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STABILITY ANALYSIS OF LINEAR NONAUTONOMOUS LARGE SCALE SYSTEMS WITH UNCERTAIN PARAMETERS

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Abstract

Stabilization of dynamical systems is a very important problem and has received great attention. The solution of such problems can be achieved for linear autonomous small scale (centralized) systems with specific parameters using the conventional control theory. This leads to the design of a centralized controller which determines the control actions based on the centralized structure. However, with the development of modern technology, the size and complexity of the systems are increasing everyday, stabilization of large scale systems using centralized techniques is therefore not feasible. And decentralized techniques are an attractive approach for large scale systems stabilization. Our objective in this research is to investigate the stability of linear nonautonomous large scale systems with uncertain parameters. Both feedback-free and feedback control systems will be studied. The technique is based upon using a Liapunov function to disconnect and reassemble the subsystems in different ranges. So new criteria for studying and designing the finite-time or uniform stability can be developed. These criteria can also be used to design or estimate the convergence rate of the global system. In addition, since small scale (centralized) systems are subset of large scale systems, the theories developed for large scale systems will still be valid for small scale (centralized) systems. Examples are given. Application and extensions are also discussed.

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Stability Analysis of Linear Nonautonomous Large scale Systems with Uncertain Parameters

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Chapter 1

Introduction

When modern control theory [1,2] is applied to the stability analysis of dynamic systems, either for simplicity or idealization reason, it is usually handled in a centralized way. It is generally assumed that there is a single controller in the system which observes the status of the environment and then makes the best possible decisions. In spite of many successful attempts, numerous researches have shown the theory to be limited when applied to large scale physical systems [3,4,5,6,7,8] (for example, urban traffic networks, flexible manufacturing networks and power networks). Large scale systems are not simply but large versions of small scale (centralized) systems. Their essential differences are enumerated below:

(1) Presence of more than one controller or decision maker share the planning, allocation and management responsibilities, which results in decentralized control and computations.

(2) Different information is delivered to the controllers and possibly at different times.

(3) They require coordination between the operation of different controllers resulting from hierarchical or multilevel structures.

(4) The systems may operate as a "team" or in a conflicting manner. Thus, there may be a single-objective function, multi-objective function, or conflicting objective function.

Due to the importance of large scale systems, many researchers have recently addressed themselves to linear autonomous decentralized of large scale system with specific parameters. There has been an increasing number of books and publications dedicated to this subject. However, the problem of synthesizing controllers for linear nonautonomous large scale systems with uncertain parameters has not been fully analyzed. This is due to the "large", in dimensions, and "high", in complexity, of their dynamic structure. Such problems however, have great significance, consider for examples:

(1) Attempts to comprehend subtle phenomena in the environment.
(2) Sophisticated dynamic systems created due to advance of modern engineering.
(3) Physical systems functioning over a wide range of operating conditions instead of over a specific set of parameters.
(4) Increasing complexity of physical systems. (In such cases the influence of the parameter variations, time delay, time-varying and/or uncertain disturbance become important.)

Hence, an expansion of the conventional control theory in a more general framework, to account for the decentralized control of linear nonautonomous large scale systems with uncertain parameters, is necessary.

The results presented in this research are organized in the following manners:

- Chapter two discusses the previous research, in particularly, the analysis of physical systems which carry uncertain, time-varying and/or time delay parameters.

- Chapter three outlines the main goals and objective of this research.
- Chapter four contains the principle algorithm of this research. The technique is based upon using the different decomposition and aggregation procedure for the subsystems based upon the Liapunov function. As a result, we obtain low order matrix Riccati equations, producing an efficiency in establishing the stabilizing feedbacks. Also, by making use of the properties of the regular pencil and comparison principle, several stability criteria are generated. Then we can design or estimate the convergence rate of the global system.
- In chapter five, we extend the new decentralized control theory in chapter four to analyze the stability of linear bounded nonautonomous large scale systems with uncertain parameters. The basic idea is to consider the nonautonomous large scale systems with uncertain parameters as a group of autonomous subsystems with specific parameters subject to nonautonomous disturbance with uncertain parameters. By using the Liapunov function and comparison principle, we can disconnect the subsystems and reassemble them in different range. Hence, we can convert nonautonomous uncertain parameters to be specific parameters in Liapunov matrix equations. As a result, the stability criteria are easier to establish. Also, by making use of the properties of the regular pencil, we can design or estimate the convergence rate of the global system using specific parameters of low order Riccati equations.
- In chapter six, we extend further the new decentralized control theory in chapter four to analyze the stability of linear unbounded nonautonomous large scale systems with uncertain parameters.
- Chapter seven contains the numerical examples and applications of the new theories.

- A summary and conclusions are submitted in chapter eight. An outline of some outstanding problems are also presented.

Chapter 2

Literature Review

2.1. Introduction

During the past five decades, engineers have developed a number of methods for analyzing linear systems and for designing control strategies. Generally these methods are based upon the common presupposition of "centrality". Specifically, all the data available for the systems and the calculations related to the data, are considered the system as a whole. However, when considering large scale systems, the presupposition of centrality often fails to be effective either because of a lack of sufficient data or because of computing limitations. Hence, when we are considering large scale systems (for example, power networks, ecological systems and economic systems), it is helpful to introduce the concept of decentralization and decentralized control. Decentralized systems have a number of control stations. At each station, the controller observes only local system outputs and controls only local inputs. The totality of the controllers then control the large scale systems. The difficulties in the analysis and synthesis of decentralized systems however, arise due to the non-centralized structure.

2.2. Previous Investigation

There has been a large amount of researches devoted to control large scale systems in recent years. A widely used approach for solving the stability problem is decomposing the large scale systems into a number of low order interconnected local subsystems based on the Liapunov function [9,10]. In general, we can subdivide them into three main categories:

(1) Linear autonomous large scale systems with specific parameters

Different techniques [11,12,13] are now existing for the stability analysis of linear autonomous large scale systems with specific parameters. However, the most popular method is using either scalar or vector Liapunov function. In 1978, Sandell [14] surveyed the control literature of decentralization stability analysis of large scale systems. He developed a useful summary of decentralized control. Djordjevic [15] and Mahalanabis [16] suggested a vector and scalar Liapunov approach respectively. These theories depended on the choice of a system of vector or scalar Liapunov function. It is a trial and error procedure. As such, it is most effective for low order systems. Also, Popchev [17] proposed a computer algorithm which can decentralize the feedback control of the interconnected systems. However, difficulties can arise in large scale systems when we try to determine whether a dominant factor will influence the stability of the systems. The convergence rate of the global system is also difficult to control.

(2) Linear nonautonomous large scale system with specific parameters

Stability conditions for linear autonomous systems have been studied extensively for a long time. It is well known that a linear system with constant parameters is asymptotically stable, if and only if all of the eigenvalues of the system matrix have negative real parts [18]. However, this is not the case for linear non-autonomous systems [19 20 21]. Indeed, most non-autonomous systems can not be solved in close-forms. Hence, for non-autonomous systems, it is desirable to be able to investigate differential equations solutions without explicitly determining them. The second method of Liapunov is a basic tool for such studies. In general, it is not easy to find an appropriate Liapunov function. Sinha [22], Jones [23], Xu [24], Shrivastava [25,26], Abdel-Ramhan [27], and Shi [28] have tried different kinds of approaches to locate a suitable Liapunov function for a given system. However, no matter how the functions are chosen, these methods still exhibit three common shortcomings: First, for a multi-dimensional interconnected large scale systems, it is difficult to choose an appropriate Liapunov function. Second, it is hard to find the suitable feedback controller to stabilize the global system and determine or design its convergence or divergence rate. Third, since there are both constant and time-varying parameters, the system may not be uniformly stable. These methods are not convenient for finite-time stability analysis.

(3) Linear autonomous large scale systems with uncertain parameters

The system parameters in most practical problems are usually not known with great precision. At best, the ranges of values of the parameters are known. Hence, it is reasonable to explore the system stability when the parameters assume all values in their ranges. There have been a number of studies of systems with uncertain parameters. Calahan [29], ,Yeung [30] and Karl [31] have studied the characteristic polynomials of such systems. Their theories are applicable to linear systems with associated transfer functions. Xu [32] studied the special system matrix with negative diagonal elements and non-negative off-diagonal elements. Bialas [33] explored the transformation technique from the interval matrix to specific parameters matrix. Zhou [34] considered using the norm to analyze the stability of the interval matrix. However, these methods still exhibit two common shortcomings: First, for a large scale systems, it is difficult to find the feedback controller to stabilize the large scale systems. Second, it is hard to design or determine the convergence or divergence rate of the global system.

2.3. What does the previous investigation tell us ?

It is clear that none of the prior studies have attempted to study the stability of linear nonautonomous large scale systems with uncertain parameters. However, these systems are extremely important due to our advance modern technology. Previous methods have their own limitations in extending their present theories to analyze these systems, and they also have three common short comings:

(1) For a multi-dimensional interconnected large scale systems, it is difficult to choose an appropriate Liapunov function.
(2) It is hard to find the suitable feedback controller to stabilize the global system and determine or design its convergence or divergence rate.
(3) Since there are both constant and time-varying parameters, the system may not be uniformly stable. These methods are not convenient for finite-time stability analysis.

Therefore, a new concept need to be created in order to by pass the previous ideas so that we can study linear nonautonomous large scale systems with uncertain parameters more effectively.

Chapter 3

Purpose and Objective

The main goal of this research reported hereafter is to combine the scalar Liapunov function, the comparison principle and the properties of the regular pencil to study the stability of linear nonautonomous large scale systems with uncertain parameters. The objectives of the study include the followings:

(1) To develop a new decentralized control theory to analyze the stability of linear autonomous large scale systems with specific parameters.

(2) To extend the new decentralized control theory to investigate the stability of linear nonautonomous large scale systems with uncertain parameters.

(3) To estimate or design the convergence rate of the global linear nonautonomous large scale systems with uncertain parameters.

Chapter 4

Linear Autonomous Large Scale Systems with Specific Parameters

4.1. Decoupling the Interconnected systems

Given the linear autonomous large scale systems with specific parameters modeled by the following equations:

$$\dot{\overline{X}}_i = [A_{ii}]\overline{X}_i + \left\{ \sum_{j=1, j\neq i}^{N} [A_{ij}]\overline{X}_j \right\} + \{[B_{ii}][U_i]\} \tag{1}$$

for $i = 1, 2, \ldots, N$ with $X_i = (X_1^{(i)}, \cdots, X_n^{(i)}) \in R^{n_i}$ being a vector state and U_i being the local feedback controller of the subsystems. The summation term represents the effects of interaction among the ith and the other subsystems. A_{ii}, B_{ii} and A_{ij} are real matrices with specific parameters and have appropriate dimensions. It is assumed that the (A_{ii}, B_{ii}) are completely controllable.

Consider first the feedback-free interconnected systems

$$\dot{\overline{X}}_i = [A_{ii}]\overline{X}_i + \left\{ \sum_{j=1, j\neq i}^{N} [A_{ij}]\overline{X}_j \right\} \tag{2}$$

We generate a quadratic Liapunov function $V_i(X_i)$ for each subsystem

$$V_i(X_i) = X_i^T P_i X_i \tag{3}$$

where P_i is a real, symmetric, constant and positive definite matrix. Hence, the Liapunov function for the whole system (2) is [35]:

$$V(X) = \sum_{i=1}^{N} V_i(X_i)$$

or

$$V(X) = \sum_{i=1}^{N} X_i^T P_i X_i \tag{4}$$

The derivative of this function will be

$$\dot{V}(X) = \sum_{i=1}^{N} \dot{V}_i(X_i)$$

$$= \sum_{i=1}^{N} (\dot{X}_i^T P_i X_i + X_i^T P_i \dot{X}_i)$$

or

$$\dot{V}(X) = \sum_{i=1}^{N} X_i^T(A_{ii}^T P_i + P_i A_{ii})X_i + \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2X_i^T P_i A_{ij} X_j \tag{5}$$

It is possible to show that the interaction terms satisfy the following bounds (see Appendices 1 and 2).

$$\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2X_i^T P_i A_{ij} X_j \leq \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} (X_i^T P_i P_i X_i + X_j^T A_{ij}^T A_{ij} X_j) \tag{6}$$

$$\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2X_i^T P_i A_{ij} X_j \geq \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} - (X_i^T P_i P_i X_i + X_j^T A_{ij}^T A_{ij} X_j) \tag{7}$$

By substituting Equations (6) and (7) into (5), we obtain the following inequalities

$$\sum_{i=1}^{N} X_i^T q_i X_i \leq \dot{V}(X) \leq \sum_{i=1}^{N} X_i^T Q_i X_i \tag{8}$$

where
$$q_i = A_{ii}^T P_i + P_i A_{ii} - \eta_i P_i P_i - \sum_{j=1, j \neq i}^{N} A_{ji}^T A_{ji} \tag{9}$$

and
$$Q_i = A_{ii}^T P_i + P_i A_{ii} + \eta_i P_i P_i + \sum_{j=1, j \neq i}^{N} A_{ji}^T A_{ji} \tag{10}$$

where η_i is the number of non-zero A_{ij} matrix for $j = 1,2,\ldots,N$ and $j \neq i$. This leads us to the following theorems.

Theorem 1

In Equation (10), if at least one of the Q_i is a negative definite matrix and the others are either negative definite or negative semidefinite matrices for $i = 1,2,\ldots,N$, the interconnected systems which are governed by Equation (2) are asymptotically stable.

Proof:

The Liapunov function $V(X) > 0$ is chosen by the designer. From the constraint which governs the Q_i for $i = 1,2,\ldots,N$, it is assured that $\dot{V}(X) < 0$. Hence, from the stability criteria of the second method of Liapunov, the interconnected systems are asymptotically stable.

