials that the action of the 'X-ray knife' was not the same as that of an ordinary knife. When observed 24 hours after irradiation, some animals showed tuberosities at the site of irradiation, others seemed normal, others had broken intwo with a survival of either the anterior, the posterior, or both. The observations 24 hours after irradiation are tabulated in Table I and those 8 days after irradiation, in Table II.

Table I

Appearance of Tubifex tubifex 2[illegible] hours after approximately 3 somites in the regions indicated had been irradiated.

	Region irradiated		
	Anterior tip	Heart region	Middle section
Normal animals	23	69	67
Dead animals	25	13	15
Animals having tuberosities (Fig.2)	44	11	7
Animals showing progressive disintegration	11	16	1
Seperated animals with dead posteriors (Fig.3)	0	0	27
Seperated animals with dead anteriors (Fig. 3)	0	0	5
Seperated animals with both parts living (Fig.3)	0	0	24
Totals	103	109	146

Table II

Appearances of Tubifex tubifex 3 days after approximately 3 somites in the regions indicated had been irradiated.

	Regions irradiated		
	Anterior tip	Heart region	Middle section
Normal animals	27	69	41
Dead animals	28	18	34
Animals having tuberosities (Fig. 2)	45	10	7
Animals showing progressive disintegration	3	12	0
Seperated animals with dead posteriors (Fig. 3)	0	0	44
Seperated animals with dead anteriors (Fig. 3)	0	0	4
Seperated animals with both parts living (Fig. 3)	0	0	16
Totals	103	109	146

Fig. 2. Tuberosities formed on Tubifex tubifex due to the irradiation of approximately 3 somites with soft X-rays. Left, on the anterior; middle, on the heart region; right, on the middle region.

The data in Table II were obtained independently of those in Table I. Thus, for example, the seven animals which showed tuberosities in their middle sections as listed in Table I are not necessarily the same seven listed in Table II, because some of those listed in the first table died, some of the normal ones showed subsequent tuberosities which are listed in Table II, and some tuberosities were absorbed in the interval between the 24 hour and 3 day observations.

Part II. Differential action of X-rays when entire animal is irradiated. Determination of a lethal curve.- The results of the previous part indicated that the posterior of the animal was more easily injured than the anterior so that the authors were lead to investigate what would happen when the entire animals were irradiated. A convenient way to get assorted doses appeared to be to operate the X-ray tube at a rather low voltage and current and expose groups of animals for various periods of time. This is the same procedure which is used for obtaining data for a lethal curve, so the latter is included. One way to plot a lethal curve is to indicate the percentage of animals killed as a function of the time of irradiation. A better way, when the total number of animals used is not large, is that suggested by Reed (1936) which is to plot the probability of dying as a function of the time of irradiation. The data for plotting such a curve are given in Table III and the curve in Figure 4.

Among the 150 animals examined 24 hours after irradiation there were four which had dead posterior and living anterior regions. When the same group was examined 3 days after irradiation there remained four whole animals alive and nine whose posterior region alone had survived and none with tuberosities.

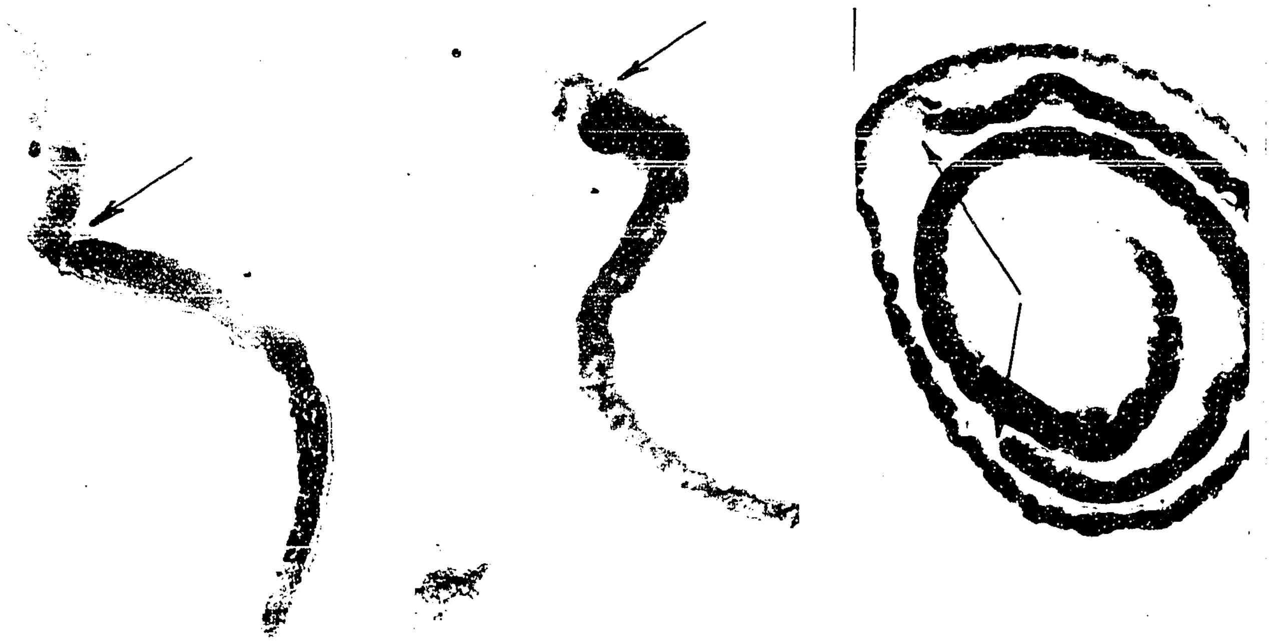


Fig. 3. Tubifex tubifex seperated due to the irradiation of approximately 3 somites with soft X-rays. Left, with death of the posterior; middle, with death of the anterior; right, with survival of both regions.

Table III

Data for plotting the lethal curve shown in Figure 4. Observations were made 24 hours after irradiation. Fifteen groups of ten animals each, making 150 in all, were used.

Minutes of irradiation	Observed number of animals			Implicit number of animals			Probability of dying
	Total	Dead	Alive	Total	Dead	Alive	$\frac{m-1}{m-n-2}$
				m-n	m	n	
1	10	0	10	57	0	57	0.02
5	10	2	8	49	2	47	0.04
10	10	3	7	44	5	39	0.13
12	10	3	7	40	8	32	0.21
15	10	8	2	41	16	25	0.40
16	10	6	4	45	22	23	0.49
18	10	6	4	47	28	19	0.59
20	10	6	4	49	34	15	0.69
26	10	7	3	52	41	11	0.78
30	10	6	4	55	47	8	0.83
35	10	7	3	58	54	4	0.92
40	10	9	1	63	62	1	0.95
45	10	10	0	73	73	0	0.99
50	10	10	0	83	83	0	0.99
55	10	10	0	93	93	0	0.99

Fig. 5 shows the appearance of two of the animals whose posterior regions had died.

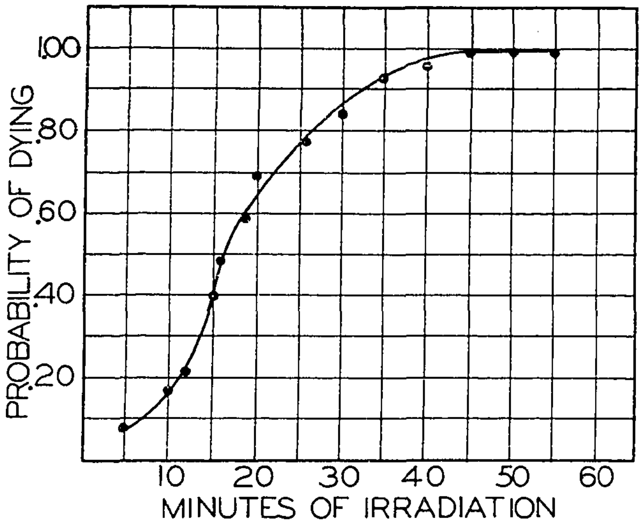


Fig. 4. Lethal curve for Tubifex tubifex irradiated with soft X-rays.

Fig. 5. Two examples of Tubifex tubifex whose posterior regions have died due to the irradiation of entire animals with soft X-rays. The arrows point in the direction of the dead regions.

Summary and Discussion

1. When approximately 3 somites of Tubifex tubifex are irradiated with an intense dose of soft X-rays applied to the anterior tip or the heart region the principal changes observed in those surviving animals which do not appear to be normal, are the formation of tuberosities (which may be absorbed later by the animal) at the place of irradiation, or disintegration progressing away from the place of irradiation.

2. When the middle section is similarly irradiated the principal change observed in the abnormal animals which survive is a breaking apart at the place of irradiation, with either both parts living or more often, the death of the posterior region only.

3. Tubifex-tubifex may be killed by irradiating the entire animals with soft X-rays. When the dose of X-rays is too small to produce 100% killing, the surviving animals show either no apparent change or the death of the posterior region only.

Although the X-ray beam, in the experiments where selected regions were being irradiated, struck not more than 3 somites, it should be pointed out that this does not mean that the adjacent regions did not receive some irradiation, for it is well known that secondary X-rays are scattered in all directions from the part irradiated. Thus the animal would be very badly injured at three somites and the adjacent somites injured to less and less degrees according to their distance from the three which are irradiated by the primary beam.

It seems that there might be two possible explanations for the results observed. The first one is that the scattered x-rays in the first part, and primary X-ray beam in the second part, acted according to the law of Bergonie and Tribondeau (1906) which states that X-rays act with greater intensity on the cells whose reproductive activity is the greater, cells whose metabolic rate is higher, and those whose morphology and function are less definitely fixed. This requires that the posterior region be the one which fulfills the requirements of the law, which is not unreasonable, but it does not seem as reasonable to expect the scattered radiation to be so potent.

