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Approved by:

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CONVERGENT FACTOR AND REGULARITY THEOREMS FOR CONVERGENT INTEGRALS

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Convergence Factor and Regularity Theorems for Convergent Integrals

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Introduction

The purpose of this study is the development of relations and theorems for convergent infinite integrals analogous to those developed for infinite series by C. N. Moore [18]*. It was desired to determine conditions necessary and sufficient that the convergence factors be regular, without unduly restricting either the integrals or the convergence factors considered. It is interesting to observe at what points the direct analogy between series and integrals holds, and at what points it breaks down.

Most of the sources drawn upon are listed in the bibliography. The most fruitful sources of inspiration were Professor Moore's book and Professor Agnew's expository article [2]. The present effort realizes but a small portion of the possible extension of these works.

I am grateful to Professor Moore for suggesting this most interesting problem, and for his guidance and help in pursuing the investigation.

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* Numbers in square brackets refer to the corresponding items listed in the bibliography.

CHAPTER I

SINGLE INTEGRALS

1. Statement of the Problems.

In this chapter we consider, first, functions f(t) which are Lebesgue integrable over every finite range (0,x), and for which the integral approaches a definite limit as $x \to \infty$. Under such conditions we shall say that "$f \varepsilon L$". Let

$$(1.1) \qquad \psi(x) = \int_0^x f(t)dt, \qquad x \geq 0, \quad f \varepsilon L,$$

$$(1.2) \qquad \psi = \int_0^\infty f(t)dt,$$

$$(1.3) \qquad |\psi(x)| \leq A < \infty, \qquad \text{all } x \geq 0.$$

It is desired to find "convergence factors" $\phi(\alpha,t)$, defined for $t \geq 0$ and for α in some convenient set $E(\alpha)$, such as to ensure the existence of the integrals

$$(1.4) \qquad \sigma(\alpha,x) = \int_0^x \phi(\alpha,t)f(t)dt, \qquad x \geq 0, \alpha \varepsilon E(\alpha), \ f \varepsilon L,$$

$$(1.5) \qquad \sigma(\alpha) = \int_0^\infty \phi(\alpha,t)f(t)dt.$$

We shall also require the existence of

$$(1.6) \qquad \sigma = \lim_{\alpha \to \alpha_0} \int_0^\infty \phi(\alpha,t)f(t)dt, \qquad f \varepsilon L, \alpha \varepsilon E'(\alpha),$$

Where α_0 is a limit point of $E(\alpha)$ not of the set, and $E'(\alpha)$ is some sub set of $E(\alpha)$ also having α_0 for limit point.

In particular, if $\sigma = \psi$ for all $f \varepsilon L$, we shall say that "ϕ is regular over L", or, briefly, "ϕ is regular". In some applications $E(\alpha)$ will be the set of real numbers and α_0 the number ∞.

It is noted that $\phi(\alpha,t) \equiv 1$ meets all the requirements made on $\phi(\alpha,t)$. Hence it is but natural to consider the existence of integrals (1.4), (1.5), (1.6) with $\phi(\alpha,t)$ replaced by $1-\phi(\alpha,t)$. Incidentally, regularity demands that

$$(1.7) \qquad \lim_{\alpha \to \alpha_0} \int_0^\infty [1-\phi(\alpha,t)]\, f(t)dt = 0, \qquad \alpha \varepsilon E'(\alpha), \quad f \varepsilon L.$$

The other problem considered is the type of convergence factor, $\theta(\alpha,t)$, required to give

$$(1.8) \qquad \lim_{\alpha \to \alpha_0} \int_0^\infty \theta(\alpha,t)\, f(t)dt = \lim_{t \to \infty} f(t),$$

where $f(t)$ is any bounded measurable function for which the right side of (1.8) exists.

2. Existence of the Transform $\sigma(\alpha)$

In order that $\sigma(\alpha,x)$ exist for every finite range $(0,x)$ and every $f \varepsilon L$, it is necessary and sufficient that $\phi(\alpha,t)$ be measurable and essentially bounded in t over $(0,x)$ for each $\alpha \varepsilon E(\alpha)$. The existence of $\sigma(\alpha)$ and σ, together with the desire for regularity of ϕ require more of $\phi(\alpha,t)$. Moreover, the convergence factors commonly used are very well-behaved. We therefore assume that $\phi(\alpha,t)$ is absolutely continuous in t over $(0,\infty)$ for each $\alpha \varepsilon E(\alpha)$, and that it is uniformly bounded; thus,

$$(2.1) \qquad |\phi(\alpha,t)| \leq B < \infty, \qquad t \geq 0, \alpha \varepsilon E(\alpha),$$

The absolute continuity of $\phi(\alpha,t)$ gives the existence almost everywhere of $\phi'_t(\alpha,t)$ for each $\alpha \varepsilon E(\alpha)$. Let us assume that $\phi'_t(\alpha,t)$ is bounded uniformly over every finite range $(0,x)$ for $\alpha \varepsilon E_x(\alpha)$, a subset of $E(\alpha)$, having α_0 as limit point, and that it converges in this subset. In order that $\phi(\alpha_0,t)$, the limit of $\phi(\alpha,t)$ as $\alpha \rightarrow \alpha_0$, be absolutely continuous also*, we assume that $\phi(\alpha,t)$ converges uniformly.

In place of (1.4) we can write

$$(2.2) \qquad \sigma(\alpha,x) = \int_0^x \phi(\alpha,t)f(t)dt = \phi(\alpha,x)\psi(x) - \int_0^x \phi_t'(\alpha,t)\psi(t)dt.$$

In order that $\sigma(\alpha,x)$ converge to a definite limit as $x\rightarrow\infty$, it is necessary and sufficient that the last two expressions in (2.2) converge to limits as $x\rightarrow\infty$. The condition analogous to that used by Moore [18] for series, is the convergence, as $x\rightarrow\infty$, of the integral

$$(2.3) \qquad \int_0^x |\phi_t'(\alpha,t)|\, dt, \qquad\qquad \alpha \varepsilon E(\alpha),$$

to some function of α defined for $\alpha \varepsilon E(\alpha)$.

The convergence of $\int_0^x |\phi_t'(\alpha,t)|\, dt$ implies that of $\int_0^x \phi_t'(\alpha,t)dt$, which in turn gives that of $\phi(\alpha,x)$, as $x\rightarrow\infty$. Since $\psi(x)\rightarrow\psi$, the second last term of (2.2) approaches a limit as $x\rightarrow\infty$. The convergence of (2.3), together with the boundedness of $\psi(t)$, ensure the convergence of the last term of (2.2). Therefore, the convergence of (2.3) is sufficient for the convergence of $\sigma(\alpha,x)$ as $x\rightarrow\infty$

In proving the necessity of this condition, we consider first the case in which $\psi(x)\rightarrow 0$ as $x\rightarrow\infty$. Assume that (2.3) does not converge, as $x\rightarrow\infty$, for some value α_1 of α. Then

$$\int_0^\infty |\phi_t'(\alpha_1,t)|\, dt = \infty .$$

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* See Zygmund [22], p.84.

There exists a number $N_1 < \infty$ such that

$$\int_0^{N_1} |\phi'_t(\alpha_1, t)| \, dt = 2.$$

Since $\phi(\alpha_1, t) \varepsilon AC$, it follows that $\phi'_t(\alpha_1, t) \varepsilon L$, and, also, $|\phi'_t(\alpha_1, t)| \varepsilon L$. Given an arbitrarily small $\epsilon > 0$, there exists an $\eta_1 > 0$, such that for any set $E \subset (0, N_1 + 1)$ of measure $< \eta_1$, we have

$$\int_E |\phi'_t(\alpha_1, t)| \, dt < \frac{\epsilon}{4}.$$

Choose any Y_1, $0 < Y_1 < 1$, such that

$$\int_{N_1}^{N_1+Y_1} |\phi'_t(\alpha_1, t)| \, dt < \frac{\epsilon}{4}.$$

Consider the function*

$$\begin{aligned} g_1(t) &= \operatorname{sgn} \phi'_t(\alpha_1, t), & 0 \leqq t \leqq N_1, \\ &= 0, & t > N_1. \end{aligned}$$

Since $\phi'_t(\alpha_1, t) \varepsilon L(0, N_1 + 1)$, so also is $g_1(t)$.

There exists ** a function $\psi_1(t) \varepsilon AC$, such that $|\psi_1(t) - g_1(t)| < \frac{\epsilon}{8}$ except in a set E_1 of measure $< \eta_1$. If $\psi_1(t)$ is not zero for $t > N_1 + Y_1/2$, make it so by joining the points $(N_1, \psi_1(N_1))$, $(N_1 + Y_1/2, 0)$ by means of a straight line. Since $g_1(t)$ is bounded by ± 1, so also is $\psi_1(t)$, and, moreover,

$$\max |\psi_1(t) - g_1(t)| \leqq 2.$$

Let $E_2 = (0, N_1 + Y_1/2) - E_1$. Then

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* $\operatorname{sgn} z = z/|z|$ if $z \neq 0$, $= 0$ if $z = 0$.

** See Titchmarsh [21] p. 376.

$$\left| \int_0^{N_1+Y_1} \phi_t' (\alpha_1, t)\, \psi_1(t) dt \right| = \left| \int_0^{N_1+Y_1} \phi_t' \{ g_1 + [\psi_1 - g_1] \} \, dt \right|$$

$$\geqq \left| \int_0^{N_1+Y_1} \phi_t' \, g_1 dt \right| - \left| \int_0^{N_1+Y_1} \phi_t' [\psi_1 - g_1] \, dt \right|$$

$$\geqq \int_0^{N_1} |\phi_t'| \, dt - \int_{E_1} |\phi_t'| \cdot |\psi_1 - g_1| \, dt - \int_{E_2} |\phi_t'| \cdot |\psi_1 - g_1| dt$$

$$\geqq 2 - 2 \int_{E_1} |\phi_t'| \, dt - \frac{\epsilon}{8} \int_0^{N_1+Y_1} |\phi_t'| \, dt$$

$$> 2 - \frac{\epsilon}{2} - \frac{\epsilon}{8} \left[2 + \frac{\epsilon}{4} \right] \qquad > 2 - \epsilon :$$

Since ϵ is arbitrary, it is clear that we can find an absolutely continuous function $\psi_1(t)$, as required, such that

$$\int_0^{N_1+Y_1} \phi_t' (\alpha_1, t)\, \psi_1(t) \, dt > 1.$$

Under the conditions supposed, there exists an $N_2 < \infty$, such that

$$\int_0^{N_2} |\phi_t' (\alpha_1, t)| \, dt = 4.$$

Consider, now, the function

$$g_2(t) = (1/2) \operatorname{sgn} \phi_t' (\alpha_1, t), \qquad N_1 + y_1 \leqq t \leqq N_2,$$

$$= 0, \qquad \text{elsewhere.}$$

Defining a $\psi_2(t)$, and a y_2, in fashion similar to that for $\psi_1(t)$, and y_1, we obtain

$$\int_{N_1+y_1}^{N_2+y_2} \phi_t' (\alpha_1, t)\, \psi_2(t) dt > 1/2.$$

Therefore,

$$\int_0^{N_2+y_2} \phi_t' (\alpha_1, t) \{ \psi_1(t) + \psi_2(t) \} \, dt$$

$$= \int_0^{N_1+y_1/2} \phi_t' \cdot \psi_1 dt + \int_{N_1+y_1/2}^{N_2+y_2} \phi_t' \cdot \psi_2 dt > 1 + 1/2.$$

If we continue this process, with

$$g_n(t) = (1/n)\ \mathrm{sgn}\ \phi'_t\,(\alpha_1,t), \qquad N_{n-1} + y_{n-1} \leq t \leq N_n ,$$

$$= 0\ , \qquad \text{elsewhere},$$

we obtain a series of "non-overlapping" functions, each absolutely continuous. If we write

$$\psi(t) = \psi_1(t) + \psi_2(t) + ---,$$

we have an absolutely continuous function which $\rightarrow 0$ as $t \rightarrow \infty$. Define now

$$f(t) = \psi'(t)\ ,$$

and consider again (2.2):

$$\sigma(\alpha_1,x) = \int_0^x \phi(\alpha_1,t)f(t)dt = \phi(\alpha_1,x)\,\psi(x) - \int_0^x \phi'_t\,(\alpha_1,t)\,\psi(t)dt.$$

The second last term $\rightarrow 0$, while the last $\rightarrow \infty$, as $x \rightarrow \infty$. Hence $\sigma(\alpha_1,x)$ diverges.

This contradiction proves the necessity of the convergence of (2.3) as $x \rightarrow \infty$, for the case $\psi(x) \rightarrow 0$. For the case $\psi(x) \rightarrow \psi \neq 0$, let us consider function f(t) as decomposed as follows:

$$f(t) = f_1(t) + f_2(t)\ ,$$

where

$$f_1(t) = \psi\ , \qquad 0 \leq t \leq 1\ ,$$

$$= 0\ , \qquad t > 1\ .$$

Then

$$\int_0^x \phi(\alpha,t)f(t)dt = \psi\int_0^1 \phi(\alpha,t)dt + \int_0^x \phi(\alpha,t)f_2(t)dt\ , \qquad x \geq 1. \tag{2.4}$$

The first integral on the right of (2.4) exists, and the second is the case $\psi(x) \to 0$ already handled.

