THEORY OF FRICTION AND ITS PART

IN THE

METAL CUTTING PROCESS

A dissertation
submitted to the faculty
of the
Graduate Department of Applied Science
of the
University of Cincinnati
in partial fulfillment
of the requirements
for the degree of

DOCTOR OF SCIENCE
1941
by

Mylon Eugene Merchant

B.S. in M.E., University of Vermont, 1936
M.E.S., University of Cincinnati, 1938

College of Engineering and Commerce
July 22, 1940

TABLE OF CONTENTS

THEORY OF FRICTION, AND ITS PART IN THE METAL CUTTING PROCESS

INTRODUCTION

The control of friction is evidently of tremendous importance to the industrial world. In spite of this, at the time the present investigation was begun, (September, 1936) relatively little was known concerning the part played by the (seemingly) many variables controlling friction.

The acute need for an understanding of the friction phenomena active in the metal cutting process was not realized until recognized by Ernst and Martellotti[1,2] in their researches on cutting. They found that the major impediment to the production of very smooth finished surfaces was the so-called "built-up edge" which usually forms on the nose of a tool taking a cut through metal. Such a built-up edge formation may be seen at the base of the chip shown in the process of removal in the photomicrograph, Fig. 1. (The tool is here not shown).

In the same figure a "fragment" of metal which has been left behind by the built-up edge as the cut was taken may be seen projecting from the "finished" surface. A profile view of a machined surface on which such fragments were left behind is shown in Fig. 2, and a plan view of a similar surface in Fig. 3. The roughness is apparent. It is by leaving a trail of such fragmentary roughness

Fig. 1. Chip being removed in planing cut, showing "built-up edge" adjacent to position occupied by tool face. Mag 100X

Fig. 2. Surface profile showing structure of fragments or "steps" left behind by built-up edge. Mag. 350X

Fig. 3. Turned surface of aluminum showing fragmentary roughness due to built-up edge. Mag. 30X

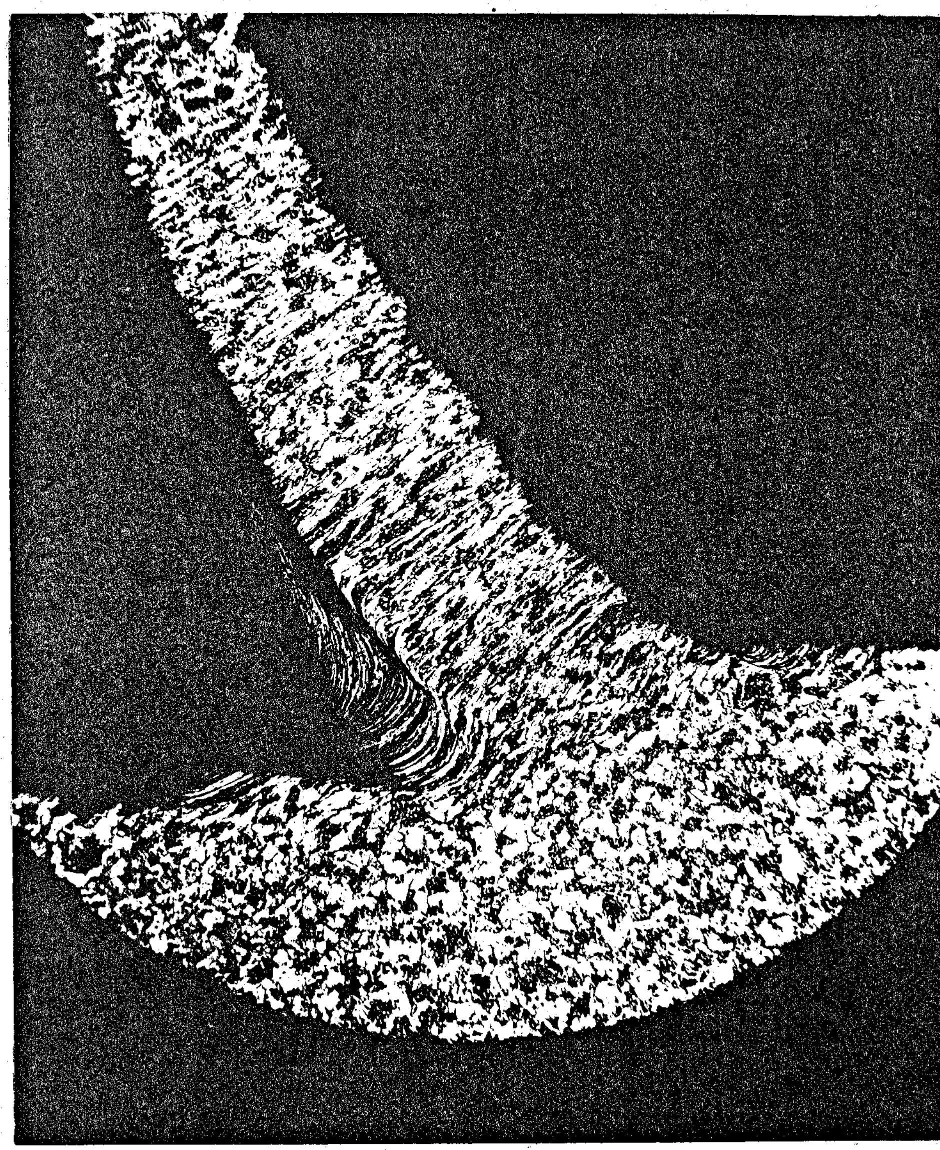


Fig. 1

Fig. 2

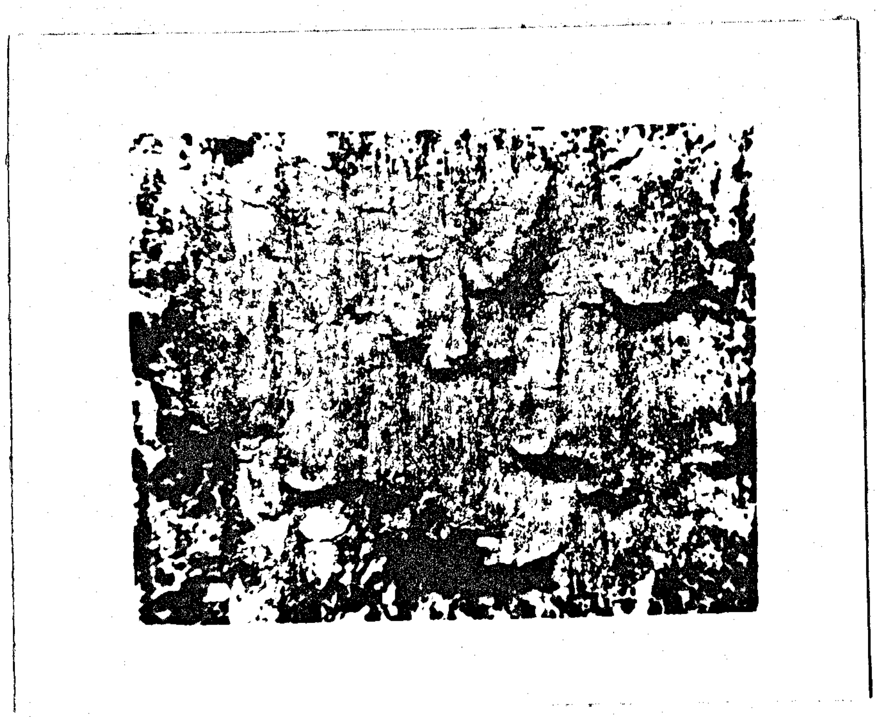


Fig. 3

behind it that the built-up edge offends, as pointed out by Ernst and Martellotti. It was apparent to them that the condition for the stability of such a stagnant mass of metal on the nose of a cutting tool must be a very high coefficient of friction between the tool face and built-up edge. From that conviction came the need for the investigation of friction and metal cutting herein described.

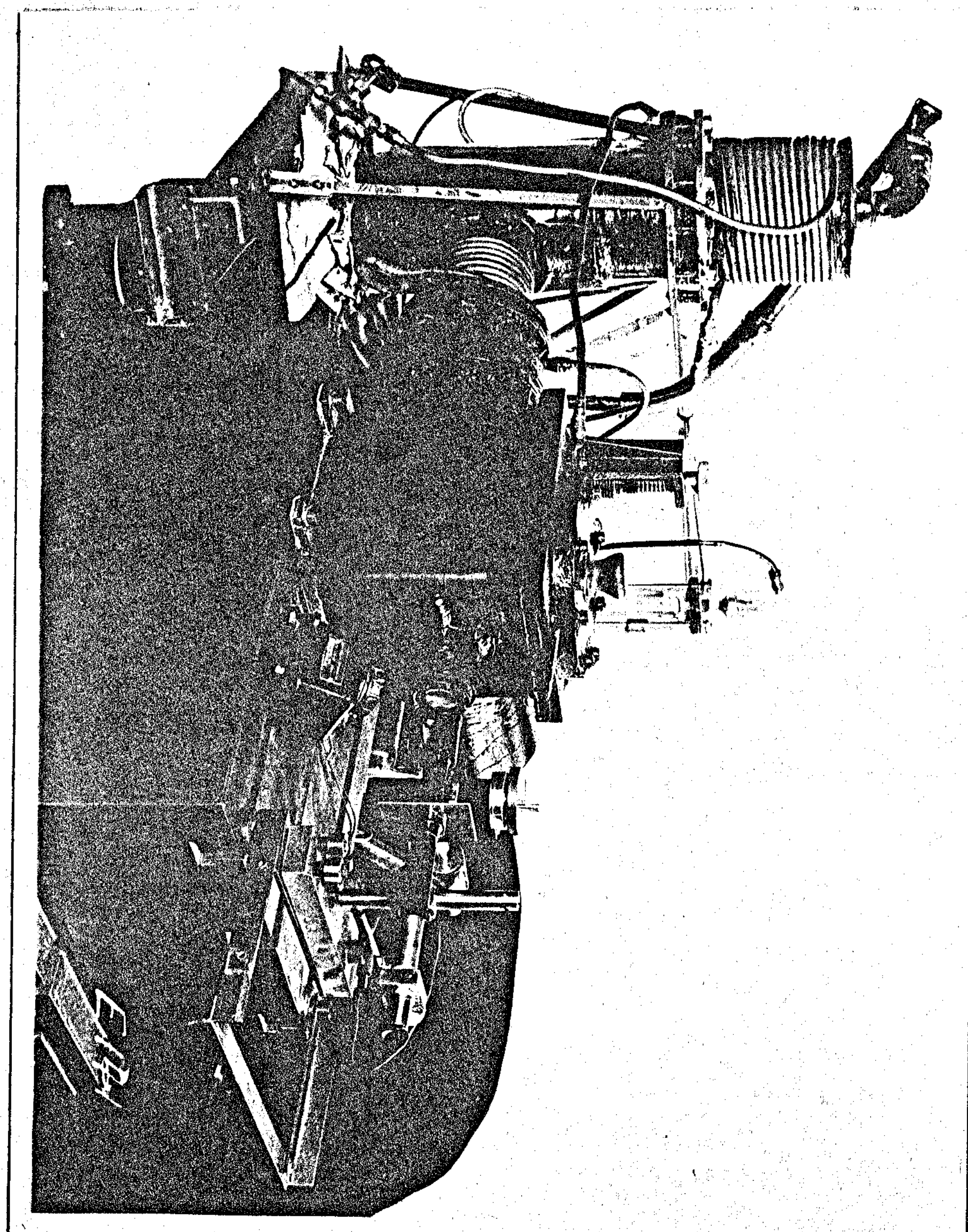
EXPERIMENTAL EQUIPMENT

The apparatus used to study static friction was especially designed for this investigation. It permits measurements of coefficients of friction to be made on small cylindrical specimens under conditions of vacuum ranging from atmospheric pressure to 10^{-4} m.m. of mercury, and conditions of temperature ranging from 80° to 350°C. It further allows variation of the applied tangential and normal forces independently of each other, a feature not possible in the usual "tilting" friction instrument. The apparatus is shown in the photograph, Fig. 4, and in the schematic drawing, Fig. 5, which also gives a detailed view of the specimen carriers. The test specimens (A) are of cylindrical form, 1/4" diameter by 7/8" long, and are mounted with their axes at right angles to each other. They are shown in place in the specimen carriers, which are mounted on the ends of two identical cantilevers (B) of 1/8" dia. Inconel rod.

Fig. 4. Photograph of friction apparatus

Fig. 5. Schematic side and front sectional views of friction apparatus, with detail of specimen carriers.

Fig.4



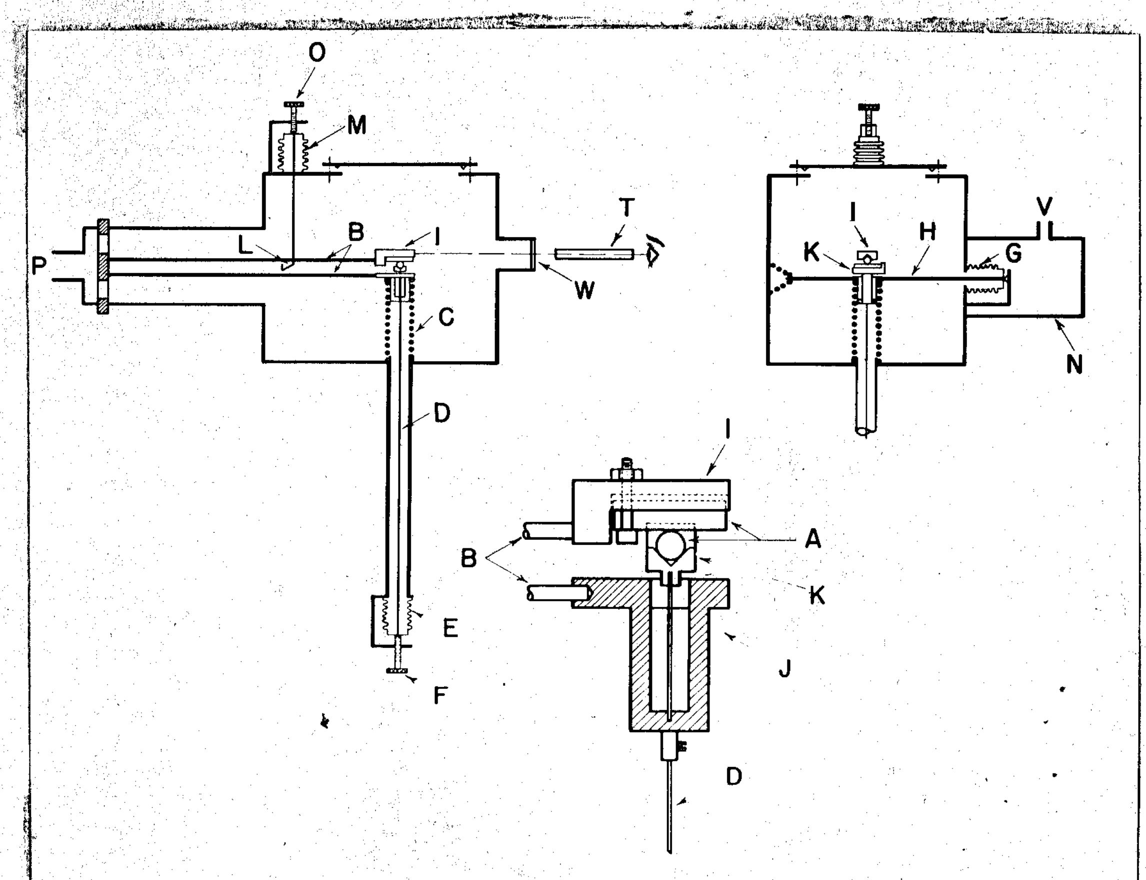


Fig. 5

It is the purpose of the upper cantilever to offer resistance to vertical and horizontal displacements of the upper specimen and carrier, and so produce the normal and tangential forces on this specimen. The amounts of the displacements, observed with the telescope (T) through the glass window (W), give a measure of these forces.