The second explanation is that the X-rays produce a toxic substance in the animal at the site of irradiation and that this moves to other parts, killing the most susceptible ones just as in the experiments of Child (1924) in which susceptibility gradients were shown by immersing animals in toxic liquids. Since no posteriors were injured when the head and heart regions were irradiated, this explanation requires that the toxin had become too diluted to be effective when it reached the posterior.

Bergonié, J., and Tribondeau, L. 1906. Intérpretation de Quelques résultats de la radiothérapie et essai de fixation d'une technique rationelle. Compt. Rend, 143: 983-985

Child, C.W. 1924. Physiological foundations of behavior. New York. pages 80-81

Reed, L.J. 1936. In Duggar, Biological effects of radiation, I. New York

Stone, R.G. 1932. Effects of X-rays on regeneration in Tubifex tubifex. Jour. Morph. 53: 389-431

-------------- 1933. Effects of X-rays upon Anterior regeneration in Tubifex tubifex. Jour. Morph. 54: 303-320

Part II

THE CONSTRUCTION AND OPERATION OF AN X-RAY INTENSITY MEASURING DEVICE

A dissertation submitted to the

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TABLE OF CONTENTS:

INTRODUCTION:

In the discussion of some of the problems arising in the preceding work, it was decided that a portable piece of apparatus for the measurement of ionization would be of great value in the soft x-ray laboratory. The center of this set-up would necessarily be a stable and precise and yet delicate current measuring device. An apparatus with the range of an electrometer, but without the most disconcerting characteristics of the latter instrument, was whata the experimenter has in mind.

After reading the articles of Penick and his references, the experimenters selected a circuit built around the electrometer tube, so named because circuits employing it are used to replaced circuits built around an electrometer.

The Western Electric Company's tube D-96475 was selected. This tube is capable of detecting six electrons per second flowing in a suitable circuit. With a sensitive wall type galvanometer, as the indicating instrument, these tubes give higher voltage sensitivity, shorter period, about equal capacitance, greater ruggedness and more convenience of operation than the electrometer.

DESIGN AND CONSTRUCTION:

Tube and Circuit Characteristics:

The circuit selected is shown in fig.1. It is a circuit first suggested by Barth but which has been modified slightly for our use.

Keeping in mind that what we desire is a device for high current amplification and measurement, we see what we need. First, if we are to make a measurement no current should flow through the galvanometer-which is our ultimate current measuring device-before the measurement is made. This is accomplished by varying R_n and R_p until the galvanometer shows no deflection. This process is called balancing and will be discussed later.

Assuming that this balancing has been accomplished, we noe seek to make a measurement. We arrange the equipment as in fig. 3, and we send a small current through R and measure the potential difference existing at the ends of R_o by the potentiometer. We shall call this difference of potential V_o. From Ohm's law

$$V_o = i_o R_o \quad ,$$

since R is in parallel with the control grid to filament resistance, there is a change of the control grid voltage

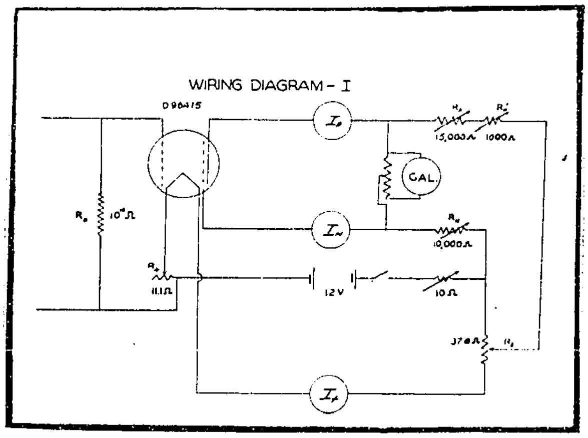


Fig. 1. The circuit initially-before modification.
The I's are the meters in the circuit.

by V_o. This change in control grid voltage is related to a change in plate current by the factor of proportionality called the mutual conductance or the grid-plate transconductance:

$$g_m = \frac{di_p}{de_g}$$

Hence the change in plate current is

$$\Delta i_p = \Delta e_g \, g_m$$

but as we saw

$$\Delta e_g = V_o = i_o R_o$$

then

$$i_p = i_o R_o g_m$$

When the plate current reaches the branch leading to the galvanometer, it will divide into two parts inversely as the resistance of the galvanometer is to R_p. The resistance of the galvanometer is 137.6 ohms, R_p is approximately 15,000 ohms. Thus about 99% of i_p goes through the galvanometer and we have to asufficient degree of exactness.

$$i_{gal.} = i_o R_o g_m . \qquad \text{b.}$$

Knowing the sensitivity of the galvanometer, for a given displacement we can calculate i_{gal}, and measuring V_o with a potentiometer gives us a method of determing g_m. And knowing g_m and either $i_{gal.}$ or i_o we can find the third since the value of R_o is gi ven by the manufacturer.

It is easy to see that we are measuring a difference of potential; tp obtain high current amplification it is necessary to make R_o as large as possible. It is useless, however, to increase R_o beyond a certain limit, because it is shunted by the grid-to-filament resistance of the tube itself. For most tubes designed for use in radio receivers this resistance is not over 10^8 ohms. To make this resistance greater expensive care must be taken in the construction so as to eliminate sources of current to the grid within the tube. One type of tube, cost $25,000, is the Western Electric's D-96475 which we are using. It has an inner space charge grid to shield the control grid from positive ions emitted by the filament. It is operated at a very low plate voltage to avoid ionization of the residual gases. The grid-to-filament resistance is approximately 10^{16} ohms. The recommended operating conditions are:

Filament voltage	1.0 volts
Plate current	85 microamperes
Inner grid current	520 "
Mutual Conductance	40 microamperes/volt
Control grid current	10^{-15} amps.
Input resistance	$10^{16}\,\Omega$
Control grid capacity to ground	3×10^{-12} farads.

However, it is well to keep in mind that there are considerable individual variations in electrometer tubes and a balance may be considered as an experiment with each tube.

Characteristics curves showing the plate current and control grid current of a typical D-96475 tube are set forth in fig. 2. The slope of the grid-current gives the grid conductance and the reciprocal of the slope is the grid resistance. The control grid is operated at -3 volts because the curve is nearly flat at that point. The slope of the plate current curve gives the mutual conductance. The curvature of the plate current curve is quite noticeable and is sufficient to cause appreciable nonlinearity if the grid voltage changes by as much as 0.1 volt.

In this construction it is well to use only the best materials. Care was given to the purchase of the resistors particularly. Although they carry very little current, still their power rating should be rather large so as to insure good thermal stability.

When the apparatus is used at the highest sensitivity it is recommened that the tube be placed in an evacuated container. For ordinary practice it is sufficient that the tube be placed in a semi-sealed container which is free of dust. Before putting the

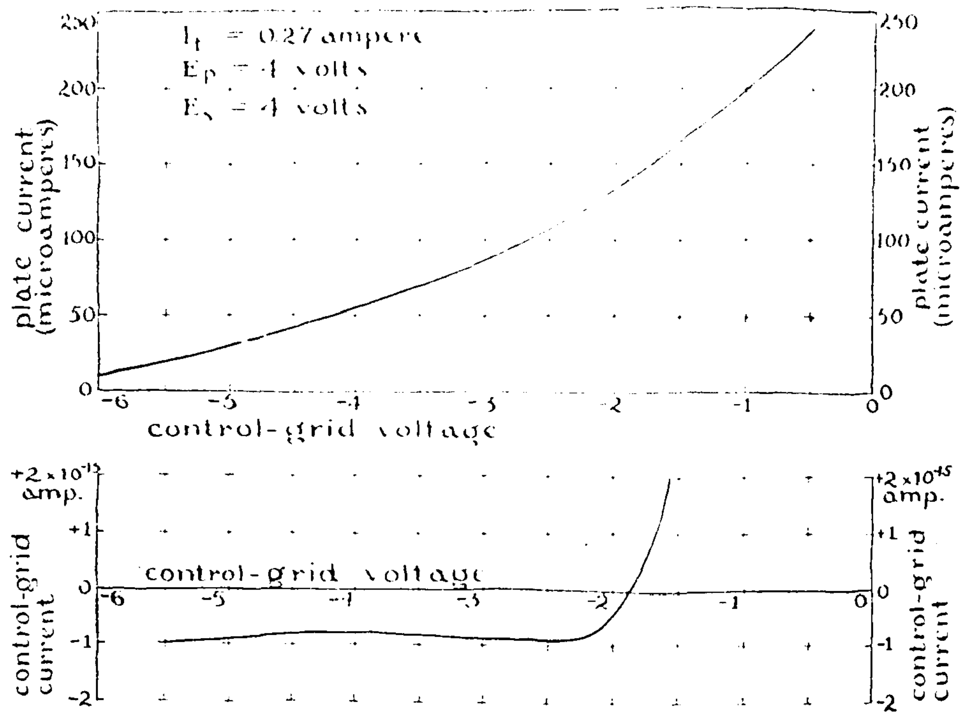


Fig. 6. Plate- and grid-current characteristics of Western Electric D-96475 tube.

Fig. 2. This figure was taken from Strong, p.414.

tube in place wipe it off with a cloth dampened with absolute alcohol and remove any stray finger prints after it is in position. The container should be of metal and well grounded. In fact all parts of the circuit should be enclosed in metallic shielding.

Balancing:

The preceding circuit, Fig. 1, was selected because of its stability under slight changes of battery voltage. Before this characteristic may be full utilized, the circuit must be balanced. The procedure the experimenters followed us:

With the galvanometer shunted to 0.1 or 0.01 of its full sensitivity and R adjusted so that the galvanometer reads zero when I_f is near its rated value, 270 ma, I_f is slowly varied by means of R_f. When the galvanometer is connected so that a positive deflection is cause by a decrease in the plate current, the deflection passes through a maximum for a value of I_f near 270 ma. If the galvanometer goes off scale before the maximum is reached it maybe brought back by a slight change in R_ρ.

