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ON THE STRONG SUMMABILITY OF THE DOUBLE FOURIER SERIES

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CONTENTS

ON THE STRONG SUMMABILITY OF THE DOUBLE FOURIER SERIES

1. Introduction. The notion of 'strong' summability (C1) of a Fourier series was introduced in 1913 by Hardy and Littlewood.[1] They designated a series as strongly summable (C1) at a point of continuity x_0 if

$$(1.1) \qquad \sum_{0}^{n} | s_m - f(x)| = o(n),$$

where s_m is the sum of the first (m + 1) terms of the Fourier series $\sum_{0}^{\infty} (a_n \cos nx + b_n \sin nx)$.

These results were further extended by Carleman,[2] Sutton,[3] and again by Hardy and Littlewood.[4]

All of this work was confined solely to the single Fourier series. The purpose of this paper is the extension of Carleman's theorem to the two dimensional case.

First, let us consider briefly the discussion of the single Fourier series case due to Carleman, and the previous work leading up to his theorem.

2. Strong Summability of the Single Fourier Series. Considering an L. integrable function developable in the Fourier series

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(1) Comptes Rendus t. 156 pp. 1307-1309
(2) Proc. London Math. Society Ser.(2) V.21 1923 pp. 483-492
(3) Proc. London Math. Society Ser.(2) V.23 1925 pp. XLVIII-LI
(4) Proc. London Math. Society Ser.(2) V. 26 1927 pp. 273-286

$$(2.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

for which the sum of the first $\omega + 1$ terms

$$(2.2) \qquad s_\omega = \frac{1}{2} a_0 + \sum_{m=1}^{\omega} (a_m \cos mx + b_m \sin mx)$$

Then for a point x_0 where

$$(2.3) \qquad S = \frac{1}{2} \left\{ f(x_0 + 0) + f(x_0 - 0) \right\}$$

exists, we have by Fejer's theorem[1]

$$(2.4) \qquad \lim_{n \to \infty} \frac{1}{n+1} \sum_{\omega=0}^{n} (s_\omega - S) = 0$$

Furthermore Lebesque has proved that 'almost everywhere'

$$(2.5) \qquad \lim_{n \to \infty} \frac{1}{n+1} \sum_{\omega=0}^{n} \left\{ s_\omega - f(x) \right\} = 0$$

and, in the case of functions of summable square, Hardy and Littlewood[2] obtained the further results, true 'almost everywhere'

$$(2.6) \qquad \lim_{n \to \infty} \frac{1}{n+1} \sum_{\omega=0}^{n} \left\{ s_\omega - f(x) \right\}^2 = 0$$

$$(2.7) \qquad \lim_{n \to \infty} \frac{1}{n+1} \sum_{\omega=0}^{n} | s_\omega - f(x) | = 0$$

Later Carleman[3] extended equation (2.7) to higher powers, showing that,

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(1) See Fourier Series and Integrals - Carslau. 3rd ed. par. 101 p. 254

(2) Comptes Rendus t 156 (1913) p. 1307

(3) Proc. Lond. Math. Soc. Ser. (2) V. 21 1923 p. 483

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\omega=0}^{n} \left| s_\omega - f(x) \right|^k = 0$$

for all positive values of k, at any value of x for which;

$$\int_0^\delta \left| f(x+t) + f(x-t) - 2f(x) \right| dt = o(\delta)$$

$$\int_0^\delta \left\{ f(x+t) + f(x-t) - 2f(x) \right\}^2 dt = O(\delta)$$

3. <u>The Development of a Function of Two Variables in the Double Fourier Series</u>. Let $f(x,y)$ be a function which is periodic of period 2π in both variables, and is L. integrable in the rectangle $(-\pi, -\pi; \pi, \pi)$. Then we may form the expansion:

$$(3.1)\quad f(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (a_{ij}\cos ix \cos jy + b_{ij}\cos ix \sin jy + c_{ij}\sin ix \cos jy + d_{ij}\sin ix \sin jy)$$

where $$a_{ij} = \frac{1}{\pi^2} \int_0^{2\pi}\int_0^{2\pi} f(x',y') \cos ix' \cos jy' \, dy' \, dx'$$

$$a_{oj} = \frac{1}{2\pi^2} \int_0^{2\pi}\int_0^{2\pi} f(x',y') \cos jy' \, dy' \, dx'$$

$$a_{io} = \frac{1}{2\pi^2} \int_0^{2\pi}\int_0^{2\pi} f(x',y') \cos ix' \, dy' \, dx'$$

$$a_{oo} = \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} f(x',y') \, dy' \, dx'$$

with similar formulae for b_{ij}, c_{ij}, and d_{ij}.

This expansion gives the following double Fourier series:

$$
(3.2)\quad
\begin{array}{c}
\begin{pmatrix} a_{11}\cos\alpha\cos\delta \\ +b_{11}\cos\alpha\sin\delta \\ +c_{11}\sin\alpha\cos\delta \\ +d_{11}\sin\alpha\sin\delta \end{pmatrix}
+
\begin{pmatrix} a_{12}\cos\alpha\cos 2\delta \\ +b_{12}\cos\alpha\sin 2\delta \\ +c_{12}\sin\alpha\cos 2\delta \\ +d_{12}\sin\alpha\sin 2\delta \end{pmatrix}
+
\begin{pmatrix} a_{13}\cos\alpha\cos 3\delta \\ +b_{13}\cos\alpha\sin 3\delta \\ +b_{13}\sin\alpha\cos 3\delta \\ +d_{13}\sin\alpha\sin 3\delta \end{pmatrix}
+\cdots \\
+ \qquad\qquad + \qquad\qquad + \\
\begin{pmatrix} +a_{21}\cos 2\alpha\cos\delta \\ +b_{21}\cos 2\alpha\sin\delta \\ +c_{21}\sin 2\alpha\cos\delta \\ +d_{21}\sin 2\alpha\sin\delta \end{pmatrix}
+
\begin{pmatrix} a_{22}\cos 2\alpha\cos 2\delta \\ +b_{22}\cos 2\alpha\sin 2\delta \\ +c_{22}\sin 2\alpha\cos 2\delta \\ +d_{22}\sin 2\alpha\sin 2\delta \end{pmatrix}
+
\begin{pmatrix} a_{23}\cos 2\alpha\cos 3\delta \\ b_{23}\cos 2\alpha\sin 3\delta \\ c_{23}\sin 2\alpha\cos 3\delta \\ d_{23}\sin 2\alpha\sin 3\delta \end{pmatrix}
+\cdots \\
+ \qquad\qquad + \qquad\qquad + \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{array}
$$

4. Cesàro Summability of the Double Fourier Series.[1] Let us define

(4.1) $$A^k_{mn} = \frac{\Gamma(m+k)}{\Gamma(k+1)\Gamma(n)} \cdot \frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)}$$

(4.2) $$S^k_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\Gamma(k+m-i)}{\Gamma(k)\Gamma(m-i+1)} \cdot \frac{\Gamma(k+n-j)}{\Gamma(k)\Gamma(n-j+1)} s_{ij}$$

where $$s_{ij} = \sum_{p=1}^{i}\sum_{q=1}^{j} a_{pq} .$$

If $\frac{S^k_{mn}}{A^k_{mn}}$ approaches a limit as m and n become infinite independently, then the double series is said to be summable (Ck) and to have a value equal to this limit.

