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I hereby recommend that the thesis prepared under my supervision by Paul Milton Pepper

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AN APPLICATION OF GEOMETRY OF NUMBERS TO A GENERALIZATION OF A CONTINUED FRACTION

A dissertation submitted to the
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PREFACE

In his Geometrie der Zahlen (p. 147 ff.), Minkowski has considered for two linear forms with real coefficients,

$$\xi = \alpha x + \beta y, \quad \eta = \gamma x + \delta y \qquad (\alpha\delta - \beta\gamma = 1),$$

the parallelograms $|\xi| \leq \rho$, $|\eta| \leq \sigma$ which have on each side a lattice point, besides those which may be at vertices, and only the origin in the interior. He found an ordered set, or "chain", of matrices $\cdots, P_{-1}, P_0, P_1, \cdots$ whose determinants are equal to 1 and in the columns of each matrix are found the coordinates of two lattice points on the corresponding parallelogram. This set has a first term if $\eta = 0$ can be satisfied by rational integers x, y, not both zero, and a last term if $\xi = 0$ can be satisfied by rational integers x, y, not both zero.

Set $P_{k-1}^{-1}P_k = T_k$, then T_k is of the form $\begin{Vmatrix} 0, & -e^{(k)} \\ e^{(k)}, & g^{(k)} \end{Vmatrix}$ where $e^{(k)} = \pm 1$ and $g^{(k)}$ is a positive integer. Then P_{k-1} transforms ξ and η into $\Xi = \lambda^{(k-1)}X + e^{(k-1)}\lambda^{(k)}Y$ and $H = -e^{(k-1)}\mu_{k-2}X + \mu_{k-1}Y$, where $\lambda^{(k-1)} > \lambda^{(k)} \geq 0$, $\mu_{k-1} > \mu_{k-2} \geq 0$, $e^{-1} = \pm 1$. The expression

$$\cfrac{1}{g^{(k)} + \cfrac{1}{g^{(k+1)} + \cfrac{1}{g^{(k+2)} + \cdots}}}$$

is the normal continued fraction development of $\lambda^{(k-1)}/\lambda^{(k)}$ and terminates or is infinite according as this ratio is rational or irrational.

If $\xi = x - \theta y$, $\eta = y$, where θ is a real number such that $0 < \theta - g < 1/2$, g being the largest integer in θ, then

$$P_0 = \begin{Vmatrix} 1, & g \\ 0, & 1 \end{Vmatrix} = \begin{Vmatrix} 0, & 1 \\ -1, & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0, & -1 \\ 1, & g \end{Vmatrix} .$$

In this case

$$\theta = g + \cfrac{1}{g^{(1)} + \cfrac{1}{g^{(2)} + \cdots}}$$

is the normal continued fraction development of θ and, in addition,

$$P_k = \begin{Vmatrix} p_{11}^{(k)}, & p_{12}^{(k)} \\ p_{21}^{(k)}, & p_{22}^{(k)} \end{Vmatrix} = \begin{Vmatrix} p_{11}^{(k-1)}, & p_{12}^{(k-1)} \\ p_{21}^{(k-1)}, & p_{22}^{(k-1)} \end{Vmatrix} \cdot \begin{Vmatrix} 0, & -(-1)^k \\ (-1)^k, & g^{(k)} \end{Vmatrix} .$$

so that $p_{12}^{(k)}/p_{22}^{(k)}$ is the kth convergent in the normal continued fraction development of θ.

In either case (whether or not $\alpha = 1$, $\beta = -\theta$) if β/α is irrational, then $|\alpha p_{12}^{(k)} + \beta p_{22}^{(k)}|$ tends monotonically to zero as k increases, whereas, at the same time, $|p_{12}^{(k)}|, |p_{22}^{(k)}|$ are relatively prime integers which become infinite with k. The ratios $-p_{12}^{(k)}/p_{22}^{(k)}$, $(k = 0, 1, \cdots, \infty)$ are rational fractions in their lowest terms and they tend toward β/α as a limit. If β/α is rational, then $\alpha p_{12}^{(k)} + \beta p_{22}^{(k)}$ actually vanishes for some value $\bar{k}$ of k, since the continued fraction terminates.

Similarly, going toward the other end of the chain, the value of $|\gamma p_{11}^{(k)} + \delta p_{21}^{(k)}|$ tends monotonically toward zero and actually becomes equal to zero if and only if γ/δ is rational.

We consider in this paper the three dimensional analogue for which $\xi = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3$, $\eta = \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3$, $\zeta = \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3$, where α_{ij} are real numbers such that no one of the equations $\xi = 0$, $\eta = 0$, or $\zeta = 0$ is satisfied by integers x_1, x_2, x_3, not all zero. The regions corresponding to the parallelograms are now parallelepipeds.

In this connexion, there are two problems:

(1) To develop a process for obtaining all the parallelepipeds $|\xi| \leqq \rho$, $|\eta| \leqq \sigma$, $|\zeta| \leqq \tau$, of such a nature that on each face there is a lattice point other than those which may be on the edges, and such that the origin is the only lattice point in the interior. (That is, to obtain a chain of third order matrices.)

(2) (a) To determine sequences of lattice points for which one of the linear forms tends to zero.

(b) To determine sequences of lattice points for which two of the linear forms tend simultaneously toward zero.

Minkowski (*Annales de l'École Normale Supérieure*, Ser. 3, XIII, pp. 41-60) stated the results for problem (1) without giving proofs. This problem is solved in Chapters II and III of this thesis.

In Chapter IV both parts of problem (2) are solved and a second algorithm is given which facilitates the determination of sequences of lattice points satisfying the conditions of part (b).

CHAPTER I

INTRODUCTION

Hermite[1] has given an elegant proof of a theorem of Tchebychef. This theorem we now propose to generalize following the methods that have their beginnings in the *Geometrie der Zahlen* of Minkowski. This remarkable work was dedicated to Hermite, as its inspiration was derived from the above mentioned paper among others of the very celebrated French mathematician. Professor Hilbert in his memorial eulogy of Minkowski[2] makes the following quotation from Hermite regarding the work of Minkowski, "Ch. Hermite, damals der Senior der französischen Mathmatiker, hatte von Anbeginn die zahlentheoretischen Arbeiten Minkowskis mit höchstem Interesse und lebhaftester Freude verfolgt. Es ist rührend, wie rückhaltlos er die Vorzüge der Minkowskischen Methode gegenüber seinen eignen Entwicklungen anerkennt, als Minkowski ihm die eben besprochenen Resultate mitteilt. *Au premier*

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1. Ch. Hermite, *OEuvre*, III, p. 513.

2. H. Minkowski, *Gesammelte Abhandlungen*, I, p. xiv.

coup d'oeil j'ai reconnu, so schreibt Ch. Hermite in einem der an Minkowski gerichteten Briefe, *que vous avez été bien au delà de mes recherches en nous ouvrant dans le domaine arithmétique des voies toutes nouvelles*. Und in einem zwei Jahre späteren Briefe vom November 1892 heiszt es: *Je me sens rempli d'étonnement et de plaisir devant vos principes et vos résultats, ils m'ouvrent comme un monde arithmétique entièrement nouveau, où les questions fondamentales de notre science sont traitées avec un éclatant succès auquel tous les géomètres rendront hommage. Vous voulez bien, Monsieur, -- et je vous en suis sincèrement reconnaissant -- rapporter à mes anciennes recherches le point de départ de vos beaux travaux, mais vous les avez tant dépassées qu'elles ne gardent plus d'autre mérite que d'avoir ouvert la voie dans laquelle vous êtes entré*." Hermite was much too modest regarding his own work.

Minkowski stated without proof, the following theorem[3]:

Lorsque $\{a, g, \ell\}$ *est un parallélépipède extrême*[4]

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3. *Ibid.*, pp. 281-282, "Zur Theorie der Kettenbrüche." First published in *Annales Scientifiques de l'École Normale Supérieure*, Ser. 3, XIII, 1896, pp. 41-60.

4. See below, Definition 5, *extreme parallelepiped*, or *äuszerstes Parallelepipedum*.

pour ξ, η, ζ, on aura toujours

(1) $$ag\ell < \Delta ;$$

$\{a, g, \ell\}$ contient exactement un point du réseau sur chaque face et ces points du réseau ont des coordonnées égales et de signes contraires pour les faces à l'opposite les unes des autres.

On peut alors, et cela d'une seule manière, trouver trois points du réseau

$$r,\ s,\ t;\ r',\ s',\ t';\ r'',\ s'',\ t''$$

sur trois faces non opposées

$$\xi = \varepsilon a, \quad \eta = \varepsilon' g, \quad \zeta = \varepsilon'' \ell,$$

tels que

$$\varepsilon \varepsilon' \varepsilon'' = +1$$

et tels que, lorsque le système $\varepsilon\xi, \varepsilon'\eta, \varepsilon''\zeta$ est transformé par l'effet de

$$P = \left\| \begin{matrix} r, & r', & r'' \\ s, & s', & s'' \\ t, & t', & t'' \end{matrix} \right\| \qquad \text{en} \quad \Phi = \left\| \begin{matrix} a, & \pm b, & \pm c \\ \pm f, & g, & \pm h \\ \pm j, & \pm k, & \ell \end{matrix} \right\|$$

les grandeurs $a, b, c, f, g, h, j, k, \ell$ soient toutes positives, leurs signes étant répresentés par un des six systèmes suivants:

I	II	III	IV	V	VI
+ + +	+ − −	+ − −	+ + −	+ − +	+ − −
− + −	+ + +	− + −	− + +	+ + −	− + −
− − +	− − +	+ + +	+ − +	− + +	− − +

Le déterminant de P sera, dans les cas I-V, égal à 1 et dans le cas VI restant égal à zéro, et l'on aura

(2) $a > b,\ a > c;\quad g > h,\ g > f;\quad \ell > j,\ \ell > k;$

et de plus, dans les cas indiqués ci-dessous par leurs numéros d'ordre, on aura les conditions suivantes:

I	II	III
$b + c > a$	$h + f > g$	$j + k > \ell$
$f > h$ ou $j > k$	$k > j$ ou $b > c$	$c > b$ ou $h > f$

IV	V
$b > c$ ou $h > f$ ou $j > k$	$c > b$ ou $f > h$ ou $k > j$

VI
$b + c = a,\ h + f = g,\ j + k = \ell.$

Using geometric concepts which have been employed in other connections by Minkowski, we shall prove this theorem.[5] Besides, we shall derive completely the

5. Marie Zeisel proved this theorem and gave a development of two parts of the first algorithm. He stated the third algorithm in its correct form and gave an example to illustrate the case in which Minkowski's algorithm failed. (*Kaiserliche Akademie der Wissenschaften (Wien), Sitzungsberichte*, CXXVI, 1917, "Zur Minkowski'schen Parallelepipedapproximation.") (Footnote continued on next page.)

algorithms stated by Minkowski in the above cited paper. A correction is made in the last algorithm. We shall also state and prove several other theorems which give a clearer understanding of the entire situation.

Several concepts and definitions necessary in this work will now be introduced. A Cartesian coordinate system in a three dimensional space will be used, the coordinates represented by x_1, x_2, x_3.

Definition 1. A lattice is the totality of all points whose coordinates are integers.

Definition 2. A lattice point is, then, any one of the points of the lattice.

By employing Minkowski's results for lattice octahedrons we shall shorten one portion of the proof given by Marie Zeisel; also we shall give a complete detailed derivation of all three algorithms.

All proofs in this paper had been worked out by me and presented in seminar at the University of Cincinnati before I discovered the existence of this paper by Marie Zeisel.

As is to be expected, certain parts of my research are similar to the corresponding portions of the above mentioned paper; but wherever possible, I have applied the more general theory developed by Minkowski. To his proofs of this more general theory I refer in this first chapter as I introduce the necessary concepts and definitions.

Having proved the above theorem of Minkowski and having derived the various algorithms, I correlate the material by means of some existence theorems.

Definition 3. If ξ, η, ζ are three independent linear expressions with real coefficients homogeneous in the real variables x_1, x_2, x_3 and having a determinant $\Delta \neq 0$, then $|\xi| \leqq \rho$, $|\eta| \leqq \sigma$, $|\zeta| \leqq \tau$, where ρ, σ, τ are three arbitrary positive constants, is a parallelepiped with center of symmetry at the origin and faces parallel in pairs to the planes $\xi = 0$, $\eta = 0$, and $\zeta = 0$. Such a parallelepiped shall be represented by the symbol $\{\rho, \sigma, \tau\}$. The quantities ρ, σ, τ will be called the parameters of the parallelepiped.

Definition 4. Further, if $\{\rho, \sigma, \tau\}$ contains no lattice point other than the origin in its interior it shall be called free.

Definition 5. The parallelepiped $\{\rho, \sigma, \tau\}$ shall be called extreme if it is free and if, at the same time, it has on each of its faces at least one lattice point other than those which may be contained in the edges. The existence of at least one extreme parallelepiped will be established in Chapter II. An extreme parallelepiped is, then, such that no one of its parameters can be increased without introducing lattice points into the interior of the parallelepiped.

Definition 6. The operation of taking a smaller positive value for ρ shall be called lowering the ξ-faces, and that of taking a larger value for ρ shall be called raising the ξ-faces. Similar definitions will hold for lowering and raising the η- and ζ-faces.

In all that follows, one restricts ξ, η, and ζ to be independent linear forms of such a nature that no one of the equations $\xi = 0$, $\eta = 0$, or $\zeta = 0$ is satisfied by any lattice point except the origin, and it may be assumed without essential restriction that $\Delta > 0$. Due to the restriction placed on the type of linear forms, there can be at most one lattice point in any plane ξ = const., η = const., or ζ = const. There are exactly six lattice points on the surface of each extreme parallelepiped, one and only one on each face.

Definition 7. If $\{\rho, \sigma, \tau\}$ is an arbitrary extreme parallelepiped for ξ, η, and ζ, the lattice points on the η- and ζ-faces must lie in different planes ξ = const., no one of these planes being $\xi = 0$. Lower the ξ-faces until they touch a first pair of lattice points in the η- or ζ-faces. If these lattice points are the ones in the η-faces, the η-faces can now be raised and leave the parallelepiped still free. On the other hand, if these points are in the ζ-faces, then the ζ-faces can be raised instead. Raise the possible pair of faces until the parallelepiped becomes extreme. The new extreme parallelepiped thus uniquely determined will be called the ξ-neighbor of $\{\rho, \sigma, \tau\}$. Similar definitions are made for an η- and a ζ-neighbor of $\{\rho, \sigma, \tau\}$. It will be seen that the restrictions on ξ, η, and ζ require that there be an infinity of extreme parallelepipeds, if the existence of a single one be established. For each one, without exception, will possess three neighbors; the sequence obtained by taking successive ξ-neighbors, starting form an arbitrary extreme parallelepiped, is a sequence in which no two of the elements are the same, for the first parameter is less in each element than in the one which precedes it. The totality of the extremes belonging to ξ, η, and ζ is called the chain of extreme parallelepipeds for ξ, η, and ζ.

Definition 8. An octahedron, symmetric about the origin and having each vertex at a lattice point, will be called a lattice octahedron[6] if it contains on its surface and in its interior no lattice points other than the origin and its vertices.

For such lattice octahedrons, the determinant of coordinates of three vertices $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, no two of which

6. H. Minkowski, Diophantische Approximationen, p. 97.

are symmetric to each other with respect to the origin, is ± 1 or ± 2. If the determinant is ± 2, then each point of the form

$$p = (\gamma_1 + \frac{\delta}{2})p_1 + (\gamma_2 + \frac{\delta}{2})p_2 + (\gamma_3 + \frac{\delta}{2})p_3, \quad (\gamma_i, \delta, \text{ integers})$$

is a lattice point.[7]

Definition 9. If $f(x_1, x_2, x_3)$ is a function satisfying the following conditions

(1) $f(x_1,x_2,x_3) > 0 \qquad ((x_1,x_2,x_3) \neq (0,0,0))$,

(2) $f(tx_1,tx_2,tx_3) = tf(x_1,x_2,x_3) \qquad (t \geqq 0)$,

(3) $f(x_1 + y_1, x_2 + y_2, x_3 + y_3) \leqq f(x_1,x_2,x_3) + f(y_1,y_2,y_3)$,

(4) $f(-x_1,-x_2,-x_3) = f(x_1,x_2,x_3)$,

then $f(x_1,x_2,x_3) \leqq 1$ defines a convex body R with center of symmetry at the origin.

The function $f(x_1,x_2,x_3)$ is to be called a generalized distance function. Minkowski has shown the existence of such a function for an arbitrary convex body with center of symmetry at the origin.[8]

If a lattice octahedron has all its vertices on the surface of R and if three of these vertices non-coplanar

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7. Ibid., p. 100.

8. Geometrie der Zahlen, p. 36. Symmetry about the origin was not assumed in the general theory, consequently, condition (4) does not occur there.

with the origin are denoted by $A_1:(a_{11},a_{12},a_{13})$, $A_2:(a_{21},a_{22},a_{23})$, and $A_3:(a_{31},a_{32},a_{33})$, then the transformation

$$\begin{aligned} x_1 &= a_{11}X_1 + a_{21}X_2 + a_{31}X_3 \\ x_2 &= a_{12}X_1 + a_{22}X_2 + a_{32}X_3 \quad (\text{det.}|a_{ij}| = \pm 1 \text{ or } \pm 2), \\ x_3 &= a_{13}X_1 + a_{23}X_2 + a_{33}X_3 \end{aligned}$$

transforms $f(x_1,x_2,x_3)$ into $F(X_1,X_2,X_3)$ which satisfies the conditions (1) - (4).

Minkowski has proved the following results:[9]

1) If the determinant $|a_{ij}| = \pm 1$, then R contains in its interior no lattice points (x_1,x_2,x_3) other than the origin, if and only if

$$F(X_1,X_2,X_3) \geqq 1$$

for each of the twenty-two following sets of values for (X_1,X_2,X_3):

$(0, 1, \pm 1)$, $(1, 0, \pm 1)$, $(1, \pm 1, 0)$, $(1, \pm 1, \pm 1)$
$(1, \pm 1, \pm 2)$, $(1, \pm 2, \pm 1)$, $(2, \pm 1, \pm 1)$,

where all possible combinations of signs are to be taken.

2) If the determinant $|a_{ij}| = \pm 2$, than R contains in its interior no lattice points other than the origin, if and only if

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9. Diophantische Approximationen, pp. 102-103, and Gesammelte Abhandlungen, II, p. 13.

$$F(X_1,X_2,X_3) \geqq 1$$

for each of the four following sets of values for (X_1,X_2,X_3)

$$\left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right).$$

CHAPTER II

BASIC PROPERTIES OF EXTREME PARALLELEPIPEDS

Because of its extreme length the theorem of Minkowski stated in Chapter I will be here divided into several theorems. Unless specifically stated otherwise, all theorems will be for ξ, η, ζ of the special type assumed above.

Theorem 2.1 In a free parallelepiped for ξ, η, and ζ the inequality $\rho\sigma\tau < \Delta$ is always satisfied.[1]

The parallelepiped $|\xi| \leqq \rho$, $|\eta| \leqq \sigma$, $|\zeta| \leqq \tau$ is a convex body with center of symmetry at the origin. The volume of such a parallelepiped is $V = 2^3\rho\sigma\tau/\Delta$.[2] Suppose $\rho\sigma\tau > \Delta$, then $V > 2^3$ and the parallelepiped, from a theorem on convex bodies,[3] must contain at least two lattice points other than the origin in its interior. But such is not the case for a free parallelepiped, therefore, $\rho\sigma\tau \leqq \Delta$. Suppose $\rho\sigma\tau = \Delta$, then $V = 2^3$ and

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1. To prove this theorem H. Monkowski refers to another theorem first stated in a letter to Charles Hermite (H. Minkowski, Gesammelte Abhandlungen, p. 266 ff.), and later published in his Geometrie der Zahlen, pp. 102-106.

2. H. Minkowski, Geometrie der Zahlen, p. 68.

3. Ibid., p. 76.

the parallelepiped, if it contains no lattice points other than the origin in its interior, must contain at least fourteen lattice points on its surface.[4] It is impossible to distribute these fourteen lattice points on the six faces of the parallelepiped without having at least one face with two or more lattice points on it. But this is contrary to the hypothesis that none of the planes $\xi = 0$, $\eta = 0$, and $\zeta = 0$ contains other lattice points than the origin.[5] Therefore, the theorem is proved.

It is by this theorem that the existence of at least one extreme parallelepiped can be established. Let $\{\rho_0, \sigma_0, \tau_0\}$ be an arbitrary parallelepiped for ξ, η, and ζ, then $\{\theta\rho_0, \theta\sigma_0, \theta\tau_0\}$ where $0 < \theta < 1$ is a parallelepiped similar to the original and simarly placed about the origin. If one decreases θ toward zero, the three altitudes and also the volume of the parallelepiped tend to zero. When θ is equal to a sufficiently small constant θ_0 the parallelepiped will lie entirely within the cube with center at the origin and edges parallel to

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4. Ibid., p. 85.

5. See Theorem 2.2 below.

the three axes. The parallelepiped $\{\theta_0\rho_0, \theta_0\sigma_0, \theta_0\tau_0\}$ is then free, since it can contain no lattice point other than the origin in its interior. Keeping the last two parameters constant, increase the first parameter of the parallelepiped $\{\rho, \theta_0\sigma_0, \theta_0\tau_0\}$ continuously from $\rho = \theta_0\rho_0$ until the parallelepiped first contains on the ξ-faces lattice points which are not on the η- or ζ-faces; the parallelepiped remains free. Let this corresponding value for ρ be ρ_1. Such a number ρ_1 exists, for the increase of the first parameter to the value $\rho = \Delta/(\theta_0^2\sigma_0\tau_0)$ insures that the parallelepiped is no longer free. Then $\rho_1 < \Delta/(\theta_0^2\sigma_0\tau_0)$. Increase the second parameter of $\{\rho_1, \sigma, \theta_0\tau_0\}$ continuously from $\sigma = \theta_0\sigma_0$ until the η-faces first contain lattice points which are in neither the ξ- nor the ζ-faces, the parallelepiped remaining free. Let σ_1 be the corresponding value of the second parameter σ. Then $\sigma_1 < \Delta/(\theta_0\rho_1\tau_0)$. Finally, increase the third parameter of $\{\rho_1, \sigma_1, \tau\}$ continuously from $\tau = \theta_0\tau_0$ until the ζ-faces first contain lattice points which are in neither the ξ- nor the η-faces, the parallelepiped remaining free. Let τ_1 be the new value of the third parameter τ. Then $\tau_1 < \Delta/(\rho_1\sigma_1)$, and the parallelepiped $\{\rho_1, \sigma_1, \tau_1\}$ is extreme as well as free.