If the value of I_f for maximum is not within a few percent of the rated value of the tube, and adjustment of R_ρ and R_2 will bring the balance point to

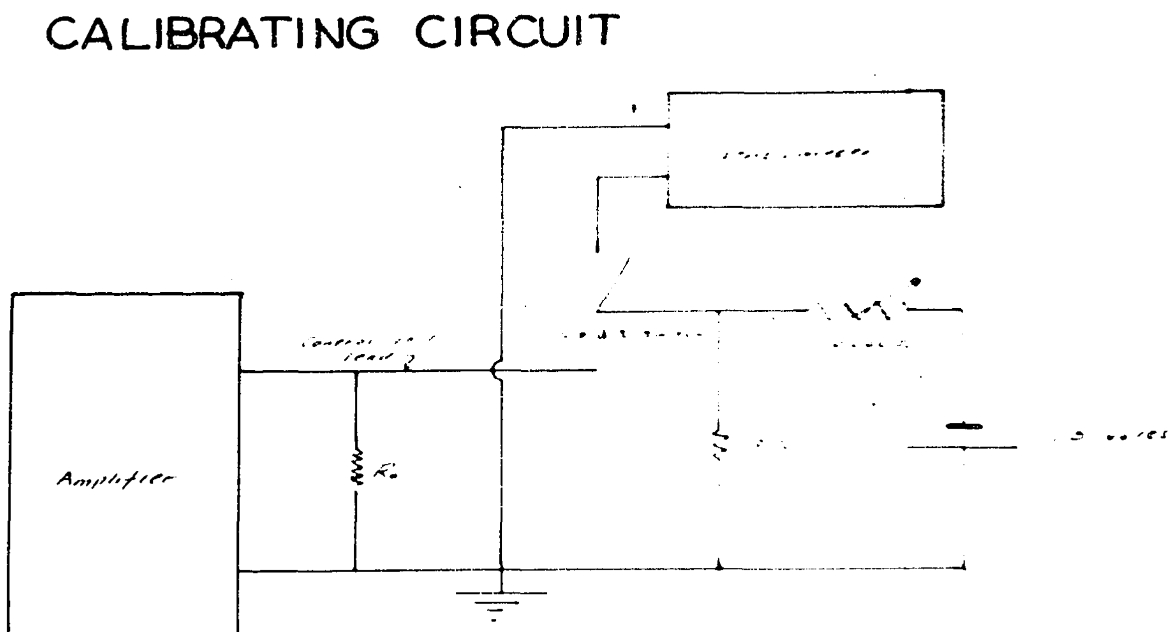


Fig. 3. The potentiometer is mounted on the table with the amplifier. The battery is an ordinary dry cell.

to a different value for I_f. The adjustment is finally made with the galvanometer at full sensitivity. Each adjustment will require a few minutes for a new thermal equilibrium to be established.

Often in our work a simpler satisfactory balance was gotten by setting I_f near the recommended value. After a few minutes for equilibrium to be established, I_f was turned precisely to 270 ma. Then R_w was turned about 2/3 full and R_f adjusted until a balance was had. No galvanometer was in the circuit in the initial work and an insensitive one was used for the final part. Of course the last adjusting is done with the wall type galvanometer as before. The control grid lead was grounded during the balancing.

During the balancing some valuable rules were stumbled upon, for completeness we list here:

1. Solder as many connections as possible, especially the battery terminals. Use only rosin flux.
2. Keep the top of the batteries clean and having them in a shielded box is excellent.
3. It is advantageous to use the batteries only on the middle portion of the discharge curve.
4. The control grid lead should be short and well

shielded and insulated

5. Calcium chloride crystals may be used in the tube container to keep out moisture.

Testing for Linearity:

The circuit had been balanced taking the necessary precautions. For input the experimenter devised the circuit shown on the following page. By varying the variable resistance a wide range of potentials could be impressed across R_o. And in accordance with our fundamental relation:

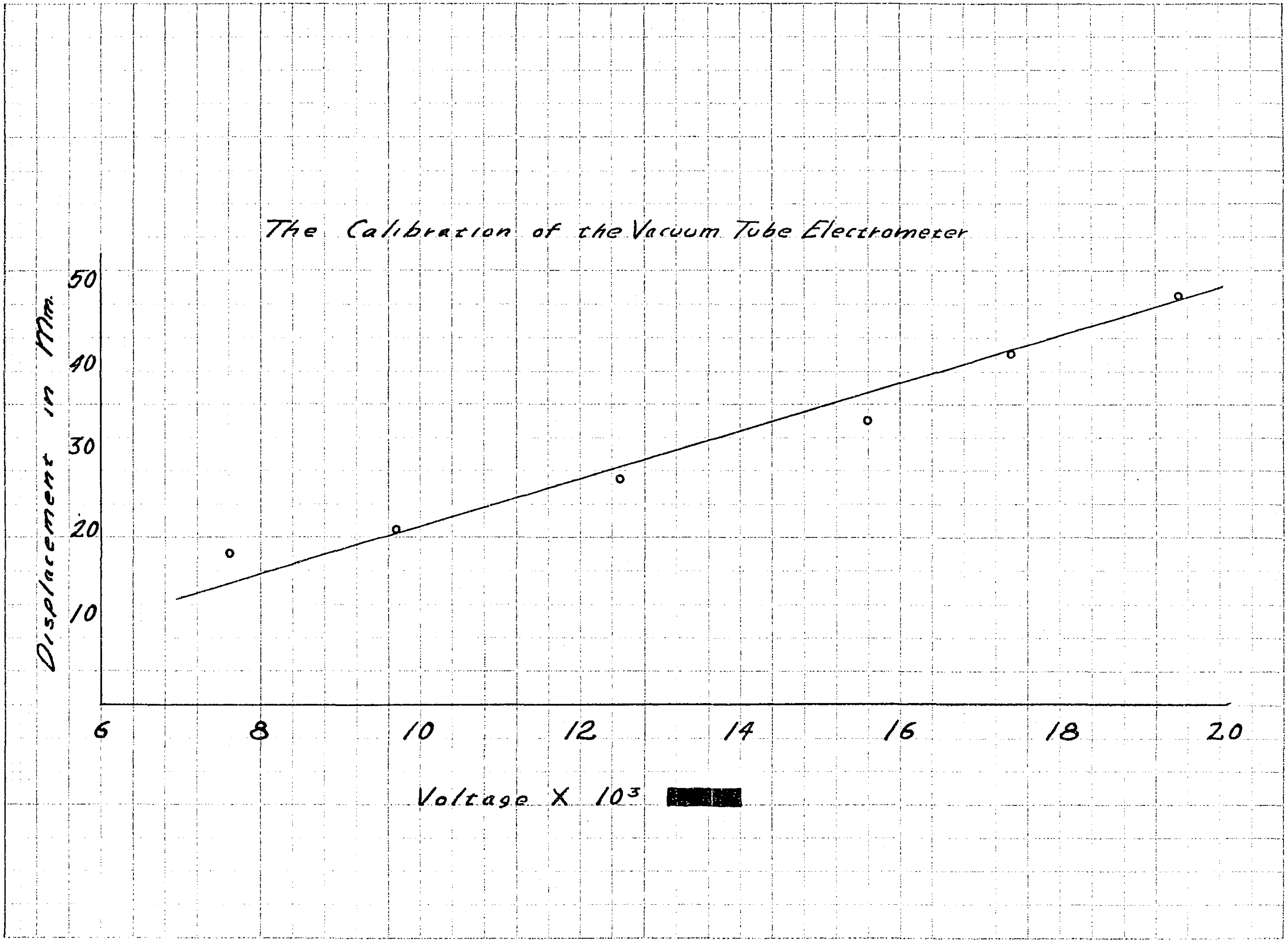
$$i_{gal} = i_o R_o g_m$$

a current would flow through the galvanometer. The galvanometer was a Leeds and Northup wall type galvanometer with a narrow coil. The sensitivity was

$$1.15 \times 10^{-8} \text{ amps/mm}$$

The data obtained was:

Potentiometer reading volts	Deflection mm
0.0017	5
0.0076	18
0.0097	21
0.0125	27
0.0156	34
0.0174	42
0.0195	45


The Calibration of the Vacuum Tube Electrometer
Displacement in Mm.
50
40
30
20
10
6
8
10
12
14
16
18
20
Voltage X 10³ [redacted]

In taking these data the single pole double throw switch was first turned to the potentiometer then to the amplifier and back to the potentiometer. The graph following this page exhibits the linearity of these data.

As far as the actual current through R_o is concerned, let us consider the first entry

$$V_o = 0.0017$$

$$i_o = \frac{0.0017}{1.05 \times 10^{10}} = 16.2 \times 10 \quad \text{amps/5 divisions.}$$

$$i_o' = 3.24 \times 10^{-14} \text{amps/mm}$$

Hence 16.2×10^{-14} amps went through R_o although we could detect 1/5 this amount. By raising R_o to 10^{12}, i_o could be pushed down to 10^{-16} amps.

The resistor R_o was bought from the S.S. White Dental Manufacturing Co., 10 East 40th Street, New York, and we have had to accept their rating. It is supposed to be correct to one part in a hundred. ($R_o = 1.05 \times 10^{10} \Omega$)

MODIFICATION FOR IONIZATION MEASUREMENTS:

The essential modifications are set forth in fig. 4. The most important item is the mounting of the tube and ionization chamber together. This idea occurred to us in order to keep the control grid lead short. After a more thorough study of the literature we found that Bearden and others had used similar arrangements for different purposes.

