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I hereby recommend that the thesis prepared under my supervision by Charles Everett Rhodes

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CONCERNING THE DOUBLE POISSON INTEGRAL
AND ITS DERIVATIVES

A dissertation submitted to the

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by

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INTRODUCTION

The Single Poisson Integral has been studied in connection with Fourier Series and the solution of Dirichlet's Problem for the circle. If we are given a function f(x) defined along the circumference of the unit circle about the origin, Poisson's Integral

$$F(r,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(u-x)+r^2} f(u)\,du$$

defines a continuous function of two variables everywhere inside the circle. It has been shown[1] that the $\lim_{r\to 1} F(r,x) = \frac{1}{2}\left[f(x+0) + f(x-0)\right]$ as the point (r,x) approaches the point (1,x) on the circumference, provided the function f(x) is bounded over the entire circumference and the limit on the right exists. If f(x) is continuous, the $\lim_{\substack{r\to 1\\ u\to x}} F(r,u) = f(x)$ as the point (r,u) approaches the point (1,x). Fatou[2] considered the $\lim_{r\to 1} F(r,x)$ for a point at which f(x) was unbounded. He also considered the derivatives $F_x(r,x)$ and $F_{xx}(r,x)$, and gave certain sufficient conditions that their limits should approach the corresponding derivatives f'(x) and f"(x). By means of these results, together with some theorems on Lebesgue integration, he was able to show that the $\lim_{r\to 1} F(r,x) = f(x)$ almost everywhere. Furthermore

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[1] Picard, Traite D'Analyse, 2nd ed. vol. I, pages 268-275

[2] Fatou, Acta Mathematica, vol. 30 (1906) pages 335-400
Series Trigonometrique et Series de Taylor

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he was able to prove Parseval's Theorem for Single Fourier Series under the sole assumption that the given function was bounded and integrable in the sense of Lebesgue.

The Double Poisson Integral is a direct extension, and is defined by

$$F(r,\rho,x,y) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(u-x)+r^2}\cdot\frac{1-\rho^2}{1-2\rho\cos(v-y)+\rho^2}\,f(u,v)\,du\,dv$$

Kustermann* has shown that the $\lim_{r,\rho\to 1} F(r,\rho,x,y) =$ $\frac{1}{4}\left[f(x+0,y+0)+f(x+0,y-0)+f(x-0,y+0)+f(x-0,y-0)\right]$ provided the function $f(x,y)$ is bounded over the fundamental square and the limit on the right exists. Making use of this in the following work, we have extended Fatou's results to the case of the Double Poisson Integral. In proving Parseval's Theorem, it was found expedient to add certain restrictions which appear to be inherent to the method used in treating the Double Poisson Integral itself.

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*Kustermann, Uber Fouriersche Doppelreihen und das Poissonsche Doppelintegral, Munich, 1913, pages 39-44.

SECTION I

I. Notation. To simplify the writing of certain expressions which occur repeatedly in this work, we shall use the following notation thruout.

$$H = \frac{1-r^2}{2\pi[1-2r\cos(u-x)+r^2]}, \quad J = \frac{1-\rho^2}{2\pi[1-2\rho\cos(v-y)+\rho^2]}$$

The symbols $\bar{H}$ and $\bar{J}$ denote the expressions for H and J when x and y are put equal to zero.

$$F(r,\rho,x,y) = \iint_R HJf(u,v)\,du\,dv$$

$$\bar{F}(r,\rho) = F(r,\rho,0,0) = \iint_R \bar{H}\bar{J}f(u,v)\,du\,dv$$

R denotes that portion of the plane of integration for which $-\pi \leq u \leq \pi$, $-\pi \leq v \leq \pi$. This will sometimes be divided up into two portions, R_1, a small square for which $-\delta \leq u \leq \delta$, $-\delta \leq v \leq \delta$, and R_2 which denotes the remaining part of R.

It is to be understood thruout that all derivative symbols denote the generalized derivative as defined on page 10, and all integral signs denote Lebesgue integration.

2. The limit as $r,\rho \to 1$ of $\bar{F}$. We first take up the study of the Double Poisson Integral of a function, $f(x,y)$, which is periodic and integrable over the region R. If the

period in both x and y is 2π, the origin may be conveniently taken as a perfectly general point. Kustermann* has shown that if $f(x,y)$ is bounded, the limit $\bar{F}(r,\rho)$ depends only upon the value of the integral taken over an arbitrarily small, but fixed square about the origin. We wish to substitute a slightly more general condition than that of boundedness. The argument can best be shown by a geometric picture. Draw a circle of radius I, and take the point (I, O) (polar coordinates) as the midpoint of an arc of length 2δ.

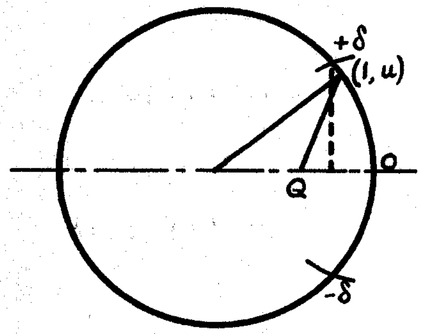

Fig. I

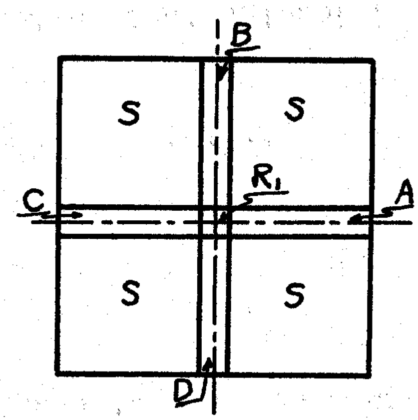

Fig. 2

$I - 2r\cos u + r^2$ is the square of the distance from a point $Q(r,o)$ to a point (I,u). If $|u| > \delta$, then $I - 2r\cos u + r^2 > \sin^2\delta$ for all r. Similarly, for $|v| > \delta$, $I - 2\rho\cos v + \rho^2 > \sin^2\delta$ for all ρ. Let us divide up the fundamental square into sections as shown in Fig. 2. The origin lies in the center of R_1, which has dimensions 2δ.

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*Uber Fouriersche Doppelreihen und das Poissonsche Doppelintegral, Munich, I9I3 page 4I.

Then

$$\left|\iint_S \bar{H}\bar{J}f(u,v)\,du\,dv\right| \leq \frac{(1-r^2)(1-\rho^2)}{4\pi^2\sin^4\delta}\left|\iint_S f(u,v)\,du\,dv\right|$$

If the integral on the right exists, then the $\lim_{r,\rho\to 1}$ of the integral on the left must be zero.

$$\left|\iint_A \bar{H}\bar{J}f(u,v)\,du\,dv\right| \leq \frac{1-r^2}{2\pi\sin^2\delta}\left|\iint_A \bar{J}f(u,v)\,du\,dv\right|$$

When the double integral exists, the repeated integrals exist and equal the double integral.

$$\iint_A \bar{J}f(u,v)\,du\,dv = \int_{-\delta}^{+\delta}\bar{J}dv\int_{\delta}^{\pi} f(u,v)\,du$$

