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ON THE SUMMABILITY OF DOUBLE FOURIER SERIES

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ON THE SUMMABILITY OF DOUBLE FOURIER SERIES

PART I

On the Logarithmic Mean of Double Fourier Series

The wide application of single and double Fourier series to problems in mathematical physics has justified a close investigation of their summability by various methods, as well as of their convergence. One of the most interesting and significant of the problems relating to the summability of a Fourier series by Cesàro means has been the so-called Cesàro summability problem which has been solved, but only in a modified form, by Hardy and Littlewood* for a single Fourier series. For the Fourier development of an integrable function bounded near the point under consideration they have derived a simple condition which is necessary for summability by Cesàro means of any order and sufficient for summability by Cesàro means of every positive order. A similar theorem for double series was first considered by Merriman,† and later by Gergen and Littauer.‡

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*Hardy and Littlewood, Solution of the Cesàro summability problem for power series and Fourier series, Mathematische Zeitschrift, vol. 19 (1924), pp. 67-96.

†G. M. Merriman, A set of necessary and sufficient conditions for the Cesàro summability of double series, Annals of Mathematics, (2), vol. 29 (1928), pp. 343-354.

‡J. J. Gergen and S. B. Littauer, Continuity and summability for double Fourier series, Transactions of the American Mathematical Society, vol. 38 (1935), pp. 401-435.

The fact that simple bounded functions exist which do not satisfy the conditions derived by Hardy and Littlewood, but whose Fourier developments are summable by Riesz's logarithmic means suggested the desirability of finding a general theorem, analogous to that of Hardy and Littlewood, concerning summability by Riesz's logarithmic means. Hardy* has proved a theorem of this type for single Fourier series. He has derived a condition which is both necessary and sufficient for the summability of a single Fourier series by Riesz's logarithmic means. It is the purpose of Part I of the present paper to prove a similar theorem for double Fourier series. Here only sufficient conditions for the summability of a double Fourier series by logarithmic means are derived. The author has not been able to show that these conditions are also necessary.

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*G. H. Hardy, The summability of a Fourier series by logarithmic means, Quarterly Journal of Mathematics, Oxford Series, vol. 2 (1931), pp. 107-112.

1. <u>Definitions</u>. Consider the double series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\,, \tag{1.1}$$

and the simple series

$$\begin{aligned} &\sum_{m=0}^{\infty} a_{mn} && \text{(every } n\text{)},\\ &\sum_{n=0}^{\infty} a_{mn} && \text{(every } m\text{)}, \end{aligned} \tag{1.2}$$

represented by any row or column of the double series (1.1). We define the following partial sums:

$$\begin{aligned} s_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}, && \\ s^{C}_{mn} &= \sum_{i=0}^{m} a_{in} && \text{(every } n\text{)},\\ s^{R}_{mn} &= \sum_{j=0}^{n} a_{mj} && \text{(every } m\text{)}; \end{aligned} \tag{1.3}$$

$$\begin{aligned} \sigma_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} s_{ij}\,, && \\ \sigma^{C}_{mn} &= \sum_{i=0}^{m} s^{C}_{in} && \text{(every } n\text{)},\\ \sigma^{R}_{mn} &= \sum_{j=0}^{n} s^{R}_{mj} && \text{(every } m\text{)}; \end{aligned} \tag{1.4}$$

$$\begin{aligned} S_{mn} &= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{s_{ij}}{ij}\,, && \\ S^{C}_{mn} &= \sum_{i=1}^{m} \frac{s_{in}}{i} && \text{(every } n\text{)},\\ S^{R}_{mn} &= \sum_{j=1}^{n} \frac{s_{mj}}{j} && \text{(every } m\text{)}. \end{aligned} \tag{1.5}$$

The double series (1.1) is said to be summable by Riesz's logarithmic mean, or summable (R,1), if the double limit

$$\lim_{m,n\to\infty} \frac{S_{mn}}{\log m \log n}$$

exists.

Similarly, if the single ~~double~~ limits

$$\lim_{m\to\infty} \frac{S^C_{mn}}{\log m} \quad \text{(every } n); \qquad \lim_{n\to\infty} \frac{S^R_{mn}}{\log n} \quad \text{(every } m),$$

exist, then the simple series (1.2) are said to be summable (R,1) for every n and for every m respectively.

2. Statement of the Theorem. It is the purpose of Part I of this paper to prove the following theorem.

THEOREM. *Let $f(x,y)$ be a function integrable in the sense of Lebesgue in the region $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, and let*

$$\phi(u,v) = \frac{1}{4}\Big[f(x+u,y+v) + f(x-u,y+v) + f(x+u,y-v) + f(x-u,y-v) - 4S\Big].$$

If

$$\Phi(x,y) = \int_0^x\int_0^y |\phi(u,v)|\, du\, dv = o\left(xy \log \frac{1}{x} \log \frac{1}{y}\right),$$

$$(2.1) \quad \Phi_1(x,y) = \int_0^x |\phi(u,y)|\, du = o\left(x \log \frac{1}{x}\right), \qquad (0 \leq y \leq \pi),$$

$$\Phi_2(x,y) = \int_0^y |\phi(x,v)|\, dv = o\left(y \log \frac{1}{y}\right), \qquad (0 \leq x \leq \pi),$$

a sufficient condition that the Fourier development of $f(x,y)$

be summable (R,1) at the point (x_0, y_0) to the sum S and that each of the simple series represented by the rows and columns of this double Fourier series be summable (R,1) at this same point is that

$$\Phi(x,y) = \int_x^\pi \int_y^\pi \frac{\phi(u,v)}{uv}\, du\, dv = o\left(\log\frac{1}{x}\log\frac{1}{y}\right),$$

$$(2.2) \qquad \Phi_1(x,y) = \int_x^\pi \frac{\phi(u,y)}{u}\, du = o\left(\log\frac{1}{x}\right) \qquad (0 \leqq y \leqq \pi),$$

$$\Phi_2(x,y) = \int_y^\pi \frac{\phi(x,v)}{v}\, dv = o\left(\log\frac{1}{y}\right) \qquad (0 \leqq x \leqq \pi),$$

as $x, y \to 0$.

For the sake of simplicity we shall assume throughout this part of the paper that the function $f(x,y)$ is even-even as well as integrable (L). We shall assume further that the Fourier coefficients a_{mn} of $f(x,y)$ are zero whenever m or n or both are zero, and that $x_0 = y_0 = 0$ and $S = 0$. Then $\phi(x,y) = f(x,y)$, and the Fourier development of $f(x,y)$ can be expressed by the relation

$$f(x,y) \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \cos mx \cos ny,$$

where the symbol $\sim$ denotes formal correspondence.

Before proceeding to the proof of this theorem we shall state and prove a series of lemmas.

3. The Auxiliary Functions $\chi(x,y)$, $\chi_1(x,y)$, $\chi_2(x,y)$.

LEMMA 1. If $f(x,y)$ is even-even and integrable (L), then the functions

$$\chi(x,y) = \frac{1}{4}\int_x^\pi\int_y^\pi f(u,v)\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv,$$

$$(3.1)\quad \chi_1(x,y) = \frac{1}{2}\int_x^\pi f(u,y)\cot\frac{u}{2}\,du, \qquad (0 \leqq y \leqq \pi),$$

$$\chi_2(x,y) = \frac{1}{2}\int_y^\pi f(x,v)\cot\frac{v}{2}\,dv, \qquad (0 \leqq x \leqq \pi),$$

<u>are also even-even and integrable</u>.

For we have

$$\chi(-x,y) = \frac{1}{4}\int_{-x}^\pi\int_{+y}^\pi f(u,v)\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv$$

$$= \frac{1}{4}\left\{\int_{-x}^0\int_y^\pi + \int_0^x\int_y^\pi\right\} f(u,v)\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv$$

$$+\chi(x,y).$$

If, in the first term in brackets, we make the change of variable $u' = -u$, we get, since $f(-x,y) = f(x,y)$ by hypothesis,

$$\frac{1}{4}\int_x^0\int_y^\pi f(-u',v)\cot\left(-\frac{u'}{2}\right)\cot\frac{v}{2}\,(-du')\,dv$$

$$= -\frac{1}{4}\int_0^x\int_y^\pi f(u,v)\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv,$$

and therefore $\chi(-x,y) = \chi(x,y)$. Similarly, $\chi(x,-y) = \chi(x,y)$. Also,

$$\chi_1(-x,y) = \frac{1}{2}\int_{-x}^\pi f(u,y)\cot\frac{u}{2}\,du$$

$$= \frac{1}{2}\left\{\int_{-x}^0 + \int_0^x\right\} f(u,y)\cot\frac{u}{2}\,du + \chi_1(x,y).$$

If, in the first term in brackets, we make the change of variable $u' = -u$, we get

$$\frac{1}{2}\int_x^0 f(-u',y)\cot\left(-\frac{u'}{2}\right)(-du') = -\frac{1}{2}\int_0^x f(u,y)\cot\frac{u}{2}\,du,$$

and therefore $\chi_1(-x,y) = \chi_1(x,y)$. Since $f(u,-y) = f(u,y)$, it is evident that $\chi_1(x,-y) = \chi_1(x,y)$. In a similar manner we can show that $\chi_2(x,y)$ is even-even.

To show that $\chi(x,y)$ is integrable, we write

$$\begin{aligned}
&\int_0^\pi\int_0^\pi |\chi(x,y)|\ dx\ dy \\
&\qquad \leqq \frac{1}{4}\int_0^\pi\int_0^\pi\left(\int_x^\pi\int_y^\pi |f(u,v)|\cdot\left|\cot\frac{u}{2}\right|\cdot\left|\cot\frac{v}{2}\right|\ du\ dv\right)dx\ dy \\
&\qquad \leqq \int_0^\pi\int_0^\pi\left(\int_x^\pi\int_y^\pi \frac{|f(u,v)|}{uv}\ du\ dy\right)dx\ dy \\
&\qquad \leqq \int_0^\pi\int_0^\pi \frac{|f(u,v)|}{uv}\left(\int_0^u\int_0^v dx\ dy\right)du\ dv \\
&\qquad = \int_0^\pi\int_0^\pi |f(u,v)|\ du\ dv.
\end{aligned}$$

To show that $\chi_1(x,y)$ is integrable write

$$\begin{aligned}
\int_0^\pi\int_0^\pi |\chi_1(x,y)|\ dx\ dy &\leqq \frac{1}{2}\int_0^\pi\int_0^\pi\left(\int_x^\pi |f(u,y)|\cdot\left|\cot\frac{u}{2}\right|\ du\right)dx\ dy \\
&\leqq \int_0^\pi\int_0^\pi\left(\int_x^\pi \frac{|f(u,y)|}{u}\ du\right)\ dx\ dy \\
&\leqq \int_0^\pi\int_0^\pi \frac{|f(u,y)|}{u}\left(\int_0^u dx\right)dy\ du \\
&= \int_0^\pi\int_0^\pi |f(u,y)|\ du\ dy,
\end{aligned}$$

which shows that $\chi_1(x,y)$ is integrable when $f(x,y)$ is integrable. Similarly, if $f(x,y)$ is integrable, $\chi_2(x,y)$ is shown to be integrable.

LEMMA 2. <u>Let</u> $f(x,y)$ <u>be a function which satisfies the relations</u> (2.2). <u>Then</u>

$$\chi(x,y) = o(\log \tfrac{1}{x} \log \tfrac{1}{y}), \qquad \underline{\text{when}}\ x,y \to 0;$$

$$\text{(3.2)} \qquad \chi_1(x,y) = o(\log \tfrac{1}{x}), \qquad \underline{\text{when}}\ x \to 0,\ (0 \leqq y \leqq \pi);$$

$$\chi_2(x,y) = o(\log \tfrac{1}{y}), \qquad \underline{\text{when}}\ y \to 0,\ (0 \leqq x \leqq \pi),$$

<u>where</u> $\chi(x,y)$, $\chi_1(x,y)$, <u>and</u> $\chi_2(x,y)$ <u>are</u> <u>defined</u> <u>as</u> <u>in</u> (3.1).

Now,

$$\begin{aligned}\chi(x,y) &= \int_x^\pi \int_y^\pi f(u,v)\, \tfrac{1}{4} \cot \tfrac{u}{2} \cot \tfrac{v}{2}\, du\, dv \\ &= \int_x^\pi \int_y^\pi \frac{f(u,v)}{uv}\, du\, dv + o(1) \\ &= \Psi(x,y) + o(1) \\ &= o(\log \tfrac{1}{x} \log \tfrac{1}{y})\end{aligned}$$

as $x,y \to 0$, by the first of equations (2.2). Also

$$\begin{aligned}\chi_1(x,y) &= \tfrac{1}{2} \int_x^\pi f(u,y) \cot \tfrac{u}{2}\, du \\ &= \int_x^\pi \frac{f(u,y)}{u}\, du + o(1) \\ &= \Psi_1(x,y) + o(1) \\ &= o(\log \tfrac{1}{x}), \qquad (0 \leqq y \leqq \pi),\end{aligned}$$

as $x \to 0$, by the second of equations (2.2). Similarly, the third of equations (3.1) can be proved if we make use of the third of equations (2.2).