Corollary 1.1.

In Equation (10), if all the Q_i are negative semidefinite matrices for $i = 1,2,\ldots,N$, the interconnected systems which are governed by Equation (2) are stable.

Theorem 2

In Equation (10), if all the q_i are positive definite matrices for $i = 1,2,\ldots,N$, the interconnected systems which are governed by Equation (2) is unstable.

Proof:

The Liapunov function $V(X) > 0$ is chosen by the designer. From the constraint which governed the q_i for $i = 1,2,\ldots,N$, it is assured that $\dot{V}(X) > 0$. Hence from the stability criteria of the second method of Liapunov, the interconnected systems are unstable.

4.2. Feedback control and stability analysis

Consider now the stabilization of the large scale systems arising from Equation (1). The states X_i and U_i are available for feedback and the controllers are selected as follows:

$$U_i = -\frac{1}{\mu_i} R_i^{-1} B_{ii}^T P_i X_i \tag{11}$$

where $i = 1,2,..,N$. The problem is to choose the feedback elements U_i so as to ensure the stability of the interconnected systems and its convergence or divergence rate. This motivates the following theorems.

Theorem 3

If there exists a positive definite matrix P_i for a positive value of μ_i in the Riccati Equation (12) for i = 1,2,...,N, where H_i is a positive definite matrix chosen by the designer and η_i is the number of non-zero A_{ij} matrix for j = 1,2,...,N and j ≠ i. Then Equation (11) is a stabilizing feedback controller.

$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + \eta_i P_i P_i + \sum_{j=1,j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{12}$$

Proof:

From Equations (11) and (1), we have

$$\dot{X}_i = (A_{ii} - \frac{1}{\mu_i} B_{ii} R_i^{-1} B_{ii}^T P_i) X_i + \sum_{j=1,j\neq i}^{N} A_{ij} X_j \tag{13}$$

Following the same procedures of decoupling the interconnected systems as in Section 4.1., we can decompose Equation (13) and obtain the inequalities:

$$-\sum_{i=1}^{N} X_i^T h_i X_i \leq \dot{V}(X) \leq -\sum_{i=1}^{N} X_i^T \mu_i H_i X_i \tag{14}$$

where
$$-h_i = A_{ii}^T P_i + P_i A_{ii} - \eta_i P_i P_i - \sum_{j=1,j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{15}$$

and
$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + \eta_i P_i P_i + \sum_{j=1,j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{16}$$

If we iterate a positive value of μ_i in Equation (16) and find P_i is a positive definite matrix for i =1,2,..,N, then V > 0 and $\dot{V} < 0$. From the stability criteria of the second method of Liapunov, the interconnected systems are then asymptotically stable. Hence Equation (11) represents a stabilizing feedback.

Dividing Equation (14) by V(X), we obtain

$$\frac{-(X_1^T h_1 X_1 + \cdots + X_N^T h_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \le \frac{\dot{V}(X)}{V(X)} \le \frac{-(X_1^T \mu_1 H_1 X_1 + \cdots + X_N^T \mu_N H_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \qquad (17)$$

Theorem 4

Given Equation (17), we can modify the inequalities to have the form

$$\frac{V(X_o)}{\beta_M} e^{\lambda_m (t - t_o)} \le X^T X \le \frac{V(X_o)}{\beta_m} e^{\lambda_M (t - t_o)} \qquad (18)$$

where λ_M is the maximum eigenvalue of $-\mu_i H_i P_i^{-1}$ for $i = 1,2,\ldots,N$,

λ_m is the minimum eigenvalue of $-h_i P_i^{-1}$ for $i = 1,2,\ldots,N$,

β_M is the maximum eigenvalue of P_i for $i = 1,2,\ldots,N$,

β_m is the maximum eigenvalue of P_i for $i = 1,2,\ldots,N$,

and $X^T = (X_1 \;\; X_2 \;\; \cdots \;\; X_N)$.

Proof:

By using the properties of the regular pencil of quadratic form [36,37], we can show that

$$\lambda_M \ge \frac{-(X_i^T \mu_i H_i X_i)}{X_i^T P_i X_i} \qquad (19)$$

and
$$\lambda_m \leq \frac{-(X_i^T h_i X_i)}{X_i^T P_i X_i} \tag{20}$$

for i = 1,2,...,N. Since $X_i^T P_i X_i$ is a positive definite function, from Equation (19), we have

$$\lambda_M X_i^T P_i X_i \geq -X_i^T \mu_i H_i X_i \tag{21}$$

Therefore,
$$\lambda_M X_1^T P_1 X_1 \geq -X_1^T \mu_1 H_1 X_1 \tag{22}$$

$$\cdots$$

$$\lambda_M X_N^T P_N X_N \geq -X_N^T \mu_N H_N X_N \tag{23}$$

By adding the left and right sides separately and then rearranging, we obtain

$$\lambda_M \geq \frac{-(X_1^T \mu_1 H_1 X_1 + \cdots + X_N^T \mu_N H_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \tag{24}$$

In a similar procedure, from Equation (20) we will obtain

$$\lambda_m X_i^T P_i X_i \leq -X_i^T h_i X_i \tag{25}$$

Therefore,
$$\lambda_m X_1^T P_1 X_1 \leq -X_1^T h_1 X_1 \tag{26}$$

$$\cdots$$

$$\lambda_m X_N^T P_N X_N \leq -X_N^T h_N X_N \tag{27}$$

By adding the left and right sides separately and rearranging, we obtain

$$\lambda_m \leq \frac{-(X_1^T h_1 X_1 + \cdots + X_N^T h_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \tag{28}$$

From Equations (14), (24) and (28), we can obtain a modified inequalities in the form of Equation (29).

$$\lambda_m \leq \frac{\dot{V}(X)}{V(X)} \leq \lambda_M \tag{29}$$

Let X be X_o when t is t_o. Then by integrating Equation (29), we obtain the bounded exponential inequalities for the global system.

$$V(X_o)e^{\lambda_m(t - t_o)} \leq V(X) \leq V(X_o)e^{\lambda_M(t - t_o)} \tag{30}$$

or $$V(X_o)e^{\lambda_m(t - t_o)} \leq (X_1^T P_1 X_1 + X_2^T P_2 X_2 + \cdots + X_N^T P_N X_N) \leq V(X_o)e^{\lambda_M(t - t_o)} \tag{31}$$

Since, $$\beta_m X^T X \leq (X_1^T P_1 X_1 + X_2^T P_2 X_2 + \cdots + X_N^T P_N X_N) \leq \beta_M X^T X \tag{32}$$

hence, $$\frac{V(X_o)}{\beta_M} e^{\lambda_m(t - t_o)} \leq X^T X \leq \frac{V(X_o)}{\beta_m} e^{\lambda_M(t - t_o)} \tag{33}$$

Corollary 2.1.

For feedback-free interconnected systems governed by Equation (2) and decoupled in a similar manner, we will obtain the inequalities analogous to Equation (33). Hence, the global system is asymptotically stable, if

$$\lambda_M < 0 \tag{34}$$

Proof:

From Equation (33), we have

$$X^T X \leq \frac{V(X_o)}{\beta_m} e^{\lambda_M (t - t_o)} \tag{35}$$

The criterion of Equation (34) implies that

$$\operatorname*{Limit}_{t \to \infty} X^T X \leq 0 \tag{36}$$

Therefore, the global system is asymptotically stable.

Corollary 2.2.

For feedback-free interconnected systems governed by Equation (2) and decoupled in a similar manner, we will obtain the inequalities analogous to Equation (33). Hence, the global system is stable, if

$$\lambda_M = 0 \tag{37}$$

Proof:

From Equation (33), we have

$$X^T X \leq \frac{V(X_o)}{\beta_m} e^{\lambda_M (t - t_o)} \tag{38}$$

The criterion of Equation (37) implies that

$$X^TX \leq X_o^TX_o \leq K \tag{39}$$

where K is a positive number. Hence, the global system is stable.

Corollary 2.3.

For feedback-free interconnected systems governed by Equation (2) and decoupled in a similar manner, we will obtain the inequalities analogous to Equation (33). Hence, the global system is unstable, if

$$\lambda_m > 0 \tag{40}$$

Proof:

From Equation (33), we have

$$X^TX \geq \frac{V(X_o)}{\beta_M} e^{\lambda_m(t - t_o)} \tag{41}$$

The criterion of Equation (40) implies that

$$\operatorname*{Limit}_{t \to \infty} X^TX \geq \infty \tag{42}$$

Hence the global system is unstable.

Theorem 5

For feedback control interconnected systems governed by Equation (13), we can design the upper bound convergence rate of the norm of the asymptotically stable global system by choosing λ_M, $H_i = 2P_i$ and $\beta_m > \psi$ for i = 1,2,...N in Equation (16) and (18) respectively.

Proof:

By substituting $H_i = 2P_i$ in Equation (16), we obtain

$$-2\mu_i P_i = A_{ii}^T P_i + P_i A_{ii} + \eta_i P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{43}$$

Hence, if $\mu_i > 0$, there exists a positive definite matrix P_i in Equation (43) for i = 1,2,...N, from Theorem 3, we can assure that the global system is asymptotically stable. By using the properties of Theorem 4 and letting $\lambda_M = \mu_M = \mathrm{Max}\{\mu_1, \cdots, \mu_N\}$, from Equation (33), we obtain the upper bound convergence rate equation of the global system which is

$$X^T X \leq \frac{V(X_o)}{\beta_m} e^{2\lambda_M (t - t_o)} \tag{44}$$

From the giving constraints,

we get,

$$\mathrm{NORM} = \sqrt{X^T X} \leq \sqrt{\frac{V(X_o)}{\psi}} e^{\lambda_M (t - t_o)} \tag{45}$$

Theorem 6

For feedback control interconnected systems governed by Equation (13), we can design the lower bound convergence rate of the norm of the asymptotically stable global system by choosing λ_m, $h_i = 2\mu_i P_i$ and $\beta_m > \psi$ for $i = 1,2,\dots N$ in Equation (15), (16) and (18) respectively.

Proof:

By substituting $h_i = 2\mu_i P_i$ in Equations (15) and (16), we obtain

$$-2\mu_i P_i = A_{ii}^T P_i + P_i A_{ii} - \eta_i P_i P_i - \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{46}$$

$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + \eta_i P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{47}$$

Hence, if $\mu_i > 0$ and H_i is a positive definite matrix, there exists a positive definite matrix P_i in Equations (46) and (47) for $i = 1,2,\dots N$, from Theorem 3, we can assure that the global system is asymptotically stable. By using the properties of Theorem 4 and letting $\lambda_m = \mu_m = \text{Min}\{\mu_1, \cdots, \mu_N\}$, from Equation (33), we obtain the lower bound convergence rate of the global system which is

$$X^T X \geq \frac{V(X_o)}{\beta_M} e^{2\lambda_m(t - t_o)} \tag{48}$$

From the giving constraints,

we get,
$$\text{NORM} = \sqrt{X^T X} \geq \sqrt{\frac{V(X_o)}{\psi}}\, e^{\lambda_m(t - t_o)} \tag{49}$$

Chapter 5

Linear Bounded Nonautonomous Large Scale Systems with Uncertain Parameters

5.1 Decoupling the interconnected systems

Given the linear bounded nonautonomous large scale systems with uncertain parameters is modelled by the following equations:

$$\dot{\overline{X}}_i = [A_{ii}]\overline{X}_i + \left\{ \sum_{j=1, j\neq i}^{N} [A_{ij}]\,\overline{X}_j \right\} + \left\{ \sum_{j=1}^{N} \left[\Delta A_{ij}(t)\right]\overline{X}_j \right\} + \{[B_{ii}][U_i]\} \qquad (50)$$

where $i = 1,2,\ldots,N$ with $X_i = (X_1^{(i)}, \cdots, X_n^{(i)}) \in R^{n_i}$ being a vector state and U_i being the local feedback control of the subsystems. The summation term represents the effects of interaction among the ith and the other subsystems. A_{ii}, B_{ii} and A_{ij} are real matrices with specific parameters and have appropriate dimensions. However, the matrices $\Delta A_{ii}(t)$ and $\Delta A_{ij}(t)$ are also real but nonautonomous with uncertain parameters. It is assumed that the (A_{ii}, B_{ii}) are completely controllable.