In this paper we are concerned with summability (C1), so that k = 1 in expressions (4.1) and (4.2), thus giving

(4.3) $$S = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{s_{ij}}{mn} .$$

5. Definitions. Let us define the following functions which will be needed in the development of the theorem to be proved:

(5.1) $$\phi(t,u) = f(x+t,\ y+u) + f(x-t,\ y+u) + f(x-t,\ y-u) + f(x+t,\ y-u) - 4f(x,y)$$

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(1)See C. N. Moore "On Convergence Factors in Double Series and the Double Fourier Series - Transactions of the American Mathematical Society' Vol. XIV No. 1 pp. 73-104 Jan. 1913.

(5.2) $$F_1(t,u) = \int_0^t \sin ix\ \phi\ (x,u)dx$$

(5.3) $$F_2(t,u) = \int_0^u \sin jy\ \phi\ (t,y)dy$$

(5.4) $$F_3(t,u) = \int_0^t \sin ix\ \phi\ (x,u)dx$$

(5.5) $$F_4(t,u) = \int_0^u \sin jy\ \phi\ (t,y)dy$$

6. Statement of the Theorem. If s_{mn} is the double sum of the first $(m+1)(n+1)$ terms of the double Fourier series of an L. integrable function $f(x,y)$ and for particular values of x and y

(6.1) $$\int_0^\delta |\phi(t,u)|\ du = o(\delta)$$

(6.2) $$\int_0^\delta \{\phi(t,u)\}^2\ du = O(\delta)$$

uniformly in t for the interval $(-\pi,\pi)$, and

(6.3) $$\int_0^\xi |\phi(t,u)|\ dt = o(\xi)$$

(6.4) $$\int_0^\xi \{\phi(t,u)\}^2\ dt = O(\xi)$$

uniformly in u for the interval $(-\pi,\pi)$ and further;

(6.5) $$\int_a^b F_1(t,u)du = o(\delta)$$

uniformly in u for the interval $(-\pi,\pi)$ $0 \leqq a < b \leqq \pi$ and for $i = 1, 2, 3 \ldots\ldots,$

(6.6) $$\int_a^b F_2(t,u)dt = o(\xi)$$

uniformly in t for the interval $(-\pi,\pi)$ $0 \leqq a < b \leqq \pi$ and for $j = 1, 2, 3 \ldots\ldots,$

(6.7) $$\int_a^b F_3(t,u)du = o(\delta)$$

uniformly in u for the interval $(-\pi,\pi)$ $0 \leqq a < b \leqq \pi$ and for $i = 1, 2, 3 \ldots\ldots,$

(6.8) $$\int_a^b F_4(t,u)dt = o(\xi)$$

uniformly in t for the interval $(-\pi,\pi)$ $0 \leqq a < b \leqq \pi$ and for $j = 1, 2, 3 \ldots\ldots,$

then:

(6.9) $$\lim_{m,n\to\infty} \frac{1}{m+1} \frac{1}{n+1} \sum_{i=0}^{m} \sum_{j=0}^{n} \left| s_{ij} - f(x,y) \right|^k = 0$$

for the corresponding values of x and y, and for every positive integer k.

7. Remark. If the equation (6.9) is true for $k = k_0$, it is also true for $0 < k \leqq k_0$, for by Hölder's inequality:

(7.1) $$\sum_{i=0}^{m} \sum_{j=0}^{n} \left| s_{ij} - f(x,y) \right|^k \leqq \left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} \left| s_{ij} - f(x,y) \right|^{k_0} \right\}^{\frac{k}{k_0}} \left\{ \sum_{i=0}^{m} \sum_{j=1}^{n} \right\}^{\frac{k_0-k}{k_0}}$$

and so

$$(7.2)\quad \frac{1}{m+1}\frac{1}{n+1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s_{ij}-f(x,y)\right|^{k} \leqq \left\{\frac{1}{m+1}\frac{1}{n+1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s_{ij}-f(x,y)\right|^{k_0}\right\}^{\frac{k}{k_0}}$$

Therefore it will be sufficient to prove the theorem where k is an even positive integer $2p$. Also we may, without loss of generality, take the particular values of x and y considered to be $x = 0$, $y = 0$. We can suppose further that $f(x,y) = 0$ except for $|x| < \alpha$, $|y| < \delta$ so that the square of $f(x,y)$ will be summable throughout $(-\pi,\pi)$.

8. <u>Proof of the Theorem</u>. Let

$$(8.1)\qquad f(x,y) = f_1(x,y) + f_2(x,y)$$

where

$$f_1(x,y) = 0 \qquad \begin{array}{l} \alpha < |x| < \pi, \ \delta < |y| < \pi \\ |x| < \alpha, \ |y| > \delta \\ |x| > \alpha, \ |y| < \delta \end{array}$$

and

$$f_2(x,y) = 0 \qquad |x| < \alpha, \ |y| < \delta$$

and for the corresponding partial sums s'_{ij} and s''_{ij},

$$(8.2)\qquad s_{ij} = s'_{ij} + s''_{ij}\ .$$

Using an inequality due to Minkowski[1]

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(1) See F. Riesz - Les Systems d'equations lineaires à une infinite d'inconnues - Paris 1913 p. 145

$$\left\{\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s_{ij}-f(0,0)\right|^{2p}\right\}^{\frac{1}{2p}} \leqq \left\{\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s'_{ij}-f(0,0)\right|^{2p}\right\}^{\frac{1}{2p}} + \left\{\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s''_{ij}\right|^{2p}\right\}^{\frac{1}{2p}}.$$

Noting that, for the values $x = 0$, $y = 0$

$$\lim_{i,j\to\infty} s''_{ij} = 0,$$

we see that

$$\lim_{m,n\to\infty} \frac{1}{m+1}\frac{1}{n+1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s''_{ij}\right|^{2p} = 0.$$

Hence

$$\lim_{m,n\to\infty} \frac{1}{m+1}\frac{1}{n+1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s_{ij}-f(0,0)\right|^{2p} = 0$$

if

$$(8.3)\quad \lim_{m,n\to\infty} \frac{1}{m+1}\frac{1}{n+1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left|s'_{ij}-f(0,0)\right|^{2p} = 0.$$

We have

$$(8.4)\quad s_{ij}-f(0,0) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{\sin(i+\frac{1}{2})t}{\sin\frac{1}{2}t}\,\frac{\sin(j+\frac{1}{2})u}{\sin\frac{1}{2}u}\,\phi(t,u)\,dt\,du$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\sin it\cot\frac{t}{2}\sin ju\cot\frac{u}{2}\,\phi(t,u)\,dt\,du$$

$$+ \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\sin it\cot\frac{t}{2}\cos ju\,\phi(t,u)\,dt\,du$$

$$+ \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\sin ju\cot\frac{u}{2}\cos it\,\phi(t,u)\,dt\,du$$

$$+ \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\cos it\cos ju\,\phi(t,u)\,dt\,du.$$