The forces on the specimen are applied by raising the lower carrier until the two specimens make contact and then further raising it to apply a definite load through deflection of the upper cantilever. The lower carrier may then be traversed horizontally, dragging the upper one with it by means of the frictional resistance offered at the point of contact between the two specimens. The upper specimen will traverse with the lower until the opposition to deflection afforded by the upper cantilever reaches such a value as to overcome this frictional resistance. At this point relative slip occurs in the form of a definite return "jump" of the upper member. If the value of horizontal deflection of the upper cantilever at the point where slip occurred has been observed with the telescope, we have a measure of the force of friction, F, since in the range in which we are working the resisting force of the beam is directly proportional to its deflection. We also have a measure of the normal force, N, in the value of the vertical displacement initially measured. Since the modulus is the same for both these deflections, the coefficient of friction, $\mu = F/N$, may immediately be obtained by dividing the value of the horizontal deflection by that of the vertical. There is thus no necessity for calibration of the cantilever rod. In addition, the accuracy of the measurements is independent of temperature effects on the elastic modulus of the upper rod.

If, following the first return "jump" of the upper member, the traverse of the lower carriage is allowed to continue, a second, third, or more return slips may be observed and recorded, thus duplicating at a very slow rate the process of stick-slip which Bowden and Leben[3] have shown to be the usual mechanism of sliding. Only the first reading of this series is truly accurate, however, since a slight twisting of the lower cantilever occurs when slip has taken place. This causes the lower specimen to make a slight angle with its initial horizontal position.

The lower cantilever has been made identical with the upper in order to permit the path of the lower carrier, when traversing horizontally, to duplicate that of the upper, as it is evident that the latter will not move in a straight line. The vertical displacement of the lower carrier is produced by a coil spring (C) below it, and controlled by a wire (D) attached to the carrier and running down through a long tube to a sylphon bellows (E) which may be compressed by the screw and handwheel (F) on the outside of the vacuum chamber.

The horizontal displacement is effected by means of air pressure acting on a second sylphon bellows (G), which is coupled to the lower carrier by a rod (H). The inside of this bellows communicates with the main vacuum chamber, whereas the exterior is surrounded by an auxiliary chamber (N) connected to a separate mechanical vacuum pump through the outlet (V). If both chambers are evacuated, and air then allowed to leak into the auxiliary chamber, the lower carrier will be slowly forced to the left (as viewed through W) since the vacuum in the main chamber is unaffected.

The upper specimen carrier (I) is of the form shown in the detail drawing in Fig. 5. The specimen is carried in a V-notch and held with a clamp at one end. Its position is such that its point of contact with the lower specimen shall lie on the axis of the cantilever rod. This is done that there may be no twisting of the rod about its axis, with consequent shifting of the point of contact on the upper specimen during application of the tangential force.

The lower specimen carrier (J) has been so constructed that the moment of the forces on it about the axis of the lower rod is zero, thus fulfilling the same condition as cited for the upper carrier, above.

The actual specimen holder (E) of the lower carrier is a small V-block with a stop at one end, against which the specimen is pressed by a clamping screw (not shown) on the other end. It may be noted that the V-block specimen holder is mounted on a steel reed whose wide dimension lies in the direction of the applied tangential or frictional force. This is done to prevent any appreciable force in a direction perpendicular to the plane of the two applied forces from acting on the contact area and so introducing errors. A slight force in this direction is set up by the relative motion of the upper and lower specimen carriers during the raising of the two (incidental to the application of the normal force). However, this is eliminated by raising the upper specimen out of contact with the lower, and then gently returning it, by means of the hook (L). This hook is moved by means of a screw and handwheel (O) on the outside of the chamber, through a third sylphon bellows (M).

Since the specimens are mounted at the ends of long, thin rods, the instrument is sensitive to vibration. This is minimized by mounting the entire apparatus on Keldur pads so as to provide isolation from external disturbances. Moreover, the presence of any vibration of sufficient magnitude to introduce appreciable extraneous forces is readily detectable when observing the upper carrier through the telescope. Readings for which vibration is observed are usually rejected, although ordinarily they do not differ appreciably from those taken under conditions where no vibration is present.

The telescope with which the measurements of coefficient of friction are made may be seen mounted in front of the main chamber in the picture of the apparatus. (Fig. 4). It is sighted on a pair of scratch marks on the front of the upper specimen carrier. The micrometer slide in which the telescope is mounted is used to measure the vertical displacement of the upper rod. The horizontal deflection is followed and read with the filar micrometer eyepiece. The telescope is so mounted that the line of travel of the eyepiece crosshairs is parallel to the axis of the lower specimen, since this latter direction is the true horizontal for the measurement of forces on the specimens. The two motions of the telescope are set exactly at right angles to each other by means of the eyepiece crosshairs. Both the horizontal and vertical micrometer read to 0.0001". At the usual rate of horizontal traverse of the specimens (about 0.05" per minute), the accuracy in reading is about ± 0.0005" for both the horizontal and vertical displacements. On the basis of a vertical displacement of 0.0800", which is used as standard except where otherwise specified, in these tests, this gives an error of about ± 0.006 in the value of coefficient of friction. Thus the overall experimental error may safely be set at ± .01 for readings taken at the standard load. Since the usual values of coefficient of friction with which we are dealing are of the order of one, this value of experimental error is quite permissible.

The vacuum system employed exhausts the main chamber of the apparatus through the port (P). It consists of a two-stage oil diffusion pump backed by a Cenco Hypervac "20" mechanical pump. The former can be clearly seen, connected to the back of the friction apparatus, in the photograph (Fig. 4). The pumping fluid used is n-amyl phthalate. Friction measurements at room temperature are carried out in a vacuum of 10^{-1}m.m. of H_g supplied by the mechanical pump alone, the vacuum in this case being measured with a McLeod gage (partially visible in the background). For all friction measurements taken above room temperature a vacuum of 10^{-4} m.m. of Hg is employed. In this case the McLeod gage is cut off from the system, and the vacuum measured with an ionization gage initially calibrated with the McLeod gage.

The elevated temperatures are obtained by means of nichrome wire heaters within the main chamber. These heat the specimens and their carriers by radiation. Temperature measurement is made with a chromel-alumel thermocouple attached to the lower specimen carrier. The maximum temperature usually employed is 350°C since very much higher temperatures might permit noticeable creep in the cantilever rods at high values of stress, and also impair the sylphon bellows. No creep or set in the Inconel rod was ever detected with the telescope micrometer, even after readings which subjected them to a fiber stress of approximately 18,000 lbs/sq. inch at the maximum temperature of 350°C. The entire apparatus has been in service over 2-1/2 years.

THEORY OF STATIC CONTACT

Although the experimental results obtained with the friction apparatus described above are of intrinsic interest from a purely empirical point of view, they assume much greater importance in the light of the theory of static contact and friction presented below.

Since resistance to sliding can arise only at the points at which two surfaces actually bear on each other, the first step in the development of this theory must be a consideration of the nature of the mechanism by which surfaces of crystalline solids make contact. Ordinary surface contours of such materials are built up from portions of individual, randomly arranged crystals and therefore may be expected to be characterized, at least on a submicroscopic scale, by a succession of sharp points and angularities with extremely small radii of curvature. Incipient contact between a pair of such surfaces can bring only a few of the highest points and angularities of each to bear on portions of the other. Thus, for almost infinitesimal loads on the surfaces, the value of surface shear stress calculated for these projections from Hertz's well known equations for elastic contact should rise to values in excess of the elastic limit of the weaker material of the pair, because of the large curvature and few contact points. Therefore, it is reasonable to expect that, at appreciable loads, all but a negligible portion of the area of true contact between faces of crystalline solids (with the exception of pairs of surfaces that are extremely smooth and well fitted) will be established by plastic flow of the material. This concept will be taken as the first fundamental postulate of the theory of static friction.

A portion of the experimental results recently obtained by Bowden and Tabor[4] on the area of contact between stationary surfaces is in agreement with the above postulate, as they have indicated. They have shown qualitatively how this offers an explanation of Amontons's law. Bowden and Tabor have, however, failed to present any mechanism to account for the major portion of their experimental value, viz: That portion dealing with decreasing contact area.

In processes where an area of contact is established between two bodies by plastic flow, as for instance in the testing of the hardness of metals by indentation, it is found that the projection of the area of contact on the plane of the surface is approximately proportional to the maximum load applied normal to that plane. The constant of proportionality is known as the mean pressure hardness value of the material.

This value is a true constant of the material in the case of completely work-hardened metal, and is equal to the "pressure of fluidity" value for that metal[5].

In view of the foregoing facts and of the initial postulate, it is evident that for all cases where contact is established between ordinary surfaces of metals by application of some given load, it can be said that to a first degree of approximation

$$N = A\,H \qquad (1)$$

where: N = normal load on the plane of the contact region of the surfaces,

A = projection of the area of true contact on the plane of the contact region,

H = mean pressure surface hardness value for the softer of the two metals.

Assuming that the first postulate is valid, this equation indicates that the idea of always associating extreme static contact pressures with high loads is incorrect. The compressive stress between dry surfaces in static contact is, in general, practically independent of load, and depends only on the value of H.

Thus far the question of the relation between load and area of contact has been discussed only for the cases where the applied load is increased, or is held constant. The behavior when load is decreased is, of course, equally important. In giving thought to this problem one might at first conclude that if the mechanism by which contact is established between solids is that of plastic deformation, there should be no change whatever in an initial area of contact when load is decreased. In fact, Bikerman and Rideal[6] have made this (incorrect) conclusion the basis for an argument that all friction must be due to surface roughness only. They considered that "adhesion" or resistance to shear at the area of contact was thus proven non-existent.

The assumption that an area of contact formed by plastic deformation should not decrease when load is decreased is, in general, incorrect. This fact is readily observed in such processes as the testing of the hardness of metals by indentation. There the area of contact established at some given load between a hard indenting agent - such as a steel ball - and the metal being tested may be observed to grow smaller when the initial load is decreased[7]. The two surfaces are peeled apart by elastic forces. This process is known as "elastic recovery". It must tend to occur whenever the load is decreased on an area of contact which has been established by plastic deformation. Otherwise there would be a violation of Hertz's equations as applied to the final macroscopic surface curvature.

The first step in arriving at a quantitative relation between contact area and decreasing load lies in reaching some conclusion concerning the nature of the adhesion, if any, acting at the area of true contact between two solid surfaces which are being separated. It is a well established fact[8] that ordinary supposedly clean metal surfaces are covered to a greater or lesser extent with adsorbed films which are very difficult to remove. Ordinary cleaning with solvents is quite ineffective in removing them, the solvent itself often being adsorbed as an additional component of the film. These films consist of originally adsorbed materials, or the products of the chemical reaction of these materials with the metal, or both. The various constituents of the film are bound to the surface either by intermolecular forces, such as Van der Waals forces, or by chemical combination - "chemisorption".

In view of the presence of these films, the ratio of the area of metal-to-metal contact to the total contact area will evidently be small (perhaps negiligible) for all ordinary dry metal surfaces. Such surfaces will evidently offer little resistance to separation. The tensile forces acting at points covered by the film will surely not be great enough to produce plastic flow of the metal.

When two perfectly clean metals are brought together, however, the two solid surfaces are forced into intimate contact at scattered points by mutual plastic flow, with no separating film interposed. Continuous metal junctions are thus established between the two bodies; rapid diffusion will occur at the interfacial planes in these junctions, even at room temperature. There is good evidence to indicate that the bond thus formed is usually stronger than the weaker material of the pair[9]. Therefore, any successful attempt to separate such surfaces must produce sufficient tensile stress in these metallic "bridges" to cause plastic flow and consequent reduction of cross-sectional area, with final rupture.

The above idea of the formation of strong intermetallic junctions at points of actual metal-to-metal contact may be taken as a second fundamental postulate. It will be confirmed by the experimental data presented later in the paper.

This postulate, together with the idea of elastic recovery, furnishes the clue to the behavior of contact area when load is decreased. As has already been mentioned, elastic recovery usually must tend to peel two surfaces apart when N is reduced, in accord with Hertz's equations. (This will not be the case when both surfaces originally have exactly the same overall curvature under zero load, as for instance two very flat surfaces). Consider the case where the two surfaces are so clean that the previously postulated seizing occurs at the points of actual contact. If the total area of actual contact is only a small percentage of the apparent contact area (as given by Hertz's equations), the peeling action introduced by the elastic recovery can produce plastic flow in the scattered metallic junctions between the two surfaces when load is decreased. The tensile stress necessary to produce plastic flow in a work-hardened metal rod having extremely small cross-sectional area is approximately equal to the mean pressure hardness value, H, of the given metal. This is shown in O'Neill's[10] analysis of the Stead tensile test diagram. Therefore, for the case under discussion, the variation of contact area with decreasing load must obey equation (1) (namely N = A H) approximately.

On the other hand, if the total area of actual contact of two perfectly clean surfaces equals their apparent contact area, the elastic recovery stress will evidently be insufficient to produce any plastic reduction of the entire area. In such cases, the contact area should be practically independent of decreasing load. An appreciable tensile force will then be required to separate the two bodies.

Mention should be made of Bowden and Tabor's investigation[4] by electrical conductance tests, of the variation of the area of contact between partially clean metal cylinders on which the load was decreased. It was found that the decreasing contact area was apparently proportional to some power of N very nearly equal to one. Although the experimenters postulated that the area of contact varied by plastic flow of the metal, they failed to explain how such a phenomena could occur under conditions of decreasing load. Their results, however, appear to be in agreement with the above theoretical treatment.

THEORY OF STATIC FRICTION

Having protrayed the foregoing picture of the mechanism of contact under various conditions, it is possible to proceed directly to the derivation of a quantitative theory of static friction. This will be done for the types of contact for which equation (1) applies to a good approximation. These include practically all cases where "increased" loads are involved, and the particular "decreased load" case involving exceptionally clean surfaces and a total contact area which is considerably less than the apparent contact area.

In equation (1) we have an approximate expression for the normal load, N, on two contacting surfaces in terms of a physical constant (H) of one of the surfaces and the projection, A, of the area of true contact of the two. Since, by definition, $\mu = F/N$, a similar type of expression is needed for the force of friction, F, if μ, the coefficient of friction, is to be expressed in terms of physical constants of the surfaces. It is evident that in order to cause relative slip by application of the tangential force, F, the shear stress applied to A', the area of true contact, must equal the average shear strength, S, of that area. Thus $F'/A' = S$, where F' represents the component along A' of the resultant of F and N. These force relationships are shown in Fig. 6.

When the surfaces are rough and the tiny "hills" of one interlock with those on the other, the plane of A' evidently makes some average angle, θ , with the plane of the contact region of the surfaces. For the moment, however, let us consider the ideal case in which the surfaces are so smooth that θ is negligibly small. Then it is

Fig. 6. Force relationships at contact point between interlocking (exaggerated) rough surfaces.

