UNIVERSITY OF CINCINNATI

June 1, 1937

I hereby recommend that the thesis prepared under my supervision by William Clemmer Mitchell *entitled* On Euler Summability and one of its generalizations.

be accepted as fulfilling this part of the requirements for the degree of Ph. D.

Approved by:

Charles N. Moore

Form 668—G. S. and Ed.—1M—7-31

ON EULER SUMMABILITY
AND ONE OF ITS GENERALIZATIONS

A dissertation submitted to the

Graduate School
of the University of Cincinnati

in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

1937

by

William Clemmer Mitchell

A. B. University of Cincinnati 1928
A. M. University of Cincinnati 1932

TABLE OF CONTENTS

INDEX OF THEOREMS

Chapter I

A new approach to Euler summability

1. <u>The Euler-Knopp means</u>. In this paper we shall approach Euler summability from a somewhat different point of view from that of Knopp [1] [2]. We shall summarize the

[1]) "Uber das Eulersche Summierungsverfahren", Math. Zeit. 1922, 14-15, pp. 226-253.

[2])See also Knopp's "Theory and Application of Infinite Series", § 63.

Euler method briefly here in order to indicate our point of departure. The Euler transformation of the infinite series $\sum u_i$ is

$$\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \sum_{j=0}^{i} \binom{i}{j} u_j, \tag{1}$$

and if we define

$$S_n = \sum_{i=0}^{n} u_i, \quad S_n' = \sum_{i=0}^{n} \frac{1}{2^{i+1}} \sum_{j=0}^{i} \binom{i}{j} u_j, \tag{2}$$

we have [3])

$$S_n' = \frac{1}{2^{n+1}} \sum_{i=0}^{n} \binom{n+1}{i+1} S_i. \tag{3}$$

[3]) The following proof is given in Knopp's article.

Denoting the expression $\frac{1}{2^{m+1}}\sum_{i=0}^{m}\binom{m+1}{i+1}S_i$ by A_m, we have

$$2^{m+1}(A_m - A_{m-1}) = \sum_{i=0}^{m}\binom{m+1}{i+1}S_i - 2\sum_{i=0}^{m-1}\binom{m}{i+1}S_i$$

$$= S_m + \sum_{i=0}^{m-1}\binom{m+1}{i+1}S_i - \sum_{i=0}^{m-1}\binom{m}{i+1}S_i - \sum_{i=0}^{m-1}\binom{m}{i+1}S_i$$

$$= S_m + \sum_{i=0}^{m-1}\binom{m}{i}S_i - \sum_{i=0}^{m-1}\binom{m}{i+1}S_i$$

$$= \sum_{i=0}^{m}\binom{m}{i}S_i - \sum_{i=0}^{m}\binom{m}{i}S_{i-1}$$

$$= \sum_{i=0}^{m}\binom{m}{i}u_i = 2^{m+1}(S'_m - S'_{m-1}).$$

So

$$A_m - A_{m-1} = S'_m - S'_{m-1},$$

and since $A_0 = S'_0$, we have $S'_m = A_m$.

This is the E transformation of the sequence S_m, and Knopp defines means of positive integral orders by iteration, in a manner entirely analogous to the procedure in the Hölder process. He finds the pth power of E to be the transformation [4]) [5])

$$\text{(4)} \qquad \frac{1}{(2^p)^{m+1}}\sum_{i=0}^{m}\binom{m+1}{i+1}(2^p-1)^{m-i}S_i,$$

which he uses as the definition of means of non-integral orders.

[4]) This can be shown by mathematical induction. We have

$$S''_m = \frac{1}{2^{m+1}}\sum_{i=0}^{m}\binom{m+1}{i+1}\frac{1}{2^{i+1}}\sum_{j=0}^{i}\binom{i+1}{j+1}S_j$$

$$= \frac{(m+1)!}{2^{m+1}}\sum_{j=0}^{m}\frac{S_j}{(j+1)!}\sum_{i=j}^{m}\frac{1}{2^{i+1}(m-i)!\,(i-j)!}$$

$$= \frac{1}{2^{m+1} \cdot 2^{m+1}} \sum_{j=0}^{m} \frac{(m+1)!\, S_j}{(m-j)!\,(j+1)!} \sum_{i=0}^{m-j} \frac{(m-j)!\, 2^{m+1}}{(m-j-i)!\, i!\, 2^{i+j+1}}$$

$$= \frac{1}{(2^2)^{m+1}} \sum_{j=0}^{m} \binom{m+1}{j+1} 3^{m-j} S_j,$$

and assuming that

$$S_m^{(p-1)} = \frac{1}{(2^{p-1})^{m+1}} \sum_{i=0}^{m} \binom{m+1}{i+1} (2^{p-1}-1)^{m-i} S_i,$$

we have

$$S_m^{(p)} = \frac{1}{2^{m+1}} \sum_{i=0}^{m} \binom{m+1}{i+1} \frac{1}{(2^{p-1})^{i+1}} \sum_{j=0}^{i} \binom{i+1}{j+1} (2^{p-1}-1)^{i-j} S_j$$

$$= \frac{1}{2^{m+1} \cdot (2^{p-1})^{m+1}} \sum_{j=0}^{m} \frac{(m+1)!\, S_j}{(m-j)!\,(j+1)!} \sum_{i=j}^{m} \frac{(m-j)!\,(2^{p-1}-1)^{i-j}(2^{p-1})^{m+1}}{(m-i)!\,(i-j)!\,(2^{p-1})^{i+1}}$$

$$= \frac{1}{(2^p)^{m+1}} \sum_{j=0}^{m} \binom{m+1}{j+1} (2^p-1)^{m-j} S_j.$$

[5]) If S_m had been defined as $\sum_{i=0}^{m-1} u_i$ $(m>0)$ and $S_0 = 0$, we would have had for the pth transformation

$$\frac{1}{(2^p)^m} \sum_{i=0}^{m} \binom{m}{i} (2^p-1)^{m-i} S_i.$$

This is equivalent to changing from the series $u_0 + u_1 + \cdots$ to the series $0 + u_0 + u_1 + \cdots$, or, as we shall say, from the sequence S_m to the sequence S_{m-1}. Knopp has shown [1]) that if one of these series is summable E, the other is also, to the same sum.

2. Definitions and formulas. Starting with (3), we define the integral orders of Euler summability in a manner analogous to the Cesàro method, and whereas the Hölder and Cesàro methods are equivalent for all orders, Knopp's integral orders are not only equivalent to, but his means are identical to, certain of ours. Our integral orders include all of his, with interpolations besides. [6]) We set

[6]) We shall deal only with real orders of summability, yet our means of real order must

be interpreted as means of certain complex orders in the Knopp system, as will be shown in 3.

$$S_n = u_0 + u_1 + \cdots + u_n,$$

$$s_n^{(0)} = (u_0 + u_1 + \cdots + u_n)/(n+1)!,$$

(5)
$$s_n^{(1)} = \frac{s_0^{(0)}}{n!} + \frac{s_1^{(0)}}{(n-1)!} + \cdots + \frac{s_n^{(0)}}{0!},$$

$$s_n^{(2)} = \frac{s_0^{(1)}}{n!} + \frac{s_1^{(1)}}{(n-1)!} + \cdots + \frac{s_n^{(1)}}{0!},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$s_n^{(r)} = \frac{s_0^{(r-1)}}{n!} + \frac{s_1^{(r-1)}}{(n-1)!} + \cdots + \frac{s_n^{(r-1)}}{0!},$$

and say that $\sum u_i$ is summable E_r to S if

$$\lim_{n\to\infty} \frac{(n+1)!\, s_n^{(r)}}{(r+1)^{n+1}} = S.$$

Just as in the case of the Cesàro method, the definition can be put into a form having a meaning for non-integral and negative r. We put

(6)
$$a_n^{(r)} = \frac{(r+1)^n}{n!}, \quad s_n^{(r)} = \sum_{i=0}^{n} s_i^{(0)} a_{n-i}^{(r-1)},$$

$$S_n^{(r)} = \frac{s_n^{(r)}}{a_{n+1}^{(r)}} \qquad (r \neq -1),$$

and say the series is summable E_r to S provided [7] [8]

(7)
$$\lim_{n\to\infty} S_n^{(r)} = S.$$

7) Note that E_0 is convergence, E_{-1} is not defined, and $S_n^{(2^r-1)}$ is precisely (4). If we write $S_{n-2}=x_n$ $(n>1)$, $S_{-1}=0$, then $S_{n-2}^{(r-1)}$ is the exponential mean

$$\sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} \frac{(r-1)^{n-i}}{r^{n-i}} x_i,$$

defined in Miss Julia Dale's paper "Some properties of the exponential mean", American Journal of Mathematics, vol. 47 (1925), pp. 71-90.

8) If we had defined $s_n^{(0)} = \sum_{i=0}^{n} u_i/n!$, $S_n^{(r)} = s_n^{(r)}/a_n^{(r)}$, the other definitions being the same, our $S_n^{(2^p-1)}$ would have been the pth transformation of Knopp as given in[5]). In this connection see the generalization in Chapter IV.

That this is the same as (5) for integral r is seen from the second of the two formulas[9])

(8) $$a_n^{(r+s+1)} = \sum_{i=0}^{n} a_i^{(r)} a_{n-i}^{(s)} \qquad \text{(all real r and s)},$$

(9) $$s_n^{(r+s+1)} = \sum_{i=0}^{n} s_i^{(r)} a_{n-i}^{(s)} \qquad \text{(all real r and s)},$$

9) The analogy with the Cesàro method is here quite apparent. For if we had defined $a_n^{(s)}=\binom{n+s}{s}$ and $s_n^{(0)}=S_n$, leaving the other definitions in (6) the same, $S_n^{(s)}$ would have been the Cesàro mean, and the formulas (8) and (9) the well known formulas

$$\binom{n+r+s+1}{n} = \sum_{i=0}^{n} \binom{i+r}{r}\binom{n-i+s}{s},$$

$$s_n^{(r+s+1)} = \sum_{i=0}^{n} s_i^{(r)} \binom{n-i+s}{s}.$$

which may be proved as follows.

We have

$$\sum_{i=0}^{m} a_i^{(r)} a_{m-i}^{(s)} = \sum_{i=0}^{m} \frac{(r+1)^i (s+1)^{m-i}}{(m-i)!\, i!}$$

$$= \frac{(r+s+2)^m}{m!} = a_m^{(r+s+1)},$$

and

$$\sum_{i=0}^{m} s_i^{(r)} a_{m-i}^{(s)} = \sum_{i=0}^{m} a_{m-i}^{(s)} \sum_{j=0}^{i} s_j^{(0)} a_{i-j}^{(r-1)}$$

$$= \sum_{j=0}^{m} s_j^{(0)} \sum_{i=j}^{m} a_{m-i}^{(s)} a_{i-j}^{(r-1)} = \sum_{j=0}^{m} s_j^{(0)} \sum_{i=0}^{m-j} a_{m-j-i}^{(s)} a_i^{(r-1)}$$

$$= \sum_{j=0}^{m} s_j^{(0)} a_{m-j}^{(r+s)} = s_m^{(r+s+1)}.$$

Thus formulas (8) and (9) are established.

We now prove the following:

<u>Theorem</u> I. <u>The product of the two transformations</u> E_r <u>and</u> E_s <u>is given by the formula</u> $E_r E_s = E_{rs+r+s}$. This fact may be expressed also by[10])

$$S_m^{(r)}\left(S_m^{(s)}\right) = S_m^{(rs+r+s)} \qquad (r,s \neq -1). \tag{10}$$

[10]) For Hölder summability the analogous theorem is $H_r H_s = H_{r+s}$.