If we let

$$\varphi_1(t,u) \equiv \frac{1}{\pi^2} \sin it \cot \frac{t}{2} \sin ju \cot \frac{u}{2} \phi(t,u)$$

$$\varphi_2(t,u) \equiv \frac{1}{\pi^2} \sin it \cot \frac{t}{2} \cos ju \, \phi(t,u)$$

$$\varphi_3(t,u) \equiv \frac{1}{\pi^2} \sin ju \cot \frac{u}{2} \cos it \, \phi(t,u)$$

$$\varphi_4(t,u) \equiv \frac{1}{\pi^2} \cos it \cos ju \, \phi(t,u)$$

equation (8.4) can be written in the form

$$\begin{aligned} s_{ij} - f(0,0) = & \int_0^\delta\int_0^\varepsilon \varphi_1(t,u)\,dt\,du + \int_\delta^\pi\int_0^\varepsilon \varphi_1(t,u)\,dt\,du \\ & + \int_0^\delta\int_\varepsilon^\pi \varphi_1(t,u)\,dt\,du + \int_\delta^\pi\int_\varepsilon^\pi \varphi_1(t,u)\,dt\,du \\ & + \int_0^\delta\int_0^\varepsilon \varphi_2(t,u)\,dt\,du + \int_\delta^\pi\int_0^\varepsilon \varphi_2(t,u)\,dt\,du \\ & + \int_0^\delta\int_\varepsilon^\pi \varphi_2(t,u)\,dt\,du + \int_\delta^\pi\int_\varepsilon^\pi \varphi_2(t,u)\,dt\,du \\ & + \int_0^\delta\int_0^\varepsilon \varphi_3(t,u)\,dt\,du + \int_\delta^\pi\int_0^\varepsilon \varphi_3(t,u)\,dt\,du \\ & + \int_0^\delta\int_\varepsilon^\pi \varphi_3(t,u)\,dt\,du + \int_\delta^\pi\int_\varepsilon^\pi \varphi_3(t,u)\,dt\,du \\ & + \int_0^\pi\int_0^\pi \varphi_4(t,u)\,dt\,du \end{aligned}$$

$$= \alpha_{ij}^{(1)} + \alpha_{ij}^{(2)} + \alpha_{ij}^{(3)} + \alpha_{ij}^{(4)} + \beta_{ij}^{(1)} + \beta_{ij}^{(2)} + \beta_{ij}^{(3)} +$$

$$\beta_{ij}^{(4)} + \gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} + \gamma_{ij}^{(3)} + \gamma_{ij}^{(4)} + \delta_{ij}$$

Now using the Minkowski inequality again

$$\left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left| s_{ij} - f(0,0) \right|^{2p} \right\}^{\frac{1}{2p}} \leqq \sum_{r=1}^{4} \left[\left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \alpha_{ij}^{(r)} \right|^{2p} \right\}^{\frac{1}{2p}} + \left\{ \sum_{i=o}^{m} \sum_{j=o}^{m} \left| \beta_{ij}^{(r)} \right|^{2p} \right\}^{\frac{1}{2p}} + \left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \gamma_{ij}^{(r)} \right|^{2p} \right\}^{\frac{1}{2p}} \right] + \left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \delta_{ij} \right|^{2p} \right\}^{\frac{1}{2p}} .$$

We will now treat the following double sums in order

$$\sum_{i=o}^{m} \sum_{j=o}^{n} \left| \alpha_{ij}^{(r)} \right|^{2p}, \quad \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \beta_{ij}^{(r)} \right|, \quad \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \gamma_{ij}^{(r)} \right| \quad r=1,2,3,4.$$

and $\sum_{i=o}^{m} \sum_{j=o}^{n} \left| \delta_{ij} \right|$.

Consider first

$$(i) \quad \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \alpha_{ij}^{(1)} \right|^{2p}, \quad \alpha_{ij}^{(1)} = \frac{1}{\pi^2} \int_o^{\delta} \int_o^{\varepsilon} \sin it \cot \frac{t}{2} \sin ju \cot \frac{u}{2} \phi(t,u)\, dt\, du.$$

We can simplify the integrand

$$\left|\sin it \cot \frac{t}{2}\right| < i \left|t \cot \frac{t}{2}\right| < Ai$$

$$\left|\sin ju \cot \frac{u}{2}\right| < j \left|u \cot \frac{u}{2}\right| < Bj$$

where A and B represent maximum values of $\left|t \cot \frac{t}{2}\right|$ and $\left|u \cot \frac{u}{2}\right|$ in the interval $(0,\pi)$.

Using the first mean value theorem for integrals,

$$(8.5) \qquad \left|\alpha_{ij}^{(1)}\right| < \frac{ABij}{\pi^2} \int_0^\delta \int_0^\xi \left|\phi(t,u)\right| dt\, du,$$

and hence

$$(8.6) \qquad \left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left|\alpha_{ij}^{(1)}\right|^{2p} \right\}^{\frac{1}{2p}} \leqq \left\{ m^{2p+1}\, n^{2p+1} \right\}^{\frac{1}{2p}} o(\delta)$$

$$= (m^{1+\frac{1}{2p}}\, n^{1+\frac{1}{2p}})\, o(\delta).$$

For the next sum

$$(ii) \qquad \sum_{i=o}^{m} \sum_{j=o}^{n} \left|\alpha_{ij}^{(2)}\right|^{2p}, \quad \alpha_{ij}^{(2)} = \frac{1}{\pi^2} \int_\delta^\pi \int_0^\xi \sin it \cot \frac{t}{2}$$

$$\sin ju \cot \frac{u}{2}\, \phi(t,u)\, dt\, du,$$

we have the inequality (8.5)

$$\left|\alpha_{ij}^{(2)}\right| < \frac{ABij}{\pi^2} \int_\delta^\pi \int_0^\xi \left|\phi(t,u)\right| dt\, du,$$

and hence

$$(8.7) \qquad \left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left|\alpha_{ij}^{(2)}\right|^{2p} \right\}^{\frac{1}{2p}} \leqq (m^{1+\frac{1}{2p}}\, n^{1+\frac{1}{2p}})\, o(\xi).$$

Turning now to

$$\text{(iii)} \quad \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \alpha_{ij}^{(3)} \right|^{2p}, \quad \alpha_{ij}^{(3)} = \frac{1}{\pi^2} \int_o^\delta \int_\varepsilon^\pi \sin it \cot \frac{t}{2}$$

$$\sin ju \cot \frac{u}{2} \phi(t,u)\, dt\, du,$$

we have from the inequality (8.5)

$$\left| \alpha_{ij}^{(3)} \right| < \frac{ABij}{\pi^2} \int_o^\delta \int_\varepsilon^\pi \left| \phi(t,u) \right| dt\, du,$$

and hence

$$(8.8) \quad \left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \alpha_{ij}^{(3)} \right|^{2p} \right\}^{\frac{1}{2p}} \leqq \left(m^{\frac{1}{2p}}\, n^{1+\frac{1}{2p}}\right) o(\delta).$$