Let us develop $\chi(x,y)$, $\chi_1(x,y)$, and $\chi_2(x,y)$ in double Fourier series in the following way:

$$\chi(x,y) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn} b_{mn} \cos mx \cos ny,$$

$$\chi_1(x,y) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn} c_{mn} \cos mx \cos ny,$$

$$\chi_2(x,y) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn} d_{mn} \cos mx \cos ny,$$

where

$$(3.3) \qquad \lambda_m = \begin{cases} 1/2, & \text{if } m = 0, \\ 1, & \text{if } m > 0, \end{cases} \qquad \lambda_{mn} = \lambda_m \cdot \lambda_n .$$

We can then prove the following lemma.

LEMMA 3. <u>The Fourier coefficients</u> b_{mn}, c_{mn}, d_{mn} $(m > 0, n > 0)$ <u>of the functions</u> $\chi(x,y)$, $\chi_1(x,y)$, $\chi_2(x,y)$, <u>respectively, are expressible in terms of the Fourier coefficients</u> a_{mn} <u>of</u> $f(x,y)$ <u>as follows</u>:

$$(3.4) \qquad \begin{aligned} b_{mn} &= \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{m-i,n-j}\, a_{ij}, && m,n > 0; \\ c_{mn} &= \frac{1}{m}\sum_{i=1}^{m} \lambda_{m-i}\, a_{in}, && m > 0, \text{ every } n; \\ d_{mn} &= \frac{1}{n}\sum_{j=1}^{n} \lambda_{n-j}\, a_{mj}, && n > 0, \text{ every } m. \end{aligned}$$

We have by definition,

$$\begin{aligned} b_{mn} &= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \chi(u,v) \cos mu \cos nv \, du\, dv \\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\left(\int_u^{\pi}\int_v^{\pi} f(x,y) \cot\frac{x}{2} \cot\frac{y}{2}\, dx\, dy\right) \\ &\qquad \cdot \cos mu \cos nv \, du\, dv \\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\left(\int_0^{x}\int_0^{y} \cos mu \cos nv \, du\, dv\right) f(x,y) \\ &\qquad \cdot \cot\frac{x}{2} \cot\frac{y}{2}\, dx\, dy \end{aligned}$$

$$= \frac{1}{mn\pi^2}\int_0^\pi\int_0^\pi f(x,y)\ \sin mx \cot \frac{x}{2}\ \sin ny \cot \frac{y}{2}\ dx\ dy$$

$$= \frac{4}{mn\pi^2}\int_0^\pi\int_0^\pi f(x,y)\ \frac{\sin mx \cos \frac{x}{2}}{2 \sin \frac{x}{2}}\ \frac{\sin ny \cos \frac{y}{2}}{2 \sin \frac{y}{2}}\ dx\ dy.$$

Now by an elementary trigonometric formula,

$$\sin mx \cos \frac{x}{2} = \sin\ (2m + 1)\frac{x}{2} - \sin \frac{x}{2} \cos mx,$$

and therefore

$$\frac{\sin mx \cos \frac{x}{2}}{2 \sin \frac{x}{2}} = \frac{\sin (2m+1)\frac{x}{2}}{2 \sin \frac{x}{2}} - \cos mx/2$$

$$= \sum_{i=1}^{m-1} \cos ix + \frac{1}{2}\cos mx - \frac{1}{2}$$

$$= \sum_{i=1}^{m} \lambda_{m-i} \cos ix - \frac{1}{2}.$$

Hence

$$b_{mn} = \frac{4}{mn\pi^2}\int_0^\pi\int_0^\pi f(x,y)\left[\sum_{i=1}^{m} \lambda_{m-i}\cos ix - \frac{1}{2}\right]\left[\sum_{j=1}^{n}\lambda_{n-j}\cos jy - \frac{1}{2}\right] dx\ dy$$

$$= \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{m-i,n-j}\left(\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\cos ix \cos jy\ dx\ dy\right)$$

$$- \frac{1}{2mn}\sum_{i=1}^{m}\lambda_{m-i}\left(\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\cos ix\ dx\ dy\right)$$

$$- \frac{1}{2mn}\sum_{j=1}^{n}\lambda_{n-j}\Big(\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\cos jy\ dx\ dy$$

$$+ \frac{1}{4mn}\left(\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\ dx\ dy\right)$$

$$= \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{m-i,n-j} \; a_{ij}, \qquad m,n > 0.$$

Similarly, making use of the same trigonometric formula, we get

$$c_{mn} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \chi_1(u,v) \cos mu \cos nv \, du \, dv$$

$$= \frac{2}{\pi^2} \int_0^\pi \int_0^\pi \left(\int_u^\pi f(x,v) \cot \frac{x}{2} \, dx \right) \cos mu \cos nv \, du \, dv$$

$$= \frac{2}{\pi^2} \int_0^\pi \int_0^\pi \left(\int_0^x \cos mu \, du \right) f(x,v) \cot \frac{x}{2} \cos nv \, dx \, dv$$

$$= \frac{2}{m\pi^2} \int_0^\pi \int_0^\pi f(x,v) \cot \frac{x}{2} \sin mx \cos nv \, dx \, dv$$

$$= \frac{4}{m\pi^2} \int_0^\pi \int_0^\pi f(x,v) \left[\sum_{i=1}^{m} \lambda_{m-i} \cos ix - \frac{1}{2} \right] \cos nv \, dx \, dv$$

$$= \frac{1}{m} \sum_{i=1}^{m} \lambda_{m-i} \left(\frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x,v) \cos nv \, dx \, dv \right)$$

$$- \frac{1}{2m} \left(\frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x,v) \cos nv \, dx \, dv \right)$$

$$= \frac{1}{m} \sum_{i=1}^{m} \lambda_{m-i} \, a_{in}, \qquad (m > 0, \; n = 1,2,\cdots).$$

In an entirely analogous manner we can prove the third of the equations (3.3).

LEMMA 4. <u>The partial sums</u> s_{mn}, s_{mn}^{C} (<u>every</u> n), s_{mn}^{R} (<u>every</u> m) <u>are expressible in the following way in terms of the Fourier coefficients of</u> $\chi(x,y)$, $\chi_1(x,y)$, <u>and</u> $\chi_2(x,y)$:

$$\frac{s_{mn}}{mn} = b_{mn} + \frac{1}{2n} c_{mn} + \frac{1}{2m} d_{mn} + o\left(\frac{1}{mn}\right);$$

$$(3.5) \qquad \frac{s^C_{mn}}{m} = c_{mn} + o\left(\frac{1}{m}\right) \qquad \text{(every n)};$$

$$\frac{s^R_{mn}}{n} = d_{mn} + o\left(\frac{1}{n}\right) \qquad \text{(every m)}.$$

We have by the first of the definitions (1.3)

$$\frac{s_{mn}}{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{a_{ij}}{mn}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{a_{ij}}{mn} + \frac{1}{2}\sum_{i=1}^{m-1} \frac{a_{in}}{mn} + \frac{1}{2}\sum_{j=1}^{n-1} \frac{a_{mj}}{mn} + \frac{a_{mn}}{mn}$$

$$= \left[\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{a_{ij}}{mn} + \frac{1}{2}\sum_{i=1}^{m-1} \frac{a_{in}}{mn} + \frac{1}{2}\sum_{j=1}^{n-1} \frac{a_{mj}}{mn} + \frac{a_{mn}}{4mn}\right]$$

$$+ \frac{1}{2n}\left[\sum_{i=1}^{m-1} \frac{a_{in}}{m} + \frac{a_{mn}}{2m}\right] + \frac{1}{2m}\left[\sum_{j=1}^{n-1} \frac{a_{mj}}{n} + \frac{a_{mn}}{4mn}\right] + \frac{a_{mn}}{mn}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{m-i,n-j} \frac{a_{ij}}{mn} + \frac{1}{2}\sum_{i=1}^{m} \lambda_{m-i} \frac{a_{in}}{mn}$$

$$+ \frac{1}{2}\sum_{j=1}^{n} \lambda_{n-j} \frac{a_{mj}}{mn} + \frac{a_{mn}}{4mn}$$

$$= b_{mn} + \frac{1}{2n} c_{mn} + \frac{1}{2m} d_{mn} + o\left(\frac{1}{mn}\right);$$

by Lemma 3. Also, by the second of the definitions (1.3), and by Lemma 3,

$$\frac{s^C_{mn}}{m} = \frac{1}{m}\sum_{i=1}^{m} a_{in}$$

$$= \frac{1}{m}\sum_{i=1}^{m} \lambda_{m-i}\, a_{in} + \frac{a_{mn}}{2m}$$

$$= c_{mn} + o\left(\frac{1}{m}\right),$$

Similarly, the third of the relations (3.5) follows.

4. Some Lemmas on Partial Sums. Let us now make the following definitions for the Fourier coefficients of $\chi(x,y)$, $\chi_1(x,y)$, and $\chi_2(x,y)$:

$$(4.1)\qquad t_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} \lambda_{ij}\, b_{ij}, \qquad \tau_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} t_{ij};$$

$$(4.2)\qquad \begin{aligned} (t_1)_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} \lambda_{ij}\, c_{ij}, & (\tau_1)_{mn} &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (t_1)_{ij}; \\ (t_1)^C_{mn} &= \sum_{i=0}^{m} \lambda_{in}\, c_{in} & (\tau_1)^C_{mn} &= \sum_{i=0}^{m-1} (t_1)^C_{in} \\ &\text{(every } n), & &\text{(every } n); \end{aligned}$$

$$(4.3)\qquad \begin{aligned} (t_2)_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} \lambda_{ij}\, d_{ij}, & (\tau_2)_{mn} &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (t_2)_{ij}; \\ (t_2)^R_{mn} &= \sum_{j=0}^{n} \lambda_{mj}\, d_{mj} & (\tau_2)^R_{mn} &= \sum_{j=0}^{n-1} (t_2)^R_{mj} \\ &\text{(every } m), & &\text{(every } m). \end{aligned}$$

Making use of these definitions and those contained in §1, the following well known integral expressions can readily be derived:

$$(4.4)\qquad \begin{aligned} \sigma_{mn} &= \frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi f(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy, \\ \sigma^C_{mn} &= \frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi f(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\cos ny\,dx\,dy, \\ \sigma^R_{mn} &= \frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi f(x,y)\cos mx\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy, \end{aligned}$$

$$(4.5)\qquad \tau_{mn} = \frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi \chi(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy,$$

$$(4.6)\quad \begin{aligned} (\tau_1)_{mn} &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi_1(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\;\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy \\ (\tau_1)^C_{mn} &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi_1(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,\cos ny\,dx\,dy \end{aligned}$$

$$(4.7)\quad \begin{aligned} (\tau_2)_{mn} &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi_2(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\;\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy \\ (\tau_2)^R_{mn} &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi_2(x,y)\,\cos mx\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy. \end{aligned}$$

For example, we derive the expression for σ_{mn}. By definition,

$$\begin{aligned} \sigma_{mn} &= \sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\sum_{p=1}^{i}\sum_{q=1}^{j} a_{pq}\Big) \\ &= \frac{4}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\sum_{p=1}^{i}\sum_{q=1}^{j}\int_0^\pi\int_0^\pi f(x,y)\cos px\,\cos qy\,dx\,dy\Big) \\ &= \frac{4}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^\pi\int_0^\pi f(x,y)\sum_{p=0}^{i}\sum_{q=0}^{j}\lambda_{pq}\cos px\,\cos qy\,dx\,dy \\ &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\sin(2i+1)x/2}{2\sin x/2}\;\frac{\sin(2j+1)y/2}{2\sin y/2}\,dx\,dy \\ &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\;\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy \end{aligned}$$

since $a_{oo} = a_{po} = a_{oq} = 0$ for every p and every q. The other expressions in (4.4)-(4.7) may be derived in precisely the same manner.

We are now ready to prove some lemmas.

