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I hereby recommend that the thesis prepared under my supervision by Harry H. Denman

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MACROSCOPIC ELECTRODYNAMICS

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by

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Introduction

Macroscopic electrodynamics may be regarded as based on Maxwell's field equations, the constitutive relations, and the Lorentz force equation. Some authors[1] start their treatments of macroscopic electrodynamics by assuming these equations; others[2] regard them as generalizations from the classical experimental laws of Coulomb, Ampère, Faraday, etc. In either case, they are accepted as fundamental equations of macroscopic electrodynamics, for they have been successfully applied to a wide range of physical phenomena.

The microscopic field equations of Maxwell and Lorentz must also be considered well established, since many of the laws of electrodynamics can be deduced from them. The attempt by Lorentz[3] to make these equations invariant with respect to transformations between inertial coordinate systems first led to the Lorentz transformation equations for the coordinates. These transformation equations are also derived in the special theory of relativity. There seems to be a close connection between microscopic electrodynamics and special relativity, which lends support to both theories. The microscopic field equations have also been generalized and quantized with a certain degree of success in formulating quantum electrodynamics.

Thus we have two sets of field equations, both of which are accepted as governing electromagnetic phenomena in their particular domains. However, since the macroscopic world is assumed to be composed of interacting microscopic systems, it would seem that the macroscopic equations and constitutive relations should be derivable from the microscopic equations. Since we can test the macroscopic equations much more directly than the microscopic, we might regard such a derivation as a primary requirement for our acceptance of the microscopic equations.

Such derivations have been carried out in a number of different ways by various workers in this field. Lorentz[4] regards matter as composed of different types of particles distinguished by the types of fields that they produce. He assigns these particles certain special characteristics and makes other simplifying assumptions. To obtain macroscopic quantities, he averages the microscopic quantities over a "physically infinitely small" volume element of matter, i.e., one which contains a very great number of particles, but which is very small compared to the dimensions of the material body considered. The transformations of these results between inertial coordinate systems are not calculated. Dällenbach[5], however, though using the same model of matter as developed by Lorentz and using many of the assumptions and conditions imposed by Lorentz, employs a four-dimensional formalism and averages the microscopic quantities over an invariant region in four-space, so that his results are in tensor form.

A different approach is used by Van Vleck.[6] Basically, he assumes that the space averages of the microscopic potentials are the macroscopic potentials. He then considers the effects of molecules on the microscopic potentials. By various approximations and assumptions, and by interchanging certain operations which are not generally commutable, he also produces Maxwell's equations and the constitutive relations. By working in a coordinate system at rest with respect to the molecules, he ignores the effects of moving matter, and thus the results are not applicable to media moving with respect to the observer.

As several authors have pointed out,[7,8] these methods either explicitly or implicitly exclude certain higher order effects due to electric quadrupole, octopole, and higher moments, as well as magnetic multipole moments of higher order than the magnetization. These effects are not negligible, as they have also pointed out, when considering certain problems, e.g., the equation of state of a liquid.

Therefore we shall consider here three problems. One is to derive Maxwell's equations and the constitutive relations from the microscopic equations without extensive or unwarranted assumptions. Another is to determine the effects of the higher order electric and magnetic moments on the macroscopic equations. Lastly, we would like the resultant equations to be in tensor form so that they may be applied to any inertial coordinate system.

In this work we shall make the usual assumptions of the (special) relativistic microscopic electrodynamics of point charges, and we shall postulate the existence of definite groups of these point charges (molecules). Then, after determining the action function of such a system, we shall apply the principle of least action (the variational method). This yields a set of microscopic field equations which includes effects due to the molecules (including the higher order effects mentioned above). By means of an averaging process we then obtain Maxwell's equations and the constitutive relations, which also include effects due to the higher order electric and magnetic moments.

Microscopic Electrodynamics[9]

The classical microscopic viewpoint in electrodynamics, which we shall adopt, is that matter is entirely composed of two entities- the charged particles and the electromagnetic field. We regard the charges as points, and assume that the trajectories of such particles are either known as functions of time or can be measured or calculated to any desired degree of accuracy. Thus we ignore quantum mechanical effects. We also ignore gravitational and other forces, and consider only electrical forces.

The microscopic electromagnetic field is characterized by the vector and scalar potentials $\vec{a}$ and φ, or the field vectors $\vec{e}$ and $\vec{h}$ (the arrows indicate three-vectors). The relations between the potentials and the field vectors are

$$\vec{e} = -\nabla\varphi - \frac{1}{c}\frac{\partial\vec{a}}{\partial t} \qquad (1)$$

$$\vec{h} = \nabla\times\vec{a} \quad . \qquad (2)$$

The action function W of such a microscopic system may be written as the sum of three parts

$$W = \int L\,dt = W_p + W_f + W_{pf} \quad , \qquad (3)$$

where L is the Lagrangian function of the system,
W_p is the action function of the particles alone,
W_f is the action function of the field alone,
and W_{pf} is the interaction between the particles and the field.

Since we regard the field as produced by the charges, plus any external field which might be present, the interactions between the particles are contained in W_{pf}. W_p may be written

$$W_p = -\sum_s m_s c^2 \int \beta_s^{-1} dt$$

where m_s is the mass of the s^{th} particle,
$\beta_s = (1 - v_s^2/c^2)^{-1/2}$,
the summation over s extends over all the particles in the system,
and the limits of integration are points on the world line of the particle.

This expression for W_p is already in a form dictated by the requirements of the special theory of relativity, rather than the form corresponding to Newtonian mechanics. For W_f we have (in Gaussian units, which we shall use throughout)

$$W_f = 1/8\pi \int (e^2 - h^2) dVdt \qquad (4)$$

where dV is the element of volume in three-space, i.e., dV = dxdydz.

Finally, for W_{pf} we have

$$W_{pf} = \sum_s e_s \int \left[\frac{1}{c} \vec{v}_s \cdot \vec{a}(\vec{r}_s,t) - \varphi(\vec{r}_s,t)\right] dt \qquad (5)$$

where the potentials $\vec{a}(\vec{r}_s,t)$ and $\varphi(\vec{r}_s,t)$ are the values of the potentials evaluated at the position of the s^{th} particle, $\vec{r}_s = (x_s, y_s, z_s)$.

If we now require that these results be consistent with the special theory of relativity, the total action function W of the system, and therefore W_p, W_f, and W_{pf} also, must be invariant with respect to transformations between inertial coordinate systems. We obtain

$$W_p = -\sum_s m_s c \int ds_s \qquad (6)$$

where ds is the element of the world line of the s^{th} particle.