It is seen at once how Theorem 2.1 applies to prove also the existence of the three neighbors of a given extreme parallelepiped. This theorem insures that the parameters do not become infinite. The three neighbors, however, do not always exist when one of the equations $\xi = 0$, $\eta = 0$, or $\zeta = 0$ is satisfied by other lattice points besides the origin; for in this case diminution of one of the parameters to zero may be necessary before another can be increased; consequently, when one decreases this parameter to zero, the entire domain is situated in a plane and has no interior, both of the other parameters can then be increased indefinitely. The region corresponding to the neighbor is no longer a parallelepiped but an entire plane.

Theorem 2.2 *The parallelepiped* $\{a, g, \ell\}$ *contains, besides the origin, exactly six lattice points, one on each face, on opposite faces lattice points with coordinates the negatives of each other.*

Since ξ, η, and ζ are homogeneous linear forms, the absolute value of any one of these linear forms is the same for a point $\mathfrak{p}$ with coordinates (p_1, p_2, p_3) as it is for the point $-\mathfrak{p}$ with coordinates $(-p_1, -p_2, -p_3)$.

Therefore, if $\mathfrak{p}$ is a lattice point on the surface of $\{a, g, \ell\}$, so also is $-\mathfrak{p}$.

There are six faces to the parallelepiped and each can contain at most one lattice point; for suppose two lattice points $\mathfrak{p}$ and $\mathfrak{q}$ were in the same face, then the point $\mathfrak{p} - \mathfrak{q}$ with coordinates $(p_1 - q_1, p_2 - q_2, p_3 - q_3)$ would make one of the linear forms ξ, η, or ζ vanish. But this is contrary to the hypothesis that the origin is the only lattice point in any of the planes $\xi = 0$, $\eta = 0$, or $\zeta = 0$.

From the definition of an extreme parallelepiped, each face must contain at least one lattice point which is in no other face. Therefore, each face contains exactly one lattice point, and this lattice point is thus not on an edge. The whole surface then contains exactly six lattice points.

Theorem 2.3 One can always select from the lattice points on $\{a, g, \ell\}$ in one and only one way three lattice points $\mathfrak{p}_1: (p_1^{(1)}, p_2^{(1)}, p_3^{(1)})$, $\mathfrak{p}_2: (p_1^{(2)}, p_2^{(2)}, p_3^{(2)})$, and $\mathfrak{p}_3: (p_1^{(3)}, p_2^{(3)}, p_3^{(3)})$ on respective faces $\xi = e_1 a$, $\eta = e_2 g$, and $\zeta = e_3 \ell$,[6] not opposite one another, so that $e_1 e_2 e_3 = +1$ and so that

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6. The quantities e_1, e_2, e_3, are equal to ± 1, since the three planes concerned are those containing faces of $\{a, g, \ell\}$.

when the matrix of coefficients of the linear forms $e_1\xi$, $e_2\eta$, $e_3\zeta$ is transformed by

$$P = \begin{Vmatrix} p_1^{(1)}, & p_1^{(2)}, & p_1^{(3)} \\ p_2^{(1)}, & p_2^{(2)}, & p_2^{(3)} \\ p_3^{(1)}, & p_3^{(2)}, & p_3^{(3)} \end{Vmatrix} \quad \underline{\text{into}}\ \Phi = \begin{Vmatrix} a, & \pm b, & \pm c \\ \pm f, & g, & \pm h \\ \pm j, & \pm k, & \ell \end{Vmatrix},$$

then the numbers a, b, c, f, g, h, j, k, *and* ℓ *will be all positive and the matrix* Φ *will have one of the following six systems of signs:*

I	II	III	IV	V	VI
+ + +	+ − −	+ − −	+ + −	+ − +	+ − −
− + −	+ + +	− + −	− + +	+ + −	− + −
− − +	− − +	+ + +	+ − +	− + +	− − + ;

and also, $a > b$, $a > c$; $g > f$, $g > h$; $\ell > j$, $\ell > k$.

In the proof of this theorem several lemmas will be used.

Lemma 2.1 No two of the six lattice points on the surface of an extreme parallelepiped for ξ, η, and ζ can lie in the same one of the octants formed by the planes $\xi = 0$, $\eta = 0$, $\zeta = 0$.

Assume two of these six points, $\mathfrak{r}$ and $\mathfrak{s}$, are in one and the same octant. Denote by $\xi(\mathfrak{r})$, $\xi(\mathfrak{s})$, $\eta(\mathfrak{r})$, $\eta(\mathfrak{s})$, $\zeta(\mathfrak{r})$, and $\zeta(\mathfrak{s})$ the values of ξ, η, and ζ for $\mathfrak{r}$ and $\mathfrak{s}$. Then

$$\xi(\mathfrak{r}) = e_1\kappa_1, \quad \eta(\mathfrak{r}) = e_2\lambda_1, \quad \zeta(\mathfrak{r}) = e_3\mu_1,$$
$$\xi(\mathfrak{s}) = e_1\kappa_2, \quad \eta(\mathfrak{s}) = e_2\lambda_2, \quad \zeta(\mathfrak{s}) = e_3\mu_2,$$

where $e_i = \pm 1$ $(i = 1, 2, 3)$ and

$$0 < \kappa_i \leqq a, \qquad 0 < \lambda_i \leqq g, \qquad 0 < \mu_i \leqq \ell \qquad (i = 1, 2).$$

Due to the distributive property of linear forms one has

$$\xi(\mathfrak{r}-\mathfrak{s}) = e_1(\kappa_1-\kappa_2), \ \eta(\mathfrak{r}-\mathfrak{s}) = e_2(\lambda_1-\lambda_2), \ \zeta(\mathfrak{r}-\mathfrak{s}) = e_3(\mu_1-\mu_2).$$

From Theorem 2.2 the points $\mathfrak{r}$ and $\mathfrak{s}$ are not in the same plane ξ = const., η = const., or ζ = const. It would follow, then, that

$$0 \leqq |\xi(\mathfrak{r}-\mathfrak{s})| < a, \ 0 \leqq |\eta(\mathfrak{r}-\mathfrak{s})| < g, \ 0 \leqq |\zeta(\mathfrak{r}-\mathfrak{s})| < \ell.$$

Since $\mathfrak{r}$ and $\mathfrak{s}$ were assumed distinct points, it is seen that $\mathfrak{r} - \mathfrak{s}$ is not the origin. A lattice point $\mathfrak{r} - \mathfrak{s}$, not the origin, could then be within the extreme parallelepiped, but this contradicts the definition of such a parallelepiped. Therefore, the assumption that two lattice points could be in the same octant is untenable and the lemma is proved.

Lemma 2.2 If $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, -\mathfrak{r}_1, -\mathfrak{r}_2, -\mathfrak{r}_3$ are the six lattice points on the surface of the extreme parallelepiped $\{a, g, \ell\}$, then there is a unique way of choosing three lattice points $\mathfrak{q}_1$, $\mathfrak{q}_2$, and $\mathfrak{q}_3$ from these six so that $\mathfrak{q}_1$ is in the plane $\xi = a$, $\mathfrak{q}_2$ in the plane $\eta = g$, and $\mathfrak{q}_3$ in the plane $\zeta = \ell$. In the matrix

$$\Psi = \begin{Vmatrix} \xi(q_1), & \xi(q_2), & \xi(q_3) \\ \eta(q_1), & \eta(q_2), & \eta(q_3) \\ \zeta(q_1), & \zeta(q_2), & \zeta(q_3) \end{Vmatrix} = \begin{Vmatrix} a, & \pm b, & \pm c \\ \pm f, & g, & \pm h \\ \pm j, & \pm k, & \ell \end{Vmatrix},$$

where

$$a = |\xi(q_1)|,\ b = |\xi(q_2)|,\ c = |\xi(q_3)|,$$
$$f = |\eta(q_1)|,\ g = |\eta(q_2)|,\ h = |\eta(q_3)|,$$
$$j = |\zeta(q_1)|,\ k = |\zeta(q_2)|,\ \ell = |\zeta(q_3)|,$$

the system of signs must be one of the following twenty-four:

1	2	3	4	5	6
+ + +	+ + -	+ - +	+ - -	+ - -	+ - -
+ + -	+ + +	+ + -	+ + -	+ + +	+ + -
+ - +	+ - +	+ + +	+ + +	+ - +	+ - +

7	8	9	10	11	12
+ + +	+ + -	+ - +	+ - +	+ - +	+ - -
+ + -	+ + +	+ + +	+ + -	+ + +	+ + +
- + +	- + +	- + +	- + +	- - +	- - +

13	14	15	16	17	18
+ + +	+ + -	+ + +	+ + +	+ - +	+ - -
- + -	- + -	- + +	- + -	- + +	- + -
- + +	- + +	- - +	- - +	- - +	- - +

19	20	21	22	23	24
+ + -	+ + -	+ + +	+ + -	+ - +	+ - -
- + +	- + -	- + +	- + +	- + +	- + -
+ + +	+ + +	+ - +	+ - +	+ + +	+ + + ;

<u>also</u>, $a > b$, $a > c$; $g > f$, $g > h$; $\ell > j$, $\ell > k$.

The first statement in Lemma 2.2 follows directly from Theorem 2.2, the second statement is a consequence of Lemma 2.1, and the inequalities at the end of the lemma are a consequence of the definition of an extreme parallelepiped and the restrictions imposed on ξ, η, and ζ.

The twenty-four systems of signs can be subdivided into six classes as follows:

Choose e_1, e_2, and e_3 equal to $+1$ or -1 independently of each other; then $e_1\xi(e_1q_1) = \xi(q_1)$, $e_2\eta(e_2q_2) = \eta(q_2)$, and $e_3\zeta(e_3q_3) = \zeta(q_3)$. Let $p_1 = e_1q_1$, $p_2 = e_2q_2$, and $p_3 = e_3q_3$. The matrix

$$\Phi = \begin{Vmatrix} e_1\xi(p_1), & e_1\xi(p_2), & e_1\xi(p_3) \\ e_2\eta(p_1), & e_2\eta(p_2), & e_2\eta(p_3) \\ e_3\zeta(p_1), & e_3\zeta(p_2), & e_3\zeta(p_3) \end{Vmatrix} = \begin{Vmatrix} e_1e_1\xi(q_1), & e_1e_2\xi(q_2), & e_1e_3\xi(q_3) \\ e_2e_1\eta(q_1), & e_2e_2\eta(q_2), & e_2e_3\eta(q_3) \\ e_3e_1\zeta(q_1), & e_3e_2\zeta(q_2), & e_3e_3\zeta(q_3) \end{Vmatrix}$$

differs from the matrix Ψ only in the sign attached to each element.

<u>Lemma 2.3</u> <u>For each of the twenty-four systems of signs in the matrix</u> Ψ, <u>integers</u> e_1, e_2, <u>and</u> e_3 <u>can be chosen in one and only one way so that</u> $e_1e_2e_3 = +1$ <u>and so that the matrix</u> Φ <u>has one of the six following systems of signs:</u>

I	II	III	IV	V	VI
+ + +	+ − −	+ − −	+ + −	+ − +	+ − −
− + −	+ + +	− + −	− + +	+ + −	− + −
− − +	− − +	+ + +	+ − +	− + +	− − +

Six of the twenty-four systems are identical with these six. A brief inspection will show what values to take for e_1, e_2, and e_3 in the other eighteen cases. The following tabulation gives in each case the e_i which have to be taken negative and gives also the system of signs which the matrix Φ assumes under this selection of e_i.

1. e_2, e_3, VI; 2. e_1, e_3, III; 3. e_1, e_2, II; 4. e_1, e_3, IV;
5. e_1, e_2, V; 6. e_2, e_3, I; 7. e_2, e_3, III; 8. e_1, e_3, VI;
9. e_1, e_3, I; 10. none, V; 11. e_2, e_3, IV; 12. none, II;
13. e_1, e_2, IV; 14. e_1, e_3, II; 15. e_1, e_3, V; 16. none, I;
17. e_1, e_2, III; 18. none, VI; 19. e_1, e_2, I; 20. e_2, e_3, V;
21. e_2, e_3, II; 22. none, IV; 23. e_1, e_2, VI; 24. none, III.

The six systems of signs I – VI will be called the <u>normal systems</u>.

Regard the transformation of a matrix by this process as an operation on the matrix apart from any consideration in connection with the problem. It is at once seen that if the matrix Φ is operated upon by the same transformation (i.e., the same set of e_i) as that which transformed Ψ into Φ, then Φ will be transformed back into Ψ.

More generally, if a transformation with one choice of e_i is followed by a transformation with a different choice e_i', the total effect is that of one transformation in which the choice e_i'' is to be taken as $e_i'' = e_i e_i'$ (i = 1, 2, 3). It appears at once that any choice of the e_i other than those listed above transforms Ψ into a matrix which has none of the six normal systems, but which has a different one of the eighteen other systems. For example, the system 1 is transformed into the system 23 by $e_2 = 1$, $e_1 = e_3 = -1$.

It is noteworthy that the lemma could just as well have been stated that <u>there is a unique way of selecting integers</u> e_1, e_2, <u>and</u> e_3 <u>so that the product</u> $e_1 e_2 e_3 = -1$ <u>and so that the matrix</u> Φ <u>has one of the six normal systems of signs</u>. This is obvious if one notes that changing the signs of all the e_i at one and the same time does not affect the matrix Φ.

The extreme parallelepipeds are to be classified according to the normal system of signs in Φ, that is, according to six distinct types of distributions of the lattice points on the surface of the parallelepiped.

Theorem 2.3 is found to be proved in the combined results of Lemmas 2.2 and 2.3, for when the $\mathfrak{q}_i$ and the e_i are uniquely determined, so also are the $\mathfrak{p}_i$.

Theorem 2.4 According as the normal system of signs in Φ is the system I, II, ···, or VI, the elements of the matrix Φ satisfy the following conditions listed under the corresponding numeral:

I	II	III
$b + c > a$,	$f + h > g$,	$j + k > \ell$,
$f > h$ or $j > k$	$k > j$ or $b > c$	$c > b$ or $h > f$

IV	V	VI
$b > c$ or $h > f$ or $j > k$	$c > b$ or $f > h$ or $k > j$	$b+c=a$, $f+h=g$, $j+k=\ell$;

among the two or three conditions separated by the word "or", at least one must be satisfied each time.

The proofs for the individual cases follow seriatim:

I. In this case one has

$$\begin{aligned} e_1\xi(\mathfrak{p}_1) &= a & e_1\xi(\mathfrak{p}_2) &= b, & e_1\xi(\mathfrak{p}_3) &= c,\\ e_2\eta(\mathfrak{p}_1) &= -f, & e_2\eta(\mathfrak{p}_2) &= g, & e_2\eta(\mathfrak{p}_3) &= -h,\\ e_3\zeta(\mathfrak{p}_1) &= -j, & e_3\zeta(\mathfrak{p}_2) &= -k, & e_3\zeta(\mathfrak{p}_3) &= \ell. \end{aligned}$$

The values of ξ, η, and ζ for the lattice point $\mathfrak{p} = \mathfrak{p}_2 + \mathfrak{p}_3$ are

$$\begin{aligned} \xi(\mathfrak{p}) &= \xi(\mathfrak{p}_2) + \xi(\mathfrak{p}_3) = e_1(b + c),\\ \eta(\mathfrak{p}) &= \eta(\mathfrak{p}_2) + \eta(\mathfrak{p}_3) = e_2(g - h),\\ \zeta(\mathfrak{p}) &= \zeta(\mathfrak{p}_2) + \zeta(\mathfrak{p}_3) = e_3(-k + \ell). \end{aligned}$$

This point $\mathfrak{p}$ is obviously not the origin since $|\xi(\mathfrak{p})|$ has the value $b + c > 0$. Then the point $\mathfrak{p}$ cannot be in the interior of the extreme parallelepiped. Therefore, since η and ζ have absolute values less than g and ℓ, respectively, it follows that $|\xi(\mathfrak{p})| \geqq a$, i. e., $b + c \geqq a$. Suppose that $b + c = a$, then $\xi(\mathfrak{p}) = e_1 a$, but this is the value of $\xi(\mathfrak{p}_1)$. There is at most one lattice point in the plane $\xi = e_1 a$, hence $\mathfrak{p} = \mathfrak{p}_1$. This requires that $\eta(\mathfrak{p}) = \eta(\mathfrak{p}_1) = -e_2 f$, whereas it has been seen that $\eta(\mathfrak{p}) = e_2(g - h)$; then $g - h = -f$. But this is impossible since $g > h$ and $f > 0$ by virtue of Theorem 2.3. Thus the condition $b + c > a$ is proved.

The values of ξ, η, and ζ for the lattice point $\mathfrak{p} = -\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3$ are

$$\xi(\mathfrak{p}) = -\xi(\mathfrak{p}_1) + \xi(\mathfrak{p}_2) + \xi(\mathfrak{p}_3) = e_1(-a + b + c),$$
$$\eta(\mathfrak{p}) = -\eta(\mathfrak{p}_1) + \eta(\mathfrak{p}_2) + \eta(\mathfrak{p}_3) = e_2(f + g - h),$$
$$\zeta(\mathfrak{p}) = -\zeta(\mathfrak{p}_1) + \zeta(\mathfrak{p}_2) + \zeta(\mathfrak{p}_3) = e_3(j - k + \ell).$$

Since $a > b$, $a > c$ and $b + c > a$, it follows that $a > b + c - a > 0$, i. e., $a > |\xi(\mathfrak{p})| > 0$. Then $\mathfrak{p}$ is not the origin and, consequently, is not in the interior of the parallelepiped. From the relations $g > h$ and $\ell > k$ it follows that $f + g - h > 0$ and $j - k + \ell > 0$, so that either $f + g - h \geqq g$ or $j - k + \ell \geqq \ell$ must hold in order to insure that the point $\mathfrak{p}$ be not interior to the parallelepiped. Consequently, at least one of the inequalities $f \geqq h$ and $j \geqq k$ must be satisfied. Neither equality sign can hold; for suppose $f = h$, then since no two lattice points can lie in the same plane η = const., it would follow that $\mathfrak{p}_1 = \mathfrak{p}_3$. But this is contradictory to the fact that $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ are three distinct points. A similar contradiction is reached if the assumption $j = k$ is made. The necessity of the conditions in I is then proved.

II. Let α_{ij} be the element in the ith row and the jth column of the matrix of coefficients of the linear forms ξ, η, and ζ. Set $\xi' = \alpha_{22}x_1' + \alpha_{23}x_2' + \alpha_{21}x_3'$, $\eta' = \alpha_{32}x_1' + \alpha_{33}x_2' + \alpha_{31}x_3'$, and $\zeta' = \alpha_{12}x_1' + \alpha_{13}x_2' + \alpha_{11}x_3'$; this has the effect of renaming the axes and the linear forms as follows: $\xi' = \eta$, $\eta' = \zeta$, $\zeta' = \xi$, $x_1' = x_2$, $x_2' = x_3$, and $x_3' = x_1$. The determinat of the matrix of coefficients of ξ', η', and ζ' is the same as that of ξ, η, and ζ.

The extreme parallelepiped $\{a, g, \ell\}$ for ξ, η, and ζ is the extreme parallelepiped $\{g, \ell, a\}'$ for ξ', η', and ζ'. Theorems 2.1, 2.2, and 2.3 apply in such a way as to yield matrices P' and Φ' of the same natures as P and Φ. Since the normal system of signs in Φ is II, then that in Φ' is I. The conditions $b' + c' > a'$; $f' > h'$ or $j' > k'$ which hold for Φ' can be translated by means of the relations between ξ, η, ζ and ξ', η', and ζ' into $f + h > g$; $k > j$ or $b > c$ for Φ.

III. If the normal system of signs in Φ is III, one sets $\xi' = \zeta$, $\eta' = \xi$, $\zeta' = \eta$; $x_1' = x_3$, $x_2' = x_1$, $x_3' = x_2$. In the same manner as in II one obtains the conditions: $j + k > \ell$; $c > b$ or $h > f$.

IV. In this case

$$\begin{array}{lll} e_1\xi(\mathfrak{p}_1) = a, & e_1\xi(\mathfrak{p}_2) = b, & e_1\xi(\mathfrak{p}_3) = -c, \\ e_2\eta(\mathfrak{p}_1) = -f, & e_2\eta(\mathfrak{p}_2) = g, & e_2\eta(\mathfrak{p}_3) = h, \\ e_3\zeta(\mathfrak{p}_1) = j, & e_3\zeta(\mathfrak{p}_2) = -k, & e_3\zeta(\mathfrak{p}_3) = \ell. \end{array}$$

For the lattice point $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3$ one has

$$\xi(\mathfrak{p}) = \xi(\mathfrak{p}_1) + \xi(\mathfrak{p}_2) + \xi(\mathfrak{p}_3) = e_1(a + b - c),$$

$$\eta(\mathfrak{p}) = \eta(\mathfrak{p}_1) + \eta(\mathfrak{p}_2) + \eta(\mathfrak{p}_3) = e_2(-f + g + h),$$

$$\zeta(\mathfrak{p}) = \zeta(\mathfrak{p}_1) + \zeta(\mathfrak{p}_2) + \zeta(\mathfrak{p}_3) = e_3(j - k + \ell).$$

This point $\mathfrak{p}$ is not the origin since $a > c$ yields $|\xi(\mathfrak{p})| = a + b - c > 0$. All three algebraic sums in the parentheses on the right in the equalities above are positive. Since the point $\mathfrak{p}$ does not lie in the interior of the parallelepiped, at least one of the following three inequalities must hold:

$$a + b - c \geqq a, \quad -f + g + h \geqq g, \text{ or } j - k + \ell \geqq \ell.$$

The equality sign can hold in no one of the cases, for the assumption that it may hold leads to a contradiction similar to that in the preceding cases. Therefore, $b > c$ or $h > f$ or $j > k$.

V. In this case set $\xi' = \xi$, $\eta' = \zeta$, $\zeta' = \eta$; $x_1' = x_1$, $x_2' = x_3$, $x_3' = x_2$. The matrix Φ' has the normal system of signs IV. The conditions $b' > c'$ or $h' > f'$ or $j' > k'$ for Φ' become $c > b$ or $k > j$ or $f > h$ for Φ.

