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I hereby recommend that the thesis prepared under my supervision by Edward H. Gallagher, O.P.

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BOREL SUMMABILITY OF FOURIER SERIES OF CONTINUOUS FUNCTIONS

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by

Edward H. Gallagher, O.P.

A. B. St. Thomas College 1933

M. S. Catholic University 1941

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CHAPTER I.

INTRODUCTION.

1) Background of the Problem.

The Cesaro summability method will sum the Fourier series of every continuous function to its proper value $f(x_0)$ at the point x_0, if $f(x)$ is the function under consideration. This is part of Fejer's theorem [Zygmund, 12, p.45].

The Borel summability method is often more powerful than the Cesaro method, especially with reference to power series outside their region of convergence. Yet the Borel method will not sum the Fourier series of every continuous function. This is deduced from the following theorem:

"If the sequence

$$u_n(x) = \int_a^b x(t)\, y_n(t)\, dt$$

is bounded for every bounded, or even only continuous function x, then

$$\int_a^b |y_n(t)|\, dt$$

is $O(1)$ ". [Zygmund, 12, p. 99].

It can be shown that

$$B\left[f(x)\right] = \frac{1}{\pi}\int_0^{\pi} \psi(t)\, e^{-\frac{1}{2}yt^2}\, \frac{\sin yt}{t}\, dt$$

as $y \to \infty$, is a necessary and sufficient condition for Borel summability of $f(x)$, where $\psi = \{f(x+t) + f(x-t)\}$, and $f(x)$ is integrable (L) and periodic.

It is also possible to prove that

$$\frac{2}{\pi}\int_0^{\pi} \frac{e^{-\frac{1}{2}yt^2}\,|\sin yt|}{t}\,dt$$

does not remain bounded as $y \to \infty$. This integral corresponds to the sequence y_n.

Consequently, $B\,[f(x)]$ is unbounded for some continuous function $f(x)$, and so not every continuous function has a Borel summable Fourier series.

This is the approach by Moore, [Moore, 7, pp. 284 - 288], who then mentions an example of such a continuous function. The example used is Fejer's :

$$\theta(x) = \sum_{n=1}^{\infty} \frac{\sin 2^{n^3} x}{n^2} \qquad 0 \leq x \leq \pi$$

which was constructed as an example of a continuous function whose Fourier series was divergent at $x = 0$. Fejer proved this divergence [Fejer, 2, pp. 1 - 5]. A more detailed proof of divergence was given by Picard [Picard, 8, pp. 294 - 299].

Moore states that $\theta(x)$ is not summable (B). In order to show a comparison between convergence and summability (B) he modifies $\theta(x)$ so that it has a Fourier series still divergent, but now Borel summable.

The new function is

$$E(x) = \sum_{n=1}^{\infty} \frac{\alpha(x,n)}{n^2} \qquad 0 \leq x \leq \pi$$

where

$$\alpha(x,n) = 2^{n^3/4}\, x \sin 2^{3n^3/4} \qquad (0 \leq x \leq 2^{-n^3/4})$$

$$\alpha(x,n) = \sin 2^{n^3} x \qquad (2^{-n^3/4} < x \leq \pi).$$

It would be worthwhile to have the detailed proof of this, unless it would be possible to construct a simpler example. It was at Dr. Moore's suggestion that this work on Schwarz' example was begun.

Schwarz' functinn, which follows here, was likewise constructed as an example of a continuous function with a divergent Fourier series at a point. It can be shown that this example has a non - Borel summable Fourier series. We then show that the function, with a slight modification, has a Fourier series that is still divergent, but summable (B).

This example has an advantage over that of Fejer by being defined "graphically" and less artificially. Schwarz used this example in his lectures, and later gave permission to Sachse, [Sachse, 9, p. 245], to use it in his article on Fourier series [Sachse, 9, pp. 271 - 274].

We shall present first the example as Hardy - Rogosinski define it, then, as Sachse has it. It will be clear how Hardy- Rogosinski have simplified it. Their definition is the one that we shall use here.

2) Schwarz Example,

as Hardy - Rogosinski Define it.