Thus we complete the necessity part of the theorem:

Theorem 1 *A necessary and sufficient condition that*

$$\sigma(\alpha, x) = \int_0^x \phi(\alpha, t) f(t) dt, \qquad f \varepsilon L, \alpha \varepsilon E(\alpha),$$

converge as $x \to \infty$, is that

$$(2.5) \qquad \int_0^\infty |\phi_t'(\alpha, t)| \, dt < K(\alpha),$$

where $K(\alpha)$ is a positive function of α defined for $\alpha \varepsilon E(\alpha)$.

3. *Application to Bounded Functions.*

Given any bounded measurable function f(t) , defined for $t \geq 0$, and for which

$$(3.1) \qquad \lim_{t \to \infty} f(t) = L$$

exists, it is desired to find convergence factors $\theta(\alpha, t)$ to ensure the convergence of

$$(3.2) \qquad \tau(\alpha, x) = \int_0^x \theta(\alpha, t) f(t) dt, \qquad x \geqq 0, \quad \alpha \varepsilon E(\alpha),$$

$$(3.3) \qquad \tau(\alpha) = \int_0^\infty \theta(\alpha, t) f(t) dt, \qquad \alpha \varepsilon E(\alpha),$$

$$(3.4) \qquad \tau = \lim_{\alpha \to \alpha_0} \int_0^\infty \theta(\alpha, t) f(t) dt, \qquad \alpha \varepsilon E'(\alpha).$$

In particular, " θ is regular over B" if $\tau = L$ for every $f \varepsilon B$.

It is noted here, and will be elaborated on in §6, that this is essentially the problem treated by Agnew [2].

We see that θ must be Lebesgue integrable in t for each α. A sufficient condition, since f is bounded, for the existence of $\tau(\alpha)$ is the convergence, as $x \to \infty$, of

$$\int_0^x |\theta(\alpha,t)|\, dt.$$

Suppose this condition is not necessary. Then

$$\int_0^\infty |\theta(\alpha_1,t)|\, dt = \infty$$

for some value $\alpha_1 \varepsilon E(\alpha)$. Therefore there exists a sequence of numbers $\lambda_0 = 0$, λ_1, λ_2, - - -, λ_n, - - - with $\lambda_n \to \infty$, such that

$$\int_{\lambda_n}^{\lambda_{n+1}} |\theta(\alpha_1,t)|\, dt > n+1.$$

Define f(t) as follows:

$$f(t) = (1/(n+1)) \operatorname{sgn} \theta(\alpha_1,t), \qquad \lambda_n \leqq t \leqq \lambda_{n+1}$$

This $f \varepsilon B$, and $\lim_{t\to\infty} f(t) = 0$. But

$$\int_0^{\lambda_{n+1}} \theta(\alpha_1,t)\, f(t)dt = \sum_{i=0}^{n} \int_{\lambda_i}^{\lambda_{i+1}} \frac{|\theta(\alpha_1,t)|}{i+1} dt > n.$$

Therefore $\tau(\alpha_1) = \infty$, and this contradiction proves the theorem:

Theorem 2 *A necessary and sufficient condition for the convergence, as $x \to \infty$, of the integral*

$$\tau(\alpha,x) = \int_0^x \theta(\alpha,t) f(t)dt, \qquad f \varepsilon B, \quad \alpha \varepsilon E(\alpha),$$

is that

$$(3.4) \qquad \int_0^\infty |\theta(\alpha,t)|\, dt < K(\alpha),$$

where $K(\alpha)$ is a positive function of α defined for $\alpha \varepsilon E(\alpha)$.

4 Conditions Sufficient for Regularity

We desire that

$$(4.1) \qquad \sigma = \lim_{\alpha \to \alpha_0} \int_0^\infty \phi(\alpha, t) f(t)dt = \psi = \int_0^\infty f(t)dt, \qquad \alpha \varepsilon e(\alpha),$$

where $e(\alpha)$ is some subset of $E(\alpha)$, having α_0 as limit point. We shall prove the sufficiency of the following four conditions:

$$(4.2) \qquad \int_0^\infty |\phi'_t(\alpha, t)| \, dt < K(\alpha), \qquad \alpha \varepsilon E(\alpha),$$

$$(4.3) \qquad \int_0^\infty |\phi'_t(\alpha, t)| \, dt < K, \qquad \alpha \varepsilon E'(\alpha),$$

$$(4.4) \qquad \lim_{\alpha \to \alpha_0} \phi(\alpha, t) = 1, \qquad 0 \leqq t < \infty, \alpha \varepsilon E''(\alpha),$$

$$(4.5) \qquad \lim_{\alpha \to \alpha_0} \int_0^x |\phi'_t(\alpha, t)| \, dt = 0, \qquad 0 \leqq x < \infty, \alpha \varepsilon E'''(\alpha),$$

where $E'(\alpha)$, $E''(\alpha)$, $E'''(\alpha)$ are subsets of $E(\alpha)$ having α_0 for limit point, and K is an absolute constant.

The first three of these conditions are the analogues of the three conditions required by Moore [18]. In our work we have found that the fourth condition must be included.

Consider first the case $\psi(x) \to 0$, as $x \to \infty$. From (2.2) we have

$$\sigma(\alpha, x) = \int_0^x \phi(\alpha, t) f(t)dt = \phi(\alpha, x)\psi(x) - \int_0^x \phi'_t(\alpha, t)\psi(t)dt.$$

Hence

$$(4.6) \qquad \sigma(\alpha) = \int_0^\infty \phi(\alpha, t) f(t)dt = -\int_0^\infty \phi'_t(\alpha, t)\psi(t)dt.$$

From (4.3), for $\alpha \varepsilon E'(\alpha)$, there is an $M < \infty$, such that

$$\int_M^\infty |\phi'_t(\alpha,t)|\,dt < \frac{\epsilon}{2A}, \qquad \alpha \,\varepsilon\, E'(\alpha).$$

From (4.5), there is a set $E'''(\alpha)$ such that

$$\int_0^M |\phi'_t(\alpha,t)|\,dt < \frac{\epsilon}{2A}, \qquad \alpha \,\varepsilon\, E'''(\alpha).$$

Then for $\alpha \,\varepsilon\, E'(\alpha)\cdot E'''(\alpha)$, we have

$$|\sigma(\alpha)| \leqq \int_0^M |\phi'_t(\alpha,t)|\cdot|\psi(t)|\,dt + \int_M^\infty |\phi'_t(\alpha,t)|\cdot|\psi(t)|\,dt$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\sigma(\alpha) \longrightarrow 0$ as $\alpha \rightarrow \alpha_0$, with α in some suitably chosen subset of $E(\alpha)$.

Next we consider the case $\psi(x) \rightarrow \psi \neq 0$, as $x \rightarrow \infty$. As before, we write

$$f(t) = f_1(t) + f_2(t),$$

with

$$f_1(t) = \psi, \qquad 0 \leqq t \leqq 1,$$

$$= 0, \qquad t > 1.$$

Then, for $x \geqq 1$,

$$\int_0^x \phi(\alpha,t)f(t)\,dt = \psi\int_0^1 \phi(\alpha,t)\,dt + \int_0^x \phi(\alpha,t)f_2(t)\,dt \tag{4.7}$$

The first term on the right of (4.7) tends to ψ as $\alpha \rightarrow \alpha_0$, in view of condition (4.4). The last term in (4.7) tends to zero by the proof just completed. Therefore, if we choose $e(\alpha) = E'(\alpha)\cdot E''(\alpha)\cdot E'''(\alpha)$, we can say that the conditions (4.2) – (4.5) are sufficient for the convergence to ψ as $\alpha \rightarrow \alpha_0$ in $e(\alpha)$, of the expression $\sigma(\alpha)$.

5 Necessity of the Conditions

We proved the necessity of (4.2) in §2. To investigate (4.4) we consider the function

$$f(t) = 1, \qquad 0 \leqq t \leqq h < \infty,$$

$$= 0, \qquad t > h.$$

This leads to the necessity of

(5.1) $$\lim_{\alpha \to \alpha_0} \int_0^h \phi(\alpha, t)\,dt = h, \qquad 0 \leqq h < \infty, \ \alpha \varepsilon E''(\alpha),$$

or, what is the same thing,

(5.2) $$\lim_{\alpha \to \alpha_0} \int_0^h \left[1 - \phi(\alpha, t) \right] dt = 0 .$$

By the restrictions we have imposed on $\phi(\alpha, t)$, we can write*

$$\int_0^h \lim_{\alpha \to \alpha_0} \left[1 - \phi(\alpha, t) \right] dt = 0,$$

and, therefore,

$$\int_0^h \left[1 - \phi(\alpha_0, t) \right] dt = 0 .$$

Since $1 - \phi(\alpha_0, t)$ is absolutely continuous, and h is arbitrary, we must have**

(5.3) $$1 - \phi(\alpha_0, t) \equiv 0 .$$

The necessity of (4.4) is thus proved.

By our suppositions concerning $\phi_t'(\alpha, t)$, we can write***

(5.4) $$\lim_{\alpha \to \alpha_0} \int_0^x | \phi_t'(\alpha, t) |\,dt = \int_0^x | \phi_t'(\alpha_0, t) |\,dt, \qquad \alpha \varepsilon E_x(\alpha).$$

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* See Franklin [9] p.402

** See Titchmarsh [21] p.360

***See McShane [16] p.169

Since $\phi(\alpha_0,t) \equiv 1$, and the convergence to $\phi(\alpha_0,t)$ is uniform, the derivative $\phi'_t(\alpha_0,t)$ must be zero. Therefore, the right side of (5.4) is zero, and this proves the necessity of (4.5).

If the condition (4.3) does not hold for any subset of $E(\alpha)$, having α_0 as limit point, we can select from $E(\alpha)$ a sequence $\{\alpha_n\}$, approaching α_0, such that

$$\int_0^{x_n} |\phi'_t(\alpha_n,t)|\, dt > n, \qquad n = 1, 2, 3, ---,$$

for a proper choice of x_n.

Choose $y_n > x_n$, and define

$$g(t) = \frac{1}{\sqrt{n}} \operatorname{sgn} \phi'_t(\alpha_n,t), \qquad 0 \leqq t \leqq y_n.$$

Choose α_m from $\{\alpha_n\}$ such that

$$\int_0^{y_n} |\phi'_t(\alpha_m,t)|\, dt < \frac{1}{8},$$

which is possible from the necessity of (4.5) already proved.

Let x_m, $(m > n)$, be the corresponding value for which

$$P = \int_0^{x_m} |\phi'_t(x_m,t)|\, dt > m.$$

Take $y_m > x_m$ and such that

$$\int_{y_m}^{\infty} |\phi'_t(\alpha_m,t)|\, dt < \frac{1}{8},$$

which is possible in view of the necessity of (4.2) already established. We now define

$$g(t) = \frac{1}{\sqrt{m}} \operatorname{sgn} \phi'_t(\alpha_m,t), \qquad y_n \leqq t \leqq y_m.$$

Let $\eta > 0$ be such, that for any set $E \subset (0,y_m)$ of measure $< \eta$, we have

$$\int_E |\phi_t' (\alpha_m, t)| \, dt < \frac{1}{8}.$$

Let $\psi(t)$ be an absolutely continuous function such that*

$$|\psi(t) - g(t)| < \frac{1}{8P}, \qquad 0 \quad t \quad y,$$

except on a set E_1 of measure $< \eta$. Let $E_2 = (0, y_m) - E_1$.

Moreover,

$$\max |\psi(t) - g(t)| \leqq \frac{2}{\sqrt{n}}.$$

Let $\psi(t)$, $g(t)$ at present be both undefined for $t > y_m$, merely restricted to be less than 1 in absolute value.

Now

$$\left|\int_0^\infty \phi_t' (\alpha_m, t)\, \psi(t) dt\right| = \left|\int_0^\infty \phi_t' (\alpha_m, t)\left[g(t) + \{\psi(t) - g(t)\}\right] dt\right|$$

$$\geqq \left|\int_0^\infty \phi_t' (\alpha_m, t) g(t) dt\right| - \left|\int_0^\infty \phi_t' (\alpha_m, t) \{\psi(t) - g(t)\} dt\right|$$

$$\geqq \left|\int_{y_n}^{y_m} \{\phi_t'\}\{g\} dt\right| - \left|\int_0^{y_n} \{\phi_t'\}\{g\} dt\right| - \left|\int_{y_m}^\infty \{\phi_t'\}\{g\} dt\right.$$

$$- \left|\int_0^{y_m} \{\phi_t'\}\{\psi - g\} dt\right| - \left|\int_{y_m}^\infty \{\phi_t'\}\{\psi - g\} dt\right|$$

$$> \frac{1}{\sqrt{m}} \int_0^{y_m} |\phi_t'| \, dt - \frac{1}{\sqrt{m}} \int_0^{y_n} |\phi_t'| \, dt - \int_0^{y_n} |\phi_t'| dt - \int_{y_m}^\infty |\phi_t'| \, dt$$

$$- \frac{1}{8P} \int_{E_2} |\phi_t'| \, dt - \frac{2}{\sqrt{n}} \int_{E_1} |\phi_t'| \, dt - 2 \int_{y_m}^\infty |\phi_t'| \, dt$$

$$> \sqrt{m} - \frac{1}{8\sqrt{m}} - \frac{1}{8} - \frac{1}{8} - \frac{1}{8} - \frac{1}{4\sqrt{n}} - \frac{1}{4}$$

$$> \sqrt{m} - 1.$$

Continuing in this fashion, we define an absolutely continuous

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* See Titchmarsh [21] p.376

Function $\psi(t)$, convergent to zero, for which the left side of (4.6), namely,

$$\int_0^\infty \phi(\alpha,t)f(t)dt \;=\; -\int_0^\infty \phi_t'(\alpha,t)\,\psi(t)dt\,,$$

increases indefinitely over a set of values of α, having α_0 as a limit point. Thus we have a contradiction, and the necessity of (4.3) is established.