We have

$$S_m^{(r)}(S_m^{(s)}) = \frac{s_m^{(r)}(S_m^{(s)})}{a_{m+1}^{(r)}}$$

$$= \frac{1}{a_{m+1}^{(r)}} \sum_{i=0}^{m} a_{m-i}^{(r-1)} s_i^{(0)}(S_i^{(s)})$$

$$= \frac{1}{a_{m+1}^{(r)}} \sum_{i=0}^{m} a_{m-i}^{(r-1)} a_{i+1}^{(0)} S_i^{(s)}$$

$$= \frac{1}{a_{m+1}^{(r)}} \sum_{i=0}^{m} \frac{a_{m-i}^{(r-1)} a_{i+1}^{(0)}}{a_{i+1}^{(s)}} s_i^{(s)}$$

$$= \frac{1}{a_{m+1}^{(r)}} \sum_{i=0}^{m} \frac{a_{m-i}^{(r-1)} a_{i+1}^{(0)}}{a_{i+1}^{(s)}} \sum_{j=0}^{i} s_j^{(0)} a_{i-j}^{(s-1)}$$

$$= \frac{1}{a_{m+1}^{(r)}} \sum_{j=0}^{m} s_j^{(0)} \sum_{i=j}^{m} \frac{a_{i-j}^{(s-1)} a_{m-i}^{(r-1)} a_{i+1}^{(0)}}{a_{i+1}^{(s)}}$$

$$= \frac{a_{m+1}^{(0)}}{a_{m+1}^{(r)} a_{m+1}^{(s)}} \sum_{j=0}^{m} s_j^{(0)} \sum_{i=j}^{m} s_{i-j}^{(s-1)} s_{m-i}^{(rs+r-1)}$$

$$= \frac{1}{a_{m+1}^{(rs+r+s)}} \sum_{j=0}^{m} s_j^{(0)} s_{m-j}^{(rs+r+s-1)} ,$$

from which (10) follows immediately.

3. Relationship with the Knopp system. We now examine the relationship between our system and that of Knopp. The r in (6) and the p in (4) are connected by the relation

$$r = 2^p - 1,$$

or

$$p = \frac{\log(r+1)}{\log 2} = \frac{\log|r+1| + i\pi n}{\log 2},$$

where the integer n is even or odd according as $r+1$ is positive or negative. As p ranges from 0 to $-\infty$, r ranges from 0 to -1. As r ranges from -1 to $-\infty$, p can never be real. It assumes all complex values whose purely imaginary part is an odd multiple of $i\pi/\log 2$. Knopp has shown that his complex orders are not regular.

4. <u>Variation of the Borel method corresponding to</u> $r < -1$. Knopp (Footnote [20]), § 6) proved that every E-summable series of any positive order whatever is also B-summable to the same sum. We now show that this is true for $r > -1$ (Viz. for Knopp's negative as well as his positive orders), and that a certain variation of the Borel mean affords a corresponding theorem for $r < -1$. We prove

<u>Theorem</u> II. <u>If</u> $E_r - \lim S_n = S$ <u>then</u>

$$\lim_{x \to \infty} e^{-x} \sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = S \qquad (r > -1),$$

$$\lim_{x \to -\infty} e^{-x} \sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = S \qquad (r < -1).$$

We have [4])

$$\sum_{n=0}^{\infty} S_{n-1} \frac{x^n}{n!} = \sum_{n=1}^{\infty} x^n s_{n-1}^{(0)},$$

$$= \sum_{m=1}^{\infty} x^m \sum_{i=0}^{m-1} s_i^{(r)} a_{m-1-i}^{(-r-1)}$$

$$= \sum_{m=0}^{\infty} x^{m+1} \sum_{i=0}^{m} s_i^{(r)} a_{m-i}^{(-r-1)}.$$

") Note that the power series involved are convergent for all x provided $\sum_{i=0}^{\infty} u_i$ is summable E_r.

But the latter expression is the Cauchy product of two power series, viz. is equal to

$$x\left(\sum_{m=0}^{\infty} a_m^{(-r-1)} x^m\right)\left(\sum_{m=0}^{\infty} s_m^{(r)} x^m\right).$$

Consequently

$$\sum_{m=0}^{\infty} S_{m-1} \frac{x^m}{m!} = e^{-rx} \sum_{m=0}^{\infty} S_m^{(r)} \frac{[(r+1)x]^{m+1}}{(m+1)!}$$

$$= e^{-rx} \sum_{m=0}^{\infty} S_{m-1}^{(r)} \frac{[(r+1)x]^m}{m!} \qquad (S_{-1}^{(r)} = 0).$$

Therefore

$$\lim_{x\to\infty} e^{-x} \sum_{m=0}^{\infty} S_{m-1} \frac{x^m}{m!}$$

$$= \lim_{x\to\infty} e^{-x} \sum_{m=0}^{\infty} S_{m-1}^{(r)} \frac{x^m}{m!} \qquad (r > -1),$$

and

$$\lim_{x\to-\infty} e^{-x} \sum_{m=0}^{\infty} S_{m-1} \frac{x^m}{m!}$$

$$= \lim_{x\to\infty} e^{-x} \sum_{m=0}^{\infty} S_{m-1}^{(r)} \frac{x^m}{m!} \qquad (r < -1).$$

From this the theorem follows immediately when we note that we can replace S_{n-1} by S_n if we replace $S_{n-1}^{(\lambda)}$ by $S_{n-1}^{(\lambda)}$ (S_{n+1}), and that the latter approaches S by virtue of Theorem V (See footnote [15])).

Chapter II

Some applications of the Toeplitz Theorem

5. Relative power of the different orders. We now show that the power of E_r is monotonic increasing for $r > -1$ and decreasing for $r < -1$, discuss the effect of omitting or prefixing a first term in a summable series, and derive some necessary conditions for summability E_r. We shall need the Toeplitz Theorem [12], viz.

[12] See Knopp's "Infinite Series", p. 73, Theorem 5.

Theorem III. If $\lim x_n = \xi$, and $a_{\mu,\nu}$ satisfy

(a) $\lim_{n\to\infty} a_{n,i} = 0$ (for every i)

(11) (b) $|a_{n,0}| + |a_{n,1}| + \cdots + |a_{n,n}|$ is bounded for all n

(c) $\lim_{n\to\infty} (a_{n,0} + a_{n,1} + \cdots + a_{n,n}) = 1$,

then

$$\lim_{n\to\infty} \sum_{i=0}^{n} a_{n,i}\, x_i = \xi.$$

We now prove

Theorem IV. If $S_n^{(r)} \to S$ then $S_n^{(r+k)} \to S$ providing either $r < -1$ and $k \leq 0$ or $r > -1$ and $k \geq 0$.

We have

(12)
$$S_n^{(r+k)} = \frac{\sum_{i=0}^{n} \left(\frac{s_i^{(r)}}{a_{i+1}^{(r)}}\right) a_{n-i}^{(k-1)}\, a_{i+1}^{(r)}}{a_{n+1}^{(r+k)}}$$
$$= \sum_{i=0}^{n} a_{n,i}\, x_i \qquad \begin{pmatrix} r+k \neq -1 \\ r \neq -1 \end{pmatrix},$$

where

$$a_{m,i} = \frac{a^{(k-1)}_{m-i}\, a^{(r)}_{i+1}}{a^{(r+k)}_{m+1}}, \quad x_i = S^{(r)}_i .$$

Thinking of i as fixed, we have

$$a_{m,i} = \frac{\dfrac{k^{m-i}}{(m-i)!}\,\dfrac{(r+1)^{i+1}}{(i+1)!}}{\dfrac{(r+k+1)^{m+1}}{(m+1)!}} \sim \frac{\left(\dfrac{r+1}{k}\right)^{i+1}}{(i+1)!}\cdot\frac{m^{i+1}}{\left(\dfrac{r+k+1}{k}\right)^{m+1}},$$

so that (a) of (11) is satisfied if and only if

$$|r+k+1| > |k|.$$

This is also the condition that (c) be satisfied, since [13]

$$\sum_{i=0}^{m} a_{m,i} = \frac{a^{(r+k)}_{m+1} - a^{(k-1)}_{m+1}}{a^{(r+k)}_{m+1}} = 1 - \left(\frac{k}{r+k+1}\right)^{m+1}.$$

[13] Noting that $\sum\limits_{i=0}^{m} a^{(r)}_{m-i}\, a^{(s)}_{i+1} = \sum\limits_{i=1}^{m+1} a^{(r)}_{m+1-i}\, a^{(s)}_{i} = a^{(r+s+1)}_{m+1} - a^{(r)}_{m+1}$.

Testing condition (b) we see that

$$\sum_{i=0}^{m} |a_{m,i}| = \frac{\sum\limits_{i=0}^{m} |a^{(k-1)}_{m-i}|\,|a^{(r)}_{i+1}|}{|a^{(r+k)}_{m+1}|} = \frac{\sum\limits_{i=0}^{m} a^{(|k|-1)}_{m-i}\, a^{(|r+1|-1)}_{i+1}}{a^{(|r+k+1|-1)}_{m+1}}$$

$$= \frac{a^{(|k|+|r+1|-1)}_{m+1} - a^{(|k|-1)}_{m+1}}{a^{(|r+k+1|-1)}_{m+1}}$$

$$= \left(\frac{|k| + |r+1|}{|r+k+1|} \right)^{m+1} - \left| \frac{k}{r+k+1} \right|^{m+1}.$$

Consequently a necessary and sufficient condition that (a), (b), and (c) all be satisfied is

$$|k| + |r+1| = |r+k+1|, \tag{13}$$

it being understood, of course, that r and $r+k$ are $\neq -1$. This says that k and $r+1$ must have the same sign, and therefore completes the proof of Theorem IV.

6. <u>Relation between the summability of $\sum_{i=0}^{\infty} u_i$ and that of $\sum_{i=1}^{\infty} u_i$.</u> We now examine the effect of omitting or adjoining a first term to a summable series. We shall prove

<u>Theorem V. If $\sum_{i=0}^{\infty} u_i$ is summable E_r then $\sum_{i=1}^{\infty} u_i$ is also, and to the correct sum, but the converse is true only for</u> $r > -\frac{1}{2}$. [14] [15] We write

[14] Knopp [1] discussed this for summability E_1.

[15] In terms of sequences this says the existence of E_r-lim S_{m+1} ⊃ the existence of E_r-lim S_m, but the existence of E_r-lim S_{m+1} ⊃ the existence of E_r-lim S_m only if $r > -\frac{1}{2}$.

$$S_m' = u_1 + u_2 + \cdots + u_{m+1}, \tag{14}$$

so that $S_m' = S_{m+1} - u_0$. Then

$$S_m^{(r)}(S'_m) = \frac{\sum_{i=0}^{m} \frac{S_{i+1}}{(i+1)!} a_{m-i}^{(r-1)} - u_0 \sum_{i=0}^{m} a_{i+1}^{(0)} a_{m-i}^{(r-1)}}{a_{m+1}^{(r)}}$$

$$= \frac{\sum_{i=1}^{m+1} \frac{S_i}{i!} a_{m+1-i}^{(r-1)} - u_0 \left(a_{m+1}^{(r)} - a_{m+1}^{(r-1)}\right)}{a_{m+1}^{(r)}}$$

$$= \frac{\sum_{i=0}^{m+1} \frac{S_i}{(i+1)!} a_{m+1-i}^{(r-1)} - u_0 a_{m+1}^{(r-1)} - u_0 \left(a_{m+1}^{(r)} - a_{m+1}^{(r-1)}\right)}{a_{m+1}^{(r)}}$$

$$= \frac{\sum_{i=0}^{m+1} s_i^{(0)} (i+1)\, a_{m+1-i}^{(r-1)}}{a_{m+1}^{(r)}} - u_0$$

$$= \frac{(m+2) \sum_{i=0}^{m+1} s_i^{(0)} a_{m+1-i}^{(r-1)} - r \sum_{i=0}^{m} s_i^{(0)} a_{m-i}^{(r-1)}}{a_{m+1}^{(r)}} - u_0$$

$$= \frac{(m+2)\, s_{m+1}^{(r)} - r\, s_m^{(r)}}{a_{m+1}^{(r)}} - u_0$$

$$= (r+1)\, S_{m+1}^{(r)} - r\, S_m^{(r)} - u_0 .$$

Thus the first half of the theorem is proved. We now establish the formula

(16) $$\frac{\sum_{i=0}^{m+1} \frac{u_{i-1}}{i!} a_{m+1-i}^{(r)}}{a_{m+1}^{(r)}} = \left(\frac{r}{r+1}\right)^m \sum_{i=0}^{m} \left(\frac{r+1}{r}\right)^i \left(S_i^{(r)} - S_{i-1}^{(r)}\right) \qquad \left(\begin{matrix} u_{-1} = 0 \\ S_{-1}^{(r)} = 0 \end{matrix}\right).$$