"Suppose N_r is an odd integer, at least three, and

$n_0 = 1, \quad n_r = N_1 N_2 \ldots N_r, \quad a_r > 0, \quad \sum a_r < \infty,$

$a_r \log N_r \to \infty$. We define $f(t)$ in(the closed interval)

$(0, \pi)$ by

$$f(0) = 0$$

$$f(t) = a_r \sin n_r t \qquad (\pi/n_r \leq t \leq \pi/n_{r-1}),$$

and by evenness and periodicity elsewhere".

[Hardy - Rogosinski, 4, p. 50] .

3) Schwarz Example, as Originally Defined.

Let $f(\beta)$ be defined in $(0, \pi/2)$ by

$$f(0) = 0$$

$$f(\beta) = c_\lambda \sin [\lambda] \beta$$

where $c_1, c_2, \ldots, c_\mu, \ldots$ is a series decreasing to zero.

$[\lambda] = 1 \cdot 3 \cdot 5 \cdots (2\lambda + 1)$ for $\lambda = 1, 2, 3, \ldots$

and $\lim \mu = \infty$.

[Sachse, 9, pp. 271 - 274].

Hobson [Hobson, 5, pp. 545 - 548] uses this function as originally defined, except for using an interval (0,a). He shows more than divergence. In fact, he proves that the function has a Fourier series that is infinitely oscillating, by showing that

$$\int_0^a \varphi(z) \frac{\sin(2n+1)z}{z} \, dz$$

becomes infinite for certain increasing values of n ; and that the integral approaches 0 for another set of increasing values of n .

4) Outline of Thesis.

As we mentioned, we shall use the Schwarz example as Hardy-Rogosinski defined it.

In Chapter II, the proof of the divergence of the Fourier series of the Schwarz function $f(x)$ is taken from Hardy - Rogosinski, with the details added.

In Chapter III, Moore's criterion for Borel summability is developed. Acknowledgement is made of the use of Dr. Moore's own notes on the problem.of the criterion.

In Chapter IV, proof is given that the Fourier series of the Schwarz function is not summable (B).

In Chapter V, this function is slightly modified to a function called $F(x)$. It is shown that $F(x)$ has a divergent Fourier series.

In Chapter VI, the Fourier series of $F(x)$ is proved to be Borel summable.

CHAPTER II.

SCHWARZ EXAMPLE, A CONTINUOUS FUNCTION WITH A DIVERGENT FOURIER SERIES.

The proof that this example has a divergent Fourier series is taken from Hardy-Rogosinski, [H-R., 4, p.50]. Details are added in the proof.

2.1. Example of Schwarz.

Suppose N_r is an odd integer, at least 3, and $n_0 = 1$,

$n_r = N_1 N_2 \ldots N_r$, $N_r \to \infty$, $a_r > 0$, $\sum a_r < \infty$, $a_r \log N_r \to \infty$.

All these conditions are satisfied, e.g., if $a_r = r^{-2}$, $n_r = 3^{r^4}$, and $N_r = n_r / n_{r-1} = 3^{4r^3 - 6r^2 - 4r - 1}$. f(x) is defined in the closed interval $(0, \pi)$ by

$$f(0) = 0$$

$$f(x) = a_r \sin n_r x \qquad \pi/n_r \leq x \leq \pi/n_{r-1}$$

and by evenness and periodicity elsewhere.

Then f(x) is continuous in the closed interval $(-\pi, \pi)$. The only point we need consider is $x = 0$, since the function is obviously continuous at the other points. For continuity, f(x) must approach $f(0) = 0$, as x approaches $+0$. $|f(x)| = |a_r \sin n_r x| \leq a_r$ which approaches zero, if we let $x = 1/n_r$, as r becomes infinite.

F(x) is of bounded variation in any sub-interval which does not include zero, so that its Fourier series is uniformly convergent in any such closed sub-interval. ("If f is of bounded variation, the Fourier series of f converges at every point x to the value $\frac{1}{2}\{f(x+0) + f(x-0)\}$. If f is in addition continuous at every point of a (closed) interval $I = (a,b)$, the Fourier series is uniformly convergent in I". [Zygmund, 12, p.25].