LEMMA 5. *For the partial sums* τ_{mn}, $(\tau_1)_{mn}$, $(\tau_1)^C_{mn}$, (*every* n), $(\tau_2)_{mn}$, $(\tau_2)^R_{mn}$ (*every* m), *as defined by* (4.1), (4.2), *and* (4.3), *we have the following relationships:*

$$
(4.8)\qquad
\begin{aligned}
\tau_{mn} &= o(mn \log m \log n),\\
\tau_{m1} &= o(m \log m \log n),\\
\tau_{1n} &= o(n \log m \log n),
\end{aligned}
$$

$$
(4.9)\qquad
\begin{aligned}
(\tau_1)_{mn} &= o(mn \log m \log n),\\
(\tau_1)_{m1} &= o(m \log m \log n),\\
(\tau_1)_{1n} &= o(n \log m \log n),
\end{aligned}
$$

$$
(4.10)\qquad
\begin{aligned}
(\tau_2)_{mn} &= o(mn \log m \log n),\\
(\tau_2)_{m1} &= o(m \log m \log n),\\
(\tau_2)_{1n} &= o(n \log m \log n),
\end{aligned}
$$

as m, n $\to \infty$;

$$
(4.11)\qquad
\begin{aligned}
(\tau_1)^C_{mn} &= o(m \log m), &&\text{(every n)},\\
(\tau_1)^C_{mn} &= o(\log m) &&\text{(every n)},
\end{aligned}
$$

and also as n *becomes infinite; and*

$$
(4.12)\qquad
\begin{aligned}
(\tau_2)^R_{mn} &= o(n \log n), &&\text{(every m)},\\
(\tau_2)^R_{mn} &= o(\log n), &&\text{(every m)},
\end{aligned}
$$

and also as m *becomes infinite.*

Let us first consider equations (4.8). From (4.5) and Lemma 5, we have

$$\tau_{mn} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi o(\log\frac{1}{x}\log\frac{1}{y})\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy$$

$$= \frac{1}{\pi^2}\left\{\int_0^{1/m}\int_0^{1/n} + \int_{1/m}^{\pi}\int_0^{1/n} + \int_0^{1/m}\int_{1/n}^{\pi} + \int_{1/m}^{\pi}\int_{1/n}^{\pi}\right\}$$

$$\cdot\, o(\log\frac{1}{x}\log\frac{1}{y})\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy$$

$$= \frac{m^2n^2}{\pi^2}\int_0^{1/m}\int_0^{1/n} o(\log\frac{1}{x}\log\frac{1}{y})\,dx\,dy$$

$$+ \frac{n^2}{\pi^2}\int_{1/m}^{\pi}\int_0^{1/n} o(\log\frac{1}{x}\log\frac{1}{y})\,\frac{dx}{x^2}\,dy$$

$$+ \frac{m^2}{\pi^2}\int_0^{1/m}\int_{1/n}^{\pi} o(\log\frac{1}{x}\log\frac{1}{y})\,dx\,\frac{dy}{y^2}$$

$$+ \frac{1}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi} o(\log\frac{1}{x}\log\frac{1}{y})\,\frac{dx}{x^2}\,\frac{dy}{y^2}$$

$$= o(m^2n^2\frac{1}{mn}\log m\log n) + o(n^2\frac{m}{n}\log m\log n)$$

$$+ o(m^2\frac{n}{m}\log m\log n) + o(mn\log m\log n)$$

$$= o(mn\log m\log n).$$

In a similar manner we can verify the other two of the relations (4.8). From the first of equations (4.6) and by Lemma 2, we have

$$(\tau_1)_{mn} = \left(\frac{1}{\pi}\int_0^\pi o(\log\frac{1}{x})\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,dx\right)\left(\frac{1}{\pi}\int_0^\pi\frac{\sin^2 ny/2}{\sin^2 y/2}\,dy\right).$$

Now, Hardy has shown that*

*G. H. Hardy, loc. cit.

$$\frac{1}{\pi}\int_0^{\pi} o(\log \frac{1}{x}) \frac{\sin^2 mx/2}{\sin^2 x/2}\, dx = o(m \log m),$$

and it is well known that

$$\frac{1}{n\pi}\int_0^{\pi} \frac{\sin^2 ny/2}{\sin^2 y/2}\, dy = 1.$$

Thus we see immediately that

$$(\tau_1)_{mn} = o(m \log m)\; n = o(mn \log m \log n).$$

In an analogous manner the other two of the relations (4.9) and the three relations (4.10) follow.

Next, we have from the second of equations (4.6) and by Lemma 2 and making use of Hardy's work again,

$$(\tau_1)^C_{mn} = \left(\frac{1}{\pi}\int_0^{\pi} o(\log \frac{1}{x}) \frac{\sin^2 mx/2}{\sin^2 x/2}\, dx\right)\left(\frac{\lambda_n}{\pi}\int_0^{\pi} \cos ny\, dy\right)$$

$$= o(m \log m), \qquad \text{(every } n),$$

and it is evident that

$$(\tau_1)^C_{1n} = o(\log m), \qquad \text{(every } n).$$

We also notice that

$$(\tau_1)^C_{mn} = o(m \log m); \qquad (\tau_1)^C_{1n} = o(\log m), \text{ as } m,n \to \infty.$$

Similarly, relations (4.12) follow and we also have

$$(\tau_2)^R_{mn} = o(n \log n); \qquad (\tau_2)^R_{m1} = o(\log n), \text{ as } m,n \to \infty.$$

LEMMA 6. <u>We have the following identities for the partial sums defined in §1:</u>

$$(m+1)(n+1)S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i} \frac{s_{pj}}{p}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{q=1}^{j} \frac{s_{iq}}{q} + \sigma_{mn}$$

(4.13)
$$(m+1)S^{C}_{mn} = \sum_{i=1}^{m} S^{C}_{in} + \sigma^{C}_{mn} \qquad (\underline{\text{every}}\ n),$$

$$(n+1)S^{R}_{mn} = \sum_{j=1}^{n} S^{R}_{mj} + \sigma^{R}_{mn} \qquad (\underline{\text{every}}\ m).$$

In order to verify, for example, the first of equations (4.13), we merely make use of the definitions of S_{mn} and σ_{mn}:

$$(m+1)(n+1)\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{s_{ij}}{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}\sum_{q=1}^{j}\frac{s_{pq}}{pq}$$

$$- \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}\frac{s_{pj}}{pj} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{q=1}^{j}\frac{s_{iq}}{q}$$

$$= (m+1)(n+1)\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{s_{ij}}{ij} - \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{i=p}^{m}\sum_{j=q}^{n}\frac{s_{pq}}{pq}$$

$$- \sum_{p=1}^{m}\sum_{j=1}^{n}\sum_{i=p}^{m}\frac{s_{pj}}{p} - \sum_{i=1}^{m}\sum_{q=1}^{n}\sum_{j=q}^{n}\frac{s_{iq}}{q}$$

$$= (m+1)(n+1)\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{s_{ij}}{ij} - (m+1)(n+1)\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{s_{pq}}{pq}$$

$$+ (m+1)\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{s_{pq}}{p} + (n+1)\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{s_{pq}}{q} - \sum_{p=1}^{m}\sum_{q=1}^{n} s_{pq}$$

$$+ (m+1)\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{s_{pj}}{p} + (n+1)\sum_{i=1}^{m}\sum_{q=1}^{n}\frac{s_{iq}}{q} - \sum_{p=1}^{m}\sum_{q=1}^{n} s_{pj}$$

$$= \sum_{i=1}^{m}\sum_{q=1}^{n} s_{iq} = \sigma_{mn}.$$

The other two relations can be verified in a similar manner.

LEMMA 7. <u>If</u> s_{mn}, s^{C}_{mn} (<u>every</u> n), s^{R}_{mn} (<u>every</u> m), are <u>defined</u> <u>as</u> <u>in</u> §1, <u>then</u>

$$\sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} = o(mn \log m \log n),$$

<u>as</u> $m,n \to \infty$; <u>and</u>

$$\sum_{i=1}^{m} s^{C}_{in} = o(m \log m), \quad \text{as } m \to \infty \ (\text{every } n);$$

(4.15)

$$\sum_{j=1}^{n} s^{R}_{mj} = o(n \log n), \quad \text{as } n \to \infty \ (\text{every } m).$$

We shall consider first the relations (4.14). By Lemma 4,

$$s_{ij} = \sum_{p=1}^{i} \sum_{q=1}^{j} b_{pq} + \sum_{p=1}^{i} \sum_{q=1}^{j} \frac{c_{pq}}{2q} + \sum_{p=1}^{i} \sum_{q=1}^{j} \frac{d_{pq}}{2p} + \sum_{p=1}^{i} \sum_{q=1}^{j} o\left(\frac{1}{pq}\right)$$

$$= t_{ij} - t_{io} - t_{oj} + \frac{1}{2} \sum_{q=1}^{j} \left[(t_1)^{C}_{iq} - (t_1)^{C}_{oq} \right] \frac{1}{q}$$

$$+ \frac{1}{2} \sum_{p=1}^{i} \left[(t_2)^{R}_{pj} - (t_2)^{R}_{po} \right] \frac{1}{p}$$

$$+ o(\log i \log j),$$

recalling the definitions in (4.1), (4.2), and (4.3). Now, since

$$(t_1)^{C}_{oq} = \frac{1}{2} \lambda_q c_{oq} = O(1) \qquad (\text{every } q),$$

$$(t_2)^{R}_{po} = \frac{1}{2} \lambda_p d_{po} = O(1) \qquad (\text{every } p),$$

it is evident that

$$s_{ij} = t_{ij} - t_{io} + \cancel{t_{oj}} + \frac{1}{2} \sum_{q=1}^{j} (t_1)^{C}_{iq} + O(1) \sum_{q=1}^{j} \frac{1}{q}$$

$$+ \frac{1}{2} \sum_{p=1}^{i} (t_2)^{R}_{pj} + O(1) \sum_{p=1}^{i} \frac{1}{p} + o(\log i \log j)$$

$$= t_{ij} - t_{i0} - t_{0j} + \frac{1}{2}\sum_{q=1}^{j}(t_1)^C_{iq}\,\frac{1}{q} + \frac{1}{2}\sum_{p=1}^{i}(t_2)^R_{pj}\,\frac{1}{p}$$

$$+ o(\log i \log j).$$

Therefore, if we again make use of the definitions in (4.1), (4.2), and (4.3), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n} t_{i0} - \sum_{i=1}^{m}\sum_{j=1}^{n} t_{0j}$$

$$+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{q=1}^{j}(t_1)^C_{iq}\,\frac{1}{q} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}(t_2)^R_{pj}\,\frac{1}{p}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} o(\log i \log j)$$

$$= \tau_{m+1,n+1} - (n+1)\,\tau_{m+1,1} - (m+1)\tau_{1,n+1} + (m+n+1)\,\tau_{1,1}$$

$$+ \frac{1}{2}\sum_{j=1}^{n}\sum_{q=1}^{j}\frac{1}{q}\left[(\tau_1)^C_{m+1,q} - (\tau_1)^C_{1,q}\right]$$

$$+ \frac{1}{2}\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{1}{p}\left[(\tau_2)^R_{p,n+1} - (\tau_2)^R_{p,1}\right. + o(mn \log m \log n).$$

We now apply Lemma 5 and obtain thereby

$$S_{ij} = o(mn \log m \log n) + (n+1)\,o(m \log m \log n)$$

$$+ (m+1)\,o(n \log m \log n) + \frac{1}{2}\sum_{j=1}^{n}\sum_{q=1}^{j}\frac{1}{q}\,o(m \log m)$$

$$+ \frac{1}{2}\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{1}{p}\,o(n \log n)$$

$$= o(mn \log m \log n),$$

since all the relations in Demma 5 hold when $m,n \to \infty$.

Let us now consider the first of relations (4.15). We have by definition (1.5) and Lemma 4,

$$s^C_{in} = \sum_{p=1}^{i} \frac{s^C_{pn}}{p}$$

$$= \sum_{p=1}^{i} c_{pn} + \sum_{p=1}^{i} o(1/p)$$

$$= (t_1)^C_{in} + o(\log i), \qquad \text{(every n)},$$

making use of the definition of $(t_1)^C_{mn}$ given in (4.2). Hence, by Lemma 5,

$$\sum_{i=1}^{m} s^C_{in} = \sum_{i=1}^{m} \left[(t_1)^C_{in} + o(\log i)\right]$$

$$= (\tau_1)^C_{mn} + o(m \log m)$$

$$= o(m \log m) \qquad \text{(every n)},$$

and also as $n \to \infty$.

The second of the relations (4.15) can be proved in an analogous manner.

The lemma is now completely proved.