Consider first the feedback-free interconnected systems

$$\dot{\overline{X}}_i = [A_{ii}]\overline{X}_i + \left\{ \sum_{j=1, j\neq i}^{N} [A_{ij}]\,\overline{X}_j \right\} + \left\{ \sum_{j=1}^{N} \left[\Delta A_{ij}(t)\right]\overline{X}_j \right\} \qquad (51)$$

We generate a quadratic Liapunov function $V_i(X_i)$ for each subsystem

$$V_i(X_i) = X_i^T P_i X_i \tag{52}$$

where P_i is a real, symmetric, constant and positive definite matrix. Hence, the Liapunov function for the whole system (51) is [35]:

$$V(X) = \sum_{i=1}^{N} V_i(X_i) \tag{53}$$

or

$$V(X) = \sum_{i=1}^{N} X_i^T P_i X_i \tag{54}$$

The derivative of this function will be

$$\dot{V}(X) = \sum_{i=1}^{N} \dot{V}_i(X_i) = \sum_{i=1}^{N} (\dot{X}_i^T P_i X_i + X_i^T P_i \dot{X}_i) \tag{55}$$

or

$$\dot{V}(X) = \sum_{i=1}^{N} X_i^T (A_{ii}^T P_i + P_i A_{ii}) X_i + \left\{ \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} 2 X_i^T P_i A_{ij} X_j \right\} + \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} 2 X_i^T P_i \Delta A_{ij}(t) X_j \right\} \tag{56}$$

It is possible to show that the interaction terms satisfy the following bounds (see Appendices 1 to 4).

$$\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} 2 X_i^T P_i A_{ij} X_j \leq \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} (X_i^T P_i P_i X_i + X_j^T A_{ij}^T A_{ij} X_j) \tag{57}$$

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} 2X_i^TP_iA_{ij}X_j \geq \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} -(X_i^TP_iP_iX_i + X_j^TA_{ij}^TA_{ij}X_j) \tag{58}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^TP_i\Delta A_{ij}(t)X_j \leq \sum_{i=1}^{N}\sum_{j=1}^{N}\left\{K_{ij}X_i^TP_iP_iX_i + X_j^T\Big(\sum_{n=1}^{K_{ij}} D_{nij}^{Max\,T}D_{nij}^{Max}\Big)X_j\right\} \tag{59}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^TP_i\Delta A_{ij}(t)X_j \geq \sum_{i=1}^{N}\sum_{j=1}^{N}-\left\{K_{ij}X_i^TP_iP_iX_i + X_j^T\Big(\sum_{n=1}^{K_{ij}} D_{nij}^{Max\,T}D_{nij}^{Max}\Big)X_j\right\} \tag{60}$$

Substituting Equations (57) to (60) into (56), we obtain the following inequalities

$$\sum_{i=1}^{N} X_i^Tq_iX_i \leq \dot{V}(X) \leq \sum_{i=1}^{N} X_i^TQ_iX_i \tag{61}$$

where $$Q_i = A_{ii}^TP_i + P_iA_{ii} + \Big(\eta_i + \sum_{j=1}^{N} K_{ij}\Big)P_iP_i + \sum_{j=1,j\neq i}^{N} A_{ji}^TA_{ji} + \sum_{j=1}^{N}\sum_{n=1}^{K_{ji}} D_{nji}^{Max\,T}D_{nji}^{Max} \tag{62}$$

and $$q_i = A_{ii}^TP_i + P_iA_{ii} - \Big(\eta_i + \sum_{j=1}^{N} K_{ij}\Big)P_iP_i - \sum_{j=1,j\neq i}^{N} A_{ji}^TA_{ji} - \sum_{j=1}^{N}\sum_{n=1}^{K_{ji}} D_{nji}^{Max\,T}D_{nji}^{Max} \tag{63}$$

where η_i is the number of non-zero A_{ij} matrix for $j = 1,2,\ldots,N$ and $j \neq i$. This leads us to the following theorems.

Theorem 7

In Equation (62), if at least one of the Q_i is a negative definite matrix and the others are either negative definite or negative semidefinite matrices for i = 1,2,...,N, the interconnected systems which are governed by Equation (51) are asymptotically stable.

Proof:

The Liapunov function $V(X) > 0$ is chosen by the designer. From the constraint which governs the Q_i for i = 1,2,...,N, it is assured that $\dot{V}(X) < 0$. Hence, from the stability criteria of the second method of Liapunov, the interconnected systems are asymptotically stable.

Corollary 7.1.

In Equation (62), if all the Q_i are negative semidefinite matrices for i = 1,2,...,N, the interconnected systems which are governed by Equation (51) are stable.

Theorem 8

In Equation (63), if all the q_i are positive definite matrices for i = 1,2,...,N, the interconnected systems which are governed by Equation (51) is unstable.

Proof:

The Liapunov function $V(X) > 0$ is chosen by the designer. From the constraint which governed the q_i for i = 1,2,...,N, it is assured that $\dot{V}(X) > 0$. Hence from the stability criteria of the second method of Liapunov, the interconnected systems are unstable.

5.2 Feedback control and stability analysis

Consider now the stabilization of the large scale systems arising from Equation (50). The states X_i and U_i are available for feedback and the controller are selected as follows:

$$U_i = -\frac{1}{\mu_i} R_i^{-1} B_{ii}^T P_i X_i \tag{64}$$

where i = 1,2,..,N. The problem is to choose the feedback elements U_i so as to ensure the stability of the interconnected systems and its convergence rate. This motivates the following theorems.

Theorem 9

If there exists a positive definite matrix P_i for a positive value of μ_i in the Riccati Equation (65) for i = 1,2,...,N, where H_i is a positive definite matrix chosen by the designer and η_i is the number of non-zero A_{ij} matrix for j = 1,2,...,N and j ≠ i. Then Equation (64) is a stabilizing feedback controller.

$$\begin{aligned} -\mu_i H_i = {} & A_{ii}^T P_i + P_i A_{ii} + (\eta_i + \sum_{j=1}^{N} K_{ij}) P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} + \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} \\ & - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \end{aligned} \tag{65}$$

Proof:

From Equations (15) and (1), we have

$$\dot{X}_i = \left(A_{ii} - \frac{1}{\mu_i} B_{ii} R_i^{-1} B_{ii}^T P_i \right) X_i + \left\{ \sum_{j=1, j\neq i}^{N} A_{ij} X_j \right\} + \left\{ \sum_{j=1}^{N} \Delta A_{ij}(t) X_j \right\} \tag{66}$$

Following the same procedures of decoupling the interconnected systems as in Section 5.1, we can decompose Equation (66) and obtain the inequalities:

$$-\sum_{i=1}^{N} X_i^T h_i X_i \leq \dot{V}(X) \leq -\sum_{i=1}^{N} X_i^T \mu_i H_i X_i \tag{67}$$

where
$$-h_i = A_{ii}^T P_i + P_i A_{ii} - \left(\eta_i + \sum_{j=1}^{N} K_{ij}\right) P_i P_i - \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} - \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max\,T} D_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{68}$$

and
$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + \left(\eta_i + \sum_{j=1}^{N} K_{ij}\right) P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} + \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max\,T} D_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{69}$$

If we iterate a positive value of μ_i in Equation (69) and find P_i is a positive definite matrix for i =1,2,..,N, then $V > 0$ and $\dot{V} < 0$. From the stability criteria of the second method of Liapunov, the interconnected systems are then asymptotically stable when $D_{nii} = D_{nii}^{Max}$. In order to prove the stability condition still valid when $D_{nii} \leq D_{nii}^{Max}$, let us arrange Equation (69). So, we obtain

$$-\mu_i H_i = \left[A_{ii}^T P_i + P_i A_{ii} + \Big(\eta_i + \sum_{j=1}^{N} K_{ij}\Big) P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \right] + \left[\sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} \right] \tag{70}$$

Since $\left[\sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} \right]$ is a diagonal matrix with positive diagonal elements, hence, from the properties of extremizing quadratic forms : The Min-max Principle [38], we get

$$X_i^T \left[\sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} \right] X_i \geq X_i^T \left[\sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nii}^T D_{nii} \right] X_i \tag{71}$$

It implies

$$-\mu_i X_i^T H_i X_i \geq X_i^T \left[A_{ii}^T P_i + P_i A_{ii} + \Big(\eta_i + \sum_{j=1}^{N} K_{ij}\Big) P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \right] X_i + X_i^T \left[\sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nii}^T D_{nii} \right] X_i \tag{72}$$

This completes the proof.

Dividing Equation (67) by V(X), we obtain

$$\frac{-(X_1^T h_1 X_1 + \cdots + X_N^T h_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \leq \frac{\dot{V}(X)}{V(X)} \leq \frac{-(X_1^T \mu_1 H_1 X_1 + \cdots + X_N^T \mu_N H_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \tag{73}$$

Theorem 10

Given Equation (73), we can modify the inequalities to have the form

$$\frac{V(X_o)}{\beta_M}e^{\lambda_m(t-t_o)} \le X^TX \le \frac{V(X_o)}{\beta_m}e^{\lambda_M(t-t_o)} \tag{74}$$

where λ_M is the maximum eigenvalue of $-\mu_iH_iP_i^{-1}$ for i = 1,2,...,N,

λ_m is the minimum eigenvalue of $-h_iP_i^{-1}$ for i = 1,2,...,N,

β_M is the maximum eigenvalue of P_i for i = 1,2,...,N,

β_m is the maximum eigenvalue of P_i for i = 1,2,...,N,

and $X^T = (X_1 \;\; X_2 \;\; \cdots \;\; X_N)$.

Proof:

By using the properties of the regular pencil of quadratic form [37,38], we can show that

$$\lambda_M \ge \frac{-(X_i^T\mu_iH_iX_i)}{X_i^TP_iX_i} \tag{75}$$

and

$$\lambda_m \le \frac{-(X_i^Th_iX_i)}{X_i^TP_iX_i} \tag{76}$$

for i = 1,2,...,N. Since $X_i^TP_iX_i$ is a positive definite function, from Equation (75), we have

$$\lambda_MX_i^TP_iX_i \ge -X_i^T\mu_iH_iX_i \tag{77}$$

Therefore, $$\lambda_M X_1^T P_1 X_1 \geq -X_1^T \mu_1 H_1 X_1 \tag{78}$$

$$\cdots$$

$$\lambda_M X_N^T P_N X_N \geq -X_N^T \mu_N H_N X_N \tag{79}$$

By adding the left and right sides separately and then rearranging, we obtain

$$\lambda_M \geq \frac{-(X_1^T \mu_1 H_1 X_1 + \cdots + X_N^T \mu_N H_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \tag{80}$$

In a similar procedure, from Equation (76) we will obtain

$$\lambda_m X_i^T P_i X_i \leq -X_i^T h_i X_i \tag{81}$$

Therefore, $$\lambda_m X_1^T P_1 X_1 \leq -X_1^T h_1 X_1 \tag{82}$$

$$\cdots$$

$$\lambda_m X_N^T P_N X_N \leq -X_N^T h_N X_N \tag{83}$$

By adding the left and right sides separately and rearranging, we obtain

$$\lambda_m \leq \frac{-(X_1^T h_1 X_1 + \cdots + X_N^T h_N X_N)}{X_1^T P_1 X_1 + \cdots + X_N^T P_N X_N} \tag{84}$$

From Equations (73), (80) and (84), we can obtain a modified inequalities in the form of Equation (85).

$$\lambda_m \leq \frac{\dot{V}(X)}{V(X)} \leq \lambda_M \tag{85}$$

Let X be X_o when t is t_o. Then by integrating Equation (85), we obtain the bounded exponential inequalities for the global system.

$$V(X_o)e^{\lambda_m(t-t_o)} \le V(X) \le V(X_o)e^{\lambda_M(t-t_o)} \tag{86}$$

or $$V(X_o)e^{\lambda_m(t-t_o)} \le (X_1^T P_1 X_1 + X_2^T P_2 X_2 + \cdots + X_N^T P_N X_N) \le V(X_o)e^{\lambda_M(t-t_o)} \tag{87}$$

since, $$\beta_m X^T X \le (X_1^T P_1 X_1 + X_2^T P_2 X_2 + \cdots + X_N^T P_N X_N) \le \beta_M X^T X \tag{88}$$

Hence, $$\frac{V(X_o)}{\beta_M}e^{\lambda_m(t-t_o)} \le X^T X \le \frac{V(X_o)}{\beta_m}e^{\lambda_M(t-t_o)} \tag{89}$$

Corollary 10.1.

For feedback-free interconnected systems governed by Equation (51) and decoupled in a similar manner, we will obtain the inequalities analogous to Equation (89). Hence, the global system is asymptotically stable, if

$$\lambda_M < 0 \tag{90}$$

Proof:

From Equation (89), we have

$$X^T X \le \frac{V(X_o)}{\beta_m}e^{\lambda_M(t-t_o)} \tag{91}$$

The criterion of Equation (90) implies that

$$\underset{t\to\infty}{\text{Limit}}\ X^TX \le 0 \tag{91}$$

Therefore, the global system is asymptotically stable.

Corollary 10.2.

For feedback-free interconnected systems governed by Equation (51) and decoupled in a similar manner, we will obtain the inequalities analogous to Equation (89). Hence, the global system is stable, if

$$\lambda_M = 0 \tag{93}$$

Proof:

From Equation (89), we have

$$X^TX \le \frac{V(X_o)}{\beta_m} e^{\lambda_{M}(t - t_o)} \tag{94}$$

The criterion of Equation (93) implies that

$$X^TX \le X_o^TX_o \le K \tag{95}$$

where K is a positive number. Hence, the global system is stable.

Corollary 10.3.

For feedback-free interconnected systems governed by Equation (51) and decoupled in a similar manner, we will obtain the inequalities analogous to Equation (89). Hence, the global system is unstable, if

$$\lambda_m > 0 \tag{96}$$

Proof:

From Equation (89), we have

$$X^T X \geq \frac{V(X_o)}{\beta_M} e^{\lambda_m (t - t_o)} \tag{97}$$

The criterion of Equation (96) implies that

$$\operatorname*{Limit}_{t \to \infty} X^T X \geq \infty \tag{98}$$

Hence the global system is unstable.

Theorem 11

For feedback control interconnected systems governed by Equation (50), we can design the upper bound convergence rate of the norm of the asymptotically stable global system by choosing λ_M, $H_i = 2P_i$ and $\beta_m > \psi$ for $i = 1,2,\ldots N$ in Equation (69) and (74) respectively.

Proof:

By substituting $H_i = 2P_i$ in Equation (69), we obtain

$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + (\eta_i + \sum_{j=1}^{N} K_{ij}) P_i P_i + \sum_{j=1,j\neq i}^{N} A_{ji}^T A_{ji} + \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \qquad (99)$$

Hence, if $\mu_i > 0$, there exists a positive definite matrix P_i in Equation (99) for i = 1,2,...N, from Theorem 9, we can assure that the global system is asymptotically stable. By using the properties of Theorem 10 and letting $\lambda_M = \mu_M = \text{Max}\{\mu_1, \cdots, \mu_N\}$, from Equation (74), we obtain the upper bound convergence rate equation of the global system which is

$$X^T X \leq \frac{V(X_o)}{\beta_m} e^{2\lambda_M (t - t_o)} \qquad (100)$$

From the giving constraints,

we get,

$$\text{NORM} = \sqrt{X^T X} \leq \sqrt{\frac{V(X_o)}{\psi}} e^{\lambda_M (t - t_o)} \qquad (101)$$

Theorem 12

For feedback control interconnected systems governed by Equation (50), we can design the lower bound convergence rate of the norm of the asymptotically stable global system by choosing λ_m, $h_i = 2\mu_i P_i$ and $\beta_m > \psi$ for i = 1,2,...N in Equation (68), (69) and (74) respectively.