2.2. <u>Convergence condition for Fourier Series;</u> For the Fourier series of $f(x)$ to converge, the partial sum

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\sin(n+\frac{1}{2})t}{2\sin t/2}\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\sin nt}{t}\,dt + o(1)$$

must converge as $n \longrightarrow \infty$, [Zygmund, 12, p.20, and p. 24] . Since our function is even and periodic

$$s_n(x) = \frac{2}{\pi}\int_0^{\pi} f(x+t)\,\frac{\sin nt}{t}\,dt$$

If $s_n(x)$ is not convergent for certain values of n that become infinite, that will suffice to prove the divergence of s_n, as $n \to \infty$. Values of $n = n_k$, $k = 1,2,\ldots$, are chosen.

2.3. <u>Schwarz example has divergent Fourier Series.</u> At $x = 0$, we shall show that

$$J(k) = \int_0^{\pi} f(0+t)\,\frac{\sin n_k t}{t}\,dt \longrightarrow \infty, \text{ as } k \to \infty.$$

Substituting for $f(x)$,

$$J(k) = \sum_{r=1}^{\infty} a_r \int_{\pi/n_r}^{\pi/n_{r-1}} \frac{\sin n_r t \sin n_k t}{t}\,dt$$

(2.31)
$$= \left(\sum_{r=1}^{k-1} a_r + \sum_{k+1}^{\infty} a_r\right)\int_{\pi/n_r}^{\pi/n_{r-1}} \frac{\sin n_r t \sin n_k t}{t}\,dt + a_k\int_{\pi/n_k}^{\pi/n_{k-1}} \frac{(\sin n_k t)^2}{t}\,dt$$

We can show that the integral in the sum terms is uniformly bounded, and so the sums will be bounded. In the first sum, writing $\lambda_r = \pi/n_r$, and since $r < k$,

$$\int_{\lambda_r}^{\lambda_{r-1}} \frac{\sin n_r t \sin n_k t}{t}\,dt = \frac{1}{2}\int_{\pi/n_r}^{\pi/n_{r-1}} \frac{\cos(n_k-n_r)t - \cos(n_k+n_r)t}{t}\,dt$$

$$= \frac{1}{2}\int_{\frac{\pi}{n_r}(n_k-n_r)}^{\frac{\pi}{n_{r-1}}(n_k-n_r)} \frac{\cos u}{u}\,du - \frac{1}{2}\int_{\frac{\pi}{n_r}(n_k+n_r)}^{\frac{\pi}{n_{r-1}}(n_k+n_r)} \frac{\cos u}{u}$$

$$(2.32)\quad = \frac{1}{2}\int_1^{\frac{\pi}{n_{r-1}}(n_k-n_r)} \frac{\cos u}{u}\,du - \frac{1}{2}\int_1^{\frac{\pi}{n_r}(n_k-n_r)} \frac{\cos u}{u}\,du - \frac{1}{2}\int_1^{\frac{\pi}{n_{r-1}}(n_k+n_r)} \frac{\cos u}{u}\,du + \frac{1}{2}\int_1^{\frac{\pi}{n_r}(n_k+n_r)} \frac{\cos u}{u}\,du$$

For $\cos u / u$ to be integrable, none of these limits can be zero. Actually none is zero, e.g., since $r < k$, and $n_r / n_{r-1} = N_r \geq 3$,

$$\frac{\pi}{n_r}(n_k - n_r) = \pi\left(\frac{n_k}{n_r} - 1\right) \geq \pi(N_r - 1) \geq 2\pi.$$

When the upper limit increases indefinitely, each of these last integrals exists since $\int_1^{\infty} \frac{\cos u}{u}\,du$ [Titchmarsh, 11, p.21] is convergent. Thus

$$\sum_{r=1}^{k-1} a_r \int_{\lambda_r}^{\lambda_{r-1}} \frac{\sin n_r t \,\sin n_k t}{t}\,dt$$

is bounded, since the integral is uniformly bounded for $r < k$.