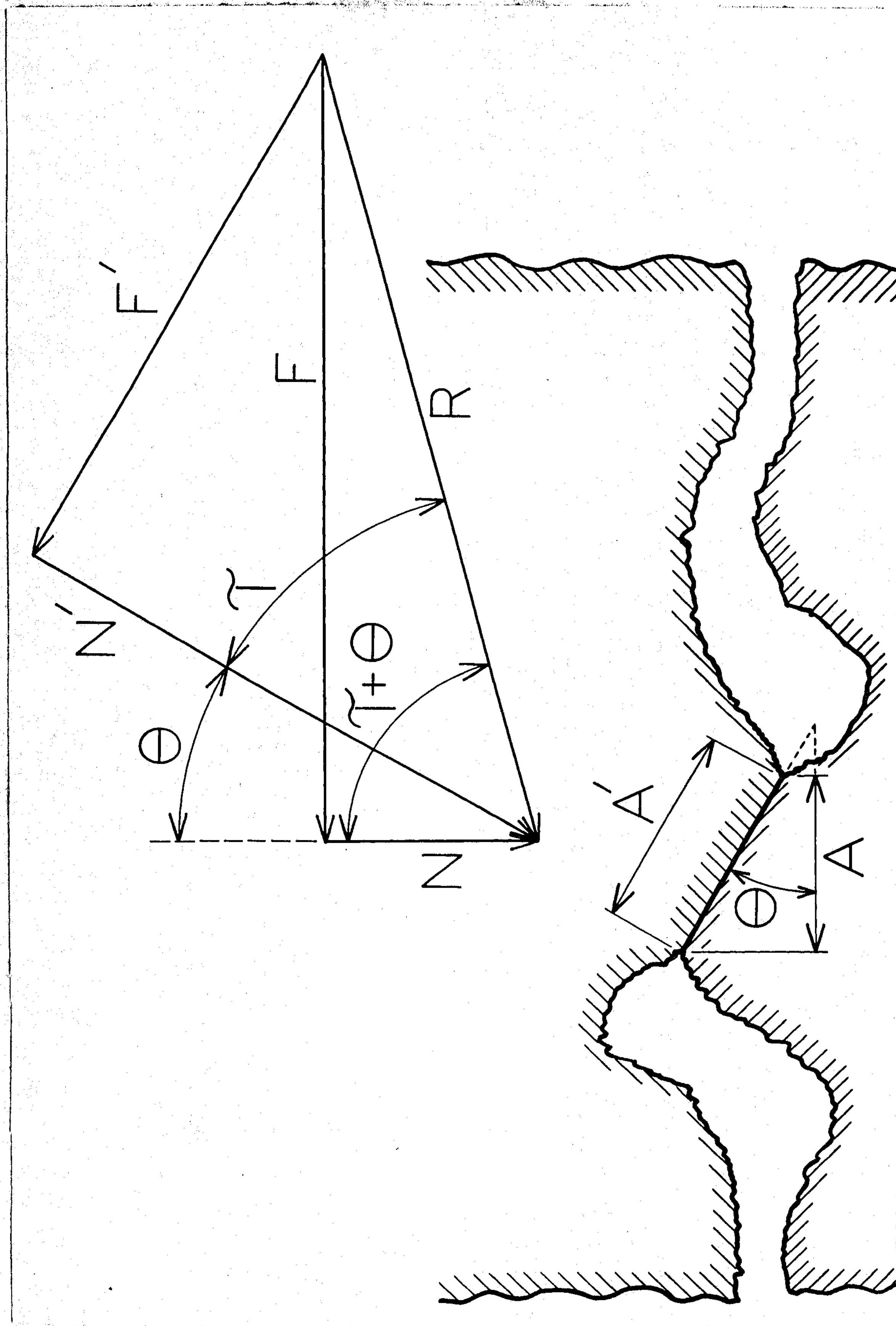


Fig. 6

evident from Fig. 6 that $F' \rightarrow F$ and $A' \rightarrow A$. Therefore:

$$\frac{F'}{A'} = \frac{F}{A} = S$$

or $$F = A\,S \qquad (2)$$

Now, by definition, $\mu = F/N$

and therefore, from equations (1) and (2), it follows that

$$\lim_{\theta \rightarrow 0} \mu = \frac{A\,S}{A\,H} = \frac{S}{H} \qquad (3)$$

In Appendix A to this paper it is shownthat for the general case where θ is appreciable, the expression for coefficient of friction takes the approximate form

$$\mu = S/H + \tan\theta \qquad (4)$$

This has been presented in a previous brief publication[11] as the general equation of static friction. Although this equation is an approximation, it illustrates very well the separate parts played by adehsion (represented by S/H) and by surface roughness (represented by tan θ) in the mechanism of static friction. The expression is in agreement with the empirical Amontons-Coulomb laws of friction, so long as S, H, and θ remain good constants of the surfaces.

Several interesting conclusions may be drawn from equation (4), considering first the case in which $S \rightarrow 0$ and θ is variable, and later the case in which $\theta \rightarrow 0$ and S is variable.

The first case is observable when solid surfaces are covered with an adsorbed film which permits no metal-to-metal contact but which possesses a shear strength of practically zero (i.e. a perfect boundary lubricant). For this case we have, from equation (4)

$$\lim_{S \to 0} \mu = \tan \theta \qquad (5)$$

The obvious conclusion from this result is that, for solids covered with an excellent boundary lubricant, the static coefficient of friction is a measure of surface finish. It may be employed, with judgment, to specify that entity.

Equation (5), minus the indication that it is a limiting case, is the expression proposed by Bikerman and Rideal[6] for the coefficient of static friction. . Their failure to realize that it is a limiting case results from their previously mentioned incorrect conclusion concerning the variation of contact area with decreasing load.

The more interesting case for discussion is that in which S has an appreciable value and θ is negligible either because of smoothness of surface or because of non-interlocking of surface irregularities. Then, from equation (4) we obtain

$$\lim_{\theta \to 0} \mu = S/H \qquad (3)$$

The quantity S has already beendefined as the average shear strength of the area of true contact of two surfaces.

The structure to be sheared ordinarily consists of an adsorbed film (of the type previously discussed) and any possible metal-to-metal "bridges" formed at points not protected by the film. The value of S is thus the appropriate weighted average of the shear strengths of the film and the metal-to-metal bridges if any of the latter exist. The value of H is the surface hardness of the metal alone. These conclusions are in agreement with the work of Hardy and his colleagues[12] on the static friction of smooth boundary lubricated surfaces. They found the coefficient of friction to be a function of both the lubricant and the type of material on which the lubricant was used.

When surfaces are perfectly clean, the shear strength becomes that of the resulting metal-to-metal union only. If the contacting bodies are of identical material, S must have a value equal to the static shear strength of the given metal. This last conclusion follows from our second postulate. A quantitative evaluation of the ratio S/H under these two special conditions will be made, for the case of completely work-hardened, elementary metals

The quantitative value of H for a completely work-hardened metal is its maximum mean pressure hardness (or its "pressure of fluidity" value) as pointed out earlier in the paper. Pressure of fluidity values as given by O'Neill[13] will thus be substituted for H in forming the ratios.

The value to be substituted for S is not arrived at as easily. The shear strength desired is given by that value of stress necessary to initiate slip on a given plane through the completely work-hardened metal. Since there is no satisfactory experimental data obtainable definitely establishing this quantity as a physical constant of the material, in the manner in which the maximum value of H has been established, the value for S must be arrived at by theoretical considerations. This necessitates that some assumption be made as to the nature of the process by which shear in metals occurs. An hypothesis may be arrived at by the following line of thought.

In observing the shear process one is impressed by the freedom with which glide occurs during the "slip" portion of each "stick-slip" cycle. The action resembles that which would be observed if a fluid lubricant were instantaneously present on the plane of shear. This suggests the possibility that during slip the metal itself, in the group of adjacent shear planes, is in a state very similar to a liquid. It is only necessary that this liquid-like property be manifested in the direction of slip. The shear planes may retain their rigidity in all other directions. At the instant of slip, the only influence present which would be able to "melt" the metal on a set of planes is the shear stress. The foregoing suggests that the process of shear may be analagous to the process of pressure melting, and may be treated by the thermodynamic methods applicable to pressure melting. This idea may be taken as a third fundamental postulate, as follows:

The process of slip or shear may be treated as one of pressure melting in one dimension.

In order to obtain from this postulate a quantitative expression for shear strength, the following approximations must be made:

(1) The density of a metal is taken to be independent of temperature and pressure.

(2) The heat of fusion of a metal is taken to be independent of temperature.

(3) The "heat of fusion" for a "one-dimensional melting" process is taken to be one-third that of the usual three-dimensional fusion process.

Use must also be made of the fact that the stress or pressure exerted on the solid in the direction of shear at the instant of melting is not exerted on the resulting liquid. As soon as sufficient shear stress has been applied to the solid to cause the liquid state to appear, free slip can occur. From the above approximations and fact, thermodynamic considerations then lead to the expression

$$S = .427\ (L/3)\ \rho \ \ln T_m/T \qquad (6)$$

where: S = static internal slip stress of homegenous metal at the temperature T, kg/mm^2.

T = observation temperature or temperature of the metal, degrees absolute

L = latent heat of fusion of the metal in the particular crystalline modification which is stable at T, cal/gm

ρ = density of the metal at T, gm/cm^3

T_m = melting point of the metal in the particular crystalline modification which is stable at T, degrees absolute.

The complete derivation of equation (6) is given in Appendix B to this paper. It may be pointed out that the equation is merely an expression for the value of shear stress which must be applied to a metal to lower its melting point to the temperature T (usually room temperature). The "pressure melting by shear" which it describes must not be confused with the process of "pressure melting by compression" of the type often discussed in connection with the mechanism of sliding on ice[14].

The evident implication of the postulated mechanism of shear is that any kinetic shear process in solids must commence with a steady increase of stress up to the point at which melting occurs, followed by a sudden slip at the instant liquid appears, with consequent reduction in applied force. This reduction will permit resolidification to occur and stress to increase once more, allowing repetition of the cycle so long as sufficient applied force is maintained. In other words, all shear processes in solids should consist of a succession of slip-stick cycles. Bowden and Leben's experimental analysis of the nature of sliding shows it to be made up of a succession of slip-stick cycles whenever the surfaces are poorly lubricated. Since sliding is merely another kinetic shear process, this slip-stick behavior is to be expected whenever solid, metal-to-metal contact is free to occur.

A point to be noted concerning equation (6) is that it represents the maximum static shear strength of homogeneous metal, since it makes use of physical constants of the bulk material, rather than those corresponding to some certain direction in a single crystal. It will not be applicable to cases where shear is free to follow preferred slip planes within the metal. For the case of completely work-hardened metal this possibility of shear on preferred planes is evidently at a minimum, particularly if the areas in shear are small. Thus equation (6) is well suited for use in calculating values of static shear strength for the tiny areas of contact and well defined planes of shear occurring in the contact region of two clean solids of identical metal. However, in view of the approximations made in deriving this equation, it may be expected that the absolute values of S which it gives are less trustworthy than the relative values from metal to metal.

TABLE I

Calculated values of $S/H_{max.}$ for clean, completely work-hardened metal surfaces at room temperature

Metal	Crystal structure at room temperature	S kg/mm²	$H_{max.}$ kg/mm²	$S/H_{max.}$
Al	Face-centered cubic	42	42	1.00
Cu	" " "	96	100	.96
Ni	" " "	160	160	1.00
Pb	" " "	7	7	1.00
Fe	Body-centered cubic	130	130	1.00
Bi	Rhombohedral hexagonal	10.3	~ 20*	~ .5
Cd	Close-packed hexagonal	11.3	31	.36
Mg	" " "	13.5	40	.34
Zn	" " "	21	70	.30

*Value uncertain. See Reference 13.

In Table I values of S calculated from equation (6) for various metals are given. These values correspond to T = 293° absolute. The values of L used in the calculation are taken from the data listed as "Directly measured" in the U. S. Bureau of Mines Bulletin 393, "Heats of Fusion of Inorganic Substances" by K. K. Kelly. The values of H listed in the table correspond to O'Neill's values for pressure of fluidity for the various metals, as already stated. The resulting values of $S/H_{max.}$ as given have one noticeable point of interest. The value of the ratio is evidently in the vicinity of unity for all the cubic metals listed, and in the vicinity of 1/3 for all the close-packed hexagonal metals listed. One might expect to find some relation between H and S, since the hardness phenomenon is one involving internal shear of the metal. However, the relation appears to be surprisingly simple here. This point must be investigated further to determine whether the apparent correspondence between crystal types and $S/H_{max.}$ ratios is actual or coincidental.

EXPERIMENTAL RESULTS

The main discussion of experimental results in this paper is limited to the case of friction between specimens which are both of the same metal. A short discussion of the behavior of pairs of specimens made up two different metals is included at the end of this section. The experimental results cited are based on data obtained from over 5000 readings, taken over a period of two and one-half years.

From the foregoing theory it may be seen that surface cleanliness and surface roughness should influence the variables determining the coefficient of friction. Since these two factors are difficult to control, their effects were investigated first.

The first metal specimens tested were prepared by careful polishing with dry metallographic papers followed by treatment with an excess of rouge with a trace of water, on broadcloth. The polishing with rouge was continued until the point was reached at which the surface of the specimen was suddenly dry and rouge-free. The polished piece was then handled with forceps and the working surface touched with nothing other than "clean" tissue paper. Specimens of various metals thus prepared gave coefficients of friction of 0.1 to 0.2 - values of the same magnitude as those usually given for the friction of supposedly dry, "clean" surfaces. The points of contact were not appreciably scratched or torn by sliding. Evidently a lubricating film covered much of the area of actual contact.

In the next series of tests the rouge-polished surfaces were neither wiped with tissue nor touched with anything after polishing, but were quickly transferred to the vacuum chamber of the friction apparatus, whereupon the values of coefficient of friction obtained were found to be generally of the order of unity for cubic metals, and one-half for hexagonal metals - values of the same magnitude as the calculated ratios S/H of the preceding section. These surfaces were very clean, as indicated by the fact that when viewed under a microscope they were seen to be severely torn and scored at the points of contact by even the very first slide, as shown in Fig. 7. Evidently metat-to-metal contact occurred over a large percentage of the total contact area. This latter behavior corresponds to the type of friction of interest in connection with the dry cutting of metals. Therefore, it was decided that the remainder of the investigation should be carried out on surfaces clean enough to give this thorough "seizing".

Fig. 7. Torn, scored contact points produced by a single slide of two very clean copper friction specimens. Top, upper specimen. Bottom, lower specimen. Mag. 125X

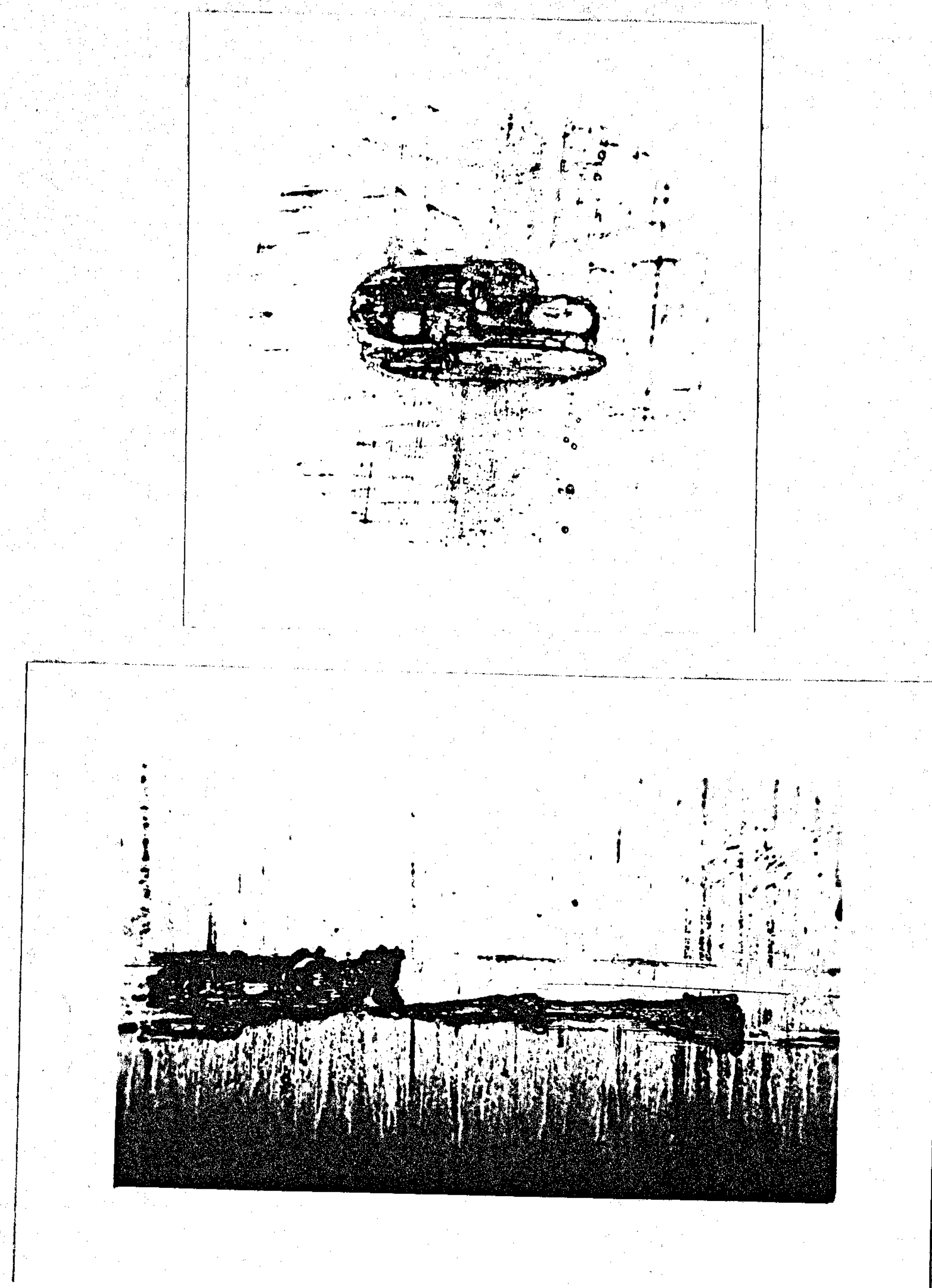


Fig.7

The seizing produces rough, torn surfaces when sliding occurs, so the effect of this roughness on friction was studied next. Pairs of specimens of various metals were carefully rendered as smooth and clean as possible by the rouge technique. The coefficient of friction was recorded for the very first slip of the unscratched surfaces of each pair. Succeeding readings were taken at the same contact points, now rough and torn. Four or five values of μ for the first slip (smooth surface) were obtained at various sets of contact points on a given pair of specimens and averaged. This value was compared with the average coefficient of friction for slips other than the first (rough surface) obtained with the same pair. Values for several such pairs are given in Table II.