Proof:

By substituting $h_i = 2\mu_i P_i$ in Equations (68) and (69), we obtain

$$-2\mu_i P_i = A_{ii}^T P_i + P_i A_{ii} - (\eta_i + \sum_{j=1}^{N} K_{ij}) P_i P_i - \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} - \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \qquad (102)$$

$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + (\eta_i + \sum_{j=1}^{N} K_{ij}) P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} + \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \qquad (103)$$

Hence, if $\mu_i > 0$ and H_i is a positive definite matrix, there exists a positive definite matrix P_i in Equations (102) and (103) for i = 1,2,...N, from Theorem 9, we can assure that the global system is asymptotically stable. By using the properties of Theorem 10 and letting $\lambda_m = \mu_m = \mathrm{Min}\{ \mu_1, \cdots, \mu_N \}$, from Equation (74), we obtain the lower bound convergence rate of the global system which is

$$X^T X \geq \frac{V(X_o)}{\beta_M} e^{2\lambda_m (t - t_o)} \qquad (104)$$

From the giving constraints,

we get,

$$\mathrm{NORM} = \sqrt{X^T X} \geq \sqrt{\frac{V(X_o)}{\psi}}\, e^{\lambda_m (t - t_o)} \qquad (105)$$

Chapter 6

Linear Unbounded Nonautonomous Large Scale Systems with Uncertain Parameters

Given the linear nonautonomous large scale systems with uncertain parameters is modelled by the following equations:

$$\dot{\bar{X}}_i = [A_{ii}]\bar{X}_i + \left\{\sum_{j=1,j\neq i}^{N} [A_{ij}]\bar{X}_j\right\} + \left\{\sum_{j=1}^{N} \left[\Delta A_{ij}(t)\right]\bar{X}_j\right\} + \left\{\sum_{j=1}^{N} \left[\Delta\Omega_{ij}(t)\right]\bar{X}_j\right\} + \{[B_{ii}][U_i]\} \qquad (106)$$

where i = 1,2,...,N with $X_i = (X_1^{(i)}, \cdots, X_n^{(i)}) \in R^{n_i}$ being a vector state, U_i being the local feedback control of the subsystems. The summation term represents the effects of interaction among the ith and the other subsystems. A_{ii}, B_{ii} and A_{ij} are real matrices with specific parameters and $\Delta A_{ij}(t)$ is a real matrix with bounded time-varying and uncertain parameters. However, the matrices $\Delta\Omega_{ij}(t)$ are also real matrix but with unbounded time-varying and uncertain parameters for i = 1,2,...,N. It is assumed that the (A_{ii}, B_{ii}) are completely controllable.

Consider now the states X_i and U_i are available for feedback and the controllers are selected as follows:

$$U_i = -\frac{1}{\mu_i}R_i^{-1}B_{ii}^{T}P_iX_i - \frac{1}{\varepsilon_i}r_i^{-1}B_{ii}^{T}P_iX_i \qquad (107)$$

where i = 1,2,..,N. The problem is to choose the feedback elements U_i so as to ensure the stability of the interconnected systems and its convergence rate. Let us investigate them in details.

From Equations (106) and (107), we have

$$\dot{X}_i = \left(A_{ii} - \frac{1}{\mu_i} B_{ii} R_i^{-1} B_{ii}^T P_i \right) X_i + \left\{ \sum_{j=1, j\neq i}^{N} A_{ij} X_j \right\} + \left\{ \sum_{j=1}^{N} \Delta A_{ij}(t) X_j \right\}$$
$$+ \left\{ \sum_{j=1}^{N} \Delta\Omega_{ij}(t) X_j - \frac{1}{\varepsilon_i} B_{ii} r_i^{-1} B_{ii}^T P_i X_i \right\} \qquad (108)$$

We generate a quadratic Liapunov function $V_i(X_i)$ for each subsystem

$$V_i(X_i) = X_i^T P_i X_i \qquad (109)$$

where P_i is a real, symmetric, constant and positive definite matrix. Hence, the Liapunov function for the whole system (108) is [35]:

$$V(X) = \sum_{i=1}^{N} V_i(X_i) \qquad (110)$$

or

$$V(X) = \sum_{i=1}^{N} X_i^T P_i X_i \qquad (111)$$

The derivative of this function will be

$$\dot{V}(X) = \sum_{i=1}^{N} \dot{V}_i(X_i)$$

$$= \sum_{i=1}^{N} (\dot{X}_i^T P_i X_i + X_i^T P_i \dot{X}_i) \qquad (112)$$

or $$\dot{V}(X) = \sum_{i=1}^{N} X_i^T(A_{ii}^T P_i + P_i A_{ii})X_i + \left\{\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2X_i^T P_i A_{ij} X_j\right\}$$

$$+ \left\{\sum_{i=1}^{N} \sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j\right\} + + \left\{\sum_{i=1}^{N} \sum_{j=1}^{N} 2X_i^T P_i \Delta\Omega_{ij}(t) X_j\right\}$$

$$- \left\{\sum_{i=1}^{N} X_i^T(\frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i)X_i\right\} - \left\{\sum_{i=1}^{N} X_i^T(\frac{2}{\varepsilon_i} P_i B_{ii} r_i^{-1} B_{ii}^T P_i)X_i\right\} \qquad (113)$$

It is possible to show that the interaction terms satisfy the following bounds (see Appendices 1 to 6).

$$\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2X_i^T P_i A_{ij} X_j \leq \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} (X_i^T P_i P_i X_i + X_j^T A_{ij}^T A_{ij} X_j) \qquad (114)$$

$$\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2X_i^T P_i A_{ij} X_j \geq \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} - (X_i^T P_i P_i X_i + X_j^T A_{ij}^T A_{ij} X_j) \qquad (115)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \left\{K_{ij} X_i^T P_i P_i X_i + X_j^T(\sum_{n=1}^{K_{ij}} D_{n_{ij}}^{Max^T} D_{n_{ij}}^{Max})X_j\right\} \qquad (116)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j \geq \sum_{i=1}^{N} \sum_{j=1}^{N} -\left\{K_{ij} X_i^T P_i P_i X_i + X_j^T(\sum_{n=1}^{K_{ij}} D_{n_{ij}}^{Max^T} D_{n_{ij}}^{Max})X_j\right\} \qquad (117)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} 2X_i^T P_i \Delta\Omega_{ij}(t) X_j \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \left\{M_{ij} X_i^T P_i P_i X_i\right\} + \sum_{j=1}^{N} \left\{X_j^T(\sum_{i=1}^{N} \sum_{n=1}^{M_{ij}} E_{n_{ij}}^{Max^T} E_{n_{ij}}^{Max} X_j\right\} \qquad (118)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} 2X_i^T P_i \Delta\Omega_{ij}(t) X_j \geq -\sum_{i=1}^{N} \sum_{j=1}^{N} \left\{M_{ij} X_i^T P_i P_i X_i\right\} - \sum_{j=1}^{N} \left\{X_j^T(\sum_{i=1}^{N} \sum_{n=1}^{M_{ij}} E_{n_{ij}}^{Max^T} E_{n_{ij}}^{Max} X_j\right\} \qquad (119)$$

Substituting Equations (114) to (119) into (113), we obtain the following inequalities

$$\sum_{i=1}^{N} X_i^T q_i X_i \le \dot{V}(X) \le \sum_{i=1}^{N} X_i^T Q_i X_i \tag{120}$$

where

$$Q_i = A_{ii}^T P_i + P_i A_{ii} + \left\{ \eta_i + \sum_{j=1}^{N} (K_{ij} + M_{ij}) \right\} P_i P_i + \sum_{j=1, j \neq i}^{N} A_{ji}^T A_{ji} + \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max}$$

$$+ \sum_{j=1}^{N} \sum_{n=1}^{M_{ji}} E_{nji}^{Max^T} E_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i - \frac{2}{\varepsilon_i} P_i B_{ii} r_i^{-1} B_{ii}^T P_i \tag{121}$$

and

$$q_i = A_{ii}^T P_i + P_i A_{ii} - \left\{ \eta_i + \sum_{j=1}^{N} (K_{ij} + M_{ij}) \right\} P_i P_i - \sum_{j=1, j \neq i}^{N} A_{ji}^T A_{ji} - \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max}$$

$$- \sum_{j=1}^{N} \sum_{n=1}^{M_{ji}} E_{nji}^{Max^T} E_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i - \frac{2}{\varepsilon_i} P_i B_{ii} r_i^{-1} B_{ii}^T P_i \tag{122}$$

where η_i is the number of non-zero A_{ij} matrix for $j = 1,2,\ldots,N$.

Since $\sum_{i=1}^{N} \sum_{n=1}^{M_{ji}} \left[E_{nji}^{Max^T} E_{nji}^{Max} \right]$ is a diagonal matrix with positive diagonal elements, hence,from the properties of extremizing quadratic forms. We get

$$\beta_i(t) X_i^T [I] X_i \ge X_i^T \sum_{i=1}^{N} \sum_{n=1}^{M_{ji}} \left[E_{nji}^{Max^T} E_{nji}^{Max} \right] X_i \tag{123}$$

and $$-\beta_i(t)X_i^T[I]\,X_i \leq -X_i^T\sum_{i=1}^{N}\sum_{n=1}^{M_{ji}}\left[E_{nji}^{Max^T}E_{nji}^{Max}\right]X_i \tag{124}$$

where $\beta_i(t)$ is a positive scalar. Its magnitude is equal or greater than the maximum diagonal entities of $\sum_{i=1}^{N}\sum_{n=1}^{M_{ji}}\left[E_{nji}^{Max^T}E_{nji}^{Max}\right]$. Also, from the above inequalities, we can modify (121) and (122) as:

$$\begin{aligned} Q_i = {} & A_{ii}^TP_i + P_iA_{ii} + \left\{\eta_i + \sum_{j=1}^{N}(K_{ij}+M_{ij})\right\}P_iP_i + \sum_{j=1,j\neq i}^{N} A_{ji}^TA_{ji} + \sum_{j=1}^{N}\sum_{n=1}^{K_{ji}} D_{nji}^{Max^T}D_{nji}^{Max} \\ & + \beta_i(t)[I] - \frac{2}{\mu_i}P_iB_{ii}R_i^{-1}B_{ii}^TP_i - \frac{2}{\varepsilon_i}P_iB_{ii}r_i^{-1}B_{ii}^TP_i \end{aligned} \tag{125}$$

and

$$\begin{aligned} q_i = {} & A_{ii}^TP_i + P_iA_{ii} - \left\{\eta_i + \sum_{j=1}^{N}(K_{ij}+M_{ij})\right\}P_iP_i - \sum_{j=1,j\neq i}^{N} A_{ji}^TA_{ji} - \sum_{j=1}^{N}\sum_{n=1}^{K_{ji}} D_{nji}^{Max^T}D_{nji}^{Max} \\ & - \beta_i(t)[I] - \frac{2}{\mu_i}P_iB_{ii}R_i^{-1}B_{ii}^TP_i - \frac{2}{\varepsilon_i}P_iB_{ii}r_i^{-1}B_{ii}^TP_i \end{aligned} \tag{126}$$

This motivates the following theorem.

Theorem 13

If we choose $r_i^{-1} = \beta_i\sigma_i^{-1}$, there exists a positive definite matrix P_i for a positive value of μ_i and ε_i in the Riccati Equation (127) and (128) for i = 1,2,...,N, where H_i and G_i is a positive definite matrix chosen by the designer and η_i is the number of non-zero A_{ij} matrix for j = 1,2,...,N. Then Equation (107) is a stabilizing feedback controller.

$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + \left\{ \eta_i + \sum_{j=1}^{N} (K_{ij} + M_{ij}) \right\} P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} + \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{127}$$

$$-\varepsilon_i G_i = [I] - \frac{2}{\varepsilon_i} P_i B_{ii} \sigma_i^{-1} B_{ii}^T P_i \tag{128}$$

Proof:

Let

$$Q_i = -\mu_i H_i - \beta_i \varepsilon_i G_i \tag{129}$$

From Equation (125) and (129), we assume

$$-\mu_i H_i = A_{ii}^T P_i + P_i A_{ii} + \left\{ \eta_i + \sum_{j=1}^{N} (K_{ij} + M_{ij}) \right\} P_i P_i + \sum_{j=1, j\neq i}^{N} A_{ji}^T A_{ji} + \sum_{j=1}^{N} \sum_{n=1}^{K_{ji}} D_{nji}^{Max^T} D_{nji}^{Max} - \frac{2}{\mu_i} P_i B_{ii} R_i^{-1} B_{ii}^T P_i \tag{130}$$

and

$$-\beta_i(t) \varepsilon_i G_i = \beta_i(t)[I] - \frac{2}{\varepsilon_i} P_i B_{ii} r_i^{-1} B_{ii}^T P_i \tag{131}$$

Since $r_i^{-1} = \beta_i \sigma_i^{-1}$, we can modify Equation (131) as

$$-\varepsilon_i G_i = [I] - \frac{2}{\varepsilon_i} P_i B_{ii} \sigma_i^{-1} B_{ii}^T P_i \tag{132}$$

If we iterate a positive value of μ_i and ε_i in Equation (130) and (132), there exists a positive definite matrix P_i for i =1,2,..,N, then $V > 0$ and $\dot{V} < 0$. From the stability criteria of the second method of Liapunov, the interconnected systems are then asymptotically stable. Hence Equation (107) represents a stabilizing feedback controller.