Likewise, when $r > k$, and $\lambda_r = \pi/n_r$, the second sum of (2.31):

$$(2.33)\quad \sum_{r=k+1}^{\infty} a_r \int_{\lambda_r}^{\lambda_{r-1}} \frac{\sin n_r t \,\sin n_k t}{t}\,dt$$

$$= \frac{1}{2}\sum a_r \int_{\lambda_r}^{\lambda_{r-1}} \frac{\cos(n_r - n_k)t - \cos(n_k + n_r)t}{t}\,dt$$

is seen to be bounded. As before, we have,

$$\int_1^{\lambda_{r-1}(n_r-n_k)} \frac{\cos u}{u}\,du - \int_1^{\lambda_r(n_r-n_k)} \frac{\cos u}{u}\,du + \int_1^{\lambda_r(n_r+n_k)} \frac{\cos u}{u}\,du - \int_1^{\lambda_{r-1}(n_r+n_k)} \frac{\cos u}{u}\,du$$

The smallest of these upper limits is greater than $2\pi/3$. Since $r > k$,

$$\frac{\pi}{n_r}(n_r - n_k) = \pi\left(1 - \frac{n_k}{n_r}\right) \geq \pi\left(1 - \frac{1}{N_r}\right) \geq \pi\left(1 - \tfrac{1}{3}\right).$$

Thus the integral of (2.33),
$$\int_{\lambda_r}^{\lambda_{r-1}} \frac{\sin n_r t \sin n_k t}{t}\, dt$$
is uniformly bounded for all $r > k$.

We have seen that the first two terms of $J(k)$, (2.31), are bounded, where
$$J(k) = \left(\sum_{r=1}^{k-1} a_r + \sum_{k+1}^{\infty} a_r\right) \int_{\lambda_r}^{\lambda_{r-1}} \frac{\sin n_r t \sin n_k t}{t}\, dt + a_k \int_{\lambda_k}^{\lambda_{k-1}} \frac{(\sin n_k t)^2}{t}\, dt$$

This last integral is not bounded, as k increases indefinitely.

$$(2.34) \qquad \int_{\pi/n_k}^{\pi/n_{k-1}} \frac{(\sin n_k t)^2}{t}\, dt = \int_{\pi/n_k}^{\pi/n_{k-1}} \frac{1 - \cos 2 n_k t}{2t}\, dt$$
$$= \frac{1}{2} \log\left(\frac{\pi}{n_{k-1}} \Big/ \frac{\pi}{n_k}\right) - \frac{1}{2} \int_{2\pi}^{2\pi N_k} \frac{\cos u}{u}\, du$$
$$= \frac{1}{2} \log N_k - \frac{1}{2} \int_{2\pi}^{2\pi N_k} \frac{\cos u}{u}\, du$$

This last term is bounded, but $\frac{1}{2} a_k \log N_k$ becomes infinite, as $k \to \infty$, by the assumption in the problem (2.1).

Consequently, $J(k)$ becomes infinite with k, and so does $s_n(x)$ with n, and the Fourier series of $f(x)$, as defined above, is divergent at $x = 0$, a point of continuity.

CHAPTER III.

THE CONDITION FOR BOREL SUMMABILITY OF A FOURIER SERIES.

3.1. Fourier series. Let $f(x)$ be Lebesgue integrable, and a periodic function. It has a Fourier series development, which may or may not be convergent.

$$
\begin{aligned}
f(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{ \tfrac{1}{2} + \sum_{n=1}^{\infty} (\cos nt \cos nx + \sin nt \sin nx) \right\} dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{ \tfrac{1}{2} + \sum_{1}^{\infty} \cos n(x-t) \right\} dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) \left\{ \tfrac{1}{2} + \sum_{1}^{\infty} \cos nt \right\} dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} \left\{ f(x+t) + f(x-t) \right\} \left\{ \tfrac{1}{2} + \sum_{n=1}^{\infty} \cos nt \right\} dt
\end{aligned}
$$

since $f(x)$ is periodic. Writing $\psi(x,t)$ for $\{f(x+t)+f(x-t)\}$, we have

$$
(3.11) \qquad f(x) = \frac{1}{\pi}\int_{0}^{\pi} \psi(x,t) \left\{ \tfrac{1}{2} + \sum_{n=1}^{\infty} \cos nt \right\} dt.
$$

3.2. Borel Summability. A sum $\sum_{n=0}^{\infty} u_n$ is said to be Borel summable by the integral method if

$$
\lim \int_{0}^{y} e^{-y} \sum_{n=0}^{\infty} \frac{u_n y^n}{n!} \, dy
$$

exists as continuous $y \to \infty$. [Bromwich, 1, p. 267;] or [Szasz, 10, p. 29], or [Borel, Lecons sur les Series Divergentes, 2nd. edition, (1928), p. 122].