Almost everywhere there exists $\int_{\delta}^{\pi} f(u,v)\,du = \varphi(v)$ which is an integrable function. If $\varphi(v)$ is bounded for $-\delta \leq v \leq \delta$, then $\int_{-\delta}^{\delta} J\varphi(v)\,dv$ is part of a regular single Poisson integral, and $\lim_{\rho\to 1}\int_{-\delta}^{\delta} J\varphi(v)\,dv = \varphi(y)$. But the $\lim_{r\to 1}\frac{1-r^2}{2\pi\sin^2\delta} = 0$. Hence $\lim_{r,\rho\to 1}\iint_A \bar{H}\bar{J}f(u,v)\,du\,dv = 0$. In a similar fashion, we can show that the limits of this integral over the regions B, C, and D are also zero. These results can be stated as

<u>Lemma I</u>. If $f(x,y)$ is an integrable function over the region R, and if for an arbitrarily small $\delta > 0$, the integrals

$$\int_{-\pi}^{-\delta} f(u,v)\,du, \quad \int_{\delta}^{\pi} f(u,v)\,du, \quad \int_{-\pi}^{-\delta} f(u,v)\,dv, \quad \int_{+\delta}^{\pi} f(u,v)\,dv$$

are bounded for $-\delta \leq v \leq \delta$, and $-\delta \leq u \leq \delta$, then

$$\lim_{r,\rho\to 1} \iint_{R_2} \bar{H}\bar{J} f(u,v)\,du\,dv = 0.$$

3. <u>The limit $\bar{F}$ when $\lim_{x,y\to 0} f(x,y) = +\infty$</u>. We next show that if the function $f(x,y)$ satisfies the conditions of Lemma I, and $\lim_{x,y\to 0} f(x,y) = +\infty$ for all methods of approach, the value of the Double Poisson Integral $\bar{F}(r,\rho)$ will increase indefinitely as r, ρ approach I. By hypothesis, there exists for any arbitrarily large value N a number δ, such that $f(u,v) > N$ for $|u| < 2\delta$, $|v| < 2\delta$. By Lemma I, we need consider the limiting values of the integral only over the region R_1. Referring to Fig. 3, take a point M' such that $I - r < \delta$. With M' as center and radius $\sqrt{2}(I - r)$, describe arcs cutting the circle in P and Q. Call arc $MP = x'$.

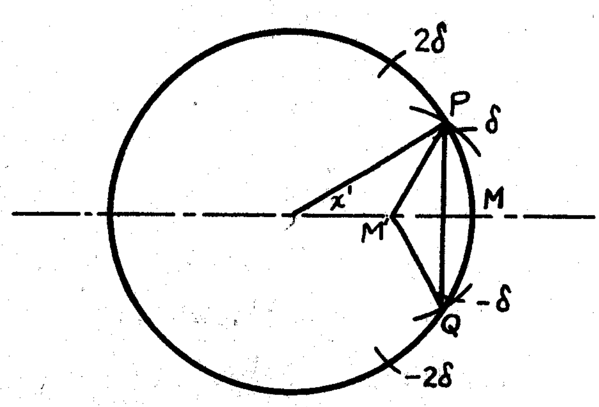

Fig. 3

Then

$$(1) \qquad \cos x' = \frac{1 + r^2 - 2(1 - r)^2}{2r} = 1 - \frac{(1 - r)^2}{2r}.$$

Chord PQ $= 2 \sin x'$

$$= 2(1 - r)\frac{1}{\sqrt{r}}\left[1 - \frac{(1 - r)^2}{4r}\right]^{\frac{1}{2}} \cdot \; \frac{1}{\sqrt{r}}\left[1 - \frac{(1 - r)^2}{4r}\right]^{\frac{1}{2}}$$

$$= \left[1 + \frac{1 - r}{2} + \cdots\cdots\right]\left[1 - \frac{(1 - r)^2}{8r} - \cdots\cdots\right]$$

Chord PQ $= 2(1 - r) + \epsilon'$ and arc PQ $= 2x' = 2(1 - r) + \epsilon$ where ϵ' and ϵ both approach zero in such a way that the limit $\lim\limits_{r \to 1} \frac{\epsilon'}{(1-r)^2}$ and the limit $\lim\limits_{r \to 1} \frac{\epsilon}{(1-r)^2}$ are finite. Hence, for sufficiently small values of $1 - r$, $x' < 2\delta$. Also for $|u| < x'$, we have from (1), $1 - 2r \cos u + r^2 < 2(1 - r)^2 = \overline{M'P}^2$, and so

$$\bar{H} = \frac{1 - r^2}{2\pi\left[1 - 2r\cos u + r^2\right]} > \frac{1 - r^2}{4\pi(1 - r)^2} = \frac{1 + r}{4\pi(1 - r)}$$

$$> \frac{1}{4\pi(1 - r)}.$$

Similarly we can obtain a value $y' < 2\delta$, such that $2y' = 2(1 - \rho) + \eta$ where η approaches zero with $(1 - \rho)^2$, and

$\bar{J} > \dfrac{1}{4\pi(1-\rho)}$ for $|v| < y'$.

$$\int_{-2\delta}^{2\delta}\int_{-2\delta}^{2\delta} \bar{H}\bar{J} f(u,v)\,du\,dv > \int_{-x'}^{x'}\int_{-y'}^{y'} \bar{H}\bar{J} f(u,v)\,du\,dv > \int_{-x'}^{x'}\int_{-y'}^{y'} \frac{N\,du\,dv}{16\pi^2(1-r)(1-\rho)}$$

$$= \frac{N}{4\pi^2}\left[1 + \frac{\epsilon}{2(1-r)} + \frac{\eta}{2(1- \quad)} + \frac{\epsilon\eta}{(1-r)(1-\rho)}\right]$$

The limit of this as r, ρ approach 1 is $\dfrac{N}{4\pi^2}$. Since N was arbitrarily large, $\lim\limits_{\lambda,\rho\to 1} \bar{F}(r,\rho) = +\infty$. Hence we have

<u>Theorem I.</u> If $f(x,y)$ is a function satisfying the conditions of Lemma I, and if $\lim\limits_{x,y\to 0} f(x,y) = +\infty$ for all methods of approach,

$$\lim_{\lambda,\rho\to 1} \bar{F}(r,\rho) = +\infty.$$

A similar theorem, of course, would hold if $\lim\limits_{x,y\to 0} f(x,y) = -\infty$ for all methods of approach.

SECTION II

<u>The Derivatives of $F(r,\rho,x,y)$</u>. We now propose to consider the limits as r,ρ, approach I, of the partial derivatives with respect to x and y of the function $F(r,\rho,x,y)$ defined by the Double Poisson Integral for the given function $f(x,y)$. Certain sufficient conditions will be established under which the derivatives of $F(r,\rho,x,y)$ will approach the corresponding derivatives of $f(x,y)$. As before, we shall use the origin as a perfectly general point.