We have

$$\frac{1}{n+1}\sum_{i=0}^{n}\frac{u_i}{i!}a_{n-i}^{(r-1)} = \sum_{i=0}^{n+1}\frac{u_{i-1}}{i!}a_{n+1-i}^{(r-1)}$$

$$-\frac{r}{n+1}\sum_{i=0}^{n}\frac{u_{i-1}}{i!}a_{n-i}^{(r-1)} \qquad (u_{-1}=0),$$

so that

$$\sum_{i=0}^{n} r^{n-i}\, i!\sum_{j=0}^{i}\frac{u_j}{j!}a_{i-j}^{(r-1)}$$

$$= \sum_{i=0}^{n} r^{n-i}(i+1)!\left[\sum_{j=0}^{i+1}\frac{u_{j-1}}{j!}a_{i+1-j}^{(r-1)} - \frac{r}{i+1}\sum_{j=0}^{i}\frac{u_{j-1}}{j!}a_{i-j}^{(r-1)}\right]$$

$$= (n+1)!\sum_{j=0}^{n+1}\frac{u_{j-1}}{j!}a_{n+1-j}^{(r-1)} - r\,n!\sum_{j=0}^{n}\frac{u_{j-1}}{j!}a_{n-j}^{(r-1)}$$

$$+ \sum_{i=1}^{n} r^{n-i+1}\, i!\sum_{j=0}^{i}\frac{u_{j-1}}{j!}a_{i-j}^{(r-1)}$$

$$- \sum_{i=0}^{n-1} r^{n-i+1}\, i!\sum_{j=0}^{i}\frac{u_{j-1}}{j!}a_{i-j}^{(r-1)}$$

$$= (n+1)!\sum_{j=0}^{n+1}\frac{u_{j-1}}{j!}a_{n+1-j}^{(r-1)},$$

from which it follows that

$$(17)\quad \frac{\sum_{i=0}^{n+1}\frac{u_{i-1}}{i!}a_{n+1-i}^{(r-1)}}{a_{n+1}^{(r)}} = \left(\frac{r}{r+1}\right)^{n}\sum_{i=0}^{n}\left(\frac{r+1}{r}\right)^{i}\frac{\sum_{j=0}^{i}\frac{u_j}{j!}a_{i-j}^{(r-1)}}{(r+1)\,a_i^{(r)}}.$$

Also

$$S_m^{(\lambda)} - S_{m-1}^{(\lambda)} = \frac{\sum_{i=0}^{m} \frac{S_i}{(i+1)!} a_{m-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}} - \frac{\frac{\lambda+1}{m+1} \sum_{i=0}^{m-1} \frac{S_i}{(i+1)!} a_{m-1-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}}$$

$$= \frac{\frac{S_m}{(m+1)!} + \sum_{i=0}^{m-1} \frac{S_i}{(i+1)!} a_{m-i}^{(\lambda-1)} - \frac{\lambda+1}{m+1} \sum_{i=0}^{m-1} \frac{S_i}{(i+1)!} a_{m-1-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}}$$

$$= \frac{\frac{S_m}{(m+1)!} + \frac{1}{m+1} \sum_{i=0}^{m-1} \frac{S_i}{i!} a_{m-i}^{(\lambda-1)} - \frac{1}{m+1} \sum_{i=0}^{m-1} \frac{S_i}{(i+1)!} a_{m-1-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}}$$

$$= \frac{\frac{1}{m+1} \sum_{i=0}^{m} \frac{S_i}{i!} a_{m-i}^{(\lambda-1)} - \frac{1}{m+1} \sum_{i=1}^{m} \frac{S_{i-1}}{i!} a_{m-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}}$$

$$= \frac{\frac{1}{m+1} \sum_{i=0}^{m} \frac{u_i}{i!} a_{m-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}} = \frac{\sum_{i=0}^{m} \frac{u_i}{i!} a_{m-i}^{(\lambda-1)}}{(\lambda+1)\, a_m^{(\lambda)}} \qquad (S_{-1} = 0).$$

(16) follows from (17) and (18) immediately. Now we can prove the second half of the theorem. We write

$$S''_m = u + S_{m-1} \qquad (S_{-1} = 0).$$

Then

$$S_m^{(\lambda)}(S'_m) = \frac{u \sum_{i=0}^{m} a_{i+1}^{(0)} a_{m-i}^{(\lambda-1)} + \sum_{i=1}^{m} \frac{S_{i-1}}{(i+1)!} a_{m-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}}$$

$$= \frac{u\left(a_{m+1}^{(\lambda)} - a_{m+1}^{(\lambda-1)}\right)}{a_{m+1}^{(\lambda)}} + \frac{\sum_{i=1}^{m} \frac{S_{i-1}}{(i+1)!} a_{m-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}},$$

so that

$$(19)\qquad \begin{aligned} S_m^{(r)} - S_m^{(r)}(S_m'') &= \frac{\sum_{i=0}^{m} \frac{u_i}{(i+1)!}\, a_{m-i}^{(r-1)}}{a_{m+1}^{(r)}} - u + u\left(\frac{r}{r+1}\right)^{m+1} \\ &= \frac{\sum_{i=0}^{m+1} \frac{u_{i-1}}{i!}\, a_{m+1-i}^{(r)}}{a_{m+1}^{(r)}} - u + u\left(\frac{r}{r+1}\right)^{m+1} \quad (u_{-1}=0) \\ &= \left(\frac{r}{r+1}\right)^{m} \sum_{i=0}^{m} \left(\frac{r+1}{r}\right)^{i} \left(S_i^{(r)} - S_{i-1}^{(r)}\right) \\ &\quad - u + u\left(\frac{r}{r+1}\right)^{m+1} \qquad (S_{-1}^{(r)}=0), \end{aligned}$$

the last step following from (16). We shall show that for $0 \neq r > -\frac{1}{2}$ [16] the right side of (19) approaches $-u$ as n becomes infinite provided $S_m^{(r)} \to S$, and for $r \leq -\frac{1}{2}$ we can give an example to show that S_m'' may not be limitable E_r. To do the former we need only show that

$$(20)\qquad \left(\frac{r}{r+1}\right)^{m} \sum_{i=0}^{m} \left(\frac{r+1}{r}\right)^{i} \epsilon_i \longrightarrow 0,$$

where $r > -\frac{1}{2}$ and $\epsilon_m \to 0$. If $-\frac{1}{2} < r < 0$ we can write $k = -r$, and (20) becomes

$$(-1)^m \left(\frac{k}{1-k}\right)^m \sum_{i=0}^{m} \left(\frac{1-k}{k}\right)^i \eta_i \longrightarrow 0,$$

where $\eta_i = (-1)^i \epsilon_i$, so it is sufficient to prove

$$\rho\, \frac{\sum_{i=0}^{m} \rho^i \epsilon_i}{\rho^{m+1}} \longrightarrow 0,$$

where $\rho > 1$. This can be proved by application of the

[16] The case $r=0$, being convergence, is already taken care of.

Toeplitz Theorem. [17] There remains now only the exhibition

[17] Applying the Toeplitz theorem (Theorem III), we have
$$a_{m,i} = \frac{\rho^i}{\rho^{m+1}}, \quad x_i = \epsilon_i.$$
(a) is obviously satisfied, and (b) is satisfied if (c) is, since the ρ's are positive. We need only show then that
$$\lim_{m \to \infty} \frac{\sum_{i=0}^{m} \rho^i}{\rho^{m+1}}$$
exists, since then we could multiply by the proper constant to make it 1. We have
$$\sum_{i=0}^{m} \rho^i < \int_0^{m+1} \rho^x dx = \frac{\rho^{m+1}}{\log \rho} - \frac{1}{\log \rho},$$
so that $\frac{1}{\rho^{m+1}} \sum_{i=0}^{m} \rho^i$ is bounded above. It is monotonic increasing since
$$\frac{\sum_{i=0}^{m+1} \rho^i}{\rho^{m+2}} - \frac{\sum_{i=0}^{m} \rho^i}{\rho^{m+1}} = \frac{\sum_{i=0}^{m+1} \rho^i - \sum_{i=1}^{m+1} \rho^i}{\rho^{m+2}} = \frac{1}{\rho^{m+2}} > 0.$$

of a series summable E_r $(r \leq -\frac{1}{2})$ which loses its summability on having a term prefixed, in order to complete the proof of Theorem V. The first example in the next section is of this nature.

7. Some examples of summable series. The series

$$1 - r + r^2 - \cdots + (-r)^m + \cdots \tag{21}$$

is summable E_r $(r \neq -1)$ to the sum $1/(1+r)$, for we have

$$S_m = \frac{1-(-r)^{m+1}}{1+r},$$
$$S_m^{(r)} = \frac{\sum_{i=0}^{m} \frac{1-(-r)^{i+1}}{(1+r)(i+1)!} a_{m-i}^{(r-1)}}{a_{m+1}^{(r)}}$$

$$= \frac{\sum_{i=0}^{m} a^{(0)}_{i+1}\, a^{(\lambda-1)}_{m-i} - \sum_{i=0}^{m} a^{(-\lambda-1)}_{i+1}\, a^{(\lambda-1)}_{m-i}}{(1+\lambda)\; a^{(\lambda)}_{m+1}}$$

$$= \frac{a^{(\lambda)}_{m+1} - a^{(\lambda-1)}_{m+1} - a^{(-1)}_{m+1} + a^{(\lambda-1)}_{m+1}}{(1+\lambda)\; a^{(\lambda)}_{m+1}}$$

$$= \frac{1}{1+\lambda}.$$

Consequently for the series (21) we have from (19)

$$S^{(\lambda)}_m - S^{(\lambda)}_m(S''_m) = \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^m - u + u\left(\frac{\lambda}{1+\lambda}\right)^{m+1}.$$

When $r \leqq -\frac{1}{2}$ the right side will oscillate or become infinite except in the case where $u = -r^{-1}$. This shows that the prefixing of a term may destroy the summability (Of order $\leqq -\frac{1}{2}$) of a series. Note that when $r < -1$, the series (21) is divergent to $+\infty$, yet it is summable E_λ to the quotient which when formally expanded gives the series.

Another example which illustrates the behavior of the negative orders is the following. The series $0+0+0+\cdots$ can be written as $(1-1)+(2r-2r)+(3r^2-3r^2)+\cdots$. Removing the parentheses and regrouping the terms, we get

(22) $$1-(1+2r)+(2r+3r^2)-(3r+4r^3)+\cdots.$$

This series is summable E_λ to 0, for,

$$S_m = (-1)^m(m+1)\lambda^m, \quad s^{(0)}_m = \frac{(-\lambda)^m}{m!} = a^{(-\lambda-1)}_m,$$

so that

$$S_m^{(\lambda)} = \frac{\sum_{i=0}^{m} a_i^{(-\lambda-1)} a_{m-i}^{(\lambda-1)}}{a_{m+1}^{(\lambda)}} = \frac{a_m^{(-1)}}{a_{m+1}^{(\lambda)}} = 0.$$

For a positive r, series (22) is an alternating one. For $-1 < r < 0$ its terms eventually become negative.[18] It is,

[18] We have $u_m = (-1)^m[nr^{m-1} + (n+1)r^m] = (-r)^m\left[\frac{n}{r} + n + 1\right]$
$= n(-r)^m\left[\frac{1+r}{r} + \frac{1}{n}\right].$
When $-1 < r < 0$, $(-r)^m$ is positive and $\frac{1+r}{r}$ is negative. So u_m will be negative for sufficiently large n.

however, convergent, as must be the case from Theorem IV. For example, when $r = -\frac{1}{2}$, we have

$$0 = 1 + 0 - \frac{1}{4} - \frac{2}{8} - \frac{3}{16} - \frac{4}{32} - \cdot\ \cdot\ \cdot\ .$$

When $r \leq -1$,[19] the terms are all positive, and the series

[19] The general term is
$(-1)^m[m\lambda^{m-1} + (m+1)\lambda^m] = (-\lambda)^m\left[m\left(\frac{\lambda+1}{\lambda}\right) + 1\right].$
and when $r < -1$ we have $\frac{\lambda+1}{\lambda} > 0$.

diverges to $+\infty$. For example, when $r = -2$, we have

$$1 + 3 + 8 + 20 + \cdot\ \cdot\ \cdot + 2^{m-1}(n+2) + \cdot\ \cdot\ \cdot\ .$$

Although the means of negative order would seem to be quite powerful as evidenced by these two examples, it is easy to see that they are unsuited for more delicate work. For example, the means of order $< -\frac{1}{2}$ for the series

$$1 + 0 + 0 + 0 + \cdot\ \cdot\ \cdot$$

oscillate between $+\infty$ and $-\infty$ as $n \to \infty$. For, we have

$$S_n = 1, \quad s_n^{(0)} = \frac{1}{(n+1)!} = a_{n+1}^{(0)},$$

so that

$$S_n^{(r)} = \frac{\sum_{i=0}^{n} a_{i+1}^{(0)} a_{n-i}^{(r-1)}}{a_{n+1}^{(r)}}$$

$$= \frac{a_{n+1}^{(r)} + a_{n+1}^{(r-1)}}{a_{n+1}^{(r)}} = 1 + \left(\frac{r}{r+1}\right)^{n+1}.$$

So $S_n^{(r)} \to 1$ if $r > -\frac{1}{2}$, $S_n^{(r)}$ oscillates between 0 and 2 if $r = -\frac{1}{2}$, $S_n^{(r)}$ oscillates infinitely if $r < -\frac{1}{2}$. However, a negative order (< -1) can never treat a sequence so badly that it can't be restored by a suitable negative order, as is seen from Theorem I.