VI. In this case

$$\begin{aligned} e_1\xi(p_1) &= a, & e_1\xi(p_2) &= -b, & e_1\xi(p_3) &= -c, \\ e_2\eta(p_1) &= -f, & e_2\eta(p_2) &= g, & e_2\eta(p_3) &= -h, \\ e_3\zeta(p_1) &= -j, & e_3\zeta(p_2) &= -k, & e_3\zeta(p_3) &= \ell. \end{aligned}$$

For the lattice point $p = p_1 + p_2 + p_3$ one has

$$\begin{aligned} \xi(p) &= \xi(p_1) + \xi(p_2) + \xi(p_3) = -e_1(-a + b + c), \\ \eta(p) &= \eta(p_1) + \eta(p_2) + \eta(p_3) = -e_2(f - g + h), \\ \zeta(p) &= \zeta(p_1) + \zeta(p_2) + \zeta(p_3) = -e_3(j + k - \ell). \end{aligned}$$

Since $a > b > 0$, $a > c > 0$; $g > f > 0$, $g > h > 0$; $\ell > j > 0$, and $\ell > k > 0$, it follows that $|\xi(p)| < a$, $|\eta(p)| < g$, $|\zeta(p)| < \ell$. This requires that the lattice point p be in the interior of the parallelepiped and identifies it with the origin, the only such lattice point. Therefore, all of the quantities $\xi(p)$, $\eta(p)$, and $\zeta(p)$ are zero, that is to say, $b + c = a$, $f + h = g$, and $j + k = \ell$.

Any matrix Φ will be called <u>normal</u> if its system of signs is normal and if, at the same time, its elements satisfy the conditions $a > b$, $a > c$; $g > f$, $g > h$; $\ell > j$, $\ell > k$ and also the conditions which by Theorem 2.4 correspond to its normal system of signs. A normal matrix Φ will be called <u>normal of type</u> I, II, $\cdots$, or VI, according as its normal system of signs is I, II, $\cdots$, or VI.

<u>Theorem 2.5</u> <u>If Φ is normal of one of the types</u> I - V, <u>then the determinant of</u> P <u>is</u> + 1; <u>if it is normal of type</u> VI, <u>the determinant of</u> P <u>equals zero</u>.

Denote the determinat of Φ by $|\Phi|$, and that of P by D. From the definition of Φ, it follows that $\Delta D = |\Phi|$. Since the elements of P are all integers or zero, the determinant D is an integer or zero.

The proof that $D = 0$ when Φ is normal of type VI. Hére one has

$$|\Phi| = \Delta D = \begin{vmatrix} a & -b, & -c \\ -f, & g, & -h \\ -j, & -k, & \ell \end{vmatrix},$$

where $b + c = a$, $f + h = g$, $j + k = \ell$. Then $|\Phi| = 0$, and since $\Delta > 0$, D must equal zero.

The proof that $D = 1$ when Φ is normal of type I - V follows.

Lemma 2.4 *If Φ is normal of type* I, II, $\cdots$, *or* V, *then* $4 \geq D \geq 1$.

Since the proof of the lemma follows the same lines for all five types, the details will be given for only type I. Here one has

$$|\Phi| = \Delta D = \begin{vmatrix} a, & b, & c \\ -f, & g, & -h \\ -j, & -k, & \ell \end{vmatrix}$$

$$= ag\ell - ahk + fb\ell + fck + jbh + jgc,$$

where $a > b > 0$, $a > c > 0$; $g > f > 0$, $g > h > 0$; $\ell > j > 0$, $\ell > k > 0$. Then $ag\ell > ahk > 0$, $ag\ell > fb\ell > 0$, $ag\ell > fck > 0$, $ag\ell > jbh > 0$, $ag\ell > jgc > 0$, so that $5ag\ell > |\Phi| > 0$. From Theorem 2.1 and the definition of an extreme parallelepiped, one obtains $ag\ell < \Delta$, which leads to $5 > D > 0$. But D is an integer, therefore, $4 \geq D \geq 1$. The lemma is thus proved.

Use is made of theorems concerning lattice octahedrons (cf. Definition 8, Chapter I) to prove that $D \neq 2$, 3, or 4. The octahedron with the six points $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, $-\mathfrak{p}_1$, $-\mathfrak{p}_2$, $-\mathfrak{p}_3$ as vertices has the origin as its center of symmetry and all its vertices are lattice points. Since the entire octahedron except its vertices is in the interior of the extreme parallelepiped $\{a, g, \ell\}$, this octahedron is then by definition a *lattice octahedron*. It follows at once that D = 1 or 2. If D = 2, than all the points of the form $\mathfrak{p} = (\gamma_1 + \frac{\delta}{2})\mathfrak{p}_1 + (\gamma_2 + \frac{\delta}{2})\mathfrak{p}_2 + (\gamma_3 + \frac{\delta}{2})\mathfrak{p}_3$, ($\gamma_i$, δ integers) are lattice points. This alternative does not occur here as one will see.

If Φ is normal of type I, consider the point for which $\gamma_1 = 0$, $\gamma_2 = \gamma_3 = -1$, $\delta = 1$. Namely

$$\mathfrak{p} = \frac{1}{2}\mathfrak{p}_1 - \frac{1}{2}\mathfrak{p}_2 - \frac{1}{2}\mathfrak{p}_3.$$

Then

$$|\xi(\mathfrak{p})| = \frac{1}{2}|a - b - c| < a,$$
$$|\eta(\mathfrak{p})| = \frac{1}{2}|-f - g + h| < g,$$
$$|\zeta(\mathfrak{p})| = \frac{1}{2}|-j + k - \ell| < \ell.$$

Hence $\mathfrak{p}$ is in the interior of the parallelepiped. Since $D \neq 0$, it follows that $\mathfrak{p}$ is not the origin, and,consequently, is not a lattice point. Therefore D = 1.

If Φ is normal of type II, it follows in the same way that the point for which $\gamma_2 = 0$, $\gamma_1 = \gamma_3 = -1$, $\delta = 1$ is not a lattice point; consequently, D is again equal to 1. Likewise for type III, $\gamma_3 = 0$, $\gamma_1 = \gamma_2 = -1$, $\delta = 1$ yields a point which is not a lattice point. Hence again D = 1. In types IV and V, the point for which $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\delta = 1$ is not a lattice point. Hence D = 1 and the theorem is proved.

<u>Theorem 2.6</u> <u>If one can find three lattice points</u> $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ <u>and three integers</u> e_1, e_2, <u>and</u> e_3 <u>for which</u> $e_1 e_2 e_3 = 1$, <u>such that the matrix</u> P <u>of the coordinates of</u> $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ <u>has a determinant</u> + 1, <u>and such that</u> P <u>transforms the matrix of the three forms</u> $e_1\xi$, $e_2\eta$, $e_3\zeta$ <u>into a normal matrix</u> Φ <u>of type</u> I, II, $\cdots$, <u>or</u> V, <u>than the matrix</u> P <u>is one of those matrices associated by Theorem</u> 2.3 <u>with the extreme parallelepipeds in the chain for</u> ξ, η, <u>and</u> ζ.

One can regard the matrix P as a transformation introducing a new coordinate system (X_1, X_2, X_3) where

$$x_i = p_i^{(1)}X_1 + q_i^{(2)}X_2 + q_i^{(3)}X_3 \qquad (i = 1, 2, 3).$$

Since the determinant of P is 1 the coefficients $q_i^{(j)}$ in P^{-1} are integers and one has

$$x_i = q_i^{(1)}X_1 + q_i^{(2)}X_2 + q_i^{(3)}X_3 \qquad (i = 1, 2, 3).$$

It appears at once that lattice points in (X_1, X_2, x_3) are lattice points in (x_1, x_2, x_3), and conversely. The three points $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ have respective coordinates (1, 0, 0), (0, 1, 0), and (0, 0, 1) in the system (X_1, X_2, X_3). It is obvious that the octahedron having $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, $-\mathfrak{p}_1$, $-\mathfrak{p}_2$, $-\mathfrak{p}_3$ as vertices is a lattice octahedron in (X_1, X_2, X_3). Since the lattice is transformed into itself by P (or P^{-1}), this octahedron is also a lattice octahedron in (x_1, x_2, x_3).

Consider the parallelepiped $\{a, g, \ell\}$ where a, g, and ℓ are the elements in the principal diagonal of Φ. Since Φ is a normal matrix, one has from the definition, $a > b$, $a > c$, $g > f$, $g > h$, $\ell > j$, $\ell > k$. Consequently, p_1, p_2, p_3 are on the surface of $\{a, g, \ell\}$, not merely in the planes of the faces. This parallelepiped is a convex body with center of symmetry at the origin. The function of generalized distance which corresponds to this parallelepiped is

$$f(x_1, x_2, x_3) \equiv \max(|\xi/a|, |\eta/g|, |\zeta/\ell|).$$

Applying P as is indicated in Definition 9, one obtains

$$F(X_1, X_2, X_3) = \max(|\Xi/a|, |H/g|, |Z/\ell|),$$

where Ξ, H, and Z are the respective transforms of $e_1\xi$, $e_2\eta$, $e_3\zeta$. Hence the coefficients in Ξ are the elements of the first row of the matrix Φ, those of H are the elements of the second row, and those of Z are the elements of the third row.

When the coordinates of the points listed in 1) of Definition 9, (see p. 10) are substituted into $F(X_1, X_2, X_3)$ it is seen, upon recognition of the conditions on the elements of a normalized matrix, that $\{a, g, \ell\}$ is free. This parallelepiped then satisfies the definition of an extreme parallelepiped for ξ, η, ζ and the theorem is proved.

CHAPTER III

DEVELOPMENT OF THE MINKOWSKI ALGORITHM

The purpose of this chapter is to develop the algorithms necessary for finding all the parallelepipeds in the chain for ξ, η, ζ, taking for a starting point an arbitrary extreme parallelepiped for which P has a determinant equal to 1.

<u>Theorem 3.1</u> <u>When one starts with an arbitrary extreme parallelepiped for ξ, η, and ζ, he can secure the totality of all existing extreme parallelepipeds by the successive formation of all possible neighbors</u>.

The demonstration of this theorem was given by Minkowski.[1] It is an obvious consequence of this theorem that the transition from one arbitrary extreme parallelepiped to another can be effected by a finite number of passages from neighbor to neighbor.

<u>Theorem 3.2</u> <u>Under a suitable renaming of the axes</u>

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1. H. Minkowski, <u>Gesammelte Abhandlungen</u> I, p. 283, "Zur Theorie der Kettenbrüche."

<u>and the linear forms,</u>[2] <u>the desired</u> ξ-, η-, <u>or</u> ζ-<u>neighbor becomes the</u> ξ'-<u>neighbor and in such a way that the matrix</u> Φ' <u>is always a normal matrix of type</u> I, II, $\cdots$, V <u>in which the inequality</u> $b' > c'$ <u>is satisfied</u>:

Condition	Transformation	The type of Φ' is in column headed by type of Φ.					
		I	II	III	IV	V	VI
ξ-neighbor, $c > b$	$\xi = \xi'$, $\eta = \zeta'$, $\zeta = \eta'$ $x_1 = x_1'$, $x_2 = x_3'$, $x_3 = x_2'$	I	III	II	V	IV	VI
η-neighbor, $f > h$	$\xi = \eta'$, $\eta = \xi'$, $\zeta = \zeta'$ $x_1 = x_2'$, $x_2 = x_1'$, $x_3 = x_3'$	II	I	III	V	IV	VI
η-neighbor, $h > f$	$\xi = \zeta'$, $\eta = \xi'$, $\zeta = \eta'$ $x_1 = x_3'$, $x_2 = x_1'$, $x_3 = x_2'$	III	I	II	IV	V	VI
ζ-neighbor, $j > k$	$\xi = \eta'$, $\eta = \zeta'$, $\zeta = \xi'$ $x_1 = x_2'$, $x_2 = x_3'$, $x_3 = x_1'$	II	III	I	IV	V	VI
ζ-neighbor, $k > j$	$\xi = \zeta'$, $\eta = \eta'$, $\zeta = \xi'$ $x_1 = x_3'$, $x_2 = x_2'$, $x_3 = x_1'$	III	II	I	V	IV	VI

In the chain of parallelepipeds for ξ, η, and ζ, those parallelepipeds for which the corresponding matrix P has a determinant $D = 1$ are to be designated as parallelepipeds of <u>first kind</u>. The parallelepipeds for which $D = 0$ will be called parallelepipeds of <u>second kind</u>.

<u>Theorem 3.3</u> <u>The neighbors of a parallelepiped of second kind are all parallelepipeds of first kind</u>.

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2. This same type of renaming was done in the proofs of Cases II, III, and V of Theorem 2.4.

Rename the axes and linear forms as indicated in the table, so that the neighbor in question becomes the ξ'-neighbor with $b' > c'$. Then since $\bar{\Phi}$ is of the type VI, so also is $\bar{\Phi}'$. From Theorems 2.3 and 2.4, one has $a' > b'$, $a' > c'$, $g' > f'$, $g' > h'$, $\ell' > j'$, $\ell' > k'$; $b' + c' = a'$, $f' + h' = g'$, $j' + k' = \ell'$. If $\mathfrak{p}'_1$, $\mathfrak{p}'_2$, $\mathfrak{p}'_3$ are the three lattice points of P', in the order of their occurence in the columns of P', then the ξ'-neighbor has on its surface the four lattice points $\mathfrak{p}'_2$, $\mathfrak{p}'_3$, $-\mathfrak{p}'_2$, $-\mathfrak{p}'_3$, and, besides, two other lattice points which will be designated by $\bar{\mathfrak{p}}'_2$, and $-\bar{\mathfrak{p}}'_2$. (The subscript 2 is used here because the coordinates of $\bar{\mathfrak{p}}'_2$ take their place in the second column of the matrix $\bar{P}'$.[3]) Since $b' > c'$, the coordinates of $\mathfrak{p}'_2$ or $-\mathfrak{p}'_2$ will occupy the first column in the matrix $\bar{P}'$, and those of $\mathfrak{p}'_3$ or $-\mathfrak{p}'_3$, the third column.

Under the assumption that the neighbor is of the second kind, the matrix $\bar{\bar{\Phi}}'$ for this neighbor then becomes

$$\bar{\bar{\Phi}}' = \begin{Vmatrix} b', & -\bar{b}', & -c' \\ -g', & \bar{g}', & -h' \\ -k', & -\bar{k}', & \ell' \end{Vmatrix} \quad \text{where} \quad \begin{aligned} b' &= \bar{b}' + c', \\ \bar{g}' &= g' + h', \\ \ell' &= k' + \bar{k}'. \end{aligned}$$

But from above, $\ell' = k' + j'$, therefore, $\bar{k}' = j'$. Since

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3. The matrix $\bar{P}'$ is the matrix of lattice points on the neighbor with coordinates referred to the axes as renamed above; $\bar{\bar{\Phi}}'$ is the normal matrix for the neighbor under this reference.

there is only one lattice point, either p_1' or $-p_1'$ for which $\zeta = j'$, one has $\bar{p}_2' = \pm p_1'$; this yields $a' = \bar{b}' = b' - c'$. But this is impossible because $b' > c' > 0$ and $a' > b'$. Thus the theorem is proved.

To secure the ξ-neighbor when $b > c$, one lowers the ξ-faces to the lattice points in the faces $\eta = \pm g$ (one has $\xi = \pm b$ for these points) and subsequently raises the η-faces to the first pair of lattice points in neither the ξ- nor ζ-faces; the first parameter is decreased, the second is increased, and the third is left unchanged. For the ξ-neighbor $\{a_1, g_1, \ell_1\}$ of $\{a, g, \ell\}$ one has then $a_1 = b < a$, $g_1 > g$, and $\ell_1 = \ell$. Since there is at most one lattice point in each of the planes $\xi =$ const., $\eta =$ const., it follows that two of the lattice points whose coordinates are in the matrix $\bar{P}$ for the neighbor are $\bar{p}_1 = d_1 p_2$, $\bar{p}_3 = d_3 p_3$ where $d_1 = \pm 1$, $d_3 = \pm 1$.

If $\{a, g, \ell\}$ is a parallelepiped of the first kind, then P is a matrix of integral elements with a determinant 1. Then the inverse P^{-1} of the transformation P exists and has integers as elements; and the determinant of P^{-1} is 1. Set $T = P^{-1}\bar{P}$. Being the product of two matrices whose elements are integers, T is itself such a matrix. Moreover, the determinant of T is equal to that of $\bar{P}$. Of one denotes

by $q_i^{(j)}$ the cofactor of $p_i^{(j)}$ in P, one has $\|q_j^{(i)}\| \cdot \|\bar{p}_i^{(j)}\| = \|t_i^{(j)}\|$.
From the relations $\bar{p}_i^{(1)} = d_1 p_i^{(2)}$, $\bar{p}_i^{(3)} = d_3 p_i^{(3)}$ (i = 1, 2, 3)
it follows that

$$T = \begin{Vmatrix} 0, \sum_{k=1}^{3} q_k^{(1)}\bar{p}_k^{(2)}, 0 \\ d_1, \sum_{k=1}^{3} q_k^{(2)}\bar{p}_k^{(2)}, 0 \\ 0, \sum_{k=1}^{3} q_k^{(3)}\bar{p}_k^{(2)}, d_3 \end{Vmatrix}.$$

Then $\sum_{k=1}^{3} q_k^{(1)}\bar{p}_k^{(2)} = -d_1 d_3 = \pm 1$ or 0 according as the neighbor $\{a_1, g_1, \ell_1\}$ is of first or of second kind. Set $\sum_{k=1}^{3} q_k^{(1)}\bar{p}_k^{(2)} = \kappa d_2$, $\sum_{k=1}^{3} q_k^{(2)}\bar{p}_k^{(2)} = md_2$, $\sum_{k=1}^{3} q^{(3)}\bar{p}^{(2)} = nd_2$, where $d_2 = -d_1 d_3$ and where $\kappa = 1$ or 0 according as $\{a_1, g_1, \ell\}$ is of first or second kind. Then

$$\bar{P} = PT = \begin{Vmatrix} d_1 p_1^{(2)}, & d_2(\kappa p_1^{(1)} + mp_1^{(2)} + np_1^{(3)}), & d_3 p_1^{(3)} \\ d_1 p_2^{(2)}, & d_2(\kappa p_2^{(1)} + mp_2^{(2)} + np_2^{(3)}), & d_3 p_2^{(3)} \\ d_1 p_3^{(2)}, & d_2(\kappa p_3^{(1)} + mp_3^{(2)} + np_3^{(3)}), & d_3 p_3^{(3)} \end{Vmatrix}$$

The integers κ, m, and n are not all zero, for this case $\bar{\mathfrak{p}}_2$ would be the origin, which is impossible.

Then

$$\bar{\mathfrak{p}}_2 = d_2(\kappa \mathfrak{p}_1 + m\mathfrak{p}_2 + n\mathfrak{p}_3)$$

and

$$\xi(\bar{\mathfrak{p}}_2) = e_1 d_2(\kappa a \pm mb \pm nc),$$
$$\eta(\bar{\mathfrak{p}}_2) = e_2 d_2(\pm \kappa f + mg \pm nh),$$
$$\zeta(\bar{\mathfrak{p}}_2) = e_3 d_2(\pm \kappa j \pm mk + n\ell),$$

where the system of signs is that of the normal system of signs in Φ.

Due to Theorem 2.1, the relations $a_1 = b$, $g_1 < g$, $\ell_1 = \ell$ yield the condition $g_1 < \frac{\Delta}{b\ell}$. The lattice point $\bar{\mathfrak{p}}_2$ is to be found among those lattice points of the form $\mathfrak{p} = \kappa\mathfrak{p}_1 + m\mathfrak{p}_2 + n\mathfrak{p}_3$ ($\kappa = \pm 1$ or 0; m, n integers; $|\kappa| + |m| + |n| \geqq 1$) which satisfy

$$|\xi| < b, \quad |\eta| < \frac{\Delta}{b\ell}, \quad |\zeta| < \ell,$$

and, in particular, $\pm\bar{\mathfrak{p}}_2$ will be the two such points which make $|\eta|$ a minimum.[4] It is clear, then, that if the simultaneous inequalities just stated can be solved for the finite number of solutions which they possess, then $\bar{\mathfrak{p}}_2$ can be determined except for sign. The sign can be later determined to make $\bar{\bar{\Phi}}$ a normal matrix.

The matrix T for $\bar{\Phi}$ normal of the various types I – V is given in the following table, the derivations will be carried out later.

ALGORITHM FOR THE ξ-NEIGHBOR ($b > c$).

I: 1] $j > k$,

$$T = \begin{Vmatrix} 0, & -1, & 0 \\ 1, & 1, & 0 \\ 0, & 0, & 1 \end{Vmatrix}, \quad +, +, +;\ V.$$

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4. There are only two lattice points for which $|\eta|$ will assume this minimum value, otherwise $\eta = 0$ would be satisfied by lattice points other than the origin.

2] $j < k$,

$$T = \begin{Vmatrix} 0, & 1, & 0 \\ 1, & -1, & 0 \\ 0, & -1, & -1 \end{Vmatrix}, \quad +, -, -; \text{ III.}$$

The Roman numeral denotes the normal system for $\bar{\bar{\Phi}}$; the three signs are the signs of the products $\bar{e}_i e_i$ $(i = 1, 2,3)$, the e_i being those for $\bar{\Phi}$ in Theorem 2.3 and the $\bar{e}_i$ those for $\bar{\bar{\Phi}}$.

II and V:

The upper sign is for Case II; the lower sign for Case V.

$$M \equiv \left[\frac{G}{F}\right], \; N \equiv \left[\frac{\pm H}{F}\right], \; u \equiv a - Mb - Nc, \; v \equiv \pm j + Mk - N\ell.$$

Here F, G, and H are the signed minors of f, g, and $\pm$ h in $\bar{\Phi}$. The square brackets denote the largest integer $\leqq$ the quantity within them.

	m	n	δ
1) $u < c$, $\quad v > k$	$M - 1$	$N + 1$	$+ 1$
2) $u < b - c$, $v < 0$	M	$N - 1$	$- 1$
3) $u < b$, $\quad v > 0$, but not 1)	M	N	$- 1$
4) $u < b$, $\quad v < 0$, but not 2)	M	N	$+ 1$
5) $u > b$, $\quad v > 0$	M	$N + 1$	$+ 1$
6) $u > b$, $\quad v < 0$	$M + 1$	N	$- 1$

$$T = \begin{Vmatrix} 0, & \mp\delta, & 0 \\ \pm\delta, & \mp\delta m, & 0 \\ 0, & -\delta n, & 1 \end{Vmatrix}, \quad \begin{array}{c} \mp\delta,\ \mp\delta,\ +1; \\ \delta = +1,\ \mathrm{I};\ \delta = -1,\ \mathrm{IV}. \end{array}$$

III: 1] $a + c < 2b$

$$T = \begin{Vmatrix} 0 & 1, & 0 \\ 1, & 1, & 0 \\ 0, & -1, & -1 \end{Vmatrix}, -, +, -; \mathrm{II}.$$

2] $a + c > 2b$

$$T = \begin{Vmatrix} 0, & 0, & 0 \\ -1, & 1, & 0 \\ 0, & -1, & 1 \end{Vmatrix}, +, +, +; \mathrm{VI}.$$

IV: 1] $a < 2b,\ f < h,\ j + k < \ell$

$$T = \begin{Vmatrix} 0, & -1, & 0 \\ 1, & 1, & 0 \\ 0, & 0, & 1 \end{Vmatrix}, +, +, +; \mathrm{II}.$$

2] $a > 2b$ or $f > h$ or $j + k > \ell$

$$T = \begin{Vmatrix} 0, & 0, & 0 \\ -1, & 1, & 0 \\ 0, & 1, & -1 \end{Vmatrix}, -, +, -; \mathrm{VI}.$$

ALGORITHM FOR THE ξ-NEIGHBOR OF THE ξ-NEIGHBOR IN III 2], IV 2].