This completes the proof of the theorem:

Theorem 3 Necessary and sufficient conditions that

(1.5) $$\sigma(\alpha) = \int_0^\infty \phi(\alpha,t)f(t)dt$$

converge in E (α), whenever

(1.2) $$\psi = \int_0^\infty f(t)dt$$

is convergent, and approach the value ψ as $\alpha \to \alpha_0$, are that the convergence factor $\phi(\alpha,t)$ satisfy the conditions (4.2), (4.3), (4.4), and (4.5).

6 Regularity for Bounded Functions

We desire that

(6.1) $$\lim_{\alpha\to\alpha_0} \int_0^\infty \theta(\alpha,t)f(t)dt = \lim_{x\to\infty} f(x), \qquad f \in B\,,\ \alpha \in E'(\alpha).$$

Agnew [2] considered just this problem. One of his conditions on $\theta(\alpha,t)$ is equivalent to

(6.2) $$\lim_{\alpha\to\alpha_0} \int_0^h \theta(\alpha,t)f(t)dt = 0\,,$$

for every $0 \leqq h < \infty$ and $f \in B$, for α in some set $E'(\alpha) \subset E(\alpha)$.

We suppose that $\theta(\alpha,t)$ is continuous in t for all α and converges boundedly.. We are then able to modify the condition (6.2) so as to obtain the direct analogue of the condition for series used by Moore [18]. The condition in question becomes our condition (6.5). One is reminded by conditions (6.4), (6.5), (6.6) of the conditions on "quasi-positive" kernels*.

Agnew proved the theorem which we now state. We could refer to his paper for the proof, but it is better that we develop a proof which will have simple extensions to the double- and multimple- integral cases.

The theorem in question is

Theorem 4 Necessary and sufficient conditions that

(6.1) $$\lim_{\alpha \to \alpha_0} \int_0^\infty \theta(\alpha, t) f(t)\,dt = \lim_{x \to \infty} f(x),$$

for all functions f(x), bounded and measurable, for which the right side of (6.1) exists, are

(6.3) $$\int_0^\infty |\theta(\alpha,t)|\,dt < K_1(\alpha), \qquad \alpha \varepsilon E(\alpha),$$

(6.4) $$\int_0^\infty |\theta(\alpha,t)|\,dt < K_1, \qquad \alpha \varepsilon E'(\alpha),$$

(6.5) $$\lim_{\alpha \to \alpha_0} \theta(\alpha,t) = 0, \qquad 0 \leqq t < \infty,\ \alpha \varepsilon E''(\alpha),$$

(6.6) $$\lim_{\alpha \to \alpha_0} \int_0^\infty \theta(\alpha,t)\,dt = 1, \qquad \alpha \varepsilon E'''(\alpha).$$

Consider first the case where $f(x) \to 0$ as $x \to \infty$. Let

$$|f(x)| < B$$

for all $x \geqq 0$. There exists a number $N < \infty$, such that for $x \geqq N$, we have

---

* See Zygmund [22] p.45

$$|f(x)| < \frac{\epsilon}{2K_1} .$$

In virtue of condition (6.5), there is a set $E''(\alpha)$, with limit point α_0, such that

$$|\theta(\alpha,t)| < \frac{\epsilon}{2BN}, \qquad \alpha \in E''(\alpha) .$$

Then, for $\alpha \in E'(\alpha)\cdot E''(\alpha)$,

$$\begin{aligned} \left|\int_0^\infty \theta(\alpha,t)f(t)dt\right| &\leqq \int_0^N |\theta(\alpha,t)|\cdot|f(t)|\,dt + \int_N^\infty |\theta(\alpha,t)|\cdot|f(t)|\,dt \\ &< B\int_0^N |\theta(\alpha,t)|\,dt + \frac{\epsilon}{2K_1}\int_N^\infty |\theta(\alpha,t)|\,dt \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon . \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{\alpha \to \alpha_0} \int_0^\infty \theta(\alpha,t)f(t)dt = 0 , \qquad \alpha \in e(\alpha) ,$$

where $e(\alpha)$ is a suitably chosen set.

Consider now $f(x) \to L \neq 0$. Letting

$$f(x) = L + f_1(x) ,$$

we obtain

$$(6.7) \qquad \int_0^\infty \theta(\alpha,t)f(t)dt = L\int_0^\infty \theta(\alpha,t)dt + \int_0^\infty \theta(\alpha,t)f_1(t)dt.$$

The last term on the right of (6.7) is the case already considered and tends to zero as $\alpha \to \alpha_0$. The first term on the right of (6.7) tends to L as $\alpha \to \alpha_0$ because of the condition (6.6).

Thus, the left side of (6.1) tends to the limit reached by the right side whenever $\theta(\alpha,t)$ satisfies the conditions (6.3), (6.4), (6.5), and (6.6).

To show the conditions to be necessary, we consider the convergence of

$$\int_0^\infty \theta(\alpha, t)\, \psi(t)dt ,$$

where $\psi(t)$ is a bounded absolutely continuous function, instead of the function f(t) merely bounded as in the requirements of this section. Let

$$\phi(\alpha, t) = \int_t^\infty \theta(\alpha, x)dx.$$

Then

$$\phi'_t(\alpha, t) = -\theta(\alpha, t).$$

If we define f(t) by means of

$$f(t) = \psi'(t) ,$$

we can write the identity (2.2) in the form

(6.8) $$\int_0^x \theta(\alpha, t)\, \psi(t)dt = \sigma(\alpha, x) - \phi(\alpha, x)\, \psi(x) \cdot$$

Consider first the case where $\psi(x) \longrightarrow 0$ as $x \longrightarrow \infty$. We see that the convergence of the left side of (6.8) can be made to depend on the conditions prescribed for the convergence of $\sigma(\alpha, x)$.

The condition

$$\int_0^\infty |\phi'_t(\alpha, t)|\, dt < K , \qquad \alpha \varepsilon E'(\alpha) ,$$

goes directly into condition (6.4). The condition

$$\lim_{\alpha \to \alpha_0} \int_0^x |\phi'_t(\alpha, t)|\, dt = 0, \qquad \alpha \varepsilon E''(\alpha) ,$$

which depends on

$$\lim_{\alpha \to \alpha_0} \phi'_t(\alpha, t) = 0 , \qquad 0 \leqq t \leqq x, \ \alpha \varepsilon E_x(\alpha) ,$$

goes over to condition (6.5).

The condition (6.6) is shown to be necessary by considering a function $\psi(t)$ identically a constant.

Since the conditions (6.3) – (6.6) are necessary when the function considered is bounded and absolutely continuous, it is obvious that they are also necessary when we consider a more general class of functions, namely, the class of functions bounded and measurable. Therefore, the conditions (6.3) – (6.6) are necessary and sufficient that (6.1) should hold.

In later work, we shall have to refer to Theorem 4 in which condition (6.5) is modified. Let us write instead of (6.5),

$$(6.5)' \qquad \lim_{\alpha \to \alpha_0} \int_0^x |\theta(\alpha, t)|\, dt = 0, \qquad 0 \leqq x < \infty, \alpha \varepsilon E_x(\alpha),$$

where $E_x(\alpha)$ is a subset of $E(\alpha)$, depending on x, and having α_0 as a limit point. This condition, instead of (6.5), together with (6.4) and (6.6), can be shown to be sufficient. The necessity of it follows at once from the necessity of

$$\lim_{\alpha \to \alpha_0} \int_0^x |\phi_t'(\alpha, t)|\, dt = 0 \qquad \alpha \varepsilon E_x(\alpha),$$

instead of the more restrictive

$$\lim_{\alpha \to \alpha_0} \phi_t'(\alpha, t) = 0.$$

CHAPTER II

DOUBLE INTEGRALS

1 Statement of the Problems.

Corresponding to the single-integral case we are to consider functions $f(x,y)$ which are Lebesgue integrable over every rectangle $0 \leqq x \leqq p$, $0 \leqq y \leqq q$, and for which the integral exists as either p, or q, or both (independently), increase without limit. Let

$$(1.1)\qquad \begin{aligned} S(p,q) &= \int_0^p \int_0^q f(x,y)\,dx\,dy\,, \qquad p,q \geqq 0\,,\ f \in L\,, \\ &= \int_0^q \int_0^p f(x,y)\,dy\,dx\,, \end{aligned}$$

and suppose that

$$(1.2)\qquad |S(p,q)| \leqq A < \infty\,, \qquad \text{all } p,q \geqq 0\,.$$

Let us write

$$(1.3)\qquad S = \int_0^\infty \int_0^\infty f(x,y)\,dx\,dy.$$

After the fashion of Chapter I, we desire convergence factors $\phi(x,y;\alpha)$ defined for $x,y \geqq 0$, and α in some set $E(\alpha)$, having α_0 as a limit point of the set but not of it. These convergence factors must ensure the existence of the integrals

$$(1.4)\qquad \sigma(p,q;\alpha) = \int_0^p \int_0^q \phi(x,y;\alpha)f(x,y)\,dx\,dy, \qquad E(\alpha)\,,$$

$$(1.5)\qquad \sigma(\infty,q;\alpha) = \int_0^\infty \int_0^q \phi(x,y;\alpha)f(x,y)\,dx\,dy, \qquad E(\alpha)\,,$$

$$(1.6)\qquad \sigma(p,\infty;\alpha) = \int_0^p \int_0^\infty \phi(x,y;\alpha)f(x,y)\,dx\,dy, \qquad E(\alpha)\,,$$

$$(1.7)\qquad \sigma(\alpha) = \int_0^\infty \int_0^\infty \phi(x,y;\alpha)f(x,y)\,dx\,dy, \qquad E(\alpha)\,,$$

and

$$(1.8) \qquad \sigma = \lim_{\alpha \to \alpha_0} \int_0^\infty\int_0^\infty \phi(x,y;\alpha)f(x,y)\,dx\,dy, \qquad \alpha \in E'(\alpha),$$

where $E'(\alpha)$ is $E(\alpha)$ or a subset of $E(\alpha)$ having α_0 for a limit point.

As before of particular interest is the case $\sigma = S$, when we say that "ϕ is regular".

We shall assume that $\phi(x,y;\alpha)$ has all the properties necessary for the manipulations which follow. We might be tempted to assume ϕ uniformly bounded; thus:

$$(1.9) \qquad |\phi(x,y;\alpha)| < B < \infty, \qquad \text{all } x,y \geq 0, \quad \alpha \in E(\alpha).$$

But we see later that we do not need quite as much restriction on ϕ.

The other problem considered in this Chapter is analogous to the second one of Chapter I. As in Chapter I, we shall designate the convergence factor sought by θ, in this case $\theta(x,y;\alpha)$. This will be discussed more fully in sections 4 and 7.

2 Existence of the Transform

It will be useful to assume that $\phi(x,y;\alpha)$ converges uniformly in x and y as $\alpha \to \alpha_0$. By taking $\phi(x,y;\alpha)$ absolutely continuous we are assured that $\phi(x,y;\alpha_0)$ is also absolutely continuous.

We assume that $\phi(x,y;\alpha)$ possesses first partial derivatives $\phi'_x(x,y;\alpha)$ and $\phi'_y(x,y;\alpha)$ for all $\alpha \in E(\alpha)$, which are absolutely continuous in y and x respectively. Then $\phi''_{xy}(x,y;\alpha)$ exists almost everywhere. Suppose that, for every square $0 \leq x,y \leq z < \infty$, there is a subset $E_2(\alpha)$ of $E(\alpha)$, having α_0 as limit point, such that $\phi''_{xy}(x,y;\alpha)$ converges boundedly for x and y in this square as $\alpha \to \alpha_0$ in $E_2(\alpha)$. This is the same condition as required in Chapter I.