The large average deviation of individual readings about the mean, noted in this table, must be expected of very clean surfaces as will be explained later.

Although in general there is a noticeable difference between the average values obtained for the first slides and those for succeeding slides, as given in Table II, the discrepancy is about as often positive as negative. This fact, plus the observed large magnitudes of the values of μ, make it clear that the effect of surface

TABLE II

INFLUENCE OF SURFACE ROUGHNESS ON THE COEFFICIENT OF FRICTION OF VERY CLEAN SURFACES

METAL PAIR	AVERAGE μ FIRST SLIDE (POLISHED SURFACE)	AVERAGE μ SUCCEEDING SLIDES (ROUGH, TORN SURFACE)
AL	1.15	1.21
CU	1.28	1.38
FE	1.34	1.20
NI	1.38	1.40
SB	.67	.50
AVERAGE DEVIATION OF INDIVIDUAL READINGS	$\sim \pm 10\%$	$\sim \pm 10\%$

roughness on the friction of these clean cylindrical specimens is negligible. This is equivalent to saying that in this case the value of θ is negligible, or that the planes of shear are nearly parallel to the general surface. Stated in this last form, the experimental result is one which perhaps might be expected for these clean surfaces which seize wherever they touch each other. As regards theory, the result indicates that equation (3), namely $\mu = S/H$, is a sufficiently accurate form of the general equation of static friction for use in the case of the friction of these clean, cylindrical metal specimens. This equation will, therefore, be used as exact in all further discussion of the experimental results.

In view of the above results, preparation of specimens by the difficult polishing, rouge-cleaning method was abandoned, and all further tests were made on specimens turned with a clean, dry cutting tool at slow speed in a lathe. This simple method of preparation insured a surface free from contaminating films of any sort except those of adsorbed or chemically combined gases. (The matter of the extreme cleanliness of a freshly machined surface will be discussed in more detail later in the paper.) Each specimen was transferred to the vacuum chamber of the friction apparatus with flame-cleaned forceps as soon as machined. The specimens were allowed to touch nothing which could transfer an organic film to the portion of the surface on which contact was to occur.

Friction specimens prepared by the above method gave individual values of coefficient of friction which exhibited the same average deviation from the mean (approximately ±10%) as had been characteristic of the clean specimens prepared by polishing. This behavior must be expected with very clean surfaces, for with each particular engagement of the surfaces the orientation of crystals in the metal at the contact points will be considerably different. Thus the shear strength must vary considerably from reading to reading. However, for several groups of a large number of readings each, the average shear strength should be the same. In practice it was found that groups of eight or more readings taken on a given set of specimens usually yielded average values of coefficient of friction agreeing within an amount little larger than the experimental error of the instrument (approximately ±.01).

Furthermore, the average values of the coefficients of friction of different pairs of specimens of the same metal machined under the same conditions usually agreed within limits not greatly exceeding the above amount. Preparation of the specimens by machining, therefore, afforded good control of the variables S and H.

The next step was to investigate the validity of Amontons's law for these clean surfaces. Coefficients of friction for several metals were determined at various loads. The permissible range of loads over which the instrument is capable of making accurate measurements of values of μ is limited, but over a large portion of that range the coefficient of friction was found to be entirely independent of N for the metals tested. At very small values of load, μ was observed to fall off from the constant figure found at higher loads. Amontons's law evidently holds for these clean surfaces except at very small values of load. The results obtained for Fe and for Sb are plotted in Fig. 8.

It should be made clear that these determinations of μ versus N were not made under "decreased" load conditions. Results obtained by decreasing the load from a higher value will be presented later in the paper. The coefficients of friction discussed above were all obtained at values of the load which were the maximum applied in each case.

In view of the dependence of μ on N at very small

Fig. 8. Coefficient of static friction versus load, for very clean specimens of iron and of antimony.

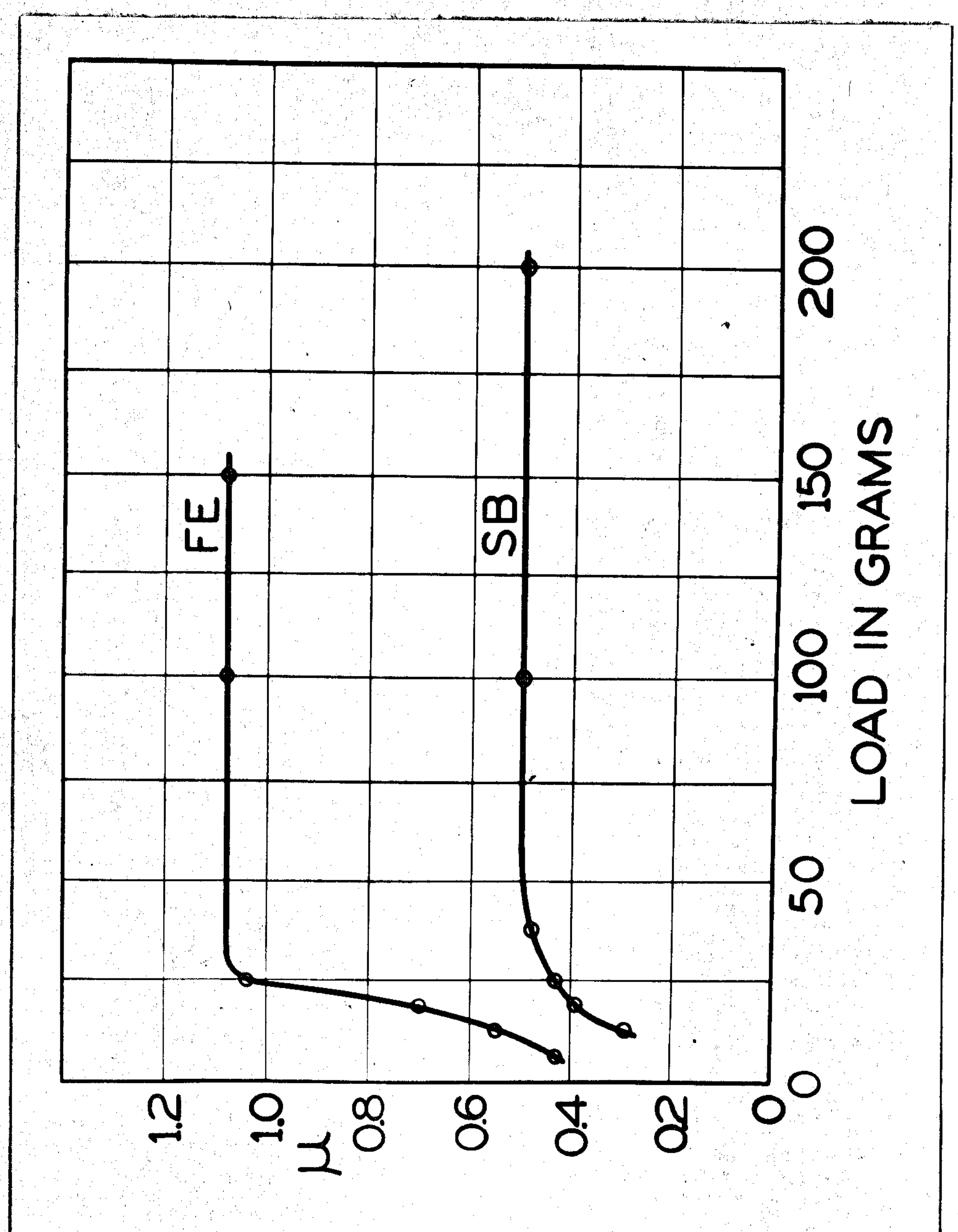


Fig. 8

loads, a large value of N corresponding to a vertical deflection of 0.0800" of the upper specimen carrier (about 100 grams) was adopted as standard. All values of coefficient of friction given in this paper were taken at this value of N, unless otherwise stated.

Although at very small loads μ decreases as N decreases, this fact does not indicate that in the range where such behavior occurs the area of contact of the specimens is produced by elastic deformation. A contact area produced by this means must, according to Hertz's equations, be approximately proportional to some power, x, of the normal force, where x is less than one. If this is the case, the force of friction will also be proportional to N^x. Because x is less than one, F/N or μ should <u>increase</u> as load is decreased, if contact is elastic. The opposite behavior is observed.

On the other hand, the results obtained do not necessarily contradict the hypothesis that the area of contact is produced by plastic deformation, for it is quite possible that at very small loads with their correspondingly minute contact areas, either S or H, or both, are no longer good constants of the metal, (i.e. the laws of macroscopic plastic deformation and shear may fail at these low loads). However, the fact that μ is constant at higher loads offers qualitative confirmation of the assumption that the contacting surfaces are then deformed plastically.

Quantitative confirmation of this hypothesis, as well as of the two additional ones made during the development of the theory of friction, is found in a comparison of the coefficients of friction obtained for various metals with their calculated values of $S/H_{max.}$ previously listed in Table I. Since the process of machining the specimens produces a severe cold-working of the surface, rather complete work-hardening is to be expected for those metals which do not anneal appreciably at room temperature. The values of S and H used in calculating the ratios $S/H_{max.}$ are those for completely work-hardened metal. Therefore, the best agreement between observed coefficients of friction and the calculated values of $S/H_{max.}$ should be found for those metals which anneal only slightly. The comparison made in Table III shows this actually to be the case.

Due to the presence of the adsorbed or combined gas films on the surfaces tested, the experimental values of μ given in the table may be somewhat less than the absolute values which would be given by perfectly clean, outgassed surfaces of the same hardness. However, relative

TABLE III

COMPARISON OF AVERAGE OBSERVED VALUES OF μ

WITH THE

THEORETICAL VALUES OF $S/H_{max.}$ FROM TABLE I

METAL PAIR	CRYSTAL STRUCTURE AT ROOM TEMPERATURE	MELTING POINT, DEGREES ABSOLUTE	CALCULATED $S/H_{max.}$ (FULLY WORK-HARDENED METAL)	AVERAGE OBSERVED μ, ROOM TEMPERATURE
AL	Face-centered cubic	931.7	1.00	1.05
CU	" " "	1357	.96	1.35
NI	" " "	1725	1.00	1.10
PB	" " "	600.5	1.00	~2.2
FE	Body-centered cubic	1803	1.00	1.10
MO	" " "	2895	-	1.25

TABLE III (continued)

METAL PAIR	CRYSTAL STRUCTURE AT ROOM TEMPERATURE	MELTING POINT, DEGREES ABSOLUTE	CALCULATED $S/H_{max.}$ (FULLY WORK-HARDENED METAL)	OBSERVED μ, ROOM TEMPERATURE
BI	Rhombohedral hexagonal	544.1	~.5*	.86
SB	" " "	903.1	-	.53
CD	Close-packed hexagonal	594	.36	.83
MG	" " "	923	.34	.39
ZN	" " "	692.6	.30	.85
CO	" " "	-	-	.35

* See Table I

values of μ from metal-to-metal are probably quite reliable, just as are the corresponding values of S given by equation (6).

Comparing the values in the last two columns of Table III, it may be seen that the observed average coefficient of friction of each given metal either approximately equals or else exceeds the value of S/H calculated for that metal. The difference between the observed and the calculated values is very large only for those metals of the group which are known to anneal considerably at room temperature after severe plastic deformation, namely Pb, Bi, Cd, and Zn. Applying the rough rule that the relaxation point of a metal lies at approximately one-third of its absolute melting point, it may be seen that the above four metals are the only ones of the group for which that point lies considerably below room temperature.

The above observation suggests that heating the other metals (after machining) to a temperature greater than one-third of their absolute melting point should produce a considerable increase in their coefficient of friction if the observed behavior is due to annealing. Upon making tests at various temperatures up to 350°C (in a vacuum of less than one micron of mercury) it was found that for those metals whose relaxation points lie slightly above room temperature, the coefficients of friction rose rapidly with rising temperature. For those metals whose relaxation points lie above 350°C, and thus could not be reached in the present apparatus, the coefficient of friction remained independent of temperature. The results obtained for Cu, Mo, and Mg, are shown in Fig. 9. Annealing a metal before machining had no effect on its coefficient of friction, since the machining process always cold-worked the surface.

For those cubic metals (Al and Cu) whose relaxation temperature could be exceeded by heating, the final high value of μ obtained at 350°C was retained upon cooling to room temperature. There was no appreciable lowering of the coefficient of friction by the cold-working produced by each slip of the specimens. This behavior may be attributed to the large capacity of cubic metals for cold-work. However, for Mg, of hexagonal structure, the high value of μ obtained at 350°C decreased after the very first slip following cooling to a value of the order of that observed at room temperature before heating. This is in agreement with the small capacity of hexagonal metals for cold work.

The behavior of the coefficient of friction with change in temperature cannot readily be attributed to

Fig. 9. Typical curves of coefficient of static friction versus temperature, for copper, molbdenum, and magnesium.

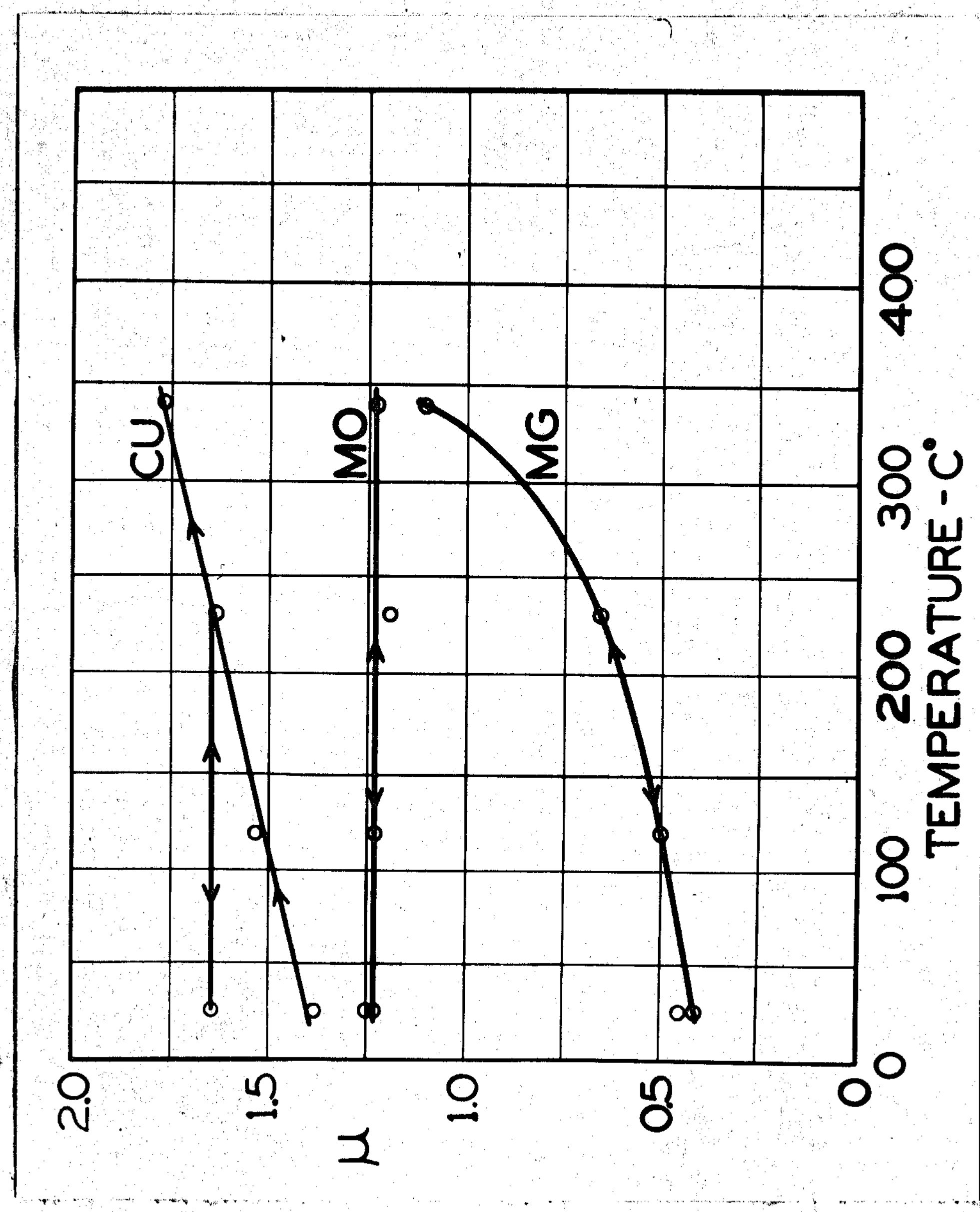


Fig.9

the removal of an adsorbed gas film by the high temperature vacuum treatment. After cooling those metals which maintain their high value of μ at room temperature an exposure to the atmosphere equivalent to that experienced during the original preparation of the specimens leaves the coefficients of friction unchanged.