Figs. 5 and 6 show the completed apparatus in position in the soft x-ray laboratory. It is believed that the pictures and diagrams are fairly self-explanatory.

Since copper is more likely to be free of raiodactive contaminations than brass, the ionization chamber is constructed of the former element. The tube container is constructed of 3mm lead sheet. The insulator is of bakelite with an amber in amberiod center.

No quantitative results have been taken yet. But in preliminary experiments with soft x-rays the apparatus showed more than sufficient sensitivity with great stability.

It is obvious that with slight modifications, this apparatus could be used for many other current measuring jobs.

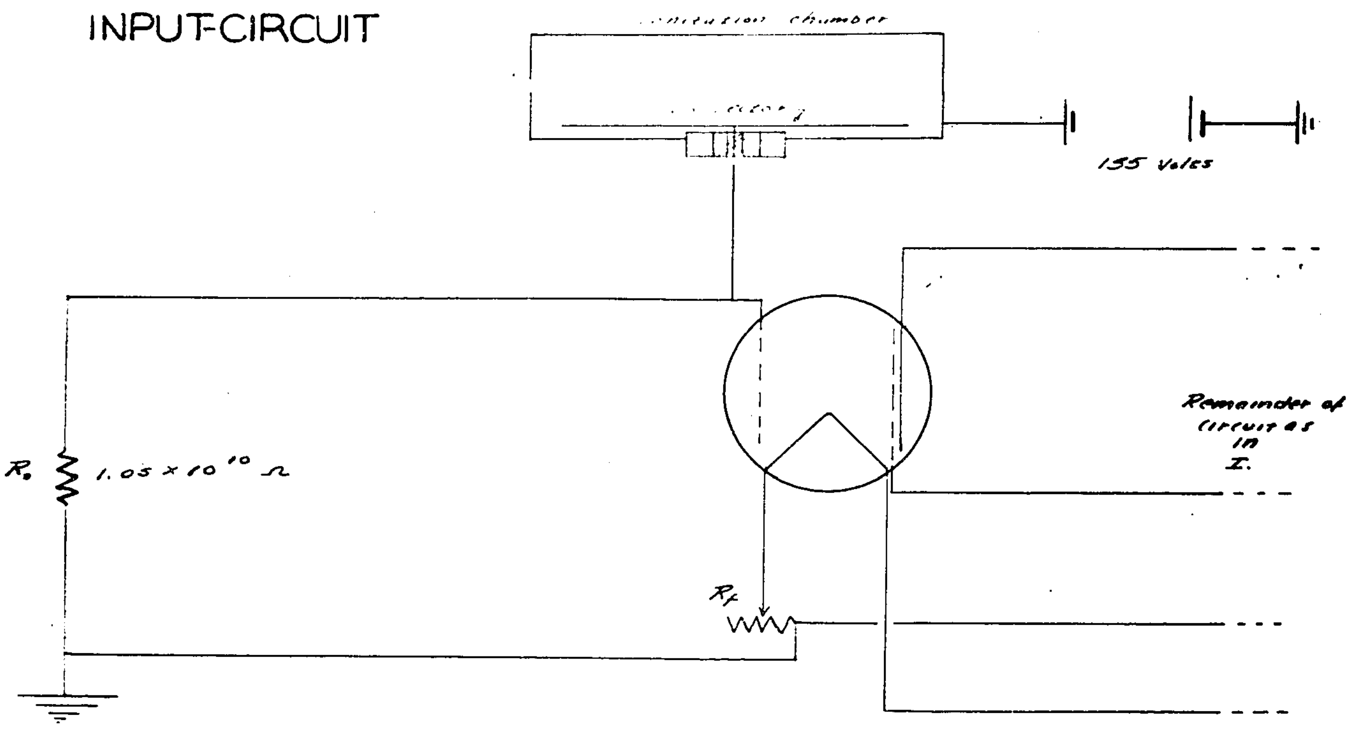


Fig. 4. The ionization chamber is 16x4 cm. The entire unit is mounted together. The batteries are ordinary B-batteries.

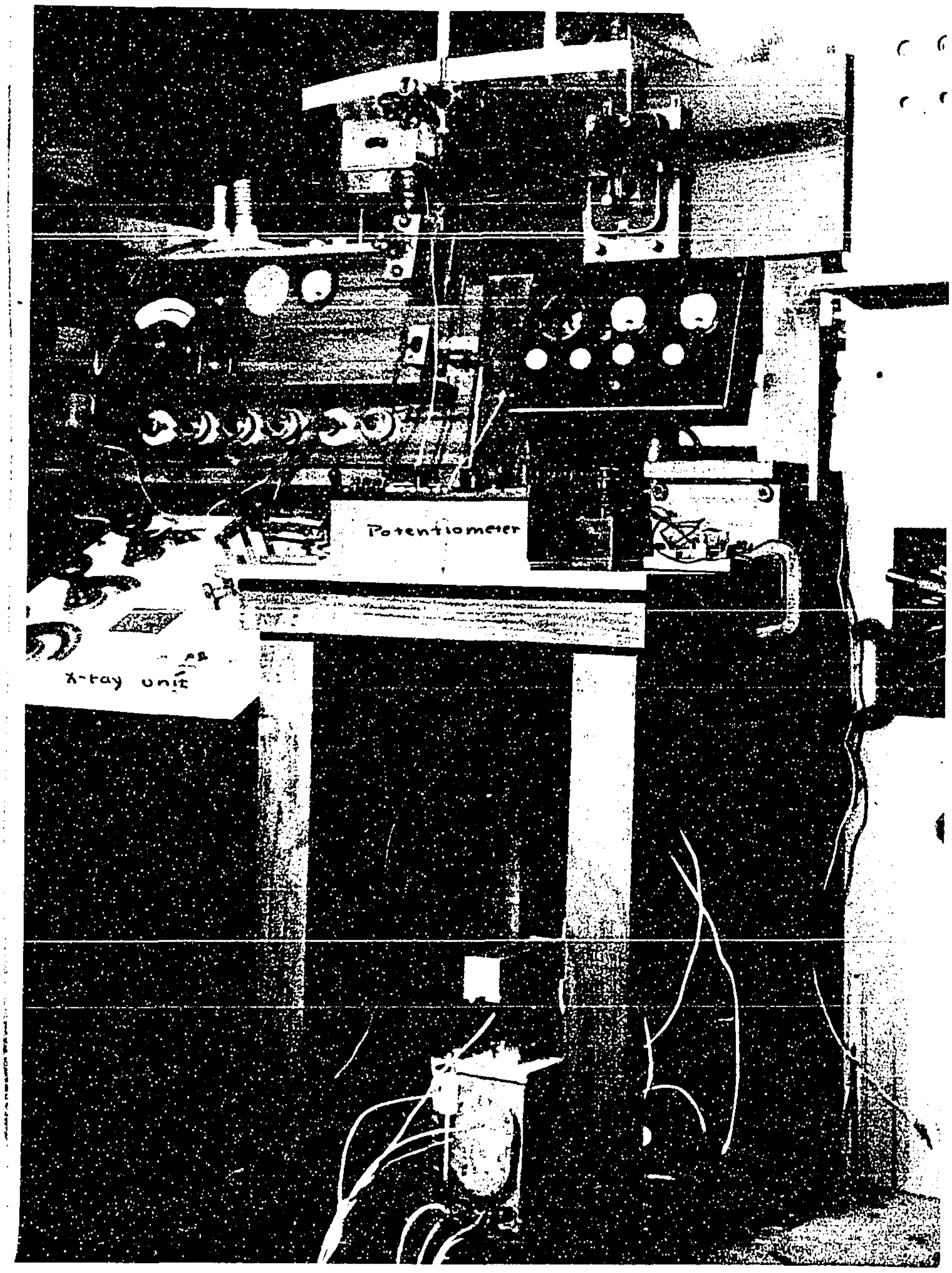


Fig. 5. This is a view of the entire apparatus in the soft x-ray laboratory. At the time this picture was made one of the meters had been removed. The storage batteries are placed on the lower shelf.