We have,

$$(2.1) \qquad \int_q^S \phi''_{xy}(x,y;\alpha)dy = \phi'_x(x,S;\alpha) - \phi'_x(x,q;\alpha), \qquad 0 \leqq q \leqq S,$$

and

$$(2.2) \qquad \int_p^r \phi''_{xy}(x,y;\alpha)dx = \phi'_y(r,y;\alpha) - \phi'_y(p,y;\alpha), \qquad 0 \leqq p \leqq r,$$

On integrating (1.4) by parts twice, the second time with an inversion of the order of integration, we obtain

$$\begin{aligned}
\sigma(p,q;\alpha) &= \int_0^p\int_0^q \phi(x,y;\alpha)f(x,y)dxdy \\
&= \phi(p,q;\alpha)\int_0^p\int_0^q f(x,y)dxdy - \int_0^p \phi'_x(x,q;\alpha)\left\{\int_0^x\int_0^q f(z,y)dz\,dy\right\}dx \\
&\quad - \int_0^q \phi'_y(p,y;\alpha)\left\{\int_0^p\int_0^y f(x,u)dx\,du\right\}dy \\
&\quad + \int_0^p\int_0^q \phi''_{xy}(x,y;\alpha)\left\{\int_0^x\int_0^y f(z,u)dzdu\right\}dxdy
\end{aligned}$$

$$\begin{aligned}
(2.3) \qquad &= \phi(p,q;\alpha)S(p,q) - \int_0^p \phi'_x(x,q;\alpha)S(x,q)dx \\
&\quad - \int_0^q \phi'_y(p,y;\alpha)S(p,y)dy + \int_0^p\int_0^q \phi''_{xy}(x,y;\alpha)S(x,y)dxdy.
\end{aligned}$$

From the form of the last term in (2.3), we are led to the condition

$$(2.4) \qquad \int_0^\infty\int_0^\infty \left|\phi''_{xy}(x,y;\alpha)\right| dxdy < K(\alpha), \qquad \alpha \in E(\alpha),$$

where $K(\alpha)$ is a positive function of α, defined in $E(\alpha)$. This condition is analogous to that used in Chapter I.

The satisying of condition (2.4) leads at once to the convergence of $\int_0^p\int_0^q \phi''_{xy}(x,y;\alpha)dxdy$ as $p,q \to \infty$. We impose on $\phi(x,y;\alpha)$ the further conditions

$$(2.5) \qquad \lim_{y\to\infty} \int_0^\infty |\phi'_x(x,y;\alpha)|\, dx = 0\,, \qquad \alpha \varepsilon E(\alpha)\,,$$

and

$$(2.6) \qquad \lim_{x\to\infty} \int_0^\infty |\phi'_y(x,y;\alpha)|\, dy = 0\,, \qquad \alpha \varepsilon E(\alpha)\,.$$

We note that these conditions are somewhat different from the ones used by Moore [18] for series. For example, the direct analogue of the series condition, in place of (2.5), would have required

$$\lim_{y\to\infty} \phi'_x(x,y;\alpha) = 0\,.$$

Such a condition would have been sufficient, but not necessary.

If (2.5) holds, it follows that

$$\lim_{y\to\infty} \int_{x_1}^{x_2} \phi'_x(x,y;\alpha)dx = 0, \qquad x_1, x_2 \gtreqless 0\,,$$

or,

$$\phi(x_1,\infty;\alpha) = \phi(x_2,\infty;\alpha)\,, \qquad x_1, x_2 \gtreqless 0\,.$$

Using both (2.4) and (2.5), we then obtain

$$\int_0^p\int_0^\infty \phi''_{xy}(x,y;\alpha)dxdy = \phi(p,\infty;\alpha) - \phi(0,\infty;\alpha) - \phi(p,0;\alpha) + \phi(0,0;\alpha)$$

$$= \phi(0,0;\alpha) - \phi(p,0;\alpha)\,.$$

In virtue of these conditions, therefore, $\phi(p,0;\alpha)$ is defined for all $p \gtreqless 0$. Likewise, by use of (2.6) and (2.4), we arrive at the existence of $\phi(0,q;\alpha)$ for all $q \gtreqless 0$.

Again, because of (2.4), we can write

$$\int_0^p\int_0^q \phi''_{xy}(x,y;\alpha)dxdy = \phi(p,q;\alpha) - \phi(p,0;\alpha) - \phi(0,q;\alpha) + \phi(0,0;\alpha).$$

Therefore $\phi(p,q;\alpha)$ exists for all $p,q \gtreqless 0$. Thus we see that the condition (1.9) can be replaced by a less-restrictive one, namely,

$$(2.7) \qquad |\phi(x,y;\alpha)| < B(\alpha), \qquad \alpha \, \varepsilon \, E(\alpha).$$

In proving the sufficiency, for the existence of $\sigma(\alpha)$, of the conditions (2.4), (2.5) and (2.6), let us consider first the case where $S(p,q) \longrightarrow 0$ as $p,q \rightarrow \infty$. Given an arbitrary $\epsilon > 0$, there exist $p_0 < \infty$, $q_0 < \infty$, such that for $p > p_0$, $q > q_0$, we have

$$|S(p,q)| < \frac{\epsilon}{3B(\alpha)},$$

$$\int_0^\infty |\phi'_x(x,q;\alpha)|\,dx < \frac{\epsilon}{3A},$$

and

$$\int_0^\infty |\phi'_y(p,y;\alpha)|\,dy < \frac{\epsilon}{3A}.$$

Therefore, for $p > p_0$, $q > q_0$, (2.3) gives us

$$\left|\sigma(p,q;\alpha) - \int_0^p\int_0^q \phi''_{xy}(x,y;\alpha)S(x,y)dxdy\right| < \epsilon.$$

Since ϵ is arbitrary, it follows that

$$(2.8) \qquad \lim_{p,q\to\infty}\sigma(p,q;\alpha) = \int_0^\infty\int_0^\infty \phi''_{xy}(x,y;\alpha)S(x,y)dxdy$$

The boundedness of $S(x,y)$ and the holding of (2.4) ensure the existence of the right side of (2.8).

To consider the case $S(p,q) \longrightarrow S \neq 0$, we write

$$f(x,y) = f_1(x,y) + f_2(x,y),$$

where

$$f_1(x,y) = S, \qquad 0 \leqq x,y \leqq 1,$$
$$= 0, \qquad \text{elsewhere.}$$

Then, for $p,q \geqq 1$, we have

$$(2.9)\quad \int_0^p\int_0^q \phi(x,y;\alpha)f(x,y)dxdy = S\int_0^1\int_0^1 \phi(x,y;\alpha)dxdy + \int_0^p\int_0^q \phi(x,y;\alpha)f_2(x,y)dxdy.$$

The first integral on the right of (2.9) exists, and the second is the case $S(p,q) \longrightarrow 0$ as $p,q \to \infty$, already considered.

Therefore, the conditions (2.4), (2.5) and (2.6) are sufficient for the existence of $\sigma(\alpha)$, for all $f(x,y)$ such that $S(x,y)$ is bounded and convergent.

3 Necessity of the Conditions.

Suppose that (2.4) is not true. Then for some value α_1 of α we have

$$\int_0^\infty\int_0^\infty |\phi''_{xy}(x,y;\alpha_1)|\, dxdy = \infty .$$

Therefore, there exists an $N_1 < \infty$, such that

$$\int_0^{N_1}\int_0^{N_1} |\phi''_{xy}(x,y;\alpha_1)|\, dxdy = 2.$$

Consider now the function

$$g_1(x,y) = \operatorname{sgn} \phi''_{xy}(x,y;\alpha_1), \qquad 0 \leqq x,y \leqq N_1,$$
$$= 0, \qquad \text{elsewhere.}$$

The integrability of ϕ''_{xy} gives that of g_1.

Given $\epsilon > 0$, choose a $z_1 > 0$, such that

$$\int_0^{N_1+z_1}\int_0^{N_1+z_1} |\phi''_{xy}(x,y;\alpha_1)|\, dxdy < 2 + \epsilon .$$

Let $\eta > 0$ be such that for any set E, contained in the square $0 \leq x,\ y \leq N_1 + z_1$, of measure $< \eta$, we have

$$\iint_E |\phi''_{xy}(x,y;\alpha_1)|\ dxdy < \frac{\epsilon}{4}\ .$$

By the analogue of the theorem used in I, § 2, there exists a function $S_1(x,y)$, with

$$S_1(x,y) = \int_0^x \int_0^y f_1(s,t)dsdt, \tag{3.1}$$

such that

$$|S_1(x,y) - g_1(x,y)| < \frac{\epsilon}{4}\ , \qquad 0 \leq x,\ y \leq N_1+z_1,$$

except on a set E_1 of measure less than η. As in I, § 2, we make $S_1(x,y)$ zero for $x,y \geq N_1 + z_1/2$. In particular, $S_1(x,N_1 + z)$ and $S_1(N_1 + z_1, y)$ are zero, regardless of the values of x,y.

Substituting $f_1(x,y)$, as given by (3.1), in (2.3), yields

$$\begin{aligned}
|\sigma(N_1 + z_1, N_1 + z_1; \alpha) &= \left| 0 - 0 - 0 + \int_0^{N_1+z_1}\int_0^{N_1+z_1} \phi''_{xy}(x,y;\alpha)S_1(x,y)dxdy \right| \\
&= \left| \int_0^{N_1+z_1}\int_0^{N_1+z_1} \phi''_{xy}(x,y;\alpha_1)\left\{g_1(x,y) + [S_1(x,y) - g_1(x,y)]\right\} dxdy \right| \\
&\geq \left| \int_0^{N_1+z_1}\int_0^{N_1+z_1} \phi''_{xy}\ g_1\ dxdy \right| - \left| \int_0^{N_1+z_1}\int_0^{N_1+z_1} \phi''_{xy}[S_1 - g_1|\ dxdy \right| \\
&\geq \int_0^{N_1}\int_0^{N_1} |\phi''_{xy}|\ dxdy - \iint_{E_0} |\phi''_{xy}| \cdot \frac{\epsilon}{4}\ dxdy - \iint_{E_1} |\phi''_{xy}| \cdot 2dxdy \\
&> 2 - \frac{\epsilon}{4}(2 + 6) - 2 \cdot \frac{\epsilon}{4} > 1\ .
\end{aligned}$$

We continue this process by considering the number $N_2 < \infty$, such that

$$\int_0^{N_2}\int_0^{N_2} |\phi''_{xy}(x,y;\alpha_1)|\ dxdy = 4,$$

and the function $g_2(x,y)$ defined by

$$
\begin{aligned}
g_2(x,y) &= 0\,, && 0 \leqq x,\ y \leqq N_1 + z_1\,, \\
&= 0\,, && x \text{ or } y > N_2 \text{ (or both)}, \\
&= \tfrac{1}{2}\operatorname{sgn} \phi''_{xy}(x,y;\alpha_1)\,, && \text{elsewhere.}
\end{aligned}
$$

By choosing a function $S_2(x,y)$ in a way similar to that for $S_1(x,y)$, with a corresponding z_2, we shall obtain

$$\sigma(N_2 + z_2\,,\ N_2 + z_2\,;\alpha_1) > 1 + \tfrac{1}{2}\,.$$

Thus, as in Chapter I, we arrive at a contradiction, and this proves the necessity of condition (2.4).

Suppose that (2.5) is not true. Then there is a constant $C > 0$, and a value α_1 of α, such that for some infinite sequence $\{q_n\}$, $q_n - q_{n-1} \geqq 1$, $q_n \rightarrow \infty$, we have

$$\int_0^\infty \left| \phi'_x\,(x,q_n;\alpha_1) \right| dx > C\,.$$

If this is true, there exists a corresponding sequence $\{p_n\}$, $p_n \geqq p_{n-1}$, $p_n \rightarrow \infty$, such that

$$\int_0^{p_n - 1} \left| \phi'_x\,(x,q_n;\alpha_1) \right| dx > \tfrac{3}{4}\,C\,.$$

Define

$$
\begin{aligned}
g(x,y) &= (-1)^n \operatorname{sgn} \phi'_x\,(x,q_n;\alpha_1), && 0 \leqq x \leqq p_n - 1,\ y = q_n, \\
&= 0\,, && \text{elsewhere.}
\end{aligned}
$$

By the method adopted in Chapter I, we can obtain a function $S_n(x)$ absolutely continuous in x for each q_n, approximating the function $g(x,y)$, bounded by $\pm$ 1, zero for $x \geqq p_n$, and such that

$$\left| \int_0^{p_n} \phi'_x(x,q_n;\alpha_1)\, S_n(x)\ dx \right| > C/2\,.$$

Define the absolutely continuous function S(x,y) as any convenient surface, bounded by ± 1, which has the traces $S_n(x)$ in the planes $y = q_n$. We note that the integral

$$\int_0^{p_n} \phi'_x\,(x,q_n;\alpha_1)S(x,q_n)dx$$

is alternately $> \frac{c}{2}$ and $< -\frac{c}{2}$.

From (2.3) we have

$$\sigma(p_n,q_n;\alpha_1) = \phi(p_n,q_n;\alpha_1)S(p_n,q_n) - \int_0^{p_n}\phi'_x\,(x,q_n;\alpha_1)S(x,q_n)dx$$

$$(3.2) \qquad -\int_0^{q_n}\phi'_y\,(p_n,y;\alpha_1)S(p_n,y)dy + \int_0^{p_n}\int_0^{q_n}\phi''_{xy}(x,y;\alpha_1)S(x,y)dxdy.$$

From our method of choosing S(x,y), we have $S(p_n,y) = 0$, which makes the first and third terms on the right of (3.2) zero. The last term converges to a definite limit as p_n, $q_n \to \infty$, in virtue of the condition (2.4) and the boundedness of S(x,y). The left side of (3.2) also must approach a definite limit. The oscillation of the second term on the right of (3.2) between limits greater than $\frac{c}{2}$ and less than $-\frac{c}{2}$ provides a contradiction. Therefore it is seen that condition (2.5) is necessary.