If we define the microscopic charge density ρ_m and the microscopic current density $\vec{j}$ for the point charges by

$$\rho_m = \sum_s e_s \delta(\vec{r}-\vec{r}_s) \qquad (7)$$

$$\text{and} \quad \vec{j} = \sum_s e_s \frac{\vec{v}_s}{c} \delta(\vec{r}-\vec{r}_s) \ , \qquad (8)$$

it can be shown[10] that $j_\alpha = (\vec{j}, i\rho_m)$ is a four-vector(Roman indices run from 1 to 3, and indicate vector, dyadic, etc. components. Greek indices run from 1 to 4 and indicate four-tensor components; e.g., the position vector $x_\alpha = (x,y,z,ict)$ is a four-vector. Since we shall consider only rectangular inertial coordinate systems, we shall not distinguish between covariant and contravariant tensor components. We shall also adopt the Einstein summation convention for repeated indices in a term, except for summations over s and n). For W_{pf} we get

$$W_{pf} = \frac{1}{ic}\int j_\alpha \varphi_\alpha d\Omega \qquad (9)$$

where φ_α is the four-vector potential $(\vec{a}, i\varphi)$,
and $d\Omega$ is the four-dimensional volume element, i.e.,

$$d\Omega = dx_1 dx_2 dx_3 dx_4 = dxdydzd(ict).$$

If we define the anti-symmetric second rank tensor $f_{\alpha\beta}$ by

$$f_{\alpha\beta} = \frac{\partial\varphi_\beta}{\partial x_\alpha} - \frac{\partial\varphi_\alpha}{\partial x_\beta} \ , \qquad (10)$$

we find that W_f may be expressed in the invariant form

$$W_f = -\frac{1}{16\pi ic}\int f_{\alpha\beta} f_{\alpha\beta} d\Omega \ . \qquad (11)$$

The four-dimensional region of integration in Eq.(9) and (11) is the invariant region of four-space between two infinite space-like hypersurfaces. Thus the action function for the microscopic system may be expressed in the invariant form

$$W = -\sum_{s} m_s c \int ds_s + \frac{1}{ic} \int j_\alpha \varphi_\alpha d\Omega - \frac{1}{16\pi ic} \int f_{\alpha\beta} f_{\alpha\beta} d\Omega \quad . \qquad (12)$$

To obtain the microscopic field equations, we assume that the trajectories of the particles are given and that only the field is varied. From the principle of least action, $\delta W = 0$; since the trajectories of the particles are known, $\delta W_p = 0$, and in the usual way we obtain

$$\frac{1}{4\pi} \frac{\partial f_{\alpha\beta}}{\partial x_\beta} = j_\alpha \quad . \qquad (13)$$

The first three components of this equation give, in three-vector form,

$$\nabla \times \vec{h} - \frac{1}{e} \frac{\partial \vec{e}}{\partial t} = 4\pi \vec{j} \quad , \qquad (14)$$

while the fourth component gives

$$\nabla \cdot \vec{e} = 4\pi \rho_m \quad . \qquad (15)$$

We also obtain, directly from the definition of $f_{\alpha\beta}$, that

$$\frac{\partial f_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial f_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial f_{\gamma\alpha}}{\partial x_\beta} = 0 \quad . \qquad (16)$$

In three-vector form, this gives the equations

$$\nabla \times \vec{e} + \frac{1}{c} \frac{\partial \vec{h}}{\partial t} = 0 \qquad (17)$$

and

$$\nabla \cdot \vec{h} = 0 \quad . \qquad (18)$$

These can also be obtained directly from the defining equations of $\vec{e}$ and $\vec{h}$ in terms of the potentials $\vec{a}$ and φ.

These field equations (14),(15),(17), and (18), which are called the Maxwell-Lorentz equations, enable us to calculate the field (e and h) when $\vec{j}$ and ρ_m are given.

If we keep the field fixed, and vary the trajectories of the particles to give $\delta W = 0$, we obtain

$$mc\frac{du_\alpha}{ds} = \frac{e}{c} f_{\alpha\beta} u_\beta ,$$

where $u_\alpha = \frac{dx_\alpha}{ds}$.

In three-vector form, we obtain for each particle

$$\frac{d\vec{P}}{dt} = e(\vec{e} + \frac{\vec{v}}{c} \times \vec{h}) \tag{19}$$

where $\vec{P}$ is the kinetic momentum of the particle,

i.e., $\vec{P} = \beta m \vec{v}$,

e is the charge of the particle,

and $\vec{v}$ is the three-vector velocity.

This is the Lorentz force equation for the motion of a particle in its external field.

Macroscopic Electrodynamics

Maxwell's equations (the macroscopic field equations) may be written, in three-vector form, for points in material media at which the properties of the media do not change abruply, as

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad (20)$$

$$\nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = 4\pi\vec{J} \qquad (21)$$

$$\nabla\cdot\vec{B} = 0 \qquad (22)$$

$$\nabla\cdot\vec{D} = 4\pi\rho\ . \qquad (23)$$

In these equations, as in the microscopic case, we assume that the macroscopic charge and current densities ρ and $\vec{J}$ are known functions of space and time, and these equations, with certain boundary conditions, determine the field. Unfortunately, the macroscopic field is characterized by four vector quantities $\vec{B}$,$\vec{D}$,$\vec{E}$,and $\vec{H}$, and the above equations do not determine $\vec{B}$,$\vec{D}$,$\vec{E}$,and $\vec{H}$ uniquely.

The use of the above four field vectors infers the existence of satisfactory definitions for these quantities. From the macroscopic Lorentz force equation

$$\vec{F} = \rho(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}) \qquad (24)$$

where $\vec{F}$ is the force on an unit volume element of matter with charge density ρ ,
and $\vec{v}$ is the macroscopic velocity of this charged volume element of matter,

we can give $\vec{B}$ and $\vec{E}$ operational definitions in terms of forces on charged matter. In special circumstances, such as in electrostatics and magnetostatics, we can give $\vec{D}$ and $\vec{H}$ simple operational definitions, but for the general electromagnetic field, their meanings are somewhat obscure.

In any case, we can not determine the macroscopic field completely from Maxwell's equations, given ρ and $\vec{J}$ and the boundary conditions. Certain auxiliary relationships between the field vectors, called the constitutive relations, are normally introduced which we use with Maxwell's equations to determine the field. These constitutive relations are

$$\vec{D} = \vec{E} + 4\pi\vec{P} \tag{25}$$

$$\vec{B} = \vec{H} + 4\pi\vec{M} \tag{26}$$

where $\vec{P}$ is the polarization or dipole moment per
unit volume of the material,
and $\vec{M}$ is the magnetization per unit volume.

If we regard these equations as defining $\vec{P}$ and $\vec{M}$, then we have two more unknowns and two more equations, which does not remove the indeterminancy of the field equations. But if we have separate definitions of $\vec{P}$ and $\vec{M}$, then we may use the above equations to define $\vec{D}$ and $\vec{H}$. Experimentally, it appears that for many homogeneous isotropic media, $\vec{P}$ is parallel to $\vec{E}$ and $\vec{M}$ is parallel to $\vec{B}$, so that Eq.(25) and (26) can be written simply as

$$\vec{D} = \epsilon\vec{E} \tag{27}$$

$$\vec{B} = \mu\vec{H} \tag{28}$$

where ϵ and μ are constants called respectively the
dielectric constant and permeability of the medium.

In using Maxwell's equations and these constitutive relations, we note that although the polarization and magnetization appear in Eq.(25) and (26), at no point do the higher order electric and magnetic effects appear. In addition, a question arises concerning the effect on the above results if the medium moves with respect to the observer.