The Borel exponential definition of summability is:

$$\lim_{t\to\infty} e^{-t} \sum_{n=0}^{\infty} s_n \frac{t^n}{n!} .$$

The integral method is more general, although the two are equivalent for series whose n-th term is o(1), [Szasz, 10, pp. 31,32], and so for Fourier series. The Riemann - Lebesgue theorem shows that the n-th term of the Fourier series is o(1) :

"If $f(x)$ is integrable over (a,b), then as $\lambda \to \infty$

$$\int_a^b f(x) \cos \lambda x \, dx \to 0, \qquad \int_a^b f(x) \sin \lambda x \, dx \to 0 \text{"}.$$

[Titchmarsh, 11, p.403] . And, from above (3.1), the general term of the Fourier series is

$$a_n \cos nx + b_n \sin nx =$$

$$\cos nx \int_{-\pi}^{\pi} f(t) \cos nt \, dt + \sin nx \int_{-\pi}^{\pi} f(t) \sin nt \, dt .$$

<u>3.3. Application of Borel integral definition</u>. The Borel integral method will be used here. Although not as simple as the exponential method, it is interesting in itself. Comparison might be made between the result of this section and that obtained using the exponential method, (as given, e.g, in Zygmund, 12, p.186).

It was seen that the Fourier series associated with integrable f(x) was (3.11):

$$f(x) = \frac{1}{\pi}\int_0^{\pi} \psi(x,t) \left\{\tfrac{1}{2} + \sum_{n=1}^{\infty} \cos nt\right\} dt.$$

This series will be Borel summable to s if

$$(3.31)\quad B\left[f(x)\right] = \lim_{y\to\infty} \frac{1}{\pi}\int_0^y\int_0^\pi \psi(x,t)\, e^{-y}\left\{\tfrac{1}{2}+\sum_{n=1}^{\infty}\frac{y^n}{n!}\cos nt\right\} dt\, dy$$

has the limit s, as continuous $y \to \infty$.

Since this double integral exists, if the iterated integral does, and since $\frac{1}{2} + \sum_{1}^{\infty} \frac{y^n}{n!} \cos nt$ is the real part (signified by R)

of $\left\{e^{y(\cos t + i \sin t)} - \frac{1}{2}\right\}$, we write

$$B\left[f(x)\right] = \frac{1}{\pi}\int_0^\pi \psi(x,t)dt \int_0^y e^{-y}\, R\left[e^{y(\cos t + i\sin t)} - \tfrac{1}{2}\right] dy$$

as $y \to \infty$. Simplifying,

$$R\left[\int_0^y\left\{e^{-y(1-\cos t - i\sin t)} - \frac{e^{-y}}{2}\right\}dy\right]$$

$$= R\left[-\frac{e^{-y(1-\cos t - i\sin t)}}{(1-\cos t - i\sin t)} + \frac{e^{-y}}{2} + \frac{1}{(1-\cos t - i\sin t)} - \tfrac{1}{2}\right]$$

$$= R\left[A + B + C + D\right].$$

$R\left[C\right]$ is equal to $\frac{1}{2}$, and so the last two terms are zero.. Since $R\left[B\right] = e^{-y}/2 = o(1)$, and since $\int_0^\pi \psi(x,t)\cdot o(1)\, dt$ is itself $o(1)$, as $y \to \infty$, we may neglect this term.

$$R[A] = R\left[-\frac{e^{-y(1-\cos t - i\sin t)}}{(1-\cos t - i\sin t)}\right]$$

$$= -R\left[\frac{e^{-y(1-\cos t)}\, e^{iy\sin t}}{(1-\cos t - i\sin t)} \cdot \frac{1-\cos t + i\sin t}{1-\cos t + i\sin t}\right]$$

$$R[A] = -e^{-y(1-\cos t)} R\left[\frac{\cos(y\sin t)+i\sin(y\sin t)}{2(1-\cos t)}(1-\cos t+i\sin t)\right]$$

$$= -\frac{e^{-y(1-\cos t)}}{2(1-\cos t)}\left[(1-\cos t)\cdot\cos(y\sin t)-(\sin t)\cdot\sin(y\sin t)\right]$$