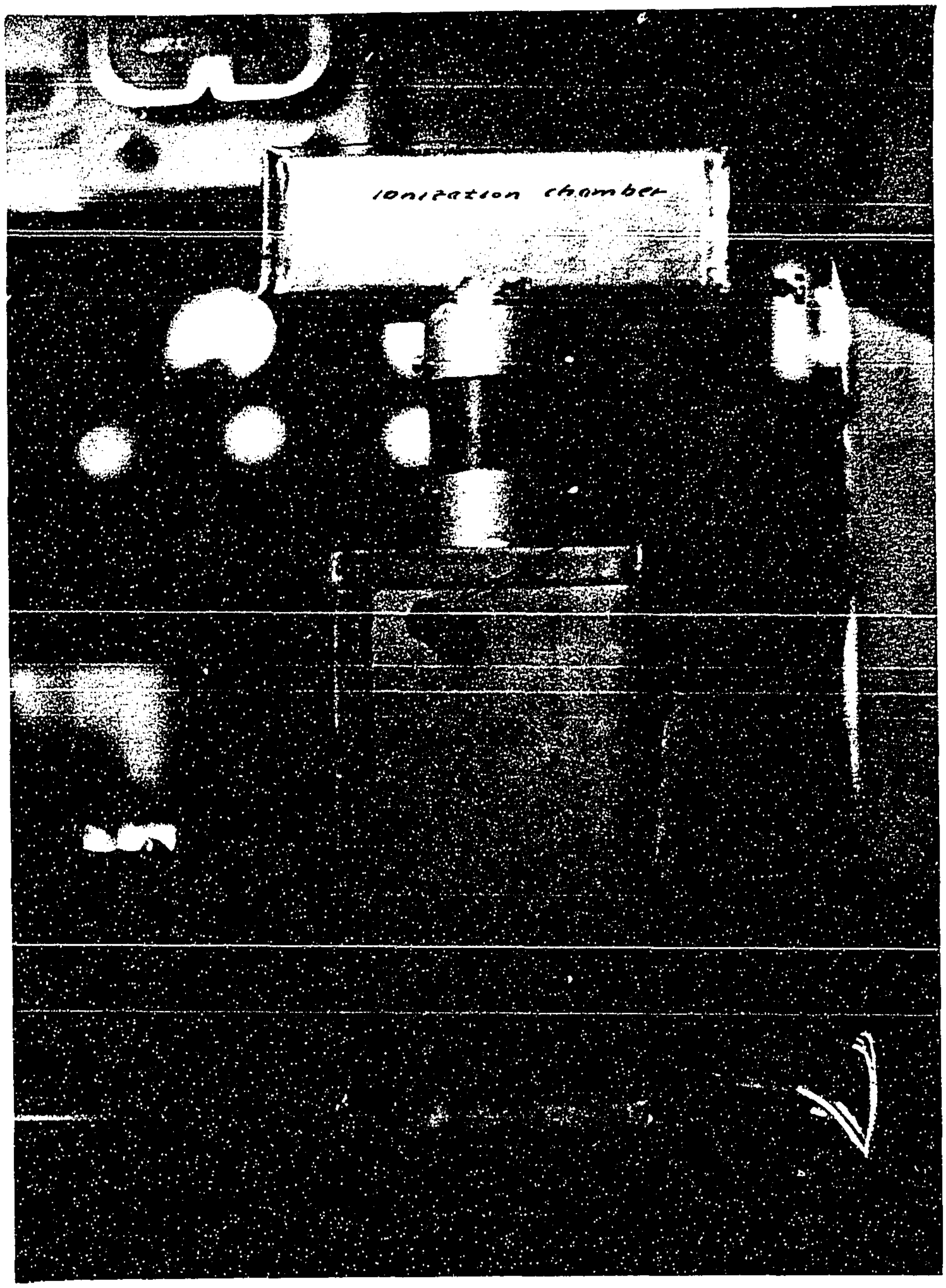
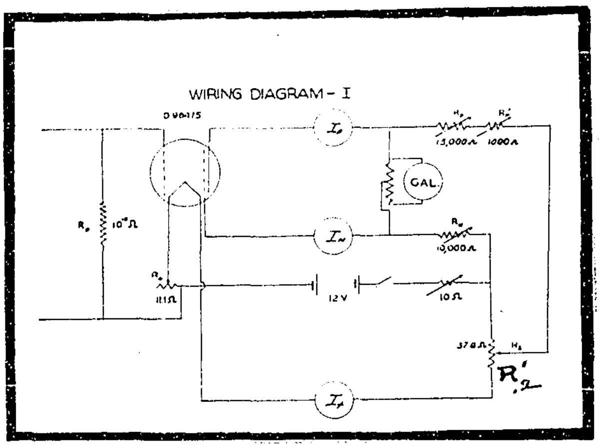


Fig. 6. The ionization chamber and tube container.

MATHEMATICAL RELATIONS IN BALANCING:

In balancing the problem is to determine such values of our circuit resistances so that the tube shall be operated with the desired voltage, the current through the galvanometer shall be approximately zero and the galvanometer current shall be approximately constant for small variations of the filament current. The equations may easily be deduced from the following circuit:



We shall use the symbols:

e_p = the plate voltage.

e_n = the inner grid voltage.

e_g = the control grid voltage.

e_b = the battery voltage.

That is we shall use these in addition to those given on the circuit. From the circuit we have immediately:

1. $e_s = -i_f r_f$

2. $e_b = e_f + i_f(R_f + R_4 + R_2)$

3. $e_p = e_f + i_f R_2' - i_p R_p$; $\quad R_p = (R_p + R_p')$

4. $e_n = e_f + i_f R_2 - i_n R_n$

We impose the condition

$$R_p \Delta i_p = R_n \Delta i_n$$

Computing the changes in e_p and e_n for small changes in i_f we have:

$$\Delta e_p = \frac{\partial e_f}{\partial i_f} \Delta i_f + R_2' \Delta i_f - R_p \left(\frac{\partial i_p}{\partial i_f} \Delta i_f + \frac{\partial i_p}{\partial e_s} \Delta e_s + \frac{\partial i_p}{\partial e_n} \Delta e_n + \frac{\partial i_p}{\partial e_p} \Delta e_p \right)$$

Transposing and collecting we are lead to:

$$\left(1 + R_p \frac{\partial i_p}{\partial e_p}\right) \Delta e_p = \left(\frac{\partial e_f}{\partial i_f} + R_2' - R_p \frac{\partial i_p}{\partial i_f} \right) \Delta i_f - R_p \frac{\partial i_p}{\partial e_s} \Delta e_s - R_p \frac{\partial i_p}{\partial e_n} \Delta e_n$$

Since

$$\Delta e_n = \frac{\partial e_f}{\partial i_f} \Delta i_f + R_2' \Delta i_f - R_n \left(\frac{\partial i_s}{\partial i_f} \Delta i_f + \frac{\partial i_n}{\partial e_s} \Delta e_s + \frac{\partial i_n}{\partial e_n} \Delta e_n \right) - R_n \frac{\partial i_n}{\partial e_p} \Delta e_p$$

The condition of balance may be written:

$$\Delta e_p = \Delta e_n$$

Substituting the values found for these increments in the preceding equation with the substitutions:

$$g_{pp} = \frac{\partial i_p}{\partial e_p} \qquad g_{sn} = \frac{\partial i_n}{\partial e_s} \qquad K_{fn} = \frac{\partial i_n}{\partial i_f}$$

$$g_{sp} = \frac{\partial i_p}{\partial e_s} \qquad g_{np} = \frac{\partial i_p}{\partial e_n} \qquad K_{fp} = \frac{\partial i_p}{\partial i_f}$$

$$g_{nn} = \frac{\partial i_n}{\partial e_n} \qquad g_{pn} = \frac{\partial i_n}{\partial e_p} \qquad r_f = \frac{\partial e_f}{\partial i_f}$$

we have

$$(1 + R_p g_{pp}) \Delta e_p = (r_f + R_2' - R_p K_{fp}) \Delta i_f - R_p g_{sp} \Delta e_s - R_p g_{np} \Delta e_n$$

Since $\Delta e_g = -\Delta i_f R_f$

We have

$$(1 + R_p g_{pp})\Delta e_p = (r_f + R_z' - R_p K_{fp} + R_p R_f g_{gp})\Delta i_f - R_p g_{np}\Delta e_n$$

In order to simplify further we introduce

$$R_p(g_{pp} + g_{np}) = R_p g_p$$

$$R_n(g_{nn} + g_{pn}) = R_n g_n$$

Our equations become then

$$(1 + R_p g_p)\Delta e_p = (r_f + R_z' - R_p K_{fp} + R_p g_{gp} R_f)\Delta i_f$$

$$(1 + R_n g_n)\Delta e_n = (r_f + R_z - R_n K_{fn} + R_n g_{gn} R_f)\Delta i_f$$

Substituting in $\Delta e_p = \Delta e_n$ we have

$$5. \quad \frac{1 + R_p g_p}{1 + R_n g_n} = \frac{r_f + R_z' - R_p K_{fp} + R_p g_{gp} R_f}{r_f + R_z - R_n K_{fn} + R_n g_{gn} R_f}$$

This gives us a fifth equation. Since there are six unknowns and only five equations one of the unknowns must be fixed arbitrarily. This arbitrary element gives the circuit a degree of flexibility lacking in some of the other circuits.

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Part III

ON THE QUANTIZATION OF MASS

A dissertation submitted to the

Graduate School
of the University of Cincinnati

in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

1939

by

Herman Russell Branson

B.S. Virginia State College 1936

TABLE OF CONTENTS:

1. Introduction

The underlying idea of this investigation is the following: If the Dirac equation is solved for the term involving the mass of the electron, then it is seen that the operator

$$\frac{i\hbar}{c}\left\{ \frac{\alpha}{c}\frac{\partial\psi}{\partial t} - \left(\beta_1\frac{\partial}{\partial x} + \beta_2\frac{\partial}{\partial y} + \beta_3\frac{\partial}{\partial z}\right)\right\} \qquad 1.1$$

where α and β's are matrices of the Dirac type, can be regarded as representing the electron mass m. We are then justified in looking for eigen values of this operator, which would be the possible values of the electron mass. However the fact that the Dirac equation for a free electron has solutions for an arbitrary value of m shows that in ordinary flat space-time of the special relativity theory all values of m are possible. This is analogous to the fact that in an infinite space a free particle can have arbitrary energy.

In a box of finite dimensions,as is well known, the energy is quantized. This occurs as the result of the boundary conditions that have to be imposed on ψ at the boundaries of x, y, and z. Accordingly if one could impose boundary conditions at the boundaries of time, t, one would expect the existence of a discrete spectrum for m. A somewhat similar situation occurs if a curved space-time is finite (re-entrant) although not bounded. The boundary conditions are then replaced by the periodicity conditions. The possible values of m would then depend upon the radius of the universe, constants c and h, and a quantum number. The idea, that the mass of the electron depends upon the kind of universe it is in, is attractive, as well as the corollary that the electrons of various rest masses may be different states of essentially the same particle.[1] It seemed therefore worthwhile to try to solve the Dirac equation in various cosmological spaces.