This question is usually answered by imposing the assumption of the special theory of relativity that physical laws have the same form in all inertial coordinate systems. When this is required of the field equations, and the Lorentz transformation equations for the coordinates are used, we obtain the transformation equations for the field quantities. These results are shown quite simply if we put the field equations in four-tensor form, since they will then have the same form in all inertial coordinate systems, and by examining the tensor character of the electromagnetic field quantities we shall see how they transform. We introduce the macroscopic four-vector potential $\Phi_\alpha = (\vec{A}, i\Phi)$, where $\vec{A}$ is the usual vector potential and Φ the scalar potential, where

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{29}$$

and $$\vec{B} = \nabla\times\vec{A} \tag{30}$$

Then if we define $F_{\alpha\beta}$ by $F_{\alpha\beta} = \frac{\partial\Phi_\beta}{\partial x_\alpha} - \frac{\partial\Phi_\alpha}{\partial x_\beta}$, we obtain (31)

$$F_{\alpha\beta} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \tag{32}$$

From its definition $F_{\alpha\beta}$ is an anti-symmetric tensor of second rank, and the first and third of Maxwell's equations, Eq.(20) and (22), become, in tensor form,

$$\frac{\partial F_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x_\beta} = 0 \, . \qquad (33)$$

If we introduce another anti-symmetric second rank tensor $G_{\alpha\beta}$ defined by[11]

$$G_{\alpha\beta} = \begin{pmatrix} 0 & H_z & -H_y & -iD_x \\ -H_z & 0 & H_x & -iD_y \\ H_y & -H_x & 0 & -iD_z \\ iD_x & iD_y & iD_z & 0 \end{pmatrix} ,$$

then the second and fourth of Maxwell's equations, Eq.(21) and (23), have the invariant form

$$\frac{\partial G_{\alpha\beta}}{\partial x_\beta} = 4\pi J_\alpha \qquad (34)$$

where J_α is the four-vector $(\vec{J}, i\rho)$.

These relations are sufficient to make Maxwell's equations invariant to a Lorentz transformation. They also give that $G_{\alpha\beta} - F_{\alpha\beta}$ must be an anti-symmetric second rank tensor. If we assume that the constitutive equations (25) and (26) hold in one coordinate system and let $4\pi M_{\alpha\beta} = G_{\alpha\beta} - F_{\alpha\beta}$ (35), then

$$M_{\alpha\beta} = \begin{pmatrix} 0 & -M_z & M_y & -iP_x \\ M_z & 0 & -M_x & -iP_y \\ -M_y & M_x & 0 & -iP_z \\ iP_x & iP_y & iP_z & 0 \end{pmatrix} \qquad (36)$$

and thus the constitutive relations (25) and (26) hold in every inertial coordinate system. We see that $\vec{P}$ and $\vec{M}$ are not absolute quantities; i.e., although an observer at rest with respect to the medium (the proper observer) can separate the polarization and magnetization densities, these quantities for an observer moving with respect to the medium will in general be compounded from both the polarization and magnetization densities of the proper observer. However, the simpler equations (27) and (28) hold strictly only in the proper coordinate system of the material, and are not relativistically invariant.

Derivation of the Macroscopic Equations

From the microscopic point of view, a material medium is composed of a very large number of point charges which are, in general, moving rapidly and in complex trajectories in an Euclidean three-space and which interact with the microscopic electromagnetic field in accordance with the Maxwell-Lorentz field equations and the Lorentz force equation. Macroscopically we observe certain characteristics of the medium such as the polarization and magnetization. Since these characteristics are involved in the macroscopic equations, if we wish to derive such equations from the microscopic description of the medium, then we must modify this description in some way.

The properties of material media, such as polarization and magnetization, seem to be basically associated with certain groupings of the point charges which remain somewhat stable in time (i.e., they remain in a small region about the center of gravity of the group). The charges making up the group may be in rapid motion about their center of gravity which may itself be in motion. Let us introduce such groupings, which might be atoms, molecules, or groups of molecules (we shall hereafter call them molecules), into the microscopic description of the medium and determine the effect on the action function of the system, from which, using the principle of least action, we can derive field equations which will include effects due to the molecules.

In Eq.(12) we give the expression for the action function

W of a microscopic system. Of course, W and its parts are not altered in value by the introduction of molecular groupings, but they can be rewritten. First, let us separate the charged particles into two types: the free charges not associated with any of the molecules, which we may call conduction charges, and the charges which make up the molecules. Then W_p as given in Eq.(6) is unchanged, where the summation extends over all the charges of both types. W_f may also be left in the form given in Eq.(11).

However, we shall separate W_{pf} into two parts

$$W_{pf} = W_{cf} + W_{mf} \quad , \tag{37}$$

where W_{cf} is the interaction between the field and the free or conduction charges,

and W_{mf} is the interaction between the field and the molecular charges.

From the expression for W_{pf} of Eq.(5), we see that W_{cf} has the forms

$$W_{cf} = \sum_s e_s \int [\frac{\vec{v}_s}{c} \cdot \vec{a}(\vec{r}_s,t) - \varphi(\vec{r}_s,t)]\, dt \tag{38}$$

$$\text{or} \quad W_{cf} = \frac{1}{ic} \int j_c^\alpha \varphi_\alpha \, d\Omega \quad , \tag{39}$$

where the expressions for $j_c^\alpha = (\vec{j}_c, i\rho_c')$ are the same as those given in Eq.(7) and (8) except that the summation extends over the conduction charges only.

For W_{mf}, we divide the field four-potential φ_α into two parts: the first part $\varphi_\alpha^e = (\vec{a}^e, i\varphi^e)$ due to all the charges and fields external to the molecule, and the second part φ_α' due to the other charges of that molecule. Then the interaction term $W_{m_n f}$ for the charges of the n^{th} molecule is

$$W_{m_n f} = \int L^e_{m_n f}\, dt + W'_{m_n f} = \sum_s e_s \int [\frac{\vec{V}_s}{c} \cdot \vec{a}^e(\vec{r}_{ns}, t) - \varphi^e(\vec{r}_{ns}, t)]\, dt + W'_{m_n f} \qquad (40)$$

where $L^e_{m_n f}$ is the Lagrangian function for the interaction between the molecular charges and the external field,

and $W'_{m_n f}$ is the interaction of the charges of the n^{th} molecule with the field produced by the other charges of that molecule. Of course, this term could be put in the same form as the first term, with the internal potentials substituted for the external, but, as we shall see, there is no need for writing $W'_{m_n f}$ in its expanded form.

In Eq.(40), the summation index s goes over the charges in the n^{th} molecule, and $\vec{r}_{ns}$ is the position vector of the s^{th} charge in the n^{th} molecule. We shall obtain W_{mf} by summing $W_{m_n f}$ over n.

From the definition of the external field, it has no singularities in the region occupied by the molecule. Thus the potentials $\vec{a}^e(\vec{r}_{ns}, t)$ and $\varphi^e(\vec{r}_{ns}, t)$ may be expanded about an arbitrary point inside the molecule, and such a series will converge. For definiteness, we choose this point to be the center of gravity of the molecule, to which we assign the position vector $\vec{r}_n(t)$. Dropping the superscript e denoting the external field and the subscript n in $\vec{r}_{ns}$, we have, for the vector potential term,

$$\sum_s e_s \frac{\vec{v}_s}{c}\cdot\vec{a}(\vec{r}_s,t) = \frac{1}{c}\sum_s e_s v_s^i\Big[a_i(\vec{r}_n,t) + \frac{\partial a_i}{\partial x_j}(\vec{r}_s-\vec{r}_n)_j + \frac{1}{2!}\frac{\partial^2 a_i}{\partial x_j\,\partial x_k}(\vec{r}_s-\vec{r}_n)_j(\vec{r}_s-\vec{r}_n)_k + \ldots\ldots\Big] ,$$

where all the partial derivitives are evaluated at $\vec{r}_n$.