LEMMA 8. <u>Let</u> $f(x,y)$ <u>be a function which satisfies the relation</u> (2.1), <u>and let</u> s_{ij} <u>be defined as in</u> (1.3), <u>where the</u> a_{ij} <u>are the Fourier coefficients of</u> $f(x,y)$. <u>Then</u>

$$\text{(4.16)} \quad \begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{p=1}^{i} \frac{s_{pj}}{p} &= o(mn \log m \log n) \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{q=1}^{j} \frac{s_{iq}}{q} &= o(mn \log m \log n), \end{aligned}$$

<u>as</u> $m,n \to \infty$.

For, we have by definition (1.3),

$$\frac{s_{pj}}{p} = \sum_{r=1}^{p} \sum_{q=1}^{j} \frac{a_{rq}}{p}$$

$$= \sum_{q=1}^{j} \left(\frac{\sum_{r=1}^{p} \lambda_{p-r}\, a_{rq}}{p} + \frac{a_{pq}}{2p} \right)$$

$$= \sum_{q=1}^{j} c_{pq} + \sum_{q=1}^{j} \frac{a_{pq}}{2p},$$

by Lemma 4, so that

$$\sum_{p=1}^{i} \frac{s_{pj}}{p} = \sum_{p=1}^{i} \sum_{q=1}^{j} c_{pq} + \sum_{p=1}^{i} \sum_{q=1}^{j} \frac{a_{pq}}{2p}$$

$$= (t_1)_{ij} - (t_1)_{i0} - (t_1)_{0j} + (t_1)_{00} + \sum_{p=1}^{i} \sum_{q=1}^{j} \frac{a_{pq}}{2p},$$

where $(t_1)_{ij}$ is defined as in (4.2). Therefore, making use of the definition of $(\tau_1)_{mn}$, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{p=1}^{i} \frac{s_{pq}}{p} = \sum_{i=1}^{m} \sum_{j=1}^{n} (t_1)_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} (t_1)_{i0}$$

$$- \sum_{i=1}^{m} \sum_{j=1}^{n} (t_1)_{0j} + \sum_{i=1}^{m} \sum_{j=1}^{n} (t_1)_{00}$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{q=1}^{j} \sum_{p=1}^{i} \frac{a_{pq}}{2p}$$

$$= (\tau_1)_{m+1,n+1} - (n+1)\,(\tau_1)_{m+1,1}$$

$$- (m+1)\,(\tau_1)_{1,n+1} + (m+n+1)\,(\tau_1)_{1,1}$$

$$+ mn\, O(1) + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{p=1}^{i} \sum_{q=1}^{j} \frac{a_{pq}}{2p},$$

and hence by Lemma 5

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}\frac{s_{pj}}{p} = o(mn \log m \log n) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}\sum_{q=1}^{j}\frac{a_{pq}}{2p}.$$

Now let us consider the second term of the right-hand side of the relation above. We have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}\sum_{q=1}^{j}\frac{a_{pq}}{p}$$

$$= \frac{4}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}\sum_{q=1}^{j}\int_0^{\pi}\int_0^{\pi} f(x,y)\,\frac{\cos px}{p}\,\cos qy\,dx\,dy$$

$$= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y)\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{\cos px}{p}\,\frac{\sin^2 ny/2}{2\sin^2 y/2}\,dx\,dy$$

$$= \frac{8}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y)\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{\cos px}{p}\,\frac{\sin^2 ny/2}{y^2}\,dx\,dy$$

$$+ \frac{2}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y)\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{\cos px}{p}\left[\frac{1}{\sin^2 y/2} - \frac{4}{y^2}\right]\sin^2 ny/2\,dx\,dy$$

$$= J_1 \quad J_2 .$$

Let us first consider the integral J_1; we have

$$|J_1| \leq \frac{8}{\pi^2}\int_0^{\pi}\int_0^{\pi}|f(x,y)|\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{\cos px}{p}\,\frac{\sin^2 ny/2}{y^2}\,dx\,dy$$

$$= \frac{8}{\pi^2}\int_0^{\pi}\int_0^{1/n}|f(x,y)|\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{|\cos px|}{p}\,\frac{\sin^2 ny/2}{y^2}\,dx\,dy$$

$$+ \frac{8}{\pi^2}\int_0^{\pi}\int_{1/n}^{\pi}|f(x,y)|\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{|\cos px|}{p}\,\frac{\sin^2 ny/2}{y^2}\,dx\,dy$$

$$= J_1' + J_2'' .$$

Now, making use of the third of relations (2.1), it is seen that

$$J_1' = O(1)\int_0^{\pi}\int_0^{1/n}|f(x,y)|\sum_{i=1}^{m}\sum_{p=1}^{i}\frac{1}{p}\,n^2\,dx\,dy$$

$$= O(n^2 m \log m) \int_0^\pi \Phi_2 (x, \tfrac{1}{n})\, dx$$

$$= O(n^2 m \log m)\; o(\tfrac{1}{n} \log n)$$

$$= o(mn \log m \log n),$$

and

$$J_1'' = O(1) \int_0^\pi \sum_{i=1}^{m} \sum_{p=1}^{i} \frac{1}{p} \left(\int_{1/n}^\pi \frac{|f(x,y)|}{y^2}\, dy \right) dx$$

$$= O(m \log m) \int_0^\pi \left(\int_{1/n}^\pi \frac{\frac{\partial}{\partial y} \Phi_2(x,y)}{y^2}\, dy \right) dx$$

$$= O(m \log m) \int_0^\pi \left[\int_0^{1/n} \frac{\partial}{\partial y} \left(\frac{\Phi_2(x,y)}{y^2} \right) dy + 2 \int_{1/n}^\pi \frac{\Phi_2(x,y)}{y^3}\, dy \right] dx$$

$$= O(m \log m) \int_0^\pi \left[\frac{\Phi_2(x,\pi)}{\pi^2} - n^2 \Phi_2(x, \tfrac{1}{n}) + 2 \int_{1/n}^\pi \frac{o(\log \frac{1}{y})}{y^2}\, dy \right] dx$$

$$= O(m \log m) \int_0^\pi \left[\Phi_2(x,\pi) + o(n \log n) \right] dx$$

$$= O(m \log m) \int_0^\pi \Phi_2 (x,\pi)\, dx + o(mn \log m \log n)$$

$$= O(m \log m)\; \Phi(\pi,\pi) + o(mn \log m \log n)$$

$$o(mn \log m \log n).$$

We therefore have

$$J_1 = o(mn \log m \log n).$$

Next, we consider J_2 and obtain

$$|J_2| \leq \frac{2}{\pi^2}\int_0^\pi\int_0^\pi |f(x,y)| \sum_{i=1}^{m}\sum_{p=1}^{i} \frac{|\cos px|}{p} \sin^2 ny/2 \, O(1)\, dx\, dy$$

$$= O(1)\int_0^\pi\int_0^\pi |f(x,y)| \sum_{i=1}^{m}\sum_{p=1}^{i}\frac{1}{p}\, dx\, dy$$

$$= O(m \log m)$$

$$= o(mn \log m \log n).$$

Thus we have proved that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{p=1}^{i}\sum_{q=1}^{j}\frac{a_{pq}}{p} = o(mn \log m \log n),$$

and consequently that

$$\sum_{i=1}^{m}\sum_{p=1}^{i}\sum_{j=1}^{n}\frac{s_{pj}}{p} = o(mn \log m \log n).$$

Similarly, since

$$\frac{s_{iq}}{q} = \sum_{p=1}^{i} d_{pq} + \sum_{p=1}^{i}\frac{a_{pq}}{q},$$

we can show that the second of the relations (4.16) holds.

LEMMA 9. Let $f(x,y)$ be a function which satisfies the relations (2.1), and let σ_{mn}, σ_{mn}^{C} (every n), σ_{mn}^{R} (every m) be defined as in (1.4), where the a_{mn} are the Fourier coefficients of $f(x,y)$. Then

$$\sigma_{mn} = o(mn \log m \log n) \qquad \text{as } m,n \to \infty;$$

(4.17) $$\sigma_{mn}^{C} = o(m \log m) \qquad \text{as } m \to \infty \text{ (every } n);$$

$$\sigma_{mn}^{R} = o(n \log n) \qquad \text{as } n \to \infty \text{ (every } m).$$

In order to prove the first of relations (4.17) we use the expressions for σ_{mn} given in (4.4):

$$\sigma_{mn} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy$$

$$= \frac{16}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y)\,\frac{\sin^2 mx/2}{x^2}\,\frac{\sin^2 ny/2}{y^2}\,dx\,dy + o(1)$$

$$= J + o(1),$$

where

$$|J| \leqq \frac{16}{\pi^2}\left\{\int_0^{1/m}\int_0^{1/n} + \int_{1/m}^{\pi}\int_0^{1/n} + \int_0^{1/m}\int_{1/n}^{\pi} + \int_{1/m}^{\pi}\int_{1/n}^{\pi}\right\} \cdot |f(x,y)|\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,\frac{\sin^2 ny/2}{\sin^2 y/2}\,dx\,dy$$

$$= J_1 + J_2 + J_3 + J_4,$$

Let us first consider the integral J_1 for which we have

$$J_1 \leqq \frac{16}{\pi^2}\int_0^{1/m}\int_0^{1/n}|f(x,y)|\,\frac{m^2x^2/4}{x^2}\,\frac{n^2y^2/4}{y^2}\,dx\,dy$$

$$= \frac{m^2n^2}{\pi^2}\int_0^{1/m}\int_0^{1/n}|f(x,y)|\,dx\,dy$$

$$= \frac{m^2n^2}{\pi^2}\,\Phi\,(1/m,1/n)$$

$$= m^2n^2\;o\left(\frac{1}{mn}\log m \log n\right).$$

$$= o(mn \log m \log n),$$

by the first of equations (2.1).

Next using the second of equations (2.1) we get

$$J_2 = \frac{16}{\pi^2}\int_{1/m}^{\pi}\int_0^{1/n} |f(x,y)| \frac{1}{x^2}\frac{n^2y^2/4}{y^2}\, dx\, dy$$

$$= \frac{4n^2}{\pi^2}\int_{1/m}^{\pi}\int_0^{1/n} |f(x,y)| \frac{1}{x^2}\, dx\, dy$$

$$= \frac{4n^2}{\pi^2}\int_{1/m}^{\pi} \frac{\Phi_2(x,\frac{1}{n})}{x^2}\, dx$$

$$= n^2 o(\frac{1}{n}\log n)$$

$$= o(n \log n)\ (-1/\pi + m)$$

$$= o(mn \log n)$$

$$= o(mn \log m \log n).$$

Similarly, we can show that

$$J_3 = o(mn \log m \log n).$$

Finally, using the first of equations (2.1) again, we have

$$J_4 = \frac{16}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi} |f(x,y)| \frac{1}{x^2y^2}\, dx\, dy$$

$$= \frac{16}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi} \frac{\partial^2\Phi(x,y)}{\partial x\,\partial y}\frac{1}{x^2y^2}\, dx\, dy$$

$$= \frac{16}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi} \frac{\partial^2}{\partial x\,\partial y}\left(\frac{\Phi(x,y)}{x^2y^2}\right) dx\, dy$$

$$+ \frac{32}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi} \frac{\partial}{\partial x}\left(\frac{\Phi(x,y)}{x^2}\right)\frac{1}{y^3}\, dx\, dy$$

$$+ \frac{32}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi} \frac{\partial}{\partial y}\left(\frac{\Phi(x,y)}{y^2}\right)\frac{1}{x^3}\, dx\, dy$$

$$+ \frac{64}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi}\frac{\Phi(x,y)}{x^3y^3}\,dx\,dy$$

$$= \frac{16}{\pi^2}\left[\frac{\Phi(\pi,\pi)}{\pi^4} - m^2n^2\,\Phi(1/m,1/n)\right]$$

$$+ \frac{32}{\pi^2}\int_{1/m}^{\pi}\left[\frac{\Phi(\pi,y)}{\pi^2} - \frac{\Phi(1/m,y)}{1/m^2}\right]\frac{1}{y^3}\,dy$$

$$+ \frac{32}{\pi^2}\int_{1/m}^{\pi}\left[\frac{\Phi(x,\pi)}{\pi^2} - \frac{\Phi(x,1/n)}{1/n^2}\right]\frac{1}{x^3}\,dx$$

$$+ \frac{64}{\pi^2}\int_{1/m}^{\pi}\int_{1/n}^{\pi}\frac{\Phi(x,y)}{x^3y^3}\,dx\,dy$$

$$= o(mn\log m\log n)$$

$$+\int_{1/n}^{\pi}\left[\frac{o(\log\frac{1}{y})}{y^2} + \frac{o(m\log m\log\frac{1}{y})}{y^2}\right]dy$$