The upper sign is for Case III 2], the lower for IV 2].

1] $b - c > c$:

$$M \equiv [\pm G/F],\ N \equiv [(\pm G + H)/F];$$

$$u^{o} = b - c,\ u' = c;\ v^{o} = \ell,\ v' = \ell - k;$$

$$u \equiv a - Mu^{o} - Nu',\ v \equiv -j + Mv' - Nv^{o}$$

		m	n	δ
1) $u < u'$,	$v > v'$	M − 1	N + 1	+ 1
2) $u < u'$,	$v' > v > 0$	M	N + 1	− 1
3) $u > u'$,	$v > 0$	M	N + 1	+ 1
4) $u < u^{o}$,	$v < 0$	M	N	+ 1
5) $u > u^{o}$,	$v < 0$	M + 1	N + 1	− 1

$$T = \begin{Vmatrix} 0, & \mp\delta, & 0 \\ -1, & -\delta m, & 0 \\ \pm 1, & \pm\delta(m-n), & \mp\delta \end{Vmatrix} \quad \begin{matrix} \pm 1,\ -\delta,\ \mp\delta; \\ \delta = +1,\ \mathrm{V};\ \delta = -1,\ \mathrm{III}. \end{matrix}$$

2] $b - c < c$:

$$M \equiv [(\pm K + L)/J],\ N \equiv [\pm K/J];$$

$$u^{o} = c,\ u' = b - c;\ v^{o} = g + h,\ v' = h;$$

$$u \equiv a - Mu^{o} - Nu',\ v \equiv -f + Mv' - Nv^{o}.$$

$$T = \begin{Vmatrix} 0, & 0, & \mp\delta \\ 0, & -\delta, & -\delta n \\ \mp 1, & \pm\delta, & \mp\delta(m-n) \end{Vmatrix}, \quad \begin{matrix} \pm 1,\ -\delta,\ \mp\delta; \\ \delta = +1,\ \mathrm{IV};\ \delta = -1,\ \mathrm{II}. \end{matrix}$$

<u>When</u> $b > c$ <u>the</u> η-<u>neighbor of the</u> ξ-<u>neighbor is</u> $\{a, g, \ell\}$ <u>itself</u>.

ALGORITHM FOR THE ζ-NEIGHBOR OF THE ξ-NEIGHBOR IN III 2], IV 2].

1] $k < \ell - k$:

$$M \equiv [-H/F],\ N \equiv [(\mp G - H)/F];$$

$$u^o = \ell - k,\ u' = k;\ v^o = b,\ v' = b - c;$$

$$\bar{u} \equiv j - Mu^o - Nu',\quad \bar{v} \equiv -a + Mv' - Nv^o.$$

$$T = \begin{Vmatrix} 0, & \mp\delta, & 0 \\ \delta, & -\delta(m-n), & -1 \\ 0, & \pm\delta m, & \pm 1 \end{Vmatrix},\quad \begin{matrix} \mp\delta,\ -\delta,\ \pm 1; \\ \delta = +1,\ \text{IV};\ \delta = -1,\ \text{I}. \end{matrix}$$

2] $k > \ell - k$:

In case III:

$$T = \begin{Vmatrix} -1, & 0, & 0 \\ -1, & -1, & 1 \\ 1, & 1, & 0 \end{Vmatrix},\quad \begin{matrix} -,\ -,\ +; \\ \text{V}. \end{matrix}$$

In case IV:

1) $j > k$:

$$T = \begin{Vmatrix} -1, & 0, & 0 \\ 0, & 1, & 1 \\ 1, & 1, & 0 \end{Vmatrix},\quad \begin{matrix} -,\ +,\ -; \\ \text{II}. \end{matrix}$$

2) $j < k$:

$$T = \begin{Vmatrix} 1, & 0, & 0 \\ 0, & -1, & 1 \\ 0, & -1, & 0 \end{Vmatrix},\quad \begin{matrix} +,\ -,\ -; \\ \text{V}. \end{matrix}$$

The derivations are taken up seriatim as indicated by the Roman numerals below.

I.

The relations to be satisfied by the integers κ, m, and n are :

(1) $|\kappa a + mb + nc| < b,$

(2) $|-\kappa j - mk + n\ell| < \ell,$ $\quad (|\kappa| + |m| + |n| \geqq 1,$ $\kappa = \pm 1$ or $0).$

(3) $|-\kappa f + mg - nh| = \min.$

One has from Theorems 2.3 and 2.4 the additional conditions $a > b > 0$, $a > c > 0$; $g > f > 0$, $g > h > 0$; $\ell > j > 0$, $\ell > k > 0$; $b + c > a$; and $f > h$ or $j > k$. Besides these the assumption $b > c$ is made.

Rewrite (1), (2), and (3) as

(1′) $U \equiv \kappa a + mb + nc = \lambda_1 b,$

(2′) $V \equiv -\kappa j - mk + n\ell = \lambda_2 \ell,$ $\quad (|\lambda_i| < 1, \lambda_i \neq 0).$

(3′) $W \equiv -\kappa f + mg - nh$ to have minimum absolute value.

Since $\kappa = \pm 1$ or 0 for all solutions with which we are concerned, solve (1′) and (2′) for m and n in terms of κ, a, b, c, etc.:

$$m = \kappa \frac{a\ell + cj}{-(b\ell + ck)} + \frac{\lambda_1 b\ell - \lambda_2 c\ell}{b\ell + ck} = \kappa \frac{G}{F} + \nu_1,$$

$$n = \kappa \frac{ak - bj}{-(b\ell + ck)} + \frac{-\lambda_2 b\ell - \lambda_1 bk}{b\ell + ck} = \kappa \frac{H}{F} + \nu_2,$$

where F, G, and H are the cofactors of $-f$, g, and $-h$, respectively, in Φ, and ν_1 and ν_2 are the above expressions involving λ_1 and λ_2. It is at once apparent from the conditions on b, c, ℓ, k, λ_1, and λ_2 that $|\nu_i| < 2 \quad (i = 1, 2).$

If $\kappa = 0$, then $m = \nu_1$, $n = \nu_2$ and since m and n are integers one has $|m| \leqq 1$, $|n| \leqq 1$. Among the m and n satisfying these conditions $m = 0$, $n = 0$ is the only pair of values which satisfy (1) and (2). But this would make $\bar{p}_2$ be the origin which is impossible, since $\bar{p}_2$ is a point on the surface of an extreme parallelepiped. Therefore $\kappa = \pm 1$. In the terminology introduced above, one can state this result as follows:

If $b > c$ *and if* $\bar{\Phi}$ *is a normalized matrix of type* I, *then the* ξ-*neighbor of* $\{a, g, \ell\}$ *is always of the first kind.*

If $\kappa = 1$, one secures bounds for $\frac{G}{F}$ and $\frac{H}{F}$. From Theorem 2.4, $b + c > a$, hence

$$-3 < \frac{G}{F} < 0, \quad -1 < \frac{H}{F} < 1.$$

Use of these in conjunction with the bounds of ν_1 and ν_2, and the fact that m and n are integers, yields

$$-4 \leqq m \leqq 1, \quad -2 \leqq n \leqq 2.$$

Substitution of these values in (1) and (2) shows that none but the following are solutions:

(i)	$m = -3$, $n = 1$,	$\|W\| = f + 3g + h$;
(ii)	$m = -2$, $n = 0$,	$\|W\| = f + 2g$;
(iii)	$m = -2$, $n = 1$,	$\|W\| = f + 2g + h$;
(iv)	$m = -1$, $n = -1$,	$\|W\| = f + g - h$;
(v)	$m = -1$, $n = 0$,	$\|W\| = f + g$;
(vi)	$m = -1$, $n = 1$,	$\|W\| = f + g + h$;

each of which can be a solution under certain restrictions on a, b, c, etc.

The smallest value of $|W|$ occurs for (iv), but (iv) satisfies (2) only if $j < k$, (1) being satisfied always by (iv). Thus when $j < k$, then $m = -1$, $n = -1$ satisfy (1), (2), and (3).

The next smallest value of $|W|$ occurs for (v), and (v) always satisfies both (1) and (2). Hence if $j > k$, then $m = -1$, $n = 0$, satisfy (1), (2), and (3).

The use of $\kappa = -1$ will merely give the two solutions $j < k$: $\kappa = -1$, $m = 1$, $n = 1$; $j > k$: $\kappa = -1$, $m = 1$, $n = 0$, which are the negatives of the above solutions, hence essentially the same.

It remains to determine d_1, d_2, d_3, $\bar{e}_1 e_1$, $\bar{e}_2 e_2$, $\bar{e}_3 e_3$, and the system of signs in $\bar{\Phi}$.

[1] $j > k$ $\qquad \bar{\mathfrak{p}}_2 = d_2(\mathfrak{p}_1 - \mathfrak{p}_2)$

$$T = \begin{Vmatrix} 0, & d_2, & 0 \\ d_1, & -d_2, & 0 \\ 0, & 0, & d_3 \end{Vmatrix}, \quad \bar{P} = PT = \begin{Vmatrix} d_1 p_1^{(2)}, & d_2(p_1^{(1)} - p_1^{(2)}), & d_3 p_1^{(3)} \\ d_1 p_2^{(2)}, & d_2(p_2^{(1)} - p_2^{(2)}), & d_3 p_2^{(3)} \\ d_1 p_3^{(2)}, & d_2(p_3^{(1)} - p_3^{(2)}), & d_3 p_3^{(3)} \end{Vmatrix}$$

Let $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ be the e_i for $\bar{P}$ and e_1, e_2, e_3 those for P, then

$$\bar{\Phi} = \begin{Vmatrix} \bar{e}_1 d_1 \xi(\mathfrak{p}_2), & \bar{e}_1 d_2(\xi(\mathfrak{p}_1) - \xi(\mathfrak{p}_2)), & \bar{e}_1 d_3 \xi(\mathfrak{p}_3) \\ \bar{e}_2 d_1 \eta(\mathfrak{p}_2), & \bar{e}_2 d_2(\eta(\mathfrak{p}_1) - \eta(\mathfrak{p}_2)), & \bar{e}_2 d_3 \eta(\mathfrak{p}_3) \\ \bar{e}_3 d_1 \zeta(\mathfrak{p}_2), & \bar{e}_3 d_2(\zeta(\mathfrak{p}_1) - \zeta(\mathfrak{p}_2)), & \bar{e}_3 d_3 \zeta(\mathfrak{p}_3) \end{Vmatrix}$$

$$\overline{\overline{\Phi}} = \left\| \begin{matrix} \bar{e}_1e_1d_1b, & \bar{e}_1e_1d_2(a-b), & \bar{e}_1e_1d_3c \\ \bar{e}_2e_2d_1g, & \bar{e}_2e_2d_2(-f-g), & -\bar{e}_2e_2d_3h \\ -\bar{e}_3e_3d_1k, & \bar{e}_3e_3d_2(-j+k), & \bar{e}_3e_3d_3\ell \end{matrix} \right\|$$

Since the diagonal terms must be positive one has $d_1 = \bar{e}_1e_1$, $d_2 = -\bar{e}_2e_2$, $d_3 = \bar{e}_3e_3$ and

$$\overline{\overline{\Phi}} = \left\| \begin{matrix} (\bar{e}_1e_1)^2b, & -(\bar{e}_1e_1)(\bar{e}_2e_2)(a-b), & (\bar{e}_1e_1)(\bar{e}_3e_3)c \\ (\bar{e}_2e_2)(e_1e_1)g, & (\bar{e}_2e_2)^2(f+g), & -(\bar{e}_2e_2)(\bar{e}_3e_3)h \\ -(\bar{e}_3e_3)(e_1e_1)k, & (\bar{e}_3e_3)(\bar{e}_2e_2)(j-k), & (\bar{e}_3e_3)^2\ell \end{matrix} \right\| .$$

Since $\bar{e}_1\bar{e}_2\bar{e}_3 = 1$ and $e_1e_2e_3 = 1$, it follows that $(\bar{e}_1e_1)(\bar{e}_2e_2)(\bar{e}_3e_3) = 1$. Then Lemmas 2.2 and 2.3 afford a way of selecting the unique products $\bar{e}_1e_1$, $\bar{e}_2e_2$, $\bar{e}_3e_3$ which make $\overline{\overline{\Phi}}$ a normal matrix; namely, $\bar{e}_1e_1 = 1$, $\bar{e}_2e_2 = 1$, $\bar{e}_3e_3 = 1$, and the system of signs in $\overline{\overline{\Phi}}$ is V. Hence $d_1 = 1$, $d_2 = -1$, $d_3 = 1$, $\bar{p}_1 = p_2$, $\bar{p}_2 = -p_1 + p_2$, $\bar{p}_3 = p_3$.

2] $j < k$, $\bar{p}_2 = d_2(p_1 - p_2 - p_3)$.

The same procedure as in part 1] leads to

$$\overline{\overline{\Phi}} = \left\| \begin{matrix} \bar{e}_1e_1d_1b, & -\bar{e}_1e_1d_2(-a+b+c), & \bar{e}_1e_1d_3c \\ \bar{e}_2e_2d_1g, & -\bar{e}_2e_2d_2(f+g-h), & -\bar{e}_2e_2d_3h \\ -\bar{e}_3e_3d_1k, & -\bar{e}_3e_3d_2(j-k+\ell), & \bar{e}_3e_3d_3\ell \end{matrix} \right\| ,$$

hence $d_1 = \bar{e}_1e_1$, $d_2 = -\bar{e}_2e_2$, $d_3 = \bar{e}_3e_3$. Replace these values in $\overline{\overline{\Phi}}$ and noting that $b + c > a$ apply Lemmas 2.2 and 2.3 to obtain $\bar{e}_1e_1 = 1$, $\bar{e}_2e_2 = -1$, $\bar{e}_3e_3 = -1$, and the system of signs in $\overline{\overline{\Phi}}$ to be III. Hence $d_1 = 1$, $d_2 = 1$,

$d_3 = -1,\ \bar{p}_1 = p_2, \bar{p}_2 = p_1 - p_2 - p_3,\ \bar{p}_3 = -p_3.$

II and V.

If V is transformed, Cases II and V can be solved simultaneously. The conditions (1), (2), and (3) to be satisfied in the respective cases are

II

$$\begin{aligned} &(1') \quad |\kappa a - mb - nc| < b, \\ &(2') \quad |-\kappa j - mk + n\ell| < \ell, \\ &(3') \quad |\kappa f + mg + nh| = \text{min.}, \end{aligned} \qquad \left(|\kappa| + |m| + |n| \geqq 1,\ \kappa = \pm 1 \text{ or } 0\right).$$

V

$$\begin{aligned} &(\bar{1}) \quad |\bar{\kappa} a - \bar{m}b + \bar{n}c| < b, \\ &(\bar{2}) \quad |-\bar{\kappa} j + \bar{m}k + \bar{n}\ell| < \ell, \\ &(\bar{3}) \quad |\bar{\kappa} f + \bar{m}g - \bar{n}h| = \text{min.}, \end{aligned} \qquad \left(|\bar{\kappa}| + |\bar{m}| + |\bar{n}| \geqq 1,\ \bar{\kappa} = \pm 1 \text{ or } 0\right).$$

For V set $\kappa = \bar{\kappa}$, $m = \bar{m}$, $n = -\bar{n}$ and rewrite (1), (2), and (3) as

(1) $\quad U \equiv \kappa a - mb - nc = \lambda_1 b,$

(2) $\quad V \equiv \pm \kappa j + mk - n\ell = \lambda_2 \ell, \qquad (|\lambda_i| < 1,\ \lambda_i \neq 0)$

(3) $\quad W \equiv \kappa f + mg + nh$ to be as small as possible in absolute value.

The upper sign in (2) is for II and the lower sign for V. One has also $F = b\ell + ck$, $G = a\ell \mp cj$, $H = \pm ak + bj$. Solve (1) and (2) for m and n to obtain

$$m = \kappa \frac{a\ell \mp cj}{b\ell + ck} + \frac{-\lambda_1 b\ell + \lambda_2 c\ell}{b\ell + ck} = \kappa \frac{G}{F} + \nu_1,$$
$$n = \kappa \frac{\pm bj + ak}{b\ell + ck} + \frac{-\lambda_2 b\ell - \lambda_1 bk}{b\ell + ck} = \kappa \frac{\pm H}{F} + \nu_2.$$

If $\kappa = 0$, one has as before $|m| \leq 1$, $|n| \leq 1$, and of these, only $m = 0$, $n = 0$ is a solution of (1) and (2). But $\bar{\mathfrak{p}}_2$ is not the origin, therefore:

<u>If</u> $b > c$ <u>and if</u> Φ <u>is a normal matrix of type</u> II <u>or</u> V, <u>then the</u> ξ-<u>neighbor of</u> $\{a, g, \ell\}$ <u>is always of the first kind</u>.

Numerical bounds for G/F and $\pm H/F$ independent of a, b, c, etc., cannot be secured as in I, hence the solution will have to be expressed in terms of these two quantities.

Set $M \equiv [G/F]$, $N \equiv [\pm H/F]$, where the square brackets have the same meaning as before.

By determinant theory one obtains from Φ the relations

$$a - \frac{G}{F}b - \frac{\pm H}{F}c = 0,\quad f + \frac{G}{F}g + \frac{\pm H}{F}h = \frac{\Delta}{F},\quad \pm j + \frac{G}{F}k - \frac{\pm H}{F}\ell = 0.$$

Set $u \equiv a - Mb - Nc$, $v \equiv \pm j + Mk - N$, $w \equiv f + Mg + Nh$, then

$$0 < u < b + c,\quad - k < v < \ell,\quad \frac{\Delta}{F} - (g + h) < w < \frac{\Delta}{F}.$$

From the above expressions for m and n one obtains

$$[\kappa G/F] + [\nu_1] \leq m \leq [\kappa G/F] + [\nu_1] + 1,$$
$$[\pm \kappa H/F] + [\nu_2] \leq n \leq [\pm \kappa H/F] + [\nu_2] + 1,$$

where $-2 \leqq [\nu_i] \leqq 1$. The solutions of (1) and (2) for $\kappa = -1$ are the negatives of the solutions for $\kappa = +1$, due to the homogeneity of the expressions in κ, m, and n. Set $\kappa = 1$. Since m is an integer, either G/F and ν_1 are both integers or neither is an integer. When both are integers, it is seen that $-m = -\frac{G}{F} - \nu_1 = -\left[\frac{G}{F}\right] + [-\nu_1]$ so that $-M - 2 \leqq -m \leqq -M + 1$. When neither is an integer, it is seen that $m = \frac{G}{F} + \nu_1 = \left[\frac{G}{F}\right] + [\nu_1] + 1$ so that $M - 1 \leqq m \leqq M + 2$. Similar results follow for n and one has

$$M - 1 \leqq m \leqq M + 2, \quad N - 1 \leqq n \leqq N + 2.$$

Of the sixteen possible combinations, many can be eliminated by using the bounds on u and v derived above. For example, if $m = M + 2$, $n = N - 1$, then $U = a - b(M + 2) - c(N - 1) = u - 2b + c$, $V = \pm j + k(M + 2) - \ell(N - 1) = v + 2k + \ell$, so that $-2b + c < U \quad -b + 2c$, $k + \ell < V < 2k + 2\ell$. Then condition (2), namely $|V| < \ell$, is not satisfied.

After all possible rejections are made in this way there remain the cases:

	Solutions of (1), (2)	Bounds for U and V	W	Conditions for solution
(i)	$m = M + 2$ $n = N$	$-2b < U < -b + c$ $k < V < \ell + 2k$	$w + 2g$	$u < b$ $v < \ell - 2k$
(ii)	$m = M + 1$ $n = N$	$-b < U < c$ $0 < V < k + \ell$	$w + g$	$u > 0$ $v < \ell - k$
(iii)	$m = M + 1$ $n = N + 1$	$-b - c < U < 0$ $-\ell < V < k$	$w + g + h$	$u > c$ $v > -k$
(iv)	$m = M + 1$ $n = N + 2$	$-b - 2c < U < -c$ $-2\ell < V < k - \ell$	$w + g + 2h$	$u > 2c$ $v > \ell - k$
(v)	$m = M$ $n = N - 1$	$c < U < b + 2c$ $\ell - k < V < 2\ell$	$w - h$	$u < b - c$ $v < 0$
(vi)	$m = M$ $n = N$	$0 < U < b + c$ $-k < V < \ell$	w	$u < b$ $v < \ell$
(vii)	$m = M$ $n = N + 1$	$-c < U < b$ $-\ell - k < V < 0$	$w + h$	$u > 0$ $v > 0$
(viii)	$m = M - 1$ $n = N + 1$	$b - c < U < 2b$ $-\ell - 2k < V < -k$	$w - g + h$	$u < c$ $v > k$

It is now a question of the minimum of $|W|$. From the definitions of M and N and the conditions $a > b > c > 0$, $\ell > j$, one obtains $M \geqq 0$, $N \geqq -1$. Then $w \geqq f - h$ for all solutions, so that except possibly for (v), $W \geqq f - g > -g$. But the definition of the extreme parallelepiped $\{a, g, \ell\}$ requires that there be no solutions of $|\xi| < a$, $|\zeta| < \ell$, for which $|\eta| \leqq g$; hence no solutions of (1) and (2) for which $|W| \leqq g$. Then for all solutions except (v), one has $W > g$. For (v) one has $W = w - h$, but whenever

(v) is a solution of (1) and (2) so also is (vi). It was just shown that when (vi) is a solution, $w > g$, so that for (v) $W = w - h > g - h > 0$; consequently, $w - h > g$. Hence $|W| = W$ for all solutions.