The question of whether the orders < -1 could give results inconsistent with those > -1 is one of great importance. We have not been able to settle it yet but expect to return to it in the future.

8. <u>Some necessary conditions for Euler summability</u>.

We now prove

<u>Theorem</u> VI. <u>If</u> $\sum_{i=0}^{\infty} u_i$ <u>is summable</u> E_r <u>then</u>

(a) $\quad S_n^{(r-k)} = o\left(\left[\frac{k+r+1}{k-r-1}\right]^{n+1}\right) \quad (|k| + |r+1| = |r+k+1|)$

(23) (b) $\quad u_n^{(r-k)'} = o\left(\left[\frac{k+r+1}{k-r-1}\right]^{n+1}\right) \quad (|k| + |r+1| = |r+k+1|)$

(c) $\quad E_{\frac{k}{r-k+1}}$ - lim $u_{n+1}^{(r-k)'} = 0 \quad (|k| < |r+1|)$,

<u>where</u> $u_n^{(r)'}$ <u>denotes the general term of the</u> rth <u>transformed series</u>.

We have

$$S_n^{(r-k)} = \frac{\sum_{i=0}^{n} s_i^{(r)}\, a_{n-i}^{(-k-1)}}{a_{n+1}^{(r-k)}} \qquad (r-k \neq -1).$$

Putting

$$s_n^{(r)} = S\, a_{n+1}^{(r)} + \epsilon_n\, a_{n+1}^{(r)} \qquad (r \neq -1),$$

where $\epsilon_n \to 0$, we have

$$\begin{aligned} S_n^{(r-k)} &= S\, \frac{a_{n+1}^{(r-k)} - a_{n+1}^{(-k-1)}}{a_{n+1}^{(r-k)}} + \frac{\sum_{i=0}^{n} \epsilon_i\, a_{i+1}^{(r)}\, a_{n-i}^{(-k-1)}}{a_{n+1}^{(r-k)}} \\ &= S\left[1 - \left(\frac{k}{k-r-1}\right)^{n+1}\right] \\ &\quad + \left(\frac{k+r+1}{k-r-1}\right)^{n+1} \frac{\sum_{i=0}^{n} (-1)^{i+1} \epsilon_i\, a_{i+1}^{(r)}\, a_{n-i}^{(k-1)}}{a_{n+1}^{(r+k)}}. \end{aligned}$$

Consequently

$$\begin{aligned} \frac{S_n^{(r-k)}}{\left(\frac{k+r+1}{k-r-1}\right)^{n+1}} &= S\left[\left(\frac{k-r-1}{k+r+1}\right)^{n+1} - \left(\frac{k}{k+r+1}\right)^{n+1}\right] \\ &\quad + \frac{\sum_{i=0}^{n} (-1)^{i+1} \epsilon_i\, a_{i+1}^{(r)}\, a_{n-i}^{(k-1)}}{a_{n+1}^{(r+k)}}. \end{aligned}$$

As seen from (12) and (13), the right side $\to 0$ as $n \to \infty$ provided k and $r+1$ have the same sign. So the condition (a) for summability E_r is established. For $r = k \geqq 0$ this result was obtained in different form by Knopp. [20] In

[20]) "Über das Eulersche Summierungsverfahren II". Mathematische Zeitschrift, vol. 18, 1923, pp. 125-156. Putting $r = 2^k - 1$ in our condition (a), we get the third condition in Knopp's Theorem 4.

particular, if we put $r = k = 1$, we see that summability E_1 requires that $S_m = o(3^m)$ [21].

[21]) See Knopp's "Infinite Series", p. 512, Theorem 1.

We have

$$u_m^{(r)'} = S_m^{(r)} - S_{m-1}^{(r)} \qquad (S_{-1}^{(r)} = 0).$$

The prime is used to distinguish from the $u_m^{(r)}$ as defined later. Then we have from (18)

$$\text{(24)} \qquad u_m^{(r)'} = \frac{\sum_{i=0}^{m} \frac{u_i}{i!} a_{m-i}^{(r-1)}}{(r+1)\, a_m^{(r)}}.$$

The inverse of this is

$$\text{(25)} \qquad u_m = (r+1)\, m! \sum_{i=0}^{m} u_i^{(r)'} a_i^{(r)} a_{m-i}^{(-r-1)},$$

which we show as follows. We have

$$\begin{aligned}
&(r+1) m! \sum_{i=0}^{m} u_i^{(r)'} a_i^{(r)} a_{m-i}^{(-r-1)} \\
&\quad = (r+1) m! \sum_{i=0}^{m} a_i^{(r)} a_{m-i}^{(-r-1)} \frac{1}{(r+1) a_i^{(r)}} \sum_{j=0}^{i} \frac{u_j}{j!} a_{i-j}^{(r-1)} \\
&\quad = m! \sum_{j=0}^{m} \frac{u_j}{j!} \sum_{i=j}^{m} a_{m-i}^{(-r-1)} a_{i-j}^{(r-1)} \\
&\quad = u_m\,!
\end{aligned}$$

Further, we have from (24) and (25)

$$u_{m+1}^{(r-k)'} = \frac{\sum_{i=0}^{m+1} \frac{u_i}{i!} a_{m+1-i}^{(r-k-1)}}{(r-k+1)\, a_{m+1}^{(r-k)}}$$

$$= \frac{\sum_{i=0}^{m+1} \frac{1}{i!} a_{m+1-i}^{(r-k-1)} (r+1) i! \sum_{j=0}^{i} u_j^{(r)'} a_j^{(r)} a_{i-j}^{(-r-1)}}{(r-k+1)\, a_{m+1}^{(r-k)}}$$

$$= \frac{r+1}{r-k+1} \frac{\sum_{j=0}^{m+1} u_j^{(r)'} a_j^{(r)} \sum_{i=j}^{m+1} a_{m+1-i}^{(r-k-1)} a_{i-j}^{(-r-1)}}{a_{m+1}^{(r-k)}}$$

$$= \frac{r+1}{r-k+1} \frac{\sum_{j=0}^{m+1} u_j^{(r)'} a_j^{(r)} \sum_{i=0}^{m-j+1} a_{m-j+1-i}^{(r-k-1)} a_i^{(-r-1)}}{a_{m+1}^{(r-k)}}$$

$$= \frac{r+1}{r-k+1} \frac{\sum_{j=0}^{m+1} u_j^{(r)'} a_j^{(r)} a_{m+1-j}^{(-k-1)}}{a_{m+1}^{(r-k)}}$$

$$= \frac{r+1}{r-k+1} \frac{u_0^{(r)'} a_{m+1}^{(-k-1)} + \sum_{j=0}^{m} u_{j+1}^{(r)'} a_{j+1}^{(r)} a_{m-j}^{(-k-1)}}{a_{m+1}^{(r-k)}}$$

$$= \frac{r+1}{r-k+1} \left[u_0^{(r)'} \left(\frac{k}{k-r-1} \right)^{m+1} + \left(\frac{k+r+1}{k-r-1} \right)^{m+1} \frac{\sum_{j=0}^{m} (-1)^{j+1} u_{j+1}^{(r)'} a_{j+1}^{(r)} a_{m-j}^{(k-1)}}{a_{m+1}^{(r+k)}} \right].$$

Thus, precisely as we obtained the necessary condition (a) we see that (b) is also necessary. In particular when $k=r$, we have

$$(27) \qquad u_m = O\left((2r+1)^m\right) \qquad (|r-1| + |r+1| = |2r|).$$

This result for $r \geq 0$ is Knopp's second condition in different form (See footnote [10])).

Also, we have from (26),

$$u_{m+1}^{(r-k)'} = \frac{r+1}{r-k+1} \, \frac{u_0^{(r)'} a_{m+1}^{(-k-1)} + \sum_{j=0}^{m} u_{j+1}^{(r)'} a_{j+1}^{(r)} a_{m-j}^{(-k-1)}}{a_{m+1}^{(r-k)}}$$

$$= \frac{r+1}{r-k+1}\left[u_0^{(r)'}\left(\frac{k}{k-r-1}\right)^{m+1} + \frac{(r+1)^{m+1}\sum_{j=0}^{m}\frac{u_{j+1}^{(r)'}}{(j+1)!}\, a_{m-j}^{\left(-\frac{k+r+1}{r+1}\right)}}{a_{m+1}^{(r-k)}}\right]$$

$$= \frac{r+1}{r-k+1}\left[u_0^{(r)'}\left(\frac{k}{k-r-1}\right)^{m+1} + \frac{\sum_{j=0}^{m} s_j^{(0)}(u_{j+1}^{(r)'})\, a_{m-j}^{\left(-\frac{k+r+1}{r+1}\right)}}{a_{m+1}^{\left(-\frac{k}{r+1}\right)}}\right]$$

$$= \frac{r+1}{r-k+1}\left[u_0^{(r)'}\left(\frac{k}{k-r-1}\right)^{m+1} + S_m^{\left(-\frac{k}{r+1}\right)}(u_{m+1}^{(r)'})\right].$$

Now if we replace r by $r-k$ and k by $-k$, we have

$$(28) \qquad u_{m+1}^{(r)'} = \frac{r-k+1}{r+k+1}\left[u_0^{(r-k)'}\left(\frac{k}{r+1}\right)^{m+1} + S_m^{\left(\frac{k}{r-k+1}\right)}(u_{m+1}^{(r-k)'})\right].$$

Condition (e) follows from this. In particular when $r = k$, we have [22])

$$E_\lambda\text{-lim } u_m = 0 \qquad (r > -\tfrac{1}{2}). \tag{29}$$

[22]) Note that the $n+1$ has been replaced by n by virtue of Theorem V.

This result for $r \geqq 0$ is Knopp's first condition (See footnote [10])).

Chapter III

A product theorem

9. <u>The Cauchy product of two infinite series</u>. Cauchy proved that if $\sum u_i$ and $\sum v_i$ are both absolutely convergent then their Cauchy product, $\sum w_i$, is convergent to the correct sum. Mertens showed that this is the case even if only one of the two convergent series is absolutely convergent, and Abel had already shown that if all three of the series converged, say to U, V, and W respectively, then $W = UV$. Knopp has proved theorems (see footnote [20], § 2) analogous to the Abel and Mertens theorems for E_r-summability of positive orders. Cesàro proved that if $\sum u_i$ is summable C_r to U and $\sum v_i$ is summable C_s to V, then the Cauchy product is summable C_{r+s+1} to UV. We have pointed out the analogy between E_r-summability and the Cesàro method, and in the proof of the product theorem of this chapter we find the similarity still striking, although in the case of Euler summability the Cauchy product is replaced by a new definition of the product of two series.