Thus finally we have,

$$(3.32)\quad B[f(x)] = \frac{1}{\pi}\int_0^{\pi}\psi(x,t)\,\frac{e^{-y(1-\cos t)}}{2(1-\cos t)}(\sin t)\sin(y\sin t)\,dt$$

$$-\frac{1}{\pi}\int_0^{\pi}\psi(x,t)\,e^{-y(1-\cos t)}\cos(y\sin t)\,dt$$

as $y\to\infty$. This last integral is $o(1)$ by the Riemann-Lebesgue which states that $\int_a^b \varphi(t)\cos\lambda t\,dt = o(1)$, as $\lambda\to\infty$, if $\varphi(t)$ is integrable. Here $\varphi(t)$ is $\psi(x,t)\,e^{-y(1-\cos t)}$, and this is integrable, since $\psi(x,t) = \{f(x+t)+f(x-t)\}$. $\lambda = y\sin t\to\infty$, as $y\to\infty$, for any fixed value of $\sin t$, except $t = 0$. (3.32) then becomes

$$B[f(x)] = \frac{1}{\pi}\int_0^{\pi}\psi(x,t)\,\frac{e^{-y(1-\cos t)}}{2(1-\cos t)}\sin t\sin(y\sin t)\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\psi(x,t)\,\frac{e^{-y(1-\cos t)}}{2\sin\frac{1}{2}t}\cos\tfrac{1}{2}t\sin(y\sin t)\,dt$$

$$(3.33)\quad B[f(x)] = \frac{1}{\pi}\int_0^{\pi/2}\{f(x+2t)+f(x-2t)\}\frac{e^{-y(1-\cos t)}}{\sin t}\cos t\sin(y\sin 2t)\,dt$$

as $y\to\infty$.

This may be simplified to

$$B\left[f(x)\right] = \frac{1}{\pi}\int_0^{\pi/2}\left\{f(x+2t) + f(x-2t)\right\}\frac{e^{-2yt^2}}{t}\sin 2yt\,dt \qquad y \to \infty$$

as Lorch [Lorch, 6, pp. 460 - 462] does for the essentially similar integral obtained by using the Borel exponential method on Fourier series.

We may assume $|f(x)| \leq 1$, without any loss of generality.

1) $\cos t / \sin t$ may be replaced in the limit by $1/t$.

Since $\dfrac{1}{t} - \dfrac{\cos t}{\sin t} = \dfrac{\sin t - t\cos t}{t\sin t}$

and since $\sin t - t\cos t \leq t^3/3$, and $1/t\sin t \leq \pi/2t^2$, $(0 \leq t \leq \pi/2)$,

$$\left|\int_0^{\pi/2} e^{-y(1-\cos t)}\left(\frac{1}{t} - \frac{\cos t}{\sin t}\right)\sin(y\sin t)\,dt\right|$$

$$\leq \frac{\pi}{6}\int_0^{\pi/2} t e^{-y(1-\cos t)}\,dt = \frac{\pi}{6}\int_0^{\pi/2} t e^{-2y\sin^2 t}\,dt$$

$$\leq \frac{\pi}{6}\int_0^{\pi/2} t e^{-\frac{8}{\pi^2}yt^2}\,dt = o(1) \quad \text{as } y \to \infty$$

2) Writing $(1-\cos t) = 2\sin^2 t$, and since $e^s - 1 \leq s e^s$, for $s \geq 0$, where $s = 2y(t^2 - \sin^2 t)$, we may replace $e^{-2y\sin^2 t}$ by e^{-2yt^2}:

$$\left|\int_0^{\pi/2}\left(e^{-2y\sin^2 t} - e^{-2yt^2}\right)\frac{\sin(y\sin t)}{t}\,dt\right|$$

$$= \left|\int_0^{\pi/2}\frac{e^{2y(t^2-\sin^2 t)} - 1}{e^{2yt^2}}\cdot\frac{\sin(y\sin t)}{t}\,dt\right| \leq$$

$$\leq \int_0^{\pi/2} 2y\left(\frac{t^2-\sin^2 t}{t}\right) e^{-2y\sin^2 t}\left|\sin(y\sin t)\right| dt$$

$$\leq \frac{2y}{6}\int_0^{\pi/2} t^3 e^{-\frac{8}{\pi^2}yt^2} dt = o(1) \quad \text{as } y \to \infty.$$

We have used $\sin t \leq t$; and $0 \leq t - \sin t \leq \frac{t^3}{6}$.