The reasonable conclusions consistent with all the preceding results are the following:

(1) When no annealing occurs, S and H must be the same function of temperature. (This function of temperature is evidently logarithmic, according to equation (6).)

(2) When annealing occurs H decreases apace, in accord with common experience, whereas the value of S at which noticeable slip takes place apparently does not change greatly from the quantity given by equation (6). All experimental data obtained are in agreement with this picture.

The first conclusion is one which might be expected in view of the similarity between the processes involved in pure shear and in indentation deformation. The behavior ascribed to S in the second conclusion can be attributed to the fact that a small amount of creep always goes on at the contact areas during the time the tangential force is being applied. The process is confined to an extremely thin layer of metal at the contact points by the sharp maximum which occurs there in the stress distribution. The creep is thus sufficient to continuously strain-harden the metal in this thin layer and raise its shear strength to a value exceeding the stress. The process can continue until the layer is completely work-hardened and the maximum value of S thereby attained. At this point noticeable slip must occur.

This creep process has been carefully observed for several of the metals tested. The creep curve obtained for a pair of copper specimens first annealed at 350^{0} C in the vacuum chamber is shown in Fig. 10. In the case of lead, annealing occurs at such a high rate that the creep becomes very great, making it difficult to determine exactly at what value of tangential force an actual jump occurs. It is for this reason that only an approximate average value can be given for the coefficient of friction of Pb in Table III.

All of the experimental values of coefficent of friction reported above have been taken under "increased" load conditions. According to the theoretical discussion

Fig. 10. Total creep occurring at contact points of annealed copper specimens during application of tangential force, before actual slip.

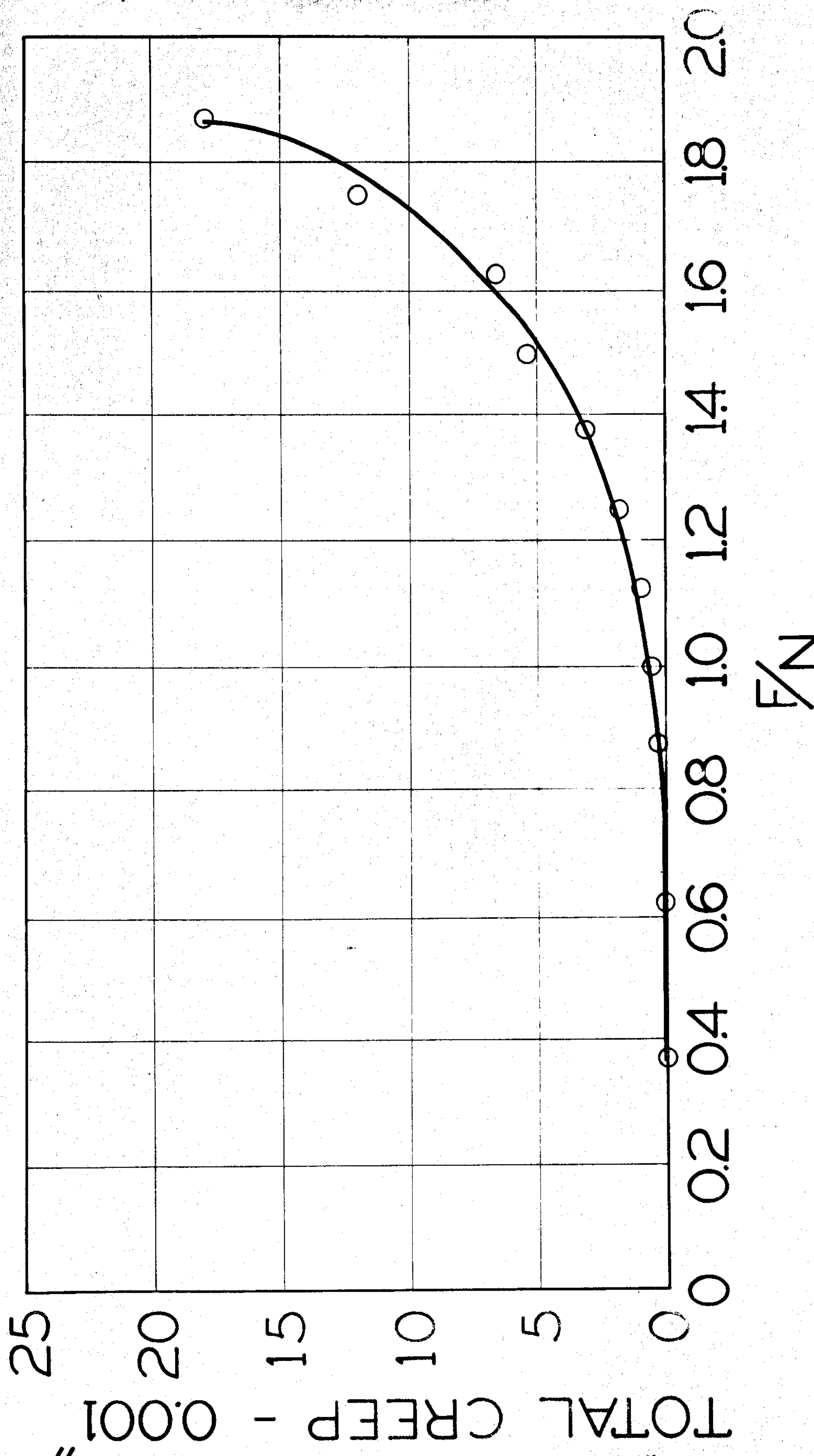


Fig.10

earlier in the paper, coefficients of friction of clean metal surfaces taken under "decreased" load conditions should be equal to, or greater than, those obtained by the usual method. Experimental values of μ for specimens of several different metals have been obtained by slowly decreasing the normal force from an initial value of 100 grams while maintaining a small tangential force (one equal to about 50% of that which would be required to produce slip at the 100 gram load). The value of "decreased" load at which slip occurred was carefully observed in each case. The average values of μ thus obtained were compared with the average coefficients of friction of the same specimens taken by the usual method. The results, presented in Table IV, are seen to be in agreement with the prediction from theory noted above.

TABLE IV

COMPARISON OF VALUES OF μ OBTAINED BY DECREASING THE NORMAL FORCE WITH THOSE OBTAINED BY THE USUAL METHOD

MATERIAL	INITIAL LOAD ON SURFACES, GRAMS	AVERAGE μ, DECREASING LOAD	AVERAGE μ, INITIAL LOAD
AL	100	1.00	1.01
CU	100	1.39	1.38
FE	100	1.04	1.05
ZN	100	1.04	.86
PB	100	"negative"	"positive"

The value of μ obtained for lead under decreasing load conditions was "negative" in the sense that the direction of the normal force had reversed so that the entire contact area was in tension when slip finally occurred.

Coefficients of friction obtained by decreasing the load were also observed for contact areas that were established by permitting one or more slips, or stick-slip cycles, to occur first at the initial load. These values of μ were always much larger than the coefficients of friction obtained by the usual method. This result cannot be attributed to an increase in the initial contact area as a result of the sliding of the surfaces, for the average value of coefficient of friction obtained in the usual way from the second or third "sticks" of a set of three stick-slip cycles never differs greatly from that obtained from the initial, or usually observed, "stick." For example, a specimen of copper gave an average value of μ = 1.33 for the usual, or initial readings. The readings corresponding to the second "stick", at the same load, gave an average value of μ = 1.42. However, the readings corresponding to the second "stick", taken by decreasing the load, gave an average value of μ of roughly six. Therefore, it appears that contact areas formed by sliding the surfaces together tangentially may have little elastic stress of the type tending to separate them in the direction of their common normal. This fact is evidently of importance in an analysis of kinetic friction.

All of the experimental results presented up to this point have been taken on specimens for which both members of a contacting pair were of the same material. Considerable work has been done also on pairs consisting of two different metals.

When specimens of two different metals are brought into contact, the softer surface must be deformed plastically by the harder. The value of H for such a pair must then be the actual mean pressure surface hardness of the softer of the two metals. It may be calculated by dividing the value of S for that metal by its coefficient of friction. The shear strength of the area of contact of the pair must be that of the weakest material present. It will be equal to the value of S of the metal whose shear strength is the lower, unless the transition alloy formed at the surfaces of union between the two bodies has a lower shear strength than either of the two original materials. Neglecting the effect of surface roughness, it follows that the coefficient of friction of a mixed pair must be either equal to or less than the number obtained by dividing the value of S for the weaker metal by the actual surface hardness H of the softer metal.

TABLE V

COEFFICIENTS OF FRICTION
OF
MIXED PAIRS OF CLEAN METALS

A - Pairs forming solid solutions at room temperature

PAIR	PREDICTED μ	OBSERVED μ
AL - FE	1.05	1.05
AL - ZN	.85	.82
CO - FE	-	.54*
CO - CU	.90	.89
CO - AL	1.05	1.01
CU - CD	.83	.85
CU - ZN	.85	.86
ZN - FE	.85	.85
ZN - SB	.85	.85

* This value used to calculate S for CO

B - Pairs almost mutually insoluble at room temperature

PAIR	PREDICTED μ	OBSERVED μ
CD - AL	$<$ .83	.57
CD - BI	$<$ .83	.79
CD - FE	$<$ .83	.64
CD - ZN	$<$ .83	.62
CU - FE	$<$ 1.35	1.05
ZN - BI	$<$ .86	.70

The values of coefficient of friction for various mixed pairs of metals tested to date are given in Table V. It may be seen that the above prediction concerning the value of μ is fully confirmed. The values listed in the column headed "PREDICTED μ" are in each case the ratio of the S value of the weaker member of the pair to the actual surface hardness of the softer member. In the case of CO, the value of S for the hexagonal form evidently cannot be calculated from equation (6). The value of μ for the CO - FE pair is therefore used to calculate S for CO. This turns out to be S = 64 kg/mm^2, giving H = 183 kg/mm^2. This S value is then used in calculating the predicted μ for CO - CU.

The pairs of metals for which the transition layer at the surface of union is weaker in shear than either of the two original materials evidently have inherent anti-scoring properties. The shear stress at their surface of union is less than that necessary to produce shear or scoring of the adjacent metal. The data in Table V show that this property was exhibited by those pairs which are characterized by almost complete mutual insolubility in the solid state at the given temperature. By a simple reasoning process it may be shown that nearly all pairs of metals so characterized may be expected to have anti-scoring or anti-seizing properties to a greater or lesser degree.

When two clean metals which are almost completely insoluble in the solid state are brought into contact, the transition layer "alloy" at their surface of union is necessarily of an "unnatural" percentage composition. As a result of the opposition to mixing, this layer is in a highly stressed condition. If the two materials have greater solubility in the liquid than in the solid state (and this is usually the case) then the given percentage composition of the transition layer in the liquid state will not be as unnatural as in the solid form; it may even be of a permissible value. The resulting instantaneous "insolubility stress" in the liquid alloy formed at melting will then be less than that in the solid.

In this case, therefore, the applied stress necessary to produce melting of the transition layer (i.e. a stress equal to the shear strength) will be less than that of the unalloyed materials, in which no "insolubility stresses" are present to aid the applied stress.

A semi-empirical treatment of this question, employing thermodynamic methods, has been found to lead to the same conclusion. Accordingly it may be said that pairs of metals having a negligible solid solubility at a given temperature generally will exhibit, at least to some extent, anti-scoring properties at that temperature. This will not be the case if the solubility of the liquids is equal to or less than that of the solids.

FRICTION AND THE MECHANISM OF CHIP FORMATION

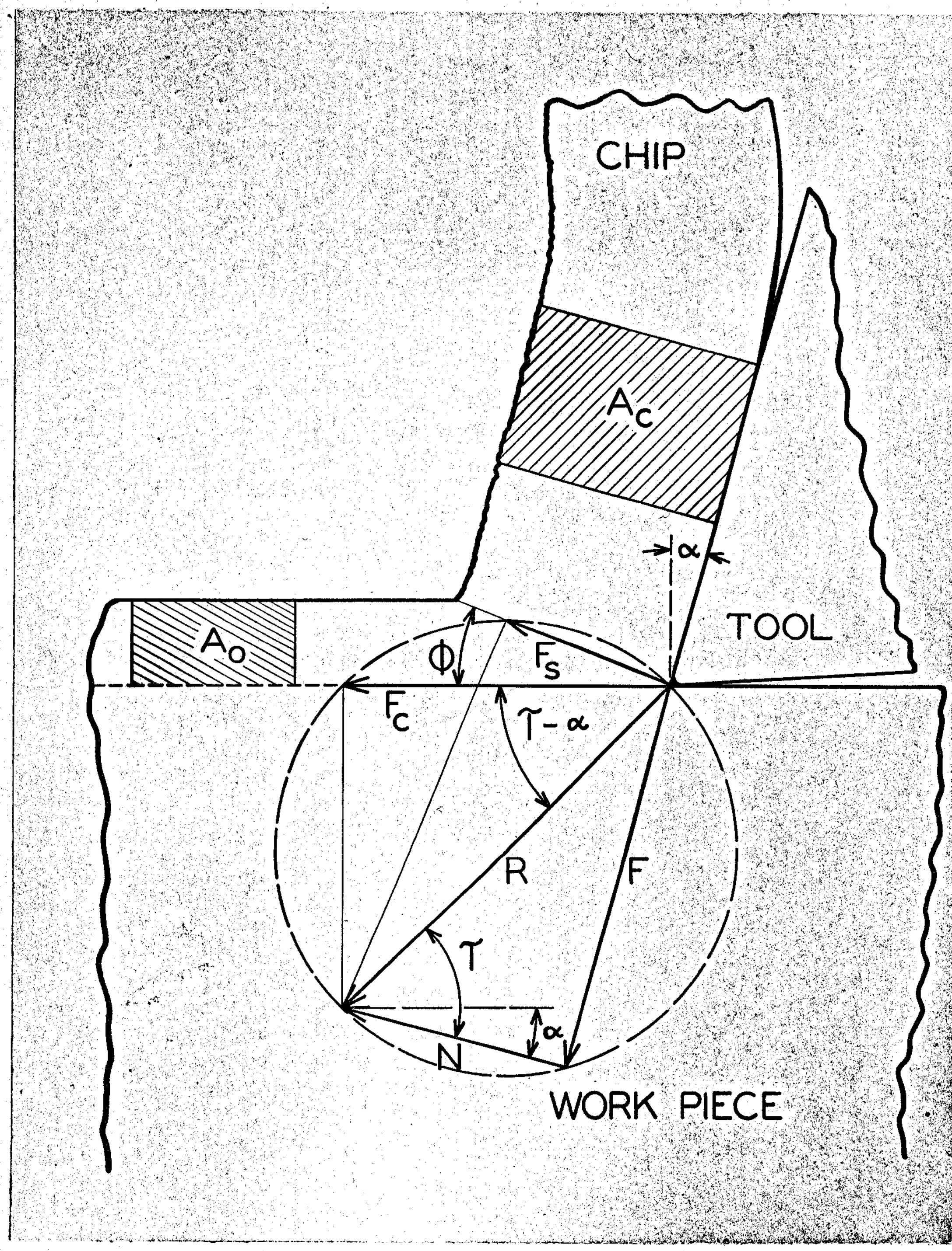
The preceding sections have presented a quantitative theory of the mechanism of static contact and of static friction. This theory also has applications to the kinetic friction of clean surfaces, because of the fact that for such surfaces the mechanism of sliding must be a stick-slip process. The apparent importance of friction as a variable in the metal cutting process has been emphasized in the introduction to this paper. Therefore, it is evident that a quantitative theory showing the part played by friction in the mechanism of chip formation, in the efficiency of metal removal, and in the production of smooth surfaces should be an interesting and useful companion piece to the preceding discussion. Such a quantitative theory is in the process of development. The results obtained to date will be presented here.