Chapter 7

Examples

Example 1

Consider the feedback control linear autonomous small scale (centralized) system with uncertain parameters whose governing dynamic equations are given by:

$$\dot{X} = \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1+A & 5 \\ -4 & 0.5+B & 2 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U \qquad (131)$$

where $-2 \le A \le 2$ and $-1 \le B \le 1$ (132)

If we choose $\mu = 0.1$ (133)

$$R = \begin{bmatrix} 2.2 & 0 \\ 0 & 2.2 \end{bmatrix} \qquad (134)$$

and $H = 2P$ (135)

By solving Equation (99) with Equations (133), (134) and (135), we obtain

$$P = \begin{bmatrix} .01168 & .024 & .0064 \\ .024 & .75 & .02657 \\ .0064 & .02657 & .01231 \end{bmatrix} \qquad (136)$$

Since P is a positive definite matrix, according to the stability criteria of Theorem 11, we can conclude that the feedback system is asymptotically stable. Also, from Theorem 10, the norm of the global system is bounded in the design interval. Let X be X_o when t is t_o, we get

$$1.15\sqrt{V(X_o)}e^{-8.135(t-t_o)} \leq \sqrt{X^TX} \leq 13\sqrt{V(X_o)}e^{-0.1(t-t_o)} \tag{137}$$

Without loss of generality, we assume $t = 0$ when $X^T = [\,1 \;\; 1 \;\; 1\,]$. Then by computer simulation of the feedback control centralized system, we get

If A = -2 and B = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	1.392	11.401	0.000
2	0.540	10.316	0.000
3	0.307	9.334	0.000
4	0.255	8.446	0.000
5	0.129	7.642	0.000
6	0.048	6.915	0.000
7	0.040	6.257	0.000
8	0.027	5.662	0.000
9	0.011	5.123	0.000
10	0.006	4.635	0.000

If A = -2 and B = 1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	1.283	11.401	0.000
2	0.559	10.316	0.000
3	0.249	9.334	0.000
4	0.218	8.446	0.000
5	0.135	7.642	0.000
6	0.053	6.915	0.000
7	0.030	6.257	0.000
8	0.025	5.662	0.000
9	0.014	5.123	0.000
10	0.005	4.635	0.000

If A = 2 and B = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	3.477	11.401	0.000
2	1.276	10.316	0.000
3	1.140	9.334	0.000
4	0.976	8.446	0.000
5	0.453	7.642	0.000
6	0.371	6.915	0.000
7	0.329	6.257	0.000
8	0.157	5.662	0.000
9	0.122	5.123	0.000
10	0.111	4.635	0.000

If $A = 2$ and $B = 1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	3.109	11.401	0.000
2	1.134	10.316	0.000
3	0.837	9.334	0.000
4	0.800	8.446	0.000
5	0.422	7.642	0.000
6	0.234	6.915	0.000
7	0.239	6.257	0.000
8	0.148	5.662	0.000
9	0.068	5.123	0.000
10	0.069	4.635	0.000

The above figures show that the actual norm of the global system is bounded by our predicted interval. If we plot Equation (137) and the computer simulation of the global system with different values of A and B, we have

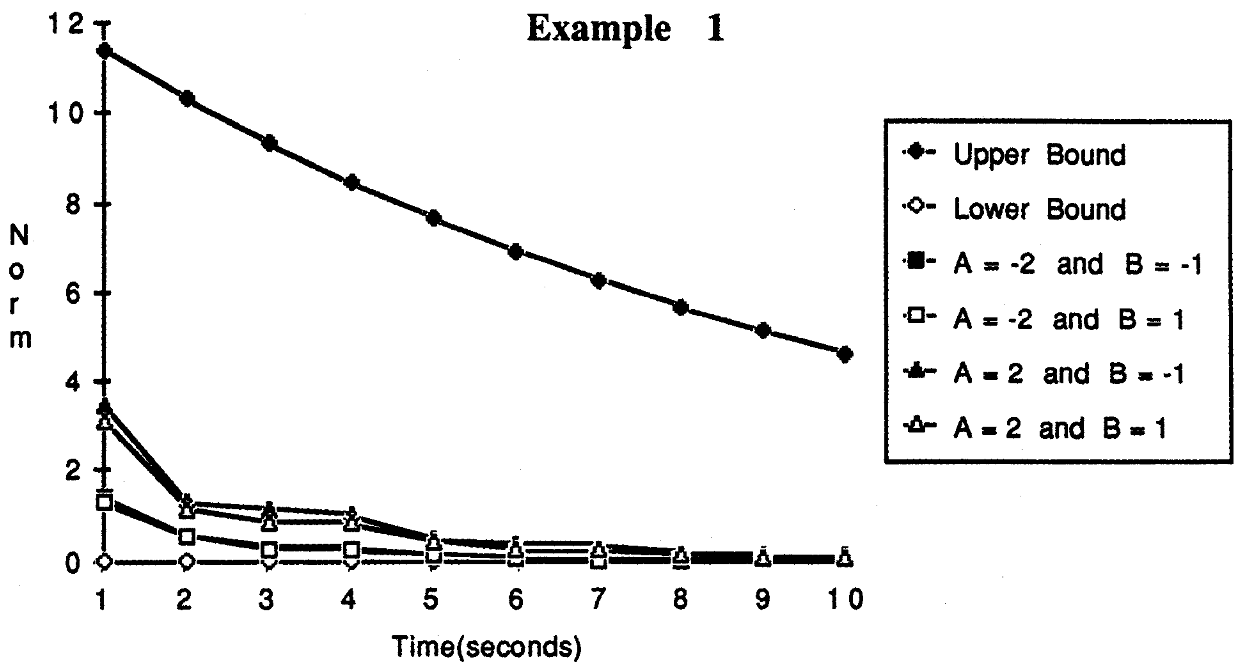

Example 1
12
10
8
6
4
2
0
Norm
1
2
3
4
5
6
7
8
9
10
Time(seconds)
Upper Bound
Lower Bound
A = -2 and B = -1
A = -2 and B = 1
A = 2 and B = -1
A = 2 and B = 1

Figure 7.1.

Example 2

Consider the feedback control linear autonomous large scale systems with uncertain parameters whose governing dynamic equations are given by:

$$\dot{X}_1 = \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 & 5 \\ -4 & 0.5 & 2 \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & B & 0 \end{bmatrix} X_2 + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U_1 \qquad (138)$$

$$\dot{X}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & D & 0 \end{bmatrix} X_1 + \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 & 5 \\ -4 & 0.5 & 2 \end{bmatrix} X_2 + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U_2 \qquad (139)$$

where $-2 \leq A \leq 2,\ -1 \leq B \leq 1,\ -2 \leq C \leq 2$ and $-1 \leq D \leq 1.$ (140)

If we choose $\mu_1 = \mu_2 = 0.1$ (141)

$$R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (142)$$

and $H_1 = H_2 = 2P_1 = 2P_2$ (143)

By solving Equation (99) with Equations (141), (142) and (143), we obtain

$$P_1 = P_2 = \begin{bmatrix} 0.025 & 0.0447 & 0.01392 \\ 0.0447 & 1.2158 & 0.06195 \\ 0.01392 & 0.06195 & 0.02728 \end{bmatrix} \qquad (144)$$

Since, P_1 and P_2 are positive definite matrices, according to the stability criteria of Theorem 11, we can conclude that the feedback system is asymptotically stable. Also, from Theorem 10, the norm of the global system is bounded in the designed interval. Let X be X_o when t is t_o, we get

$$0.905\sqrt{V(X_o)}e^{-5.885(t-t_o)} \leq \sqrt{X^TX} \leq 9.09\sqrt{V(X_o)}e^{-0.1(t-t_o)} \quad (145)$$

Without loss of generality, we assume t = 0 when $X_1^T = [\,1\ 1\ 1\,]$ and $X_2^T = [\,1\ 1\ 1\,]$. Then by computer simulation of the feedback control large scale systems, we get

If A = -2, B = -1, C = 2 and D = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	2.890	10.071	0.003
2	1.237	9.112	0.000
3	0.837	8.245	0.000
4	0.713	7.461	0.000
5	0.357	6.751	0.000
6	0.167	6.108	0.000
7	0.158	5.527	0.000
8	0.102	5.001	0.000
9	0.044	4.525	0.000
10	0.032	4.094	0.000

If A = -0.78, B = 0.75, C = 2 and D = -0.55

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	3.492	10.071	0.003
2	1.532	9.112	0.000
3	0.923	8.245	0.000
4	0.892	7.461	0.000
5	0.557	6.751	0.000
6	0.256	6.108	0.000
7	0.219	5.527	0.000
8	0.176	5.001	0.000
9	0.093	4.525	0.000
10	0.053	4.094	0.000

If A = 1.38, B = -0.45, C = 2 and D = 0.0

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	5.463	10.071	0.003
2	1.997	9.112	0.000
3	1.296	8.245	0.000
4	1.574	7.461	0.000
5	1.095	6.751	0.000
6	0.556	6.108	0.000
7	0.486	5.527	0.000
8	0.467	5.001	0.000
9	0.281	4.525	0.000
10	0.154	4.094	0.000

If $A = 2$, $B = 1$, $C = 2$ and $D = 1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	5.641	10.071	0.003
2	1.783	9.112	0.000
3	1.239	8.245	0.000
4	1.518	7.461	0.000
5	0.959	6.751	0.000
6	0.646	6.108	0.000
7	0.380	5.527	0.000
8	0.369	5.001	0.000
9	0.306	4.525	0.000
10	0.149	4.094	0.000

The above figures show that the actual norm of the global system is bounded by our predicted interval. If we plot Equation (145) and the computer simulation of the global system with different values of A, B, C and D, we have

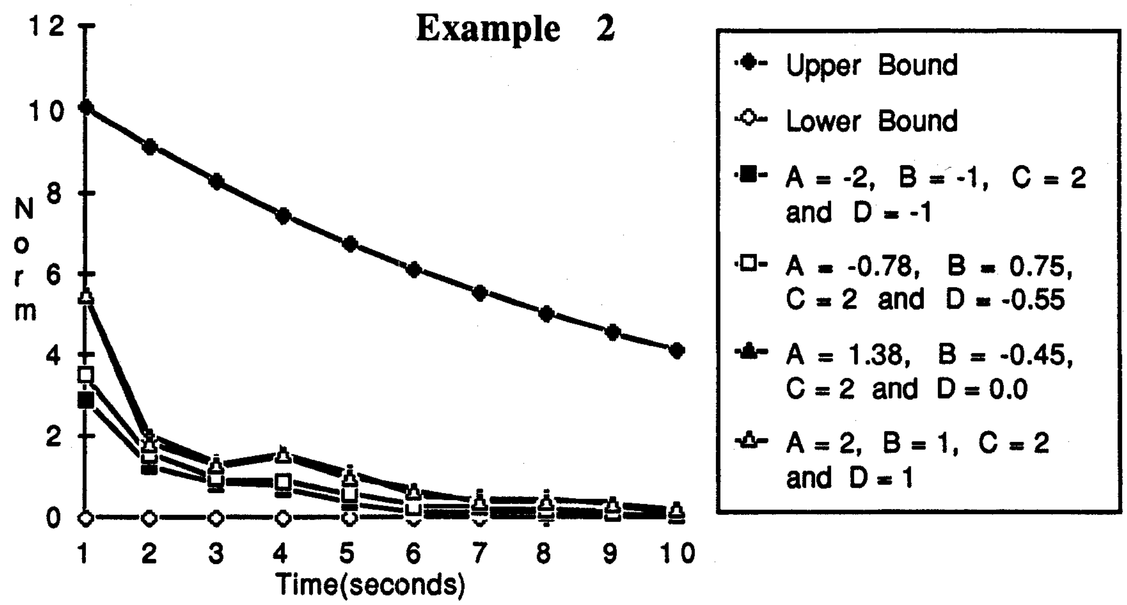
Example 2
12
10
8
6
4
2
0
Norm
1
2
3
4
5
6
7
8
9
10
Time(seconds)
◆ Upper Bound
◇ Lower Bound
■ A = -2, B = -1, C = 2 and D = -1
□ A = -0.78, B = 0.75, C = 2 and D = -0.55
▲ A = 1.38, B = -0.45, C = 2 and D = 0.0
△ A = 2, B = 1, C = 2 and D = 1

Figure 7.2.

Example 3

Consider the feedback-free linear nonautonomous large scale systems with non-cyclic time-varying and uncertain parameters whose governing dynamic equations are given by:

$$\dot{X}_1 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} X_1 + \begin{bmatrix} 0 & A\cos t \\ 0 & 0 \end{bmatrix} X_2 + \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} X_3 \qquad (146)$$

$$\dot{X}_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} X_1 + \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} X_2 + \begin{bmatrix} 0 & B\sin t \\ 0 & 0 \end{bmatrix} X_3 \qquad (147)$$

$$\dot{X}_3 = \begin{bmatrix} 0 & Ce^{0.0018t} \\ 0 & 0 \end{bmatrix} X_2 + \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} X_3 \qquad (148)$$

where $-0.5 \le A \le 0.5, \ -0.5 \le B \le 0.5$ and $-0.2 \le C \le 0.2.$ (149)

Let us investigate the finite-time global stability and its transient response for $0 < t < 500$. From the constraint on the t interval, we get

$$E_{mij}^{Max} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} \qquad (150)$$

where i or j = 1,2,3 but $i \neq j$.