$$+\int_{1/m}^{\pi}\left[\frac{o(\log\frac{1}{x})}{x^2} + \frac{o(n\log\frac{1}{x}\log n)}{x^2}\right]dx$$

$$+\int_{0}^{\pi}\int_{1/n}^{\pi}\frac{o(\log\frac{1}{x}\log\frac{1}{y})}{x^2y^2}\,dx\,dy$$

$$= o(mn\log m\log n) + m\int_{1/n}^{\pi}\frac{o(\log m\log\frac{1}{y})}{y^2}\,dy$$

$$+ n\int_{1/m}^{\pi}\frac{o(\log\frac{1}{x}\log n)}{x^2}\,dx$$

$$+ \int_{1/m}^{\pi}\int_{1/n}^{\pi}\frac{o(\log\frac{1}{x}\log\frac{1}{y})}{x^2y^2}\,dx\,dy$$

$$= o(mn\log m\log n).$$

We therefore have

$$J = o(mn\log m\log n),$$

In order to prove that the second of equations (4.17) holds, we use the integral expression for σ_{mn}^C as given in (4.4). We have

$$\sigma_{mn}^C = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\cos ny\,dx\,dy$$

$$= O(1)\int_0^\pi\int_0^\pi f(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,dx\,dy$$

$$= O(1)\left\{\int_0^{1/m}\int_0^\pi + \int_{1/m}^\pi\int_0^\pi\right\} f(x,y)\,\frac{\sin^2 mx/2}{\sin^2 x/2}\,dx\,dy$$

$$= O(1)\,m^2\int_0^{1/m}\int_0^\pi f(x,y)\,dx\,dy$$

$$+ O(1)\int_{1/m}^\pi\int_0^\pi \frac{f(x,y)}{x^2}\,dx\,dy.$$

Now making use of the second of relations (2.1), we get

$$\sigma_{mn}^C = O(1)\,m^2\int_0^\pi \Phi_1(1/m,y) + O(1)\int_0^\pi\int_{1/m}^\pi \frac{\frac{\partial}{\partial x}\Phi_1(x,y)}{x^2}\,dx\,dy$$

$$= o(m\log m) + O(1)\int_0^\pi\int_{1/m}^\pi \frac{\partial}{\partial x}\left(\frac{\Phi_1(x,y)}{x^2}\right)dx\,dy$$

$$+ O(1)\int_0^\pi\int_{1/m}^\pi \frac{\Phi_1(x,y)}{x^3}\,dx\,dy + o(m\log m)$$

$$+ O(1)\int_0^\pi \Phi_1(\pi,y)\,dy + O(1)\,m^2\int_0^\pi \Phi_1\,(1/m,y)\,dy$$

$$+ O(1)\int_0^\pi\int_{1/m}^\pi \frac{o(\log\frac{1}{x})}{x^2}\,dx\,dy$$

$$= o(m\log m) + O(1) + o(\frac{m^2}{m}\log m) + o(m\log m)$$

$$= o(m\log m) \qquad \text{(every } n\text{)}.$$

Similarly, the third of the relations (4.17) can be proved.

We are now ready to prove the theorem stated in §2.

5. Proof of the Theorem. In order to prove the theorem we must show that from the hypothesis (2.2) it follows that

$$S_{mn} = o(\log m \log n);$$

$$S^{C}_{mn} = o(\log m), \qquad \text{(every } n);$$

$$S^{R}_{mn} = o(\log n), \qquad \text{(every } m).$$

If we consider the expressions (4.13) in Lemma 6, and make use of Lemmas 7,8, and 9, we obtain the above relations immediately.

PART II

On the Cesàro Summability of Double Fourier Series

We now turn to a problem concerning Cesàro summability of double Fourier series. The theorems in Part II are not concerned with deriving conditions under which a series is summable by Cesàro means of given order, nor with finding the actual sum of the series. We are rather concerned with investigating the rapidity with which the rth Cesàro mean $(0 < r < 1)$ of a double Fourier series approaches a given sum when the function considered satisfies certain conditions. It is also shown that values connected with the function under consideration, other than the value of the function itself at a given point, can be represented by means of the rth Cesàro mean of the Fourier development of the function.

The results contained in this part of the paper are generalizations of results obtained by Szász* who considered the case of a single Fourier series. In an earlier paper Szász proved similar results for the arithmetic mean of a single Fourier series.†

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*Otto Szász, Über die Cesàroschen Mittel Fourierscher Reihen, Acta Litterarum ac Scientiarum, vol. 3 (1927), pp. 38-48.

†Otto Szasz, Über die arithmetischen Mittel Fourierscher Reihen, Acta Mathematica, vol. 48 (1926), pp. 353-362.

6. <u>Definitions and Notation</u>. Let us consider a function $f(x,y)$ periodic of periods 2π and integrable (L). Its Fourier development may be written in the form

$$(6.1)\quad f(x,y) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\lambda_{mn}}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\, P_{mn}(x,y,u,v)\, du\, dv,$$

where the symbol $\sim$ denotes formal expansion,

$$P_{mn}(x,y,u,v) = \cos\,(m(u-x))\,\cos\,(n(v-y)),$$

and λ_{mn} is defined as in Part I, (3.3). The Cesàro mean of the rth order of this series, where $o < r < 1$, may be expressed in the form

$$(6.2)\quad \frac{S_{mn}^{(r)}(x,y)}{C_{mn}^{(r)}} = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\Big[f(x+u,y+v) + f(x-u,y+v)$$

$$f(x+u,y-v) + f(x-u,y-v)\Big]\, H_{mn}(u,v)\, du\, dv$$

where

$$C_m^{(r)} = \frac{\Gamma(m+r)}{\Gamma(r+1)\,\Gamma(m)}, \qquad C_{mn}^{(r)} = C_m^{(r)}\cdot C_n^{(r)},$$

and $H_m(u) = H$

$$H_m(u) = \frac{1}{C_m^{(r)}}\sum_{i=0}^{m-1} C_{m-i}^{(r-1)}\,\frac{\sin\,(2i-1)u/2}{\sin u/2},$$

$$H_{mn}(u,v) = H_m(u)\cdot H_n(v).$$

Since

$$\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} H_{mn}(u,v)\, du\, dv = 1,$$

we have

$$(6.3)\qquad \frac{S_{mn}^{(r)}(x,y)}{C_{mn}^{(r)}} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(u,v)\, H_{mn}(u,v)\, du\, dv,$$

where $\phi(u,v)$ is defined as in Part I, §1.

7. A Special Case. We shall first prove a theorem which indicates the rapidity with which the rth Cesàro mean $(0 < r < 1)$ of the series (6.1) as given in (6.2) approaches a suitable chosen quantity S.

THEOREM 1. *Let $f(x,y)$ be a function which is integrable (L), and let $S_{mn}^{(r)}(x,y)/C_{mn}^{(r)}$ be the Cesàro mean of the rth order of its Fourier series, where $0 < r < 1$. If, for a suitable S,*

$$(7.1)\qquad \Phi(x,y) = \int_0^x\int_0^y |\phi(u,v)|\, dy\, dv = o(x^{1+\alpha}y^{1+\beta}),\ \begin{cases} 0 \leqq \alpha \leqq r \\ 0 \leqq \beta \leqq r \end{cases},$$

as $x,y \to +0$, and if

$$\Phi_1(x,y) = \int_0^x |\phi(u,y)|\, dy = O(x^{1+\alpha}) \quad \text{as } x \to +0,$$
$$(0 \leqq \alpha \leqq r,\ 0 \leqq y \leqq \pi),$$

$$(7.2)\qquad \Phi_2(x,y) = \int_0^y |\phi(x,v)|\, dv = O(y^{1+\beta}) \quad \text{as } y \to +0,$$
$$(0 \leqq x \leqq \pi,\ 0 \leqq \beta \leqq r),$$

then

$$(7.3)\qquad \frac{S_{mn}^{(r)}(x,y)}{C_{mn}^{(r)}} - S = \begin{cases} o(m^{-\alpha}n^{-\beta}), & (0<\alpha<r,\ 0<\beta<r); \\ o(m^{-r}n^{-\beta}\log m), & (0<\beta<r); \\ o(m^{-\alpha}n^{-r}\log n), & (0<\alpha<r); \\ o(m^{-r}n^{-r}\log m \log n), & \end{cases}$$

as $m,n \to \infty$.

Let us consider the absolute value of the expression on the left-hand side of (7.3). From (6.3) we have

$$(7.4)\quad \left| \frac{S_{mn}^{(r)}(x,y)}{C_{mn}^{(r)}} - s \right| = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi |\phi(u,v)|\ |H_m(u)|\ |H_n(v)|\ du\, dv.$$

In view of the hypothesis (7.1) as well as (7.2) we can determine a positive $\delta < 1$ corresponding to a given positive $\epsilon < \pi$, such that

$$(7.5)\qquad u^{-1-\alpha}\, v^{-1-\beta}\, \Phi(u,v) < \epsilon, \qquad (u,v < \delta),$$

while at the same time

$$(7.6)\qquad \begin{aligned} u^{-1-\alpha}\, \Phi_1(u,v) &< A_1, && (u < \delta,\ 0 \leqq v \leqq \pi),\\ v^{-1-\beta}\, \Phi_2(u,v) &< A_1, && (0 \leqq u \leqq \pi,\ v < \delta), \end{aligned}$$

where A_1 is a positive constant.* The integral on the right-hand side of (7.4) may then be written as the sum of four integrals

$$(7.7)\qquad \int_0^\delta\int_0^\delta + \int_0^\delta\int_\delta^\pi + \int_\delta^\pi\int_0^\delta + \int_\delta^\pi\int_\delta^\pi = I_1 + I_2 + I_3 + I_4,$$

where the expression under the integral sign is the same as in (7.4). Let us first consider the integral I_1. M. Riesz †

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*Throughout the remainder of the paper we shall indicate positive constants by A_1, A_2, A_3, ...

† M. Riesz, Sur la sommation des séries de Fourier, Acta Litteraurm ac Scientarum, vol. 1 (1923), p. 110.

has show that for $0 < r < 1$ and $0 < u < \pi$,

(7.8) $$|H_m(u)| < mA_2$$

(7.9) $$|H_m(u)| < m^{-\nu} u^{-\nu-1} A_3,$$

where A_2 and A_3 depend only on r. Let us choose m and n, depending on the ϵ we have already chosen, such that $m > 1/\delta$, $n > 1/\delta$. We may then write

$$I_1 = \int_0^{1/m}\int_0^{1/n} + \int_{1/m}^{\delta}\int_0^{1/n} + \int_0^{1/m}\int_{1/n}^{\delta} + \int_0^{1/m}\int_0^{1/n}$$

$$= I_1' + I_2' + I_3' + I_4' ,$$

where the expression under the integral sign is the same as in (7.4). In view of (7.8) and (7.5) we have

$$I_1' < mn\, \Phi(1/m,1/n)\, A_2^2$$

so that

(7.10) $$I_1' < m^{-\alpha} n^{-\beta} \epsilon A_2^2.$$

If we make use of both (7.8) and (7.9) and integrate by parts with respect to u, we get

$$I_2' < m^{-\nu} n A_2 A_3 \int_{1/m}^{\delta}\int_0^{1/n}|\phi(u,v)|\, u^{-\nu-1} \, du\, dv$$

$$< m^{-\nu} n A_2 A_3 \Big[\delta^{-\nu-1}\int_{1/m}^{\delta}\int_0^{1/n} |\phi(u,v)| \, du\, dv$$

$$+ (r+1)\int_{1/m}^{\delta}\int_0^{1/n}\Big(u^{-\nu-2}\int_{1/m}^{u}|\phi(r,v)|\, dr\Big)\, du\, dv\Big].$$

Now since the integrand in $\Phi(u,v)$ is positive, it is evident that its integral taken over any part of the region of integration indicated in (7.1) is less than its integral over the entire region. It is also clear that the same is true for $\Phi_1(u,v)$ and $\Phi_2(u,v)$, and we shall make use of this fact later on in the proof of the theorem. We therefore have

$$I_2' < A_2 A_3 m^{-\nu} n \left[\delta^{-\nu-1} \Phi(\delta, 1/n) + (r+1) \int_{1/m}^{\delta} u^{-\nu-2} \Phi(u,1/n)\, du \right]$$

$$< A_2 A_3 m^{-\nu} n \left[\epsilon\, \delta^{-\nu+\alpha} n^{-1-\beta} + (r+1)\, \epsilon\, n^{-1-\beta} \int_{1/m}^{\delta} u^{-\nu+\alpha-1}\, du \right]$$

so that

$$(7.11) \quad I_2' < A_2 A_3 m^{-\nu} n^{-\beta} \epsilon \left[m^{-\alpha+\nu} + (r+1) \int_{1/m}^{\delta} u^{-\nu+\alpha-1}\, du \right].$$