10. <u>Definitions and formulas</u>. Before stating the product theorem we introduce some notations and derive some formulas. $\sum_{i=0}^{\infty} u_i$ and $\sum_{i=0}^{\infty} v_i$ are said to be summable E_r to U and V respectively provided

$$\lim_{n\to\infty} U_n^{(r)} = U \quad\text{and}\quad \lim_{n\to\infty} V_n^{(r)} = V,$$

where

$$U_n = \sum_{i=0}^{n} u_i, \quad V_n = \sum_{i=0}^{n} v_i,$$

$$u_n^{(0)} = \frac{U_n}{(n+1)!}, \quad v_n^{(0)} = \frac{V_n}{(n+1)!},$$

$$u_n^{(r)} = \sum_{i=0}^{n} u_i^{(0)} a_{n-i}^{(r-1)}, \quad v_n^{(r)} = \sum_{i=0}^{n} v_i^{(0)} a_{n-i}^{(r-1)},$$

$$U_n^{(r)} = \frac{u_n^{(r)}}{a_{n+1}^{(r)}} \ (r \neq -1), \quad V_n^{(r)} = \frac{v_n^{(r)}}{a_{n+1}^{(r)}} \ (r \neq -1).$$

We have

$$u_n^{(-1)} = \sum_{i=0}^{n} u_i^{(0)} a_{n-i}^{(-2)} = \sum_{i=0}^{n} \frac{U_i}{(i+1)!} a_{n-i}^{(-2)}$$

$$= \sum_{i=0}^{n} a_{i+1}^{(0)} a_{n-i}^{(-2)} \sum_{j=0}^{i} u_j$$

$$= \sum_{j=0}^{n} u_j \sum_{i=j}^{n} a_{i+1}^{(0)} a_{n-i}^{(-2)}.$$

But

$$\sum_{i=j}^{n} a_{i+1}^{(0)} a_{n-i}^{(-2)} = \frac{(-1)^n}{n+1} a_j^{(-2)} a_{n-j}^{(0)}, \tag{31}$$

as we now show by mathematical induction. For $j = 0$ we have

$$\sum_{i=0}^{n} a_{i+1}^{(0)} a_{n-i}^{(-2)} = -a_{n+1}^{(-2)},$$

or

$$\sum_{i=0}^{m+1} a_i^{(0)} a_{m+1-i}^{(-2)} = 0,$$

which is true by virtue of (8) and (6). Then assuming the formula true for j, we subtract $a_{j+1}^{(0)} a_{m-j}^{(-2)}$ from both sides, and get

$$\sum_{i=j+1}^{m} a_{i+1}^{(0)} a_{m-i}^{(-2)} = \frac{(-1)^m}{m+1} a_j^{(-2)} a_{m-j}^{(0)} - a_{j+1}^{(0)} a_{m-j}^{(-2)}$$

$$= \frac{(-1)^m}{m+1} a_{j+1}^{(-2)} a_{m-j-1}^{(0)},$$

which completes the inductive proof. Thus

$$u_m^{(-1)} = \frac{(-1)^m}{m+1} \sum_{j=0}^{m} a_j^{(-2)} a_{m-j}^{(0)} u_j. \tag{32}$$

We now prove

$$u_m = m! \sum_{i=0}^{m} \frac{i+1}{(m-i)!} u_i^{(-1)}, \tag{33}$$

which is the inverse of (32). We need to show that

$$u_m = m! \sum_{i=0}^{m} \frac{i+1}{(m-i)!} \cdot \frac{(-1)^i}{i+1} \sum_{j=0}^{i} a_j^{(-2)} a_{i-j}^{(0)} u_j$$

$$= m! \sum_{j=0}^{m} a_j^{(0)} u_j \sum_{i=j}^{m} a_{i-j}^{(-2)} a_{m-i}^{(0)}$$

$$= m! \sum_{j=0}^{m} a_j^{(0)} u_j \sum_{i=0}^{m-j} a_i^{(-2)} a_{m-j-i}^{(0)}$$

$$= n!\, a_n^{(0)} u_n + n! \sum_{j=0}^{n-1} a_j^{(0)} u_j a_{n-j}^{(-1)},$$

the last step coming from the use of (8). Since $a_{n-j}^{(-1)} = 0$ $(j \neq n)$ this is obviously true.

We now establish the relation

(34) $$\sum_{i=0}^{n} u_i^{(r)} v_{n-i}^{(s)} = \sum_{i=0}^{n} u_i^{(r+k)} v_{n-i}^{(s-k)} \qquad \text{(all real } r,s,k).$$

We have

$$\sum_{i=0}^{n} u_i^{(r)} v_{n-i}^{(s)} = \sum_{i=0}^{n} v_{n-i}^{(s)} \sum_{j=0}^{i} u_j^{(r+k)} a_{i-j}^{(-k-1)}$$

$$= \sum_{j=0}^{n} u_j^{(r+k)} \sum_{i=j}^{n} a_{i-j}^{(-k-1)} v_{n-i}^{(s)}$$

$$= \sum_{j=0}^{n} u_j^{(r+k)} \sum_{i=0}^{n-j} a_i^{(-k-1)} v_{n-j-i}^{(s)}$$

$$= \sum_{j=0}^{n} u_j^{(r+k)} v_{n-j}^{(s-k)},$$

by virtue of (9).

11. A new definition of the product of two infinite series. We now prove

Theorem VII. If $\sum_{i=0}^{\infty} u_i$ is summable E_r to U and $\sum_{i=0}^{\infty} v_i$ is summable E_s to V, then $\sum_{i=0}^{\infty} w_i$, where

$$(35) \qquad w_n = \sum_{\mu+\nu \leq n} \frac{(n+1)!(2n+2-\mu-\nu)(-1)^{n-\mu-\nu} u_\mu v_\nu}{\mu!\,\nu!\,(n-\mu-\nu)!(n+1-\mu)(n+1-\nu)},$$

is summable E_{r+s+1} to UV provided r and s are both > -1 or both < -1.

The series $\sum_{i=0}^{\infty} w_i$ will be summable E_{r+s+1} providing the series $0 + w_0 + w_1 + \cdots$ is, as seen from Theorem V. We now show that the latter series is summable E_{r+s+1}, and we shall use a dash to distinguish our notation referring to it. For example,

$$(36) \qquad \overline{W}_0 = 0, \quad \overline{W}_n = W_{n-1} \ (n > 0).$$

We have

$$W_n = \sum_{i=0}^{n} (i+1)! \sum_{\mu+\nu \leq i} \frac{(2i+2-\mu-\nu)(-1)^{i-\mu-\nu} u_\mu v_\nu}{\mu!\,\nu!\,(i-\mu-\nu)!(i+1-\mu)(i+1-\nu)}$$

$$= \sum_{i=0}^{n} (i+1)! \sum_{\mu=0}^{i} \frac{u_\mu}{\mu!(i+1-\mu)} \sum_{\nu=0}^{i-\mu} \frac{v_\nu (2i+2-\mu-\nu)(-1)^{i-\mu-\nu}}{\nu!(i+1-\nu)(i-\mu-\nu)!}$$

$$= \sum_{\mu=0}^{n} \frac{u_\mu}{\mu!} \sum_{i=\mu}^{n} \frac{(i+1)!}{(i+1-\mu)} \sum_{\nu=0}^{i-\mu} \frac{v_\nu (2i+2-\mu-\nu)(-1)^{i-\mu-\nu}}{\nu!(i+1-\nu)(i-\mu-\nu)!}$$

$$(37) \qquad = \sum_{\mu=0}^{n} \frac{u_\mu}{\mu!} \sum_{i=0}^{n-\mu} \frac{(i+\mu+1)!}{i+1} \sum_{\nu=0}^{i} \frac{v_\nu (2i+2+\mu-\nu)(-1)^{i-\nu}}{\nu!(i+1+\mu-\nu)(i-\nu)!}$$

$$= \sum_{\mu=0}^{n} \frac{u_\mu}{\mu!} \sum_{\nu=0}^{n-\mu} \frac{v_\nu}{\nu!} \sum_{i=\nu}^{n-\nu} \frac{(i+\mu+1)!\,(2i+2+\mu-\nu)(-1)^{i-\nu}}{(i+1)(i+1+\mu-\nu)(i-\nu)!}$$

$$= \sum_{\mu=0}^{n} \frac{u_\mu}{\mu!} \sum_{\nu=0}^{n-\mu} \frac{v_\nu}{\nu!} \sum_{i=0}^{n-\mu-\nu} \frac{(i+\mu+\nu+1)!\,(2i+2+\mu+\nu)(-1)^{i}}{(i+\nu+1)(i+\mu+1)\,i!}.$$

But

$$\sum_{i=0}^{n-\mu-\nu} \frac{(i+\mu+\nu+1)!\,(2i+2+\mu+\nu)\,(-1)^{i}}{(i+\nu+1)(i+\mu+1)\,i!}$$

(38)
$$= (n+2)!\,\mu! \sum_{i=0}^{n-\mu-\nu} \frac{(-1)^i}{(n-\nu-i+1)!\,(i+\nu+1)\;i!}\;,$$

as we now show. It is easily seen to be true for $n=0$, and $n=\mu+\nu$. [23]) Using mathematical induction we need

[23]) It is necessary to observe this since the assumption that it holds for n, for all μ, ν such that $\mu+\nu \leqq n$, will not enable us to include in our induction the case $\mu+\nu = n+1$.

to show that

$$\frac{(n+2)!\,(2n-\mu-\nu+4)(-1)^{n-\mu-\nu+1}}{(n-\mu+2)(n-\nu+2)(n-\mu-\nu+1)!}$$

$$= (n+3)!\,\mu! \sum_{i=0}^{n+1-\mu-\nu} \frac{(-1)^i}{(n-\nu-i+2)!\,(i+\nu+1)\,i!}$$

$$- (n+2)!\,\mu! \sum_{i=0}^{n-\mu-\nu} \frac{(-1)^i}{(n-\nu-i+1)!\,(i+\nu+1)\,i!}$$

$$= \frac{(n+3)!\,\mu!\,(-1)^{n-\mu-\nu+1}}{(\mu+1)!\,(n-\mu+2)\,(n-\mu-\nu+1)!}$$

$$+ (n+2)!\,\mu! \sum_{i=0}^{n-\mu-\nu} \frac{(-1)^i}{(i+\nu+1)\,i!}\left[\frac{n+3}{(n-\nu-i+2)!} - \frac{1}{(n-\nu-i+1)!}\right],$$

or

$$\frac{(n+2)!\,(-1)^{n-\mu-\nu+1}}{(n-\mu+2)\,(n-\mu-\nu+1)!}\left[\frac{2n-\mu-\nu+4}{n-\nu+2} - \frac{n+3}{\mu+1}\right]$$

$$= (n+2)!\,\mu! \sum_{i=0}^{n-\mu-\nu} \frac{(-1)^i}{(i+\nu+1)\,i!}\left[\frac{n+3}{(n-\nu-i+2)!} - \frac{1}{(n-\nu-i+1)!}\right],$$

or

$$\frac{(n+2)!\,(-1)^{n-\mu-\nu}}{(n-\mu-\nu)!\,(\mu+1)(n-\nu+2)} = (n+2)!\,\mu! \sum_{i=0}^{n-\mu-\nu} \frac{(-1)^i}{(n-\nu-i+2)!\,i!},$$

or

$$\sum_{i=0}^{n-\mu-\nu} a^{(0)}_{n-\nu-i+2}\, a^{(-2)}_i = a^{(-2)}_{n-\mu-\nu}\, a^{(0)}_{\mu+1}\, \frac{1}{n-\nu+2}.$$

That this is true follows from the formula

$$(39) \quad \sum_{i=0}^{n} a^{(-2)}_i a^{(0)}_{n+m+1-i} = a^{(-2)}_n a^{(0)}_m \frac{1}{n+m+1} \qquad (m \geqq 0),$$

which we now prove. We have

$$\sum_{i=0}^{n} a_i^{(-2)} a_{n+m+1-i}^{(0)}$$

$$= \sum_{i=0}^{n+m+1} a_i^{(-2)} a_{n+m+1-i}^{(0)} - \sum_{i=n+1}^{n+m+1} a_i^{(-2)} a_{n+m+1-i}^{(0)}$$

$$= a_{n+m+1}^{(-1)} - \sum_{i=n}^{n+m} a_{i+1}^{(-2)} a_{n+m-i}^{(0)}$$

$$= a_{n+m+1}^{(-1)} + (-1)^{n+m} \sum_{i=n}^{n+m} a_{i+1}^{(0)} a_{n+m-i}^{(-2)}$$

$$= 0 + \frac{a_n^{(-2)} a_m^{(0)}}{n+m+1},$$

the second step coming from (8) and the last step from (6) and (31).