I. <u>The special function.</u> We first need to show that without loss of generality, we can take

$$f(o,o) = f(x,o) = f(o,y) = f_x(o,o) = f_y(o,o)$$
$$= f_{xx}(o,o) = f_{yy}(o,o) = o.$$

Let $g(x,y)$ be the given function, and let

$$g(x,o) = \lim_{y\to+o} \tfrac{1}{2}\left[g(x,+y) + g(x,-y)\right]$$

$$(2)\quad g(o,y) = \lim_{x\to+o} \tfrac{1}{2}\left[g(+x,y) + g(-x,y)\right]$$

$$g(o,o) = \lim_{x,y\to+o} \tfrac{1}{4}\left[g(+x,+y)+g(-x,+y)+g(+x,-y)+g(-x,-y)\right]$$

If the original function, $g(x,y)$ is defined otherwise along the axes, we can arbitrarily redefine it as above, since the axes constitute an areal point set of measure zero. The value of the integral of any function is unaltered when the values of the function are changed at a set of points of measure zero. We define

(3) $\quad f(x,y) = g(x,y) - g(x,o) - g(o,y) + g(o,o).$

Then $\quad f(o,o) = f(x,o) = f(o,y) = o.$

We shall use the generalized derivatives, defined as follows:

$$f_x(o,o) = \lim_{h\to+o} \frac{f(h,o) - f(-h,o)}{2h} = o$$

$$f_y(o,o) = \lim_{k\to+o} \frac{f(o,k) - f(o,-k)}{2k} = o$$

(4)

$$f_{xx}(o,o) = \lim_{h\to+o} \frac{f(h,o) + f(-h,o) - 2f(o,o)}{h^2} = o$$

$$f_{yy}(o,o) = \lim_{k\to+o} \frac{f(o,k) - f(o,-k) - 2f(o,o)}{k} = o$$

Multiplying (3) by HJ, and integrating over R, we get

$$(5)\quad \iint_R HJf(u,v)\,du\,dv = \iint_R HJg(u,v)\,du\,dv - \iint_R HJg(u,o)\,du\,dv$$

$$- \iint_R HJg(o,v)\,du\,dv + \iint_R HJg(o,o)\,du\,dv$$

If the integrand functions on the right are integrable, the repeated integrals will exist, and equal the double integrals. Therefore

$$(6)\quad \iint_R HJf(u,v)\,du\,dv = \iint_R HJg(u,v)\,du\,dv - \int_{-\pi}^{\pi} Hg(u,o)\,du \times \int_{-\pi}^{\pi} J\,dv$$

$$-\int_{-\pi}^{\pi} H\,du \times \int_{-\pi}^{\pi} Jg(o,v)\,dv + g(o,o) \times \int_{-\pi}^{\pi} H\,du \times \int_{-\pi}^{\pi} J\,dv.$$

Now since the function HJ is uniformly continuous in u,v,x,y for $r, \rho < I$* we can differentiate under the integral sign, and write

$$(7)\quad F_\chi(r,\rho,x,y) = \iint_R H_\chi Jg(u,v)\,du\,dv - \int_{-\pi}^{\pi} H\,g(u,o)\,du \times \int_{-\pi}^{\pi} J\,dv$$

$$-\int_{-\pi}^{\pi} H_\chi\,du \times \int_{-\pi}^{\pi} Jg(o,v)\,dv + g(o,o) \times \int_{-\pi}^{\pi} H_\chi\,du \times \int_{-\pi}^{\pi} J\,dv.$$

Fatou has proved† that if the generalised derivative exists, $\lim_{\lambda \to 1} \int_{-\pi}^{\pi} \bar{H}_\chi g(u,o)\,du = g_\chi(o,o)$. $\int_{-\pi}^{\pi} \bar{J}g(o,v)\,dv$ is a regular Poisson Integral, and approaches $g(o,o)$ as ρ approaches I. If we can prove that there exists $\lim_{\lambda,\rho \to o} \bar{F}_\chi(r,\rho) = o = f_\chi(o,o)$, then it follows that the only remaining term in (7) must also approach a limit, L. Setting $x = y = o$ in (7), and taking the $\lim_{\lambda,\rho \to 1}$, we get

$$(8)\quad o = L - g_\chi(o,o) \times I - o \times g(o,o) + g(o,o) \times o \times I.$$

Hence, $L = g_\chi(o,o)$, and

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* Carslaw, Introduction to the Theory of Fourier's Series and Integrals, 3rd ed. page 190. Here the theorem is stated for integration over a linear point set, but the same identical proof applies to integration over a plane point set.
† Acta Mathematica, vol. 30, page 347.

(9) $$\underset{r,\rho\to 1}{\text{limit}} \iint_R \bar{H}_x \bar{J} g(u,v)\,du\,dv = g_x(o,o).$$

An exactly similar argument would show that

(10) $$\underset{r,\rho\to 1}{\text{limit}} \iint_R \overline{HJ}_y\, g(u,v)\,du\,dv = g_y(o,o).$$

The argument for differentiating under the integral sign in (6) will also apply to (7), and hence

(11) $$F_{xx}(r,\rho,x,y) = \iint_R H_{xx} J g(u,v)\,du\,dv - \int_{-\pi}^{\pi} H_{xx} g(u,o)\,du \times \int_{-\pi}^{\pi} J\,dv$$
$$- \int_{-\pi}^{\pi} H_{xx}\,du \times \int_{-\pi}^{\pi} J g(o,v)\,dv + g(o,o) \times \int_{-\pi}^{\pi} H_{xx}\,du \times \int_{-\pi}^{\pi} J\,dv.$$

Fatou has also proved* that if the generalized derivative exists, $\underset{r\to 1}{\text{limit}} \int_{-\pi}^{\pi} \bar{H}_{xx} g(u,o)\,du = g_{xx}(o,o)$. If we prove that there exists $\underset{r,\rho\to 1}{\text{limit}}\ \bar{F}_{xx}(r,\rho) = o = f_{xx}(o,o)$, then it follows as before that the term $\iint_R \bar{H}_{xx} \bar{J} g(u,v)\,du\,dv$ must approach a limit M. Setting $x = y = o$, and, taking the limits of the terms in (11), we get

(12) $o = M - g_{xx}(o,o) \times 1 - o \times g(o,o) + g(o,o) \times o \times 1.$

Therefore

(13) $$M = \underset{r,\rho\to 1}{\text{limit}} \iint_R \bar{H}_{xx} \bar{J} g(u,v)\,du\,dv = g_{xx}(o,o).$$

Similarly

(14) $$\underset{r,\rho\to 1}{\text{limit}} \iint_R \overline{HJ}_{yy}\, g(u,v)\,du\,dv = g_{yy}(o,o).$$

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* loc. cit. page 352.

2. The limits of the derivatives of $\bar{F}(r,\rho)$ as (r,ρ,o,o) approaches (I,I,o,o). We shall now prove that if the generalized derivative $f(x,y)$ exists in slightly modified form, $\lim_{r,\rho\to 1} F(r,\rho,o,o) = o$. In taking the limit, we first set $x=y=o$, and then let r,ρ approach I. This corresponds to what is termed radial approach in the case of the Single Poisson Integral. Differentiating under the integral sign, as in (7), we have

$$(15)\qquad \frac{\partial \bar{F}}{\partial x} = \iint_R \bar{H}\bar{J}\, \frac{2r\sin u}{1-2r\cos u + r^2}\, f(u,v)\,du\,dv$$

Since $\bar{H}$, $\bar{J}$ are even functions of u and v, while the fraction is an odd function of u, we can write (15) in the form

$$(16)\qquad \frac{\partial \bar{F}}{\partial x} =$$

$$\int_0^\pi\!\!\int_0^\pi \bar{H}\bar{J}\,\frac{2r\sin u}{1-2r\cos u + r^2}\left[f(u,v)-f(-u,v)+f(u,-v)-f(-u,-v)\right]du\,dv$$

$$(17)\qquad \frac{2r\sin u}{1-2r\cos u + r^2} = \frac{4r\sin\frac{u}{2}\cos\frac{u}{2}}{(1-r)^2+4r^2\sin^2\frac{u}{2}} \le \frac{1}{\tan\frac{u}{2}}\,.$$

$$(18)\qquad \left|\frac{\partial \bar{F}}{\partial x}\right| \le$$

$$\int_0^{\pi}\int_0^{\pi} \bar{H}\bar{J}\cdot 4 \left| \frac{f(u,v)-f(-u,v)}{2u} + \frac{f(u,-v)-f(-u,-v)}{2u} \right| \frac{\frac{u}{2}}{\tan\frac{u}{2}}\, du\, dv.$$