Similarly, we can show that

$$(7.12) \quad I_3' < A_2 A_3 m^{-\alpha} n^{-\nu} \epsilon \left[n^{-\beta+\nu} + (r+1) \int_{1/n}^{\delta} v^{-\nu+\beta-1}\, dv \right].$$

Now, if we make use of (7.9) and integrate by parts twice, we get

$$I_4' < m^{-\nu} n^{-\nu} A_3^2 \int_{1/m}^{\delta} \int_{1/n}^{\delta} |\phi(u,v)|\, u^{-\nu-1} v^{-\nu-1}\, du\, dv$$

$$\leq m^{-\nu} n^{-\nu} A_3^2 \left[\delta^{-\nu-1} \int_{1/m}^{\delta} \int_{1/n}^{\delta} v^{-\nu-1} |\phi(u,v)|\, du\, dv \right.$$

$$\left. + (r+1) \int_{1/m}^{\delta} \int_{1/n}^{\delta} \left(u^{-\nu-2} v^{-\nu-1} \int_{1/m}^{u} |\phi(r,v)|\, dr \right) du\, dv \right.$$

$$< m^{-\nu} n^{-\nu} A_3^2 \left[\delta^{-\nu-1} \int_{1/n}^{\delta} v^{-\nu-1} \Phi_1(\delta, v)\, dv \right.$$

$$\left. + (r+1) \int_{1/m}^{\delta} \int_{1/n}^{\delta} u^{-\nu-2} v^{-\nu-1} \Phi_1(u,v)\, du\, dv \right]$$

$$= m^{-\nu} n^{-\nu} A_3^2 \left[\delta^{-2\nu-2} \int_{1/n}^{\delta} \Phi_1(\delta, v)\, dv \right.$$

$$+ (r+1)\, \delta^{-\nu-1} \int_{1/n}^{\delta} \left(v^{-\nu-2} \int_{1/n}^{v} \Phi_1(\delta, s)\, ds \right) dv$$

$$+ (r+1)\, \delta^{-\nu-1} \int_{1/m}^{\delta}\int_{1/n}^{\delta} u^{-\nu-2}\; \Phi_1(u,v)\; du\; dv$$

$$+ (r+1)^2 \int_{1/m}^{\delta}\int_{1/n}^{\delta} u^{-\nu-2} v^{-\nu-2}\; \Phi(u,v)\; du\; dv$$

$$< m^{-\nu} n^{-\nu} A_3^2 \Big[\delta^{-2\nu-2}\, \Phi(\delta,\delta)$$

$$+ (r+1)\, \delta^{-\nu-1} \int_{1/n}^{\delta} v^{-\nu-2} \Phi(\delta, v)\; dv$$

$$+ (r+1)\, \delta^{-\nu-1} \int_{1/m}^{\delta} u^{-\nu-2} \Phi(u,\delta)\; du$$

$$+ (r+1)^2 \int_{1/m}^{\delta}\int_{1/n}^{\delta} u^{-\nu-2}\; v^{-\nu-2}\, \Phi(u,v)\; du\; dv$$

$$< m^{-\nu} n^{-\nu} \epsilon\, A_3^2 \Big[\delta^{-\nu+\alpha}\, \delta^{-\nu+\beta}$$

$$+ (r+1)\; \delta^{-\nu+\alpha} \int_{1/n}^{\delta} v^{-\nu+\beta-1}\; dv$$

$$+ (r+1)\, \delta^{-\nu+\beta} \int_{1/m}^{\delta} u^{-\nu+\alpha-1}\; du$$

$$+ (r+1)^2 \int_{1/m}^{\delta}\int_{1/n}^{\delta} u^{-\nu+\alpha-1}\; v^{-\nu+\beta-1}\; du\; dv \Big].$$

Hence,

$$I_4' < m^{-\nu} n^{-\nu} \epsilon\; A_3^2 \Big[m^{\nu-\alpha}\, n^{\nu-\beta}$$

$$+ (r+1)\; m^{\nu-\alpha} \int_{1/n}^{\delta} v^{-\nu+\beta-1}\; dv$$

$$(7.13) \qquad + (r+1)\; n^{\nu-\beta} \int_{1/m}^{\delta} u^{-\nu+\alpha-1}\; du$$

$$+ (r+1)^2 \int_{1/m}^{\delta}\int_{1/n}^{\delta} u^{-\nu+\alpha-1}\; v^{-\nu+\beta-1}\; du\; dv \Big].$$

Now, if we consider the four inequalities (7.10), (7.11), (7.12), and (7.13), which we have just obtained, we see that

$$I_1 < \epsilon\; A_4 \Big[\; m^{-\alpha}\; n^{-\beta} + (r+1)\; m^{-\nu}\; n^{-\beta} \int_{1/m}^{\delta} u^{-\nu+\alpha-1} du$$

$$+ (r+1)\; m^{-\alpha}\; n^{-\nu} \int_{1/n}^{\delta} v^{-\nu+\beta-1} dv$$

$$+ (r+1)^2 m^{-\nu} n^{-\nu} \int_{1/m}^{\delta}\int_{1/m}^{\delta} u^{-\nu+\alpha-1} v^{-\nu+\beta-1}\, du\, dv \Big].$$

We therefore have

$$(7.14)\quad I_1 = \begin{cases} \epsilon\, m^{-\alpha} n^{-\beta} \dfrac{2\nu-\alpha+1}{\nu-\alpha}\, \dfrac{2\nu-\beta+1}{\nu-\beta}\, A_4, & \begin{cases} 0 < \alpha < r, \\ 0 < \beta < r, \end{cases} \\[2ex] \epsilon\, m^{-\nu} n^{-\beta} \big[1 + (r+1)\log m\big] \dfrac{2\nu-\beta+1}{\nu-\beta}\, A_4, & \\ & (0 < \beta < r), \\[2ex] \epsilon\, m^{-\alpha} n^{-\nu} \dfrac{2\nu-\alpha+1}{\nu-\alpha} \big[1 + (r+1)\log n\big] A & \\ & (0 < \alpha < r), \\[2ex] \epsilon\, m^{-\nu} n^{-\nu} \big[1 + (r+1)\log m\big]\big[1 + (r+1)\log n\big] A_4. & \end{cases}$$

We shall now consider the integral I_2 in (7.7). We shall write I_2 as the sum of two integrals,

$$I_2 = \Big\{ \int_0^{1/m}\int_\delta^\pi + \int_{1/m}^{\delta}\int_\delta^\pi \Big\} |\phi(u,v)|\ |H_m(u)|\,|H_\nu(v)|\ du\, dv,$$

$$< m\, n^{-\nu} \delta^{-\nu-1}\ \Phi\,(1/m,\pi)\, A_2\, A_3 + m^{-\nu} n^{-\nu} \delta^{-\nu-1}\ A_3^2$$

$$\cdot \int_{1/m}^{\delta} u^{-\nu-1}\ \Phi_2\,(u,\pi)\, du$$

in view of the inequalities (7.8) and (7.9). Now, integrating the last integral by parts with regard to u we obtain

$$\int_{1/m}^{\delta} u^{-\nu-1}\ \Phi_2(u,\pi)\, du = \delta^{-\nu-1} \int_{1/m}^{\delta} \Phi_2\,(u,\pi)\, du$$

$$+ (r+1) \int_{1/m}^{\delta} \Big(u^{-\nu-2} \int_{1/m}^{u} \Phi_2(r,\pi)\, dr \Big)\, du$$

$$< \delta^{-\nu-1}\ \Phi\ (\delta,\pi) + (r+1) \int_{1/m}^{\delta} u^{-\nu-2}\ \Phi\ (u,\pi)\, du.$$

We therefore have, if we make use of the second of the inequalities (7.6),

$$I_2 < m\, n^{-\nu} \delta^{-\nu-1}\ \Phi(1/m,\pi)\ A_2\ A_3 + m^{-\nu} n^{-\nu} \delta^{-2\nu-2} \Phi(\delta,\pi)\ A_3^2$$

$$+ m^{-\nu}\ n^{-\nu} \delta^{-\nu-1}\ (r+1)\ A_3^2 \int_{1/m}^{\delta} u^{-\nu-2} \Phi(u,\pi)\ du.$$

Hence,

$$I_2 < m^{-\alpha}\ n^{-\nu} \delta^{-\nu-1}\ A_1\ A_2\ A_3 + m^{-\alpha}\ n^{-\nu} \delta^{-\nu-1} A_1\ A_3^2$$

$$+ m^{-\nu}\ n^{-\nu} \delta^{-\nu-1}\ (r+1)\ A_1\ A_3^2 \int_{1/m}^{\delta} u^{-\nu+\alpha-1}\ du$$

$$< m^{-\alpha}\ n^{-\nu}\ A_4 + m^{-\nu}\ n^{-\nu}\ (r+1)\ A_4 \int_{1/m}^{\delta} u^{-\nu+\alpha-1}\ du,$$

and therefore

$$(7.15) \qquad I_2 < \begin{cases} m^{-\alpha}\ n^{-\nu} \dfrac{2\nu-\alpha+1}{\nu-\alpha} A_4, & (\alpha < r), \\ m^{-\nu}\ n^{-\nu} \left[1 + (r+1) \log m\right] A_4. \end{cases}$$

Similarly, it can be shown that

$$(7.16) \qquad I_3 < \begin{cases} m^{-\nu}\ n^{-\beta} \dfrac{2\nu-\beta+1}{\nu-\beta}\ A_5 & (\beta < r), \\ m^{-\nu}\ n^{-\nu} \left[1 + (r+1) \log n\right] A_5. \end{cases}$$

Finally, we consider the integral I_4 in (7.7). From the inequality (7.9) we have

$$I_4 < m^{-\nu}\ n^{-\nu}\ A_4^2 \int_{\delta}^{\pi}\int_{\delta}^{\pi} |\phi(u,v)|\ u^{-\nu-1}\ v^{-\nu-1}\ du\ dv$$

$$< m^{-\nu}\ n^{-\nu} \delta^{-2\nu-2}\ \Phi(\pi,\pi)\ A_4^2,$$

and hence

(7.17) $$I_4 < m^{-r} n^{-r} A_6.$$

Now, if we combine the inequalities (7.14), (7.15), (7.16), and (7.18), we obtain the following relations:

$$\left| \frac{S^{(r)}_{mn}(x,y)}{C^{(r)}_{mn}} - S \right| < \begin{cases} m^{-\alpha} n^{-\beta} \epsilon \; \dfrac{2r-\alpha+1}{r-\alpha} \; \dfrac{2r-\beta+1}{r-\beta} \; A_7, \\ \qquad\qquad (\alpha < r, \beta \quad r), \\ m^{-r} n^{-\beta} \epsilon \log m \; \dfrac{2r-\beta+1}{r-\beta} \; A \;, \; (\beta < r), \\ m^{-\alpha} n^{-r} \epsilon \; \dfrac{2r-\alpha+1}{r-\alpha} \log n \; A_7, \; (\alpha < r), \\ m^{-r} n^{-r} \epsilon \log m \log n \; A_7, \end{cases}$$

for $m = m(\epsilon)$, $n = n(\epsilon)$. It is now evident that the relations (7.3) hold.

8. <u>The General Case</u>. We shall now prove a more general theorem involving not only the sum of the series, but certain other quantities connected with the function.