Now using (38) we see that (37) becomes

$$W_m = (m+2)! \sum_{\mu=0}^{m} u_\mu \sum_{\nu=0}^{m-\mu} \frac{v_\nu}{\nu!} \sum_{i=0}^{m-\mu-\nu} \frac{a_i^{(-2)}}{(m-\nu-i+1)!\,(i+\nu+1)}$$

$$= (m+2)! \sum_{\mu=0}^{m} u_\mu \sum_{\nu=\mu}^{m} v_{\nu-\mu} a_{\nu-\mu}^{(0)} \sum_{i=\nu}^{m} \frac{a_{i-\nu}^{(-2)}}{(m+\mu-i+1)!\,(i-\mu+1)}$$

$$= (m+2)! \sum_{\mu=0}^{m} u_\mu \sum_{i=\mu}^{m} \frac{1}{(m+\mu-i+1)!\,(i-\mu+1)} \sum_{\nu=\mu}^{i} v_{\nu-\mu} a_{\nu-\mu}^{(0)} a_{i-\nu}^{(-2)}$$

$$= (m+2)! \sum_{\mu=0}^{m} u_\mu \sum_{i=\mu}^{m} \frac{1}{(m+\mu-i+1)!\,(i-\mu+1)} \sum_{\nu=0}^{i-\mu} v_\nu a_\nu^{(0)} a_{i-\mu-\nu}^{(-2)}$$

$$= (n+2)! \sum_{\mu=0}^{n} u_\mu \sum_{i=\mu}^{n} \frac{1}{(i+1)!\,(n-i+1)} \sum_{\nu=0}^{n-i} v_\nu\, a_\nu^{(0)}\, a_{n-i-\nu}^{(-2)}$$

$$= (n+2)! \sum_{i=0}^{n} \frac{1}{(i+1)!} \sum_{\mu=0}^{i} u_\mu \frac{(-1)^{n-i}}{n-i+1} \sum_{\nu=0}^{n-i} v_\nu\, a_\nu^{(-2)}\, a_{n-i-\nu}^{(0)},$$

and from (30) and (32) we have

(40) $$W_n = (n+2)! \sum_{i=0}^{n} u_i^{(0)}\, v_{n-i}^{(-1)}.$$

From (36) and (40) we have

(41) $$\overline{w}_n^{(0)} = \frac{W_{n-1}}{(n+1)!} = \sum_{i=0}^{n-1} u_i^{(0)}\, v_{n-1-i}^{(-1)} \qquad (\overline{w}_0^{(0)} = 0).$$

So

$$\overline{w}_n^{(\lambda+s+1)} = \sum_{i=0}^{n} \overline{w}_i^{(0)}\, a_{n-i}^{(\lambda+s)}$$

$$= \sum_{i=1}^{n} a_{n-i}^{(\lambda+s)} \sum_{j=0}^{i-1} u_j^{(0)}\, v_{i-1-j}^{(-1)}$$

$$= \sum_{i=1}^{n} a_{n-i}^{(\lambda+s)} \sum_{j=1}^{i} u_{j-1}^{(0)}\, v_{i-j}^{(-1)}$$

$$= \sum_{j=1}^{n} u_{j-1}^{(0)} \sum_{i=j}^{n} a_{n-i}^{(\lambda+s)}\, v_{i-j}^{(-1)}$$

$$= \sum_{j=1}^{n} u_{j-1}^{(0)} v_{n-j}^{(r+s)}$$

$$= \sum_{i=0}^{n-1} u_i^{(r)} v_{n-1-i}^{(s)},$$

the first step following from (30), the second from (41), the fifth from (9), and the last from (34). Thus

$$\overline{W}_{n+1}^{(r+s+1)} = \frac{\sum_{i=0}^{n} u_i^{(r)} v_{n-i}^{(s)}}{a_{n+2}^{(r+s+1)}}.$$

The following theorem [24]) is a special case of Theorem X proved in the next chapter.

<u>Theorem</u> VIII. <u>If</u>

$$\frac{u_n^{(r)}}{a_{n+1}^{(r)}} \to U \quad \underline{\text{and}} \quad \frac{v_n^{(s)}}{a_{n+1}^{(s)}} \to V,$$

<u>where</u> r <u>and</u> s <u>are either both</u> > -1 <u>or both</u> < -1, <u>then</u>

$$\frac{\sum_{i=0}^{n} u_i^{(r)} v_{n-i}^{(s)}}{a_{n+2}^{(r+s+1)}} \to UV.$$

[24]) This is analogous to the lemma used in proving the Cesàro theorem on the Cauchy product. (See footnote [1]), p. 490.)

Theorem VII follows from this immediately.

We saw that while the summability of the series $0 + w_0 + w_1 + \cdots$ implies that of $w_0 + w_1 + \cdots$, the converse is not true when $r + s + 1 < -\frac{1}{2}$. So when r and s are < -1 a little more can be said than is said by this product theorem. This is taken care of in the more general product theorem of the next chapter.

Chapter IV

Generalization of the foregoing method

12. Sannia's generalization of the Borel method. It is possible to destroy the Borel summability of a series by prefixing a term to it. Sannia [25]) conceived the idea of

[25]) "Nuovo Metodo Di Sommazione Delle Serie: Estensione Del Metodo Di Borel", Rendiconti Di Palermo, XLII, 1917, p. 303.

defining the summabilities of $\cdots, \sum_{i=1}^{\infty} u_i, \sum_{i=0}^{\infty} u_i,$ $0 + \sum_{i=0}^{\infty} u_i, 0+0+\sum_{i=0}^{\infty} u_i, \cdots,$ respectively to be different orders of Borel summability. This enabled him to prove a Cauchy product theorem for Borel summability precisely analogous to the Cesàro theorem. We follow Sannia's example and extend each B_r transformation to different orders, although as seen from Theorem V, we get orders of different power only when $r \leq -\frac{1}{2}$. This enables us to prove a more general product theorem than that of the last chapter.

13. Definitions and formulas. We define (for k any integer)

$$0 = a_{-1}^{(r)} = a_{-2}^{(r)} = \cdots,$$

$$U_m = \sum_{i=0}^{m} u_i,$$

$$u_m^{(k,0)} = U_m\, a_{m+k}^{(0)},$$

$$u_n^{(k,r)} = \sum_{i=0}^{n} u_i^{(k,0)} \quad a_{n-i}^{(r-1)} \quad , ^{26)}$$

$$U_n^{(k,r)} = \frac{u_n^{(k,r)}}{a_{n+k}^{(r)}} \quad \left(\begin{matrix} r \neq -1 \\ n+k \geqq 0 \end{matrix} \right).$$

26) Note that $u_n^{(k,r)} = 0 \ (n+k<0)$.

Then if $\lim_{n \to \infty} U_n^{(k,r)} = U$, we say $\sum_{i=0}^{n} u_i$ is summable $E_r^{(k)}$ to the sum U. [27]

27) Note that $E_r^{(1)}$ is the same as E_r in the preceding, and that $E_r^{(0)}$ is the method mentioned in footnote [5].

We now prove the formula:

$$(43) \qquad u_n^{(k,r+s+1)} = \sum_{i=0}^{n} u_i^{(k,r)} a_{n-i}^{(s)} .$$

We have

$$\sum_{i=0}^{n} u_i^{(k,r)} a_{n-i}^{(s)} = \sum_{i=0}^{n} a_{n-i}^{(s)} \sum_{j=0}^{i} u_j^{(k,0)} a_{i-j}^{(r-1)}$$

$$= \sum_{j=0}^{n} u_j^{(k,0)} \sum_{i=j}^{n} a_{i-j}^{(r-1)} a_{n-i}^{(s)}$$

$$= \sum_{j=0}^{n} u_j^{(k,0)} a_{n-j}^{(\lambda+s)} = u_n^{(k,\lambda+s+1)}.$$

We now derive a formula for $u_n^{(k,-1)}$ in terms of the u's. We have

$$u_n^{(k,-1)} = \sum_{i=0}^{n} u_i^{(k,0)} a_{n-i}^{(-2)} = \sum_{i=0}^{n} U_i a_{i+k}^{(0)} a_{n-i}^{(-2)}$$

$$= \sum_{i=0}^{n} a_{i+k}^{(0)} a_{n-i}^{(-2)} \sum_{j=0}^{i} u_j$$

$$= \sum_{j=0}^{n} u_j \sum_{i=j}^{n} a_{i+k}^{(0)} a_{n-i}^{(-2)}.$$

But

$$(44) \quad \sum_{i=j}^{n} a_{i+k}^{(0)} a_{n-i}^{(-2)} = \begin{cases} 0 & (k < -n) \\ 1 & (k = -n) \\ \dfrac{a_{j+k-1}^{(0)} a_{n-j}^{(-2)}}{n+k} & (k > -n), \end{cases}$$

as seen from (42) and (31). So

$$(45)\quad \begin{cases} u_n^{(k,-1)} = 0 & (k < -n) \\ u_n^{(-n,-1)} = U_n & \\ u_n^{(k,-1)} = \dfrac{1}{n+k}\displaystyle\sum_{j=0}^{n} u_j a_{j+k-1}^{(0)} a_{n-j}^{(-2)} & (k > -n). \end{cases}$$

Also, we have

$$(46)\quad \begin{aligned} \sum_{i=0}^{n} u_i^{(k,r)} v_{n-i}^{(l,s)} &= \sum_{i=0}^{n} v_{n-i}^{(l,s)} \sum_{j=0}^{i} u_j^{(k,r+p)} a_{i-j}^{(-p-1)} \\ &= \sum_{j=0}^{n} u_j^{(k,r+p)} \sum_{i=j}^{n} a_{i-j}^{(-p-1)} v_{n-i}^{(l,s)} \\ &= \sum_{j=0}^{n} u_j^{(k,r+p)} v_{n-j}^{(l,s-p)}. \end{aligned}$$

We now prove

<u>Theorem</u> IX. <u>If a series is summable</u> $E_r^{(k)}$ <u>then it is summable</u> $E_r^{(k-1)}$ <u>but the converse statement is true only if</u> $r > -\frac{1}{2}$.[28])

We write

$$\bar{u}_0 = 0, \quad \bar{u}_n = u_{n-1} \quad (n > 0).$$

[28]) The proof of this theorem brings out the similarity of the method ($r \leq -\frac{1}{2}$) with Sannia's generalization of the Borel Method. See [25]).

Then

$$\bar{u}_n^{(k-1,r)} = \sum_{i=0}^{n} \bar{u}_i^{(k-1,0)} a_{n-i}^{(r-1)} = \sum_{i=1}^{n} u_{i-1}^{(k,0)} a_{n-i}^{(r-1)}$$

$$= \sum_{i=0}^{m-1} u_i^{(k,0)} \; a_{m-1-i}^{(\lambda-1)} \; = \; u_{m-1}^{(k,\lambda)} .$$

So

$$(47) \qquad \bar{U}_m^{(k-1,\lambda)} = \frac{\bar{u}_m^{(k-1,\lambda)}}{a_{m+k-1}^{(\lambda)}} = \frac{u_{m-1}^{(k,\lambda)}}{a_{m-1+k}^{(\lambda)}} = U_{m-1}^{(k,\lambda)} .$$

We suppose first that $k \geqq 2$. (47) says that the $E_\lambda^{(k-1)}$ transformation for the series $0 + u_0 + u_1 + \cdots$ is equal to the $E_\lambda^{(k)}$ transformation for the series $u_0 + u_1 + u_2 + \cdots$. By iteration of this process we see that the $E_\lambda^{(k)}$ transformation for the series $u_0 + u_1 + u_2 + \cdots$ is the E_λ (or $E_\lambda^{(1)}$) transformation for the series

$$0 + 0 + \cdot \quad \cdot \quad \cdot + 0 + u_0 + u_1 + \cdot \quad \cdot \quad \cdot ,$$

where there are $k-1$ zeros. Then the $E_\lambda^{(k-1)}$ transformation for the series $u_0 + u_1 + u_2 + \cdots$ is the E_λ transformation for

$$0 + \cdot \quad \cdot \quad \cdot + 0 + u_0 + u_1 + \cdot \quad \cdot \quad \cdot ,$$

where there are $k-2$ zeros. So when $k \geqq 2$ the above theorem follows by virtue of Theorem V.