The proof of the necessity of condition (2.6) is entirely analogous to the of (2.5),and need not be shown here. This completes the proof of the theorem:

<u>Theorem 1</u> <u>Necessary and sufficient conditions that</u>

$$\sigma(p,q;\alpha) = \int_0^p\int_0^q \phi(x,y;\alpha)f(x,y)dxdy$$

converge to a definite limit as $p,q \to \infty$, for all functions f(x,y) satisfying condition (1.2) are that

$$(2.4) \qquad \int_0^\infty\int_0^\infty|\phi''_{xy}(x,y;\alpha)|\,dxdy < K(\alpha), \qquad \alpha \in E(\alpha),$$

$$(2.5) \qquad \lim_{y \to \infty} \int_0^\infty \left| \phi'_x(x,y;\alpha) \right| dx = 0, \qquad \alpha \,\varepsilon\, E(\alpha),$$

and

$$(2.6) \qquad \lim_{x \to \infty} \int_0^\infty \left| \phi'_y(x,y;\alpha) \right| dy = 0, \qquad \alpha \,\varepsilon\, E(\alpha).$$

4. Application to Bounded Functions of Two Variables.

Given any bounded, measurable function $f(x,y)$, defined for $x \geqq 0$, $y \geqq 0$, and for which

$$(4.1) \qquad \lim_{x,y \to \infty} f(x,y) = A,$$

we desire to discover the properties of convergence factors $\theta(x,y;\alpha)$, defined for $x \geqq 0$, $y \geqq 0$, $\alpha \,\varepsilon\, E(\alpha)$, to ensure the existence of the integrals

$$(4.2) \qquad \tau(p,q;\alpha) = \int_0^p \int_0^q \theta(x,y;\alpha) f(x,y)\,dx\,dy, \quad p,q \geqq 0, \ f \,\varepsilon\, B,$$

$$(4.3) \qquad \tau(\infty,q;\alpha) = \int_0^\infty \int_0^q \theta(x,y;\alpha) f(x,y)\,dx\,dy,$$

$$(4.4) \qquad \tau(p,\infty;\alpha) = \int_0^p \int_0^\infty \theta(x,y;\alpha) f(x,y)\,dx\,dy,$$

$$(4.5) \qquad \tau(\alpha) = \int_0^\infty \int_0^\infty \theta(x,y;\alpha) f(x,y)\,dx\,dy,$$

$$(4.6) \qquad \tau = \lim_{\alpha \to \alpha_0} \int_0^\infty \int_0^\infty \theta(x,y;\alpha) f(x,y)\,dx\,dy.$$

The case of regularity of θ —— $\tau = A$ for all $f \,\varepsilon\, B$, as $\alpha \to \alpha_0$, in some set $E'(\alpha)$ —— will be considered later. We shall be concerned at the moment with the existence of (4.2) – (4.5). Let

$$|f(x,y)| \leqq B, \qquad \text{All } x,y \geqq 0.$$

We assume that $\theta(x,y;\alpha)$ is Lebesgue integrable over every finite rectangle, for each $\alpha \,\varepsilon\, E(\alpha)$. This condition is necessary and sufficient for the existence of (4.2).

A condition we investigate is the finiteness of the integral

$$\int_0^p \int_0^q |\theta(x,y;\alpha)| \; dxdy$$

as $p,q \to \infty$. The boundedness of $f(x,y)$, together with the finiteness of this integral, for all $\alpha \in E(\alpha)$, are easily seen to be sufficient for the existence of the integrals (4.3) – (4.5).

Suppose this condition is not necessary. Then, for some value α_1 of α, we have

$$\int_0^\infty \int_0^\infty |\theta(x,y;\alpha_1)| \; dxdy = \infty .$$

There exists a sequence of numbers $\{\lambda_n\}$, such that

$$\int_0^{\lambda_n} \int_0^{\lambda_n} |\theta(x,y;\alpha_1)| \; dxdy = n^2 .$$

If we consider the function $f(x,y)$ defined as follows:

$$f(x,y) = (1/n) \operatorname{sgn} \theta(x,y;\alpha_1), \quad \begin{cases} \lambda_{n-1} \leqq x \leqq \lambda_n , 0 \leqq y < \lambda_n \\ 0 \leqq x < \lambda_n , \lambda_{n-1} \leqq y < \lambda_n \\ \lambda_{n-1} \leqq x < \lambda_n , \lambda_{n-1} \leqq y < \lambda_n \end{cases}$$

and substitute this in (4.2), with p,q taking on successively the values λ_n, λ_n, we have

$$\begin{aligned} \tau(\lambda_n, \lambda_n; \alpha_1) &= \int_0^{\lambda_n} \int_0^{\lambda_n} \theta(x,y;\alpha_1) f(x,y) dxdy \\ &= \sum_{i=1}^{n} \frac{2i-1}{i} > n. \end{aligned}$$

But, $f(x,y) \to 0$ as $x,y \to \infty$. Therefore, if $\tau(p,q;\alpha_1) \to \infty$ as $p,q \to \infty$ we have a contradiction. Hence we have proved

Theorem 2 A necessary and sufficient condition for the convergence, as $p,q \to \infty$, of

$$\tau(p,q;\alpha) = \int_0^p\int_0^q \theta(x,y;\alpha)f(x,y)dxdy,$$

whenever lim f(x,y) exists, is that

$$\int_0^\infty\int_0^\infty |\theta(x,y;\alpha)|\ dxdy < K_1(\alpha), \tag{4.7}$$

where $K_1(\alpha)$ is a positive function of α defined for $\alpha \varepsilon E(\alpha)$.

5. Conditions Sufficient for Regularity

We desire that

$$\sigma = \lim_{\alpha \to \alpha_0} \int_0^\infty\int_0^\infty \phi(x,y;\alpha)f(x,y)dxdy = \int_0^\infty\int_0^\infty f(x,y)dxdy = S, \quad \alpha \varepsilon e(\alpha), \tag{5.1}$$

where $e(\alpha)$ is some subset of $E(\alpha)$ having α_0 as limit point. Let us investigate the sufficiency of the following conditions:

$$\int_0^\infty\int_0^\infty |\phi''_{xy}(x,y;\alpha)|\ dxdy < K(\alpha), \qquad E(\alpha), \tag{5.2}$$

$$\lim_{y \to \infty} \int_0^\infty |\phi'_x(x,y;\alpha)|\ dx = 0, \qquad E(\alpha), \tag{5.3}$$

$$\lim_{x \to \infty} \int_0^\infty |\phi'_y(x,y;\alpha)|\ dy = 0, \qquad E(\alpha), \tag{5.4}$$

$$\int_0^\infty\int_0^\infty |\phi''_{xy}(x,y;\alpha)|\ dxdy < K, \qquad E'(\alpha), \tag{5.5}$$

$$\lim_{\alpha \to \alpha_0} \int_0^q \left\{ \int_0^\infty |\phi''_{xy}(x,y;\alpha)|\ dx \right\} dy = 0, \qquad 0 \leqq q < \infty; \quad E^{(q)}(\alpha), \tag{5.6}$$

$$\lim_{\alpha \to \alpha_0} \int_0^p \left\{ \int_0^\infty |\phi''_{xy}(x,y;\alpha)|\ dy \right\} dx = 0, \qquad 0 \leqq p < \infty; \quad E^{(p)}(\alpha), \tag{5.7}$$

$$\lim_{\alpha \to \alpha_0} \phi(x,y;\alpha) = 1, \qquad x,y \geqq 0; \quad E''(\alpha). \tag{5.8}$$

If $S(p,q) \to 0$ as $p,q \to \infty$, by the method of §2, we have

$$\sigma(\alpha) = \int_0^\infty\int_0^\infty \phi(x,y;\alpha)f(x,y)dxdy = \int_0^\infty\int_0^\infty \phi''_{xy}(x,y;\alpha)S(x,y)dxdy. \tag{5.9}$$

In view of (5.5), for $\alpha \varepsilon E'(\alpha)$, there is an $N < \infty$, such that, for $p,q > N$, we have

$$\left| \int_0^\infty \int_0^\infty |\phi''_{xy}(x,y;\alpha)|\,dxdy - \int_0^p \int_0^q |\phi''_{xy}(x,y;\alpha)|\,dxdy \right| < \frac{\epsilon}{2A}, \qquad E'(\alpha).$$

Then

$$\left| \sigma(\alpha) - \int_0^p \int_0^q \phi''_{xy}(x,y;\alpha) S(x,y)\,dxdy \right| < \frac{\epsilon}{2} .$$

Fix p. From (5.7), there exists a set $E'''(\alpha)$, such that for $\alpha \varepsilon E'''(\alpha)$,

$$\int_0^p \left\{ \int_0^\infty |\phi''_{xy}(x,y;\alpha)|\,dy \right\} dx < \frac{\epsilon}{2A}, \qquad E'''(\alpha).$$

Therefore,

$$\left| \int_0^p \int_0^q \phi''_{xy}(x,y;\alpha) S(x,y)\,dxdy \right| < \frac{\epsilon}{2}, \qquad E'''(\alpha).$$

Hence, for $\alpha \varepsilon E'(\alpha) \cdot E'''(\alpha)$, we have

$$|\sigma(\alpha)| < \epsilon .$$

Since ϵ is arbitrary, it follows that

$$\lim_{\alpha \to \alpha_0} \sigma(\alpha) = 0 = S ,$$

and regularity is obtained in this case.

The case $S(p,q) \longrightarrow S \neq 0$ is handled as in previous work. We let

$$f(x,y) = f_1(x,y) + f_2(x,y) ,$$

where

$$\begin{aligned} f_1(x,y) &= S , && 0 \leqq x,\ y \leqq 1 , \\ &= 0 , && \text{elsewhere.} \end{aligned}$$

Then, for $p,q \gtreqless 1$,

$$\int_0^p\int_0^q \phi(x,y;\alpha)f(x,y)dxdy = S\int_0^1\int_0^1 \phi(x,y;\alpha)dxdy + \int_0^p\int_0^q \phi(x,y;\alpha)f_1(x,y)dxdy$$

The first integral on the right converges to S as $\alpha \to \alpha_0$, for $\alpha \epsilon E''(\alpha)$, because of condition (5.8). The last integral $\longrightarrow 0$ by the work above.

Therefore, in all cases, the conditions (5.2) – (5.8) are sufficient for regularity.

6 Necessity of the Conditions.

The necessity of conditions (5.2), (5.3), and (5.4) was proved in a preceding section. To prove the necessity of (5.8), we consider the function

$$\begin{aligned} f(x,y) &= 1 , && 0 \leqq x \leqq h,\ 0 \leqq y \leqq k, \\ &= 0 , && \text{elsewhere.} \end{aligned}$$

Then,

$$\lim_{\alpha \to \alpha_0} \int_0^h\int_0^k \phi(x,y;\alpha)dxdy = hk , \qquad \alpha \epsilon E''(\alpha).$$

This reduces to

$$\lim_{\alpha \to \alpha_0} \int_0^h\int_0^k \left[1-\phi(x,y;\alpha)\right] dxdy = 0 .$$

By the same reasoning as used in Chapter I, since h and k are arbitrary, we conclude that

$$\lim_{\alpha \to \alpha_0} \left[1 - \phi(x,y;\alpha)\right] \equiv 0 , \qquad 0 \leqq x,\ y < \infty ,$$

which completes the proof of the necessity of (5.8).

We have required that $\phi''_{xy}(x,y;\alpha)$ converge boundedly for x,y in a square $0 \leqq x,y \leqq z$ and $\alpha \varepsilon E_z(\alpha)$. Since the limit of $\phi(x,y;\alpha)$ is identically 1, the value of $\phi''_{xy}(x,y;\alpha_0)$ is zero, and the limit of $\phi''_{xy}(x,y;\alpha)$, for x,y in this square and $\alpha \varepsilon E_z(\alpha)$, is zero almost everywhere. Thus

$$(6.1) \qquad \lim_{\alpha \to \alpha_0} \int_0^z \int_0^z |\phi''_{xy}(x,y;\alpha)|\, dxdy = 0, \qquad \alpha \varepsilon E_z(\alpha).$$

Suppose that condition (5.5) does not hold for any subset $E'(\alpha)$ of $E(\alpha)$ having α_0 as limit point. Then there exists a sequence of α's $\{\alpha^{(m)}\}$ converging to α_0, such that

$$(6.2) \qquad F(p, \alpha^{(m)}) = \int_0^p \int_0^p |\phi''_{xy}(x,y;\alpha)|\, dxdy > m$$

for a proper choice of p.