First let $P_1 = P_2 = P_3 = \begin{bmatrix} 0.7 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$ (151)

From Equation (61), we obtain the bounded inequalities:

$$\dot{V} \ge X_1^T \begin{bmatrix} -3.85 & -0.24 \\ -0.24 & -3.52 \end{bmatrix} X_1 + X_2^T \begin{bmatrix} -3.6 & -0.24 \\ -0.24 & -4.02 \end{bmatrix} X_2 + X_3^T \begin{bmatrix} -3.1 & -0.12 \\ -0.12 & -3.76 \end{bmatrix} X_3 \qquad (152a)$$

$$\dot{V} \le X_1^T \begin{bmatrix} -1.35 & 0.24 \\ 0.24 & -2.48 \end{bmatrix} X_1 + X_2^T \begin{bmatrix} -1.6 & 0.24 \\ 0.24 & -1.98 \end{bmatrix} X_2 + X_3^T \begin{bmatrix} -2.1 & 0.12 \\ 0.12 & -2.24 \end{bmatrix} X_3 \qquad (152b)$$

From Theorem 7, we obtain the bounded interval:

$$-8.34 \le \frac{\dot{V}(X)}{V(X)} \le -1.77 \qquad (153)$$

Let X be X_o when t is 0. Then by integrating Equation (153), we have

$$V(X_o)e^{-8.34t} \le V(X) \le V(X_o)e^{-1.77t} \qquad (154)$$

which is valid for $0 < t < 500$. Hence the global system is finite-time asymptotically stable as t approaches or equals to 500. If we plot Equation (154) and the computer simulation of the global system with different values of A, B and C, we have

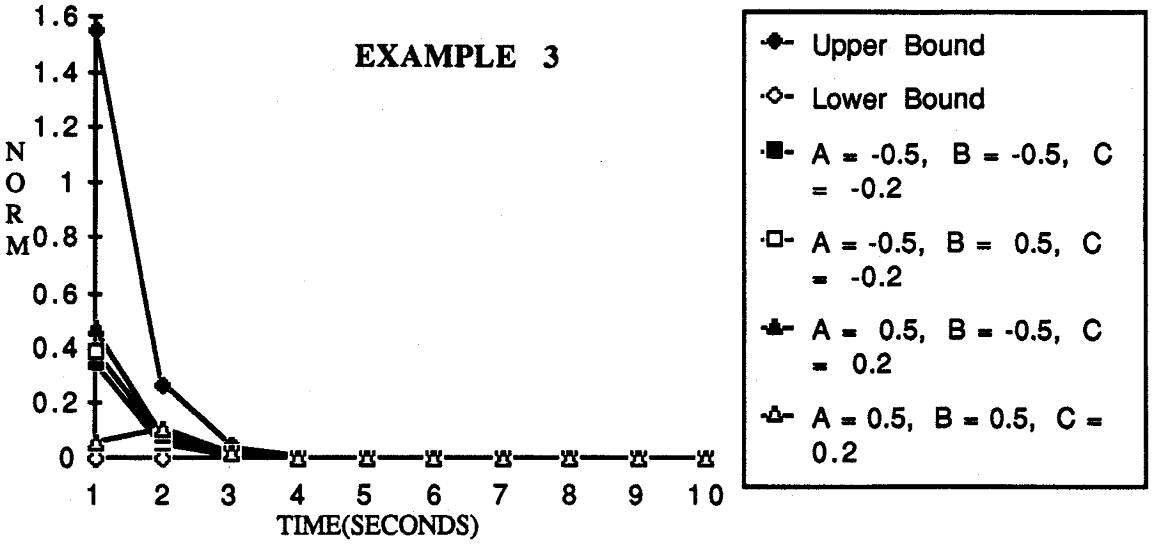

EXAMPLE 3
NORM
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
10
TIME(SECONDS)
Upper Bound
Lower Bound
A = -0.5, B = -0.5, C = -0.2
A = -0.5, B = 0.5, C = -0.2
A = 0.5, B = -0.5, C = 0.2
A = 0.5, B = 0.5, C = 0.2

Figure 7.3.

Example 4

Consider the feedback control linear nonautonomous small scale (centralized) systems with cyclic time-varying and uncertain parameters whose governing dynamic equations are given by:

$$\dot{X} = \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 + A\cos 2t & 5 \\ -4 & 0.5 + B\sin 6t & 2 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U \qquad (155)$$

where

$$-2 \leq A \leq 2 \text{ and } -1 \leq B \leq 1 \qquad (156)$$

Equation (155) has two cyclic time-varying parameters with a common period equal to π. Although Floquet's theory [39,40] is very powerful in investigating the stability of linear systems with cyclic time-varying parameters, it is not suitable for this problem. It is because A and B have infinity combinations. When feedback controller stabilizes one set of parameters, it cannot guarantee it will be true for other combinations. However, our methods will be very effective in such situations.

If we choose

$$\mu = 0.1 \qquad (157)$$

$$R = \begin{bmatrix} 2.2 & 0 \\ 0 & 2.2 \end{bmatrix} \qquad (158)$$

and

$$H = 2P \qquad (159)$$

By solving Equation (99) with Equations (157), (158) and (159), we obtain

$$P = \begin{bmatrix} .01168 & .024 & .0064 \\ .024 & .75 & .02657 \\ .0064 & .02657 & .01231 \end{bmatrix} \tag{160}$$

Since, P is a positive definite matrix, according to the stability criteria of Theorem 10, we can conclude that the feedback system is asymptotically stable. Also, from Theorem 11, the norm of the global system is bounded in the design interval. Let X be X_o when t is t_o, we get

$$1.15\sqrt{V(X_o)}e^{-8.135(t-t_o)} \leq \sqrt{X^TX} \leq 13\sqrt{V(X_o)}e^{-0.1(t-t_o)} \tag{161}$$

Without loss of generality, we assume t = 0 when $X^T = [\,1 \quad 1 \quad 1\,]$. Then by computer simulation of the feedback control centralized system, we get

If A = -2 and B = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	1.425	11.401	0.000
2	0.581	10.316	0.000
3	0.420	9.334	0.000
4	0.327	8.446	0.000
5	0.158	7.642	0.000
6	0.083	6.915	0.000
7	0.072	6.257	0.000
8	0.042	5.662	0.000
9	0.018	5.123	0.000
10	0.015	4.635	0.000

If $A = -2$ and $B = 1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	1.333	11.401	0.000
2	0.576	10.316	0.000
3	0.379	9.334	0.000
4	0.307	8.446	0.000
5	0.159	7.642	0.000
6	0.078	6.915	0.000
7	0.068	6.257	0.000
8	0.042	5.662	0.000
9	0.018	5.123	0.000
10	0.014	4.635	0.000

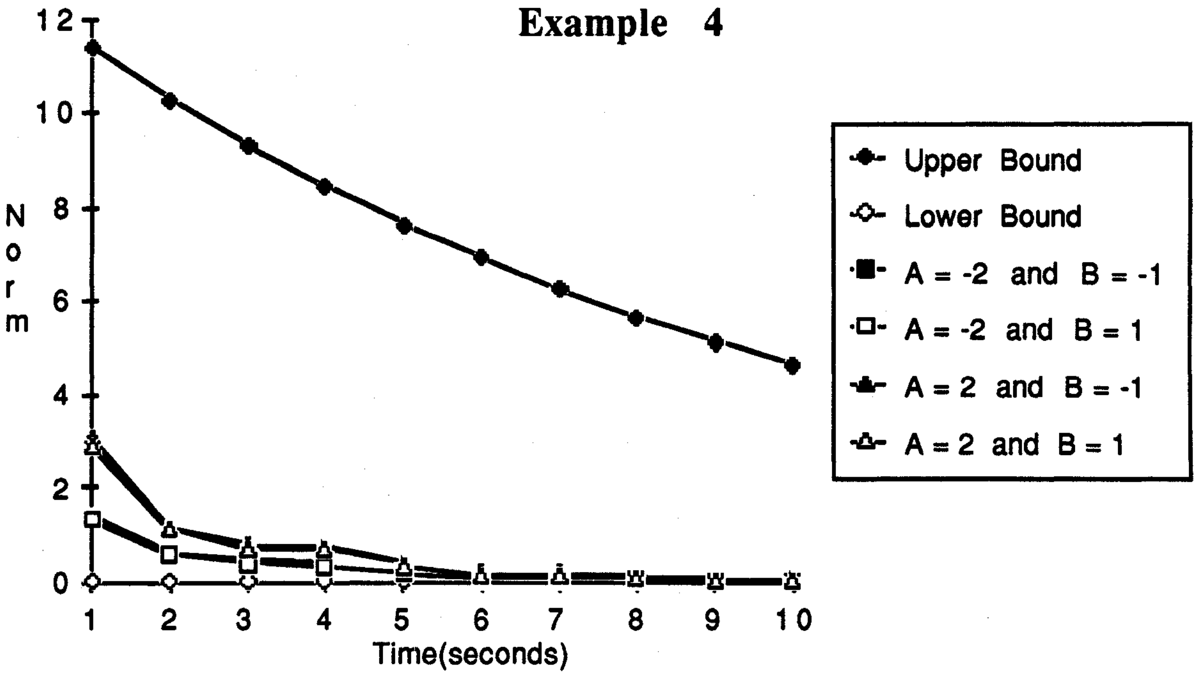
If $A = 2$ and $B = -1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	3.111	11.401	0.000
2	1.148	10.316	0.000
3	0.760	9.334	0.000
4	0.720	8.446	0.000
5	0.327	7.642	0.000
6	0.153	6.915	0.000
7	0.165	6.257	0.000
8	0.091	5.662	0.000
9	0.035	5.123	0.000
10	0.035	4.635	0.000

If $A = 2$ and $B = 1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	2.900	11.401	0.000
2	1.132	10.316	0.000
3	0.660	9.334	0.000
4	0.647	8.446	0.000
5	0.319	7.642	0.000
6	0.133	6.915	0.000
7	0.141	6.257	0.000
8	0.085	5.662	0.000
9	0.032	5.123	0.000
10	0.029	4.635	0.000

The above figures show that the actual norm of the global system is bounded by our predicted interval. If we plot Equation (161) and the computer simulation of the global system with different values of A and B, we have


Example 4
N o r m
12
10
8
6
4
2
0
1
2
3
4
5
6
7
8
9
10
Time(seconds)
Upper Bound
Lower Bound
A = -2 and B = -1
A = -2 and B = 1
A = 2 and B = -1
A = 2 and B = 1

<u>Figure 7.4.</u>

Example 5

Consider the feedback control linear nonautonomous large scale systems with cyclic time-varying and uncertain parameters whose governing dynamic equations are given by:

$$\dot{X}_1 = \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 & 5 \\ -4 & 0.5 & 2 \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A\cos t & 0 \\ 0 & B\sin 4t & 0 \end{bmatrix} X_2 + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U_1 \qquad (162)$$

$$\dot{X}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & D & 0 \end{bmatrix} X_1 + \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 & 5 \\ -4 & 0.5 & 2 \end{bmatrix} X_2 + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U_2 \qquad (163)$$

where $-2 \leq A \leq 2,\ -1 \leq B \leq 1,\ -2 \leq C \leq 2$ and $-1 \leq D \leq 1.$ (164)

If we choose $\mu_1 = \mu_2 = 0.1$ (165)

$$R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (166)$$

and $H_1 = H_2 = 2P_1 = 2P_2$ (167)

By solving Equation (99) with Equations (165), (166) and (167), we obtain

$$P_1 = P_2 = \begin{bmatrix} 0.025 & 0.0447 & 0.01392 \\ 0.0447 & 1.2158 & 0.06195 \\ 0.01392 & 0.06195 & 0.02728 \end{bmatrix} \qquad (168)$$

Since, P_1 and P_2 are positive definite matrices, according to the stability criteria of Theorem 11, we can conclude that the feedback system is asymptotically stable. Also, from Theorem 10, the norm of the global system is bounded in the design interval. Let X be X_o when t is t_o, we get

$$0.905\sqrt{V(X_o)}e^{-5.885(t-t_o)} \leq \sqrt{X^TX} \leq 9.09\sqrt{V(X_o)}e^{-0.1(t-t_o)} \qquad (169)$$

Without loss of generality, we assume t = 0 when $X_1^T = [\,1\ \ 1\ \ 1\,]$ and $X_2^T = [\,1\ \ 1\ \ 1\,]$. Then by computer simulation of the feedback control large scale systems, we get

If A = -2, B = -1, C = 2 and D = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	2.926	10.071	0.003
2	1.324	9.112	0.000
3	1.063	8.245	0.000
4	1.053	7.461	0.000
5	0.642	6.751	0.000
6	0.285	6.108	0.000
7	0.265	5.527	0.000
8	0.192	5.001	0.000
9	0.094	4.525	0.000
10	0.071	4.094	0.000

If A = -2, B = 1, C = 2 and D = 1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	2.797	10.071	0.003
2	1.296	9.112	0.000
3	0.835	8.245	0.000
4	0.848	7.461	0.000
5	0.570	6.751	0.000
6	0.245	6.108	0.000
7	0.194	5.527	0.000
8	0.164	5.001	0.000
9	0.087	4.525	0.000
10	0.048	4.094	0.000

If A = 2, B = -1, C = 2 and D = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	6.369	10.071	0.003
2	2.360	9.112	0.000
3	1.353	8.245	0.000
4	1.298	7.461	0.000
5	0.844	6.751	0.000
6	0.421	6.108	0.000
7	0.411	5.527	0.000
8	0.366	5.001	0.000
9	0.190	4.525	0.000
10	0.098	4.094	0.000