Let $\vec{r}_s-\vec{r}_n=\vec{l}_s$. Then, since $\vec{v}_s=\frac{d\vec{r}_s}{dt}$ and since $\vec{r}_s=\vec{r}_n+\vec{l}_s$,

$$\vec{v}_s = \frac{d\vec{r}_n}{dt} + \frac{d\vec{l}_s}{dt} = \vec{v}_n + \dot{\vec{l}}_s ,$$

where $\vec{v}_n$ is the velocity of the center of gravity of the nth molecule, and the dot denotes differentiation with respect to time.

Then L_{m_nf} becomes

$$\begin{aligned} L_{m_nf} = {} & \frac{1}{c}a_i(\vec{r}_n,t)\Big[v_n^i\sum_s e_s + \sum_s e_s \dot{l}_s^i\Big] + \frac{1}{c}\frac{\partial a_i}{\partial x_j}\Big[v_n^i\sum_s e_s l_s^j + \sum_s e_s \dot{l}_s^i l_s^j\Big] \\ & + \frac{1}{2!c}\frac{\partial^2 a_i}{\partial x_j\,\partial x_k}\Big[v_n^i\sum_s e_s l_s^j l_s^k + \sum_s e_s l_s^j l_s^k \dot{l}_s^i\Big] + \ldots . \\ & - \Big[\varphi(\vec{r}_n,t)\sum_s e_s + \frac{\partial\varphi}{\partial x_i}\sum_s e_s l_s^i + \frac{1}{2!}\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}\sum_s e_s l_s^i l_s^j + \ldots .\Big] \end{aligned}$$

Let

$$\sum_s e_s = q_n , \tag{41}$$

$$\sum_s e_s\vec{l}_s = \vec{P}_n , \tag{42}$$

$$\frac{1}{2!}\sum_s e_s\vec{l}_s\vec{l}_s = \vec{\vec{Q}}_n , \tag{43}$$

$$\frac{1}{3!}\sum_s e_s\vec{l}_s\vec{l}_s\vec{l}_s = \vec{\vec{\vec{O}}} \text{ , etc.} \tag{44}$$

where q_n is the net charge of the nth molecule,
$\vec{P}_n$ is its dipole moment,
$\vec{\vec{Q}}_n$ is its quadrupole moment,
$\vec{\vec{\vec{O}}}_n$ is its octopole moment, etc.

Thus the dipole moment $\vec{P}_n$ is a three-vector, $\vec{\vec{Q}}_n$ is a symmetric second rank tensor in three-space(a self-conjugate dyadic), etc. The number of arrows above a symbol denotes its tensor rank. We have also

$$\vec{P}_n = \sum_s e_s\vec{l}_s ,$$

$$\overset{\rightrightarrows}{\dot{Q}}_n = \frac{1}{2!} \sum_s e_s (\dot{\vec{l}}_s \vec{l}_s + \vec{l}_s \dot{\vec{l}}_s) \quad , \text{ etc.}$$

Thus,

$$L_{m_n f} = \frac{1}{c} a_i(\vec{r}_n, t) [v_n^i q_n + \dot{P}_n^i] + \frac{1}{c} \frac{\partial a_i}{\partial x_j} [v_n^i P_n^j + \sum_s e_s \dot{l}_s^i l_s^j]$$
$$+ \frac{1}{c} \frac{\partial^2 a_i}{\partial x_j \partial x_k} [v_n^i Q_n^{jk} + \frac{1}{2!} \sum_s e_s l_s^j l_s^k \dot{l}_s^i] + \ldots\ldots$$
$$- [\varphi(\vec{r}_n, t) q_n + \frac{\partial \varphi}{\partial x_i} P_n^i + \frac{\partial^2 \varphi}{\partial x_i \partial x_j} Q_n^{ij} + \ldots\ldots] . \qquad (45)$$

From the identity

$$\frac{\partial a_i}{\partial x_j} v_n^i P_n^j = \left(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i}\right) v_n^i P_n^j + \frac{\partial a_j}{\partial x_i} v_n^i P_n^j \quad ,$$

we obtain

$$\frac{1}{c} \frac{\partial a_i}{\partial x_j} v_n^i P_n^j = \frac{1}{c} (\vec{P}_n \times \vec{v}_n) \cdot (\nabla \times \vec{a}) + \frac{1}{c} \frac{\partial a_j}{\partial x_i} v_n^i P_n^j \qquad (46)$$

where $\nabla \times \vec{a}$ is evaluated at $\vec{r}_n$.

With the additional identity

$$\frac{1}{c} \sum_s e_s \dot{l}_s^i l_s^j = \frac{1}{c} \left[\dot{Q}_n^{ij} + \frac{1}{2} \sum_s e_s (\dot{l}_s^i l_s^j - l_s^i \dot{l}_s^j) \right] \quad ,$$

if we define the magnetization $\vec{M}_n$ of the nth molecule by

$$\vec{M}_n = \frac{1}{2c} \sum_s e_s (\vec{l}_s \times \dot{\vec{l}}_s) \quad , \qquad (47)$$

then

$$\frac{1}{c} \frac{\partial a_i}{\partial x_j} \sum_s e_s \dot{l}_s^i l_s^j = \frac{1}{c} \frac{\partial a_i}{\partial x_j} \dot{Q}_n^{ij} + (\nabla \times \vec{a}) \cdot \vec{M}_n \quad . \qquad (48)$$

By performing similar manipulations, we obtain

$$\frac{1}{c} \frac{\partial^2 a_i}{\partial x_j \partial x_k} v_n^i Q_n^{jk} = \frac{1}{c} [\nabla \cdot (\overset{\rightrightarrows}{Q}_n \times \vec{v}_n)] \cdot (\nabla \times \vec{a}) + \frac{1}{c} \frac{\partial^2 a_j}{\partial x_i \partial x_k} Q_n^{jk} v_n^i \quad , \qquad (49)$$

where $\nabla \cdot (\overset{\rightrightarrows}{Q}_n \times \vec{v}_n) = \frac{1}{2!} \sum_s e_s l_s^k (\vec{l}_s \times \vec{v}_n) \frac{\partial}{\partial x_k}$;

also,

$$\frac{1}{2!c} \frac{\partial^2 a_i}{\partial x_j \partial x_k} \sum_s e_s l_s^j l_s^k \dot{l}_s^i = \frac{1}{3c} \frac{\partial^2 a_i}{\partial x_j \partial x_k} \sum_s e_s (l_s^k \dot{l}_s^i - \dot{l}_s^k l_s^i) l_s^j$$
$$+ \frac{1}{c} \frac{\partial^2 a_i}{\partial x_j \partial x_k} \frac{\partial O_n^{ijk}}{\partial t} \quad .$$

Then if we define $\overset{\rightrightarrows}{M}_n$, the second order magnetization for the for the nth molecule, by $\vec{\vec{M}}_n = \frac{1}{3c}\sum_s e_s \vec{l}_s(\vec{l}_s\times\dot{\vec{l}}_s)$ (50), we get

$$\frac{1}{2!c}\frac{\partial^2 a_i}{\partial x_j\,\partial x_k}\sum_s e_s l_s^j l_s^k \dot{l}_s^i = (\nabla\cdot\vec{\vec{M}}_n)\cdot(\nabla\times\vec{a}) + \frac{1}{c}\frac{\partial^2 a_i}{\partial x_j\,\partial x_k}\frac{\partial O_n^{ijk}}{\partial t} . \tag{51}$$