The values of W in (viii) and (v) are the two smallest and since the conditions under which the two are solutions are mutually exclusive, (viii) is the solution of (1), (2), and (3) whenever $u < c$, $v > k$; and (v) is the solution whenever $u < b - c$, $v < 0$. The next smallest value of W is for (vi) which satisfies (1) and (2) whenever $u < b$. But (3) is not satisfied by (vi) unless neither (viii) nor (v) satisfies (1) and (2). Hence (vi) is the solution of (1), (2), and (3) if u and v satisfy the conditions for (vi) to be a solution of (1) and (2) but not the conditions for (viii) or (v) to be a solution. The continuation of this process leads to the following table which covers all possibilities:

	Conditions	m	n	Corresp.
1)	$u < c$, $v > k$	$M - 1$	$N + 1$	(viii)
2)	$u < b - c$, $v < 0$	M	$N - 1$	(v)
3), 4)	$u < b$, but not 1) or 2)	M	N	(vi)
5)	$u > b$, $v > 0$	M	$N + 1$	(vii)
6)	$u > b$, $v < 0$	$M + 1$	N	(ii)

It remains to determine T, $\bar{e}_1e_1$, $\bar{e}_2e_2$, $\bar{e}_3e_3$, and the system of signs in $\bar{\bar{\Phi}}$. Since $\bar{p}_2 = d_2(p_1 + mp_2 \pm np_3)$ one has

$$T = \left\| \begin{matrix} 0, & d_2, & 0 \\ d_1, & d_2m, & 0 \\ 0, & \pm d_2n, & d_3 \end{matrix} \right\| .$$

Following the procedure in I(see p.46) one obtains

$$\bar{\bar{\Phi}} = \left\| \begin{matrix} -\bar{e}_1e_1d_1b, & \bar{e}_1e_1d_2(a - mb - nc), & \mp \bar{e}_1e_1d_3c \\ \bar{e}_2e_2d_1g, & \bar{e}_2e_2d_2(f + mg + nh), & \pm \bar{e}_2e_2d_3h \\ \mp \bar{e}_3e_3d_1k, & \bar{e}_3e_3d_2(-j \mp mk \pm n\ell), & \bar{e}_3e_3d_3\ell \end{matrix} \right\| .$$

Recalling that $W > 0$ one has $d_1 = -\bar{e}_1e_1$, $d_2 = \bar{e}_2e_2$, $d_3 = \bar{e}_3e_3$, and

$$\bar{\bar{\Phi}} = \left\| \begin{matrix} (\bar{e}_1e_1)^2b, & (\bar{e}_1e_1)(\bar{e}_2e_2)U, & \mp (\bar{e}_1e_1)(\bar{e}_3e_3)c \\ -(\bar{e}_2e_2)(\bar{e}_1e_1)g, & (\bar{e}_2e_2)^2W, & \pm (\bar{e}_2e_2)(\bar{e}_3e_3)h \\ \pm (\bar{e}_3e_3)(\bar{e}_1e_1)k, & \mp (\bar{e}_3e_3)(\bar{e}_2e_2)V, & (\bar{e}_3e_3)^2\ell \end{matrix} \right\| .$$

1) $u < c$, $v > k$: $m = M - 1$, $n = N + 1$. Recalling that $0 < u < b + c$, $-k < v < \ell$ one has $U > 0$, $V < 0$. Lemmas 2.2 and 2.3 yield $\bar{e}_1e_1 = \mp 1$, $\bar{e}_2e_2 = \mp 1$, $\bar{e}_3e_3 = 1$; $d_1 = \pm 1$, $d_2 = \mp 1$, $d_3 = 1$; and the system of signs in $\bar{\bar{\Phi}}$ is I.

2) $u < b - c$, $v < 0$: $m = M$, $n = N - 1$; $U > 0$, $V > 0$. Lemmas 2.2 and 2.3 yield $\bar{e}_1e_1 = \pm 1$, $\bar{e}_2e_2 = \pm 1$, $\bar{e}_3e_3 = 1$, $d_1 = \mp 1$, $d_2 = \pm 1$, $d_3 = 1$, and the system of signs in $\bar{\bar{\Phi}}$

is IV.

3) $u < b$, $v > 0$, but not 1): $m = M$, $n = N$; $U > 0$, $V > 0$.

Same as 2).

4) $u < b$, $v < 0$, but not 2): $m = M$, $n = N$; $U > 0$, $V < 0$.

Same as 1).

5) $u > b$, $v > 0$: $m = M$, $n = N + 1$; $U > 0$, $V < 0$. Same as 1).

6) $u > b$, $v < 0$: $m = M + 1$, $n = N$; $U > 0$, $V > 0$. Same as 2).

III and IV.

The two cases can be combined. The conditions to be satisfied in the respective cases are:

III	IV
(1) $\lvert \bar{\kappa} a - \bar{m} b - \bar{n} c \rvert < b$,	(1) $\lvert \kappa a + mb - nc \rvert < b$,
(2) $\lvert \bar{\kappa} j + \bar{m} k + \bar{n} \ell \rvert < \ell$,	(2) $\lvert \kappa j - mk + n\ell \rvert < \ell$,
(3) $\lvert -\bar{\kappa} f + \bar{m} g - \bar{n} h \rvert = \min.$;	(3) $\lvert -\kappa f + mg + nh \rvert = \min.$

In III set $\kappa = \bar{\kappa}$, $m = -\bar{m}$, $n = \bar{n}$ and for both cases one has:

$$\begin{aligned} &(1') \quad U \equiv \kappa a + mb - nc = \lambda_1 b, \\ &(2') \quad V \equiv \kappa j - mk + n\ell = \lambda_2 \ell, \end{aligned} \qquad (\lvert\lambda_i\rvert < 1,\ \lambda_i \neq 0).$$

(3′) $W \equiv \pm\kappa f + mg + nh$ to have minimum absolute value.

The upper sign is for III and the lower for IV in (3′).

$\kappa = 0$:

Solve (1′) and (2′) for m and n,

$$m = \frac{\lambda_1 b\ell + \lambda_2 c\ell}{b\ell - ck}, \quad n = \frac{\lambda_2 b\ell + \lambda_1 bk}{b\ell - ck};$$

then

$$m - n = \lambda_1 \frac{b\ell - bk}{b\ell - ck} + \lambda_2 \frac{c\ell - b\ell}{b\ell - ck}.$$

Since $b > c$, $\ell > k$, one has $|m - n| < 2$; but m and n are integers, therefore $|m - n| \leqq 1$. Write $m = n + \varepsilon$ ($\varepsilon = 0$ or ± 1). Replace in ($1'$) and ($2'$) to obtain

$$\begin{aligned} (1'') \quad & n(b - c) + \varepsilon b = \lambda_1 b, \\ (2'') \quad & n(\ell - k) - \varepsilon k = \lambda_2 \ell, \end{aligned} \qquad (|\lambda_i| < 1,\ \lambda_i \neq 0).$$

Not both m and n are zero, hence $|n| + |\varepsilon| \geqq 1$. From ($1''$), it is evident that $n \neq 0$.

The supposition that $\varepsilon = 1$ yields $n \leqq -1$ from ($1''$) and $n \geqq 0$ from ($2''$). Therefore, $\varepsilon \neq 1$. The supposition that $\varepsilon = -1$ yields $n \geqq 1$ from ($1''$) and $n \leqq 0$ from ($2''$). Therefore $\varepsilon \neq -1$; i.e., $\varepsilon = 0$ and $m = n$.

The minimum of $|W|$ is then yielded for $m = n = \pm 1$, $|W| = g + h$. Also, $K = 0$, $m = n = 1$ always satisfies ($1'$) and ($2'$), so that this is the solution of ($1'$), ($2'$), and ($3'$), except when a smaller value of $|W|$ is yielded by another solution of ($1'$) and ($2'$) for which $K = 1$.

$K = 1$:

Solve ($1'$) for m and ($2'$) for n,

$$(1''')\quad m = \lambda_1 - \frac{a}{b} + n\frac{c}{b},$$
$$\qquad (|\lambda_i| < 1,\ \lambda_i \neq 0).$$
$$(2''')\quad n = \lambda_2 - \frac{j}{\ell} + m\frac{k}{\ell},$$

Then since m and n are integers

$m \leqq [\lambda_1] + [-a/b] + [nc/b] + 2$, $n \leqq [\lambda_2] + [-j/\ell] + [mk/\ell] + 2$.

(a) Assume $m > 0$, $n > 0$, then $m \leqq 0 - 2 + n - 1 + 2 = n - 1$, $n \leqq 0 - 1 + m - 1 + 2 = m$; but these inequalities are incompatible, hence either $m \leqq 0$ or $n \leqq 0$.

(b) Assume $n < 0$, then from $(1''')$, $m \leqq 0 - 2 - 1 + 2 = -1$.

(c) Assume $m < 0$, then from $(2''')$, $n \leqq 0 - 1 - 1 + 2 = 0$.

(d) Assume $n = 0$, then from $(1''')$, $m = \lambda_1 - a/b < 0$ or $m \leqq -1$.

(e) Assume $m = 0$, then from $(2''')$, $n = \lambda_2 - j/\ell < 1$ or $n \leqq 0$.

<u>Then in all cases one has</u> $m \leqq -1$, $n \leqq 0$ (since (d) precludes the possibility of $m = 0$, $n = 0$).

The minimum of $|W|$ will be determined next, and for this, III and IV will be treated separately.

III: From Theorem 2.4, $j + k > \ell$, $h > f$; then $(2')$ is not satisfied by $\kappa = 1$, $m \leqq -1$, $n = 0$. The two solutions for which $|W|$ is the smallest are

$m = -1$, $n = -1$: $|W| = -f + g + h > g$;

$m = -1$, $n = -2$: $|W| = -f + g + 2h > g + h$.

Of these, only the former value of $|W|$ is less than $g + h$, and the triple $\kappa = 1$, $m = -1$, $n = -1$ is to be taken whenever it is a solution of $(1')$ and $(2')$; otherwise $\kappa' = 0$, $m = 1$, $n = 1$.

That $\kappa' = 1$, $m = -1$, $n = -1$ be a solution of $(1')$ requires that $a + c < 2b$; whereas the triple is always a solution of $(2')$.

Then for III one has

1] $a + c < 2b$: $\kappa' = 1$, $m = -1$, $n = -1$;

2] $a + c > 2b$: $\kappa' = 0$, $m = 1$, $n = 1$.

<u>The ξ-neighbor is of the first kind if</u> $a + c < 2b$; <u>otherwise it is of the second kind.</u>

IV: Here $m \leqq -1$, $n \leqq 0$, and the smallest value of $|W|$ is yielded for $m = -1$, $n = 0$, namely $|W| = f + g$. For all other pairs of m and n satisfying these inequalities. $|W| \geqq f + g + h > g + h$. Then $|W|$ can be smaller than $g + h$, the value for $\kappa' = 0$, $m = 1$, $n = 1$, only if $f < h$, and if the triple $\kappa' = 1$, $m = -1$, $n = 0$ satisfies both $(1')$ and $(2')$. That $(1')$ and $(2')$ be satisfied imposes the conditions $a < 2b$, $j + k < \ell$.

Then for IV

1] $a < 2b$, $f < h$, $j + k < \ell$: $\kappa' = 1$, $m = -1$, $n = 0$;

2] $a > 2b$ or $f > h$ or $j + k > \ell$: $\kappa' = 0$, $m = 1$, $n = 1$.

<u>The ξ-neighbor is of the first kind only if</u> $a < 2b$, $f < h$, <u>and</u> $j + k < \ell$; <u>otherwise it is of the second kind.</u>

It remains to determine T, $\bar{e}_i e_i$, and the system of signs in $\bar{\bar{\Phi}}$.

III:
$$T = \left\| \begin{matrix} 0, & \kappa d_2, & 0 \\ d_1, & d_2, & 0 \\ 0, & -d_2, & d_3 \end{matrix} \right\|.$$

Then following the procedure as in I (see p. 46) one obtains

$$\bar{\Phi} = \left\| \begin{matrix} -\bar{e}_1 e_1 d_1 b, & \bar{e}_1 e_1 d_2(\kappa a - b + c), & -\bar{e}_1 e_1 d_3 c \\ \bar{e}_2 e_2 d_1 g, & \bar{e}_2 e_2 d_2(-\kappa f + g + h), & -\bar{e}_2 e_2 d_3 h \\ \bar{e}_3 e_3 d_1 k, & \bar{e}_3 e_3 d_2(\kappa j + k - \ell), & \bar{e}_3 e_3 d_3 \ell \end{matrix} \right\|.$$

Hence $d_1 = -\bar{e}_1 e_1$, $d_2 = \bar{e}_2 e_2$, $d_3 = \bar{e}_3 e_3$, and

$$\bar{\bar{\Phi}} = \left\| \begin{matrix} (\bar{e}_1 e_1)^2 b, & (\bar{e}_1 e_1)(\bar{e}_2 e_2)(\kappa a - b + c), & -(\bar{e}_1 e_1)(\bar{e}_3 e_3) c \\ -(\bar{e}_2 e_2)(\bar{e}_1 e_1) g, & (\bar{e}_2 e_2)^2(-\kappa f + g + h), & -(\bar{e}_2 e_2)(\bar{e}_3 e_3) h \\ -(\bar{e}_3 e_3)(\bar{e}_1 e_1) k, & (\bar{e}_3 e_3)(\bar{e}_2 e_2)(\kappa j + k - \ell), & (\bar{e}_3 e_3)^2 \ell \end{matrix} \right\|.$$

$\kappa = 0$: Lemmas 2.2 and 2.3 yield $\bar{e}_1 e_1 = 1$, $\bar{e}_2 e_2 = 1$, $\bar{e}_3 e_3 = 1$, and the system of signs in $\bar{\bar{\Phi}}$ is VI; then $d_1 = -1$, $d_2 = 1$, $d_3 = 1$.

$\kappa = 1$: Lemmas 2.2 and 2.3 yield $\bar{e}_1 e_1 = -1$, $\bar{e}_2 e_2 = 1$, $\bar{e}_3 e_3 = -1$, and the system of signs in $\bar{\bar{\Phi}}$ is II; then $d_1 = 1$, $d_2 = 1$, $d_3 = -1$.

IV:
$$T = \left\| \begin{matrix} 0, & \kappa d_2 & 0 \\ d_1, & -d_2, & 0 \\ 0, & (\kappa - 1) d_2, & d_3 \end{matrix} \right\|.$$

The same procedure as before leads to

$$\overline{\overline{\Phi}} = \left\| \begin{matrix} \bar{e}_1e_1d_1b, & \bar{e}_1e_1d_2(\varkappa a - b - \overline{\varkappa - 1}\, c), & -\bar{e}_1e_1d_3c \\ \bar{e}_2e_2d_1g, & \bar{e}_2e_2d_2(-\varkappa f - g + \overline{\varkappa - 1}\, h), & \bar{e}_2e_2d_3h \\ -\bar{e}_3e_3d_1k, & \bar{e}_3e_3d_2(\varkappa j + k + \overline{\varkappa - 1}\, \ell), & \bar{e}_3e_3d_3\ell \end{matrix} \right\|,$$

then $d_1 = \bar{e}_1e_1$, $d_2 = -\bar{e}_2e_2$, $d_3 = \bar{e}_3e_3$. One obtains, finally,

$\varkappa = 0$: $\bar{e}_1e_1 = -1$, $\bar{e}_2e_2 = 1$, $\bar{e}_3e_3 = -1$; the system of signs in $\overline{\overline{\Phi}}$ is VI; and $d_1 = -1$, $d_2 = -1$, $d_3 = -1$.

$\varkappa = 1$: $\bar{e}_1e_1 = 1$, $\bar{e}_2e_2 = 1$, $\bar{e}_3e_3 = 1$; the system of signs in $\overline{\overline{\Phi}}$ is II; and $d_1 = 1$, $d_2 = -1$, $d_3 = 1$.

<u>The ξ-neighbor of the ξ-neighbor in</u> III 2] <u>and</u> IV 2]:

Let P, $\overline{\Phi}$, be the matrices and e_1, e_2, e_3 the e_i of Theorem 2.3 for $\{a, g, \ell\}$ and let P^*, $\overline{\Phi}^*$, e_1^*, e_2^*, and e_3^* have the corresponding meanings for the ξ-neighbor of the ξ-neighbor. The solution follows the same lines as for the ξ-neighbor in II and V.

Theorem 3.3 shows that the determinant of P^* is 1. Since P also has a determinant 1, P^{-1} is a matrix of integers with determinant 1. Then $T^* \equiv P^{-1}P^*$ is a matrix of integers with determinant 1.

1] $b - c > c$: Since $\mathfrak{p}_1^* = \pm\bar{\mathfrak{p}}_2$, $\mathfrak{p}_3^* = \pm\bar{\mathfrak{p}}_3$, where $\bar{\mathfrak{p}}_2$ and $\bar{\mathfrak{p}}_3$ are the lattice points on the ξ-neighbor of $\{a, g, \ell\}$ as found in III 2] and IV 2], one has

$$T^* = \left\| \begin{matrix} 0, & \kappa^* d_2, & 0 \\ d_1, & m^* d_2, & 0 \\ \mp d_1, & n^* d_2, & d_3 \end{matrix} \right\|,$$

where κ^*, m^*, n^*, are integers and d_i are to determined as 1 or $-$ 1. The upper sign in the first column of the matrix T^* is for III and the lower sign for IV. Since the determinant of T^* is 1, $|\kappa^*| = 1$. Choose $\kappa^* = 1$.

Set $m^* = \pm m$, $n^* = -(m - n)$, then

$$T = \left\| \begin{matrix} 0, & d_2, & 0 \\ d_1, & \pm m d_2, & 0 \\ \mp d_1, & -(m - n) d_2, & d_3 \end{matrix} \right\|;$$

this has the effect of expressing $\mathfrak{p}_2^*$ in terms of $\mathfrak{p}_1$, $\bar{\mathfrak{p}}_2$ $(= \mathfrak{p}_2 \mp \mathfrak{p}_3)$, and $\bar{\mathfrak{p}}_3$ $(= \pm \mathfrak{p}_3)$.

Set $u^o = b - c$, $u' = c$; $v^o = \ell$, $v' = \ell - k$; $w^o = g + h$, $w' = h$, then $u^o > u' > 0$; $w^o > f > 0$, $w^o > w' > 0$; $v^o > j > 0$, $v^o > v' > 0$ (and $w' > f > 0$ in III), and

$$\xi(\mathfrak{p}_2^*) = e_1 d_2 (a - m u^o - n u'),$$
$$\eta(\mathfrak{p}_2^*) = e_2 d_2 (-f \pm m w^o \mp n w'),$$
$$\zeta(\mathfrak{p}_2^*) = e_3 d_2 (j - m v' + n v^o).$$

The conditions to be satisfied are

(1) $| a - m u^o - n u' | < u^o$,

(2) $| -j + m v' - n v^o | < v^o$,

(3) $| \mp f + m w^o - n w' | = \min.$

Rewrite (1), (2), and (3) as

$$(1') \quad U \equiv a - mu^{\circ} - nu' = \lambda_1 u^{\circ}, \qquad (|\lambda_i| < 1,\ \lambda_i \neq 0).$$
$$(2') \quad V \equiv -j + mv' - nv^{\circ} = \lambda_2 v^{\circ},$$

(3′) $W \equiv \mp f + mw^{\circ} - nw'$ to have minimum absolute value.

Solve (1′) and (2′) for m and n and evaluate the parts independent of λ_i in terms of F, G, and H, the respective cofactors of $-f$, g, and $\mp h$ in Φ.

$$m = \frac{av^{\circ} + ju'}{u^{\circ}v^{\circ} + u'v'} + \frac{-\lambda_1 u^{\circ}v^{\circ} + \lambda_2 v^{\circ}u'}{u^{\circ}v^{\circ} + u'v'} = \frac{\pm G}{F} + \gamma_1,$$

$$n = \frac{-ju^{\circ} + av'}{u^{\circ}v^{\circ} + u'v'} + \frac{-\lambda_2 u^{\circ}v^{\circ} - \lambda_1 u^{\circ}v'}{u^{\circ}v^{\circ} + u'v'} = \frac{\pm G + H}{F} + \gamma_2.$$

As before, from the conditions $u^{\circ} > u' > 0$, $v^{\circ} > v' > 0$, and $|\lambda_i| < 1$, one has $|\gamma_i| < 2$, and $M - 1 \leqq m \leqq M + 2$, $N - 1 \leqq n \leqq N + 2$.

Set $M = [\pm G/F]$, $N = [(\pm G + H)/F]$, $u \equiv a - Mu^{\circ} - Nu'$, $v \equiv -j + Mv' - Nv^{\circ}$, $w \equiv \mp f + Mw^{\circ} - Nw'$, and obtain from Φ as in II and V:

$$u^{\circ} + u' > u > 0,\quad v^{\circ} > v > -v',\quad w > \frac{\Delta}{\pm F} - w^{\circ} > -w^{\circ}.$$

(The quantities u°, u', v°, v', w°, w' here take the places of b, c, ℓ, k, g, and h, respectively, in II and V.)

The solutions of (1) and (2) in terms of the M and N defined here have precisely the same form as solutions (i) to (viii) of (1) and (2) for II and V. There is a difference, however, in the relative magnitudes of $|W|$.

	Solutions of (1) and (2)	W	Conditions for solution.
(i)	$m = M + 2$ $n = N$	$w + 2w^{o}$	$u < u^{o}$ $v < v^{o} - 2v'$
(ii)	$m = M + 1$ $n = N$	$w + w^{o}$	$u > 0$ $v < v^{o} - v'$
(iii)	$m = M + 1$ $n = N + 1$	$w + w^{o} - w'$	$u > u'$ $v > - v'$
(iv)	$m = M + 1$ $n = N + 2$	$w + w^{o} - 2w'$	$u > 2u'$ $v > v^{o} - v'$
(v)	$m = M$ $n = N - 1$	$w + w'$	$u < u^{o} - u'$ $v < 0$
(vi)	$m = M$ $n = N$	w	$u < u^{o}$ $v < v^{o}$
(vii)	$m = M$ $n = N + 1$	$w - w'$	$u > 0$ $v > 0$
(viii)	$m = M - 1$ $n = N + 1$	$w - w^{o} - w'$	$u < u'$ $v > v'$

From the conditions on w, w^{o}, w', it is apparent that $W > - w^{o}$ for each solution, except perhaps (iv), (vii), and (viii); but there are no solutions of (1) and (2) for which $|W| < w^{o}$, hence the solutions for which $W > - w^{o}$ satisfy also the condition $W > w^{o}$. It can be shown that $W > w^{o}$ even for the three possible exceptions mentioned. For, if (viii) is a solution, so also is (vi). Then $w > w^{o}$, and as a consequence, $w - w^{o} - w' > - w^{o}$ is satisfied. Hence $w - w^{o} - w' > w^{o}$. If (vii), but not (viii), is a solution, then either (vi) or (iii) is a solution, and in either event, one has $w - w' > w^o$. In a similar manner it can be shown that if (iv) is a solution, $w + w^o - 2w' > w^o$. Hence $|W| = W$.