For any particular value m_1 of m, let α_1 and p_1 be the corresponding values of α and p. Choose $q_1 > p_1$ and define

$$g(x,y) = \frac{1}{\sqrt{m_1}} \operatorname{sgn} \phi''_{xy}(x,y;\alpha_1), \qquad 0 \leqq x,y \leqq q_1$$

Let $z_1 > 0$ be such that

$$(6.3) \qquad F(q_1+z_1, \alpha_1) - F(q_1, \alpha_1) < \frac{1}{6}.$$

Let $\eta_1 > 0$ be such that for any set of points E of measure $< \eta_1$ taken from the square $0 \leqq x,y \leqq q_1 + z_1$, we have

$$(6.4) \qquad \iint_E |\phi''_{xy}(x,y;\alpha_1)|\, dxdy < \frac{1}{6}.$$

There exists an absolutely continuous function S(x,y) such that

$$(6.5) \qquad |S(x,y) - g(x,y)| < \frac{1}{6F(q_1+z_1, \alpha_1)}$$

except on a set E_1 of measure $< \eta_1$. Further

$$\max |S(x,y) - g(x,y)| \leqq \frac{2}{\sqrt{m_1}}, \qquad 0 \leqq x,y \leqq q_1 . \tag{6.6}$$

Let us require that $S(x,y)$ be zero for $x = q_1 + \frac{z_1}{2}$, $0 \leqq y \leqq q_1 + \frac{z_1}{2}$ and $0 \leqq x \leqq q_1 + \frac{z_1}{2}$, $y = q_1 + \frac{z_1}{2}$.

Now choose α_2 from the sequence $\{\alpha^{(m)}\}$ such that

$$F(q_1 + z_1, \alpha_2) < \frac{1}{6}, \tag{6.7}$$

which we can do in view of the necessity of (6.1). Let $m_2 > m_1$, and p_2 be the corresponding values of m and p for which (6.2) holds. Take $q_2 > p_2$ and such that

$$F(\infty, \alpha_2) - F(q_2, \alpha_2) = R(q_2, \alpha_2) < \frac{1}{6}, \tag{6.8}$$

which is possible in view of the necessity of (5.2).

We next define

$$g(x,y) = \frac{1}{\sqrt{m_2}} \operatorname{sgn} \phi''_{xy}(x,y;\alpha_2)$$

for

$$D(q_1 + z_1, q_2) \equiv (q_1 + z_1 \leqq x \leqq q_2,\ 0 \leqq y \leqq q_2;\ 0 \leqq x \leqq q_1 + z_1,\ q_1 + z_1 \leqq y \leqq q_2)$$

Define a z_2 and an η_2 in a fashion similar to that used for z_1 and η_1. Choose $S(x,y)$ absolutely continuous and such that

$$|S(x,y) - g(x,y)| < \frac{1}{6\, F(q_2 + z_2, \alpha_2)}, \tag{6.9}$$

over the region $D(q_1 + z_1, q_2 + z_2)$, except in a set E_2 of measure $< \eta_2$. We see that over $D(q_1 + z_1, q_2 + z_2)$,

$$\max |S(x,y) - g(x,y)| \leqq \frac{2}{\sqrt{m_2}} .$$

Making use of relations (6.3) – (6.9) for a function f(x,y) whose S(x,y) is always less than unity in absolute value, we have

$$
\begin{aligned}
& \left| \int_0^\infty \int_0^\infty \phi''_{xy}(x,y;\alpha_2) S(x,y)\,dxdy \right| \\
\geqq\ & \left| \int_0^{q_1} \int_{q_1}^{q_2} \phi''_{xy}(x,y;\alpha_2) S(x,y)\,dxdy + \int_{q_1}^{q_2} \int_0^{q_2} \phi''_{xy}(x,y;\alpha_2) S(x,y)\,dxdy \right| \\
-\ & \left| \int_0^{q_1} \int_0^{q_1} \phi''_{xy}(x,y;\alpha_2) S(x,y)\,dxdy \right| \\
-\ & \left| \int_0^\infty \int_0^\infty \phi''_{xy}(x,y;\alpha_2) S(x,y)\,dxdy - \int_0^{q_2} \int_0^{q_2} \phi''_{xy}(x,y\ \alpha_2) S(x,y)\,dxdy \right| \\
\geqq\ & \left| \int_0^{q_1} \int_{q_1}^{q_2} \phi''_{xy}(x,y;\alpha_2) g(x,y)\,dxdy + \int_{q_1}^{q_2} \int_0^{q_2} \phi''_{xy}(x,y;\alpha_2) g(x,y)\,dxdy \right| \\
-\ & \left| \int_0^{q_1} \int_{q_1}^{q_2} \phi''_{xy}(x,y;\alpha_2) \left[S - g \right] dxdy + \int_{q_1}^{q_2} \int_0^{q_2} \phi''_{xy}(x,y;\alpha_2) \left[S - g \right] dxdy \right| \\
-\ & \left| \int_0^{q_1} \int_0^{q_1} \phi''_{xy}(x,y;\alpha_2)\ S(x,y)\,dxdy \right| - R\,(q_2, \alpha_2) \\
\geqq\ & \frac{1}{\sqrt{m_2}} \int_0^{q_2} \int_0^{q_2} \left| \phi''_{xy}(x,y;\alpha_2) \right| dxdy - \frac{1}{\sqrt{m_2}} \int_0^{q_1} \int_0^{q_1} \left| \phi''_{xy}(x,y;\alpha_2) \right| dxdy \\
& - \frac{1}{6F(q_2+z_2,\alpha_2)} \int_0^{q_2} \int_0^{q_2} \left| \phi''_{xy}(x,y;\alpha_2) \right| dxdy - \frac{2}{\sqrt{m_2}} \iint_{E_2} \left| \phi''_{xy}(x,y;\alpha_2) \right| dxdy \\
& - \frac{1}{6} - \frac{1}{6} \\
>\ & \sqrt{m_2} - \frac{1}{6\sqrt{m_2}} - \frac{1}{6} - \frac{1}{3\sqrt{m_2}} - \frac{1}{6} - \frac{1}{6} \\
>\ & \sqrt{m_2} - 1.
\end{aligned}
$$

Continuing in this fashion, we define a function f(x,y), whose S(x,y) converges to zero, for which the left side of (5.9), namely,

$$\int_0^\infty \int_0^\infty \phi(x,y;\alpha) f(x,y)\,dxdy,$$

increases indefinitely over a set of values of α, having α_0 as a limit point. The choice of the marginal regions $D(q_i, q_i + z_i)$, $i = 1,2,\ldots\ldots$, enables us to define the absolutely continuous function $S(x,y)$ in a satisfactory fachion. Thus we have a contradiction, and the necessity of (5.5) is proved.

The proofs of the necessity of (5.6) and (5.7) are handled together. In view of the necessity of (5.5), there exists a $p < \infty$, such that for $\alpha \in E'(\alpha)$,

$$\int_0^\infty\int_0^\infty |\phi''_{xy}(x,y;\alpha)|\,dxdy - \int_0^p\int_0^p |\phi''_{xy}(x,y;\alpha)|\,dxdy < \frac{\epsilon}{2}\,.$$

$$\int_0^\infty\int_0^\infty |\phi''_{xy}(x,y;\alpha)|\,dxdy - \int_0^p\int_0^p |\phi''_{xy}(x,y;\alpha)|\,dxdy < \frac{\epsilon}{2}\,.$$ 1ave

From (6.1), there exists a set $E_1^{(p)}(\alpha)$ such that for $\alpha \in E_1^{(p)}(\alpha)$, we have

$$\int_0^p\int_0^p |\phi''_{xy}(x,y;\alpha)|\,dxdy < \frac{\epsilon}{2}\,.$$

$$\int_0^\infty\int_0^\infty |\phi''_{xy}(x,y;\alpha)|\,dxdy < \epsilon\,, \qquad \alpha \in e(\alpha).$$

Therefore, both conditions (5.6) and (5.7) follow.

7 Regularity for Bounded Functions.

We are interested in having

$$(7.1)\quad \tau(\infty,\infty;\alpha) = \lim_{\alpha \to \alpha_0} \int_0^\infty\int_0^\infty \theta(x,y;\alpha) f(x,y)\,dxdy = \lim_{x,y \to \infty} f(x,y).$$

Conditions analogous to those used for series, and for the single integral case, are

$$(7.2)\qquad \int_0^\infty\int_0^\infty |\theta(x,y;\alpha)|\,dxdy < K_1(\alpha), \qquad E(\alpha),$$

$$(7.3)\qquad \int_0^\infty\int_0^\infty |\theta(x,y;\alpha)|\,dxdy < K_1\,, \qquad E'(\alpha),$$

$$(7.4) \qquad \lim_{\alpha\to\alpha_0}\int_0^p\left\{\int_0^\infty|\theta(x,y:\alpha)|\,dy\right\}dx = 0\,, \qquad 0\leq p<\infty,\ E^{(p)}(\alpha),$$

$$(7.5) \qquad \lim_{\alpha\to\alpha_0}\int_0^q\left\{\int_0^\infty|\theta(x,y;\alpha)|\,dx\right\}dy = 0\,, \qquad 0\leq q<\infty,\ E^{(q)}(\alpha),$$

$$(7.6) \qquad \lim_{\alpha\to\alpha_0}\int_0^\infty\int_0^\infty\theta(x,y;\alpha)\,dxdy = 1\,, \qquad E''(\alpha).$$

Consider first the case $f(x,y) \longrightarrow 0$ as $x,y \to \infty$. Then there exists a finite N such that for x,y outside the square $0 \leq x,\ y \leq N$, $|f(x,y)| < \frac{\epsilon}{2K_1}$. This means that

$$\left|\int_0^\infty\int_0^\infty\theta(x,y;\alpha)f(x,y)dxdy - \int_0^N\int_0^N\theta(x,y;\alpha)f(x,y)dxdy\right|$$

$$= \left|\left\{\int_N^\infty\int_0^\infty+\int_N^\infty\int_N^\infty+\int_0^\infty\int_N^\infty\right\}\theta(x,y;\alpha)f(x,y)dxdy\right|$$

$$\leq \frac{\epsilon}{2K_1}\left\{\int_N^\infty\int_0^\infty+\int_N^\infty\int_N^\infty+\int_0^\infty\int_N^\infty\right\}|\theta(x,y;\alpha)|\,dxdy$$

$$< \frac{\epsilon}{2}, \qquad \alpha\in E'(\alpha).$$

We need, therefore, consider only

$$\int_0^N\int_0^N\theta(x,y;\alpha)f(x,y)dxdy.$$

Fix N. From (7.4) we can choose a set $E^{(N)}(\alpha)$ such that

$$\int_0^N\left\{\int_0^\infty|\theta(x,y;\alpha)|\,dy\right\}dx < \frac{\epsilon}{2B}, \qquad \alpha\in E^{(N)}(\alpha).$$

Hence, if we choose $\alpha \in e(\alpha) = E'(\alpha)\cdot E^{(N)}(\alpha)$, we shall have

$$\left|\int_0^\infty\int_0^\infty\theta(x,y;\alpha)f(x,y)dxdy\right| < \epsilon.$$

Thus $\tau(\infty,\infty;\alpha) \longrightarrow 0$, as $\alpha\to\alpha_0$, for $\alpha\in e(\alpha)$, a suitably chosen subset of $E(\alpha)$.

For the case where $\lim f(x,y) = A \neq 0$, consider

$$f(x,y) = A + f_1(x,y) .$$

Then

$$\lim_{x,y \to \infty} f_1(x,y) = 0 ,$$

and

$$|f_1(x,y)| < |A| + B .$$

Now

$$(7.7) \quad \int_0^\infty\int_0^\infty \theta(x,y;\alpha)f(x,y)dxdy = A\int_0^\infty\int_0^\infty \theta(x,y;\alpha)dxdy + \int_0^\infty\int_0^\infty \theta(x,y;\alpha)f_1(x,y)dxdy .$$

The first term on the right of (7.7) $\longrightarrow A$ as $\alpha \to \alpha_0$ for $\alpha \epsilon E''(\alpha)$, according to condition (7.6). The last term is the case $f(x,y) \longrightarrow 0$ already discussed. Therefore for α in some suitably chosen set

$$\lim_{\alpha \to \alpha_0} \int_0^\infty\int_0^\infty \theta(x,y;\alpha)f(x,y)dxdy = A,$$

for all cases.

The necessity of condition (7.2) has already been proved. The necessity of (7.6) follows at once by considering $f(x,y) \equiv A$.

The necessity of (7.3) – (7.5) follows readily by the procedure of Chapter I. We let

$$\phi\ (x,y;\alpha) = \int_x^\infty\int_y^\infty \theta(r,s;\alpha)dr\ ds$$

Then

$$\phi'_x\ (x,y;\alpha) = -\int_y^\infty \theta\ (x,s;\alpha)\ ds\ , \quad \phi'_y\ (x,y;\alpha) = -\int_x^\infty \theta\ (r,y;\alpha)dr,$$

and

$$\phi''_{xy}(x,y;\alpha) = \theta(x,y;\alpha) .$$

Condition (7.3) follows from

$$\int_0^\infty \int_0^\infty | \phi''_{xy}(x,y;\alpha) | \, dxdy < K,$$

and conditions (7.4) and (7.5) from

$$\lim_{\alpha \to \alpha_0} \int_0^p \left\{ \int_0^\infty | \phi''_{xy}(x,y;\alpha) | \, dy \right\} dx = 0$$

and

$$\lim_{\alpha \to \alpha_0} \int_0^q \left\{ \int_0^\infty | \phi''_{xy}(x,y;\alpha) | \, dx \right\} dy = 0 ,$$

respectively.

These conditions, therefore, as reasoned in Chapter I, are necessary for a bounded absolutely continuous function f(x,y). They are then even more necessary when the function f(x,y) is merely bounded and measurable.