Now suppose $k \leqq 0$. We write

$$\tilde{u}_m = u_{m+1} .$$

Then

$$u_m^{(k,\lambda)} = \sum_{i=0}^{m} u_i^{(k,0)} \; a_{m-i}^{(\lambda-1)}$$

$$= u_0^{(k,0)} \, a_m^{(\lambda-1)} + \sum_{i=1}^{m} (\tilde{u}_{i-1}^{(k+1,0)} + u_0 a_{i+k}^{(0)}) a_{m-i}^{(\lambda-1)}$$

$$= u_0 \sum_{i=0}^{m} a_{i+k}^{(0)} a_{m-i}^{(\lambda-1)} + \sum_{i=1}^{m} \tilde{u}_{i-1}^{(k+1,0)} a_{m-i}^{(\lambda-1)}$$

$$= u_0 a_{m+k}^{(\lambda)} + \tilde{u}_{m-1}^{(k+1,\lambda)} .$$

So

$$(48) \qquad U_m^{(k,\lambda)} = \frac{u_m^{(k,\lambda)}}{a_{m+k}^{(\lambda)}} = u_0 + \tilde{U}_{m-1}^{(k+1,\lambda)} .$$

Therefore the $E_\lambda^{(k)}$ transformation for $u_0 + u_1 + u_2 + \cdots$ is equal to u_0 plus the $E_\lambda^{(k+1)}$ transformation for $u_1 + u_2 + u_3 + \cdots$. By iteration of this process we see that the $E_\lambda^{(k)}$ transformation for

$$u_0 + u_1 + u_2 + \cdots$$

is equal to $u_0 + u_1 + \cdots + u_{-k}$ plus the E_λ transformation for the series

$$u_{1-k} + u_{2-k} + u_{3-k} + \cdots .$$

Also the $E_\lambda^{(k-1)}$ transformation for $u_0 + u_1 + u_2 + \cdots$ is equal to $u_0 + u_1 + \cdots + u_{-k} + u_{1-k}$ plus the E_λ transformation for the series

$$u_{2-k} + u_{3-k} + \cdots .$$

So when $k \leqq 0$ our theorem follows.

Now suppose $k = 1$. From (48) we have that the $E_\lambda^{(0)}$ transformation for $u_0 + u_1 + u_2 + \cdots$ is equal to u_0 plus the E_λ transformation for $u_1 + u_2 + u_3 + \cdots$. But the summability E_λ of $u_0 + u_1 + u_2 + \cdots$ implies that of $u_1 + u_2 + u_3 + \cdots$, and therefore the summability $E_\lambda^{(0)}$ of $u_0 + u_1 + u_2 + \cdots$, which completely establishes the theorem.

We now prove

<u>Theorem X.</u> <u>If</u>

$$\frac{u_m^{(k,r)}}{a_{m+k}^{(r)}} \to U \quad \underline{\text{and}} \quad \frac{v_m^{(\ell,s)}}{a_{m+\ell}^{(s)}} \to V,$$

<u>where</u> r <u>and</u> s <u>are either both</u> >-1 <u>or both</u> <-1, <u>and</u> k <u>and</u> ℓ <u>are any integers, then</u>

$$\frac{\sum_{i=0}^{m} u_i^{(k,r)} \; v_{m-i}^{(\ell,s)}}{a_{m+k+\ell}^{(r+s+1)}} \longrightarrow UV.$$

We write

$$u_m^{(k,r)} = (U+\delta_m)a_{m+k}^{(r)}, \quad v_m^{(\ell,s)} = (V+\vartheta_m)a_{m+\ell}^{(s)},$$

where $\delta_m \to 0$ and $\vartheta_m \to 0$. Then

$$\begin{aligned}
\sum_{i=0}^{m} u_i^{(k,r)} v_{m-i}^{(\ell,s)} &= \sum_{i=0}^{m}(UV + V\delta_i \\
&\qquad + U\vartheta_{m-i} + \delta_i\vartheta_{m-i})a_{i+k}^{(r)}\, a_{m-i+\ell}^{(s)} \\
&= UV\left(\sum_{i=0}^{m+k+\ell} - \sum_{i=m+k+1}^{m+k+\ell} - \sum_{i=0}^{k-1}\right) a_i^{(r)} a_{m+k+\ell-i}^{(s)} \\
&\quad + V\left(\sum_{i=0}^{m+k+\ell} - \sum_{i=m+k+1}^{m+k+\ell}\right)\delta_{i-k}\, a_i^{(r)} a_{m+k+\ell-i}^{(s)} \\
&\quad + U\left(\sum_{i=0}^{m+k+\ell} - \sum_{i=m+\ell+1}^{m+k+\ell}\right)\vartheta_{i-\ell}\, a_i^{(s)} a_{m+k+\ell-i}^{(r)} \\
&\quad + \sum_{i=0}^{m}\delta_i\,\vartheta_{m-i}\, a_{i+k}^{(r)}\, a_{m-i+\ell}^{(s)} \left(\begin{array}{l}\sum_{i=\mu}^{\nu} = 0 \text{ if } \nu<\mu, \\ 0 = \delta_{-1} = \delta_{-2} = \cdots, \\ 0 = \vartheta_{-1} = \vartheta_{-2} = \cdots \end{array}\right).
\end{aligned} \tag{49}$$

Consequently

$$\frac{\sum_{i=0}^{n} u_i^{(k,r)} v_{n-i}^{(\ell,s)}}{a_{n+k+\ell}^{(r+s+1)}} = UV\left[1 - \sum_{i=0}^{\ell-1}\left(\frac{s+1}{r+1}\right)^{i}\binom{n+k+\ell}{i}\left(\frac{r+1}{r+s+2}\right)^{n+k+\ell} - \sum_{i=0}^{k-1}\left(\frac{r+1}{s+1}\right)^{i}\binom{n+k+\ell}{i}\left(\frac{s+1}{r+s+2}\right)^{n+k+\ell}\right]$$

$$+ V\left[\frac{\sum_{i=0}^{n+k+\ell}\delta_{i-k}\, a_i^{(r)} a_{n+k+\ell-i}^{(s)}}{a_{n+k+\ell}^{(r+s+1)}} - \sum_{i=0}^{\ell-1}\left(\frac{s+1}{r+1}\right)^{i}\binom{n+k+\ell}{i}\left(\frac{r+1}{r+s+2}\right)^{n+k+\ell}\delta_{n+\ell-i}\right]$$

$$+ U\left[\frac{\sum_{i=0}^{n+k+\ell}\theta_{i-\ell}\, a_i^{(s)} a_{n+k+\ell-i}^{(r)}}{a_{n+k+\ell}^{(r+s+1)}} - \sum_{i=0}^{k-1}\left(\frac{r+1}{s+1}\right)^{i}\binom{n+k+\ell}{i}\left(\frac{s+1}{r+s+2}\right)^{n+k+\ell}\theta_{n+k-i}\right]$$

$$+ \frac{\sum_{i=0}^{n}\delta_i\,\theta_{n-i}\, a_{i+k}^{(r)}\, a_{n-i+\ell}^{(s)}}{a_{n+k+\ell}^{(r+s+1)}}.$$

So it is apparent that the limit of the left side of (49) is equal to

$$UV + \lim_{n\to\infty}\left[V\,\frac{\sum_{i=0}^{n+k+\ell}\delta_{i-k}\, a_i^{(r)} a_{n+k+\ell-i}^{(s)}}{a_{n+k+\ell}^{(r+s+1)}}\right.$$

$$\text{(50)}\qquad \left. U\,\frac{\sum_{i=0}^{n+k+\ell}\theta_{i-\ell}\, a_i^{(s)} a_{n+k+\ell-i}^{(r)}}{a_{n+k+\ell}^{(r+s+1)}} + \frac{\sum_{i=0}^{n}\delta_i\,\theta_{n-i}\, a_{i+k}^{(r)}\, a_{n-i+\ell}^{(s)}}{a_{n+k+\ell}^{(r+s+1)}}\right].$$

The first term in the brackets is equal to

$$V\left[\delta_{-k}\left(\frac{s+1}{r+s+2}\right)^{n+k+\ell} + \frac{\sum_{i=0}^{n+k+\ell-1}\delta_{i-k+1}\, a_{i+1}^{(r)}\, a_{n+k+\ell-i}^{(s)}}{a_{n+k+\ell}^{(r+s+1)}}\right],$$

which approaches zero as seen from (12) and (13). Similarly the second term in brackets in (50) approaches zero.

We now discuss the third term. We choose K so that

$$|\delta_n| < K > |\vartheta_n| \qquad \text{(all n).}$$

Then given any positive ϵ we choose N so that

$$|\delta_n| < \frac{\epsilon}{K} > |\vartheta_n| \qquad (n > \tfrac{N}{2}),$$

and we have

$$\text{(51)} \qquad \left| \frac{\sum_{i=0}^{N} \delta_i \vartheta_{N-i}\, a_{i+k}^{(r)} a_{N-i+\ell}^{(s)}}{a_{N+k+\ell}^{(r+s+1)}} \right| < K \frac{\epsilon}{K} \left| \frac{\sum_{i=0}^{N} a_{i+k}^{(r)} a_{N-i+\ell}^{(s)}}{a_{N+k+\ell}^{(r+s+1)}} \right|.$$

But the discussion of the coefficient of UV in (49) shows that the limit of the right side of (51) as $N \to \infty$ is ϵ. This completes the proof of the theorem.

14. <u>More general definition of the product of two infinite series.</u> <u>We define the</u> (k,ℓ) <u>product of</u> $u_0 + u_1 + u_2 + \cdots$ <u>and</u> $v_0 + v_1 + v_2 + \cdots$ <u>to be</u> $w_0(k,\ell) + w_1(k,\ell) + w_2(k,\ell) \cdots$, <u>where</u> $w_n(k,\ell)$ <u>is defined as follows:</u>

(1) <u>When</u> $n < -k-\ell$, $w_n(k,\ell) = 0$.

(2) <u>When</u> $n = -k-\ell$, <u>the coefficient of</u> $u_\mu v_\nu$ <u>in</u> $w_n(k,\ell)$ <u>is</u> 1 <u>provided either</u> $\nu = -\ell$ <u>or</u> $\mu = -k$, <u>or provided both</u> $\nu < -\ell$ <u>and</u> $\mu < -k$,

and is otherwise 0.

(3) When $n > -k-\ell$, the coefficient of $u_\mu v_\nu$ is

(52) (a) 1 when both $\nu \leqq -\ell$ and $\mu = n+\ell$.

(b) 1 when both $\mu \leqq -k$ and $\nu = n+k$.

(c) $\dfrac{(n+k+\ell)!}{(\mu+k)!\,(\nu+\ell)!}$ when both $\begin{cases}\mu > -k\\ \nu > -\ell\end{cases}$ and $\mu+\nu = n$.

(d) $\dfrac{(n+k+\ell-1)!\,(2n-\mu-\nu+k+\ell)(-1)^{n-\mu-\nu}}{(n-\mu+\ell)(n-\nu+k)(\mu+k-1)!\,(\nu+\ell-1)!\,(n-\mu-\nu)!}$ when

both $\begin{cases}\mu > -k\\ \nu > -\ell\end{cases}$ and $\mu+\nu < n$.

(e) 0 when otherwise.

We now prove the more general product theorem:

Theorem XI. If $\sum_{i=0}^{\infty} u_i$ is summable $E_r^{(k)}$ to U and $\sum_{i=0}^{\infty} v_i$ is summable $E_s^{(\ell)}$ to V, then $\sum_{i=0}^{\infty} w_i(k,\ell)$, where $w_n(k,\ell)$ is defined by (52), is summable $E_{r+s+1}^{(k+\ell)}$ to UV provided r and s are either both > -1 or both < -1, and where k and ℓ are integers.

We show that

$$W_n(k,\ell) = (n+k+\ell)! \sum_{i=0}^{n} u_i^{(k,0)} v_{n-i}^{(\ell,-1)} \quad (n \geqq -k-\ell), \tag{53}$$

where $u_n^{(k,0)}$ and $v_n^{(\ell,-1)}$ are given by (42) and (45) respectively. To do this we write

$$X_n = (n+k+\ell)! \sum_{i=0}^{n} u_i^{(k,0)} v_{n-i}^{(\ell,-1)} \qquad (n \geqq -k-\ell),$$
(54)
$$X_n = 0 \qquad (n < -k-\ell),$$

and show that $X_n - X_{n-1}$ is precisely the $w_n(k,\ell)$ defined in (52). The coefficient of $u_\mu v_\nu$ in X_n $(n \geqq -k-\ell)$ is the coefficient of $u_\mu v_\nu$ in

(55) $$(n+k+\ell)! \sum_{i=\mu}^{n-\nu} u_i^{(k,0)} v_{n-i}^{(\ell,-1)} \qquad (\mu+\nu \leqq n).$$

We have only the five possibilities 1) $\nu < -\ell$, 2) $\mu < -k$, 3) $\nu = -\ell$, 4) $\mu = -k$, 5) both $\mu > -k$ and $\nu > -\ell$. These are not mutually exclusive but we take these cases for the sake of symmetry.