Thus we obtain

$$\begin{aligned} L_{mf} = \sum_n L_{m_n f} = \sum_n \Big\{ & \frac{1}{c} a_i(\vec{r}_n,t)\, v_n^i q_n + \frac{1}{c}\Big[a_i(\vec{r}_n,t)\,\dot{P}_n^i + \frac{\partial a_j}{\partial x_i} v_n^i P_n^j\Big] \\ & + \frac{1}{c}(\vec{P}_n\times\vec{v}_n)\cdot(\nabla\times\vec{a}) + \frac{1}{c}\frac{\partial a_i}{\partial x_j}\dot{Q}_n^{ij} + \frac{1}{c}\frac{\partial^2 a_j}{\partial x_i\,\partial x_k} Q_n^{jk} v_n^i \\ & + \vec{M}_n\cdot\nabla\times\vec{a} + \frac{1}{c}\big[\nabla\cdot(\vec{\vec{Q}}_n\times\vec{v}_n)\big]\cdot(\nabla\times\vec{a}) + (\nabla\cdot\vec{\vec{M}}_n)\cdot(\nabla\times\vec{a}) \\ & + \frac{1}{c}\frac{\partial^2 a_i}{\partial x_j\,\partial x_k}\frac{\partial O_n^{ijk}}{\partial t} + \cdots \\ & - \Big[\phi(\vec{r}_n,t)\, q_n + \frac{\partial\varphi}{\partial x_i} P_n^i + \frac{\partial^2\varphi}{\partial x_i\,\partial x_j} Q_n^{ij} + \cdots\Big]\Big\}. \end{aligned} \tag{52}$$

We now define the following microscopic quantities

$$\rho'_{cv} = \sum_n q_n \delta(\vec{r}-\vec{r}_n) \quad , \tag{53}$$

$$\vec{j}_{cv} = \sum_n q_n \frac{\vec{v}_n}{c}\delta(\vec{r}-\vec{r}_n) \tag{54}$$

$$\vec{p} = \sum_n \vec{P}_n \delta(\vec{r}-\vec{r}_n) \tag{55}$$

$$\vec{\vec{q}} = \sum_n \vec{\vec{Q}}_n \delta(\vec{r}-\vec{r}_n) \tag{56}$$

$$\vec{f} = \frac{1}{c}\sum_n(\vec{P}_n\times\vec{v}_n)\,\delta(\vec{r}-\vec{r}_n) \tag{57}$$

$$\vec{\vec{g}} = \frac{1}{c}\sum_n(\vec{\vec{Q}}_n\times\vec{v}_n)\,\delta(\vec{r}-\vec{r}_n) \tag{58}$$

$$\vec{m} = \sum_n \vec{M}_n \delta(\vec{r}-\vec{r}_n) \tag{59}$$

$$\vec{\vec{m}} = \sum_n \vec{\vec{M}}_n \delta(\vec{r}-\vec{r}_n) \text{ ,etc.} \tag{60}$$

where ρ'_{cv} is the microscopic convection charge density due to the net charges of the molecules,

$\vec{j}_{cv}$ is the microscopic convection current density,

$\vec{p}$ is the microscopic dipole moment density, etc.

Integrating over the entire three-space and using the properties of $\delta(\vec{r}-\vec{r}_n)$, we have

$$\begin{aligned}\sum_n q_n \varphi(\vec{r}_n,t) &= \int \varphi(\vec{r},t)\, \rho'_{cv}(\vec{r},t)\, dV \\ \frac{1}{c}\sum_n q_n a_i(\vec{r}_n,t) v_n^i &= \int \vec{a}(\vec{r},t)\cdot \vec{j}_{cv}(\vec{r},t)\, dV \\ \sum_n \frac{\partial\varphi}{\partial x_i} P_n^i &= -\int \varphi\, \nabla\cdot\vec{p}\, dV\ , \quad \text{etc.}\end{aligned}$$

Since $\frac{\partial\vec{p}}{\partial t} = \sum_n \dot{\vec{P}}_n\, \delta(\vec{r}-\vec{r}_n) + \sum_n \vec{P}_n \frac{\partial}{\partial t}\delta(\vec{r}-\vec{r}_n)$

$= \sum_n \dot{\vec{P}}_n\, \delta(\vec{r}-\vec{r}_n) - \sum \vec{P}_n \vec{v}_n\cdot\nabla\delta(\vec{r}-\vec{r}_n)$, we find that

$$\frac{1}{c}\sum_n \left[\dot{P}_n^i a_i(\vec{r}_n,t) + \frac{\partial a_j}{\partial x_i} v_n^i P_n^j\right] = \frac{1}{c}\int \vec{a}\cdot\frac{\partial\vec{p}}{\partial t}\, dV \quad .$$

By means of such transformations, we may write L_{mf} in the form

$$\begin{aligned}L_{mf} = \int \vec{a}(\vec{r},t)\cdot\Big\{\vec{j}_{cv} &+ \frac{1}{c}\frac{\partial\vec{p}}{\partial t} + \nabla\times\vec{f} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{\vec{q}}) + \nabla\times\vec{m} - \nabla\times(\nabla\cdot\vec{\vec{g}}) \\ &- \nabla\times(\nabla\cdot\vec{\vec{m}}) + \frac{1}{c}\frac{\partial}{\partial t}[\nabla\cdot(\nabla\cdot\vec{\vec{\vec{o}}})] + \ldots\Big\}dV - \int\varphi(\vec{r},t)\Big[\rho'_{cv} \\ &- \nabla\cdot\vec{p} + \nabla\cdot(\nabla\cdot\vec{\vec{q}}) - \ldots\Big]dV \quad . \qquad (61)\end{aligned}$$

For the total action function W of the system, we have

$$W = W_p + W_{cf} + W_{mf} + W'_{mf} + W_f \quad .$$

In order to obtain the field equations, we shall apply the principle of least action, i.e., we minimize W with respect to variation of the external field, holding the trajectories of all the charges fixed and also the internal field. Variation of the internal field might be expected to give field equations applicable inside the molecules, but this would bring us into a situation where classical electrodynamics is known to fail, and where the basic laws are thought to be quantum mechanical. When we vary the external field with the charge trajectories and internal field fixed, we have $\delta W_p = 0$ and $\delta W'_{mf} = 0$, and therefore

$$\delta W = \delta W_{cf} + \delta W_{mf} + \delta W_f = 0 \quad .$$

Substituting the expressions obtained previously for these functions, we obtain

$$\delta W = \int \delta\vec{a}\cdot\Big\{ \vec{j}_c + \vec{j}_{cv} + \frac{1}{c}\frac{\partial\vec{p}}{\partial t} + \nabla\times\vec{f} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{\vec{q}}) + \nabla\times\vec{m} \tag{62}$$

$$- \nabla\times(\nabla\cdot\vec{\vec{g}}) - \nabla\times(\nabla\times\vec{\vec{m}}) + \ldots\ldots - \frac{1}{4\pi}(\nabla\times\vec{h} - \frac{1}{c}\frac{\partial\vec{e}}{\partial t})\Big\} dVdt$$

$$- \int \delta\varphi\Big\{ \rho_c' + \rho_{cv}' - \nabla\cdot\vec{p} + \nabla\cdot(\nabla\cdot\vec{\vec{q}}) - \ldots\ldots - \frac{1}{4\pi}\nabla\cdot\vec{e}\Big\} dVdt = 0 .$$