A start in this direction was made by Taub[2] who has made the calculations for a number of cosmological spaces. Although Taub, apparently, was not primarily interested in the eigenvalues of m, he finds that for the spaces considered by him there is no reason to assign any particular value to this number, except possibly when field is present.[3]

2. Regular Cosmological Spaces

Robertson showed that a large group of cosmological spaces admits a metric

$$ds^2 = c^2 dt^2 - R^2(t)\, du^2 \qquad 2.1$$

where du^2 is the three dimensional space metric.[4] We shall

1. Neddermeyer, Phys. Rev. 53, 102(1938)
2. Taub, "Quantum Equations in Cosmological Spaces", Phys. Rev. 51, 512(1937). Hereafter referred to as Taub.
3. Ref. 2, p. 514.
4. Robertson, "Relativistic Cosmology", Rev. Mod. Phys. 5, 62(1933).

call spaces defined by a metric of this type regular. Three cases are to be distinguished, corresponding, respectively, to the three-dimensional space having positive, zero, or negative curvature. The metric du for these three cases can be written in the forms

$$du^2 = d\alpha^2 + \sin^2\alpha\,(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad 2.2$$

$$du^2 = d\alpha^2 + \alpha^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad 2.3$$

and

$$du^2 = d\alpha^2 + \sinh^2\alpha\,(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad 2.4$$

respectively. These three are particular cases of the metric

$$du^2 = d\alpha^2 + f^2(\alpha)\,(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad 2.5$$

Throughout this work we shall treat cases where there are no fields and the potentials are put equal to zero. We us Taub's Eq. 3.1 as our generalized Dirac equation in the absence of an external electromagnetic field:

$$\left[\gamma^0\left(\frac{\partial}{\partial t} + \frac{3}{2}\frac{R'(t)}{R(t)}\right) - \frac{i\sigma^j}{R}\left(\frac{\partial}{\partial x^j} + \frac{3}{4}\frac{\partial}{\partial x^j}\log h^{1/2} + \frac{1}{4}h_{jl}\frac{\partial h^{lk}}{\partial x^k} + \sigma_l\frac{\partial\sigma^l}{\partial x^j}\right]\psi = \mu\psi. \qquad 2.6$$

in which $\gamma^j = -\dfrac{i\sigma^j}{R(t)}$

Eq. 2.6 may be written as Taub's Eq. 3.12 but where

$$H\psi = \frac{1}{R}\left[\beta_\alpha\frac{\hbar}{i}\left(\frac{\partial}{\partial\alpha} - \frac{1}{f(\alpha)}\right) + \frac{1}{f(\alpha)}\frac{\hbar}{i}\left(\beta_\theta\frac{\partial}{\partial\theta} + \frac{\beta_\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) + mc\,\alpha_4\right]\psi \qquad 2.7$$

instead of Taub's Eq. 3.13 where our β's are defined by

$$\beta_\alpha = \sin\theta\cos\phi\,\alpha_1 + \sin\theta\sin\phi\,\alpha_2 + \cos\theta\,\alpha_3$$

$$\beta_\theta = \cos\theta\cos\phi\,\alpha_1 + \cos\theta\sin\phi\,\alpha_2 - \sin\theta\,\alpha_3 \qquad 2.8$$

$$\beta_\phi = -\sin\phi\,\alpha_1 + \cos\phi\,\alpha_2$$

The treatment of the angular momentum for this equation is identical with the treatment of Taub. The solutions of Eq. 2.7 are then

$$\psi_1 = A_1(\alpha,t)\left(\frac{k+m}{2k-1}\right)^{1/2}\varphi(k-1,m) \qquad \psi_3 = A_3(\alpha,t)\left(\frac{k-m}{2k+1}\right)\varphi(k,m)$$

$$\psi_2 = -A_1(\alpha,t)\left(\frac{k-m-1}{2k-1}\right)^{1/2}\varphi(k-1,m+1) \qquad \psi_4 = A_3(\alpha,t)\left(\frac{k+m+1}{2k+1}\right)\varphi(k,m+1) \qquad 2.9$$

where A_1 and A_3 are functions of α and t, and k is the eigenvalue of the angular momentum operator, k an integer ≥ 1. Substituting these values in Eq. 2.7 we have

$$R(t)\left(\frac{\hbar}{ic}\frac{\partial}{\partial t} - mc\right)A_1 = \frac{\hbar}{i}\left(\frac{\partial}{\partial\alpha} + \frac{k}{f(\alpha)}\right)A_3$$

$$R(t)\left(\frac{\hbar}{ic}\frac{\partial}{\partial t} + mc\right)A_3 = \frac{\hbar}{i}\left(\frac{\partial}{\partial\alpha} - \frac{k}{f(\alpha)}\right)A_1 \qquad 2.10$$

We eliminate A_1 and then A_3 from this system of equations obtaining the equation satisfied by each A, namely:

$$R(t)\left(\frac{\hbar}{ic}\frac{\partial}{\partial t} \pm mc\right)R(t)\left(\frac{\hbar}{ic}\frac{\partial}{\partial t} \mp mc\right)A = \left(\frac{\hbar}{i}\right)^2\left(\frac{\partial}{\partial\alpha} \pm \frac{k}{f(\alpha)}\right)\left(\frac{\partial}{\partial\alpha} \mp \frac{k}{f(\alpha)}\right)A \qquad 2.11$$

where the upper sign goes with A_1 and the lower with A_3. Making the assumption that $A_i(\alpha,t) = A_i(\alpha)\,T_i(t)$ (i = 1 or 3) and separating variables we have the "time equations"

$$\frac{d^2T}{dt^2} + \frac{d\log R}{dt}\frac{dT}{dt} + c^2\left[\frac{W_i^2}{h^2R^2} + \frac{m^2c^2}{h^2} \mp \frac{im}{h}\frac{d\log R}{dt}\right]T = 0 \qquad 2.12$$

with $W_i = W_1$ for T_1, W_3 for T_3; and at the same time the general "radial equations"

$$\frac{d^2A}{d\alpha^2} + \frac{\pm k f'(\alpha) - k^2}{f^2(\alpha)}A + \frac{W_i^2}{\hbar^2}A = 0 \qquad 2.13$$

in which the upper sign goes with the subscript 1 and the lower with subscript 3.

3. The Radial Equation

For the case of space curvature equal to + 1, $f(\alpha) = \sin\alpha$, Eq. 2.13 becomes

$$-\hbar^2 \left[\frac{d^2}{d\alpha^2} - \frac{K^2 \mp K\cos\alpha}{\sin^2\alpha} \right] A = W^2 A \qquad 3.1$$

where as before the upper sign goes with A_1, and the lower with A_3. The solutions are given by Taub and require that

$$\left(\frac{W}{\hbar}\right)^2 = \left[n + \left(\frac{2K+1}{2}\right) \right]^2 \qquad 3.2$$

This leads to the quantization of energy, as Taub shows in the special cases considered by him.

Space Curvature = 0.

The solutions of the radial equation for this case leave W arbitrary, as the limiting case treated by Taub shows.

Space Curvature = -1.

This specification includes the Milne universe treated by Taub. The radial equation in this case is

$$-\hbar^2 \left[\frac{d^2}{d\alpha^2} - \frac{K^2 \mp K\cosh\alpha}{\sinh^2\alpha} \right] A = W^2 A \qquad 3.3$$

We see that this equation may be obtained from Eq. 3.1 by replacing h by ih and α by $i\alpha$. Taub states then "... the solutions of the Dirac equation for the 3 space of negative curvature can be obtained from that of positive curvature by replacing ρ by $i\rho$ (in our case h by ih) and α by $i\alpha$ in the solutions we have obtained." But this does not seem to be correct for in Eq. 2.2 $-\pi \leq \alpha \leq \pi$ or $0 \leq \alpha \leq 2\pi$ while in Eq. 2.4 $0 \leq \alpha \leq \infty$. With this range for there are no well-behaved functions A_1 and A_3 simultaneously satisfying Eq. 4.4 unless the eigenvalue of the angular momentum operator be equal to zero which is disallowed.

Thus we have a quantization of W only in a universe of positive space curvature. Referring to Eq. 2.10 we see that a quantization of m is most likely to result when there is a quantization of W. Hence the most promising cases to be investigated are those with positive space curvature. We shall learn, however, that even in these most favorable cases there is no quantization of m.

4. Time Curvature

The formula for the Riemannian curvature of a space[1] is

$$\mathcal{K} = \frac{R_{\alpha\beta\gamma\delta}\,\lambda_1^{\alpha}\lambda_2^{\beta}\lambda_1^{\gamma}\lambda_2^{\delta}}{(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\,\lambda_1^{\alpha}\lambda_2^{\beta}\lambda_1^{\gamma}\lambda_2^{\delta}} \qquad 4.1$$

The g's are taken from our fundamental metric Eq. 2.5. The unit vectors λ_1^{σ} and λ_2^{σ} determine the orientation in this space. For our 4-space we select

$$\lambda_1^{\sigma} = \delta_{0\sigma} \quad ; \quad \lambda_2^{\rho} = \delta_{1\rho} \qquad 4.2$$

Eq. 4.1 then gives

$$\mathcal{K} = -\frac{R''}{c^2 R} \qquad 4.3$$

Had we selected either $\lambda_2^{\rho} = \delta_{2\rho}$ or $\lambda_2^{\rho} = \delta_{3\rho}$ instead of $\lambda_2^{\rho} = \delta_{1\rho}$ we would arrive at the same equation, Eq. 4.3. Thus the

1. Robertson, p.84.

orientations determined by selecting any one of the unit space vectors with the unit vector in the time direction are equivalent. This indifference to the spatial directions permits the designation "time curvature" for the quantity thus obtained. The simplest cases are those with constant $\varkappa$. Here we may distinguish three cases:

$\varkappa > 0$: $\quad R = A \cos \lambda t$ or $= B \sin \lambda t$ $\quad$ 4,4

$\varkappa = 0$: $\quad R = at + b$ $\quad$ 4.5

$\varkappa < 0$: $\quad R = b \cosh ct/b$ or $= b \sinh ct/b$ $\quad$ 4.6

Each of these R(t) determines a specific type of universe. If we accept as physically plausible that the universe did not come into existence at the time, t =0, then the universes given by R = b sinh ct/b and R = b sin at are disallowed. However the equations with these functions have practically the same solutions as the corresponding cosh and cos equations with identical requirements on m.