From an examination of chip photomicrographs, such as the uppermost photograph in each of Figs. 14(a), (b), and (c) it may be seen that the process of chip formation is accomplished by continual shear of the metal on a plane extending from the point of the cutting tool to the work surface. This plane is shown making some angle ϕ with the work-surface in the schematic force diagram of a tool taking a planing cut, shown in Fig. 11. The magnitude of the shearing force, F_S, is evidently determined by the area, A_S, of this plane of shear and the shear strength, S', of the material being cut. The natural question arises - - "What relationship exists between the magnitude and direction of F_S and the magnitudes and directions of the other forces acting in the cutting process?".

The direction of F_S is quite evidently the direction of maximum shear stress in the metal preceding the tool nose, since for planes determined by values of ϕ either larger or smaller than the observed value, the stress is insufficient to shear the metal. The value of the shear stress S_A on planes from the point of the cutting tool to the surface may be expressed as a function of ϕ, as is done in the derivation given in appendix C (F_S as there considered is the component of R along the plane determined by any value of ϕ). As may be seen, differentiating S_A with respect to ϕ and equating to zero to obtain the maximum leads to the expression

$$\mu = \cot(2\phi - \alpha) \qquad (7)$$

Fig. 11. Force relationships in the metal cutting process.


CHIP
A_c
α
TOOL
A_o
Φ
F_s
F_c
$\tau - \alpha$
R
F
τ
α
N
WORK PIECE

where $\mu = \tan\tau$ = coefficient of friction between tool and chip,

α = true rake angle of the cutting tool.

Equation 7 gives a simple expression for the coefficient of friction in metal cutting in terms of easily measured quantities. The quantity α is a constant of the cutting tool. The quantity ϕ may be observed with a microscope during cutting, may be obtained from measurements on a photomicrograph of the chip in process of removal (such as those illustrating this paper), or may be evaluated by measurements on the chips removed in the cutting process. In the latter case it can be seen from the geometry of Fig. 11 that the ratio of the cross-sectional area, A_o, of the cut being taken to the average cross-sectional area, A_c, of the chip (which ratio may be called the "cutting ratio", r_c) is approximately equal to $\sin\phi$. The exact relationship is readily found to be

$$\cot\phi = \frac{\sec\alpha}{r_c} - \tan\alpha .$$

Equation 7 is derived for the case where no detached built-up edge is present on the nose of the tool. However, if such exists the equation is still practically exact, since the angle ϕ depends only on the <u>direction</u> of R, which is determined by the coefficient of friction <u>at the tool face</u> and the rake angle <u>of the tool</u>.

FRICTION AND EFFICIENCY OF METAL REMOVAL

In appendix D it is shown that, when no built-up edge is present in cutting,

$$P = F_c/A_0 = 2S' \cot \phi \qquad (8)$$

where F_c is the cutting force, or force exerted by the tool in the direction of travel, P is a quantity known a specific cutting pressure, and S' is the shear strength of the metal on the plane of shear. This ordinary shear strength, S', should not be confused with the maximum shear strength, S which, for a pure substance, is given by equation (6).

It has been shown by M. Kronenberg[15] that

$$V \cdot P = 396{,}000 \qquad (9)$$

where P = specific cutting pressure in lbs/in^2

V = cubic inches of metal removed in cutting, per horsepower delivered to the tool, per minute.

Therefore, combining equations 8 and 9 gives the following expression for the efficiency of metal removal in cutting, when no built-up edge is present

$$V = 198{,}000 \frac{\tan \phi}{S'} \qquad (10)$$

Thus the efficiency of metal removal is a function of the shear strength, S', of the material being cut and the shear angle, ϕ, which in turn is a function of μ and α, through equation 7. It is evident that for a given metal the cutting efficiency will be improved is S' is decreased (provided μ is not appreciably increased thereby, through the accompanying decrease in H), or if ϕ is increased. According to equation 7, ϕ may be increased by decreasing the coefficient of friction or by increasing the true rake angle of the tool.

Interesting confirmation of equation (8) may be found in an experimental result obtained by M. C. Shaw in his investigation of cutting fluids, and reported in his Master's thesis[16] to this department. Shaw found experimentally that <u>for a given size of cut</u>

$$r_c F_c = K, (\text{a constant of the material}) \qquad (11)$$

to within the accuracy of his data, over an extremely wide range of values of F_c and r_c. (The quantity, r_c, which we have called the cutting ratio is called, by him, the chip length ratio, because of his experimental method of obtaining this quantity.) Now equation 8 may be put in the form

$$F_c \tan \phi = 2S'A_o \qquad (8')$$

As already pointed out, r_c is approximately equal to $\sin \phi$, which is approximately equal to $\tan \phi$, since ϕ is seldom larger than 30° in Shaw's tests. Therefore,

$$F_c r_c \approx 2S'A_o \qquad (8'')$$

It is evident that, for a given size of cut, $2S'A_o$ is a constant of the material, indicating that the theoretically obtained equation 8 is in agreement with Shaw's empirical equation 11. Furthermore, from (8") and (11),

$$K = 2S'A_o$$

or $$S' = K/2A_o$$

For cuts having a value of A_o equal to 0.00125 sq. in. on annealed aluminum and on 1020 steel, Shaw obtained K = 21.25# for aluminum and K = 129# for 1020 steel. These give S' = 8,500#/in^2 for the shear strength of aluminum, and S' = 51,600#/in^2 for 1020 steel, in excellent agreement with usual values of shear strength for these materials. For instance, the A.S.M. "Metals' Handbook", p. 928, gives the shear strength of annealed aluminum as 9,600#/in^2. Mark's "Mechanical Engineers' Handbook", p. 541, gives the tensile strength of 1020 steel as 71,000#/in^2. Since, for carbon steels, the shear strength is approximately 75% of the tensile strength (see Kent's "Mechanical Engineers' Handbook", p. 589), this corresponds to a shear strength of 53,000#/in^2. for 1020 steel.

The derivation in appendix D which leads to equation 10 is based on Fig. 11, which pictures the case where no built-up edge is present. If such a formation exists then it may be seen that the original cutting force, F_c, will be increased by an amount corresponding, approximately, to the product of the shear strength S' of the metal by the area of the flank, or bottom surface, of the built-up edge. This area may be called A_B. The increase in cutting force which it produces may be called F_c'. Then, approximately at least,

$$F_c' = S'A_B$$

and the corresponding increase, P', in the specific cutting pressure is

$$P' = F_c'/A_o = S' A_B/A_o$$

The total specific cutting pressure, P_t, then becomes

$$P_t = P + P' = 2S' \cot \phi + S' A_B/A_o \qquad (12)$$

(Equation 12 is not intended for quantitative use)

Therefore, when a built-up edge comes into existance the specific cutting pressure is increased (and the efficiency of metal removal thereby decreased) by an amount depending on the shear strength of the material and on the ratio of the size of the built-up edge to the size of cut being taken. As is shown in the following section of this paper, a built-up edge will appear in cutting when the coefficient of friction has increased to some definite value. Therefore, an increasing value of μ not only produces an increasing value of P in equation 12, but also, at some point, causes P' to assume a value greater than zero. The coefficient of friction is therefore an important variable in determining the efficiency of metal removal in cutting.

FRICTION AND SURFACE FINISH

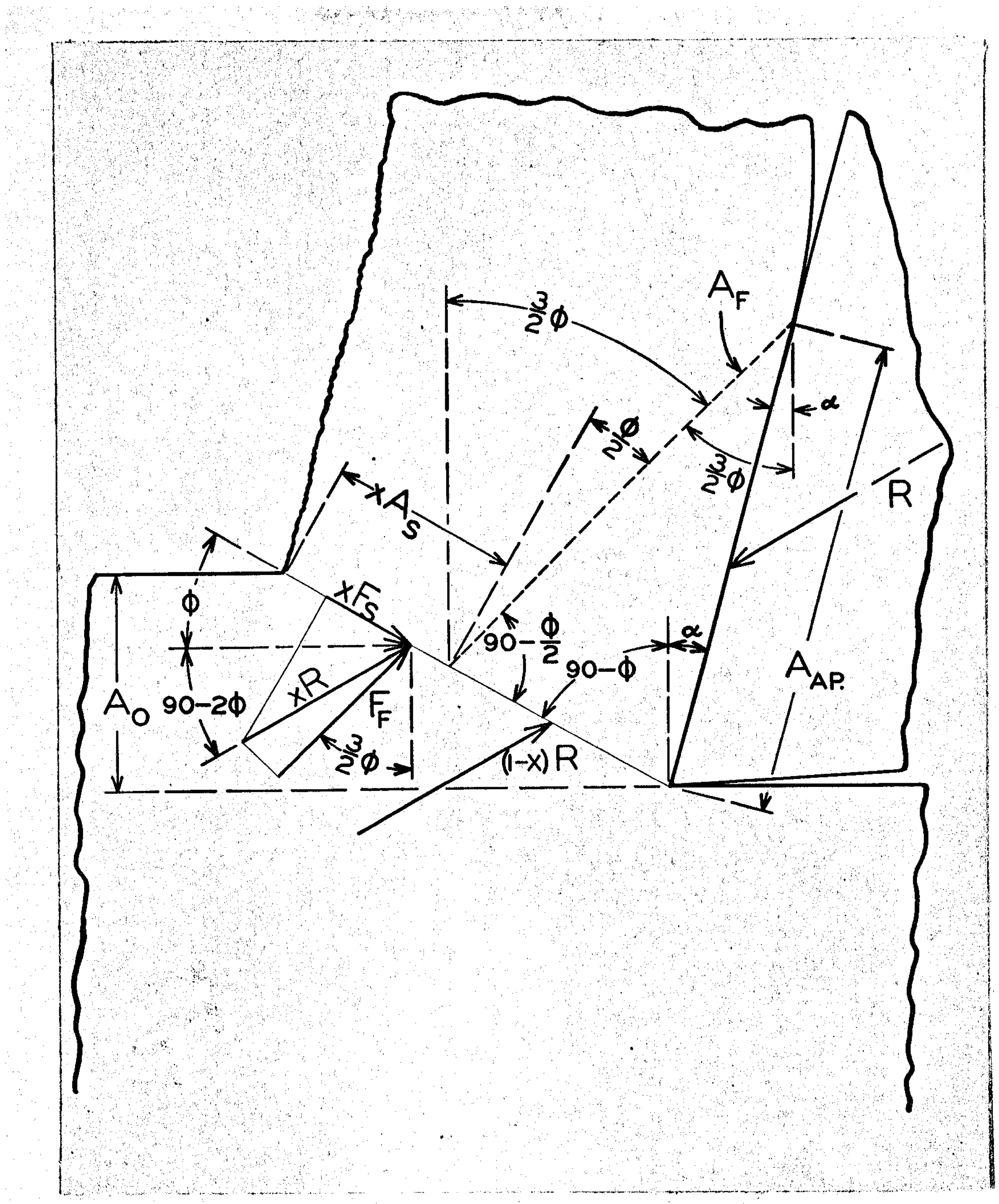
As pointed out in the introduction to this paper, the built-up edge on the nose of a cutting tool is known to be the major factor in the production of surface roughness on "finished surfaces. It was Ernst and Martellotti's conviction that the presence of the built-up edge is due to a large value of coefficient of friction between chip and tool face, as already stated. This conviction can be confirmed in a semi-quantitative manner, as follows.

It is evident that the built-up edge comes into existance when, during the "stick" portion of one of the successive "stick-slip" cycles at the tool face, the shear stress on some plane in the chip (a plane extending from the tool face down to the plane of shear) becomes equal to the shear strength of the chip metal. Failure then occurs on that plane and the "triangular" mass of metal thus severed from the chip remains nearly stationary on the tool nose. The direction of the plane of failure is approximately the direction of maximum shear stress within the chip (the chip metal is not perfectly isotropic), just as the direction of the plane of shear at the base of the chip is the direction of maximum shear stress within the work-piece. By exactly the same method as that used in deriving equation (7), it is found that the plane of maximum stress in the chip makes an angle of $3/2\phi$ with a perdendular to the surface of the work-piece, where ϕ is the angle which the plane of shear (at the base of the chip) makes with the work surface. This direction for the plane of failure is shown in Fig. 12. This figure gives a qualitative picture of the force distribution tending to produce shear in the chip. If three simplifying assumptions (which only approximate to reality) be made, a semi-quantitative expression for the stress on the plane of failure in the chip can be obtained. These assumptions, or approximations, are:

(1) The apparent area of contact, A_{ap}, between tool face and chip is taken to be proportional to the area of cut being taken, i.e. $A_{ap} = k\ A_0$. This assumption may be farther from reality than the two following, since A_{ap} is probably a function of the actual area of contact between tool and chip, which is a function, among other things, of the hardness of the chip metal and the normal force on the tool face.

(2) The distribution of the resultant force, R, along the apparent contact area, A_{ap}, is taken to be

Fig. 12. Illustrative diagram of mechanism of formation of built-up edge in metal cutting.



$\frac{3}{2}\phi$
A_F
$\frac{\phi}{2}$
α
$\frac{3}{2}\phi$
xA_S
R
ϕ
xF_S
$90-\frac{\phi}{2}$
α
$90-\phi$
$A_{AP.}$
A_O
$90-2\phi$
xR
F_F
$\frac{3}{2}\phi$
(1-x) R

uniform. This will evidently not be the case in reality, since there will be a concentration of force at the base of the chip, which gradually tapers off to zero at the uppermost point of the apparent contact area.

(3) The distribution of the resultant force, R, along the plane of shear, or base of the chip, is taken to be uniform. In all probability, this will seldom be exactly true.

Although the above assumptions are only approximations to reality, they should give us a correct qualitative picture of the part played by friction in the formation of a built-up edge in cutting. On the basis of the above three approximations and the geometry of Fig. 12, the expression for the shear stress, S_f, on the expected plane of failure is found to be, approximately,

$$S_f = S'\left[\frac{1}{k \sin^2 \phi} - \frac{\sin(3/2\,\phi - \alpha)}{\sin \phi}\right] \qquad (13)$$

where $k = \frac{A_{ap}}{A_o}$ = ratio of area of apparent contact between tool and chip to area of cut,

and S', ϕ, and α are as already defined. The derivation of equation 13 is given in appendix E.

Since the chip metal has been subjected to considerable cold-work in passing through the plane of shear, it is to be expected that S_f must ordinarily be somewhat greater than S', or S_f/S' must be somewhat greater than unity, for a built-up edge to occur.

According to equation 13, if α has a given value and k is assumed to be a constant, then S_f, the stress on the expected plane of failure in the chip will increase as ϕ is decreased, or, according to equation 7, as μ is increased. This statement is illustrated by Fig. 13, in which values of S_f/S', as calculated from equation 13, are plotted against values of μ obtained with the help of equation 7, assuming k = 2.5, and a value of 15° for the rake angle, α. It may be seen that S_f/S' rapidly climbs to a value greater than unity, with increasing μ. This qualitative result of equation 13 is in agreement with experimental observations, as will be described.

Under ordinary circumstances, the face of a tool waiting to take a cut through metal will be covered with an adsorbed film of various "oily" materials, as pointed out for the case of surfaces in general, earlier in the paper. The low shear strength of such a film insures a low value of coefficient of friction between tool face and chip, when a cut is begun. However, the avid adsorption forces of the freshly formed chip surface

Fig. 13. Qualitative relationship between shear stress in chip and coefficient of friction between chip and tool.