From the eight solutions of (1) and (2) one obtains the following table of all solutions of (1), (2), and (3):

	Conditions	m	n	Corresp.
1)	$u < u'$, $v > v'$	M - 1	N + 1	(viii)
2), 3)	$u > 0$, $v > 0$ but not 1)	M	N + 1	(vii)
4)	$u < u^o$, $v < 0$	M	N	(vi)
5)	$u > u^o$, $v < 0$	M + 1	N + 1	(iii)

The values of d_1, d_2, d_3, $e_1^*e_1$, $e_2^*e_2$, $e_3^*e_3$, and the system of signs in Φ^* will now be determined for the various solutions 1) to 5).

Since $P^* = PT^*$, one has

$$\Phi^* = \left\| \begin{matrix} \mp e_1^*e_1d_1u^o, & e_1^*e_1d_2(a - mu^o - nu'), & -e_1^*e_1d_3u' \\ e_2^*e_2d_1w^o, & e_2^*e_2d_2(-f \pm mw^o \mp nw'), & \mp e_2^*e_2d_3w' \\ \mp e_3^*e_3d_1v', & e_3^*e_3d_2(j - mv' + nv^o), & e_3^*e_3d_3v^o \end{matrix} \right\| .$$

Then $d_1 = \mp e_1^*e_1$, $d_2 = \pm e_2^*e_2$, $d_3 = e_3^*e_3$, and

$$\Phi^* = \left\| \begin{matrix} (e_1^*e_1)^2u^o, & \pm (e_1^*e_1)(e_2^*e_2)U, & -(e_1^*e_1)(e_3^*e_3)u' \\ \mp (e_2^*e_2)(e_1^*e_1)w^o, & (e_2^*e_2)^2W, & \mp (e_2^*e_2)(e_3^*e_3)w' \\ (e_3^*e_3)(e_1^*e_1)v', & \mp (e_3^*e_3)(e_2^*e_2)V, & (e_3^*e_3)^2v^o \end{matrix} \right\|$$

1) $u < u'$, $v > v'$: $U > 0$, $V < 0$. Lemmas 2.2 and 2.3 yield $e_1^*e_1 = \pm 1$, $e_2^*e_2 = -1$, $e_3^*e_3 = \mp 1$, and the system of signs in Φ^* is V. Then $d_1 = -1$, $d_2 = \mp 1$, $d_3 = \mp 1$.

2) $u < u'$, $v' > v > 0$: $U < 0$, $V < 0$. Lemmas 2.2 and 2.3 yield $e_1^*e_1 = \pm 1$, $e_2^*e_2 = 1$, $e_3^*e_3 = \pm 1$, and the system of signs in Φ^* is III. Then $d_1 = -1$, $d_2 = \pm 1$, $d_3 = \pm 1$.

3) $u > u'$, $v > 0$: $U > 0$, $V < 0$. Same as 1).

4) $u < u^o$, $v < 0$: $U > 0$, $V < 0$. Same as 1).

5) $u > u^o$, $v < 0$: $U < 0$, $V < 0$. Same as 2).

2] $b - c < c$:

Here $\mathfrak{p}_1^* = d_1\mathfrak{p}_3$ and $\mathfrak{p}_2^* = d_2(\mathfrak{p}_2 \mp \mathfrak{p}_3)$. The third point is a linear combination with integral coefficients of $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mp\mathfrak{p}_3$, and $\mathfrak{p}_3$. Set $\mathfrak{p}_3^* = d_3(\kappa p_1 \pm n(\mathfrak{p}_2 \mp \mathfrak{p}_3) + m\mathfrak{p}_3)$, then

$$T^* = \begin{Vmatrix} 0, & 0, & \kappa d_3 \\ 0, & d_2, & \pm n d_3 \\ d_1, & \mp d_2, & (m-n)d_3 \end{Vmatrix} \qquad (d_i = \pm 1).$$

But the determinant of T^* is 1, so that $|\kappa| = 1$. Choose $\kappa = 1$ as before. Then

$$\xi(\mathfrak{p}_3^*) = e_1 d_3(\, a - n\,\overline{b - c} - mc),$$
$$\eta(\mathfrak{p}_3^*) = e_2 d_3(-f \pm n\,\overline{g + h} + mh),$$
$$\zeta(\mathfrak{p}_3^*) = e_3 d_3(\, j - n\,\overline{\ell - k} + m\ell),$$

Set $u^o = c$, $u' = b - c$; $v^o = g + h$, $v' = h$; $w^o = \ell$, $w' = \ell - k$. The conditions to be satisfied by integers m and n are

(1) $U \equiv a - mu^o - nu' = \lambda_1 u^o$, $\quad (|\lambda_i| < 1,\ \lambda_i \neq 0)$.

(2) $V \equiv \pm f + mv' - nv^o = \lambda_2 v^o$,

(3) $W \equiv j + mw^o - nw'$ to have minimum absolute value.

Solve (1) and (2) for m and n, evaluating the parts independent of λ_i in terms of J, K, and L, the respective cofactors of j, ±k, and ℓ in Φ.

$$m = \frac{av^o \mp fu'}{u^ov^o + u'v'} + \frac{-\lambda_1 u^ov^o + \lambda_2 u'v^o}{u^ov^o + u'v'} = \frac{\pm K + L}{J} + \gamma_1,$$

$$n = \frac{\pm fu^o + av'}{u^ov^o + u'v'} + \frac{-\lambda_2 u^ov^o - \lambda_1 u^ov'}{u^ov^o + u'v'} = \frac{\pm K}{J} + \gamma_2.$$

Set $M = \left[(\pm K + L)/J\right]$, $N = \left[\pm K/J\right]$, then as before $|\gamma_i| < 2$ and $M - 1 \leqq m \leqq M + 2$, $N - 1 \leqq n \leqq N + 2$.

Let $u = a - Mu^o - Nu'$, $v = \pm f + Mv' - Nv^o$, $w = j + Mw^o - Nw'$, then from Φ obtain

$$u^o + u' > u > 0,\ v^o > v > -v',\ w > \frac{\Delta}{J} - w^o > -w^o.$$

One has also, $u^o > u' > 0$, $v^o > v' > 0$, $v^o > f > 0$. There are then five solutions of (1), (2), and (3) in terms of these M and N and they are of the same form as for $b - c > c$.

The only difference arises in the values of d_1, d_2, d_3, $e_1^* e_1$, $e_2^* e_2$, $e_3^* e_3$, and the system of signs in Φ^*.

Employing $P^* = PT^*$ one obtains

$$\bar{\Phi}^* = \begin{Vmatrix} -\ e_1^*e_1d_1u^\circ, & \mp\ e_1^*e_1d_2u', & e_1^*e_1d_3U \\ \mp\ e_2^*e_2d_1v', & e_2^*e_2d_2v^\circ, & \mp\ e_2^*e_2d_3V \\ e_3^*e_3d_1w^\circ, & \mp\ e_3^*e_3d_2w', & e_3^*e_3d_3W \end{Vmatrix} .$$

Then $d_1 = -\ e_1^*e_1$, $d_2 = e_2^*e_2$, $d_3 = e_3^*e_3$, and

$$\bar{\Phi}^* = \begin{Vmatrix} (e_1^*e_1)^2u^\circ, & \mp\ (e_1^*e_1)(e_2^*e_2)u', & (e_1^*e_1)(e_3^*e_3)U \\ \pm\ (e_2^*e_2)(e_1^*e_1)v', & (e_2^*e_2)^2v_0, & \mp\ (e_2^*e_2)(e_3^*e_3)V \\ -\ (e_3^*e_3)(e_1^*e_1)w^\circ, & \mp\ (e_3^*e_3)(e_2^*e_2)w', & (e_3^*e_3)^2W \end{Vmatrix} .$$

1) $u < u'$, $v > v'$: $U > 0$, $V < 0$. Lemmas 2.2 and 2.3 yield $e_1^*e_1 = \pm 1$, $e_2^*e_2 = -1$, $e_3^*e_3 = \mp 1$, and the system of signs in $\bar{\Phi}^*$ is IV. Then $d_1 = \mp 1$, $d_2 = -1$, $d_3 = \mp 1$.

2) $u < u'$, $v' > v > 0$: $U < 0$, $V < 0$. Lemmas 2.2 and 2.3 yield $e_1^*e_1 = \pm 1$, $e_2^*e_2 = 1$, $e_3^*e_3 = \pm 1$, and the system of signs in $\bar{\Phi}^*$ is II. Then $d_1 = \mp 1$, $d_2 = 1$, $d_3 = \mp 1$.

3) $u > u'$, $v > 0$: $U > 0$, $V < 0$; same as 1).

4) $u < u^\circ$, $v < 0$: $U > 0$, $V < 0$; same as 1).

5) $u > u^\circ$, $v < 0$: $U < 0$, $V < 0$; same as 2).

<u>The</u> η-<u>neighbor of the</u> ξ-<u>neighbor is</u> $\{a, g, \ell\}$ <u>itself when</u> $b > c$.

This is an immediate consequence of the definition.

<u>The</u> ζ-<u>neighbor of the</u> ξ-<u>neighbor in</u> III 2] <u>and</u> IV 2]:

1] $\ell - k > k$:

Here $p_1^* = d_1p_2$, $p_3^* = d_3(p_2 \mp p_3)$, and $p_2^* = d_2(Kp_1 \pm m(p_2 \mp p_3) \mp np_2)$.

Then

$$T^* = \left\| \begin{matrix} 0, & \kappa d_2 & 0 \\ d_1, & \pm(m-n)d_2, & d_3 \\ 0, & -md_2, & \mp d_3 \end{matrix} \right\|.$$

The determinant of T^* is 1, hence $|\kappa| = 1$. Choose $\kappa = 1$. Set $u^o = \ell - k$, $u' = k$; $v^o = b$, $v' = b - c$; $w^o = g + h$, $w' = g$; then

$$\xi(p_2^*) = e_1d_2(\ a - mv' + nv^o),$$
$$\eta(p_2^*) = e_2d_2(-f \pm mw^o \mp nw'),$$
$$\zeta(p_2^*) = e_3d_2(\ j - mu^o - nu').$$

The conditions to be satisfied are then

(1) $U \equiv\ j - mu^o - nu' = \lambda_1 u^o$,

(2) $V \equiv -a + mv' - nv^o = \lambda_2 v^o$, $\quad (|\lambda_i| < 1,\ \lambda_i \neq 0)$.

(3) $W \equiv \mp f + mw^o - mw'$ to have minimum absolute value.

Solve (1) and (2) for m and n and evaluate the parts independent of λ_i in terms of F, G, and H, the respective cofactors of $-f$, g, and $\mp h$ in Φ.

$$m = \frac{jv^o + au'}{u^ov^o + u'v'} + \frac{-\lambda_1 u^ov^o + \lambda_2 u'v^o}{u^ov^o + u'v'} = \frac{-H}{F} + \gamma_1,$$

$$n = \frac{-au^o + jv'}{u^ov^o + u'v'} + \frac{-\lambda_2 u^ov^o - \lambda_1 u^ov'}{u^ov^o + u'v'} = \frac{\mp G - H}{F} + \gamma_2.$$

Set $M = [-H/F]$, $N = [(\mp G - H)/F]$, then since $v^o > v' > 0$, $u^o > u' > 0$, one has $|\gamma_i| < 2$ and $M - 1 \leqq m \leqq M + 2$, $N - 1 \leqq n \leqq N + 2$.

Set $u = j - Mu^o - Nu'$, $v = -a + Mv' - Nv^o$, $w = \mp f + Mw^o - Nw'$; then from Φ, $u^o + u' > u > 0$, $v^o > v > -v'$, and $w > \frac{\Delta}{\pm F} - w^o > -w^o$. The solutions of (1), (2), and (3) are then, in terms of these M and N, of the same form as the five solutions in the last two previous cases.

The essential difference arises in the values of d_1, d_2, d_3, $e_1^*e_1$, $e_2^*e_2$, $e_3^*e_3$, and the system of signs in Φ^*. Employing $P^* = PT^*$ one obtains

$$\Phi^* = \left\| \begin{matrix} \mp e_1^*e_1d_1v^o, & -e_1^*e_1d_2V, & \mp e_1^*e_1d_3v' \\ e_2^*e_2d_1w', & \pm e_2^*e_2d_2W, & e_2^*e_2d_3w^o \\ \pm e_3^*e_3d_1u', & e_3^*e_3d_2U, & \mp e_3^*e_3d_3u^o \end{matrix} \right\| .$$

Then $d_1 = \mp e_1^*e_1$, $d_2 = \pm e_2^*e_2$, $d_3 = \mp e_3^*e_3$, and

$$\Phi^* = \left\| \begin{matrix} (e_1^*e_1)^2v^o, & \mp(e_1^*e_1)(e_2^*e_2)V, & (e_1^*e_1)(e_3^*e_3)v' \\ \mp(e_2^*e_2)(e_1^*e_1)w', & (e_2^*e_2)^2W, & \mp(e_2^*e_2)(e_3^*e_3)w^o \\ -(e_3^*e_3)(e_1^*e_1)u', & \pm(e_3^*e_3)(e_2^*e_2)U, & (e_3^*e_3)^2u^o \end{matrix} \right\| .$$

1) $u < u'$, $v > v'$: $U > 0$, $V < 0$. Lemmas 2.2 and 2.3 yield $e_1^*e_1 = \mp 1$, $e_2^*e_2 = -1$, $e_3^*e_3 = \pm 1$, and the system of signs in Φ^* is IV. Then $d_1 = 1$, $d_2 = \mp 1$, $d_3 = -1$.

2) $u < u'$, $v' > v > 0$: $U < 0$, $V < 0$. Lemmas 2.2 and 2.3 yield $e_1^*e_1 = \pm 1$, $e_2^*e_2 = 1$, $e_3^*e_3 = \pm 1$, and the system of signs in Φ^* is I. Then $d_1 = -1$, $d_2 = \pm 1$, $d_3 = -1$.

3) $u > u'$, $v > 0$: $U > 0$, $V < 0$; same as 1).

4) $u < u^{o}$, $v < 0$: $U > 0$, $V < 0$; same as 1).

5) $u > u^{o}$, $v < 0$: $U < 0$, $V < 0$; same as 2).

2] $k > \ell - k$:

Here $\mathfrak{p}_2^* = d_2(\mathfrak{p}_2 \mp \mathfrak{p}_3)$, $\mathfrak{p}_3^* = d_3\mathfrak{p}_2$, and $\mathfrak{p}_1^* = d_1(\kappa\mathfrak{p}_1 + m\mathfrak{p}_2 + n\mathfrak{p}_3)$. Then

$$T^* = \begin{Vmatrix} \kappa d_1, & 0, & 0 \\ md_1, & d_2, & d_3 \\ nd_1, & \mp d_2, & 0 \end{Vmatrix}.$$

The determinant of T^* is 1, hence $|\kappa| = 1$. Choose $\kappa = 1$. Then

$$\xi(\mathfrak{p}_1^*) = e_1d_1(\ a \mp mb - nc),$$
$$\eta(\mathfrak{p}_1^*) = e_2d_1(-f + mg \mp nh),$$
$$\zeta(\mathfrak{p}_1^*) = e_3d_1(\ j \pm mk + n\ell).$$

The conditions to be satisfied are

(1) $U \equiv -f + mg \mp nh = \lambda_1(g + h)$,

(2) $V \equiv j \pm mk + n\ell = \lambda_2 k$, $\quad (|\lambda_i| < 1,\ \lambda_i \neq 0)$.

(3) $W \equiv a \mp mb - nc$ to have minimum absolute value.

Solve (1) and (2) for m and n and evaluate the parts independent of λ_i in terms of A, B, and C, the respective cofactors of a, $\mp$ b, and $-$ c in Φ.

$$m = \frac{f\ell \mp hj}{g\ell + hk} + \frac{\lambda_1 g\ell + \lambda_1 h\ell \pm \lambda_2 hk}{g\ell + hk} = \frac{B}{A} + \nu_1,$$

$$n = \frac{-gj \mp fk}{g\ell + hk} + \frac{\lambda_2 gk \mp \lambda_1 gk \mp \lambda_1 hk}{g\ell + hk} = \frac{C}{A} + \nu_2.$$

Due to the conditions on λ_i and those from Theorem 2.3 on g, h, ℓ, and k, one has $|\nu_i| < 2$.

Since $b > c$, Theorem 2.4 requires that $h > f$ for III; then

$$\left.\begin{matrix}-1\\0\end{matrix}\right\} < \frac{B}{A} < \left\{\begin{matrix}1\\2\end{matrix}\right. \quad , \quad -1 < \frac{C}{A} < \left\{\begin{matrix}0\\1\end{matrix}\right. \quad ;$$

the upper line being for III, the lower for IV. Hence

$$\left.\begin{matrix}-2\\-1\end{matrix}\right\} \leqq m \leqq \left\{\begin{matrix}2\\3\end{matrix}\right. \quad , \quad -2 \leqq n \leqq \left\{\begin{matrix}1\\2\end{matrix}\right. \quad .$$

Direct substitution of these values in (1) and (2) rejects all except the following pairs which are solutions of (1) and (2):

	m	n	Case	W	Conditions for sol.
(i)	-1	0	III	$a + b$	$f < h$, $j < 2k$
(ii)	0	-1	III, IV	$a + c$	$j + k > \ell$
(iii)	0	0	III, IV	a	$j < k$
(iv)	1	-1	III	$a - b + c$	$\lvert j + k - \ell \rvert < k$
(v)	1	0	IV	$a + b$	$j < 2k$
(vi)	1	1	IV	$a + b - c$	$j + \ell < 2k$
(vii)	2	0	IV	$a + 2b$	$f + h > g$, $3k > j > k$.

The values of W are all positive. Among the solutions for III, (iv) corresponds to the smallest value of W. On the other hand, the condition $|j + k - \ell| < k$ is always satisfied as a consequence of the conditions $j + k > \ell$, $\ell > j > 0$, $\ell > k > 0$, of Theorems 2.3 and 2.4. That is, (iv) is always the solution of (1), (2), and (3) for III.

Among the solutions for IV, to (iii) corresponds the smallest value of W, hence (iii) is the solution of (1), (2), and (3) if $j < k$. When $j > k$, it is apparent that (vi) is not a solution of (1) and (2); then, to (ii) corresponds the smallest value of W. Consequently, (ii) is the solution of (1), (2), and (3) except when $j + k < \ell$. But this does not occur when $k > \ell - k$, $j > k$. That is, when $j > k$ it follows that (ii) is always the solution of (1), (2), and (3).

The determination of d_1, d_2, d_3, $e_1^* e_1$, $e_2^* e_2$, $e_3^* e_3$, and the system of signs in Φ^* is carried out below:

III;

$$T^* = \begin{Vmatrix} d_1, & 0, & 0 \\ d_1, & d_2, & d_3 \\ -d_1, & -d_2, & 0 \end{Vmatrix}.$$

Employing $P^* = PT^*$ one obtains

$$\Phi^* = \begin{Vmatrix} e_1^*e_1d_1(a - b + c), & -e_1^*e_1d_2(b - c), & -e_1^*e_1d_3b \\ e_2^*e_2d_1(-f + g + h), & e_2^*e_2d_2(g + h), & e_2^*e_2d_3g \\ e_3^*e_3d_1(j + k - \ell), & -e_3^*e_3d_2(\ell - k), & e_3^*e_3d_3k \end{Vmatrix}.$$

Hence $d_1 = e_1^*e_1$, $d_2 = e_2^*e_2$, $d_3 = e_3^*e_3$; then Lemmas 2.2 and 2.3 yield $e_1^*e_1 = -1$, $e_2^*e_2 = -1$, $e_3^*e_3 = 1$, and the system of signs in Φ^* is V. Hence $d_1 = -1$, $d_2 = -1$, $d_3 = 1$.

IV 1) $j > k$:

$$T^* = \begin{Vmatrix} d_1, & 0, & 0 \\ 0, & d_2, & d_3 \\ -d_1, & d_2, & 0 \end{Vmatrix}.$$

Employing $P^* = PT^*$ one obtains

$$\Phi^* = \begin{Vmatrix} e_1^*e_1d_1(a - c), & e_1^*e_1d_2(b - c), & e_1^*e_1d_3b \\ -e_2^*e_2d_1(f + h), & e_2^*e_2d_2(g + h), & e_2^*e_2d_3g \\ -e_3^*e_3d_1(\ell - j), & e_3^*e_3d_2(\ell - k), & -e_3^*e_3d_3k \end{Vmatrix}.$$

Then $d_1 = e_1^*e_1$, $d_2 = e_2^*e_2$, $d_3 = -e_3^*e_3$, and Lemmas 2.2 and 2.3 yield $e_1^*e_1 = -1$, $e_2^*e_2 = 1$, $e_3^*e_3 = -1$, and the system of signs in Φ^* is II. Hence $d_1 = -1$, $d_2 = 1$, $d_3 = 1$.

IV 2) $j < k$:

$$T^* = \begin{Vmatrix} d_1, & 0, & 0 \\ 0, & d_2, & d_3 \\ 0, & d_2, & 0 \end{Vmatrix}.$$

Employing $P^* = PT^*$ one obtains

$$\Phi^* = \begin{Vmatrix} e_1^*e_1d_1a, & e_1^*e_1d_2(b - c), & e_1^*e_1d_3b \\ - e_2^*e_2d_1f, & e_2^*e_2d_2(g + h), & e_2^*e_2d_3g \\ e_3^*e_3d_1j, & e_3^*e_3d_2(\ell - k), & - e_3^*e_3d_3k \end{Vmatrix} .$$

Then $d_1 = e_1^*e_1$, $d_2 = e_2^*e_2$, $d_3 = - e_3^*e_3$, and Lemmas 2.2 and 2.3 yield $e_1^*e_1 = 1$, $e_2^*e_2 = - 1$, $e_3^*e_3 = - 1$, and the system of signs in Φ^* is V. Hence $d_1 = 1$, $d_2 = - 1$, $d_3 = 1$.

CHAPTER IV

A SECOND ALGORITHM

In what follows there is given a process for obtaining the coordinates of lattice points for which $|\xi|$ and $|\eta|$ are arbitrarily small. In this chapter the same restrictions are imposed on ξ, η, and ζ, as before.