In order for this equation to be true in general, we must have

$$\frac{1}{4\pi}(\nabla\times\vec{h} - \frac{1}{c}\frac{\partial\vec{e}}{\partial t}) = \vec{j}_c + \vec{j}_{cv} + \frac{1}{c}\frac{\partial\vec{p}}{\partial t} + \nabla\times\vec{f} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{\vec{q}}) + \nabla\times\vec{m}$$

$$- \nabla\times(\nabla\cdot\vec{\vec{g}}) - \nabla\times(\nabla\cdot\vec{\vec{m}}) + \ldots\ldots \tag{63}$$

and $$\frac{1}{4\pi}\nabla\cdot\vec{e} = \rho_c' + \rho_{cv}' - \nabla\cdot\vec{p} + \nabla\cdot(\nabla\cdot\vec{\vec{q}}) - \ldots\ldots \tag{64}$$

Then, if we let $$\vec{j}_c + \vec{j}_{cv} = \vec{j}\,' \tag{65}$$

and $$\rho_c' + \rho_{cv}' = \rho' \quad , \tag{66}$$

where $\vec{j}\,'$ is the total microscopic conduction and convection current density,

and ρ' is the total microscopic conduction and convection charge density, and if we also let

$$\vec{b} = \vec{h} - 4\pi(\vec{m} + \vec{f}) + 4\pi(\nabla\cdot\vec{\vec{g}} + \nabla\cdot\vec{\vec{m}}) - \ldots\ldots \tag{67}$$

and $$\vec{d} = \vec{e} + 4\pi\vec{p} - 4\pi\,\nabla\cdot\vec{\vec{q}} + \ldots\ldots \quad , \tag{68}$$

we get $$\nabla\times\vec{b} - \frac{1}{c}\frac{\partial\vec{d}}{\partial t} = 4\pi\,\vec{j}\,' \tag{69}$$

and $$\nabla\cdot\vec{d} = 4\pi\rho' \quad . \tag{70}$$

Also, from the definitions of $\vec{e}$ and $\vec{h}$ in terms of the microscopic potentials, Eq.(1) and (2), which still apply here,

we have $$\nabla\times\vec{e} + \frac{1}{c}\frac{\partial\vec{h}}{\partial t} = 0 \tag{71}$$

and $$\nabla\cdot\vec{h} = 0 \quad . \tag{72}$$

We would like to derive the macroscopic field equations from the microscopic equations (69) to (72). The simplest procedure is to assume that the macroscopic equations and field quantities are some kind of averages over a macroscopic region of the corresponding microscopic equations and quantities. In order to obtain equations involving the averages of the microscopic field quantities from the averaged microscopic field equations, we shall have to use some kind of linear averaging process which commutes with the operators $\frac{\partial}{\partial x_\alpha}$. One such linear averaging process is integration over a "physically infinitely small" volume element of matter, as proposed by Lorentz. However, if we apply <u>any</u> linear averaging process which commutes with $\frac{\partial}{\partial x_\alpha}$ to the microscopic equations (69) to (72), we obtain

$$\nabla\times\overline{\vec{e}} + \frac{1}{c}\frac{\partial\overline{\vec{h}}}{\partial t} = 0 \qquad (73)$$

$$\nabla\times\overline{\vec{b}} - \frac{1}{c}\frac{\partial\overline{\vec{d}}}{\partial t} = 4\pi\,\overline{\vec{j}}' \qquad (74)$$

$$\nabla\cdot\overline{\vec{d}} = 4\pi\overline{\rho'} \qquad (75)$$

$$\nabla\cdot\overline{\vec{h}} = 0 \quad , \qquad (76)$$

where the bar over a microscopic quantity denotes the average of that quantity.

Since these average microscopic quantities are assumed to be the macroscopic quantities, if we compare the above equations with Maxwell's equations (20) to (23), we see that we shall be successful in our derivation if we make the identifications

$$\overline{\vec{e}} = \vec{E} \qquad (77) \qquad\qquad \overline{\vec{b}} = \vec{H} \qquad (78)$$

$$\overline{\vec{h}} = \vec{B} \qquad (79) \qquad\qquad \overline{\vec{d}} = \vec{D} \qquad (80)$$

$$\overline{\vec{j}'} = \vec{J} \qquad (81) \qquad\qquad \overline{\rho'} = \rho \qquad (82)$$

The validity of these identifications is further suggested when we note that if the microscopic Lorentz force equation is averaged and we use Eq. (77) and (79), we obtain

$$\vec{F} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) ,$$

which is the macroscopic Lorentz force equation for a charge e moving with velocity $\vec{v}$, as given in Eq.(24).

When the averaging process is applied to the microscopic constitutive equations (67) and (68), and we use Eq.(77) to (80), we have

$$\vec{H} = \vec{B} - 4\pi\left(\overline{\vec{m}} + \overline{\vec{f}}\right) + 4\pi\left(\nabla\cdot\overline{\vec{\vec{g}}} + \nabla\cdot\overline{\vec{\vec{m}}}\right) - \ldots. \qquad (83)$$

and $$\vec{D} = \vec{E} + 4\pi\,\overline{\vec{p}} - 4\pi\,\nabla\cdot\overline{\vec{\vec{q}}} + \ldots. \qquad . \qquad (84)$$

Since $\overline{\vec{p}}$ is the average of the microscopic polarization density, it is natural to equate this to the macroscopic polarization density, i.e., $\overline{\vec{p}} = \vec{P}$ (85). For the Magnetization, however, the situation is more complex. If we wish the constitutive equation (83), obtained by averaging, to be identical with Eq.(26) when we drop the higher order terms involving $\vec{\vec{g}}$, $\vec{\vec{m}}$, etc., then we must have

$$\vec{M} = \overline{\vec{m} + \vec{f}} = \overline{\vec{m}} + \overline{\vec{f}} \quad . \qquad (86)$$

In order to indicate why a term containing $\vec{f}$ appears in the macroscopic magnetization, consider the following. From its definition, $\vec{f}$ involves quantities $\vec{P}_n \times \vec{v}_n$. If we consider a dipole moving with respect to an observer, with the axis of the dipole in the direction of the velocity, the currents due to the movement of the positive and negative charges making up the dipole cancel each other, since the current is a line vector. If the dipole axis is not parallel to the velocity, then we have two equal and opposite currents which are not acting along the same line and therefore their fields do not completely cancel; the result is a magnetic field which contributes to the magnetization.

For the higher order terms, we define

$$\overset{\rightrightarrows}{Q} = \overset{\overline{\rightrightarrows}}{q} \quad , \quad \overset{\rightrightarrows}{G} = \overset{\overline{\rightrightarrows}}{g} \; , \quad \overset{\rightrightarrows}{M} = \overset{\overline{\rightrightarrows}}{m} \quad , \text{ etc.,} \tag{87}$$

where $\overset{\rightrightarrows}{Q}$ is the macroscopic quadrupole moment per unit volume, etc.