For the consideration of the next sum

$$\text{(iv)} \quad \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \alpha_{ij}^{(4)} \right|^{2p}, \quad \alpha_{ij}^{(4)} = \frac{1}{\pi^2} \int_\delta^\pi \int_\varepsilon^\pi \sin it \sin ju$$

$$\cot \frac{t}{2} \cot \frac{u}{2} \phi(t,u)\, dt\, du,$$

we will define,

$$C_{ij}(t,u) = \frac{4}{\pi^2} \int_o^t \int_o^u \sin ix \sin jy\, \phi(x,y)\, dx\, dy$$

$$C_{ij}^{(1)}(t,u) = \frac{4}{\pi^2} \sin it \int_o^u \sin jy\, \phi(t,y)\, dy$$

$$C_{ij}^{(2)}(t,u) = \frac{4}{\pi^2} \sin ju \int_o^t \sin ix\, \phi(x,u)\, dx$$

Integrating a^{4}_{ij} by parts[1] we obtain:

$$\alpha^{(4)}_{ij} = \left[\frac{1}{4}\cot\frac{t}{2}\cot\frac{u}{2}\left\{\frac{4}{\pi^2}\int_0^t\int_0^u \sin ix \sin jy\ \phi(x,y)\ dxdy\right\}\right]^{\pi,\pi}_{\delta,\xi}$$

$$-\frac{1}{16}\int_\delta^\pi\int_\xi^\pi \csc^2\frac{t}{2}\csc^2\frac{u}{2}\,C_{ij}(t,u)\,dt\,du\ +$$

$$\frac{1}{8}\int_\delta^\pi\int_\xi^\pi \cot\frac{t}{2}\csc^2\frac{u}{2}\,C^{(1)}_{ij}\,(t,u)\,dt\,du\ +$$

$$\frac{1}{8}\int_\delta^\pi\int_\xi^\pi \csc^2\frac{u}{2}\cot\frac{u}{2}\,C^{(2)}_{ij}\,(t,u)\,dt\,du$$

$$= \frac{1}{4}\cot\frac{\delta}{2}\cot\frac{\xi}{2}\,C_{ij}(\delta,\xi) - \frac{1}{16}\int_\delta^\pi\int_\xi^\pi \csc^2\frac{t}{2}\csc^2\frac{u}{2}\,C_{ij}(t,u)\,dt\,du$$

$$+\frac{1}{8}\int_\delta^\pi\int_\xi^\pi \cot\frac{t}{2}\csc^2\frac{u}{2}\,C^{(1)}_{ij}(t,u)\,dt\,du$$

$$+\frac{1}{8}\int_\delta^\pi\int_\xi^\pi \csc^2\frac{u}{2}\cot\frac{u}{2}\,C^{(2)}_{ij}(t,u)\,dt\,du,$$

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(1)

$$\int_a^x\int_b^y \frac{\partial^2 u}{\partial x\partial y}\,v\,dx\,dy = \Big[uv\Big]^{x,y}_{a,b} - \int_a^x\int_b^y \frac{\partial^2 v}{\partial x\partial y}\,u\,dx\,dy$$

$$-\int_a^x\int_b^y \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\,dx\,dy$$

$$-\int_a^x\int_b^y \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\,dx\,dy$$

providing $u = \int\int U\,dx\,dy$ and $v = \int\int V\,dx\,dy$

See Hilda Gerringer - Monatshifte für Mathematik und Physik Vo. 29 (1919) p. 182

and so

$$\left| \sum_{i=0}^{\alpha} \sum_{j=0}^{\alpha} \mu_{ij} \, a_{ij}^{(4)} \right| \leq \frac{1}{4} \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} \left| \mu_{ij} \, C_{ij}(\delta, \varepsilon) \right| \cot \frac{\delta}{2} \cot \frac{\varepsilon}{2}$$

$$+ \frac{1}{16} \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} \left| \mu_{ij} \, C_{ij}(t,u) \right| \csc^2 \frac{t}{2}$$

$$\csc^2 \frac{u}{2} \, dt \, du$$

$$+ \frac{1}{8} \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} \left| \mu_{ij} \, C_{ij}^{(1)}(t,u) \right|$$

$$\cot \frac{t}{2} \csc^2 \frac{u}{2} \, dt \, du$$

$$+ \frac{1}{8} \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} \left| \mu_{ij} \, C_{ij}^{(2)}(t,u) \right|$$

$$\csc^2 \frac{t}{2} \cot \frac{u}{2} \, dt \, du$$

for any value of μ_{ij} for which the right side exists.

Now if

$$\sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} \left| \mu_{ij} \right|^{\frac{2p}{2p-1}} = 1$$

then

$$\max_{\mu_{ij}} \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} \left| \mu_{ij} \, C_{ij} \right| = \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\alpha} \left| C_{ij} \right|^{2p} \right\}^{\frac{1}{2p}}$$

since $\Sigma\Sigma C_{ij}$ is convergent, hence

$$\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\alpha_{ij}^{(4)}\right|^{2p}\right\}^{\frac{1}{2p}} \leq \frac{1}{4}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|C_{ij}(\delta,\epsilon)\right|^{2p}\right\}^{\frac{1}{2p}}\cot\frac{\delta}{2}\cot\frac{\epsilon}{2}$$

$$+\frac{1}{16}\int_{\delta}^{\pi}\int_{\epsilon}^{\pi}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|C_{ij}(t,u)\right|^{2p}\right\}^{\frac{1}{2p}} \csc^2\frac{t}{2}\csc^2\frac{u}{2}\,dt\,du$$

(8.9)
$$+\frac{1}{8}\int_{\delta}^{\pi}\int_{\epsilon}^{\pi}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|C_{ij}^{(1)}(t,u)\right|^{2p}\right\}^{\frac{1}{2p}} \cot\frac{t}{2}\csc^2\frac{u}{2}\,dt\,du$$

$$+\frac{1}{8}\int_{\delta}^{\pi}\int_{\epsilon}^{\pi}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|C_{ij}^{(2)}(t,u)\right|^{2p}\right\}^{\frac{1}{2p}} \csc^2\frac{t}{2}\cot\frac{u}{2}\,dt\,du$$

But $C_{ij}(r,s) = \frac{4}{\pi^2}\int_0^r\int_0^s \sin it \sin ju\, \phi(t,u)\,dt\,du$ is the typical coefficient of the double Fourier sine series $\phi_{rs}(x,y)$, where

$$\phi_{r,s}(x,y) = \phi(x,y) \qquad (0 < x < r;\ 0 < y < s),$$

$$\phi_{r,s}(x,y) = 0 \qquad (r < x < \pi;\ s < y < \pi).$$

From

$$\phi_{r,s}(x,y) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij}(r,s)\sin ix \sin jy,$$

we obtain

$$\phi_{r,s}(x+v,\ y+w) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_{ij}(r,s)\sin i(x+v)\sin j(y+w).$$

Next we define

$$\phi_{r,s}^{(2)}(x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_{r,s}(v,w)\, \phi_{r,s}(x+v,\ y+w)\, dv\, dw$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ C_{ij}(r,s) \right\}^2 \cos ix \cos jy$$

Similarly

$$\phi_{r,s}^{(3)}(x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_{r,s}^{(2)}(v,w)\, \phi_{r,s}(x+v,\ y+w)\, dv\, dw$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ C_{ij}(r,s) \right\}^3 \sin ix \sin jy,$$

and in general we have

$$\phi_{r,s}^{(2p)}(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ C_{ij}(r,s) \right\}^{2p} \cos ix \cos jy$$

which for the particular value x=0, y=0, yields

$$\phi_{r,s}^{(2p)}(0,0) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ C_{ij}(r,s) \right\}^{2p} .$$