If the expression in the absolute value signs under the integral satisfies the conditions of Lemma I, (18) is a regular Double Poisson Integral whose limit as r, ρ approach 1 depends only on the behavior of the integrand function in the immediate neighborhood of the origin. By Kustermann's results* the limit of this integral is

$$(19) \qquad \lim_{u,v\to+0} \left| \frac{f(u,v)-f(-u,v)}{2u} + \frac{f(u,-v)-f(-u,-v)}{2u} \right|$$

provided this limit exists. But if this double limit exists, so also must the repeated limit, and the two will be equal†. Hence

$$(20) \qquad \lim_{u,v\to 0} \left| \frac{f(u,v)-f(-u,v)}{2u} + \frac{f(u,-v)-f(-u,-v)}{2u} \right|$$

$$= \lim_{u\to 0}\left\{ \lim_{v\to 0} \left| \frac{f(u,v)-f(-u,v)}{2u} + \frac{f(u,-v)-f(-u,-v)}{2u} \right| \right\}$$

$$= |2f_x(0,0)| = 0.$$

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* loc. cit. page 44.

† E.W.Hobson, The Theory of Functions of a Real Variable, 3rd. ed. vol. I page 407.

Therefore, $\lim\limits_{r,\rho\to 0} \bar{F}_x(r,\rho) = o = f_x(o,o)$, and we have proved

Theorem 2. If the function $f(x,y)$ is integrable over the region R, if there exist the integrals $f(x,o)dx$, $f(o,y)dy$, and if the quotient*

$$\frac{f(u,v)-f(-u,v)+f(u,-v)-f(-u,-v)}{2u}$$

satisfies the conditions of Lemma I and has a double limit as u, v approach o, the function $f(x,y)$ has a first partial derivative with respect to x at the point (o,o), and $\lim\limits_{r,\rho\to 1} \bar{F}_x(r,\rho) = f_x(o,o)$.

In an exactly analogous fashion, we can prove

Theorem 2'. If the function $f(x,y)$ is integrable over the region R, if there exist the integrals $f(x,o)dx$, $f(o,y)dy$, and if the quotient*

$$\frac{f(u,v)-f(u,-v)+f(-u,v)-f(-u,-v)}{2v}$$

satisfies the conditions of Lemma I and has a double limit as u,v approach o, the function $f(x,y)$ has a first partial derivative with respect to y at the point (o,o), and $\lim\limits_{r,\rho\to 1} \bar{F}_y(r,\rho) = f_y(o,o)$.

Let us now consider the second partial derivatives

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* In applying this condition, the form of the function $f(x,y)$ defined in (5) must be used, but we have shown that this form involves no loss of generality.

with respect to x and y of $\bar{F}(r,\rho)$. Differentiating (15) we have

$$(21)\quad \frac{\partial^2 \bar{F}}{\partial x^2} = \iint_R \bar{H} J \, \frac{4r^2 \sin^2 u + 4r^2 - 2r(1+r^2)\cos u}{[1 - 2r\cos u + r^2]^2} f(u,v)\,du\,dv$$

Since all the factors preceding f(u,v) are even functions of u and v, we can write (21) in the form

$$\frac{\partial^2 \bar{F}}{\partial x^2} = \int_0^\pi\int_0^\pi \bar{H} J \, \frac{4r^2 \sin^2 u + 4r^2 - 2r(1+r^2)\cos u}{[1 - 2r\cos u + r^2]^2} \cdot \frac{u^2 \varphi(u,v)}{u^2}\,du\,dv$$

where $\varphi(u,v) = f(u,v)+f(-u,v)+f(u,-v)+f(-u,-v)$, and the $\frac{u^2}{u^2}$ is inserted for convenience. The function

$$(22)\quad \frac{4r^2 \sin^2 u + 4r^2 - 2r(1+r^2)\cos u}{[1 - 2r\cos u + r^2]^2} \cdot u^2$$

is continuous in u and r except for the point $u = 0$, $r = 1$. We shall prove that this function remains bounded for all values of u and r, by showing that it approaches a finite limit uniformly for all methods of approach. Since the $\lim_{u \to 0} u/\sin u = 1$, we can replace sin u by u in taking the limit. Let us consider the $\lim_{u \to 0}$ along the line $1-r = mu$.

The function (22) can be written

$$\frac{4r^2\sin^2 u + 4r(1+r^2)\sin^2\frac{u}{2} - 2r(1-r)^2}{\left[(1-r)^2 + 4r\sin^2\frac{u}{2}\right]^2}\cdot u^2$$

Replacing sin u by u, 1-r by mu, we get

$$\frac{(1-mu)\left[4(1-mu)u^2 + (2-2mu+m^2u^2)u^2 - 2m^2u^2\right]u^2}{\left[m^2u^2 + (1-mu)u^2\right]^2}$$

$$= \frac{(1-mu)\left[6-6mu+m^2u^2-2m^2\right]}{\left[1-mu+m^2\right]^2}$$

The $\lim\limits_{u\to 0}$ of this fraction is

(23) $$\frac{6-2m^2}{(1+m^2)^2}$$

which is bounded for all m. Consider the difference

$$\frac{(1-mu)\left[6-6mu+m^2u^2-2m^2\right]}{\left[1-mu+m^2\right]^2} - \frac{6-2m^2}{(1+m^2)^2}$$

For $o \leqq u \leqq I$, no real values of both m and u can make $I - mu + m^2 = o$. As $m \to \infty$, the difference approaches zero, since the power of m in the denominators exceeds that in the numerators. Hence this difference remains bounded, and approaches zero uniformly in m as $u \to o$. Since (23) is bounded, we know that (22) must be bounded. Call its greatest absolute value M. Then

$$\left|\frac{\partial^2 F}{\partial \chi^2}\right| \leqq \int_0^{\pi}\int_0^{\pi} HJM \left|\frac{\varphi(u,v)}{u^2}\right| du\, dv$$

If $\dfrac{\varphi(u,v)}{u^2}$ satisfies the conditions of Lemma I, and if there exists the $\lim\limits_{u,v\to o} \dfrac{\varphi(u,v)}{u^2}$, this is a regular Poisson Integral. Hence, as in (I9)

$$(24) \qquad \lim_{\lambda,\rho\to 1} \left|\bar{F}_{\chi\chi}(r,\rho)\right| \leqq \frac{M}{4} \lim_{u,v\to o} \left|\frac{\varphi(u,v)}{u^2}\right|$$

By the same argument as in (20)

$$\lim_{u,v\to o} \frac{\varphi(u,v)}{u^2} = o = f_{\chi\chi}(o,o).$$

These results are summarized in

<u>Theorem 3.</u> If the function $f(x,y)$ is integrable over the region R, if there exist the integrals $\int_{-\pi}^{\pi} f(x,o)dx$, $\int_{-\pi}^{\pi} f(o,y)dy$, and if the quotient*

$$\frac{f(u,v)+f(u,-v)+f(-u,v)+f(-u,-v)}{u^2}$$

satisfies the conditions of Lemma I and has a double limit as u,v approach o, the function $f(x,y)$ has a second partial derivative with respect to x at the point (o,o), and $\lim_{r,\rho \to 1} \bar{F}_{xx}(r,\rho) = f_{xx}(o,o)$.