1) Suppose $\nu < -\ell$. Then $n-\nu > n+\ell$. The coefficient of v_ν in $v_{n-i}^{(\ell,-1)}$ according to (45) is

$$0 \text{ if } i > n+\ell,$$
(56) $$1 \text{ if } i = n+\ell,$$
$$0 \text{ if } i < n+\ell.$$

The latter statement follows since $a_{\nu+\ell-1}^{(0)} = 0$, which in turn follows from $\nu < -\ell$. The coefficient of u_μ in $u_i^{(k,0)}$ according to (42) is

(57) $$a_{i+k}^{(0)}.$$

Then combining (56) and (57) we see that <u>the coefficient of</u> $u_\mu v_\nu$ <u>in</u> X_n <u>is</u> 0 <u>or</u> 1 <u>according as</u> $\mu > n+\ell$ <u>or</u> $\mu \leqq n+\ell$.

2) Suppose $\mu < -k$. We have $u_i^{(k,0)} = 0$ when $i < -k$, so the expression (55) is zero when $\nu > n+k$. Suppose $\nu \leqq n+k$. The coefficient of v_ν in $v_{m-i}^{(\ell,-1)}$ is

$$\frac{a_{\nu+\ell-1}^{(0)} a_{m-\nu-i}^{(-2)}}{m-i+\ell} \qquad (i \neq n+\ell)$$

1 or 0 according as $\nu \leqq -\ell$ or $\nu > -\ell$ $(i = n+\ell)$.

So if $\nu \leqq -\ell$ the coefficient of $u_\mu v_\nu$ in X_m is 1, and if $\nu > -\ell$ it is

$$(58) \qquad (n+k+\ell)! \sum_{i=-k}^{m-\nu} a_{i+k}^{(0)} \frac{a_{\nu+\ell-1}^{(0)} a_{m-\nu-i}^{(-2)}}{n-i+\ell},$$

and this is equal to 1 as seen from the formula

$$\sum_{i=0}^{m} \frac{a_i^{(0)} a_{m-i}^{(-2)}}{n+m+1-i} = \frac{a_{n+m+1}^{(0)}}{a_m^{(0)}} \qquad (m \geqq 0),$$

which we now prove. It is easily seen to be true for $n = 0,1,2$. Using mathematical induction, we need only show that

$$\frac{1}{n+m+2} \sum_{i=0}^{m} \frac{a_i^{(0)} a_{m-i}^{(-2)}}{n+m+1-i} = \sum_{i=0}^{m+1} \frac{a_i^{(0)} a_{m+1-i}^{(-2)}}{n+m+2-i}$$

$$= \frac{a_{m+1}^{(-2)}}{n+m+2} + \sum_{i=0}^{m} \frac{a_{i+1}^{(0)} a_{m-i}^{(-2)}}{n+m+1-i},$$

or

$$\sum_{i=0}^{m} \frac{a_{m-i}^{(-2)} a_i^{(0)}}{n+m+1-i} \left[\frac{1}{n+m+2} - \frac{1}{i+1} \right] = \frac{a_{m+1}^{(-2)}}{n+m+2},$$

or

$$-\frac{1}{n+m+2}\sum_{i=0}^{m} a_{m-i}^{(-2)} a_{i+1}^{(0)} = \frac{a_{m+1}^{(-2)}}{n+m+2},$$

or

$$\sum_{i=0}^{m+1} a_{m+1-i}^{(-2)} a_{i}^{(0)} = 0,$$

which is obviously the case. So <u>the coefficient of</u> $u_\mu v_\nu$ <u>in</u> X_m <u>is</u> 0 <u>or</u> 1 <u>according as</u> $\nu > n+k$ <u>or</u> $\nu \leqq n+k$.

3) Suppose $\nu = -\ell$. Since $v_{m-i}^{(\ell,-1)} = 0$ when $i < n+\ell$, <u>the coefficient of</u> $u_\mu v_\nu$ <u>in</u> X_m <u>is</u> 1.

4) Suppose $\mu = -k$. The case $\nu \leqq -\ell$ is taken care of by cases 1) and 3). When $\nu > -\ell$ we can apply (58). So <u>the coefficient of</u> $u_\mu v_\nu$ <u>in</u> X_m <u>is</u> 1.

5) Suppose both $\mu > -k$ and $\nu > -\ell$. Then <u>the coefficient of</u> $u_\mu v_\nu$ <u>in</u> X_m <u>is</u>

$$(n+k+\ell)!\sum_{i=\mu}^{n-\nu} a_{i+k}^{(0)} \frac{a_{\nu+\ell-1}^{(0)} a_{m-\nu-i}^{(-2)}}{n-i+\ell}$$

We now summarize the preceding results in the form of a table.

Case		Coefficient of $u_\mu v_\nu$ in X_m
1) $\nu < -\ell$	If $\mu > n+\ell$	0
	If $\mu \leqq n+\ell$	1
2) $\mu < -k$	If $\nu > n+k$	0
	If $\nu \leqq n+k$	1
3) $\nu = -\ell$		1
4) $\mu = -k$		1
5) $\mu > -k$ and $\nu > -\ell$		$(n+k+\ell)!\sum_{i=\mu}^{m-\nu} a_{i+k}^{(0)} \frac{a_{\nu+\ell-1}^{(0)} a_{m-\nu-i}^{(-2)}}{m-i+\ell}$

We now find the coefficient of $u_\mu v_\nu$ in $X_n - X_{n-1}$ and show that our results coincide with the definition of $w_n(k,\ell)$ in (52). When $n-1 \geqq -k-\ell$ the coefficient of $u_\mu v_\nu$ in both X_n and X_{n-1} may be obtained readily from the above table. When $n = -k-\ell$ we have $X_n - X_{n-1} = X_n$. The following table is readily verified excepting the second part of case 5), which is worked out in detail below.

Case		Coefficient of $u_\mu v_\nu$ in $X_n - X_{n-1}$ when $n+k+\ell \geqq 1$
1) $\nu < -\ell$	If $\mu = n+\ell$	1
	Otherwise	0
2) $\mu < -k$	If $\nu = n+k$	1
	Otherwise	0
3) $\nu = -\ell$	If $\mu = n+\ell$	1
	Otherwise	0
4) $\mu = -k$	If $\nu = n+k$	1
	Otherwise	0
5) $\mu > -k$ and $\nu > -\ell$	If $\mu+\nu = n$	$(n+k+\ell)!\, a^{(0)}_{\mu+k}\, a^{(0)}_{\nu+\ell}$
	If $\mu+\nu < n$	$\dfrac{(n+k+\ell-1)!\, a^{(0)}_{\nu+\ell-1}\, a^{(0)}_{\mu+k-1}\, a^{(-2)}_{n-\mu-\nu}\,(2n-\mu-\nu+k+\ell)}{(n-\mu+\ell)(n-\nu+k)}$

In case 5) when $\mu+\nu < n$ we have

$$(n+k+\ell)!\; a^{(0)}_{\nu+\ell-1} \sum_{i=\mu}^{n-\nu} \frac{a^{(0)}_{i+k}\; a^{(-2)}_{n-\nu-i}}{n-i+\ell}$$

$$-\,(n-1+k+\ell)!\; a^{(0)}_{\nu+\ell-1} \sum_{i=\mu}^{n-1-\nu} \frac{a^{(0)}_{i+k}\; a^{(-2)}_{n-1-\nu-i}}{n-1-i+\ell}$$

$$= \frac{(n+k+\ell)!\; a^{(0)}_{\nu+\ell-1} a^{(0)}_{\mu+k} a^{(-2)}_{m-\mu-\nu}}{n-\mu+\ell}$$

$$+ (n+k+\ell-1)!\; a^{(0)}_{\nu+\ell-1} \sum_{i=\mu+1}^{m-\nu} \frac{a^{(-2)}_{m-\nu-i}}{n-i+\ell} \left[a^{(0)}_{i+k}(n+k+\ell) - a^{(0)}_{i+k-1} \right]$$

$$= \frac{(n+k+\ell)!\; a^{(0)}_{\nu+\ell-1} a^{(0)}_{\mu+k} a^{(-2)}_{m-\mu-\nu}}{n-\mu+\ell} + (n+k+\ell-1)!\, a^{(0)}_{\nu+\ell-1} \sum_{i=\mu+1}^{m-\nu} a^{(0)}_{i+k}\, a^{(-2)}_{m-\nu-i}.$$

But

$$\sum_{i=\mu+1}^{m-\nu} a^{(0)}_{i+k}\; a^{(-2)}_{m-\nu-i} = \sum_{i=\mu+k}^{m+k-\nu-1} a^{(0)}_{i+1}\; a^{(-2)}_{m+k-\nu-1-i}$$

$$= \frac{\mu+k+1}{n+k-\nu}\; a^{(0)}_{\mu+k+1}\, a^{(-2)}_{m-\mu-\nu-1},$$

as is seen from (31). So the coefficient of $u_\mu v_\nu$ in $X_m - X_{m-1}$ is

$$(n+k+\ell-1)!\; a^{(0)}_{\nu+\ell-1}\, a^{(-2)}_{\mu+k}\, a_{m-\mu-\nu} \left[\frac{n+k+\ell}{n-\mu+\ell} - \frac{n-\mu-\nu}{n+k-\nu} \right]$$

$$\frac{(n+k+\ell-1)!\; a^{(0)}_{\nu+\ell-1} a^{(0)}_{\mu+k-1} a^{(-2)}_{m-\mu-\nu} (2n-\mu-\nu+k+\ell)}{(n-\mu+\ell)(n+k-\nu)}.$$

When $n = -k-\ell$, we easily set up the following table.

Case		Coefficient of $u_\mu v_\nu$ in X_m when $n+k+\ell = 0$.
1) $\nu < -\ell$	If $\mu > -k$	0
	If $\mu \leqq -k$	1
2) $\mu < -k$	If $\nu > -\ell$	0
	If $\nu \leqq -\ell$	1
3) $\nu = -\ell$		1
4) $\mu = -k$		1

Here case 5) is ruled out since otherwise we would have $\mu + \nu > -k - \ell = n$.

From the last two tables it is evident that $X_m - X_{m-1}$ is precisely the $w_m(k, \ell)$ of (52), which fact establishes the formula (53). So we have

$$w_m^{(k+\ell,0)}(k,\ell) = W_m(k,\ell)\, a^{(0)}_{m+k+\ell} = \sum_{i=0}^{m} u_i^{(k,0)}\, v_{m-i}^{(\ell,-1)},$$

and consequently

$$\begin{aligned} w_m^{(k+\ell,\, r+s+1)}(k,\ell) &= \sum_{i=0}^{m} w_i^{(k+\ell,0)}(k,\ell)\, a_{m-i}^{(r+s)} \\ &= \sum_{i=0}^{m} a_{m-i}^{(r+s)} \sum_{j=0}^{i} u_j^{(k,0)}\, v_{i-j}^{(\ell,-1)} \\ &= \sum_{j=0}^{m} u_j^{(k,0)} \sum_{i=j}^{m} v_{i-j}^{(\ell,-1)}\, a_{m-i}^{(r+s)} \\ &= \sum_{j=0}^{m} u_j^{(k,0)}\, v_{m-j}^{(\ell,\, r+s)} \\ &= \sum_{j=0}^{m} u_j^{(k,r)}\, v_{m-j}^{(\ell,s)}, \end{aligned}$$

the latter step being due to (46). Thus

$$W_m^{(k+\ell,\, r+s+1)}(k,\ell) \quad \frac{1}{a^{(r+s+1)}_{m+k+\ell}} \sum_{i=0}^{m} u_i^{(k,r)}\, v_{m-i}^{(\ell,s)},$$

and Theorem XI follows by virtue of Theorem X.