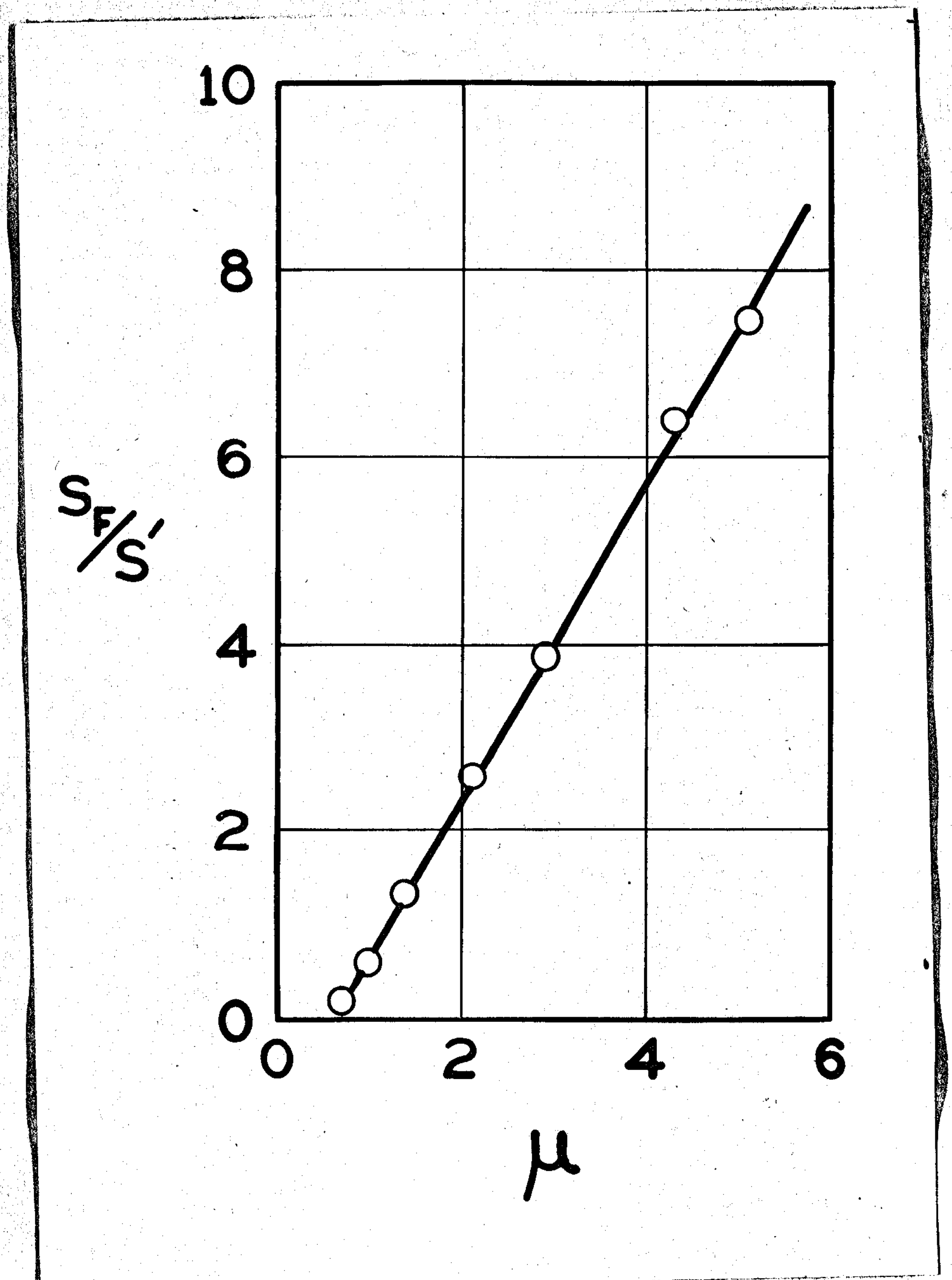


Fig. 13

which slides up the tool, progressively rob the face of its protective, low shear strength film. (It is to this fact that the friction specimens prepared by machining, described earlier in the paper, owe their great cleanliness. The nose of the tool producing such surfaces is completely purged of all organic contaminants by the very first cuts taken on such a specimen.) Because of the removal of the low shear strength film from the tool face, the coefficient of friction between chip and tool increases very rapidly as the cut progresses. When the value of μ has become sufficiently high, a built-up edge is observed to appear, just as predicted by equation 13.

The behaviour described above is illustrated by the series of photomicrographs, Figs. 14(a), (b), and (c). Each of these presents, from top to bottom, a photomicrograph of a chip being removed, a profile of the surface produced, and a plan view of the same surface, respectively. Fig. 14(a) shows the conditions existing at the very beginning of a cut on 1020 steel with a dry tool. The absence of any built-up edge and the smoothness of the resulting surface are in harmony with the initial value of 1.1 for the coefficient of friction. Fig. 14(b), taken a little later in the cut shows that shear has already commenced in the chip, as

Figs. 14. (a), (b), (c) Successive portions of dry planing cut on 1020 steel showing (from top to bottom in each case) photomicrograph of chip being removed (Mag 100X), profile of surface produced (Mag 100X), and plan view of surface (Mag 30X). In (a), μ = 1.1. In (b), μ = 2.6. In (c), μ = 3.7. Depth of cut, 0.005". Cutting speed, 5.5 in./min. Rake angle, 15°.

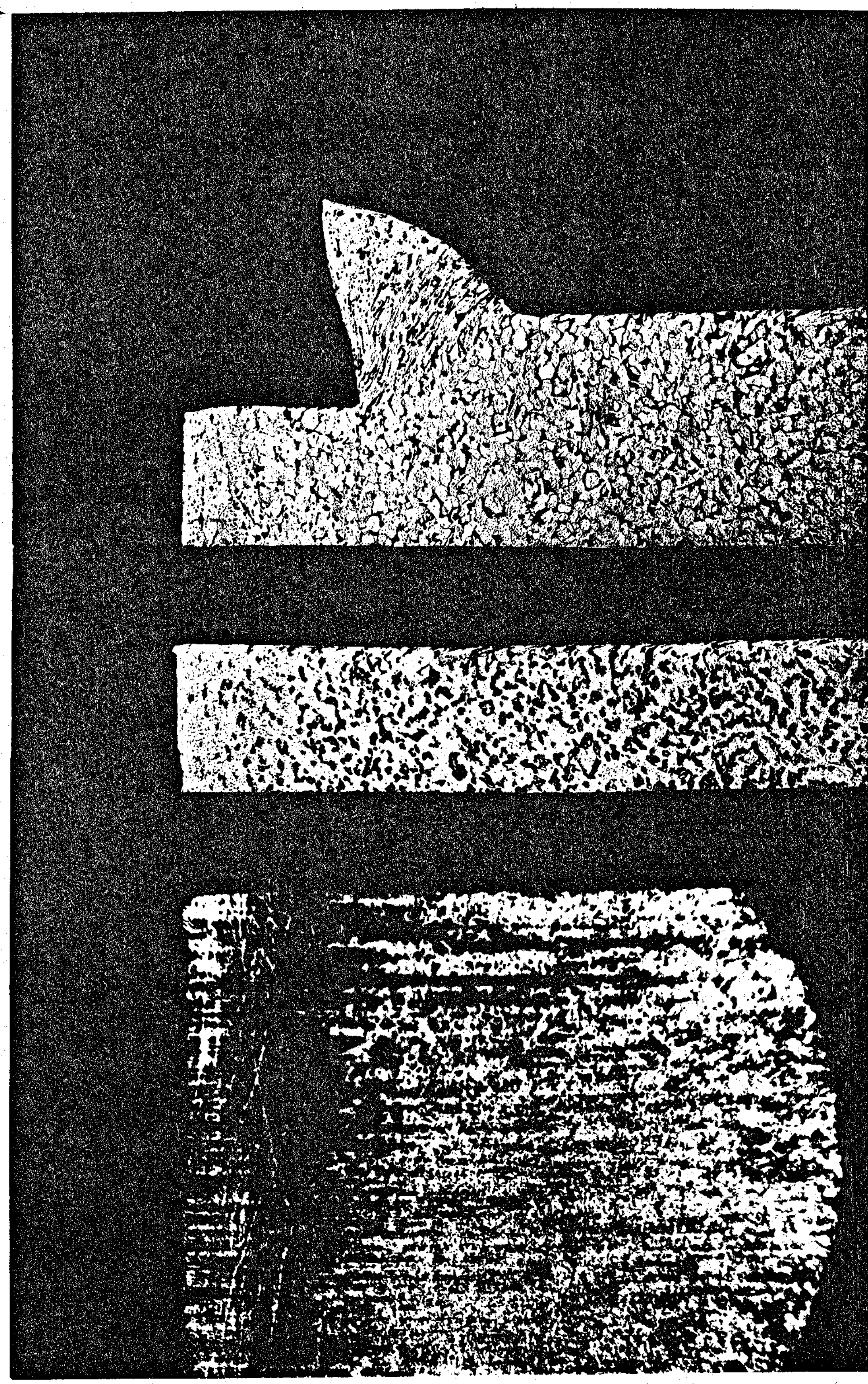


Fig. 14(a)

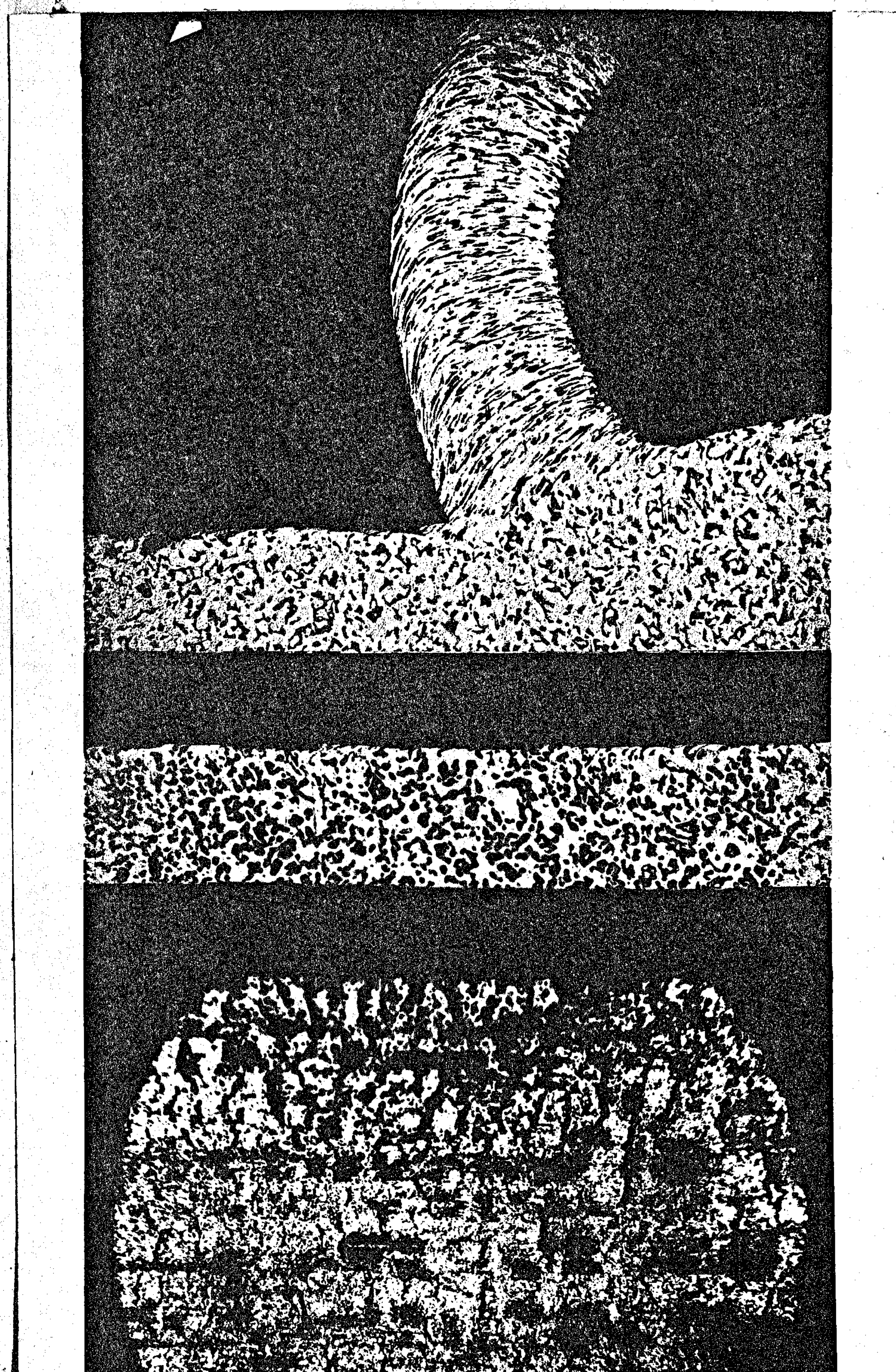


Fig. 14(b)

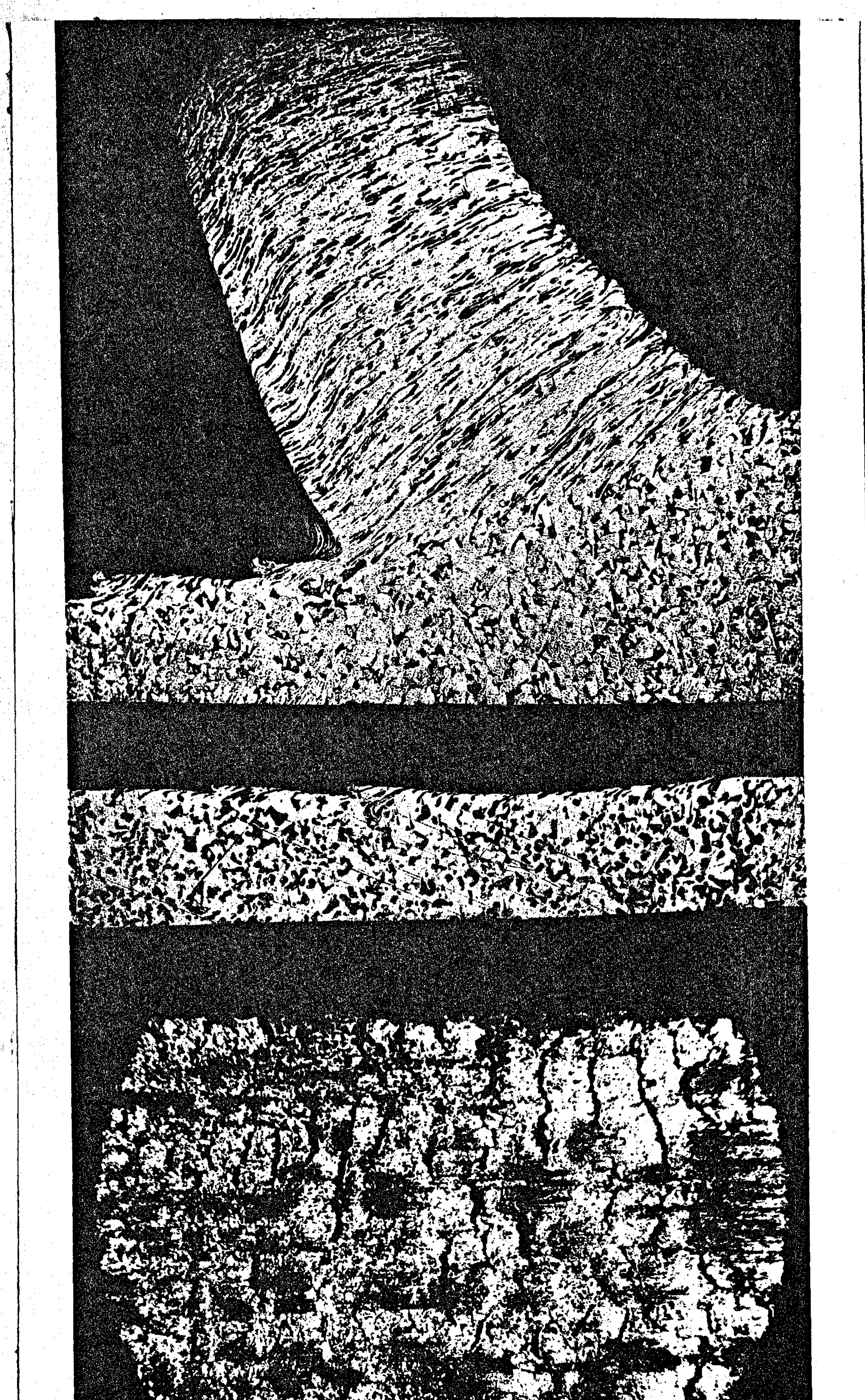


Fig. 14(c)

evidenced by the downward curvature in the lines of crystal deformation in the chip, next to the tool face. The resulting roughness of surface is evident. The coefficient of friction has here reached a value of 2.6. In Fig. 14(c) taken still later in the cut, a fully developed built-up edge is seen , and the resulting extreme roughness of surface is apparent. Here the coefficient of friction has climbed to the still higher value of 3.7. This value of μ, as well as the preceding two values, was obtained from the photomicrograph by measuring the shear angle and applying equation 7.