In an exactly analogous fashion, we can prove

<u>Theorem 3'.</u> If the function $f(x,y)$ is integrable over the region R, if there exist the integrals $\int_{-\pi}^{\pi} f(x,o)dx$, $\int_{-\pi}^{\pi} f(o,y)dy$, and if the quotient*

$$\frac{f(u,v)+f(u,-v)+f(-u,v)+f(-u,-v)}{v^2}$$

satisfies the conditions of Lemma I, and has a double limit as u,v approach o, the function $f(x,y)$ has a second partial derivative with respect to y at the point (o,o), and $\lim_{r,\rho \to 1} \bar{F}_{yy}(r,\rho) = f_{yy}(o,o)$.

For the mixed second partial, $\bar{F}_{xy}(r,\rho)$, it is convenient to use a form of function slightly different from (3). This will not restrict the generality in any way as

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* See footnote on page 15.

long as the mixed partial, $f_{xy}(x,y)$, exists. The new form is defined by

(25) $$f(x,y) = g(x,y) - xy\, g_{xy}(o,o)$$

where

$$g_{xy}(o,o) = \lim_{h,k\to 0} \frac{g(h,k)-g(-h,k)-g(h,-k)+g(-h,-k)}{4hk}$$

Therefore, $f_{xy}(o,o) = o$.

$$F(r,\rho,x,y) = \iint_R H\, J\, g(u,v)du\, dv - g_{xy}(o,o) \int_{-\pi}^{\pi} H\, udu \int_{-\pi}^{\pi} J\, vdv.$$

As before, we can differentiate under the integral sign, and write

$$F_{xy}(r,\rho,x,y) = \iint_R H_x J_y g(u,v)du\, dv - g_{xy}(o,o) \int_{-\pi}^{\pi} H_x udu \int_{-\pi}^{\pi} J_y vdv.$$

Taking the limits as r, ρ approach I, we know from Fatou's results* that $\lim_{r\to 1} \int_{-\pi}^{\pi} H\, udu = dx/dx = I$, and $\lim_{\rho\to 1} \int_{-\pi}^{\pi} J\, vdv = dy/dy = I$. If we prove that the $\lim_{r,\rho\to 1} \bar{F}_{xy}(r,\rho) = o = f_{xy}(o,o)$, it will necessarily follow that the $\lim_{r,\rho\to 1} \iint_R \bar{H}_x \bar{J}_y g(u,v)du\, dv = g_{xy}(o,o)$. Consider then

(26) $$\frac{\partial^2 \bar{F}}{\partial x\, \partial y} = \iint_R \bar{H}\bar{J} \frac{2r \sin u}{1-2r\cos u + r^2} \cdot \frac{2\rho \sin v}{1-2\rho\cos v+\rho^2} f(u,v)du\, dv$$

Since $\bar{H}$ and $\bar{J}$ are even functions, while the fractions are

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* loc. cit. page 347.

odd functions of u and v, we can write (26) as

$$\frac{\partial^2 \bar{F}}{\partial x \partial y} =$$

$$\int_0^{\pi}\!\!\int_0^{\pi} \bar{H}\bar{J}\,\frac{2r\sin u}{1-2r\cos u+r^2}\cdot\frac{2\rho\sin v}{1-2\rho\sin v+\rho^2}\Big[f(u,v)-f(-u,v)-f(u,-v)+f(-u,-v)\Big]\,du\,dv$$

Applying (17), we have

$$\left|\frac{\partial^2 \bar{F}}{\partial x \partial y}\right| \leq \int_0^{\pi}\!\!\int_0^{\pi} \bar{H}\bar{J}\cdot 16\left|\frac{f(u,v)-f(-u,v)-f(u,-v)+f(-u,-v)}{4uv}\right|\frac{\frac{uv}{4}}{\tan\frac{u}{2}\tan\frac{v}{2}}\,du\,dv$$

If the expression between the absolute value signs satisfies the conditions of Lemma I, and if the derivative $f_{xy}(o,o)$ exists, this is a regular Poisson Integral. Its limit as r, ρ approach I is $4\, f_{xy}(o,o)\ I = o = f_{xy}(o,o)$. Therefore, $\lim_{r,\rho\to 1} \bar{F}_{xy}(r,\rho) = f_{xy}(o,o)$, and we have proved

<u>Theorem 4</u>. If the function $f(x,y)$ is integrable over the region R, and possesses a mixed partial derivative at the point (o,o), and if the differential quotient

$$\frac{f(u,v)-f(-u,v)-f(u,-v)+f(-u,-v)}{4uv}$$

satisfies the conditions of Lemma I, the $\lim_{r,\rho\to 1} \bar{F}_{xy}(r,\rho)$ $= f_{xy}(o,o)$.

3. <u>Extension of Schwarz' argument to the Double Poisson Integral</u>. In the treatment of the limits in the

preceding article, we considered only the case in which the point $(r,\rho,0,0)$ approached the point $(1,1,0,0)$. In order to investigate the more general case where (r,ρ,x,y) approaches $(1,1,0,0)$, we first need to study the limit of the Double Poisson Integral itself for such methods of approach. Picard has shown* that if $f(x)$ is a bounded continuous function and has a period of 2π, $\lim\limits_{\lambda \to 1} \int_{-\pi}^{\pi} H\, f(u)\,du = f(0)$, the point (r,x) approaching the point $(1,0)$ in any manner. His treatment was due originally to Schwarz. We shall prove a similar result for the Double Poisson Integral.

If $f(x,y)$ is continuous in x and y at the point $(0,0)$, there exists a δ such that $|f(u,v) - f(0,0)| < \epsilon$ for $|u| < 2\delta$, $|v| < 2\delta$. Now consider a point (x,y) in the plane of integration (the UV plane) such that $|x| < \delta$, $|y| < \delta$. Take a cross-region with center at (x,y) as shown in Fig.4, and denote the remainder of the square by S. Divide the cross-region into parts, A,B,C,D,E as shown. If the dimensions of the square E are 2δ, the square E will lie wholly within that region where $|f(u,v) - f(0,0)| < \epsilon$. Further-

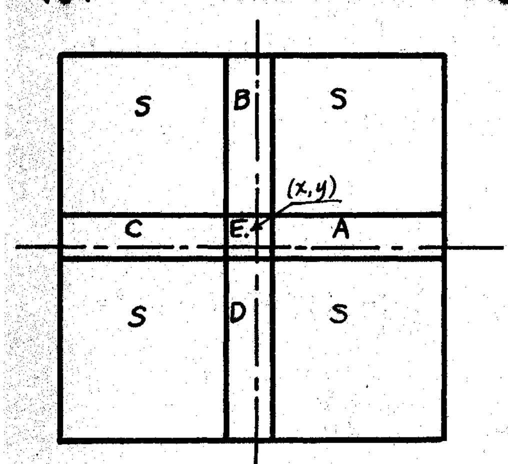

Fig. 4

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* Picard, Traite D'Analyse, 2nd ed., vol. I, page 270.

more, it will contain the point (o,o). In the regions S, A, C,

$$1 - 2r\cos(u-x) + r^2 > 2r - 2r\cos\delta = 2r(1-\cos\delta).$$

In the regions S, B, D,

$$1 - 2\rho\cos(v-y) + \rho^2 > 2\rho - 2\rho\cos\delta = 2\rho(1-\cos\delta).$$

Let M denote the maximum value of $|f(u,v)|$ thruout the region R.