Now let

$$\int_0^r\int_0^s \left| \phi(t,u) \right| dt\, du = \rho(r,s),$$

$$\int_0^r\int_0^s \left\{ \phi(t,u) \right\}^2 dt\, du = \theta(r,s)$$

Then

$$\left| \phi^{(2)}_{r,s}(x,y) \right| \leqq \frac{1}{\pi^2} \left\{ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left[\phi_{r,s}(v,w)\right]^2 dv\, dw \right.$$

$$\left. \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left[\phi_{r,s}(x+v,\ y+w)\right]^2 dv\, dw \right\}^{\frac{1}{2}}$$

$$\leqq \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \phi_{r,s}(v,w) \right|^2 dv\, dw$$

$$= \frac{4}{\pi^2}\, \theta(r,s),$$

and

$$\left| \phi^{(3)}_{r,s}(x,y) \right| \leqq \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \phi^{(2)}_{r,s}(v,w) \right| \left| \phi_{r,s}(x+v,y+w) \right| dv\, dw$$

$$\leqq \frac{4}{\pi^2}\, \theta(r,s) \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \phi_{r,s}(x+v,\ y+w) \right| dv\, dw$$

$$\leqq \left(\frac{4}{\pi^2}\right)^2 \theta(r,s)\, \rho(r,s)$$

Continuing this process, we obtain

$$\left| \phi^{(2p)}_{r,s}(x,y) \right| \leqq \left(\frac{4}{\pi^2}\right)^{2p-1} \theta(r,s) \left\{ \rho(r,s) \right\}^{2p-2}$$

Hence for the particular values $x = 0$, $y = 0$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ c_{ij}(r,s) \right\}^{2p} \leqq \left\{ \frac{4}{\pi^2} \right\}^{2p-1} \theta(r,s) \left\{ \rho(r,s) \right\}^{2p-2}$$

Now

$$\rho(r,s) < K_1 rs,$$

$$\theta(r,s) < K_2 rs.$$

Hence

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\{C_{ij}(r,s)\right\}^{2p} \leqq \left(\frac{4}{\pi}\right)^{2p-1} K_2 rs\,(K_1 rs)^{2p-2}$$

and so

$$(8.10)\quad \left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[C_{ij}(r,s)\right]^{2p}\right\}^{\frac{1}{2p}} < K_3 r^{\frac{2p-1}{2p}}\; s^{\frac{2p-1}{2p}}$$

where K_1, K_2, and K_3 are absolute constants.

Substituting in equation (8.9) the expressions (6.5), (6.6), and (8.10), we obtain the value for

$$\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\alpha_{ij}^{(4)}\right|^{2p}\right\}^{\frac{1}{2p}} \leqq \frac{k}{\delta\varepsilon}\,\delta^{\frac{2p-1}{2p}}\,\varepsilon^{\frac{2p-1}{2p}} + K\int_{\delta}^{\pi}\int^{\pi} t^{-\frac{1}{2p}-1}\,u^{-\frac{1}{2p}-1}\,dt\,du + K\,o(\delta) + K\,o(\varepsilon)$$

$$< K\;\delta^{-\frac{1}{2p}}\,\varepsilon^{-\frac{1}{2p}} + o(\delta) + o(\varepsilon)$$

where K represents the largest of the K's.

9. <u>Theorem</u> (continued). Now we will proceed to obtain inequalities for the β_{ij}'s

$$(v)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\left|\beta_{ij}^{(1)}\right|^{2p},\quad \beta_{ij}^{(1)} = \frac{1}{\pi^2}\int_0^{\delta}\int_0^{\varepsilon}\sin it\,\cot\frac{t}{2}\,\cos ju\;\phi(t,u)\,dt\,du.$$

We can simplify the integrand by using

$$\left|\sin it \cot \frac{t}{2}\right| < i \left|t \cot \frac{t}{2}\right| < Ai,$$

$$\left|\cos ju\right| \qquad < j\left|u\right| < Bj,$$

where A and B represent the maximum values of $\left|t \cot \frac{t}{2}\right|$ and $\left|u\right|$ in the interval $(0,\pi)$.

Using the first mean value theorem for integrals

(9.1) $$\beta_{ij}^{(1)} < \frac{Aij}{\pi^2} \int_0^\delta \int_0^\zeta \left|\phi(t,u)\right| \, dt\, du,$$

and hence

(9.2) $$\left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} \left|\beta_{ij}^{(1)}\right|^{2p} \right\}^{\frac{1}{2p}} < (m^{1+\frac{1}{2p}} n^{1+\frac{1}{2p}})\, o(\delta).$$

Consider next

(vi) $$\sum_{i=0}^{m} \sum_{j=0}^{n} \left|\beta_{ij}^{(2)}\right|^{2p}, \quad \beta_{ij}^{(2)} = \frac{1}{\pi^2} \int_\delta^\pi \int_0^\zeta \sin it \cot \frac{t}{2}$$

$$\cos ju\, \phi(t,u)\, dt\, du.$$

From the inequality (9.1) we have

$$\beta_{ij}^{(2)} < \frac{Aij}{\pi^2} \int_\delta^\pi \int_0^\zeta \left|\phi(t,u)\right| \, dt\, du$$

and hence

(9.3) $$\left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} \left|\beta_{ij}^{(2)}\right|^{2p} \right\}^{\frac{1}{2p}} \leq (m^{1+\frac{1}{2p}} n^{1+\frac{1}{2p}})\, o(\zeta).$$

Turning now to

(vii) $\sum_{i=0}^{m} \sum_{j=0}^{n} \left| \beta_{ij}^{(3)} \right|^{2p}$, $\beta_{ij}^{(3)} = \frac{1}{\pi^2} \int_0^\delta \int_\varepsilon^\pi \sin it \cot \frac{t}{2}$

$\cos ju\, \phi(t,u)\, dt\, du,$

we have the inequality (9.1)

$$\beta_{ij}^{(3)} < \frac{Aij}{\pi^2} \int_0^\delta \int_\varepsilon^\pi \left| \phi(t,u) \right| dt\, du,$$

and hence

$$(9.4) \quad \left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} \left| \beta_{ij}^{(3)} \right|^{2p} \right\}^{\frac{1}{2p}} \leqq (m^{1+\frac{1}{2p}} n^{1+\frac{1}{2p}})\, o(\delta).$$

For the consideration of the next sum,

(viii) $\sum_{i=0}^{m} \sum_{j=0}^{n} \left| \beta_{ij}^{(4)} \right|^{2p}$, $\beta_{ij}^{(4)} = \frac{1}{\pi^2} \int_\delta^\pi \int_\varepsilon^\pi \sin it \cos ju$

$\cot \frac{t}{2}\ \phi(t,u)\, dt\, du,$

we will define

$$D_{ij}(t,u) = \frac{4}{\pi^2} \int_0^t \int_0^u \sin ix \cos jy\, \phi(x,y)\, dx\, dy,$$

$$D_{ij}^{(1)}(t,u) = \frac{4}{\pi^2} \cos ju \int_0^t \sin ix\, \phi(x,u)\, dx.$$