From a comparison of Figs. 14(a) and (b), it seems probable that shear within the chip must have commenced at a value of coefficient of friction in the vicinity of 2. This corresponds to a hypothetical value of S_f/S' of about 2, according to Fig. 13. It appears that, in this case at least, even the quantitative results given by equation 13 are not unreasonable.

CONCLUSION

The foregoing theoretical treatment of the metal cutting process shows the great need for a low value of coefficient of friction between chip and tool, if fine finishes and maximum efficiency of metal removal are to be obtained. The theoretical treatment of friction given in this paper presents the variables controling static friction and shows the part played by each. Since sliding is ordinarily a stick-slip process, as already pointed out, it is evident that the variables of static friction will be operative in kinetic friction also. The coefficient of friction can be lowered by decreasing S and θ and by increasing H. Therefore, these qualitative findings of the friction investigation can be directly applied to the metal cutting process to obtain improved finishes and greater economy of metal removal. Preliminary laboratory tests have confirmed this idea. Practical ways and means of applying this knowledge are already being studied, as for instance in the investigation being conducted by M. C. Shaw, previously mentioned. The author hopes to be able to turn his efforts toward other attacks on this problem of reducing the coefficient of friction in metal cutting, in the near future.

ACKNOWLEDGMENTS

The author wishes to express his genuine gratitude to the Cincinnati Milling Machine Co. for the financial support and progressive research attitude which have enabled him to carry out the investigation herein described.

He gratefully acknowledges, in this work, the very helpful counsel given him by Doctors Soller and Flory, and other members of the staff of the Graduate Department of Applied Science; and by Messrs. Hans Ernst, M. Martellotti, and other members of the research department of the Cincinnati Milling Machine Co.; also, with appreciation, the genuine assistance given him by his alternates in this cooperative plan, - - Messrs. A. O. Fagerlund and M. C. Shaw.

REFERENCES

(1) Ernst and Martellotti, Mechanical Engineering, Vol. 57 (1935) Page 478

(2) American Society for Metals 1938 Symposium, "Machining of Metals" Page 1

(3) F. P. Bowden and L. Leben, Proc. Roy. Soc., Vol. 169-A, Page 371 (1939)

(4) F. P. Bowden and D. Tabor, Proc. Roy. Soc., Vol. 169-A, p. 391, (1939)

(5) H. O'Neill, "The Hardness of Metals and Its Measurement", 1934, Chapman and Hall, pp. 131-134

(6) J. J. Bikerman and E. K. Rideal, Phil. Mag. Vol. 27, Page 687, (1939)

(7) H. O'Neill, loc. cit., pp. 17, 18

(8) N. K. Adams, "Physics and Chemistry of Surfaces" second edition, 1938, Oxford, pp. 174-176

(9) N. K. Adams, ibid., pp. 219, 220

(10) H. O'Neill, loc. cit., pp. 107 and 133

(11) M. E. Merchant, Jour. App. Phys. Vol 11, No. 3, p. 230 (1940)

(12) For bibliography see N. K. Adams, loc. cit., p. 227

(13) H. O'Neill, loc. cit., p. 7; for Bi see p. 133

(14) See, for instance, F. P. Bowden and T. P. Hughes, Proc. Roy. Soc., Vol 172-A, p. 283, (1939)

(15) M. Kronenberg, manuscript: "Treatise on Milling", Cincinnati Milling Machine Co.

(16) M. C. Shaw, manuscript: "An Investigation of the Mechanism of Cutting Fluid Action", Graduate Dept. of Applied Science, University of Cincinnati

APPENDIX

APPENDIX A - Derivation of expression for μ.

Fig. 6 represents a magnified, exaggerated section through two rough contacting surfaces at an "average" point of actual contact. From the figure

$$F/N = \tan(\Upsilon + \theta)$$

But, by definition,

$$\mu = F/N$$

Therefore,

$$\mu = \tan(\Upsilon + \theta) = \frac{\tan\Upsilon + \tan\theta}{1 - \tan\Upsilon \tan\theta}$$

But, from Fig. 9

$$\tan\Upsilon = F'/N'$$

Now, from the reasoning on which the approximate equation, (1), is based, it follows that, actually,

$$N' = A'\ H$$

and, from the reasoning preceding equation (2), that

$$F' = A'\ S$$

Therefore,

$$\tan T = \frac{A'S}{A'H} = \frac{S}{H}$$

or

$$\mu = \frac{S/H + \tan \theta}{1 - S/H \tan \theta}$$

For the values of S/H and θ associated with ordinary surfaces, the denominator of the right hand side of the above equation is not greatly different from unity. Therefore, to a first approximation,

$$\mu = S/H + \tan \theta \qquad \text{(Equation 4, page 17)}$$

It is possible to show that this approximation is equivalent to assuming that the force of friction, F, has no effect on the size of the contact area.

When θ is neglegible, either of the two above equations reduces to

$$\lim_{\theta \to 0} \mu = S/H \qquad \text{(Equation 5, page 17)}$$

APPENDIX B - Derivation of therdynamic expression for S.

Designate the free energy of the solid by F_s, the free energy of the liquid by F_L, and the shear stress in the solid by p'. Then

$$dF_S = \left(\frac{\partial F_S}{\partial T}\right)_p dT + \left(\frac{\partial F_S}{\partial p'}\right)_T dp'$$

and

$$dF_L = \left(\frac{\partial F_L}{\partial T}\right)_p dT + \left(\frac{\partial F_L}{\partial p'}\right)_T dp'$$

However, since p' is a type of stress which cannot act on the liquid, as pointed out in th body of the paper, it follows that

$$\left(\frac{\partial F_L}{\partial p'}\right)_T = 0$$

When solid and liquid are in equilibrium

$$dF_S = dF_L$$

or

$$\left(\frac{\partial F_S}{\partial T}\right)_p dT + \left(\frac{\partial F_S}{\partial p'}\right)_T dp' = \frac{\partial F_L}{\partial T}_p dT$$

Now

$$\left(\frac{\partial F_S}{\partial T}\right)_p = -\sigma_S \quad \text{and} \quad \left(\frac{\partial F_L}{\partial T}\right)_p = -\sigma_L$$

where σ = entropy

Also $\left(\frac{\partial F_S}{\partial p'}\right)_T = V_S$, where V_S is the specific volume of the solid.

therefore,

$$-\sigma_S \, dT + V_S \, dp' = -\sigma_L \, dT$$

or,

$$dp' = \frac{\sigma_S - \sigma_L}{V_S} \, dT = \rho \, (\sigma_S - \sigma_L) \, dT,$$

since $1/V_S = \rho$

Now $\sigma_S - \sigma_L = - \quad = -\frac{\Delta H}{T}$

But, in accordance with the third of the three approximations stated on page 21

$\Delta H = L/3.$ Therefore,

$$dp' = -\frac{L}{3} \rho \frac{dT}{T}$$

$$\int_0^S dp' = -\int_{T_m}^{T} \frac{L}{3} \rho \frac{dT}{T}$$

Then, in accordance with the first two approximations

$$\int_0^S dp' = \frac{L}{3} \rho \int_{T_m}^{T} \frac{dT}{T}$$

or $$S = -\frac{L}{3}\rho \ln \frac{T}{Tm} = \frac{L}{3}\rho \ln \frac{Tm}{T}$$

Converting to the units used in the body of the paper gives

$$S = .427\,\frac{L}{3}\rho \ln \frac{Tm}{T}$$ (Equation 6, p. 21)

In this derivation the effect of the constant hardness pressure, H, has not been taken into account. Its only effect of S will be a shifting of the melting point, Tm, from the usual value determined at atmospheric pressure. Separate calculations have shown that if H is of the order of magnitude of S, its effect on S is negligible.

APPENDIX C. - Derivation of relation between μ and ϕ

Fig. 11 is a schematic diagram of the force system acting on the nose of a cutting tool. F_s represents the force acting along any plane of shear determined by ϕ

Let A_s = area of plane of shear

Then the shear stress on that plane is

$$S_A = \frac{F_s}{A_s}$$

But, from Fig. 11

$$F_s = R \cos(\phi + \tau - \alpha)$$

$$= R\left[\cos\phi \cos(\tau-\alpha) - \sin\phi \sin(\tau-\alpha)\right]$$

and $$A_s = \frac{A_o}{\sin\phi}$$

Therefore

$$S_A = \frac{F_s}{A_s} = \frac{R}{A_o} \sin\phi \left[\cos\phi \cos(\tau-\alpha) - \sin\phi \sin(\tau-\alpha)\right]$$

$$= \frac{R}{A_o}\left[\sin\phi \cos\phi \cos(\tau-\alpha) - \sin^2\phi \sin(\tau-\alpha)\right]$$

To find the plane on which S_A is a maximum for some given value of R, A_o, τ, and α, we set

$$0 = \frac{\partial S_A}{\partial \phi} = \frac{R}{A_o}\left[\cos^2\phi \cos(\tau-\alpha) - \sin^2\phi \cos(\tau-\alpha) - 2\sin\phi \cos\phi \sin(\tau-\alpha)\right]$$

Since $\frac{R}{A_o}$ is not zero for any finite depth of cut,

$$\cos^2\phi \cos(\tau-\alpha) - \sin^2\phi \cos(\tau-\alpha) - 2\sin\phi \cos\phi \sin(\tau-\alpha) = 0$$

$$\cos^2\phi - \sin^2\phi = \frac{2\sin\phi \cos\phi \sin(\tau-\alpha)}{\cos(\tau-\alpha)}$$

$$\frac{\cos^2\phi - \sin^2\phi}{2\sin\phi \cos\phi} = \tan(\tau-\alpha)$$

or $\cot 2\phi = \tan(\tau - \alpha)$

Therefore $90 - 2\phi = \tau - \alpha$, since τ, α, and ϕ are always acute angles.

Then $\tau = 90 - 2\phi + \alpha$

From Fig. 11 it may be seen that $F/N = \tan \tau$

Therefore

$$\mu = \tan \tau = \cot(2\phi - \alpha) \qquad \text{(Equation 7, p. 49)}$$

APPENDIX D = Derivation of expression for specific cutting pressure.

By definition

$$P = F_c/A_o$$

Now, from the geometry of Fig. 11, it may be seen that

$$F_c = R \cos(\tau - \alpha)$$

But, as shown in appendix c,

$$\tau - \alpha = 90 - 2\phi$$

$$F_c = R \cos(90 - 2\phi) = R \sin 2\phi = 2R \sin\phi \cos\phi$$

Now, from Fig. 11,

$$R = \frac{F_s}{\cos(\pi - \alpha + \phi)}$$

$$= \frac{F_s}{\cos(90 - 2\phi + \phi)}$$

$$= \frac{F_s}{\sin \phi}$$

Therefore $F_c = 2 F_s \dfrac{\sin \phi \cos \phi}{\sin \phi} = 2 F_s \cos \phi$

But $F_s = A_s S'$

where A_s is the area of the plane of shear, and S' is the shear strength of the material being cut.

From Fig. 11

$$A_s = \frac{A_o}{\sin \phi}$$

Therefore $F_s = \dfrac{A_o S'}{\sin \phi}$

and thus

$$F_c = 2 A_o S' \frac{\cos \phi}{\sin \phi} = 2 A_o S' \cot \phi$$

or $P = F_c/A_o = 2 S' \cot \phi$ (equation 8, p. 52)

APPENDIX E - Derivation of qualitative expression for shear stress on expected plane of failure in chip.

The expected plane of failure makes an angle of approximately 3/2 ϕ with a perpendicular to the work surface. This plane is represented by the dotted line extending from the tool face down to the plane of shear and denoted by A_f, in the chip diagram, Fig. 12. The shear stress on this plane is evidently

$$S_f = F_f / A_f$$

where A_f is the area of the expected plane of failure, and F_f is the component of χR along A_f, as shown in Fig. 12. From the figure

$$F_f = \chi R \cos\left[(90 - 3/2\,\phi) - (90 - 2\,\phi)\right]$$

$$= \chi R \cos \phi/2$$

Now $$\chi = \frac{\chi A_s}{A_s} = \frac{A_s - (1-\chi)A_s}{A_s} = 1 - \frac{(1-\chi)A_s}{A_s}$$

where A_s represents the area of the plane of shear, as previously. From the figure, $A_s = \dfrac{A_o}{\sin \phi}$

and $$\frac{(1-\chi)A_s}{\sin(3/2\,\phi - \alpha)} = \frac{A_{ap}}{\sin(90 - \phi/2)}$$ (law of sines)

But we have assumed $A_{sp} = k A_o$

Then $$(1-\chi)A_s = k A_o \frac{\sin(3/2\,\phi - \alpha)}{\cos \phi/2}$$

Therefore $$\chi = 1 - \frac{(1-\chi)A_s}{A_s} = 1 - \frac{k A_o \frac{\sin(3/2\,\phi-\alpha)}{\cos \phi/2}}{\frac{A_o}{\sin \phi}}$$

$$= 1 - k \frac{\sin(3/2\,\phi-\alpha) \sin \phi}{\cos \phi/2}$$

Also, from Fig. 12,

$$\chi R = \frac{\chi F_s}{\cos(90-2\phi + \phi)} = \frac{\chi F_s}{\sin \phi}$$

or $$R = \frac{F_s}{\sin \phi} = \frac{A_s S'}{\sin \phi} = \frac{A_o S'}{\sin^2 \phi}$$

since $F_s = A_s S'$ and $A_o = A_s \sin \phi$

Therefore, returning to the expression for F_f, we have

$$F_f = \chi R \cos \phi/2 = \left[1-k \frac{\sin(3/2\,\phi-\alpha)\sin \phi}{\cos \phi/2}\right] A_o S' \frac{\cos \phi/2}{\sin^2 \phi}$$

For A_f, we have from Fig. 12

$$\frac{A_f}{\sin(90-\phi+\alpha)} = \frac{A_{sp}}{\sin(90 - \phi/2)} = \frac{k A_o}{\sin(90 - \phi/2)}$$

or $$A_f = k A_o \frac{\cos(\phi - \alpha)}{\cos \phi/2}$$

Therefore

$$S_f = \frac{F_f}{A_f} = \frac{\left[1-k\,\frac{\sin\,(3/2\,\phi-\alpha)\sin\,\phi}{\cos\,\phi/2}\right]A_o S'\,\frac{\cos\,\phi/2}{\sin^2\,\phi}}{k\,A_o\,\frac{\cos\,(\phi-\alpha)}{\cos\,\phi/2}}$$

$$= S'\left[\frac{1}{k} - \frac{\sin\,(3/2\,\phi-\alpha)\,\sin\,\phi}{\cos\,\phi/2}\right]\frac{\cos^2\,\phi/2}{\sin^2\,\phi\,\cos(\phi-\alpha)}$$

Now, for usual values of ϕ and α,

$$\frac{\cos^2\,\phi/2}{\cos(\phi-\alpha)} \approx 1 \text{ and } \cos\,\phi/2 \approx 1$$

Therefore,

$$S_f \approx S'\,\frac{1}{k\,\sin^2\,\phi} - \frac{\sin\,(3/2\,\phi-\alpha)}{\sin\,\phi} \quad \text{(equation 13, p. 59)}$$