(27) $$\left|\iint_S HJf(u,v)\,du\,dv\right|$$

$$< \iint_R \frac{M(1-r^2)(1-\rho^2)}{4r\rho(1-\cos\delta)^2\,4\pi^2}\,du\,dv = \frac{M(1-r^2)(1-\rho^2)}{4r\rho(1-\cos\delta)^2}$$

(28) $$\left|\iint_A HJf(u,v)\,du\,dv\right|$$

$$< \iint_R \frac{M(1-r^2)}{2r(1-\cos\delta)2\pi}\cdot J\,du\,dv = \frac{M(1-r^2)}{2r(1-\cos\delta)}$$

Expressions similar to (28) hold for the integral taken over the regions B, C, D. The final results in (27) and (28) are independent of (x,y), and their limits as (r,ρ,x,y) approaches $(1,1,o,o)$ are zero. Hence in considering the limit of the Double Poisson Integral in the case of general approach, we need to investigate only that part inside the region E. We can write

$$F(r,\rho,x,y) = \iint_R HJf(o,o)\,du\,dv + \iint_R \left[f(u,v)-f(o,o)\right] HJ\,du\,dv.$$

Now $$\iint_R HJf(o,o)\,du\,dv = f(o,o).$$

$$\left| \iint_E \left[f(u,v)-f(o,o)\right] HJ \, du \, dv \right| < \iint_E \epsilon \, HJ \, du \, dv$$

$$< \iint_R \epsilon \, HJ \, du \, dv = \epsilon$$

for every r, ρ and for every point (x,y) having $|x| < \delta$, and $|y| < \delta$. Hence the

$$\lim_{\substack{r,\rho \to 1 \\ x,y \to 0}} \iint_E \left[f(u,v)-f(o,o)\right] HJ \, du \, dv \quad o ,$$

and the

$$\lim_{\substack{r,\rho \to 1 \\ x,y \to 0}} F(r,\rho,x,y) = f(o,o).$$

This result can be stated explicitly as

<u>Lemma 2</u>. If the function $f(x,y)$ is bounded and integrable over the region R, and if it is continuous in x and y at the point (o,o), the $\lim\limits_{\substack{r,\rho \to 1 \\ x,y \to 0}} F(r,\rho,x,y) = f(o,o)$ as (r,ρ,x,y) approaches (I,I,o,o) in any manner.

4. <u>Limit of the Derivatives of $F(r,\rho,x,y)$ as (r,ρ,x,y) approaches (I,I,o,o)</u>. In dealing with the case of general approach to the limit, it is necessary to assume that the given function and its derivatives are continuous at the point in question. This condition makes the proofs somewhat simpler, and enables us to employ the given function directly, without putting it in the form (3).

$$(29) \qquad F_x(r,\rho,x,y) = \iint_R H_x J f(u,v) \, du \, dv$$

From the definitions of H and J, we observe that

(30) $\quad H_x = -H_u \quad , \quad J_y = -J_v$. Hence

$$F_x = -\iint_R H_u J f(u,v)\,du\,dv = -\int_{-\pi}^{\pi} J\,dv \int_{-\pi}^{\pi} H_u f(u,v)\,du,$$

for, if the double integral exists, so does the repeated integral and is equal to it. If $f(u,v)$ has a partial derivative, $f_u(u,v)$, $f(u,v) = \int f_u(u,v)\,du$ almost everywhere and we can integrate by parts.

(31) $$\int_{-\pi}^{\pi} H_u f(u,v)\,du = \Big[Hf(u,v)\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} Hf_u(u,v)\,du$$

(32) $$\iint_R H_x J f(u,v)\,du\,dv = \iint_R HJf_u(u,v)\,du\,dv - \int_{-\pi}^{\pi}\Big[Hf(u,v)\Big]_{-\pi}^{\pi} J\,dv$$

If $f(u,v)$ is bounded over the region R, there exists an M such that $|f(u,v)| < M$. Now

$$\int_{-\pi}^{\pi}\Big[Hf(u,v)\Big]_{-\pi}^{\pi} J\,dv = \frac{1-r^2}{1+2r\cos x+r^2}\int_{-\pi}^{\pi} J\Big[f(\pi,v)-f(-\pi,v)\Big]\,dv$$

But

$$\left|\int_{-\pi}^{\pi} Jf(\pm\pi,v)\,dv\right| < M\int_{-\pi}^{\pi} J\,dv = M.$$

Therefore

(33) $$\lim_{\substack{r,\rho\to 1\\ x,y\to 0}} \int_{-\pi}^{\pi}\Big[Hf(u,v)\Big]_{-\pi}^{\pi} J\,dv = 0.$$

If $f(u,v)$ is bounded and integrable over the region R, and is continuous in u and v at the point (o,o), then by Lemma 2,

$$\underset{\substack{r,\rho\to 1\\ x,y\to 0}}{\text{limit}} \iint_R HJf_u(u,v)\,du\,dv = f_x(o,o).$$

Combining (29), (32), and (33), we observe that the

$$\underset{\substack{r,\rho\to 1\\ x,y\to 0}}{\text{limit}} F_x(r,\rho,x,y) = f_x(o,o).$$

This completes the proof of

<u>Theorem 5</u>. If the function $f(x,y)$, together with its first partial derivative $f_x(x,y)$, is bounded and integrable over the region R, and if $f_x(x,y)$ is continuous in x and y at the point (o,o), the

$$\underset{\substack{r,\rho\to 1\\ x,y\to 0}}{\text{limit}} F_x(r,\rho,x,y) = f_x(o,o).$$

In an exactly similar fashion, we can prove

<u>Theorem 5'</u>. If the function $f(x,y)$, together with its first partial derivative $f_y(x,y)$, is bounded and integrable over the region R, and if $f_y(x,y)$ is continuous in x and y at the point (o,o), the

$$\underset{\substack{r,\rho\to 1\\ x,y\to 0}}{\text{limit}} F_y(r,\rho,x,y) = f_y(o,o).$$

The treatment of the second partial derivative proceeds in much the same way, many of the same arguments be-applicable.

$$(34)\quad F_{xx}(r,\rho,x,y) = \iint_R H_{xx}Jf(u,v)du\ dv = \iint_R H_{uu}Jf(u,v)du\ dv$$

$$= \int_{-\pi}^{\pi} J\ dv \int_{-\pi}^{\pi} H_{uu}f(u,v)du$$

If f(u,v) has both a first and second partial derivative with respect to x almost everywhere over the region R, we can integrate by parts twice.

$$\int_{-\pi}^{\pi} H_{uu}f(u,v)du = \Big[H_u f(u,v)\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} H_u f_u(u,v)du$$

$$= \Big[H_u f(u,v)\Big]_{-\pi}^{\pi} - \Big[Hf_u(u,v)\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} Hf_{uu}(u,v)du$$

$$(35)\quad F_{xx}(r,\rho,x,y) = \int_{-\pi}^{\pi} \Big[H_u f(u,v)\Big]_{-\pi}^{\pi} J\ dv - \int_{-\pi}^{\pi} \Big[Hf_u(u,v)\Big]_{-\pi}^{\pi} J\ dv$$

$$+ \iint_R HJf_{uu}(u,v)du\ dv.$$

If $|f_u(u,v)| < M$, then by the argument for (33)

$$\lim_{\substack{r,\rho\to 1\\ x,y\to 0}} \int_{-\pi}^{\pi} \left[H f_u(u,v) \right]_{-\pi}^{\pi} J \, dv = 0.$$

$$H_u(\pm\pi,-x) = \frac{(1-r)^2 \, 2r \sin x}{2\pi\left[1 - 2r\cos x + r^2\right]^2}$$

which approaches zero as r,ρ approach 1, and x,y approach zero. Therefore,

$$\lim_{\substack{r,\rho\to 1\\ x,y\to 0}} \int_{-\pi}^{\pi} \left[H_u f(u,v) \right]_{-\pi}^{\pi} J \, dv = 0.$$