5. The Time Equation

Time Curvature < 0.

Negative time curvature occurs, for example, in considering a generalized DeSitter's[1] space-time which can be regardedas a hyperboloid

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = b^2 \qquad 5.1$$

embedded in a 5-dimensional Euclidean space, with cartesian coordinates (x_0 , x_1 , x_2 , x_3 , x_4).

A convenient coordinate system, suitable over the entire hyperboloid, may be introduced as follows: let

$$x_0 = b \sinh ct/b$$

and

$$x_4 = b \cosh ct/b \cos r/b \qquad 5.2$$

Substituting into Eq. 5.1 then gives

$$x_1^2 + x_2^2 + x_3^2 = \{ b \cosh (\tfrac{ct}{b}) \sin \tfrac{r}{b} \}^2 \qquad 5.3$$

Which suggests putting

$$x_1 = x = \cosh \frac{ct}{b} \sin \frac{r}{b} \cos \phi$$

$$x_2 = y = \cosh \frac{ct}{b} \sin \frac{r}{b} \sin \phi \qquad 5.4$$

$$x_3 = z = \cosh \frac{ct}{b} \sin \frac{r}{b}$$

The metric is obtained by substituting from Eqs. 5.2 and 5.4 into the 5-dimensional Euclidean metric, thus:

$$- ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_0^2$$

$$= - c^2 dt^2 + \cosh^2 (\tfrac{ct}{b}) \left[dr^2 + b^2 \sin^2 \tfrac{r}{b} (d\theta^2 + \sin^2\theta d\phi^2) \right] \qquad 5.5$$

In this coordinate system there is no "horizon"[2].

The spatial part of this metric is easily reduced to the form 2.2 if we put $r/b = \alpha$, and on comparing with 2.1 with see that

$$R(t) = b \cosh (ct/b) \qquad 5.6$$

1. Robertson, p.70.
2. Robertson, p.71.

The range of the variables required to comprise the entire hyperboloid are:

$$0 \le \phi \le 2\pi, \quad 0 \le \theta \le \pi, \qquad 0 \le \alpha \le \pi, \quad -\infty \le t \le \infty \qquad 5.7$$

Eq. 2.12 becomes on substituting from Eq. 5.6

$$\frac{d^2T}{dx^2} - \frac{x}{1-x^2}\frac{dT}{dx} + \left\{ \frac{\beta^2(1-x^2) + \mu^2 - i\mu x}{(1-x^2)^2} \right\} T = 0 \qquad 5.8$$

where $\beta = W/\hbar, \quad \mu = mcb/\hbar$

and $x = \tanh(ct/b)$

Well-behaved solutions of this equation are

for $W < 0$:

$$T = (1+x)^{\frac{i\mu}{2}+\beta}(1-x)^{-i\mu/2} F(-\beta, \tfrac{1}{2}-\beta-i\mu, 1-2\beta, \tfrac{2}{1+x}) \qquad 5.9$$

for $W > 0$:

$$T = (1+x)^{\frac{i\mu}{2}-\beta}(1-x)^{-i\mu/2} F(\beta, \tfrac{1}{2}+\beta-i\mu, 1+2\beta, \tfrac{2}{1+x})$$

Both requiring only that m be real.

Time Curvature = 0

Two examples of this curvature are found in the work of Taub. The first, $R(t) = a$, is his Einstein's universe; the second, $R(t) = ct$, which is equivalent to Eq. 4.5, leads to the Milne universe. Neither of these requires any limitation on m.

Time Curvature > 0

For positive time curvature, $R(t) = \cos(ct/b)$, we have

$$\frac{d^2T}{dx^2} + \frac{x}{1+x^2}\frac{dT}{dx} + \left\{ \frac{\beta^2(1+x^2) + \mu^2 + i\mu x}{(1+x^2)^2} \right\} T = 0 \qquad 5.10$$

$$\beta = \frac{W}{\hbar}$$

where $x = \tan(ct/b)$ and $\mu = bmc/h$. Satisfactory solutions of Eq. 5.10 are :

for $iW > 0$:

$$T = (1+x)^{\mu/2}(1-x)^{-i\beta-\mu/2} F\left(i\beta, \tfrac{1}{2}+i\beta-\mu, 1+2i\beta, \tfrac{2}{1+x}\right) \qquad -\tfrac{1}{2} < \frac{mcb}{h} \leq 0$$

for $iW < 0$: 5.11

$$T = (1+x)^{\mu/2}(1-x)^{i\beta+\mu/2} F\left(-i\beta, \tfrac{1}{2}-i\beta-\mu, 1-2i\beta, \tfrac{2}{1+x}\right) \qquad -1 \leq \frac{mcb}{h} < -\tfrac{1}{2}$$

Hence positive time curvature requires a quantization of m. But the conditions on W and the negative value for m renders this solution unacceptable. Thus none of the universes considered ahead require a quantization of m that is physically meaningful and acceptable.

6. A Non-Regular Universe

A non-regular universe can be defined as one whose metric is not of the form 2.5. We consider one such universe - one arising from Eddington's suggestion[1] for a suitable surface representing the universe with only a single charged particle present. This is a 4-dimensional space embedded in a 5-dimensional Euclidean space and given by

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 = a^2 \qquad 6.1$$

with t_1, t_2, and t_3 real and t_4 and t_5 imaginary. By placing $t_1 = x$, $t_2 = y$, $t_3 = z$, $t_4 = i\lambda$, and $t_5 = i\mu$ we have

$$x^2 + y^2 + z^2 = a^2 + \lambda^2 + \mu^2 \qquad 6.2$$

which shows that a^2 must be negative. The metric is

$$\begin{aligned} ds^2 &= -(dt_1^2 + dt_2^2 + dt_3^2 + dt_4^2 + dt_5^2) \\ &= (d\lambda^2 + d\mu^2) - (dx^2 + dy^2 + dz^2) \end{aligned} \qquad 6.3$$

---

1. Eddington, "The Relativity Theory of Electrons and Protons", Ch.6.

If we let

$$\begin{aligned} b^2 &= -a^2 \\ x &= b \sinh\alpha \sin\theta \cos\phi \\ y &= b \sinh\alpha \sin\theta \sin\phi \\ z &= b \sinh\alpha \cos\theta \\ \mu &= b \cosh\alpha \sin\frac{ct}{b} \\ \lambda &= b \cosh\alpha \cos\frac{ct}{b} \end{aligned} \tag{6.4}$$

we obtain for the metric

$$ds^2 = c^2 \cosh^2\alpha \, dt^2 - b^2 \left[d\alpha^2 + \sinh^2\alpha \, (d\theta^2 + \sin^2\theta \, d\phi^2) \right] \tag{6.5}$$

where

$$0 \leq \alpha \leq \infty \qquad 0 \leq \theta \leq \pi$$

$$0 \leq t \leq \frac{2\pi b}{c} \qquad 0 \leq \phi \leq 2\pi$$

t = time.

A test particle in the universe determined by this metric, which is not initially at the origin, will oscillate through the origin. After a time $2\pi b/c$ the universe returns to its original state and the particle is at its initial position ready to proceed again through its history.

For the wave equation of a particle in this universe we found it more convenient to use the relativistic form of Dirac's equation proposed by Fock[1].

$$\gamma^\sigma \left(\frac{\partial \psi'}{\partial x^\sigma} - \Gamma_\sigma \psi' \right) - \frac{i}{\hbar} mc\, \alpha_4 \psi' = 0 \tag{6.6}$$

with the definitions

$$\gamma^\sigma = \sum_k e_k \alpha_k h_k^\sigma$$

1. V. Fock, "Geometrisierung der Diraschen Theorie des Elektrons", Zeit. f. Phys. 57, 270 (1929).

$$\Gamma_\sigma = \sum_k e_k h_k$$

$$C_\ell = \frac{1}{4} \sum_{mk} \alpha_m \alpha_k e_k \gamma_{mk\ell} + i\Phi_\ell$$

$$\gamma_{ik\ell} = (\nabla_\sigma h_i^{\beta}) h_{k\beta} h_\ell^{\sigma}$$

The ∇_σ means the covariant derivative with respect to x_σ. The Φ_ℓ is a constant times the vector of potential. Since we are treating the case of no field we may set it equal to zero. The α's are Dirac matrices. The h's are n-beins or orthogonal ennuples introduced at each point of space for convenience. The h's are related to the g's by the equations

$$g^{\alpha\beta} = \sum_k e_k h_k^{\alpha} h_k^{\beta} \qquad 6.8$$

$$g_{\alpha\beta} h_k^{\beta} = h_{k\alpha} \qquad 6.9$$

Here the n-beins are designated with Latin and the coordinates with Greek indices. The e's are the Eisenhart e's introduced for convenience of dealing with an indefinite metric.