Integrating β_{ij}^{4} by parts, we obtain

$$\beta_{ij}^{(4)} = - D_{ij}(\delta,\varepsilon) \frac{1}{4} \cot \frac{\varepsilon}{2} + \frac{1}{4} \int_\delta^\pi \int_\varepsilon^\pi D_{ij}^{(1)}(t,u) \csc^2 \frac{t}{2}\, dt\, du,$$

and so

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\mu_{ij}\,\beta_{ij}^{(4)}\right| \leq \frac{1}{4}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\mu_{ij}D_{ij}(\delta,\varepsilon)\right| \cot\frac{\varepsilon}{2}$$

$$+\frac{1}{4}\int_{\delta}^{\pi}\int_{\varepsilon}^{\pi}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\mu_{ij}D_{ij}^{(1)}(t,u)\right|$$

$$\csc^2\frac{t}{2}\,dt\,du$$

for any value of μ_{ij} for which the right side exists. Now if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\mu_{ij}\right|^{\frac{2p}{2p-1}} = 1,$$

then

$$\underset{\mu_{ij}}{\text{Max.}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\mu_{ij}D_{ij}\right| = \left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|D_{ij}\right|^{2p}\right\}^{\frac{1}{2p}},$$

since $\Sigma\Sigma\, D_{ij}$ is convergent and hence

$$\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\beta_{ij}^{(4)}\right|^{2p}\right\}^{\frac{1}{2p}} \leq \frac{1}{4}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|D_{ij}(\delta,\varepsilon)\right|^{2p}\right\}^{\frac{1}{2p}}\cot\frac{\varepsilon}{2}$$

$$+\frac{1}{4}\int_{\delta}^{\pi}\int_{\varepsilon}^{\pi}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|D_{ij}^{(1)}(t,u)\right|^{2p}\right\}^{\frac{1}{2p}}$$

$$\csc^2\frac{t}{2}\,dt\,du.$$

But $D_{ij}(r,s) = \frac{4}{\pi}\int_0^r\int_0^s \sin it \cos ju\, \phi(t,u)\,dt\,du$ in the typical coefficient of the cross product double Fourier series $\phi_{r,s}(x,y)$ where

$$\phi_{r,s}(x,y) = \phi(x,y) \qquad (0 < x < r;\ 0 < y < s)$$

$$\phi_{r,s}(x,y) = 0 \qquad (r < x < \pi;\ s < y < \pi).$$

From

$$\phi_{r,s}(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} D_{ij}(r,s) \sin ix \cos jy$$

we obtain

$$\phi_{r,s}(x+v,\ y+w) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} D_{ij}(r,s) \sin i(x+v) \cos j(y+w).$$

We then define

$$\phi^{(2)}_{r,s}(x,y) \equiv \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_{r,s}(v,w)\ \phi_{r,s}(x+v,y+w)\ dv\ dw$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ D_{ij}(r,s) \right\}^2 \cos ix \cos jy.$$

Similarly

$$\phi^{(3)}_{r,s}(x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi^{(2)}_{r,s}(v,w)\ \phi_{r,s}(x+v,y+w)\ dv\ dw$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ D_{ij}(r,s) \right\}^3 \sin ix \cos jy,$$

and

$$\phi^{(4)}_{r,s}(x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi^{(3)}_{r,s}(v,w)\ \phi_{r,s}(x+v,y+w)\ dv\ dw$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ D_{ij}(r,s) \right\}^4 \cos ix \cos jy.$$

In general we have

$$\phi^{(2p)}_{r,s}(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ D_{ij}(r,s) \right\}^{2p} \cos ix \cos jy$$

which for the particular values $x = 0$, $y = 0$ yields

$$\phi_{r,s}^{(2p)}(0,0) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ D_{ij}(r,s) \right\}^{2p}$$

Now let

$$\int_0^r \int_0^s \left| \phi(t,u) \right| dt\, du = \rho(r,s),$$

$$\int_0^r \int_0^s \left\{ \phi(t,u) \right\}^2 dt\, du = \Theta(r,s).$$

Then

$$\left| \phi_{r,s}^{(2)}(x,y) \right| \leq \frac{1}{\pi^2} \left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\phi_{r,s}(v,w) \right]^2 dt\, du \right.$$

$$\left. \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\phi_{r,s}(x+v,y+w) \right]^2 dv\, dw \right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \phi_{r,s}(v,w) \right|^2 dv\, dw$$

$$= \frac{4}{\pi^2} \Theta(r,s),$$

$$\left| \phi_{r,s}^{(3)}(x,y) \right| \leq \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \phi_{r,s}^{(2)}(v,w) \right|^2 \left| \phi_{r,s}(x+v,y+w) \right| dv dw$$

$$\leq \frac{4}{\pi^2} \Theta(r,s) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \phi_{r,s}(x+v,y+w) \right. dv\, dw$$

$$\leq \left(\frac{4}{\pi}\right)^2 \Theta(r,s)\, \rho(r,s).$$

Continuing this process, we obtain

$$\left| \phi_{r,s}^{(2p)}(x,y) \right| \leq \left(\frac{4}{\pi^2}\right)^{2p-1} \Theta(r,s) \left\{ \rho(r,s) \right\}^{2p-2} .$$

Hence for the particular values x = 0, y = 0

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ D_{ij}(r,s) \right\}^{2p} \leqq \left(\frac{4}{\pi^2}\right)^{2p-1} \theta(r,s) \left\{ \rho(r,s) \right\}^{2p-2} .$$

Now

$$\rho(r,s) < K_1 rs,$$

$$\theta(r,s) < K_2 rs.$$

Hence

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ D_{ij}(r,s) \right\}^{2p} \leqq \left(\frac{4}{\pi}\right)^{2p-1} K_2 rs (K_1 rs)^{2p-2} ,$$

and so

$$\left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[D_{ij}(r,s) \right]^{2p} \right\}^{\frac{1}{2p}} < K_3\, r^{\frac{2p-1}{2p}} s^{\frac{2p-1}{2p}}$$

where K_1, K_2, and K_3 are absolute constants.

Substituting in equation (9.5), the relations (9.6) and (6.7), we obtain the following inequality

$$(9.7) \quad \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left| \beta_{ij}^{(4)} \right|^{2p} \right\}^{\frac{1}{2p}} \leqq \frac{K}{\varepsilon}\, \delta^{\frac{2p-1}{2p}} \varepsilon^{\frac{2p-1}{2p}} + K\, o(\varepsilon),$$

where K represents the largest of the K's.

10. <u>Theorem</u> (continued). Next we will obtain inequalities for the $\gamma_{ij}^{(r)}$'s.

$$(\text{ix}) \quad \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \gamma_{ij}^{(1)} \right|^{2p} \qquad \gamma_{ij}^{(1)} = \frac{1}{\pi^2} \int_o^{\delta}\int_o^{\varepsilon} \sin ju \cot \frac{u}{2}$$

$$\cos it\, \phi(t,u)\, dt\, du.$$

We can simplify the integrand by means of

$$\left|\sin ju \cot \frac{u}{2}\right| < j \left| u \cot \frac{u}{2}\right| < Aj,$$

$$\left|\cos it\right| < i \left|t\right| < Bi,$$

where A and B represent maximum values of $\left| u \cot \frac{u}{2}\right|$ and $|u|$ in the interval $(0,\pi)$.