THEOREM 2. <u>Let</u> $f(x,y)$ <u>be integrable</u> (L), <u>and let</u> $S^{(r)}_{mn}(x,y)/C^{(r)}_{mn}$ <u>be the Cesàro mean of the</u> r<u>th order of its Fourier series</u>, <u>where</u> $0 < r < 1$. <u>If, for suitable values of</u> g <u>and</u> S <u>and suitable functions</u> $\phi_1(u)$ <u>and</u> $\phi_2(v)$ <u>integrable</u> (L) <u>and such that, for</u> $0 < \alpha \leqq r$, $0 < \beta \leqq r$,

$$\phi_1(u) \quad du = O(x^{1+\alpha}), \qquad \text{as } x \to +0,$$

(8.1)

$$\phi_2(v) \quad dv = O(y^{1+\beta}), \qquad \text{as } y \to +0,$$

<u>we have</u>

(8.2)
$$\int_0^x \int_0^y |\phi(u,v) - \phi_1(u)\, v^\beta - \phi_2(v)\, u^\alpha + g\, u^\alpha v^\beta|\,du\,dv = o(x^{1+\alpha} y^{1+\beta}), \text{ as } x, y \to +0,$$

<u>while</u>

(8.3)
$$\int_0^x |\phi(u,v) - \phi_1(u)\, v^\beta - \phi_2(v)\, u^\alpha + g\, u^\alpha v^\beta|\,du = O(x^{1+\alpha}), \text{ as } x \to +0,$$
$$\int_0^y |\phi(u,v) - \phi_1(u)\, v^\beta - \phi_2(v)\, u^\alpha + g\, u^\alpha v^\beta|\,dv = O(y^{1+\beta}), \text{ as } y \to +0,$$

<u>then</u>

(8.4)
$$\lim_{m,n\to\infty} m^\alpha n^\beta \left(\frac{S_{mn}^{(r)}(x,y)}{C_{mn}^{(r)}} - S \right)$$
$$= g \frac{\Gamma(1+r)}{2^{1+\alpha}\,\Gamma(1+r-\alpha)\cos\alpha\pi/2}\ \frac{\Gamma(1+r)}{2^{1+\beta}\,\Gamma(1+r-\beta)\cos\beta\pi/2} \quad (\alpha < r,\ \beta < r),$$

<u>and</u>

(8.5)
$$\frac{S_{mn}^{(r)}(x,y)}{C_{mn}^{(r)}} - S = \begin{cases} o(m^{-r} n^{-\beta} \log m), & (\beta < r), \\ o(m^{-\alpha} n^{-r} \log n), & (\alpha < r), \\ o(m^{-r} n^{-r} \log m \log n), & \end{cases}$$

<u>as</u> $m, n \to \infty$.

Let us write for brevity

$$Q(u,v) = \phi\,(u,v) - \phi_1\,(u)\; v^{\beta} - \phi_2(v)\; u^{\alpha} + g\; u^{\alpha}\; v^{\beta},$$

$$F_1(x,y) = \int_0^x |Q(u,y)|\; du, \qquad (0 \leqq y \leqq \pi),$$

$$F_2(x,y) = \int_0^y |Q(x,v)|\; dv, \qquad (0 \leqq x \leqq \pi),$$

$$F(x,y) = \int_0^x \int_0^y |Q(u,v)|\; du\, dv,$$

wherever these expressions occur. Since $|Q(u,v)|$ is always positive, it is evident that its integral taken over any part of the region indicated above will be less than its integral over the entire region. We shall make use of this fact in the proof.

From equations (6.3) we see that

$$\begin{aligned}\frac{S_{mn}^{(r)}(x,y)}{C_{mn}^{(r)}} - S &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi Q(u,v)\; H\quad (u,v)\; du\, dv \\ &+ \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \phi_1(u)\; v^{\beta} H_{mn}(u,v)\; du\, dv \\ &+ \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \phi_2(v)\; u^{\alpha} H_{mn}(u,v)\; du\, dv \\ &- \frac{g}{\pi^2}\int_0^\pi\int_0^\pi u^{\alpha}\; v^{\beta}\; H_{mn}(u,v)\; du\, dv \\ &= \frac{1}{\pi^2}\left[I_1 + I_2 + I_3 - g\; I_4\right].\end{aligned}$$

We shall first show that

$$(8.7) \qquad I_1 = \begin{cases} o(m^{-\alpha}\; n^{-\beta}), & (\alpha < r,\ \beta < r), \\ o(m^{-r}\; n^{-\beta}\; \log m), & (\beta < r), \\ o(m^{-\alpha}\; n^{-r}\; \log n), & (\alpha < r), \\ o(m^{-r} n^{-r} \log m\; \log n), & \end{cases}$$

as $m,n \to \infty$.

Let us consider the absolute value of I_1,

$$(8.8)\qquad |I_1| \leqq \int_0^\pi\int_0^\pi |Q(u,v)|\,|H_m(u)|\,|H_n(v)|\,du\,dv.$$

Now Szász* has shown that for $o < r < 1$, $o < u < \pi$, $m = 1, 2, 3, \cdots$,

$$(8.9)\qquad (1+mu)^{r+1}\,|H_m(u)| < A_8 m,$$

where A_8 depends only on r. Thus we see that the right-hand side of (7.8) is less than

$$mnA_8^2 \int_0^\pi\int_0^\pi |Q(u,v)|\ (1+mu)^{-r-1}\ (1+nv)^{-r-1}\,du\,dv,$$

and integrating by parts with regard to u, we get

$$|I_1| < mnA_8^2\ (1+m\pi)^{-r-1}\int_0^\pi F_1(\pi,v)(1+nv)^{-r-1}\,dv$$

$$+\ 2(r+1)m^2nA_8^2\int_0^\pi\int_0^\pi F_1(u,v)(1+mu)^{-r-2}\ (1+nv)^{-r-1}\,du\,dv$$

$$< m^{-r}nA_8^2\int_0^\pi F_1(\pi,v)(1+nv)^{-r-1}\,dv$$

$$+\ 2(r+1)m^{1-\alpha}\ nA_8^2\int_0^\pi\int_0^\pi F_1(u,v)\ u^{-1-\alpha}\ (1+mu)^{-r-1}$$

$$\cdot(1+nv)^{-r-1}\,du\,dv.$$

Now integrating by parts with regard to v, we obtain

$$|I_1| < m^{-r}n^{-r}A_8^2\,F(\pi,\pi)$$

$$+2(r+1)m^{-r}\ n^{1-\beta}\ A_8^2\int_0^\pi F(\pi,v)\ v^{-\beta-1}\ (1+nv)^{\beta-r-1}\,dv$$

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*Szász, Über die Cesaroschen Mittel Fourierscher Reihen, Acta Litterarum ac Scientarum, vol. 3 (1927), p. 43.

$$+\ 2(r+1)m^{1-\alpha}\, n^{-\nu} A_8^2 \int_0^{\pi} F(u,\pi)\ u^{-\alpha-1}\ (1+mu)^{\alpha-\nu-1}\, du$$

$$(8.10)\qquad + \ 4(r+1)^2 m^{1-\alpha}\, n^{1-\beta} A_8^2$$

$$\cdot \int_0^{\pi}\int_0^{\pi} u^{-\alpha-1}\ v^{-\beta-1}\ (1+mu)^{\alpha-\nu-1}\ (1+nv)^{\beta-\nu-1}\ du\ dv.$$

The integral in the second term of the right-hand side of (8.10) may be written as the sum of two integrals,

$$\int_0^{\delta} F(\pi,v)\ v^{-\beta-1}\ (1+nv)^{\beta-\nu-1}\ dv + \int_{\delta}^{\pi} F(\pi,v)\ v^{-\beta-1}\ (1+nv)^{\beta-\nu-1} dv$$

$$< \int_0^{\delta} F(\pi,v)\ v^{-\beta-1}\ (1+nv)^{\beta-\nu-1}\ dv + (1+n\delta)\int_0^{\pi} F(\pi,v)\ v^{-\beta-1}\ dv.$$

In view of relation (8.2) we can find a positive $\delta < 1$ corresponding to a given positive $\epsilon < \pi$ such that

$$(8.11)\qquad x^{-1-\alpha}\, y^{-1-\beta} F(x,y) < \epsilon \qquad (x < \delta,\ y < \delta),$$

while from the relations (8.3)

$$(8.12)\qquad \begin{aligned} x^{-1-\alpha}\ F_1(x,y) &< A_9, \quad (x < \delta,\ 0 \leqq y \leqq \pi),\\ y^{-1-\beta}\ F_2(x,y) &< A_9, \quad (y < \delta,\ 0 \leqq x \leqq \pi), \end{aligned}$$

Thus, making use of the second of the relations (8.12), we see that the integral in the second term on the right-hand side of (8.10) is less than

$$A_9 \int_0^{\delta} (1+nv)^{\beta-\nu-1}\ dv + (n\delta)^{\beta-\nu-1}\int_0^{\pi} F(\pi,v)\ v^{-1-\beta}\ dv$$

$$< A_9 \int_0^{\delta} (1+nv)^{\beta-\nu-1}\ dv + n^{\beta-\nu-1} A_{10}.$$

Similarly, making use of the first of the relations (8.12), we can show that the integral in the third term on the right-hand side of (8.10) is less than

$$A_9 \int_0^\delta (1+mu)^{\alpha-\nu-1}\, du + m^{\alpha-\nu-1} A_{11}\,.$$

The integral in the fourth term of the right-hand side of (8.10) may be written as the sum of four integrals,

$$\int_0^\delta\int_0^\delta + \int_0^\delta\int_\delta^\pi + \int_\delta^\pi\int_0^\delta + \int_\delta^\pi\int_\delta^\pi = I_1' + I_2' + I_3' + I_4',$$

where the expression under the integral sign is the same as in (8.10). For the integral I_1' we have, from (8.11)

$$I_1' < \epsilon \int_0^\delta\int_0^\delta (1+mu)^{\alpha-\nu-1} (1+nv)^{\beta-\nu-1}\, du\, dv.$$

If we make use of the second of the relations (8.12), we find that

$$I_2' < \delta^{-1-\beta} (1+n)^{\beta-\nu-1} \int_0^\delta\int_0^\pi F(u,v)\, u^{-1-\alpha} (1+mu)^{\alpha-\nu-1}\, du\, dv$$

$$< \delta^{-1-\beta} (n\delta)^{\beta-\nu-1} A_9 \int_0^\delta\int_0^\pi (1+mu)^{\alpha-\nu-1}\, du\, dv$$

$$< n^{\beta-\nu-1} A_{12} \int_0^\delta (1+mu)^{\alpha-\nu-1}\, du,$$

and similarly, if we make use of the first of these relations, we find that

$$I_3' < A_{13}\, m^{\alpha-\nu-1} \int_0^\delta (1+nv)^{\beta-\nu-1}\, dv,$$

For the integral I_4' we have

$$I_4' < (1+m\delta)^{\alpha-\nu-1} (1+n\delta)^{\beta-\nu-1} \int_0^\pi\int_0^\pi F(u,v)\, u^{-1-\alpha}\, v^{-1-\beta}\, du\, dv$$

$$< (m\delta)^{\alpha-r-1} (n\delta)^{\beta-r-1} \int_0^{\pi}\int_0^{\pi} F(u,v)\, u^{-1-\alpha}\, v^{-1-\beta}\, du\, dv$$

$$< A_{14}\, m^{\alpha-r-1}\, n^{\beta-r-1}.$$

Combining the results which we have just obtained we see that

$$|I_1| < A_{15}\Big[m^{-r}n^{-r} + m^{1-\alpha} n^{-r}\int_0^{\delta} (1+mu)^{\alpha-r-1}\, du + m^{-r}n^{1-\beta}\int_0^{\delta} (1+nv)^{\beta-r-1}\, dv + m^{1-\alpha}\, n^{1-\beta}\epsilon \int_0^{\delta}\int_0^{\delta} (1+mu)^{\alpha-r-1} (1+nv)^{\beta-r-1} du\, dv,$$

and we therefore have

$$(8.13)\qquad |I_1| < \begin{cases} A_{16}\left[m^{-r}n^{-r} + \dfrac{m^{-\alpha}n^{-r}}{r-\alpha} + \dfrac{m^{-r}n^{-\beta}}{r-\beta} + \dfrac{m^{-\alpha}n^{-\beta}}{(r-\alpha)(r-\beta)}\right], & (\alpha < r, \beta < r), \\ A_{16}\left[m^{-r}n^{-r}(1 + \log(1+2m)) + \dfrac{m^{-r}n^{-\beta}}{r-\beta}(1 + \epsilon\log(1+2m))\right], & (\beta < r), \\ A_{16}\left[m^{-r}n^{-r}(1 + \log(1+2n)) + \dfrac{m^{-\alpha}n^{-r}}{r-\alpha}(1 + \epsilon\log(1+2n))\right], & (\alpha < r), \\ A_{16}\left[m^{-r}n^{-r}(1 + \log(1+2m) + \log(1+2n) + \epsilon\log(1+2m)(1+2n))\right], & \end{cases}$$

It is now evident that the relations (8.7) are proved.