Then Eq.(83) and (84) become

$$\vec{H} = \vec{B} - 4\pi\vec{M} + 4\pi(\nabla\cdot\overset{\rightrightarrows}{G} + \nabla\cdot\overset{\rightrightarrows}{M}) - \ldots. \tag{88}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P} - 4\pi\,\nabla\cdot\overset{\rightrightarrows}{Q} + \ldots\ldots \quad . \tag{89}$$

Thus we have derived Maxwell's field equations and a new set of constitutive equations which include higher order terms in electric and magnetic moments. If we write out the field equations, substituting for $\vec{D}$ and $\vec{H}$ from the constitutive equations (88) and (89), we have

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \tag{90}$$

$$\nabla\times\vec{B} - 4\pi\nabla\times(\vec{M} - \nabla\cdot\overset{\rightrightarrows}{G} - \nabla\cdot\overset{\rightrightarrows}{M} + \ldots) - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$- \frac{4\pi}{c}\left(\frac{\partial\vec{P}}{\partial t} - \frac{\partial}{\partial t}\nabla\cdot\overset{\rightrightarrows}{Q} + \ldots\ldots\right) = 4\pi\vec{J} \tag{91}$$

$$\nabla\cdot\vec{B} = 0 \tag{92}$$

$$\nabla\cdot\vec{E} + 4\pi\,\nabla\cdot\vec{P} - 4\pi\,\nabla\cdot(\nabla\cdot\vec{\vec{Q}}) + \ldots\ldots = 4\pi\rho \quad . \tag{93}$$

From the requirements of the special theory of relativity, these equations must have the same form in every inertial coordinate system. If we define $F_{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$ as in Eq.(32), we may write Eq.(90) and (92) above as

$$\frac{\partial F_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x_\beta} = 0 \ . \tag{94}$$

If we now define an anti-symmetric second rank tensor $M_{\alpha\beta}$ by

$$M_{\alpha\beta} = \begin{pmatrix} 0 & -M_z & M_y & -iP_x \\ M_z & 0 & -M_x & -iP_y \\ -M_y & M_x & 0 & -iP_z \\ iP_x & iP_y & iP_z & 0 \end{pmatrix} \tag{95}$$

where $\vec{M} = \vec{\vec{m}} + \vec{\vec{f}}$ and $\vec{P} = \vec{\vec{p}}$,

and if we define a third rank tensor $M_{\alpha\beta\gamma}$ by

$$M_{\alpha\beta1} = \begin{pmatrix} 0 & X_{13} & -X_{12} & iQ_{11} \\ -X_{13} & 0 & X_{11} & iQ_{12} \\ X_{12} & -X_{11} & 0 & iQ_{13} \\ -iQ_{11} & -iQ_{12} & -iQ_{13} & 0 \end{pmatrix} \qquad M_{\alpha\beta2} = \begin{pmatrix} 0 & X_{23} & -X_{22} & +iQ_{21} \\ -X_{23} & 0 & X_{21} & iQ_{22} \\ X_{22} & -X_{21} & 0 & iQ_{23} \\ -iQ_{21} & -iQ_{22} & -iQ_{23} & 0 \end{pmatrix}$$

$$M_{\alpha\beta3} = \begin{pmatrix} 0 & X_{33} & -X_{32} & iQ_{31} \\ -X_{33} & 0 & X_{31} & iQ_{32} \\ X_{32} & -X_{31} & 0 & iQ_{33} \\ -iQ_{31} & -iQ_{32} & -iQ_{33} & 0 \end{pmatrix} \qquad M_{\alpha\beta4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{96}$$

where $\vec{\vec{X}} = \vec{\vec{M}} + \vec{\vec{G}}$, and similarly for higher order moments, then we can put Eq.(91) and (93) in the tensor form

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} + \frac{\partial M_{\alpha\beta}}{\partial x_\beta} + \frac{\partial^2 M_{\alpha\beta\gamma}}{\partial x_\beta \partial x_\gamma} + \ldots\ldots = 4\pi J_\alpha \quad . \tag{97}$$

Discussion

The macroscopic field and constitutive equations for certain special cases may now be considered.

Case I: The microscopic electrostatic case.

In this case we assume there exists a coordinate system in which all of the charged particles are at rest, and that the observer is also at rest in this system. This case is characterized by the absence of currents, magnetic effects, and changes with time. The governing macroscopic field equations are then

$$\nabla \times \vec{E} = 0 \tag{98}$$

and $$\nabla \cdot \vec{D} = \nabla \cdot (\vec{E} + 4\pi\vec{P} - 4\pi\nabla \cdot \vec{\vec{Q}} + \ldots) = 4\pi\rho \; . \tag{99}$$

Since $\vec{E}$ is here irrotational, there exists a scalar potential function $\varphi(\vec{r})$ such that $\vec{E} = -\nabla\varphi$, and thus we obtain the generalization of Poisson's equation

$$\nabla^2\varphi = -4\pi[\rho - \nabla \cdot \vec{P} + \nabla \cdot (\nabla \cdot \vec{\vec{Q}}) - \ldots] \; . \tag{100}$$

Case II: The macroscopic electrostatic case.

In this case, the centers of gravity of all the molecules are at rest in their proper coordinate system(this condition actually defines the proper coordinate system) and the observer is also at rest in this coordinate system. Then $\vec{v}_n = 0$ for all the molecules, and the convection current $\vec{j}_{cv} = 0$, so that the macroscopic durrent density $\vec{J}$ is due to conduction charges only. Also, $\vec{f} = 0$, $\vec{\vec{g}} = 0$, and similar equations for higher order terms of this type. We have the usual field equations, but the constitutive relations become

$$\vec{B} = \vec{H} + 4\pi(\vec{M} - \nabla\cdot\vec{\vec{M}} + \ldots), \text{ where } \vec{M} = \bar{\vec{m}},$$

$$\text{and} \quad \vec{D} = \vec{E} + 4\pi(\vec{P} - \nabla\cdot\vec{\vec{Q}} + \ldots) .$$

Case III: Moving media.

In this case the centers of gravity of the molecules are at rest in their proper coordinate system, but it is moving with velocity $\vec{v}$ with respect to the observer. Thus $\vec{v}_n = \vec{v}$ for all the molecules. Then $\vec{j}_{cv} = \frac{\vec{v}}{c}\rho'_{cv}$ and we have also $\vec{\vec{f}} = \vec{P}\times\vec{v}$, $\vec{\vec{G}} = \vec{\vec{Q}}\times\vec{v}$, and similar equations for higher terms. While the field equations still have the same form as Eq.(20) to (23), Eq.(91) can be put in the form

$$\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = 4\pi\vec{J} + 4\pi\,\nabla\times\bar{\vec{m}} + 4\pi\,\nabla\times(\vec{P}\times\vec{v}) - \ldots \qquad .(101)$$

The term $\nabla\times(\vec{P}\times\vec{v})$ for this case was found experimentally by Röntgen, and is called by Lorentz the Röntgen current[12].