If $A = 2$, $B = 1$, $C = 2$ and $D = 1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	5.537	10.071	0.003
2	1.912	9.112	0.000
3	0.979	8.245	0.000
4	0.903	7.461	0.000
5	0.678	6.751	0.000
6	0.340	6.108	0.000
7	0.242	5.527	0.000
8	0.242	5.001	0.000
9	0.151	4.525	0.000
10	0.064	4.094	0.000

The above figure show that the actual norm of the global system is bounded by our predicted interval. If we plot Equation (169) and the computer simulation of the global system with different values of A, B, C and D, we have

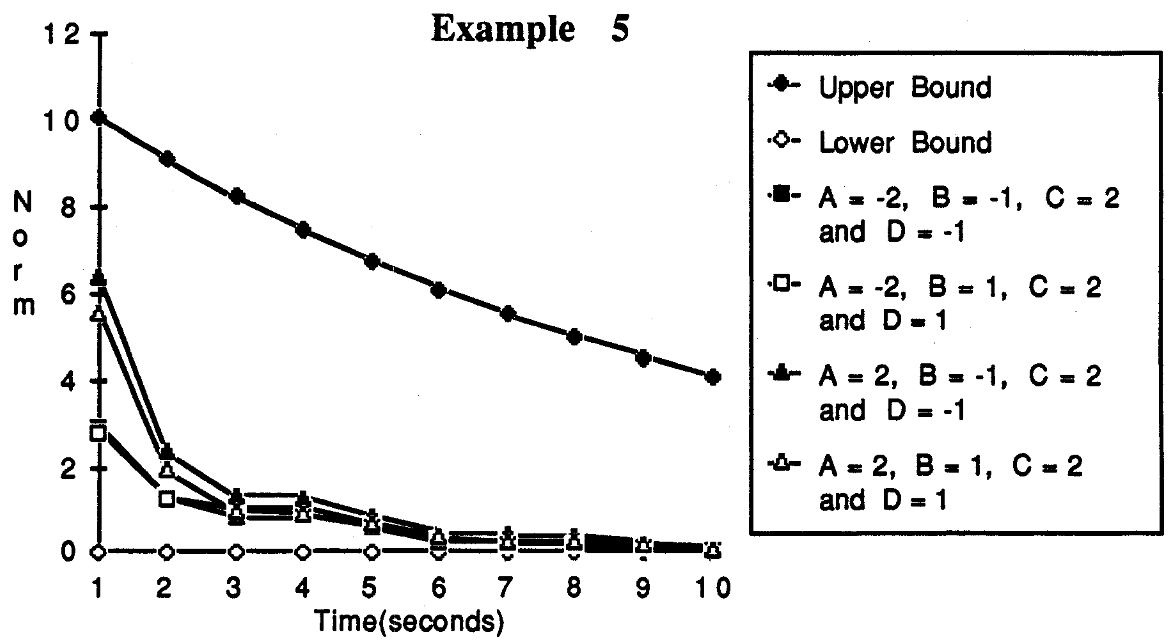

Example 5
12
10
8
6
4
2
0
Norm
1
2
3
4
5
6
7
8
9
10
Time(seconds)
Upper Bound
Lower Bound
A = -2, B = -1, C = 2 and D = -1
A = -2, B = 1, C = 2 and D = 1
A = 2, B = -1, C = 2 and D = -1
A = 2, B = 1, C = 2 and D = 1

Figure 7.5.

Example 6

Consider the feedback control nonautonomous small scale (centralized) system with non-cyclic time-varying and uncertain parameters whose governing dynamic equations are given by:

$$\dot{X} = \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 + Ae^{-0.1t}\cos 2t & 5 \\ -4 & 0.5 + B\sin t & 2 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U \qquad (170)$$

where $-2 \leq A \leq 2$ and $-1 \leq B \leq 1$ (171)

Equation (170) has non-cyclic time-varying parameters. So Floquet's theory [39,40] is not suitable for this problem. However, our methods will be very effective to such situations.

If we choose $\mu = 0.1$ (172)

$$R = \begin{bmatrix} 2.2 & 0 \\ 0 & 2.2 \end{bmatrix} \qquad (173)$$

and $H = 2P$ (174)

By solving Equation (99) with Equations (172), (173) and (174), we obtain

$$P = \begin{bmatrix} .01168 & .024 & .0064 \\ .024 & .75 & .02657 \\ .0064 & .02657 & .01231 \end{bmatrix} \qquad (175)$$

Since, P is a positive definite matrix, according to the stability criteria of Theorem 10, we can conclude that the feedback system is asymptotically stable. Also, from Theorem 11, the norm of the global system is bounded in the design interval. Let X be X_o when t is t_o, we get

$$1.15\sqrt{V(X_o)}e^{-8.135(t-t_o)} \leq \sqrt{X^TX} \leq 13\sqrt{V(X_o)}e^{-0.1(t-t_o)} \tag{176}$$

Without loss of generality, we assume $t = 0$ when $X^T = [\,1 \;\; 1 \;\; 1\,]$. Then by computer simulation of the feedback control centralized system, we get

If A = -2 and B = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	1.109	11.401	0.000
2	0.489	10.316	0.000
3	0.310	9.334	0.000
4	0.317	8.446	0.000
5	0.174	7.642	0.000
6	0.070	6.915	0.000
7	0.058	6.257	0.000
8	0.039	5.662	0.000
9	0.016	5.123	0.000
10	0.013	4.635	0.000

If $A = -2$ and $B = 1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	1.095	11.401	0.000
2	0.502	10.316	0.000
3	0.286	9.334	0.000
4	0.305	8.446	0.000
5	0.179	7.642	0.000
6	0.069	6.915	0.000
7	0.058	6.257	0.000
8	0.039	5.662	0.000
9	0.017	5.123	0.000
10	0.012	4.635	0.000

If $A = 2$ and $B = -1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	6.523	11.401	0.000
2	2.221	10.316	0.000
3	0.946	9.334	0.000
4	0.813	8.446	0.000
5	0.520	7.642	0.000
6	0.222	6.915	0.000
7	0.185	6.257	0.000
8	0.153	5.662	0.000
9	0.066	5.123	0.000
10	0.033	4.635	0.000

If $A = 2$ and $B = 1$

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	6.180	11.401	0.000
2	2.157	10.316	0.000
3	0.900	9.334	0.000
4	0.761	8.446	0.000
5	0.496	7.642	0.000
6	0.210	6.915	0.000
7	0.182	6.257	0.000
8	0.145	5.662	0.000
9	0.064	5.123	0.000
10	0.030	4.635	0.000

The above figures show that the actual norm of the global system is bounded by our predicted interval. If we plot Equation (176) and the computer simulation of the global system with different values of A and B, we have

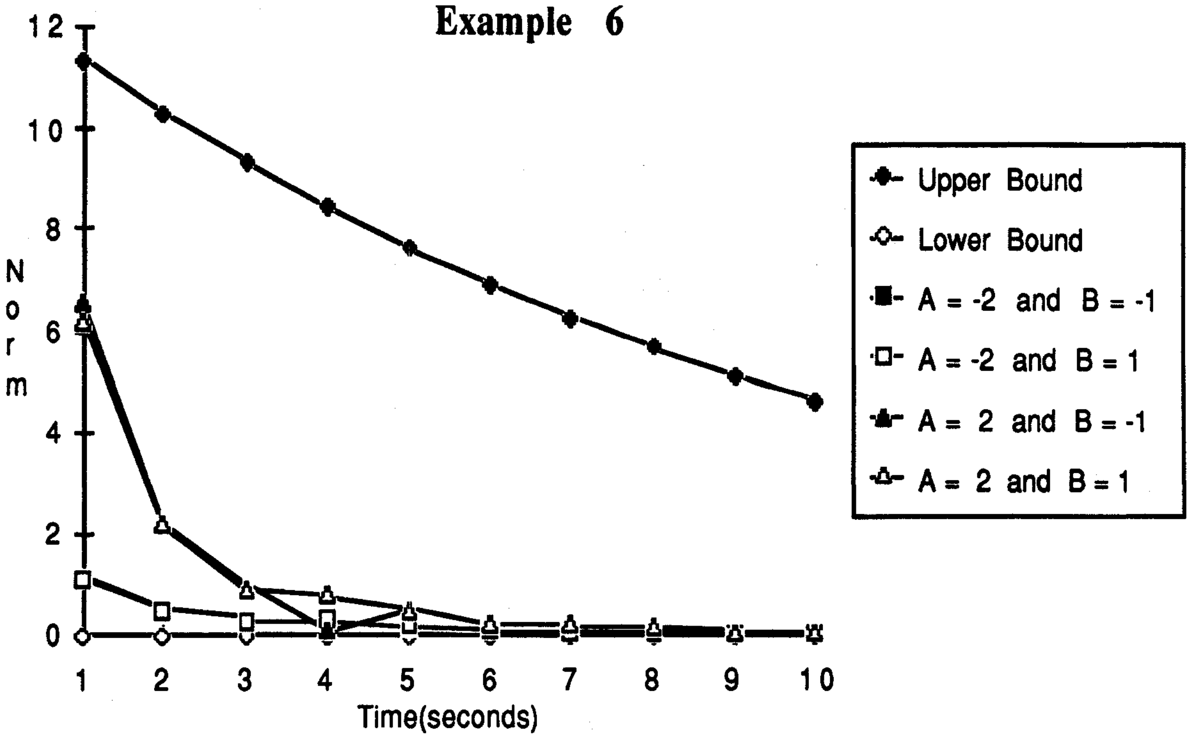

Example 6
12
10
8
6
4
2
0
Norm
1
2
3
4
5
6
7
8
9
10
Time(seconds)
Upper Bound
Lower Bound
A = -2 and B = -1
A = -2 and B = 1
A = 2 and B = -1
A = 2 and B = 1

Figure 7.6.

Example 7

Consider the feedback control nonautonomous large scale systems with non-cyclic time-varying and uncertain parameters whose governing dynamic equations are given by:

$$\dot{X}_1 = \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 & 5 \\ -4 & 0.5 & 2 \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & B\cos t & 0 \end{bmatrix} X_2 + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U_1 \qquad (177)$$

$$\dot{X}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & De^{-0.1t}\sin 3t & 0 \end{bmatrix} X_1 + \begin{bmatrix} 3 & 0.5 & 5 \\ 0.5 & 1 & 5 \\ -4 & 0.5 & 2 \end{bmatrix} X_2 + \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \\ 0.5 & 1 \end{bmatrix} U_2 \qquad (178)$$

where $-2 \leq A \leq 2,\ -1 \leq B \leq 1,\ -2 \leq C \leq 2$ and $-1 \leq D \leq 1.$ (179)

If we choose $\mu_1 = \mu_2 = 0.1$ (180)

$$R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (181)$$

and $H_1 = H_2 = 2P_1 = 2P_2$ (182)

By solving Equation (99) with Equations (180), (181) and (182), we obtain

$$P_1 = P_2 = \begin{bmatrix} 0.025 & 0.0447 & 0.01392 \\ 0.0447 & 1.2158 & 0.06195 \\ 0.01392 & 0.06195 & 0.02728 \end{bmatrix} \qquad (183)$$

Since, P_1 and P_2 are positive definite matrices, according to the stability criteria of Theorem 11, we can conclude that the feedback system is asymptotically stable. Also, from Theorem 10, the norm of the global system is bounded in the design interval. Let X be X_o when t is t_o, we get

$$0.905\sqrt{V(X_o)}e^{-5.885(t - t_o)} \le \sqrt{X^TX} \le 9.09\sqrt{V(X_o)}e^{-0.1(t - t_o)} \qquad (184)$$

Without loss of generality, we assume t = 0 when $X_1^T = [\,1 \;\; 1 \;\; 1\,]$ and $X_2^T = [\,1 \;\; 1 \;\; 1\,]$. Then by computer simulation of the feedback control large scale systems, we get

If A = -2, B = -1, C = 2 and D = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	2.864	10.071	0.003
2	1.248	9.112	0.000
3	0.813	8.245	0.000
4	0.725	7.461	0.000
5	0.394	6.751	0.000
6	0.192	6.108	0.000
7	0.172	5.527	0.000
8	0.117	5.001	0.000
9	0.057	4.525	0.000
10	0.042	4.094	0.000

If A = -0.78, B = 1, C = 2 and D = 1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	2.793	10.071	0.003
2	1.275	9.112	0.000
3	0.739	8.245	0.000
4	0.675	7.461	0.000
5	0.389	6.751	0.000
6	0.181	6.108	0.000
7	0.162	5.527	0.000
8	0.116	5.001	0.000
9	0.058	4.525	0.000
10	0.039	4.094	0.000

If A = 2, B = -1, C = 2 and D = -1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	6.476	10.071	0.003
2	2.265	9.112	0.000
3	1.473	8.245	0.000
4	1.906	7.461	0.000
5	1.250	6.751	0.000
6	0.745	6.108	0.000
7	0.580	5.527	0.000
8	0.574	5.001	0.000
9	0.399	4.525	0.000
10	0.198	4.094	0.000

If A = 2, B = 1, C = 2 and D = 1

Time(seconds)	Actual Norm	Upper Bound	Lower Bound
1	5.683	10.071	0.003
2	1.745	9.112	0.000
3	1.199	8.245	0.000
4	1.713	7.461	0.000
5	1.074	6.751	0.000
6	0.631	6.108	0.000
7	0.484	5.527	0.000
8	0.468	5.001	0.000
9	0.334	4.525	0.000
10	0.170	4.094	0.000

The above figures show that the actual norm of the global system is bounded by our predicted interval. If we plot Equation (184) and the computer simulation of the global system with different values of A, B, C and D, we have