Using the first mean value theorem for integrals,

$$(10.1) \qquad \gamma_{ij}^{(1)} < \frac{Aij}{\pi^2} \int_0^\delta\int_0^\varepsilon \left|\phi(t,u)\right| dt\, du,$$

$$(10.2) \left\{\sum_{i=0}^{m}\sum_{j=0}^{n}\left|\gamma_{ij}^{(1)}\right|^{2p}\right\}^{\frac{1}{2p}} < (m^{1+\frac{1}{2p}}\, n^{1+\frac{1}{2p}})\, o(\delta).$$

Consider now

$$(x) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}\left|\gamma_{ij}^{(2)}\right|^{2p}, \quad \gamma_{ij}^{(2)} = \frac{1}{\pi^2}\int_\delta^\pi\int_0^\varepsilon \left|\phi(t,u)\right| dt\, du.$$

From the inequality (10.1) we have

$$\gamma_{ij}^{(2)} < \frac{Aij}{\pi^2}\int_\delta^\pi\int_0^\varepsilon \left|\phi(t,u)\right| dt\, du,$$

and hence

$$(10.3) \left\{\sum_{i=0}^{m}\sum_{j=0}^{n}\left|\gamma_{ij}^{(2)}\right|^{2p}\right\}^{\frac{1}{2p}} < (m^{1+\frac{1}{2p}}\, n^{1+\frac{1}{2p}})\, o(\varepsilon)$$

Turning next to

$$(xi) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}\left|\gamma_{ij}^{(3)}\right|^{2p}, \quad \gamma_{ij}^{(3)} = \frac{1}{\pi^2}\int_0^\delta\int_\varepsilon^\pi \sin ju \cot\frac{u}{2} \cos it\, \phi(t,u)\, dt\, du,$$

we have from the inequality (10.1)

$$\gamma_{ij}^{(3)} < \frac{A_{ij}}{\pi^2} \int_0^\delta \int_\varepsilon^\pi \left| \phi(t,u) \right| dt\, du,$$

and hence

$$(10.4) \quad \left\{ \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \gamma_{ij}^{(3)} \right|^{2p} \right\}^{\frac{1}{2p}} < (m^{1+\frac{1}{2p}} n^{1+\frac{1}{2p}})\, o(\delta).$$

For the consideration of the sum

$$(xii) \quad \sum_{i=o}^{m} \sum_{j=o}^{n} \left| \gamma_{ij}^{4} \right|^{2p}, \quad \gamma_{ij}^{(4)} = \frac{1}{\pi^2} \int_\delta^\pi \int_\varepsilon^\pi \sin ju \cot \frac{u}{2} \cos it\, \phi(t,u)\, dt\, du,$$

we will define

$$E_{ij}(t,u) = \frac{4}{\pi^2} \int_o^t \int_o^u \sin jy \cos ix\, \phi(x,y)\, dx\, dy,$$

$$E_{ij}^{(1)}(t,u) = \frac{4}{\pi^2} \cos it \int_o^u \sin jy\, \phi(t,y)\, dy,$$

and integrate $\gamma_{ij}^{(4)}$ by parts, thus obtaining

$$\gamma_{ij}^{(4)} = - E_{ij}(\delta,\varepsilon) \frac{1}{4} \cot \frac{\delta}{2} + \frac{1}{4} \int_\delta^\pi \int_\varepsilon^\pi E_{ij}^{(1)}(t,u) \sin ju \csc^2 \frac{u}{2}\, dt\, du.$$

Hence

$$\left| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij} \gamma_{ij}^{(4)} \right| \leqq \frac{1}{4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left| \mu_{ij} E_{ij}(\delta,\varepsilon) \right| \cot \frac{\delta}{2} + \frac{1}{4} \int_\delta^\pi \int_\varepsilon^\pi \left| \mu_{ij} E_{ij}^{(1)}(t,u) \right| \sin ju \csc^2 \frac{u}{2}\, dt\, du$$

for any value of μ_{ij} for which the right side exists.

Now if $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\mu_{ij}|^{\frac{2p}{2p-1}} = 1$, then

$$\operatorname*{Max}_{\mu_{ij}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\mu_{ij} E_{ij}| = \left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|E_{ij}|^{2p}\right\}^{\frac{1}{2p}}$$

since $\Sigma\Sigma E_{ij}$ is convergent. Hence

$$\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\gamma_{ij}^{(4)}|^{2p}\right\}^{\frac{1}{2p}} \leq \frac{1}{4}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|E_{ij}(\delta,\varepsilon)|^{2p}\right\}^{\frac{1}{2p}} \cot\frac{\delta}{2}$$
$$+ \frac{1}{4}\int_{\delta}^{\pi}\int_{\varepsilon}^{\pi}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|E_{ij}^{(1)}(t,u)|^{2p}\right\}^{\frac{1}{2p}}$$
$$\sin ju \csc^2\frac{u}{2}\, dt\, du$$

But $E_{ij}(r,s) = \frac{4}{\pi^2}\int_0^r\int_0^s \sin ju \cos it\, \phi(t,u)\, dt\, du$ is the typical coefficient of the cross product double Fourier series $\phi_{r,s}(x,y)$, where

$$\phi_{r,s}(x,y) = \phi(x,y) \qquad (0 < x < r;\ 0 < y < s),$$

$$\phi_{r,s}(x,y) = 0 \qquad (r < x < \pi;\ s < y < \pi).$$

From

$$\phi_{r,s}(x,y) \sim \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E_{ij}(r,s) \sin ix \cos jy$$

we obtain

$$\phi_{r,s}(x+v,\ y+w) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E_{ij}(r,s) \sin i(x+v) \cos j(y+w).$$

Next define

$$\phi^{(2)}_{r,s}(x,y) \equiv \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_{r,s}(v,w)\, \phi_{r,s}(x+v,\ y+w)\, dv\, dw$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ E_{ij}\ (r,s) \right\}^2 \cos ix \cos jy.$$

Similarly

$$\phi^{(3)}_{r,s}\ (x,y) \equiv \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi^{(2)}_{r,s}\ (v,w)\ \phi_{r,s}(x+v,\ y+w)\, dv\, dw$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ E_{ij}(r,s) \right\}^3 \sin jy \cos ix,$$

and

$$\phi^{(4)}_{r,s}(x,y) \equiv \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left\{ E_{ij}(r,s) \right\}^4 \cos ix \cos jy.$$

We have in general

$$\phi^{(2p)}_{r,s}(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ E_{ij}(r,s) \right\}^{2p} \cos ix \cos jy$$

which for the particular values x = 0, y = 0, yields

$$\phi^{(2p)}_{r,s}\ (0,0) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ E_{ij}(r,s) \right\}^{2p}.$$