We have thus completed proof of the theorem:

Theorem 4 Conditions necessary and sufficient that

$$\lim_{\alpha \to \alpha_0} \int_0^\infty \int_0^\infty \theta(x,y;\alpha) f(x,y) \, dxdy = \lim_{x,y \to \infty} f(x,y), \tag{7.1}$$

for all functions f(x,y), bounded and measurable, for which the right side of (7.1) exists, are (7.2), (7.3), (7.4), (7.5) and (7.6).

CHAPTER III

Multiple Integrals

1. Extensions.

As in the chapters on single and double integrals, we are concerned with the introduction of convergence factors into convergent integrals. For convenience of notation, we shall write

$$(1.1)\qquad \begin{cases} S[p] \text{ for } S(p_1, p_2, \dots, p_m), \\ \int_0^p \text{ for } \int_0^{p_1}\int_0^{p_2}\dots\int_0^{p_m}, \\ dx \text{ for } dx_1 dx_2 \dots dx_m . \end{cases}$$

Let

$$(1.2)\qquad S[p] = \int_0^p f[x]\,dx ,$$

with

$$(1.3)\qquad |S[p]| < A < \infty, \qquad\qquad \text{all } p \geqq 0.$$

We suppose that $S[p]$ approaches a definite limit as the p's become infinite, independently or not. Finally, let

$$(1.4)\qquad S = \int_0^\infty f[x]\,dx.$$

On the introduction of the convergence factors $\phi[x;\alpha]$, with α belonging to some convenient set $E(\alpha)$, let us write

$$(1.5)\qquad \sigma[p;\alpha] = \int_0^p f[x]\phi[x;\alpha]\,dx.$$

We desire that $\sigma[p;\alpha]$ exist for all p and all $\alpha \varepsilon E(\alpha)$, and that the limit $\sigma[\alpha]$ of $\sigma[p;\alpha]$, as the p's become infinite, exist for all $\alpha \varepsilon E(\alpha)$. For the regularity case, we require that $\sigma[\alpha]$ approach the value S, as $\alpha \to \alpha_0$, where α_0 is a limit point of the set E (α), not of the set.

As in the previous cases, we shall assume that $\phi^{(n)}_{[x]}[x;\alpha]$ exists almost everywhere and is integrable; that for every hyper-square $[0,x]$ there exists a subset $E^{x}(\alpha)$, such that as $\alpha \to \alpha_0$ for $\alpha \in E^{x}(\alpha)$, $\phi^{(n)}_{[x]}[x;\alpha]$ converges boundedly; and that $\phi[x;\alpha]$ converges uniformly to $\phi[x;\alpha_0]$ as $\alpha \to \alpha_0$.

The relation analogous to II, (2.3) is

$$\begin{aligned}
\sigma[p;\alpha] \quad = \quad & \phi[p;\alpha]\, S[p] \\
& - \int_0^{p_1} \phi'_{x_1}(x_1, p_2, p_3, \dots, p_n; \alpha) S(x_1, p_2, \dots, p_n)\, dx_1 \\
& - \int_0^{p_2} \phi'_{x_2}(p_1, x_2, p_3, \dots, p_n; \alpha) S(p_1, x_2, \dots, p_n)\, dx_2 \\
& - \dots \\
& + \int_0^{p_1}\int_0^{p_2} \phi''_{x_1 x_2}(x_1, x_2, p_3, \dots, p_n; \alpha) S(x_1, x_2, p_3, p_4, \dots, p_n)\, dx_1\, dx_2 \\
& \dots \\
& + (-1)^n \int_0^{p} \phi^{(n)}_{[x]}[x;\alpha]\, S[x]\, dx.
\end{aligned} \tag{1.6}$$

2 Regularity Conditions.

We shall omit the preliminary discussion undertaken in the single and double integral cases, but shall go at once to the regularity case. The extensions of the conditions in the double integral case are seen to be

(2.1) $\int_0^{\infty} \left| \phi^{(n)}_{[x]}[x;\alpha] \right| dx < K(\alpha)$, E (α),

(2.2) $\int_0^{\infty} \left| \phi^{(n)}_{[x]}[x;\alpha] \right| dx < K$, E' (α),

(2.3) n conditions of the type

$$\lim_{y_{2-n}\to\infty} \int_0^\infty \left| \phi'_{x_1}[x;\alpha] \right| dx = 0, \qquad E(\alpha),$$

where $y_{p-n} \to \infty$ stands for $x_p \to \infty$, $x_{p+1} \to \infty$, ..., $x_n \to \infty$,

(2.4) $\binom{n}{2}$ conditions of the type

$$\lim_{y_{3-n}\to\infty} \int_0^\infty\int_0^\infty \left| \phi''_{x_1 x_2}[x;\alpha] \right| dx_1 dx_2 = 0, \qquad E(\alpha),$$

...

(2.5) n conditions of the type

$$\lim_{\alpha\to\alpha_0} \int_0^{p_1} \left\{ \int_0^\infty\int_0^\infty \cdots \int_0^\infty \left| \phi^{(n)}_{[x]}[x;\alpha] \right| dx_2 \ldots dx_n \right\} dx_1 = 0,$$

$$0 \leqq p_1 < \infty \; ; \; E^{(p_1)}(\alpha).$$

(2.6) $$\lim_{\alpha\to\alpha_0} \phi[x;\alpha] = 1, \qquad E''(\alpha).$$

By the method of Chapter II, by means of these conditions, we can show that $\phi[x;\alpha]$ is bounded by a function of α for all x. We can also show that, for $S[x] \to 0$ as $x \to \infty$,

(2.7) $$\sigma[\alpha] = \int_0^\infty \phi^{(n)}_{[x]}[x;\alpha]\, S[x]\, dx.$$

Then, by use of conditions (2.2) and (2.5), successively, we show that

$$\lim_{\alpha\to\alpha_0} \sigma[\alpha] = 0, \qquad E''(\alpha).$$

For the case $S[x] \to S \neq 0$, exactly as in the previous work we decompose $f[x]$ into two functions, and invoke condition (2.6) as well. Thus we see that the conditions (2.1) – (2.6) are sufficient that

$$\lim_{\alpha\to\alpha_0} \sigma[\alpha] = S = \int_0^\infty f[x]\, dx, \qquad E''(\alpha).$$

The necessity of these conditions is proved in the same way as for the double integral case. Thus (2.1) goes through in the same way. The necessity of conditions (2.3), (2.4), and up to (2.5) can be proved by the method of oscillations on (1.6), as was done in Chapter II. The necessity of (2.6) is easily proved by considering a function identically a constant over an arbitrary hyper-rectangle, and zero elsewhere. Then we prove the necessity of (2.2) by a proof analogous to that for the corresponding condition for double integrals. Finally, the necessity of the conditions (2.5) is shown.

We thus have the theorem:

Theorem 1 Conditions necessary and sufficient that

$$\lim_{\alpha \to \alpha_0} \int_0^\infty \phi[x;\alpha]\, f[x]\,dx = \int_0^\infty f[x]\,dx\,, \qquad \alpha \in E''(\alpha),$$

whenever the right side exists, are that $\phi[x;\alpha]$ satisfy the conditions (2.1) — (2.6).

3 Application to Bounded Functions.

As in the previous chapters, we desire that

$$(3.1) \qquad \lim_{\alpha \to \alpha_0} \int_0^\infty \theta[x;\alpha]\, f[x]\,dx = \lim_{x \to \infty} f[x] = A\,,$$

where $f[x]$ is measurable over every hyper-rectangle $0 \leq x \leq p < \infty$, and

$$(3.2) \qquad |f[x]| < B < \infty\,, \qquad \text{all } x.$$

The conditions we shall investigate in this problem, as for the double integral case, are fewer in number than those for the preceding section. They are

$$(3.3) \qquad \int_0^\infty |\theta[x;\alpha]|\,dx < K_1(\alpha), \qquad E(\alpha),$$

$$(3.4) \qquad \int_0^\infty |\theta[x;\alpha]|\,dx < K_1, \qquad E'(\quad),$$

$$(3.5) \qquad \lim_{\alpha \to \alpha_0} \int_0^{p_1} \left[\left\{\int_0^\infty \cdots \int_0^\infty\right\} |\theta[x;\alpha]|\,dx_2 \ldots dx_n\right] dx_1 = 0, \qquad E^{(p_1)}(\alpha),$$

and $(n - 1)$ more conditions of this same type (3.5); and

$$(3.6) \qquad \lim_{\alpha \to \alpha_0} \int_0^\infty \theta[x; \alpha]\, dx = 1, \qquad E''(\alpha).$$

Consider first the case $f[x] \to 0$ as $x \to \infty$. Then there is a finite N such that for x outside the hyper-square $0 \leqq x \leqq N$, we have

$$|f[x]| < \frac{\epsilon}{2K_1}, \qquad E'(\alpha).$$

We can then write

$$\left| \int_0^\infty \theta[x; \alpha]\, f[x]\, dx \right| < \left| \int_0^N \theta[x; \alpha]\, f[x]\, dx \right| + \frac{\epsilon}{2}.$$

In view of condition (3.5), there exists a set $E^{(N)}(\alpha)$ such that for α in this set, we have

$$\int_0^N \left\{ \int_0^\infty \cdots \int_0^\infty |\theta[x; \alpha]|\, dx_2 \ldots dx_n \right\} dx_1 < \frac{\epsilon}{2B}.$$

Therefore, for $\alpha \epsilon E'(\alpha) \cdot E^{(N)}(\alpha)$, we have

$$\left| \int_0^\infty \theta[x; \alpha]\, f[x]\, dx \right| < \epsilon,$$

and, since ϵ is arbitrary,

$$\lim_{\alpha \to \alpha_0} \int_0^\infty \theta[x; \alpha]\, f[x]\, dx = 0, \qquad e(\alpha)$$

where $e(\alpha)$ is some suitably chosen subset of $E(\alpha)$, having α_0 as a limit point.

The case $f[x] \to A, \neq 0$, is handled by the method of decomposition of $f[x]$ into a sum of functions, one of which is identically A, and the other $\to 0$ as $x \to \infty$. This completes the proof of the sufficiency of the conditions (3.3) to (3.6) that θ be regular.

The necessity of condition (3.6) follows at once by considering the function f identically A. To prove the necessity of the other conditions, we proceed as in the two former chapters. We let

$$\phi(x_1, \ldots, x_n; \alpha) = \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \theta(z_1, \ldots, z_n; \alpha)\, dz_1 \ldots dz_n.$$

Then

$$\phi'_{x_1}(x_1, \ldots, x_n; \alpha) = -\int_{x_2}^{\infty} \cdots \int_{x_n}^{\infty} \theta(x_1, z_2, \ldots, z_n; \alpha) dz_2 \ldots dz_n,$$

$$\ldots$$

$$\phi^{(n)}_{[x]}(x_1, \ldots, x_n; \alpha) = (-1)^n \theta(x_1, \ldots, x_n; \alpha).$$

The ϕ function we have defined must satisfy the conditions (2.1) – (2.6) of this chapter. In the employment of these conditions, the function S[x] is absolutely continuous. Since the conditions (2.1) – (2.6) are necessary for an absolutely continuous (bounded) function, they must also be necessary when the function f[x] considered is merely bounded and measurable.

Thus we complete the proof of the theorem:

Theorem 2 Conditions necessary and sufficient that

$$\lim_{\alpha \to \alpha_0} \int_0^{\infty} \theta[x;\alpha] f[x] dx = \lim_{x \to \infty} f[x], \qquad E''(\alpha),$$

are (3.3) — (3.6) of this section.

CHAPTER IV

Summation Methods

1 Definitions.

Given any function f(t) integrable over a finite range (0,x), we say that the integral $\int_0^\infty f(t)dt$ is summable X, if the transform associated with the summability method X gives a value to the infinite integral. We usually concern ourselves with regular summability methods, i.e., methods which assign the value $\int_0^\infty f(t)dt$ to the transform, whenever this original integral has a meaning.

We shall investigate the regularity of the analogues of the methods used for series. Methods considered are Cesàro, Hölder, M. Riesz, Abel, and Nörlund. We devote a special section to this last method.

2 Regularity

The existence of the limit

$$(2.1) \qquad (C,r;f) = \lim_{x \to \infty} \int_0^x \left(1 - \frac{t}{x}\right)^r f(t)dt, \qquad r \geq 0,$$

is called* (C,r) summability of $\int_0^\infty f(t)dt$. It is seen that the $\phi(\alpha, t)$ of Chapter I is given by

$$(2.2) \qquad \phi(\alpha, t) = \left(1 - \frac{t}{\alpha}\right)^r, \qquad 0 \leq t \leq \alpha,$$

$$= 0, \qquad t > \alpha,$$

where $0 < \alpha < \infty$, and $\alpha_0 = \infty$. Also,

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* See Titchmarsh [20] p.26

$$\phi'_t(\alpha,t) = -\frac{r}{\alpha}\left(1-\frac{t}{\alpha}\right)^{r-1}, \qquad 0 \leq t < \alpha,$$

$$= 0, \qquad t > \alpha.$$

We note that $\phi'_t(\alpha,t)$ is unbounded in the neighborhood of $t = \alpha$, if $r < 1$. If, however, $\alpha > 2x$, then, for $r < 1$,

$$|\phi'_t(\alpha,t)| < \frac{r}{2^r \cdot x}, \qquad 0 \leq t \leq x.$$

Thus, the existence of the set $E_x(\alpha)$, as postulated in Chapter I, is assured.