Let us now consider the integral I_2 on the right-hand side of (8.6). We shall show that

$$(8.14) \qquad I_2 = \begin{cases} o(m^{-\alpha} n^{-\beta}), & (\alpha < r, \beta < r), \\ o(m^{-r} n^{-\beta} \log m), & (\beta < r), \\ o(m^{-\alpha} n^{-r} \log n), & (\alpha < r), \\ o(m^{-r} n^{-r} \log m \log n), & \end{cases}$$

as $m, n \to \infty$. First we note that

$$(8.15) \qquad I_2 = \left(\int_0^\pi \phi_1(u)\, H_m(u)\, du \right) \left(\int_0^\pi v^\beta H_n(v)\, dv \right).$$

The absolute value of the first factor of the right-hand side of (8.5) is less than

$$mA_8 \int_0^\pi |\phi_1(u)|\, (1+mu)^{-r-1}\, du$$

by the inequality (8.9). An integration by parts gives

$$m(1+m\pi)^{-r-1} A_8 \int_0^\pi |\phi_1(u)|\, du$$

$$+ 2m^2 (r+1) A_8 \int_0^\pi \left[(1+mu)^{-r-2} \int_0^u |\phi_1(s)|\, ds \right] du$$

$$< m^{-r} A_{17} + 2m^{1-\alpha} (r+1) A_8 \int_0^\pi \left[u^{-1-\alpha} (1+mu)^{\alpha-r-1} \int_0^u |\phi_1(s)|\, ds \right] du$$

$$= m^{-r} A_{17} + 2m^{1-\alpha}(r+1) A_8 \left\{ \int_0^\delta + \int_\delta^\pi \right\}$$

$$\cdot \left[u^{-1-\alpha} (1+mu)^{\alpha-r-1} \int_0^u |\phi_1(s)|\, ds \right] du$$

$$< m^{-r} A_{17} + 2m^{1-\alpha}(r+1) A_{18} \int_0^\delta (1+mu)^{\alpha-r-1}\, du$$

$$+ 2m^{1-\alpha} (r+1)\, (1+m\delta)^{\alpha-r-1} A_{19}$$

$$< m^{-r} A_{17} + 2m^{1-\alpha}(r+1) A_{18} \int_0^\delta (1+mu)^{\alpha-r-1}\, du + m^{-r}(r+1) A_{20},$$

and therefore

$$\left|\int_0^u \phi_1(u)\ H_m(u)\ du\right| < \begin{cases} A_{21}\left[m^{-r} + \dfrac{m^{-\alpha}}{r-\alpha}\right], & (\alpha < r), \\ A_{21}\ m^{-r}\left[1 + \log\ (1+m)\right]. \end{cases}$$

It is thus evident that

$$(8.16)\qquad \int_0^u \phi_1(u)\ H_m(u)\ du = \begin{cases} o(m^{-\alpha}), & (\alpha < r), \\ o(m^{-r}\log m), \end{cases}$$

as $m \to \infty$. Szász* has shown that

$$\lim_{n\to\infty} n^{\beta}\int_0^{\pi} v^{\beta}\, H_n(v)\ dv = \frac{\pi}{2^{2+\beta}}\ \frac{\Gamma(r+1)}{\Gamma(1+r-\beta)\cos\beta\pi/2},$$

(8.17) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\beta < r),$

$$\lim_{n\to\infty} \frac{n^{r}}{\log n}\int_0^{\pi} v^{\beta}\, H_n(v)\ dv = 0,$$

so that

$$(8.18)\qquad \int_0^{\pi} v^{\beta}\, H_n(v)\ dv = \begin{cases} O(n^{-\beta}), & (\beta < r), \\ o(n^{-r}\log n), \end{cases}$$

as $n \to \infty$. By combining relations (8.16) and (8.18) we see that the relations (8.14)hold.

Similarly, we can show that

$$(8.19)\qquad I_3 = \begin{cases} o(m^{-\alpha} n^{-\beta}), & (\alpha < r,\ \beta < r), \\ o(m^{-r} n^{-\beta}\log m), & (\beta < r), \\ o(m^{-\alpha} n^{-r}\log n), & (\alpha < r), \\ o(m^{-r} n^{-r}\log m \log n), \end{cases}$$

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*Szász, Über die Cesaroschen Mittel Fourierscher Reihen Acta Litterarum ac Scientiarum, vol. 3 (1927), pp. 45-47.

as $m,n \to \infty$.

Finally, we consider the integral I_4 on the right-hand side of (8.6). It can be written as the product of two integrals

$$I_4 = \left(\int_0^{\pi} u^{\alpha} H_m(u)\, du\right)\left(\int_0^{\pi} v^{\beta} H_n(v)\, dv\right).$$

If wemake use of the relations (8.17) obtained by Szász, it is at once evident that

$$\lim_{n \to \infty} m^{\alpha} n^{\beta} I_4 = \frac{\Gamma(\nu+1)}{2^{1+\alpha}\Gamma(1-\alpha+\nu)\cos \alpha\pi/2} \; \frac{\Gamma(\nu+1)}{2^{1+\beta}\Gamma(1-\beta+\nu)\cos \beta\pi/2} \qquad (\alpha < r, \beta < r), \tag{8.20}$$

and

$$I_4 = \begin{cases} o(m^{-\nu} n^{-\beta} \log m), & (\beta < r), \\ o(m^{-\alpha} n^{-\nu} \log n), & (\alpha < r), \\ o(m^{-\nu} n^{-\nu}), \end{cases} \tag{8.21}$$

as $m,n \to \infty$.

The theorem follows immediately from the relations (8.7), (8.14), (8.19), (8.20), and (8.21).

9. *The Case of the Arithmetic Mean*. We shall now prove a theorem similar to Theorem 2 for the case when $r = 1$.

THEOREM 3. *Let* $f(x,y)$ *be a function which is integrable* (L), *and let* σ_{mn} *be the arithmetic mean of its Fourier series. If* $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, *and if for suitable values of* g *and* s *and suitable functions* $\phi_1(u)$ *and* $\phi_2(v)$ *satisfying the* conditions of Theorem 2, the relations (8.2) and (8.3) are satisfied, then

$$\lim_{m,n\to\infty} m^\alpha n^\beta(\sigma_{mn} - S) = \frac{-g}{\pi^2}\,\frac{\Gamma(\alpha)\sin\alpha\pi/2}{(1-\alpha)2^\alpha}\,\frac{\Gamma(\beta)\sin\beta\pi/2}{(1-\beta)2^\beta},$$

$$(\alpha < 1,\ \beta < 1),$$

(9.1)

$$\lim_{m,n\to\infty} \frac{m^\nu n^\beta}{\log m}(\sigma_{mn} - S) = \frac{-g}{\pi^2}\,\frac{\Gamma(\beta)\sin\beta\pi/2}{(1-\beta)2^\beta}, \quad (\beta < 1),$$

$$\lim_{m,n\to\infty} \frac{m^\alpha n^\nu}{\log n}(\sigma_{mn} - S) = \frac{-g}{\pi^2}\,\frac{\Gamma(\alpha)\sin\alpha\pi/2}{(1-\alpha)2^\alpha}, \quad (\alpha < 1),$$

$$\lim_{m,n\to\infty} \frac{m^\nu n^\nu}{\log m \log n}(\sigma_{mn} - S) = \frac{-g}{4\pi^2}.$$

It is well known that

$$\sigma_{mn} - S = \frac{1}{4mn\pi^2}\int_0^\pi\int_0^\pi \phi(u,v)\,\frac{\sin^2 mu/2}{\sin^2 u/2}\,\frac{\sin^2 nv/2}{\sin^2 v/2}\,du\,dv,$$

so that

(9.2)

$$\begin{aligned}\sigma_{mn} - S &= \frac{1}{4mn\pi^2}\int_0^\pi\int_0^\pi Q(u,v)\,\frac{\sin^2 mu/2}{\sin^2 u/2}\,\frac{\sin^2 nv/2}{\sin^2 v/2}\,du\,dv\\ &+ \frac{1}{4mn\pi^2}\int_0^\pi\int_0^\pi \phi_1(u)v^\beta\,\frac{\sin^2 mu/2}{\sin^2 u/2}\,\frac{\sin^2 nv/2}{\sin^2 v/2}\,du\,dv\\ &+ \frac{1}{4mn\pi^2}\int_0^\pi\int_0^\pi \phi_2(v)u^\alpha\,\frac{\sin^2 mu/2}{\sin^2 u/2}\,\frac{\sin^2 nv/2}{\sin^2 v/2}\,du\,dv\\ &+ \frac{-g}{4mn\pi^2}\int_0^\pi\int_0^\pi u^\alpha v^\beta\,\frac{\sin^2 mu/2}{\sin^2 u/2}\,\frac{\sin^2 nv/2}{\sin^2 v/2}\,du\,dv\\ &= \frac{1}{\pi^2}\left[I_1 + I_2 + I_3 - gI_4\right].\end{aligned}$$

Now, we note first that when r = 1 the integral I_1 of Theorem 2 is the integral I_1 of this theorem, and that the relations (8.7) still hold for r = 1, since the inequality (8.9) holds when r = 1. Similarly, for the same reason the inequality (8.9) holds when r = 1. We therefore have only to consider the integral I_4. We may write

$$(9.3)\qquad I_4 = \left(\frac{1}{2m}\int_0^{\pi}\frac{\sin^2 mu/2}{\sin^2 u/2}u^{\alpha}\,du\right)\left(\frac{1}{2m}\int_0^{\pi}\frac{\sin^2 mv/2}{\sin^2 v/2}\,v^{\beta}\,dv\right).$$

Now, if $\alpha < 1$,

$$\lim_{m\to\infty} m^{\alpha-1}\int_0^{\pi}\frac{\sin^2 mu/2}{\sin^2 u/2}\,u^{\alpha}\,du$$

(9.4)

$$= \lim_{m\to\infty} m^{\alpha-1}\int_0^{\pi}\frac{\sin^2 mu/2}{\sin^2 u/2}\,\sin^{\alpha} u\,du$$

or

$$(9.5)\qquad \lim_{m\to\infty} m^{\alpha-1}\int_0^{\pi}\frac{\sin^2 mu/2}{\sin^2 u/2}\,(u^{\alpha} - \sin^{\alpha} u)\,du = 0.$$

For since

$$u^{\alpha} - \sin^{\alpha} u < u^{2+\alpha}, \qquad (u > 0,\ 0 < \alpha < 1),$$

and

$$\frac{\sin^2 mu/2}{\sin^2 u/2} \leqq \pi^2\,\frac{\sin^2 mu/2}{u^2},$$

the integral in (9.4) is less than the integral

$$\pi^2 m^{\alpha-1}\int_0^{\pi}\sin^2 mu/2\ u^{\alpha}\,du \leqq m^{\alpha-1}\pi^2\int_0^{\pi}u^{\alpha}\,du < m^{\alpha-1}A_{22},$$

and hence can be made as small as we please by choosing m sufficiently large. We have thus proved the equality (9.4).

Now Szász* has shown that

$$\lim_{m\to\infty} \frac{m^{\alpha-1}}{2}\int_0^\pi \frac{\sin^2 mu/2}{\sin^2 u/2}\sin^\alpha u/2\,du = \frac{\Gamma(\alpha)\sin\alpha\pi/2}{(1-\alpha)2^\alpha}, \quad (\alpha<1),$$

$$\lim_{m\to\infty} \frac{1}{2\log m}\int_0^\pi \frac{\sin^2 mu/2}{\sin^2 u/2}\,du = \frac{1}{2}.$$

It is therefore evident that

$$\lim_{m\to\infty} \frac{m^{\alpha-1}}{2}\int_0^\pi \frac{\sin^2 mu/2}{\sin^2 u/2}\sin^\alpha u/2\,du = \frac{\Gamma(\alpha)\sin\alpha\pi/2}{(1-\alpha)2^\alpha}, \quad (\alpha<1),$$

$$\lim_{m\to\infty} \frac{1}{2\log m}\int_0^\pi \frac{\sin^2 mu/2}{\sin^2 u/2}\,du = \frac{1}{2}.$$

Hence we have

$$\lim_{m,n\to\infty} m^\alpha n^\beta I_4 = \frac{\Gamma(\alpha)\sin\alpha\pi/2}{(1-\alpha)2^\alpha}\;\frac{\Gamma(\beta)\sin\beta\pi/2}{(1-\beta)2^\beta}, \quad (\alpha<1,\ \beta<1),$$

$$\lim_{m,n\to\infty} \frac{m^\nu n^\beta}{\log m} I_4 = \frac{1}{2}\,\frac{\Gamma(\beta)\sin\beta\pi/2}{(1-\beta)2^\beta}, \quad (\beta<1),$$

$$\lim_{m,n\to\infty} \frac{m^\alpha n^\nu}{\log n} I_4 = \frac{\Gamma(\alpha)\sin\alpha\pi/2}{(1-\alpha)2^\alpha}\;\frac{1}{2}, \quad (\alpha<1),$$

$$\lim_{m,n\to\infty} \frac{m^\nu n^\nu}{\log m\log n} I_4 = \frac{1}{4}.$$

and, if we combine the results obtained, it is evident that the limits (9.1) exist.

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*Szász, Über die arithmetischen Mittel Fourierscher Reihen, Acta Mathematica, vol. 48 (1926), pp. 353-362.

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