__Theorem 4.1__ __There exists an infinite sequence of lattice points for which $|\xi|$ and $|\eta|$ are arbitrarily small.__

Let ε_1 and ε_2 be two arbitrary positive constants. The three linear forms $\xi' \equiv \frac{\xi\sqrt[3]{\Delta}}{\varepsilon_1}$, $\eta' \equiv \frac{\eta\sqrt[3]{\Delta}}{\varepsilon_2}$, $\zeta' \equiv \frac{\zeta\,\varepsilon_1\varepsilon_2}{\sqrt[3]{\Delta^2}}$ have the same determinant Δ as ξ, η, and ζ. From a theorem of Minkowski concerning linear forms[1] there exists, besides the origin, a lattice point for which $|\xi'| \leqq \sqrt[3]{\Delta}$, $|\eta'| \leqq \sqrt[3]{\Delta}$, and $|\zeta'| \leqq \sqrt[3]{\Delta}$. For this point, then, the inequalities $|\xi| \leqq \varepsilon_1$, $|\eta| \leqq \varepsilon_2$ are satisfied. Call this point p_1.

Since the origin is the only lattice point in the

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1. H. Minkowski, __Geometrie der Zahlen__, p. 104.

plane $\xi = 0$ (or the plane $\eta = 0$) it follows that $|\xi(p_1)| > \varepsilon_1' > 0$, $|\eta(p_1)| > \varepsilon_2' > 0$ where ε_1' and ε_2' are suitably chosen positive constants. If ε_1' and ε_2' are employed instead of ε_1 and ε_2, it is clear that there exists a second lattice point p_2 for which $|\xi(p_2)| \leqq \varepsilon_1' < \varepsilon_1$, $|\eta(p_2)| \leqq \varepsilon_2' < \varepsilon_2$. From these last inequalities it is obvious that $p_2 \neq \pm p_1$. By induction the existence is established of an infinite sequence of lattice points, all different, for which $|\xi| \leqq \varepsilon_1$, $|\eta| \leqq \varepsilon_2$. Thus the theorem is proved.

<u>Theorem 4.2</u> <u>Among the extreme parallelepipeds in the chain for ξ, η, ζ, there exists a parallelepiped</u> $\{a, g, \ell\}$ <u>for which one of the parameters, chosen in advance, is arbitrarily small</u>.

It suffices to show that there exists an extreme parallelepiped for which a is arbitrarily small. Let ε_1 and ε_2 be two arbitrary positive constants. By Theorem 2.1 the parallelepiped $\left\{\varepsilon_1, \varepsilon_2, \frac{\Delta}{\varepsilon_1 \varepsilon_2}\right\}$ is not free. Lower the ξ-faces to obtain the free parallelepiped $\left\{a, \varepsilon_2, \frac{\Delta}{\varepsilon_1 \varepsilon_2}\right\}$ for which the first parameter cannot be increased without introducing a lattice point into the interior of the parallelepiped. The hypothesis that the origin is the only lattice point in the plane $\xi = 0$

requires $a > 0$. Next raise the ζ-faces until the third parameter is equal to the largest value ℓ for which the parallelepiped remains free. Finally, raise the η-faces until the parallelepiped becomes an extreme parallelepiped $\{a, g, \ell\}$. The existence of the two numbers g and ℓ is established by Theorem 2.1. Inasmuch as $a < \varepsilon_1$, the theorem is proved.

Theorem 4.3 When a sufficiently great number of only successive ξ-neighbors are taken, an extreme parallelepiped is attained for which the first parameter is arbitrarily small. Likewise, when only successive η- or ζ-neighbors are taken, an extreme parallelepiped is attained for which, respectively, the second or the third parameter is arbitrarily small.

It suffices to prove the theorem for a sequence of successive ξ-neighbors. Denote by $\{a_0, g_0, \ell_0\}$ an arbitrary extreme parallelepiped which is to be used as a starting point. Let $\{a_i, g_i, \ell_i\}$ be the ξ-neighbor of $\{a_{i-1}, g_{i-1}, \ell_{i-1}\}$ $(i = 1, 2, \dots, \infty)$. From the definition of the ξ-neighbor it follows that $a_i > a_{i+1} > 0$, $g_{i+1} \geq g_i \geq g_0$, $\ell_{i+1} \geq \ell_i \geq \ell_0$ $(i = 0, 1, \dots, \infty)$. Consequently, $\lim_{i\to\infty} a_i \equiv a$ exists, and $a_i > a \geq 0$.

Suppose $a > 0$. The inequalities $g_i < \frac{\Delta}{a_i \ell_i} < \frac{\Delta}{a \ell_0}$,

$\ell_i < \frac{\Delta}{a_i g_i} < \frac{\Delta}{a g_o}$ are a consequence of Theorem 2.1. It would then follow that the infinite sequence of successive ξ-neighbors would be contained in the parallelepiped $\left\{a_o, \frac{\Delta}{a \ell_o}, \frac{\Delta}{a g_o}\right\}$. Denote by $p_1^{(i)}$ the lattice point whose coordinates are in the first column of the matrix P_i which, by Theorem 2.3, corresponds to $\{a_i, g_i, \ell_i\}$. Since $a_i > a_{i+1}$ the points $p_1^{(i)}$ $(i = 0, 1, \cdots, \infty)$ are all different. Each point $p_1^{(i)}$, being on the surface of an extreme parallelepiped in the sequence of successive ξ-neighbors, must be contained in $\left\{a_o, \frac{\Delta}{a \ell_o}, \frac{\Delta}{a g_o}\right\}$. But this is impossible, whence $a = 0$.

From the definition of a limit, it follows that there exists an integer N_ε such that $a_i \leqq \varepsilon$ for all $i \geqq N_\varepsilon$, the number ε being an arbitrarily chosen positive quantity. The theorem is thus proved.

Let $\{a_i, g_i, \ell_i\}$ $(i = 0, 1, \cdots, \infty)$ be an infinite sequence of extreme parallelepipeds for ξ, η, ζ, and let Φ_i and P_i $(i = 0, 1, \cdots, \infty)$ be the infinite sequences of matrices Φ and P which by Theorem 2.3 correspond to the respective extreme parallelepiped. Let a_i, b_i, $\cdots$, ℓ_i have the same meaning for Φ_i as a, b, $\cdots$, ℓ have for Φ. Denote by $p_r^{(i)}$ $(r = 1, 2, 3;\ i = 0, 1, \cdots, \infty)$ the point whose coordinates occur in the rth column of the matrix P_i.

The following lemma can then be stated.

Lemma 4.1 Among the extreme parallelepipeds in the chain for ξ, η, ζ, there exists an infinite sequence $\{a_i, g_i, \ell_i\}$ $(i = 1, \cdots, \infty)$ of first kind, all different, for which $c_i > c_{i+1}$, $h_i > h_{i+1}$ $(i = 1, \cdots, \infty)$. Likewise, there exist two other such infinite sequences of extreme parallelepipeds of first kind, one for which $b_i > b_{i+1}$, $k_i > k_{i+1}$, and another for which $f_i > f_{i+1}$, $j_i > j_{i+1}$.

Because of the symmetry of the roles taken by ξ, η, ζ, it suffices to prove the first part of the lemma. Let $\{a_0, g_0, \ell_0\}$ be an arbitrary extreme parallelepiped, and let $\{a_i, g_i, \ell_i\}$ $(i = 1, \cdots, \infty)$ be the infinite sequence of extreme parallelepipeds determined in the following manner. Because of the restrictions imposed upon ξ, η, and ζ, no one of the numbers a_0, b_0, $\cdots$, ℓ_0 is equal to zero. Consequently, Theorem 2.1 establishes the existence of at least one lattice point besides the origin in the interior of the parallelepiped $\left\{c_0, h_0, \frac{\Delta}{c_0 h_0}\right\}$. Among all the lattice points (exclusive of the origin) which may be in this parallelepiped, there is a pair of points for which $|\zeta|$ is the least. Let ℓ_1 (for which $\ell_1 > \ell_0$) be this value of $|\zeta|$, and let $\pm p_3^{(1)}$ be the two lattice points for which $|\zeta| = \ell_1$. The parallelepiped $\{c_0, h_0, \ell_1\}$ is then <u>free</u>, but it is not extreme. It is possible to raise <u>either</u> its ξ-faces <u>or</u> its η-faces without introducing lattice points into its interior; on the other hand, since $|\xi(p_3^{(0)})| = c_0$, $|\eta(p_3^{(0)})| = h_0$, $|\zeta(p_3^{(0)})| = \ell_0 < \ell_1$, it is impossible to raise both pairs of faces simultaneously without introducing $p_3^{(0)}$ and $-p_3^{(0)}$ into the interior of the parallelepiped. The raising of the ξ-faces as high as possible without introducing lattice points into the interior yields an extreme parallelepiped $\{\bar{a}_1, h_0, \ell_1\}$ for which $\bar{a}_1 < \frac{\Delta}{h_0\ell_1} < \frac{\Delta}{h_0\ell_0}$. Likewise, the raising of the η-faces as high as possible yields an extreme parallelepiped $\{c_0, \bar{g}_1, \ell_1\}$ for which $\bar{g}_1 < \frac{\Delta}{c_0\ell_1} < \frac{\Delta}{c_0\ell_0}$. It is apparent that $\{c_0, \bar{g}_1, \ell_1\}$ is <u>by definition</u> the ξ-neighbor of $\{\bar{a}_1, h_0, \ell_1\}$. Theorem 3.3 requires that at least one of these extreme parallelepipeds be of first kind. If both are of first kind, it is possible to choose <u>either</u> $a_1 \equiv \bar{a}_1$, $g_1 \equiv h_0$ <u>or</u> $a_1 \equiv c_0$, $g_1 \equiv \bar{g}_1$ in the definition of $\{a_1, g_1, \ell_1\}$. Otherwise, $\{a_1, g_1, \ell_1\}$ is defined to be that one which is of first kind. Again from the restrictions imposed on ξ, η, and ζ, no one of the constants $a_1, b_1, \cdots, \ell_1$ is zero. When $\{a_1, g_1, \ell_1\}$ is used in place of $\{a_0, g_0, \ell_0\}$, a second extreme parallelepiped $\{a_2, g_2, \ell_2\}$ of first kind can be similarly defined. The continuation of the process leads to an infinite sequence of extreme parallelepipeds $\{a_i, g_i, \ell_i\}$ of first kind for which $c_i > c_{i+1}$, $h_i > h_{i+1}$. In any sequence constructed in this way, only $\{a_0, g_0, \ell_0\}$ can be of second kind; the other $\{a_i, g_i, \ell_i\}$ are always all of first kind. The lemma is thus proved.

Such a sequence for which $c_i > c_{i+1}$, $h_i > h_{i+1}$, $\ell_i < \ell_{i+1} = \min.$ is called <u>a $\xi\eta$-sequence of extreme parallelepipeds</u>. Likewise, a sequence for which $b_i > b_{i+1}$, $g_i < g_{i+1} = \min.$, $k_i > k_{i+1}$, is called <u>a $\xi\zeta$-sequence of extreme parallelepipeds</u>. Similarly, <u>an $\eta\zeta$-sequence of extreme parallelepipeds is defined</u>.

In all that follows the results will be stated for only the $\xi\eta$- sequences, but it is to be kept in mind that like results obtain for the $\xi\zeta$- and $\eta\zeta$-sequences.

Although it may be possible at times, starting with the same $\{a_0, g_0, \ell_0\}$, to have several $\xi\eta$-sequences, it is obvious that <u>the sequence $\ell_0, \ell_1, \cdots$ of third parameters is always the same</u>. That is to say, the infinite sequence of lattice points $q_0, q_1, \cdots$ for which $\zeta(q_i) = \ell_i$ is invariable. This sequence will be called the $\xi\eta$-sequence of lattice points appertaining to $\{a_0, g_0, \ell_0\}$.

Lemma 4.2 When an arbitrary extreme parallelepiped is taken as a starting point and the $\xi\eta$-sequences of extreme parallelepipeds obtained, there is uniquely determined a $\xi\eta$-sequence of lattice points $q_1, q_2, \cdots$, all different, such that each point is on the surface of two consecutive extreme parallelepipeds of the sequence, and such that $|\xi(q_i)| = c_i$, $|\eta(q_i)| = h_i$, $\zeta(q_i) = \ell_i$.

Since $\ell_r \neq \ell_s$ $(r \neq s)$ the lattice points $q_1, q_2, \cdots$ are all different. The method by which this $\xi\eta$-sequence is obtained shows that q_i is on the surface of both $\{a_i, g_i, \ell_i\}$ and $\{a_{i+1}, g_{i+1}, \ell_{i+1}\}$.

Lemma 4.3 Let $c_1, c_2, c_3, \cdots$ and $h_1, h_2, h_3, \cdots$ be two sequences (finite or infinite) of positive terms such that $c_i > c_{i+1}$, $h_i > h_{i+1}$; let ε_1 and ε_2 be two arbitrary positive constants; and let t be an arbitrary positive integer. When

$$n \geqq 4t^2 + 2t, \quad n > \left(\frac{tc_1}{\varepsilon_1}\right)^2 + 1, \quad n > \left(\frac{th_1}{\varepsilon_2}\right)^2 + 1,$$

there exists an integer $r \leqq n - t$, such that

$$c_r - c_{r+t} < \varepsilon_1, \quad h_r - h_{r+t} < \varepsilon_2.$$

First choose n large enough to satisfy the three inequalities. (If the sequences are finite, this may not always be possible.) Then $\sqrt{n-1} > \frac{tc_1}{\varepsilon_1}$, $\sqrt{n-1} > \frac{th_1}{\varepsilon_2}$. The inequality $n \geqq 4t^2 + 2t$ $(t \geqq 1)$ requires that $n - 1 > [\sqrt{n-1}] + 1$. The number of differences $\Delta_i c \equiv c_i - c_{i+1}$ for which $\Delta_i c \geqq \frac{\varepsilon_1}{t}$ is less than $[\sqrt{n-1}] + 1$, for otherwise,

$$c_1 - c_n = \sum_{i=1}^{n-1} \Delta_i c > \left([\sqrt{n-1}] + 1 \right) \frac{\varepsilon_1}{t} \geqq c_1,$$

which is impossible. Similarly, the number of differences $\Delta_i h \equiv h_i - h_{i+1}$ for which $\Delta_i h \geqq \frac{\varepsilon_2}{t}$ is less than $[\sqrt{n-1}] + 1$. The number of indices i for which either $\Delta_i c \geqq \frac{\varepsilon_1}{t}$ or $\Delta_i h \geqq \frac{\varepsilon_2}{t}$ is never more than $2[\sqrt{n-1}]$. In order to insure that there exist t consecutive indices, r, $r+1$, $\cdots$, $r+t-1$, less than or equal to n, for which $\Delta_i c < \frac{\varepsilon_1}{t}$ $(i = r, \cdots, r+t-1)$ and at the same time $\Delta_i h < \frac{\varepsilon_2}{t}$ $(i = r, \cdots, r+t-1)$, it is sufficient that n satisfy $n - 1 > 2t[\sqrt{n-1}] + t - 1$. But this inequality is always satisfied when $n \geqq 4t^2 + 2t$. Thus the lemma is proved.

<u>Theorem 4.4</u> <u>Among the lattice points of a $\xi\eta$-sequence, there are two q_{s_m} and $q_{s_{m+1}}$ $(s_m + 2 \leqq s_{m+1})$ for which $|\xi(q_{s_m} - q_{s_{m+1}})|$ and $|\eta(q_{s_m} - q_{s_{m+1}})|$ are arbitrarily small. In particular, let ε_1 and ε_2 be two arbitrary</u>

positive constants, then whenever

$$n \geqq 6,\ n > \left(\frac{c_1}{\varepsilon_1}\right)^2 + 1,\ n > \left(\frac{h_1}{\varepsilon_2}\right)^2 + 1,$$

there exists an integer $m \leqq n - 1$ such that

$$c_{s_m} - c_{s_{m+1}} < \varepsilon_1,\ h_{s_m} - h_{s_{m+1}} < \varepsilon_2,$$

where

$$\begin{aligned} \xi(q_{s_m}) &= \delta_1 c_{s_m}, & \xi(q_{s_{m+1}}) &= \delta_1 c_{s_{m+1}}, \\ \eta(q_{s_m}) &= \delta_2 h_{s_m}, & \eta(q_{s_{m+1}}) &= \delta_2 h_{s_{m+1}}. \end{aligned} \quad (\delta_i = \pm 1)$$

The planes $\xi = 0$, $\eta = 0$, $\zeta = 0$, divide the space into octants. Each lattice point (except the origin) is in a single one of these octants. All the points of the $\xi\eta$-sequence are in the four octants for which $\xi > 0$. Let u be an arbitrary positive integer. At least one of these octants contains no less than $\left[\frac{u}{4}\right] + 1$ of the points $q_1, q_2, \cdots, q_u$ of the $\xi\eta$-sequence. Let $q_{s_1}, q_{s_2}, \cdots, q_{s_n}$ be those points of the $\xi\eta$-sequence which fall in such an octant, then $\frac{u}{4} \leqq \left[\frac{u}{4}\right] + 1 \leqq n \leqq s_n \leqq u$. Let u be chosen sufficiently large to satisfy $\frac{u}{4} \geqq 6$, $\frac{u}{4} > \left(\frac{c_1}{\varepsilon_1}\right)^2 + 1$, $\frac{u}{4} > \left(\frac{h_1}{\varepsilon_2}\right)^2 + 1$. Then $n \geqq 6$, $n > \left(\frac{c_{s_1}}{\varepsilon_1}\right)^2 + 1$, $n > \left(\frac{h_{s_1}}{\varepsilon_2}\right)^2 + 1$ and Lemma 4.3 proves the existence of the integer m. By Lemma 4.2, the points q_i and q_{i+1} are both on the surface of $\{a_{i+1}, g_{i+1}, \ell_{i+1}\}$. Application of Lemma 2.1 shows that $q_{s_m+1} \neq q_{s_{m+1}}$, hence $s_m + 2 \leqq s_{m+1}$.

Since the points of the $\xi\eta$-sequence are all different, the point $x_1 \equiv q_{s_m} - q_{s_{m+1}}$ is not the origin. Then $c_{s_m} - c_{s_{m+1}} = |\xi(x_1)| > 0$, $h_{s_m} - h_{s_{m+1}} = |\eta(x_1)| > 0$. Let δ_1 and δ_2 be two arbitrary positive constants satisfying $0 < \delta_1 < |\xi(x_1)|$, $0 < \delta_2 < |\eta(x_1)|$. When δ_1 and δ_2 are used instead of ε_1 and ε_2, it becomes apparent upon application of Theorem 4.4 that there exists among the differences of lattice points of a $\xi\eta$-sequence, a second lattice point x_2 for which $|\xi| < \delta_1$, $|\eta| < \delta_2$. By induction it is established that there exists among these differences an infinite sequence of lattice points x_1, x_2, $\cdots$, all different, for which

$$\lim_{i\to\infty} |\xi(x_i)| = 0, \quad \lim_{i\to\infty} |\eta(x_i)| = 0.$$

<u>Lemma 4.4</u> <u>Among the points of the $\xi\eta$-sequence let q_{s_1}, q_{s_2}, $\cdots$ $(s_1 < s_2 < \cdots)$ be all those which fall in a same one of the octants formed by the planes $\xi = 0$, $\eta = 0$, $\zeta = 0$; two at least of these four sequences are infinite. Let ε_1 and ε_2 be two arbitrary positive constants. Then</u>

1°: <u>there exists an integer N, dependent upon ε_1 and ε_2, of such a nature that whenever $s_m \geqq N$, the inequalities $|\xi(q_{s_m} - q_{s_{m+t}})| < \varepsilon_1$, $|\eta(q_{s_m} - q_{s_{m+t}})| < \varepsilon_2$ $(t \geqq 1)$, or</u>

2°: <u>the sequence $q_{s_1}, q_{s_2}, \cdots$ is assuredly finite. Even for a finite sequence it is possible at times to find an N satisfying the conditions of 1°.</u>

As a consequence of the relations $c_i > c_{i+1} > 0$, $h_i > h_{i+1} > 0$, the sequences $c_1, c_2, \cdots$ and $h_1, h_2, \cdots$ tend toward respective limits $c \geqq 0$ and $h \geqq 0$. The inequalities $c_i > c$ and $h_i > h$ are always satisfied. From the definition of a limit it is clear that an integer N_1 can be found such that whenever $i \geqq N_1$, the inequality $c_i - c < \varepsilon_1$ is always satisfied. Likewise, an integer N_2 can be found such that whenever $i \geqq N_2$, then $h_i - h < \varepsilon_2$. Let N be the greater of the integers N_1 and N_2, and let i and r be two integers such that $r > i \geqq N$, then $0 < c_i - c_r < c_i - c < \varepsilon_1$ and $0 < h_i - h_r < h_i - h < \varepsilon_2$,

Since $q_{s_m+1} \neq q_{s_{m+1}}$, there are at least two octants each of which contains an infinity of the points $q_1, q_2, \cdots$ the other two octants containing possibly only a finite number of these points. Suppose for a certain octant one chooses $s_m \geqq N$, then $|\xi(q_{s_m} - q_{s_{m+t}})| = c_{s_m} - c_{s_{m+t}} < \varepsilon_1$, $|\eta(q_{s_m} - q_{s_{m+t}})| = h_{s_m} - h_{s_{m+t}} < \varepsilon_2$. In case the octant under consideration contains only a finite number of points $q_1, q_2, \cdots$ it is not always possible to satisfy $s_m \geqq N$.

<u>Lemma 4.5</u> <u>In a $\xi\eta$-sequence of extreme parallelepipeds</u>

<u>the third parameters satisfy the inequality</u>

$$\ell_{i+t} > 2^{\left[\frac{t}{4}\right]} \ell_i \qquad (i,\ t = 1,\ 2,\ \cdots, \infty).$$

From the method of construction of the $\xi\eta$-sequence, it is evident that $\ell_{i+1} > \ell_i$. Among the points q_i, $q_{i+1}, \cdots$, q_{i+4} there are at least two that are in the same octant (see proof of Theorem 4.4). Let q_{s_1} and q_{s_2} be two such points. Then the lattice point $q \equiv q_{s_1} - q_{s_2}$ satisfies the relations $|\xi(q)| = c_{s_1} - c_{s_2} < c_{s_1}$, $|\eta(q)| = h_{s_1} - h_{s_2} < h_{s_1}$, $|\zeta(q)| = \ell_{s_2} - \ell_{s_1}$. Since q is not in the interior of the free parallelepiped $\{c_{s_1},\ h_{s_1},\ \ell_{s_1}\}$, the inequality $\ell_{s_2} - \ell_{s_1} \geqq \ell_{s_1+1}$ must be satisfied. Inasmuch as $\ell_{s_1+1} > \ell_{s_1}$, it is seen that $\ell_{s_2} > 2\ell_{s_1}$. From the definition of s_1 and s_2 it is clear that $i \leqq s_1 < s_2 \leqq i + 4$, from which $\ell_{i+4} > 2\ell_i$. By induction the inequality $\ell_{i+4t} > 2^t \ell_i$ is obtained. This inequality can be combined with $\ell_{i+1} > \ell_i$ to obtain the result of the lemma.