If $f_{uu}(u,v)$ is bounded and integrable over the region R, and is continuous in u and v at the point (0,0), then by Lemma 2,

$$\lim_{\substack{r,\rho\to 1\\ x,y\to 0}} \iint_R HJ f_{uu}(u,v)\,du\,dv = f_{xx}(0,0).$$

Thus we have proved

<u>Theorem 6.</u> If $f(x,y)$, together with its first and second partial derivatives with respect to x, is bounded and integrable over the region R, and if $f_{xx}(x,y)$ is continuous in x and y at the point (0,0), the

$$\lim_{\substack{r,\rho\to 1\\ x,y\to 0}} F_{xx}(r,\rho,x,y) = f_{xx}(0,0).$$

In a similar fashion, we can prove

<u>Theorem 6</u>$'$. If $f(x,y)$, together with its first and second partial derivatives with respect to y, is bounded and integrable over the region R, and if $f_{yy}(x,y)$ is continuous in x and y at the point (o,o), the

$$\lim_{\substack{\lambda,\rho\to 1\\ x,y\to 0}} F_{yy}(r,\rho,x,y) = f_{yy}(o,o).$$

We next consider the mixed partial derivative.

$$(36)\quad F_{xy}(r,\rho,x,y) = \iint_R H_x J_y f(u,v)\,du\,dv = \iint_R H_u J_v f(u,v)\,du\,dv$$

If $f(u,v)$ has a second mixed partial derivative almost everywhere over the region R, then

$$f(u,v) = \iint f_{uv}(u,v)\,du\,dv, \quad \text{and}$$

(37)

$$f_u(u,v) = \int f_{uv}(u,v)\,dv\,, \quad f_v(u,v) = \int f_{uv}(u,v)\,du,$$

exist and are integrable almost everywhere. Under this condition we can integrate by parts*.

$$(38)\quad \iint_R H_u J_v f(u,v)\,du\,dv = \Big[HJf(u,v)\Big]_{-\pi,-\pi}^{\pi,\pi} - \iint_R HJf_{uv}(u,v)\,du\,dv$$

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* H. Geiringer, Monatshefte fur Math. u. Physik, vol. 29 (1918) page 82.

$$- \iint_R H_u J f_v(u,v)\,du\,dv - \iint_R HJ_v f_u(u,v)\,du\,dv.$$

Now $\left[HJf(u,v)\right]_{-\pi,-\pi}^{\pi,\pi} =$

$$\frac{(1-r^2)(1-\rho^2)}{4\pi^2\left[1+2r\cos u+r^2\right]\left[1+2\rho\cos v+\rho^2\right]}\left[f(\pi,\pi)-f(-\pi,-\pi)\right]$$

If $f(u,v)$ is bounded, the $\lim\limits_{\substack{r,\rho\to 1\\ x,y\to 0}} \left[HJf(u,v)\right]_{-\pi,-\pi}^{\pi,\pi} = 0.$

If the derivatives f_u, f_v, f_{uv} are all bounded and integrable functions over the region R, and if they are continuous in u and v at the point (o,o), we have

$$(39)\qquad \lim_{\substack{r,\rho\to 1\\ x,y\to 0}} \iint_R HJf_{uv}(u,v)\,du\,dv = f_{xy}(0,0) \text{ by Lemma 2}$$

$$\lim_{\substack{r,\rho\to 1\\ x,y\to 0}} \iint_R H_uJf_v(u,v)\,du\,dv = -\frac{\partial}{\partial x} f_y(0,0) \text{ by Theorem 5}$$

(40)

$$\lim_{\substack{r,\rho\to 1\\ x,y\to 0}} \iint_R HJ_v f_u(u,v)\,du\,dv = -\frac{\partial}{\partial y} f_x(0,0) \text{ by Theorem 5}'$$

Because of the relations (37), we know that

(41) $\frac{\partial}{\partial x} f_y(x,y) = \frac{\partial}{\partial y} f_x(x,y) = f_{xy}(x,y)$, the generalized mixed derivative, almost everywhere. This, however, is no

assurance that they will be equal at the particular point, (o,o). Making this additional assumption of equality, and applying relations (39), (40), (41), to (38), we have

$$\begin{aligned} \lim_{\substack{\lambda,\rho\to 1\\ x,y\to 0}} F_{xy}(r,\rho,x,y) &= -f_{xy}(o,o) + f_{xy}(o,o) + f_{xy}(o,o) \\ &= f_{xy}(o,o). \end{aligned}$$

This is summarized in the

Theorem 7. If the function $f(x,y)$, together with its derivatives $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$, is bounded and integrable over the region R, if the derivatives are continuous at the point (o,o), and if

$$\frac{\partial}{\partial x} f_y(o,o) = \frac{\partial}{\partial y} f_x(o,o),$$

then the $\lim_{\substack{\lambda,\rho\to 1\\ x,y\to 0}} F_{xy}(r,\rho,x,y) = f_{xy}(o,o).$

SECTION III - APPLICATIONS

I. The $\lim_{r,\rho\to 1}$ $F(r,\rho,x,y)$ almost everywhere. By means of the preceding results of Article 2, Section II, we can prove that, for a certain class of functions, $f(x,y)$, $\lim_{r,\rho\to 1} F(r,\rho,x,y) = f(x,y)$ almost everywhere. While such a result is not applicable in determining the value of the $\lim_{r,\rho\to 1} F(r,\rho,x,y)$ for a given point (x,y), it is valuable in dealing with the $\lim_{r,\rho\to 1}$ of the integral of $F(r,\rho,x,y)$.

If the function $f(u,v)$ is integrable, there exists a bounded and continuous function, $G(u,v) = \int_a^u\int_b^v f(x,y)\,dx\,dy$, such that

$$(42)\qquad f(u,v) = G_{uv}(u,v), \qquad G_u(u,v) = \int_b^v f(u,y)\,dy,$$

$$G_v(u,v) = \int_a^u f(x,v)\,dx$$

almost everywhere. Hence we can integrate by parts as in (38).