Now let

$$\int_0^r\int_0^s \left| \phi(t,u) \right| dt\, du = \rho(r,s)$$

$$\int_0^r\int_0^s \left\{ \phi(t,u) \right\}^2 dt\, du = \Theta(r,s).$$

Then

$$\left| \phi^{(2)}_{r,s}(x,y) \right| \leq \frac{1}{\pi^2} \left\{ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \phi_{r,s}(v,w) \right|^2 dv\, dw \right.$$

$$\left. \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \phi_{r,s}(x+v,\ y+w)^2 \right. dv\, dw \right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left\{ \phi_{r,s}(v,w) \right\}^2 dv\, dw = \frac{4}{\pi^2}\, \theta(r,s),$$

and

$$\left| \phi^{(3)}_{r,s}(x,y) \right| \leq \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \phi^{(2)}_{r,s}(v,w) \right|^2 \left| \phi_{r,s}(x+v,y+w) \right| dv\, dw$$

$$\leq \frac{4}{\pi^2}\, \theta(r,s) \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_{r,s}(x+v,\ y+w) \Big| \, dv\, dw$$

$$\leq \left(\frac{4}{\pi}\right)^2 \theta(r,s)\ \rho(r,s).$$

Continuing this process we obtain

$$\left| \phi^{(2p)}_{r,s}(x,y) \right| \leq \left(\frac{4}{\pi^2}\right)^{2p-1} \theta(r,s) \left\{ \rho(r,s) \right\}^{2p-2}.$$

Hence for the particular values x = 0, y = 0

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ E_{ij}(r,s) \right\}^{2p} \leq \left(\frac{4}{\pi^2}\right)^{2p-1} \theta(r,s) \left\{ \rho(r,s) \right\}^{2p-2}$$

Now

$$\rho(r,s) < K_1 rs, \qquad \theta(r,s) < K_2 rs,$$

so that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\{ E_{ij}(r,s)\right\}^{2p} \leqq \left(\frac{4}{\pi}\right)^{2p-1} K_2 rs(K_1 rs)^{2p-2},$$

and

$$(10.6)\quad \left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\{ E_{ij}(r,s)\right\}^{2p}\right\}^{\frac{1}{2p}} < K_3\, r^{\frac{2p-1}{2p}}\, s^{\frac{2p-1}{2p}}.$$

Substituting in equation (10.5) the relations (6.8) and (10.6), we have

$$(10.7)\quad \left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\gamma_{ij}^{(4)}\right|^{2p}\right\}^{\frac{1}{2p}} \leqq \frac{K}{\delta}\,\delta^{\frac{2p-1}{2p}}\,\varepsilon^{\frac{2p-1}{2p}} + K\,o(\varepsilon)$$

$$< K\,\delta^{-\frac{1}{2p}}\,\varepsilon^{1-\frac{1}{2p}} + K\,o(\varepsilon),$$

where K represents the largest of the K's.

11. <u>Theorem</u> (continued). Lastly we will obtain an inequality for the δ_{ij}

$$(\text{xiii})\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\left|\delta_{ij}\right|^{2p}, \quad \delta_{ij} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \cos it \cos ju\ \phi(t,u)\,dt\,du$$

We see that δ_{ij} is the coefficient of the cos ix cos jy terms of the double Fourier series function $\phi(t,u)$ applying Parseval's Theorem[1]

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[1] Hobson V. 2 2nd Edition paragraph 470

$$\delta_{00}^2 + 4 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{ij}^2 = \frac{1}{\pi^2} \int_0^{\pi}\int_0^{\pi} \left\{\phi(t,u)\right\}^2 dt\, du.$$

Hence

$$\sum_{i=0}^{m} \sum_{j=0}^{n} |\delta_{ij}|^{2p} \leqq \left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} |\delta_{ij}|^2 \right\}^p = O(1),$$

and so

$$(11.1) \qquad \left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} |\delta_{ij}|^{2p} \right\}^{\frac{1}{2p}} \leqq O(1).$$

12. Theorem (conclusion). Combining the expressions (8.5), (8.6), (8.7), (8.11), and (9.2), (9.3), (9.4), (9.7), and (10.2), (10.3), (10.4), and (11.1), we find that

$$(12.1) \left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} \left| S_{ij} - f(0,0) \right|^{2p} \right\}^{\frac{1}{2p}} < m^{1+\frac{1}{2p}} n^{1+\frac{1}{2p}} o(\delta)$$
$$+ K \delta^{-\frac{1}{2p}} \varepsilon^{-\frac{1}{2p}} + K\, o(\delta)$$
$$+ K\, o(\varepsilon) + K \delta^{1-\frac{1}{2p}} \varepsilon^{-\frac{1}{2p}}$$
$$+ K \delta^{-\frac{1}{2p}} \varepsilon^{1-\frac{1}{2p}} + O(1)$$

where K represents the largest value of the various K's.

so that

$$(12.2)\quad \left\{ \frac{1}{m+1}\frac{1}{n+1} \sum_{i=o}^{m}\sum_{j=o}^{n} \left| S_{ij} - f(0,0) \right|^{2p} \right\}^{\frac{1}{2p}} < mn\,\delta\,\alpha(\delta)$$

$$+ \frac{K}{(m\delta)^{\frac{1}{2p}}(n\mathcal{E})^{\frac{1}{2p}}} + \frac{K\delta\alpha(\delta)}{m^{\frac{1}{2p}}\, n^{\frac{1}{2p}}}$$

$$+ \frac{K\mathcal{E}\,\alpha'(\mathcal{E})}{m^{\frac{1}{2p}}\, n^{\frac{1}{2p}}} + \frac{K\,\delta}{(m\delta)^{\frac{1}{2p}}(n\mathcal{E})^{\frac{1}{2p}}}$$

$$+ \frac{K\,\mathcal{E}}{(m\delta)^{\frac{1}{2p}}\,(n\varepsilon)^{\frac{1}{2p}}} + \frac{K}{m^{\frac{1}{2p}}\, n^{\frac{1}{2p}}}$$

where $\alpha(\delta) = o(1)$ when δ approaches 0 and

$\alpha(\mathcal{E}) = o(1)$ when approaches 0.

We take

$$(12.3)\qquad \delta = \frac{a}{m} \qquad\qquad \mathcal{E} = \frac{b}{n}$$

where a and b are large enough to make

$$(12.4)\quad K\,a^{-\frac{1}{2p}}\,b^{-\frac{1}{2p}} + K\,\delta\,a^{-\frac{1}{2p}}\,b^{-\frac{1}{2p}} + K\mathcal{E}\,a^{-\frac{1}{2p}}\,b^{-\frac{1}{2p}} < \frac{1}{2}\beta,$$

and m and n large enough to make

$$(12.5)\quad mn\,\delta\,\alpha(\delta) + K\,m^{-\frac{1}{2p}}\,n^{-\frac{1}{2p}}\,\delta\,\alpha(\delta)$$

$$+ K\,\mathcal{E}\,\alpha'(\mathcal{E}) + K\,m^{-\frac{1}{2p}}\,n^{-\frac{1}{2p}} < \frac{1}{2}\beta\,.$$

In the previous expression

$$\alpha(\delta) = \alpha\left(\frac{a}{m}\right),\quad \alpha(\mathcal{E}) = \alpha'\left(\frac{b}{n}\right),$$

and both approach zero as m, n approach ∞ . Hence

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \frac{1}{m+1} \frac{1}{n+1} \sum_{i=o} \sum_{j=o} \left| S_{ij} - f(0,0) \right|^{2p} \Big\} < \beta^{2p}$$

and since β is as small as we please

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=o}^{m} \sum_{j=o}^{n} \left| S_{ij} - f(0,0) \right|^{2p} = 0,$$

and the theorem is proved.