<u>Theorem 4.5</u> <u>In a $\xi\eta$-sequence of extreme parallelepipeds the inequality</u>

$$a_{i+t}\, g_{i+t} < 5 a_i\, g_i\, 2^{-\left[\frac{t}{4}\right]} \qquad (i,\ t = 1,\ 2,\ \cdots, \infty)$$

<u>is always satisfied</u>.

It has been seen (p. 29, line 10) that for an extreme parallelepiped $\{a,\ g,\ \ell\}$ of first kind the relation

$5ag\ell > \Delta \cdot D > 0$ always holds. Theorem 2.5 shows that $D = 1$ and Theorem 2.1, that $ag\ell < \Delta$. Consequently, for each parallelepiped $\{a_i, g_i, \ell_i\}$ of a $\xi\eta$-sequemce (except for the first in the case where it is of second kind) the inequality

$$\Delta > a_i g_i \ell_i > \frac{\Delta}{5}$$

holds. The application of Lemma 4.5 leads to

$$\frac{\Delta 2^{[\frac{t}{4}]}}{5a_i g_i} < 2^{[\frac{t}{4}]}\ell_i < \ell_{i+t} < \frac{\Delta}{a_{i+t}\, g_{i+t}} \quad (i,\ t = 1,\ 2,\ \cdots, \infty),$$

and from this the desired result is obtained.

Corollary *In a $\xi\eta$-sequence of extreme parallelepipeds, the inequality*

$$c_{i+t}\, h_{i+t} < 5a_i g_i\, 2^{-[\frac{t}{4}]} \qquad (i,\ t = 1,\ 2,\ \cdots, \infty)$$

is always satisfied.

This result is derived by using the inequalities $a_i > c_i$, $g_i > h_i$ $(i = 1, 2, \cdots, \infty)$ satisfied by the elements of the matrices Φ_i.

Next will be given a method for computing the coordinates of all the lattice points on the extreme parallelepipeds of a $\xi\eta$-sequence, when an arbitrary extreme parallelepiped of *first kind* is taken as a starting point. A table will be employed which serves equally well in the determination of the lattice points on the extreme parallelepipeds of an $\eta\zeta$- or $\xi\zeta$-sequence.

It suffices to exhibit a process for the evaluation of the coordinates of the lattice points on $\{a_{i+1}, g_{i+1}, \ell_{i+1}\}$ when the lattice points on $\{a_i, g_i, \ell_i\}$ (of first kind) are given.

Theorem 4.6 When Φ_i is normal of type I, III ($c_i \leq b_i$), or V, set $\xi = \eta'$, $\eta = \xi'$, $x_1 = x_2'$, $x_2 = x_1'$. Under this interchange the corresponding matrix Φ_i' is normal of the respective type II, III ($c_i' > b_i'$), or IV.

That Φ_i' is of the respective type II, III, or IV is apparent. When Φ_i is of type III, the condition $b_i > c_i$, by Theorem 2.4, requires $h_i > f_i$. Under the above interchange, h_i becomes c_i' and f_i becomes b_i'; hence the inequality $c_i' > b_i'$ is satisfied.

It suffices, then, to find an algorithm for the three cases II, III ($c > b$), IV.

There are two solutions $\pm q_{i+1}$ of $|\xi| < c_i$, $|\eta| < b_i$, $|\zeta| = \text{min.} < \Delta/c_i h_i$. Then $\ell_{i+1} = \zeta(q_{i+1}) > 0$. There are two extreme parallelepipeds $\{\tilde{a}_{i+1}, h_i, \ell_{i+1}\}$ and $\{c_i, \bar{g}_{i+1}, \ell_{i+1}\}$ which can be secured from $\{c_i, h_i, \ell_{i+1}\}$ by raising, respectively, the ξ- or the η-faces (see p.79). It is possible that one or the other of these two extreme parallelepipeds be of the second kind, but not both, for this circumstance is precluded by Theorem 3.3 and the fact that $\{c_i, \bar{g}_{i+1}, \ell_{i+1}\}$ is the ξ-neighbor of $\{\tilde{a}_{i+1}, h_i, \ell_{i+1}\}$.

Instead of associating with q_{i+1} a single extreme parallelepiped of first kind, one associates with it this pair, thus obtaining a sequence of pairs of extreme parallelepipeds. Corresponding to a given starting point, this sequence is uniquely determined. Call it a double $\xi\eta$-sequence of extreme parallelepipeds for ξ, η, and ζ.

Set $\tilde{g}_{i+1} = h_i$, $\bar{a}_{i+1} = c_i$. Let $\tilde{P}_{i+1}$ and $\tilde{\bar{\Phi}}_{i+1}$ be the P and $\bar{\Phi}$ matrices which by Theorem 2.3 are associated with $\{\tilde{a}_{i+1}, \tilde{g}_{i+1}, \ell_{i+1}\}$; let $\bar{P}_{i+1}$ and $\bar{\bar{\Phi}}_{i+1}$ be those associated with $\{\bar{a}_{i+1}, \bar{g}_{i+1}, \ell_{i+1}\}$; and, finally, let P_i and $\bar{\Phi}_i$ be the ones associated with the extreme parallelepiped $\{a_i, g_i, \ell_i\}$.

Since $\{a_i, g_i, \ell_i\}$ is of first kind, P_i has a determinant 1 and P_i^{-1} is a matrix of integers of this same nature. Let p_1, p_2, p_3 be the lattice points whose coordinates are the elements of P_i; let $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{p}_3$ be those in $\tilde{P}_{i+1}$; and let $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$ be those in $\bar{P}_{i+1}$. Since $\tilde{p}_2 = \pm p_3$ and $\bar{p}_1 = \pm p_3$, the two matrices $\tilde{T}_{i+1} \equiv P_i^{-1}\tilde{P}_{i+1}$ and $\bar{T}_{i+1} \equiv P_i^{-1}\bar{P}_{i+1}$ are matrices of the respective types

$$\widetilde{T}_{i+1} = \begin{Vmatrix} \tilde{\delta}_1\tilde{t}_{11}, & 0, & \tilde{\delta}_3 t_{13} \\ \tilde{\delta}_1\tilde{t}_{21}, & 0, & \tilde{\delta}_3 t_{23} \\ \tilde{\delta}_1\tilde{t}_{31}, & \tilde{\delta}_2, & \tilde{\delta}_3 t_{33} \end{Vmatrix} \quad \text{and } \overline{T}_{i+1} = \begin{Vmatrix} 0, & \bar{\delta}_2\bar{t}_{12}, & \bar{\delta}_3 t_{13} \\ 0, & \bar{\delta}_2\bar{t}_{22}, & \bar{\delta}_3 t_{23} \\ \bar{\delta}_1, & \bar{\delta}_2\bar{t}_{32}, & \bar{\delta}_3 t_{33} \end{Vmatrix}$$

where $\tilde{t}_{11}, \cdots, \bar{t}_{12}, \cdots, t_{33}$ are integers, and $\tilde{\delta}_h = \pm 1$, $\bar{\delta}_h = \pm 1$.

ALGORITHM FOR $\widetilde{T}_{i+1}$ AND $\overline{T}_{i+1}$.

Let A, B, $\cdots$, L denote the respective cofactors of the terms $a_i, \pm b_i, \cdots, \ell_i$ in Φ_i.

II: Set (for convenience, omit the subscript i from elements of Φ_i)

$$M_1 = [\theta J],\ M_2 = [-\theta K],\ M_3 = \theta L = \text{integer},\ \theta > 0;$$

$$|M_1| < 6 + \frac{|J|}{ch},\ |M_2| < 4 - \frac{K}{ch},\ |M_3| < 3 + \frac{L}{ch};$$

$$a = u^o,\ b = u',\ c = u'',\ u \equiv -u^oM_1 - u'M_2 + u''M_3;$$

$$f = v',\ g = v^o,\ h = v'',\ v \equiv v'M_1 - v^oM_2 + v''M_3;$$

$$j = w'',\ k = w',\ \ell = w^o,\ w \equiv -w''M_1 + w'M_2 + w^oM_3;$$

$$W \equiv -w''m_1 + w'm_2 + w^oM_3.$$

III: $(c > b)$: Set

$$M_1 = [\theta L],\ M_2 = [\theta K],\ M_3 = \theta J = \text{integer},\ \theta > 0;$$

$$|M_1| < 6 + \frac{L}{ch},\ |M_2| < 5 + \frac{K}{ch},\ |M_3| < 3 + \frac{J}{ch};$$

$$a = u^o,\ b = u',\ c = u'',\ u \equiv -u''M_1 - u'M_2 + u^oM_3;$$

$$f = v',\ g = v^o,\ h = v'',\ v \equiv v''M_1 - v^oM_2 + v'M_3;$$

$$\ell = w'',\ k = w',\ j = w^o,\ w \equiv w''M_1 + w'M_2 + w^oM_3;$$

$$W \equiv w''m_1 + w'm_2 + w^oM_3.$$

IV: Set

$$M_1 = [-\theta K],\ M_2 = [\theta J],\ M_3 = \theta L = \text{integer},\quad \theta > 0;$$

$$|M_1| < 6 + \frac{|K|}{ch},\quad |M_2| < 5 + \frac{J}{ch},\quad |M_3| < 4 + \frac{L}{ch};$$

$$g = u^{\circ},\ f = u',\ h = u'',\ u \equiv -u^{\circ}M_1 - u'M_2 + u''M_3;$$

$$b = v',\ a = v^{\circ},\ c = v'',\ v \equiv v'M_1 - v^{\circ}M_2 + v''M_3;$$

$$k = w'',\ j = w',\ \ell = w^{\circ},\ w \equiv w''M_1 + v'M_2 + w^{\circ}M_3;$$

$$W \equiv w''m_1 + w'm_2 + w^{\circ}M_3.$$

Part 1. Determination of t_{13}, t_{23}, t_{33}.

Let θ range over all positive values which make M_3 an integer and for which $|M_1|$, $|M_2|$, $|M_3|$ are less than the upper bounds given. Whenever such a triple M_1, M_2, M_3 occurs for which u and v satisfy a pair of conditions for solution in the following table, one associates with M_3 the corresponding m_1 and m_2. Among all such triples m_1, m_2, M_3 (of which there is at least one) that particular triple s_1, s_2, s_3 is selected for which $|W|$ = min.

TABLE

	m_1	m_2	Conditions for Solution	
			II, IV	III
1	M_1	M_2	$0 < u < u''$ $-v'' < v < v''$	Same as II, IV
2	M_1	M_2+1	$-u'' + u' < u < u'' + u'$ $-v'' + v^o < v < v^o$	Same as II, IV
3	M_1+1	M_2	$-u'' + u^o < u < u'' + u^o$ $-v' < v < v'' - v'$	$0 < u < 2u''$ $-v'' < v < 0$
4	M_1+1	M_2+1	$-u'' + u' + u^o < u < u' + u^o$ $-v'' - v' + v^o < v < v'' - v' + v^o$	$u' < u < u' + u''$ $v^o - 2v'' < v < v^o$

SUPPLEMENT 1

5	M_1	M_2-1	$0 < u < u'' - u'$ $-v' < v < v'' - v^o$	$0 < u < u'' - u'$ $-v'' < v < v'' - v^o$
6	M_1+1	M_2+2	$-u'' + 2u' + u^o < u < u' + u^o$ $-v'' - v' + 2v^o < v < v^o$	$2u' < u < u' + u''$ $-2v'' + 2v^o < v < v^o$

SUPPLEMENT 2

7	M_1-1	M_2+1	$0 < u < u'' + u' - u^o$ $-v'' + v' + v^o < v < v^o$
8	M_1+2	M_2	$-u'' + 2u^o < u < u' + u^o$ $-v' < v < v'' - 2v'$

II, IV, $u' > u''$, $v' > v''$: use Table alone.

II, IV, $u' < u''$, $v' > v''$: use Table combined with Supp. 1.

II, IV, $u' > u''$, $v' < v''$: use Table combined with Supp. 2.

II, IV, $u' < u''$, $v' < v''$: use Table combined with both Supps.

III: use Table combined with Supp. 1.

After s_1, s_2, s_3 are evaluated, the integers t_{13}, t_{23}, t_{33} are given in the various cases by the relations

II: $t_{13} = s_1$, $t_{23} = - s_2$, $t_{33} = s_3$;

III: $t_{13} = s_3$, $t_{23} = s_2$, $t_{33} = s_1$;

IV: $t_{13} = - s_2$, $t_{23} = s_1$, $t_{33} = s_3$.

Part 2. Determination of $\tilde{t}_{11}$, $\tilde{t}_{21}$, $\tilde{t}_{31}$

Obtain all pairs of integers y_1, y_2 which satisfy

$$y_1 t_{23} - y_2 t_{13} = 1 \text{ or } 0$$

and

$$|y_1| < 2 + \frac{|J|}{ch} + \frac{A}{h\ell_{i+1}}, \quad |y_2| < 2 + \frac{|K|}{ch} + \frac{|B|}{h\ell_{i+1}} .$$

By means of

$$|y_3| < 2 + \frac{L}{ch} + \frac{|C|}{h\ell_{i+1}}$$

and

$$|\pm fy_1 + gy_2 \pm hy_3| < h,$$

where the signs are those in the second row of Φ_i, an integer y_3 is determined to be associated with each pair y_1, y_2 to form a triple. At most two triples can correspond to any pair y_1, y_2. Among these triples, the one is sought for which

$$|ay_1 \pm by_2 \pm cy_3| = \min.,$$

$$|\pm jy_1 \pm ky_2 + \ell y_3| < \ell_{i+1} ,$$

the signs being respectively those in the first and third rows of Φ_i. <u>This triple is the solution</u> $\tilde{t}_{11}$, $\tilde{t}_{21}$, $\tilde{t}_{31}$.

<u>Part 3.</u> <u>Determination of</u> $\bar{t}_{12}$, $\bar{t}_{22}$, $\bar{t}_{32}$.

Obtain all pairs of integers y_1, y_2 which satisfy

$$y_1 t_{23} - y_2 t_{13} = 1 \text{ or } 0$$

and

$$|y_1| < 2 + \frac{|J|}{ch} + \frac{|F|}{c\ell_{i+1}}, \quad |y_2| < 2 + \frac{|K|}{ch} + \frac{G}{c\ell_{i+1}}.$$

By means of

$$|y_3| < 2 + \frac{L}{ch} + \frac{|H|}{c\ell_{i+1}}$$

and

$$|ay_1 \pm by_2 \pm cy_3| < c,$$

where the signs are those in the first row of Φ_i, an integer y_3 is determined to be associated with each pair y_1, y_2 to form a triple. At most two triples can correspond to any pair y_1, y_2. Among these triples, the one is determined for which

$$|\pm fy_1 + gy_2 \pm hy_3| = \text{min.},$$

$$|\pm jy_1 \pm ky_2 + \ell y_3| < \ell_{i+1},$$

the signs being respectively those in the second and third rows of Φ_i. <u>This triple is the solution</u> $\bar{t}_{12}$, $\bar{t}_{22}$, $\bar{t}_{32}$.

<u>Part 4.</u> <u>Determination of</u> $\tilde{\delta}_1$, $\tilde{\delta}_2$, $\tilde{\delta}_3$, $\bar{\delta}_1$, $\bar{\delta}_2$, $\bar{\delta}_3$ <u>and</u>

<u>the types of</u> $\tilde{\tilde{\Phi}}_{i+1}$, $\bar{\bar{\Phi}}_{i+1}$.

Use of the relations $\tilde{P}_{i+1} = P_i \tilde{T}_{i+1}$, $\bar{P}_{i+1} = P_i \bar{T}_{i+1}$ and Lemmas 2.2 and 2.3 yield the values of $\tilde{\delta}_1, \cdots, \bar{\delta}_3$ and the types of $\tilde{\tilde{\Phi}}_{i+1}$ and $\bar{\Phi}_{i+1}$.

The derivation of the algorithm is along the same lines in all three cases, therefore we shall show it only for Φ_i normal of type II.

The first problem is to find the solutions of $|\xi| < c$, $|\eta| < h$, $|\zeta| = \min.$ in integers. This is equivalent to finding a set of integers t_{13}, t_{23}, t_{33} which satisfy

(1) $$|ay_1 - by_2 - cy_3| < c,$$

(2) $$|fy_1 + gy_2 + hy_3| < h,$$

(3) $$|-jy_1 - ky_2 + \ell y_3| = \min.,$$

for, then, the two solutions of the former inequalities are the coordinates of the lattice points

$$\pm(t_{13}p_1 + t_{23}p_2 + t_{33}p_3).$$

Recall the definitions of u, u^o, u', u'', etc., made at the beginning of the algorithm and set

$$U \equiv -u^o m_1 - u' m_2 + u'' M_3$$

$$V \equiv v' m_1 - v^o m_2 + v'' M_3.$$

The solving of (1), (2), and (3) is equivalent to finding integers m_1, m_2, M_3 for which

(1′), (2′), (3′) $\quad |U| < u''$, $|V| < v''$, $|W| = \min.$

Determinant theory yields $Ja - Kb - Lc = 0$, $Jf + Kg + Lh = 0$, so that

$$(4) \qquad 0 < u < u^{o} + u', \quad -v' < v < v^{o}.$$

Set $\beta_1 = m_1 - M_1$, $\beta_2 = m_2 - M_2$; then β_λ = integer.

For M_3 fixed, the values of β_1 and β_2 which correspond to solutions of $(1')$ and $(2')$ are determined first.
From $(1')$ and $(2')$ one has

$$(5) \qquad \begin{aligned} -u'' &< u^{o}\beta_1 + u'\beta_2 - u < u'', \\ -v'' &< -v'\beta_1 + v^{o}\beta_2 - v < v''. \end{aligned}$$

Combine (4) and (5) to obtain the relations

$$(6) \qquad \begin{aligned} &u^{o}\beta_1 + u'\beta_2 + u'' > 0, \\ &u^{o}(1 - \beta_1) + u'(1 - \beta_2) + u'' > 0, \\ &v^{o}\beta_2 + v'(1 - \beta_1) + v'' > 0, \\ &v^{o}(1 - \beta_2) + v'\beta_1 + v'' > 0. \end{aligned}$$

From Theorem 2.3 applied to Φ_λ one has

$$(7) \qquad \begin{aligned} &u^{o} > u', \quad u^{o} > u'', \\ &v^{o} > v', \quad v^{o} > v''. \end{aligned}$$

Separate the problem into the four case $u' > u''$, $v' > v''$; $u' < u''$, $v' > v''$; $u' > u''$, $v' < v''$; and $u' < u''$, $v' < v''$, and derive from (6) and (7) the possible values for β_1 and β_2. (In III $(c > b)$ one has always $u'' > u'$, consequently, there are only two cases.)

Replace these in (5) to secure the conditions for solution. The Table and its two Supplements are a result of this solution.

To find upper bounds for $|M_1|$, $|M_2|$, and $|M_3|$, rewrite (1). (2) as

$$(1'') \qquad ay_1 - by_2 - cy_3 = \lambda_1 c,$$

$$(2'') \qquad fy_1 + gy_2 + hy_3 = \lambda_2 h, \qquad (|\lambda_\mu| < 1,\ \lambda_\mu \neq 0).$$

Use Theorem 2.1 to obtain the upper bound $\frac{\Delta}{ch}$ for the minimum in (3). Replace (3) by

$$(3'') \qquad - jy_1 - ky_2 + \ell y_3 = \lambda_3 \cdot \frac{\Delta}{ch} \quad (|\lambda_3| < 1,\ \lambda_3 \neq 0)$$

$$(3'') \qquad - jy_1 - ky_2 + \ell y_3 = \lambda_3 \cdot \frac{\Delta}{ch} \quad (|\lambda_3| < 1,\ \lambda_3 \neq 0).$$

The solutions of (1), (2), and (3) are all among the solutions of (1′), (2″), and (3″).

When these are solved for y_1, y_2, y_3 and account is taken of Theorem 2.1 and the conditions $a > b$, $a > c$; $g > f$, $g > h$, the relations

$$|m_1| < 4 + \frac{|J|}{ch}, \quad |-m_2| < 2 - \frac{K}{ch}, \quad M_3 < 3 + \frac{L}{ch},$$

are obtained. Since $|m_1 - M_1| \leqq 2$ and $|m_2 - M_2| \leqq 2$ one has the previously stated bounds for M_1, M_2, M_3.

The second problem is to find the solutions of $|\xi| = \text{min.}$, $|\eta| < h$, $|\zeta| < \ell_{i+1}$ in integers. This is equivalent to finding a set of integers $\tilde{t}_{11}$, $\tilde{t}_{21}$, $\tilde{t}_{31}$ which satisfy

$$(8) \qquad |ay_1 - by_2 - cy_3| = \text{min.},$$

$$(9) \qquad |fy_1 + gy_2 + hy_3| < h,$$

(10) $$|-jy_1 - ky_2 + \ell y_3| < \ell_{i+1} < \frac{\Delta}{ch}.$$

The same procedure as before yields the upper bounds for $|\tilde{t}_{11}|$, $|\tilde{t}_{21}|$, $|\tilde{t}_{31}|$.

On the other hand, the solving of (8), (9), and (10) is expedited by the relation $\tilde{t}_{11}t_{23} - \tilde{t}_{21}t_{13} = 1$ or 0, which is obtained from the fact that the determinant of $\tilde{T}_{i+1}$ is 1 or 0, according as $\{\tilde{a}_{i+1}, \tilde{g}_{i+1}, \ell_{i+1}\}$ is of first or second kind.

The problem of determining $\bar{t}_{12}$, $\bar{t}_{22}$, and $\bar{t}_{32}$ is exactly similar to that of obtaining $\tilde{t}_{11}$, $\tilde{t}_{21}$, and $\tilde{t}_{31}$. It is sufficient to mention the inequalities to be solved, the method being the same.

($\bar{8}$) $$|fy_1 + gy_2 + hy_3| = \min.$$

($\bar{9}$) $$|ay_1 - by_2 - cy_3| < c$$

($\overline{10}$) $$|-jy_1 - ky_2 + \ell y_3| < \ell_{i+1} < \frac{\Delta}{ch}.$$

The desired results for both the other cases, III($c > b$) and IV, are obtained in a like manner.

This algorithm has the distinct advantage of leading directly to lattice points for which two of the linear forms are as small as desired in absolute value. Although the algorithm in Chapter III will eventually lead to any extreme parallelepiped of the chain, and, in particular, to those of the $\xi\eta$-sequence, these lattice points are, in general, much more quickly obtained by the algorithm of this Chapter IV.

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(J. F. Koksma, in the work cited above, gives a much more comprehensive bibliography.)