We make the correspondence

$$\begin{aligned} x_0 &= t \\ x_1 &= \alpha \\ x_2 &= \theta \\ x_3 &= \phi \end{aligned} \qquad 6.10$$

So that

$$g_{00} = c^2 \cosh^2\alpha \qquad g_{22} = -b^2 \sinh^2\alpha$$

$$g_{11} = -b^2 \qquad g_{33} = -b^2 \sinh^2\alpha \sin^2\theta \qquad 6.11$$

$$\sqrt{-g} = c b^3 \sinh^2\alpha \cosh\alpha \sin\theta$$

From the possibilities for the h's satisfying Eq.6.8 we select

$$h^1_1 = \frac{\sin\theta\cos\phi}{b} \qquad h^2_1 = \frac{\cos\theta\cos\phi}{b\sinh\alpha}$$
$$h^1_2 = \frac{\sin\theta\sin\phi}{b} \qquad h^2_2 = \frac{\cos\theta\sin\phi}{b\sinh\alpha}$$
$$h^1_3 = \frac{\cos\theta}{b} \qquad h^2_3 = -\frac{\sin\theta}{b\sinh\alpha} \tag{6.12}$$
$$h^3_1 = -\frac{\sin\varphi}{b\sinh\alpha\sin\theta}$$
$$h^3_2 = \frac{\cos\varphi}{b\sinh\alpha\sin\theta}$$

All others are zero. The $h_{\kappa\alpha}$ are obtained from these by use of Eq.6.9. The non-vanishing Christoffel symbols are

$$\left\{ {0 \atop 1\ 0} \right\} = \tanh\alpha \qquad \left\{ {2 \atop 3\ 3} \right\} = -\sin\theta\cos\theta$$

$$\left\{ {1 \atop 0\ 0} \right\} = \frac{c^2}{b^2}\sinh\alpha\cosh\alpha \qquad \left\{ {3 \atop 2\ 3} \right\} = ctg\ \theta$$

$$\left\{ {1 \atop 2\ 2} \right\} = -\sinh\alpha\cosh\alpha \qquad \left\{ {2 \atop 1\ 2} \right\} = ctgh\ \alpha \tag{6.13}$$

$$\left\{ {1 \atop 3\ 3} \right\} = -\sinh\alpha\cosh\alpha\sin^2\theta \qquad \left\{ {3 \atop 1\ 3} \right\} = ctgh\ \alpha$$

We introduce these values into Eq. 6.7 and substitute into Eq. 6.6 and after some reduction we arrive at

$$\left\{ \frac{\beta_\alpha}{b}\left(\frac{\partial}{\partial\alpha} - \frac{1}{\sinh\alpha} + \coth\alpha \right) + \frac{1}{b\sinh\alpha}\left(\beta_\theta\frac{\partial}{\partial\theta} + \frac{\beta_\phi}{\sin\theta}\frac{\partial}{\partial\phi} \right) + \frac{i}{\hbar}mc\alpha \right\}\psi' = \frac{1}{c\cosh\alpha}\frac{\partial\psi'}{\partial t}. \tag{6.14}$$

where the β's are defined as in Eq. 2.8.

This equation may be simplified by making the substitution $\psi = \sinh\alpha\,\psi'$ so as to eliminate the $\coth\alpha$ term and eliminating the angular momentum as Taub did. The procedure is identical. The solutions are Eq. 2.9 where the A's are functions of α and t to be determined. In virtue of Eq. 2.9 we have as our system of time-radial equations

$$\left(\frac{\hbar}{ic}\frac{\partial}{\partial t} - mc\cosh\alpha\right) A_1(\alpha,t) = \frac{\hbar}{ib}\cosh\alpha\left(\frac{\partial}{\partial\alpha} + \frac{\kappa}{\sinh\alpha}\right) A_3(\alpha,t) \qquad 6.15$$

$$\left(\frac{\hbar}{ic}\frac{\partial}{\partial t} + mc\cosh\alpha\right) A_3(\alpha,t) = \frac{\hbar}{ib}\cosh\alpha\left(\frac{\partial}{\partial\alpha} - \frac{\kappa}{\sinh\alpha}\right) A_1(\alpha,t)$$

We assume as before that $A_i(\alpha,t) = A_i(\alpha)\,T_i(t)$.

Eliminating in the system 6.15 after this substitution we have for the time equations

$$\frac{T_1''}{T_1} = \frac{T_3''}{T_3} = -\frac{\omega^2}{b^2} \qquad 6.16$$

This gives

$$T_3 = T_1 = e^{-\frac{i\omega}{b}t}$$

In order that the universe be periodic it is necessary that

$$\frac{2\pi\omega b}{\hbar c}\,i = 2\pi N i \qquad 6.17$$

where N is an integer.

The radial equations may then be written

$$A_1(\alpha)\left(\frac{\hbar\beta N}{b} - mc\cosh\alpha\right) = \frac{\hbar}{ib}\cosh\alpha\left(\frac{\partial}{\partial\alpha} + \frac{\kappa}{\sinh\alpha}\right) A_3(\alpha)$$

$$A_3(\alpha)\left(\frac{\hbar\beta N}{b} + mc\cosh\alpha\right) = \frac{\hbar}{ib}\cosh\alpha\left(\frac{\partial}{\partial\alpha} - \frac{\kappa}{\sinh\alpha}\right) A_1(\alpha) \qquad 6.18$$

$\beta = -1$.

resulting in

$$(x^2-1)\frac{d^2A}{dx^2} + \left(2x - \frac{1}{x} - \frac{x^2-1}{x\pm\gamma}\right)\frac{dA}{dx} + \left\{\frac{\rho^2\gamma^2}{x^2} - \rho^2 \pm \frac{\kappa}{x\pm\gamma} - \frac{\kappa^2 \mp \kappa x}{x^2-1}\right\}A = 0 \qquad 6.19$$

$$\gamma = -\frac{n h}{mcb}\,; \quad \rho = \frac{mcb}{h}$$

with the substitution $x = \cosh\alpha$ and where the upper sign goes with A_1 and the lower with A_3.

This is a Riemann P equation with 5 singularities. Since the singularities are not elementary in the sense of Ince[1] the equation is not easily solvable. Our chief interest is in ascertaining whether this equation even in a special case has solutions - so long as that special case does not embody extreme assumptions. Hence we make a try for a solution by bringing about such a confluence of singularities that the resulting equation may be handled and, further, we attempt this solution with $k = 1$. The most obvious specialization is the confluence of singularities at ∓ 1 by putting

$$\gamma = -1 \qquad 6.20$$

We differential equation we seek to solve is then

$$(x^2-1)\frac{d^2A}{dx^2} + \left(x - \frac{1}{x} \mp 1\right)\frac{dA}{dx} + \frac{\pm x(x^2+1) - N^2(x^2-1)^2}{x^2(x^2-1)}A = 0 \qquad 6.21$$

We introduce a function F by the substitution

$$A = \frac{x^{-N}(x-1)}{(x+1)}F \qquad 6.22$$

The equation satisfied by F is

$$u(u-1)(u+1)\frac{d^2F}{du^2} + \left[(4 \mp \tfrac{1}{2})u^2 - (6 \pm \tfrac{1}{2} + 2N)u + (4 \pm 1)\right]\frac{dF}{du} +$$

$$2(2-2N)\left[u - (1 \pm \tfrac{1}{4})\right]F = 0 \qquad 6.23$$

1. E.L. Ince, Ordinaty Differential Equations, p.495

where $u = (1-x)/(1+x)$

This is in Heun's canonical form[1]:

$$x(x-1)(x-a)\frac{d^2F}{dx^2} + \left[(\alpha+\beta+1)x^2 - \{\alpha+\beta+a\gamma+(a-1)\delta\}x + a\gamma\right]\frac{dF}{dx} + \alpha\beta(x-q)F = 0 \qquad 6.24$$

where

$$F = F(a, q; \alpha, \beta, \gamma, \delta; x) = 1 + \sum \frac{G^{(\nu)}(q)}{\nu!\,\gamma(\gamma+1)\cdots(\gamma+\nu+1)}\left(\frac{x}{a}\right)^{\nu} \qquad 6.25$$

with

$$G^{(1)}(q) = q \quad ; \quad G^{(2)}(q) = \alpha\beta q^2 + \{\alpha+\beta-\delta+1+(\gamma+\delta)a\}q - a\gamma$$

and in general

$$G^{(\nu+1)}(q) = \{\nu[\alpha+\beta-\delta+\nu+(\gamma+\delta+\nu-1)a+\alpha\beta q\}G^{(\nu)}(q) - (\alpha+\nu-1)(\beta+\nu-1)(\gamma+\nu-1)\nu a\, G^{(\nu-1)}(q).$$

The characteristics of such functions are discussed by Heun[2]. A value of F giving well-behaved functions, i.e. quadratically integrable functions, A, is

$$F = F\left(-1, 1 \pm \tfrac{1}{q}; 2, 1 \mp \tfrac{1}{2}, 2 \mp \tfrac{1}{2}, 1-2N; \frac{1-x}{1+x}\right) \qquad 6.26$$

in which x ranges between 1 and ∞ and the integer

$$N \geqslant 5.$$

Eq. 6.20 may be written

$$m = Nh/bc. \qquad 6.27$$

Thus we have shown that the Dirac equation in this universe possesses solutions requiring the quantization of mass. Looking at Eq. 6.27 alternatively as giving the radius of

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1. Karl Heun, "Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten," Math. Ann. 33, 165(1889).
2. Karl Heun, "Beiträge zur Theorie Lame'chen Functionen," Math. Ann. 33, 181(1889)

the universe in terms of m, c, h and the quantum number N, where m is the rest mass of the electron, we have for the lowest mass state, $N = 5$,

$$b \sim 10^{-10} \text{cm}.$$

a value which is much too small to be interpreted as the radius of the macroscopic universe. If we were to try to make the result consistent with an acceptable value of b, for one of the suggested closed universes, it would be necessary to look upon the electron as being in a very high state, $N \sim 10^{-37}$, of the elementary particle, whose lowest mass state would be

$$m \sim 10^{-65} \text{ gms}.$$

We must keep in mind tha these interpretations are based upon the tacit assumption that Dirac's equation is corresct for all distances. Since it is known to be unacceptable for distances of the order of 10^{-13}cm., it is not unnatural to question this great extrapolation. It is one of those cases in which success would be very gratifying, but the failure is not surprising.