It is also interesting to note that the macroscopic charge and current densities ρ and $\vec{J}$ are not simply averages of the microscopic densities ρ_m and $\vec{j}$. To show this, we apply the averaging process to the microscopic field equations and use the identities (77) to (80). We obtain

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad (102) \qquad\qquad \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = 4\pi\,\bar{\vec{j}} \qquad (104)$$

$$\nabla\cdot\vec{B} = 0 \qquad (103) \qquad\qquad \nabla\cdot\vec{E} = 4\pi\,\bar{\rho}_m \qquad (105)$$

Eq.(102) and (103) are identical with Eq.(20) and (22) of Maxwell's equations. Subtracting equations (104) and (105) from (93) and (91), we have

$$\bar{\vec{j}} - \vec{J} = 4\pi\,\nabla\times(\vec{M} - \nabla\cdot\vec{\vec{G}} - \nabla\cdot\vec{\vec{M}} + \ldots) + \frac{4\pi}{c}\left(\frac{\partial\vec{P}}{\partial t} - \frac{\partial}{\partial t}\nabla\cdot\vec{\vec{Q}} + \ldots\right) \qquad (106)$$

$$\bar{\rho}_m - \rho = -4\pi\left[\nabla\cdot\vec{P} - \nabla\cdot(\nabla\cdot\vec{\vec{Q}}) + \ldots\right] \quad . \qquad (107)$$

Lorentz

Lorentz[13] explains the first order terms of such differences by noting that when we average the microscopic charge density, for example, over a macroscopic volume element, we count all the charges in the volume element, including not only the conduction and convection charges(which appear in the macroscopic charge density) but also charges from molecules which have been cut by the edges of the volume element. He shows that such charges give rise to the term $\nabla\cdot\vec{P}$ in Eq. (107). If one considers higher order effects of this kind, one might explain the terms $\nabla\cdot(\nabla\cdot\overset{\rightrightarrows}{Q})$, etc. in a similar manner. One of the advantages of this development is that if we use this averaging process of Lorentz we do not have to calculate effects due to charges in molecules cut by the edges of the volume element. In this development, only those molecules whose centers of gravity fall within the volume element contribute to the average, and they contribute their whole effect, despite the fact that some of the charges of these molecules lie in other volume elements. Thus here we are able to avoid this additional problem both for the terms in $\vec{P}$ and $\vec{M}$, for which these calculations can be carried out fairly easily, and for the higher terms involving quadrupole moments,etc., for which these calculations have not been carried out and where they appear to be considerably more complicated.

The field equations (102) to (105) have certain interesting differences from Maxwell's equations. Maxwell's equations do not completely determine the four field quantities $\vec{E}$,$\vec{B}$,$\vec{D}$,and $\vec{H}$ when ρ and $\vec{J}$ and the boundary conditions are known. Eq.(102) to (105), however, contain only $\vec{B}$ and $\vec{E}$, and these quantities will be determined by these equations when $\bar{\rho}_m$ and $\vec{\vec{j}}$ and the boundary conditions are known. From the definitions of ρ and $\vec{J}$ in terms of conduction and convection charges we see that they are not defined in terms of simple operations- the operational definition would require averaging the conduction and convection charges only, ignoring other charges which might be present in the macroscopic volume element. On the other hand, $\bar{\rho}_m$ and $\vec{\vec{j}}$ do have simple operational meanings in terms of total charge in a volume element and charge passing through a certain unit surface per unit time. Unfortunately, while it is easy to define $\bar{\rho}_m$ and $\vec{\vec{j}}$ operationally, it is not a simple matter to measure them, especially at interior points of a medium, in a moving medium, etc. But we often have some knowledge of the conduction and convection charges and currents in a particular physical situation and therefore use these approximate values in Maxwell's equations. Still, if $\bar{\rho}_m$ and $\vec{\vec{j}}$ can be determined by measurement or calculation, then Eq.(102) to (105) could be used, and the same results obtained as by use of Maxwell's equations with the generalized constitutive equations (88) and (89).

If we examine the equations which we have developed, we notice that if $\vec{\vec{Q}}$, $\vec{\vec{\vec{O}}}$,..., $\vec{\vec{M}}$, $\vec{\vec{G}}$,... are all 0, we obtain the usual macroscopic field and constitutive equations. Then the tensor $M_{\alpha\beta\gamma}$ and those of higher rank will be 0, and the field equations have the tensor form

$$\frac{\partial F_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x_\beta} = 0$$

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} + 4\pi \frac{\partial M_{\alpha\beta}}{\partial x_\beta} = 4\pi J_\alpha \ .$$

Thus, to this order of approximation, the field equations are invariant to Lorentz transformations, as required by the special theory of relativity. If we wished to include the quadrupole moment $\overset{\rightrightarrows}{Q}$ in our field equations, however, and still keep them invariant, it would be necessary to introduce $\overset{\rightrightarrows}{M}$ and $\overset{\rightrightarrows}{G}$ also, since what appears as $\overset{\rightrightarrows}{Q}$ to one observer will appear as $\overset{\rightrightarrows}{M}$ and $\overset{\rightrightarrows}{G}$ to an observer moving relatively to the first. Thus if we wish to use higher order terms in the field equations and keep them invariant, we must introduce both higher electrical and higher magnetic moments, including all those moments of the same vector, dyadic, triadic, etc. character. It seems a fortuitous accident that the magnitudes of the various effects are such that the natural approximation(i.e., neglecting $\overset{\rightrightarrows}{Q}$, $\overset{\rightrightarrows}{M}$, $\overset{\rightrightarrows}{G}$, ... in the field equations) is one which makes these approximate field equations Lorentz invariant.

Conclusions

We have shown that by using the action function as developed in microscopic electrodynamics and by introducing the concept of groups of charges, the principle of least action can be used to derive a set of microscopic field equations which include effects due to these charge groups. These equations can be averaged to give the macroscopic field equations (Maxwell's equations) and generalizations of the constitutive equations. These generalized constitutive equations contain effects of higher order electric and magnetic moments of the charge groups. No arbitrary or extensive assumptions concerning the nature of these groups were required. The definitions of the macroscopic quantities were stated in terms of averages of microscopic quantities, giving greater insight into the nature of such quantities as $\vec{M}$, $\vec{P}$, ρ, and $\vec{J}$. Finally, when the generalized constitutive equations are substituted into the field equations, the resultant field equations are put in a tensor form which is invariant to Lorentz transformation, as required by the special theory of relativity.

References

1. See, for example, J.A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941, Chap. 1.
2. M. Mason and W. Weaver, The Electromagnetic Field, University of Chicago Press, Chicago, 1929, Chap. 4.
3. H.A. Lorentz, Proc. Acad. Amsterdam 6 (1914) Reprinted in The Principle of Relativity, Dodd, Mead, New York, 1923.
4. H.A. Lorentz, Proc. Acad. Amsterdam 5, 254 (1902) Reprinted in Collected Papers, Volume 3, Martinus Nijhoff, The Hague, Holland, 1936, Page 117.
5. W. Dällenbach, Ann. der Physik 58, 523 (1919)
6. J.H. Van Vleck, The Theory of Electric and Magnetic Susceptibilities, Oxford University Press, 1932, Chap.1.
7. Reference 2, Page 33.
8. Reference 6, Page 12.
9. The material in the section "Microscopic Electrodynamics" was obtained from lecture notes taken in Dr. Podolsky's course in Electrodynamics at the University of Cincinnati.
10. B. Podolsky, Phys. Rev. 72, 624 (1947)
11. Reference 1, Page 70.
12. Reference 4.
13. Idem.