It is easily verified that the ϕ defined by (2.2) satisfies the conditions (4.2) — (4.5) of Chapter I. Since $(C, r_2 - r_1)$, for $r_2 - r_1 \geq 0$, is a regular method, we see that (C, r_1) summability implies (C, r_2) summability whenever $r_2 \geq r_1$.

Ordinary convergence is (C,0) summability. The "summability" used by Moore [17], and by Bromwich [4], is (c,1) summability.

A definition of (C,k) summability, for k a non-negative integer, analogous to that for series, involves the limit.

$$(2.3) \qquad (C,k;f) = \lim_{x \to \infty} \frac{k!\,\Gamma(x)}{\Gamma(x+k)} \int_0^x \int_0^{y_1} \int_0^{y_2} \cdots \int_0^{y_k} f(t)dt\, dy_1\, dy_2 \ldots dy_k .$$

On integrating (2.1) by parts, for r a non-negative integer, we obtain the same multiple integral as in (2.3), with the factor outside replaced by

$$\lim_{x \to \infty} \frac{r!}{x^r} .$$

Thus, for $r = k$, The representations (2.1) and (2.3) are equivalent.

Suppose g(t) is a positive monotone function of t, increasing to infinity with t. We say that $\int_0^\infty f(t)dt$ is summable (R,g,r) —— the method of M. Riesz —— when

$$(2.4) \qquad (R,g,r;f) = \lim_{x\to\infty} \int_0^x \left\{1 - \frac{g(t)}{g(x)}\right\}^r f(t)dt, \qquad r \geq 0,$$

exists. The $\phi(\alpha,t)$ defined in this case has

$$\phi_t'(\alpha,t) = -\frac{r\,g'(t)}{g(\alpha)}\left\{1 - \frac{g(t)}{g(\alpha)}\right\}^{r-1},$$

which converges boundedly almost everywhere in (0,x) for a suitable set $E^x(\alpha)$. Since this convergence almost everywhere was all that was needed in Chapter I, we can easily show that this method is regular also.

It is noted in passing that, if g(t) = t, we have the Cesàro method, which, therefore, is a special case of the M. Riesz summability method.

The analogue of the Abel method for series yields

$$(2.5) \qquad (A,f) = \lim_{\alpha\to 1} \int_0^\infty \alpha^t f(t)dt, \qquad 0 < \alpha < 1,$$

or, what is equivalent,

$$(2.6) \qquad (A,f) = \lim_{\alpha\to 0} \int_0^\infty (1-\alpha)^t f(t)dt, \qquad 0 < \alpha < 1.$$

The convergence factor α^t (or $(1-\alpha)^t$) is easily verified to satisfy the regularity conditions of Chapter I.

For the Hölder method we define

$$H^{(0)}(x) = \int_0^x f(t)dt,$$

$$H^{(1)}(x) = \frac{1}{x}\int_0^x H^{(0)}(t)dt,$$

$$\cdots$$

$$H^{(k)}(x) = \frac{1}{x}\int_0^x H^{(k-1)}(t)dt.$$

We say that $\int_0^\infty f(t)dt$ is summable (H,k) to s, if

$$\lim_{x\to\infty} H^{(k)}(x) = S.$$

The Hölder method is regular, as is seen by observing that the $H^{(k)}$ transform is the (C,1) transform of the $H^{(k-1)}$ transform, and all C methods are regular.

3 Nörlund Summability.

We proceed to define for integrals what is analogous to Nörlund means for series. Consider the function $\gamma(t)$, real or complex-valued, of the real variable t, $0 \leqq t < \infty$, $\gamma(0) \neq 0$. Let $\gamma(t) \in L(0,x)$ for all $x \geqq 0$, and set

$$\Gamma(x) = \int_0^x \gamma(t)dt. \tag{3.1}$$

For any integral $\int f(t)dt$, with $\psi(x) = \int_0^x f(t)dt$, we form

$$\sigma(x) = \frac{\int_0^x \gamma(x-t)\psi(t)dt}{\Gamma(x)} = \frac{\Psi(x)}{\Gamma(x)}. \tag{3.2}$$

If $\sigma(x) \to \sigma$ as $x \to \infty$, we say that the integral $\int_0^\infty f(t)dt$ is summable (N, γ) to the value σ. In particular, if $\sigma = \lim_{x\to\infty}\psi(x)$, whenever this limit exists, the summation method (N, γ) is regular.

Referring to the definition of $\theta(\alpha,t)$ given in Chapter I, we see that

$$\begin{aligned} \theta(\alpha,t) &= \frac{\gamma(\alpha-t)}{\Gamma(\alpha)}, & 0 \leqq t \leqq \alpha, \\ &= 0, & t > \alpha. \end{aligned} \tag{3.3}$$

The $E(\alpha)$ corresponds to the range $0 \leqq \alpha < \infty$, and $\alpha_0 = \infty$.

Condition (6.6) of Chapter I becomes

$$\int_0^\infty \theta(\alpha, t)dt = \int_0^\alpha \frac{\gamma(\alpha - t)}{T(\alpha)}\, dt = 1, \qquad \text{all } \alpha \gtrless 0,$$

and hence is satisfied. Condition (6.3) is satisfied by definition, and condition (6.4) is equivalent to

$$(3.4) \qquad \int_0^x |\gamma(t)|\, dt < K\, |T(x)|,$$

where K is a positive constant and $x \gtrless 0$. The condition (6.5) means that

$$(3.5) \qquad \lim_{\alpha \to \infty} \frac{\gamma(\alpha - t)}{T(\alpha)} = 0, \qquad \text{all } t \lesseqgtr \alpha.$$

For the case where $T(x) \longrightarrow \infty$ with x, we shall try to replace (3.5) by the condition

$$(3.6) \qquad \lim_{x \to \infty} \frac{\gamma(x)}{T(x)} = 0.$$

Set

$$(3.7) \qquad \delta(x) = |\gamma(x)|, \qquad \Delta(x) = \int_0^x \delta(t)dt, \qquad x \gtrless 0.$$

Given (3.6), for $\epsilon > 0$, there is a p_1 such that for $q \geqslant p_1$, we have

$$\left| \frac{\gamma(q)}{T(q)} \right| < \frac{\epsilon}{K}.$$

Then choose p_2 such that

$$(3.8) \qquad \left| \frac{\gamma(k)}{T(q)} \right| < \epsilon, \qquad 0 \leqslant k < p_1, \ q \gtrless p_2 \gtrless p_1.$$

For $q \gtrless k \geqslant p_1$, we have

$$(3.9) \qquad \left| \frac{\gamma(k)}{T(q)} \right| = \left| \frac{\gamma(k)}{T(k)} \right| \cdot \left| \frac{T(k)}{T(q)} \right| < \left| \frac{\gamma(k)}{T(k)} \right| \cdot \frac{\Delta(k)}{\Delta(q)/K} < K \left| \frac{\gamma(k)}{T(k)} \right| < \epsilon.$$

A combination of the inequalities (3.8) and (3.9) shows that the right-hand side of (3.3) tends to zero as α becomes infinite

We thus have the theorem:

Theorem 1. Necessary and sufficient conditions that the definition (N, γ) shall be regular are that (3.4) and (3.5) shall hold. For the case where $\Gamma(x) \to \infty$ as $x \to \infty$, (3.4) and (3.6) serve as necessary and sufficient conditions.

4. Double Integrals.

The integral analogue of Nörlund means for double series follows in similar fashion. Given a real or complex-valued function $\gamma(x,y)$ of the real variables $0 \leqq x,\ y < \infty$, $\gamma(0,0) \neq 0$, we set

$$\Gamma(x,y) = \int_0^x \int_0^y \gamma(u,v)\,du,dv. \tag{4.1}$$

For any double integral $\iint f(u,v)dudv$, with

$$S(x,y) = \int_0^x \int_0^y f(u,v)dudv, \tag{4.2}$$

write

$$\sigma(x,y) = \frac{\int_0^x \int_0^y \gamma(x-u,\ y-v)S(u,v)dudv}{\Gamma(x,y)} = \frac{\mathcal{S}(x,y)}{\Gamma(x,y)} \tag{4.3}$$

If $\sigma(x,y)$ tends to a limit σ as $x,y \to \infty$, we say that the integral (4.2) is summable $(N; \gamma)$ to the value σ.

Corresponding to the convergence factor used in Chapter II, we may write

$$\begin{aligned} \theta(u,v;\alpha) = \theta(u,v;x,y) &= \frac{\gamma(x-u,\ y-v)}{\Gamma(x,y)}, && 0 \leqq u \leqq x,\ 0 \leqq v \leqq y \\ &= 0, && \text{elsewhere.} \end{aligned}$$

The set $E(\alpha)$ corresponds to the range $x,y \geqq 0$, and α_0 has the coordinates (∞, ∞).

It is seen from (4.4) and (4.1) that condition (II – 7.6) is satisfied for all α, and hence also as $\alpha \to \alpha_0$. Condition (II - 7.2) is satisfied from definition. Condition (II - 7.3) is equivalent to

$$(4.5) \qquad \int_0^p \int_0^q |\gamma(x,y)|\, dxdy \; < \; H \, |\Gamma(p,q)| \,,$$

where H is a positive constant and $p,q \geqq 0$. The conditions (II - 7.4) and (II - 7.5) become

$$(4.6) \qquad \lim_{x,y \to \infty} \frac{\int_0^p \{ \int_0^y |\gamma(x - u,\, y - v)|\, dv \} du}{\Gamma(x,y)} = 0, \qquad 0 \leqq p < \infty,$$

and

$$(4.7) \qquad \lim_{x,y \to \infty} \frac{\int_0^q \{ \int_0^x |\gamma(x - u,\, y - v)|\, du \} dv}{\Gamma(x,y)} = 0, \qquad 0 \leqq q < \infty \,.$$

Thus we have the theorem:

Theorem 2 Conditions necessary and sufficient for the regularity of the definition $(N; \gamma)$ of this section, for double integrals whose partial sums are bounded, are the relations (4.5), (4.6) and (4.7).

5 Multiple Integrals.

Given a real or complex-valued function $\gamma[t]$ of the n variables $0 \leqq t_1, t_2, \ldots, t_n < \infty$, $\gamma[0] \neq 0$, we set

$$(5.1) \qquad \Gamma[x] = \int_0^x \gamma[t]\, dt.$$

Consider the multiple integral $\int f[t]\, dt$, with

$$(5.2) \qquad S[x] = \int_0^x f[t]\, dt.$$

Let

$$(5.3) \qquad \sigma[x] = \frac{\int_0^x \gamma[x - t]\, S[t]\, dt}{\Gamma[x]} = \frac{\mathcal{S}[x]}{\Gamma[x]}.$$

If $\sigma[x]$ tends to a limit σ as the x's become infinite, we say that the multiple integral (5.2) is summable $[N; \gamma]$ to the value σ.

Following procedure analogous to that used for single and double integrals, we note the expression which corresponds to the $\theta[t;\alpha]$ of Chapter III. Then we carry over to this case the conditions discovered to be necessary and sufficient for regularity. We have

$$(5.4) \qquad \begin{aligned} \theta[t;\alpha] &= \frac{\gamma[x - t]}{\Gamma[x]}, && 0 \leqq t \leqq x, \\ &= 0, && \text{elsewhere.} \end{aligned}$$

From the definitions (5.4) and (5.1) we see immediately that the conditions (III -3.3) and (III - 3.6) are satisfied. The condition (III - 3.4) is equivalent to

$$(5.5) \qquad \int_0^x |\gamma[t]|\, dt < H / \Gamma[x]|,$$

where H is a positive constant and $[x] \geqq 0$. The set of conditions (III - 3.5) become the conditions

$$(5.6) \qquad \lim_{y_{1-n} \to \infty} \frac{\int_0^{y_1} \left\{ \int_0^{x_2}\int_0^{x_3} \cdots \int_0^{x_n} |\gamma(x_1, x_2, \ldots, x_n)| dx_2 dx_3 \ldots dx_n \right\} dx_1}{\Gamma[x]} = 0,$$

for $p_1 \gtrless 0$, and $(n - 1)$ similar conditions.

Thus we have the theorem:

Theorem 3 The necessary and sufficient conditions that the $[N; \gamma]$ of this section be regular for multiple integrals, whose partial integrals are uniformly bounded, are the relations (5.5) and (5.6).

6. Factorable Transforms.

Corresponding to the factorable transformations of double and multiple sequences, we consider the case where the $\gamma(x,y)$ and $\gamma[x]$ take the form

$$(6.1) \qquad \gamma(x,y) = \gamma_1(x) \cdot \gamma_2(y), \qquad 0 \leqq x, y < \infty,$$

and

$$(6.2) \qquad \gamma[x] = \gamma_1(x_1) \cdot \gamma_2(x_2) \cdots\cdots \gamma_n(x_n), \qquad 0 \leqq [x] < \infty,$$

respectively.

It is readily seen that this reduces the problem of regularity to that of making both γ_1 and γ_2, in the case of double integrals, and all of $\gamma_1, \gamma_2, \ldots, \gamma_n$, in the case of multiple integrals, satisfy the conditions of Theorem 1 of this chapter.

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