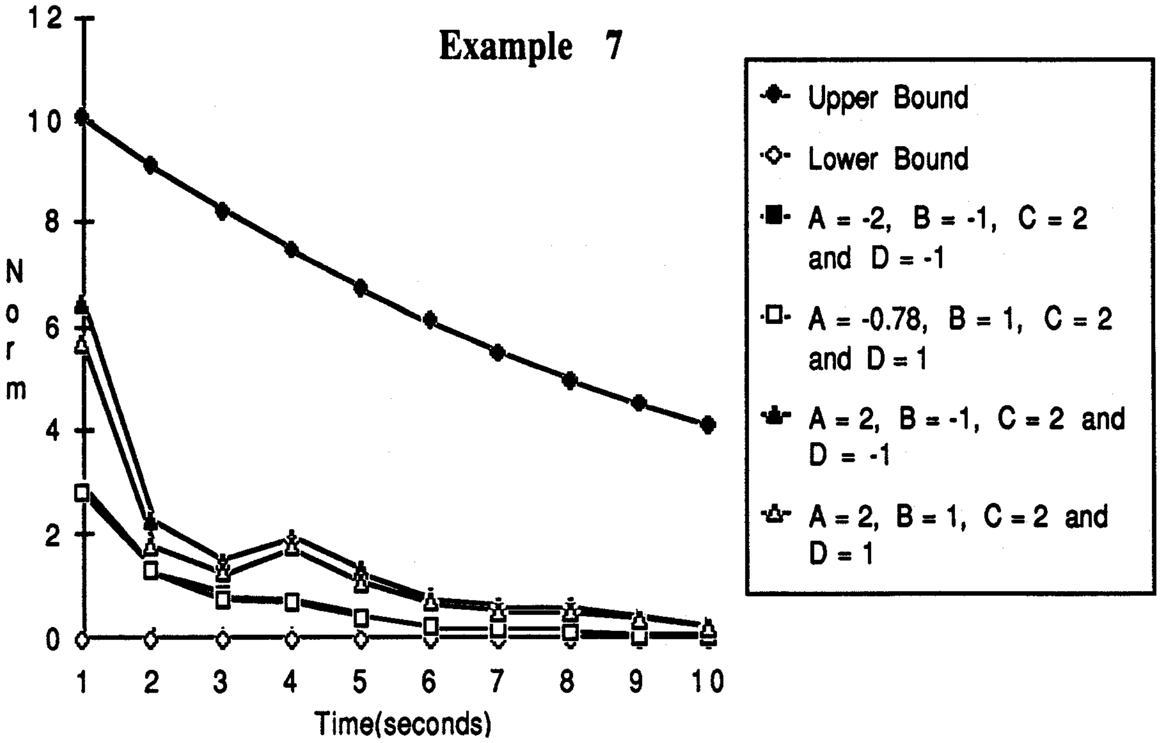

Example 7
12
10
8
6
4
2
0
Norm
1
2
3
4
5
6
7
8
9
10
Time(seconds)
Upper Bound
Lower Bound
A = -2, B = -1, C = 2 and D = -1
A = -0.78, B = 1, C = 2 and D = 1
A = 2, B = -1, C = 2 and D = -1
A = 2, B = 1, C = 2 and D = 1

Figure 7.7.

Chapter 8

Conclusions and Recommendations

8.1. Conclusions and Summaries

There is presently a great deal of interest in the decentralized control of large scale systems and the development of efficient design methods for them. Since high dimensionally and non-decentralized structure are outstanding features of the large scale systems leading to significant control difficulties. Some successful methods have been developed to analyze linear autonomous large scale systems with specific parameters. However, linear nonautonomous large scale systems with uncertain parameters have not received corresponding attention which may be due to their complicated structure. In this research, a new concept for studying their stability is created. The basic idea is to consider the global systems as a group of subsystems with specific parameters under disturbance. By using the Liapunov function and comparison principle to disconnect and reassemble the subsystems, we can develop some new stability criteria. Also, stabilizing feedback controllers are generated by solving the low order matrix equations with specific parameters. In addition, since small scale (centralized) systems are subset of large scale systems, the theories developed for large scale systems will still be valid for small scale (centralized) systems. The examples demonstrate the utility and ease of using the methods for small or large scale systems.

When designing a system, in addition to stability, it is often desirable to design the time limit to fulfill certain tolerance criteria. New stability criteria can lead us to design or estimate the global system behavior without solving a system of interconnected differential equations. Although the procedure is approximate, it will have advantages in engineering practices to design complicated models of given physical processes. Also, with the aid of symbolic software (for example, MACSYMA), we may be able to answer the stability questions symbolically and thus identify the dominant factors influencing the stability.

8.2. Recommendations for Further Research

The following areas for further research are recommended:

(1) Develop a symbolic computer program to study the stability of large scale systems symbolically.
(2) Optimize the new decentralized control theories.
(3) Expansion of present methods to study strongly coupled large scale systems.
(4) Expansion of present methods to study the non-linear large scale systems.
(5) Expansion of present methods to study nonautonomous large scale systems with unbounded time-varying parameters.

Appendices

Appendix 1

From the properties of the Cauchy-Schwarz inequality [38], we can deduce the following results:

$$\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2X_i^T P_i A_{ij} X_j \le \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \left|2X_i^T P_i A_{ij} X_j\right|$$

$$\le \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} 2\left\|X_i^T P_i\right\| \left\|A_{ij} X_j\right\|$$

$$\le \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} (X_i^T P_i P_i X_i + X_j^T A_{ij}^T A_{ij} X_j)$$

Appendix 2

From the properties of the Cauchy-Schwarz inequality [38], we can deduce the following results:

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} 2X_i^TP_iA_{ij}X_j \geq \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} -\left|2X_i^TP_iA_{ij}X_j\right|$$

$$\geq \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} -2\left\|X_i^TP_i\right\|\left\|A_{ij}X_j\right\|$$

$$\geq \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} -(X_i^TP_iP_iX_i + X_j^TA_{ij}^TA_{ij}X_j)$$

Appendix 3

Let
$$[\Delta A_{ij}(t)] = \sum_{n=1}^{K_{ij}} [D_{n_{ij}}(t)]$$

where K_{ij} is the number of non-zero column of $\Delta A_{ij}(t)$. $D_{n_{ij}}(t)$ has the same dimensions as $\Delta A_{ij}(t)$ and all the columns of $D_{n_{ij}}(t)$ have zero entry except at column n which carries the same elements as column n in $\Delta A_{ij}(t)$.

$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j = \sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i (\sum_{n=1}^{K_{ij}} D_{n_{ij}}(t)) X_j$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{K_{ij}} 2\left|X_i^T P_i D_{n_{ij}}(t) X_j\right|$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{K_{ij}} \|X_i^T P_i\| \, \|D_{n_{ij}}(t) X_j\|$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N} \left\{ K_{ii} X_i^T P_i P_i X_i + X_j^T (\sum_{n=1}^{K_{ij}} D_{n_{ij}}^T(t) D_{n_{ij}}(t)) X_j \right\}$$

Since $D_{n_{ij}}^T(t) D_{n_{ij}}(t)$ is a diagonal matrix with positive diagonal elements, hence, from the properties of extremizing quadratic forms. We get

$$X_j^T \left[\sum_{n=1}^{K_{ij}} D_{n_{ij}}^{Max^T} D_{n_{ij}}^{Max} \right] X_j \geq X_j^T \left[\sum_{n=1}^{K_{ij}} D_{n_{ij}}^T D_{n_{ij}} \right] X_j$$

Hence,
$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j \leq \sum_{i=1}^{N}\sum_{j=1}^{N} \left\{ K_{ij} X_i^T P_i P_i X_i + X_j^T (\sum_{n=1}^{K_{ij}} D_{n_{ij}}^{Max^T} D_{n_{ij}}^{Max}) X_j \right\}$$

Appendix 4

Let $$\left[\Delta A_{ij}(t)\right] = \sum_{n=1}^{K_{ij}} \left[D_{n_{ij}}(t)\right]$$

where K_{ij} is the number of non-zero column of $\Delta A_{ij}(t)$. $D_{n_{ij}}(t)$ has the same dimensions as $\Delta A_{ij}(t)$ and all the columns of $D_{n_{ij}}(t)$ have zero entry except at column n which carries the same elements as column n in $\Delta A_{ij}(t)$.

$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j = \sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Big(\sum_{n=1}^{K_{ij}} D_{n_{ij}}(t)\Big) X_j$$

$$\geq \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{K_{ij}} -2\left|X_i^T P_i D_{n_{ij}}(t) X_j\right|$$

$$\geq \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{K_{ij}} -\left\|X_i^T P_i\right\| \left\|D_{n_{ij}}(t) X_j\right\|$$

$$\geq \sum_{i=1}^{N}\sum_{j=1}^{N} -\left\{K_{ij} X_i^T P_i P_i X_i + X_j^T \Big(\sum_{n=1}^{K_{ij}} D_{n_{ij}}^T(t) D_{n_{ij}}(t)\Big) X_j\right\}$$

Since $D_{n_{ij}}^T(t) D_{n_{ij}}(t)$ is a diagonal matrix with positive diagonal elements, hence, from the properties of extremizing quadratic forms. We get

$$-X_j^T \left[\sum_{n=1}^{K_{ij}} D_{n_{ij}}^{Max^T} D_{n_{ij}}^{Max}\right] X_j \leq -X_j^T \left[\sum_{n=1}^{K_{ij}} D_{n_{ij}}^T D_{n_{ij}}\right] X_j$$

Hence, $$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j \geq \sum_{i=1}^{N}\sum_{j=1}^{N} -\left\{K_{ij} X_i^T P_i P_i X_i + X_j^T \Big(\sum_{n=1}^{K_{ij}} D_{n_{ij}}^{Max^T} D_{n_{ij}}^{Max}\Big) X_j\right\}$$

Appendix 5

Let $$[\Delta\Omega_{ij}(t)] = \sum_{n=1}^{M_{ij}} [E_{n_{ij}}(t)]$$

where M_{ij} is the number of non-zero column of $\Delta\Omega_{ij}(t)$. $E_{n_{ij}}(t)$ has the same dimensions as $\Delta\Omega_{ij}(t)$ and all the columns of $E_{n_{ij}}(t)$ have zero entry except at column n which carries the same elements as column n in $\Delta\Omega_{ij}(t)$.

$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j = \sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Big(\sum_{n=1}^{M_{ij}} \Omega_{n_{ij}}(t)\Big) X_j$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{M_{ij}} 2\left|X_i^T P_i \Omega_{n_{ij}}(t) X_j\right|$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{M_{ij}} \left\|X_i^T P_i\right\| \left\|\Omega_{n_{ij}}(t) X_j\right\|$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N} \left\{M_{ij} X_i^T P_i P_i X_i\right\} + \sum_{j=1}^{N} \left\{X_j^T \Big(\sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} E_{n_{ij}}^T(t) E_{n_{ij}}(t)\Big) X_j\right\}$$

Since $\sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} [E_{n_{ij}}^T E_{n_{ij}}]$ is a diagonal matrix with positive diagonal elements, hence, from the properties of extremizing quadratic forms. We get

$$X_j^T \sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} \left[E_{n_{ij}}^{Max\,T} E_{n_{ij}}^{Max}\right] X_j \geq X_j^T \sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} \left[E_{n_{ij}}^T E_{n_{ij}}\right] X_j$$

Hence,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta\Omega_{ij}(t) X_j \leq \sum_{i=1}^{N}\sum_{j=1}^{N} \left\{M_{ij} X_i^T P_i P_i X_i\right\} + \sum_{j=1}^{N} \left\{X_j^T \Big(\sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} E_{n_{ij}}^{Max\,T} E_{n_{ij}}^{Max} X_j\right\}$$

Appendix 6

Let $$\left[\Delta\Omega_{ij}(t)\right] = \sum_{n=1}^{M_{ij}} \left[E_{n_{ij}}(t)\right]$$

where M_{ij} is the number of non-zero column of $\Delta\Omega_{ij}(t)$. $E_{n_{ij}}(t)$ has the same dimensions as $\Delta\Omega_{ij}(t)$ and all the columns of $E_{n_{ij}}(t)$ have zero entry except at column n which carries the same elements as column n in $\Delta\Omega_{ij}(t)$.

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta A_{ij}(t) X_j &= \sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Big(\sum_{n=1}^{M_{ij}} \Omega_{n_{ij}}(t)\Big) X_j \\
&\geq -\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{M_{ij}} 2\left|X_i^T P_i \Omega_{n_{ij}}(t) X_j\right| \\
&\geq -\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{M_{ij}} \left\|X_i^T P_i\right\| \left\|\Omega_{n_{ij}}(t) X_j\right\| \\
&\geq -\sum_{i=1}^{N}\sum_{j=1}^{N} \left\{M_{ij} X_i^T P_i P_i X_i\right\} - \sum_{j=1}^{N} \left\{X_j^T \Big(\sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} E_{n_{ij}}^T(t) E_{n_{ij}}(t)\Big) X_j\right\}
\end{aligned}$$

Since $\sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} \left[E_{n_{ij}}^T E_{n_{ij}}\right]$ is a diagonal matrix with positive diagonal elements, hence, from the properties of extremizing quadratic forms. We get

$$-X_j^T \sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} \left[E_{n_{ij}}^{Max\,T} E_{n_{ij}}^{Max}\right] X_j \leq -X_j^T \sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} \left[E_{n_{ij}}^T E_{n_{ij}}\right] X_j$$

Hence,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} 2X_i^T P_i \Delta\Omega_{ij}(t) X_j \geq -\sum_{i=1}^{N}\sum_{j=1}^{N} \left\{M_{ij} X_i^T P_i P_i X_i\right\} - \sum_{j=1}^{N} \left\{X_j^T \Big(\sum_{i=1}^{N}\sum_{n=1}^{M_{ij}} E_{n_{ij}}^{Max\,T} E_{n_{ij}}^{Max} X_j\right\}$$

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