$$(43)\qquad F(r,\rho,x,y) = \iint_R HJf(u,v)\,du\,dv = \Big[HJG(u,v)\Big]_{-\pi,-\pi}^{\pi,\pi}$$

$$- \iint_R H_uJ_vG(u,v)\,du\,dv - \iint_R HJ_vG_u(u,v)\,du\,dv$$

$$- \iint_R H_uJG_v(u,v)\,du\,dv$$

Since $G(u,v)$ is bounded, $\lim_{r,\rho\to 1} \Big[HJG(u,v)\Big]_{-\pi,-\pi}^{\pi,\pi} = 0$ as before. If $G_u(u,v)$ and $G_v(u,v)$ satisfy the conditions of Theorem 2 almost everywhere,

$$\underset{r,\rho\to 1}{\text{limit}} \iint_R HJ_v G_u(u,v)\,du\,dv = -\frac{\partial}{\partial y} G_x(x,y) = f(x,y)$$

$$\underset{r,\rho\to 1}{\text{limit}} \iint_R H_u J G_v(u,v)\,du\,dv = -\frac{\partial}{\partial x} G_y(x,y) = f(x,y)$$

almost everywhere. If the differential quotient

$$\frac{G(u+h,v+k) - G(u-h,v+k) - G(u+h,v-k) + G(u-h,v-k)}{4hk}$$

satisfies the conditions of Lemma I, then by Theorem 4

$$\underset{r,\rho\to 1}{\text{limit}} \iint_R H_u J_v\, G(u,v)\,du\,dv = G_{xy}(x,y) = f(x,y)$$

almost everywhere. We have then

<u>Theorem 8.</u> If the function $f(x,y)$ is periodic and integrable over the region R; if the three differential quotients

$$\frac{G_u(u+h,v+k)-G_u(u+h,v-k)+G_u(u-h,v+k)-G_u(u-h,v-k)}{4k}$$

$$(44) \quad \frac{G_v(u+h,v+k)-G_v(u-h,v+k)+G_v(u+h,v-k)-G_v(u-h,v-k)}{4h}$$

$$\frac{G(u+h,v+k)-G(u-h,v+k)-G(u+h,v-k)+G(u-h,v-k)}{4hk}$$

satisfy the conditions of Lemma I for almost every point (u,v) in the region R; and if the double limits of the first two* exist almost everywhere, then the $\lim_{\lambda,\rho\to 1}$ $F(r,\rho,x,y) = f(x,y)$ almost everywhere as the point (r,ρ,x,y) approaches the point $(1,1,x,y)$.

This theorem could be stated somewhat more simply under the assumption that $f(x,y)$ is bounded thruout the region.R. It then follows from (42) that the differential quotients (44) are bounded, and hence must satisfy the condition of Lemma I. We have to make only the additional assumption that the double limits of the first two exist almost everywhere.

<u>2. Parseval's Theorem</u>. By means of the results of Article I, we are able to give a proof of Parseval's Theorem for Double Fourier Series. Let $f(x,y)$ be a function satisfying the conditions of Theorem 8. Then we can define the quantities

$$(45)\qquad \begin{matrix} a_{mn}= \\ b_{mn}= \\ c_{mn}= \\ d_{mn}= \end{matrix} \quad \frac{1}{\psi(m,n)\pi^2}\iint_R f(u,v)\begin{cases} \cos mu \cos nv \, du\, dv \\ \cos mu \sin nv \, du\, dv \\ \sin mu \cos nv \, du\, dv \\ \sin mu \sin nv \, du\, dv \end{cases}$$

* From the definition (42) of $G(u,v)$, the double limit of the third quotient in (44) must exist and equal $f(x,y)$ almost everywhere.

where

$$\psi(m,n) = \begin{cases} 4, \text{ if } m = n = 0 \\ 2, \text{ if } mn = 0,\ m \neq n \\ 1, \text{ if } mn > 0. \end{cases}$$

(46) $K_{mn} = a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny$

$$+ c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny.$$

The Double Fourier Series representation of the given function is

$$f(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} K_{mn}.$$

In the Poisson method of summing this series, we introduce convergence factors $r^m \rho^n$, and define

(47) $$F(r,\rho,x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r^m \rho^n K_{mn}.$$

The series (47) is absolutely and uniformly convergent thruout the region R, whenever we fix $r, \rho < 1$. Hence, using (45) and (46), we can write

(48) $F(r,\rho,x,y)$

$$= \frac{1}{\pi^2} \iint_R \left[\frac{1}{2} + \sum_{m=1}^{\infty} r^m \cos m(u-x)\right] \left[\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos n(v-y)\right] f(u,v)\, du\, dv.$$

Now

$$\frac{1}{2} + \sum_{m=1}^{\infty} r^m \cos m(u-x) = \frac{1-r^2}{2\left[1-2r \cos(u-x)+r^2\right]} = \pi H$$

$$\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos n(v-y) = \frac{1-\rho^2}{2\left[1 - 2\rho\cos(v-y)+\rho^2\right]} = \pi J$$

$$F(r,\rho,x,y) = \iint_R HJf(u,v)\,du\,dv$$

This is a regular Poisson Integral, and by Theorem 8 we know that the $\lim\limits_{r,\rho\to 1} F(r,\rho,x,y) = f(x,y)$ almost everywhere. Being uniformly convergent, (47) can be multiplied thru by $\frac{1}{\pi^2}F(r,\rho,x,y)$ and integrated termwise.

$$(49)\quad \frac{1}{\pi^2}\iint_R F^2(r,\rho,x,y)\,dx\,dy = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{r^m\rho^n}{\pi^2}\iint_R K_{mn}F(r,\rho,x,y)\,dx\,dy$$

$$\frac{r^m\rho^n}{\pi^2}a_{mn}\iint_R F(r,\rho,x,y)\cos mx\cos ny\,dx\,dy = \psi(m,n)a_{mn}^2 r^{2m}\rho^{2n}$$

as we can readily see if we multiply (45) thru by cos mx cos ny and integrate termwise. In like manner,

$$\frac{r^m\rho^n}{\pi^2}b_{mn}\iint_R F(r,\rho,x,y)\cos mx\sin ny\,dx\,dy = \psi(m,n)b_{mn}^2 r^{2m}\rho^{2n}$$

$$\frac{r^m\rho^n}{\pi^2}c_{mn}\iint_R F(r,\rho,x,y)\sin mx\cos ny\,dx\,dy = \psi(m,n)c_{mn}^2 r^{2m}\rho^{2n}$$

$$\frac{r^m\rho^n}{\pi^2}d_{mn}\iint_R F(r,\rho,x,y)\sin mx\sin ny\,dx\,dy = \psi(m,n)d_{mn}^2 r^{2m}\rho^{2n}$$

Hence

$$(50)\quad \frac{1}{\pi^2}\iint_R F^2(r,\rho,x,y)\,dx\,dy$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{mn}^2 + b_{mn}^2 + c_{mn}^2 + d_{mn}^2) r^{2m} \rho^{2n} \psi(m,n)$$

a convergent series. Therefore the integral in (50) must exist for $r, \rho < I$. But since the $\lim_{r,\rho \to 1} F^2(r,\rho,x,y) = f^2(x,y)$, there exists the $\lim_{r,\rho\to 1} \iint_R F^2(r,\rho,x,y)\,dx\,dy = \iint_R f^2(x,y)\,dx\,dy$. Therefore the series in (50) must remain convergent for $r=\rho=I$. By a theorem* on convergence factors in double series, the limit of this series must equal the value of the series with $r=\rho=I$, and

$$\frac{1}{\pi^2}\iint_R f^2(x,y)\,dx\,dy = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi(m,n)\left[a_{mn}^2 + b_{mn}^2 + c_{mn}^2 + d_{mn}^2\right].$$

We have thus arrived at Parseval's Theorem for any function which satisfies the conditions of Theorem 8. This method of proof is subject to more restrictions than the corresponding method for single series. These restrictions arise from dealing with the Double Integral over the cross region about a given point.

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* C.N.Moore, Trans. Am. Math. Soc., vol. I4, page 85. Since the series in (50) is convergent, it must, a priori, be summable. Putting $r^m\rho^n = (I-\alpha)^m(I-\beta)^n = f_{mn}(\alpha,\beta)$, all the conditions of the theorem there stated are satisfied.