3) sin(y sin 2t) may be replaced in the limit by sin 2yt :

$$\left| \int_0^{\pi/2} \frac{e^{-2yt^2}}{t}\left\{ \left|\sin 2yt\right| - \left|\sin(y\sin 2t)\right| \right\} dt \right|$$

$$\leq \int_0^{\pi/2} \frac{e^{-2yt^2}}{t} \left|\sin 2yt - \sin(y\sin 2t)\right| dt$$

$$= \int_0^{\pi/2} \frac{e^{-2yt^2}}{t}\left|\sin \tfrac{1}{2}y(2t-\sin 2t)\cdot \cos \tfrac{1}{2}(2yt+y\sin 2t)\right| dt$$

$$\leq 2y\int_0^{\pi/2} e^{-2yt^2}\left(\frac{2t-\sin 2t}{2t}\right) dt \leq \frac{4y}{3}\int_0^{\pi/2} t^2 e^{-2yt^2} dt = o(1).$$

Thus we have seen, that if $f(x)$ is an integrable (L) and periodic function, its Fourier series is Borel summable, if

$$(3.34) \quad B\left[f(x)\right] = \frac{1}{\pi}\int_0^{\pi/2} \left\{ f(x+2t) + f(x-2t) \right\} \frac{e^{-2yt^2}}{t} \sin 2yt \, dt ,$$

as continuous $y \to \infty$, aproaches a finite value, the value of $f(x)$ at the point x.

This integral is the one that Moore uses [Moore, 7, pp. 284-288], without showing its development.

It was stated that the Fourier series of $f(x)$ is summable (B), if $B\left[f(x_0)\right] \to f(x_0)$, (3.34), as continuous $y \to \infty$. Since the converse is true, this condition might be stated as a necessary and sufficient condition for summability (B) of the Fourier series of $f(x)$ at the point x.

<u>3.4. Hardy's condition for Borel summability</u>. In a paper on Divergent Series, Hardy [Hardy, 3, pp. 179-181] used the Borel exponential method, briefly simplified the integral obtained, and stated that a necessary and sufficient condition for summability(B) of a Fourier series is that

$$(3.41) \qquad \int_{d}^{\xi^{-\frac{1}{2}}} \varphi(t)\,\frac{\sin \xi t}{t}\,dt \to 0, \quad \text{as } \xi \to \infty ,$$

where $\varphi(t) = \varphi(t,x) = \frac{1}{2}\left\{f(x+t) + f(x-t) - 2s\right\} \to 0$ as $t \to 0$.

The previous condition (3.34) will be used on Schwarz' example, and mention will be made later of Hardy's condition, in Chapter VI.

CHAPTER IV.

FOURIER SERIES OF SCHWARZ EXAMPLE NOT BOREL SUMMABLE.

4.1. Schwarz example. This was defined (2.1) as:

$$f(0) = 0$$

$$f(x) = a_r \sin n_r x \qquad \pi/n_r \leq x \leq \pi/n_{r-1}$$

for x in the closed interval $(0, \pi)$, elsewhere by evenness and periodicity. Also $a_r > 0$, $\sum a_r < \infty$, $a_r \log N_r \to \infty$, $n_r = N_1 N_2 \ldots N_r$, where $N_r \geq 3$ for $r \geq 1$. $n_0 = 1$.

(4.11) An extra assumption is added, i.e., that $n_r/n_{r-1}^2 = o(1)$.

This is satisfied in the particular case mentioned (2.1); $n_r = 3^{r^4}$; since $3^{r^4} / 3^{2(r-1)^4} = o(1)$ as $r \to \infty$.

4.2. Fourier series of f(x) not summable (B). In order that the Fourier series of f(x) be Borel summable at the particular point x, (3.34),

$$B\left[f(x)\right] = \frac{1}{\pi}\int_0^{\pi}\left\{f(x+t) + f(x-t)\right\} e^{-\frac{1}{2}yt^2} \frac{\sin yt}{t}\, dt \qquad y \to \infty$$

must approach a finite limit, the value of f(x) at the point x, as continuous y becomes infinite. Since the Schwarz function f(x) is an even function, at the point x= 0,

$$B\left[f(x)\right] = \lim_{y\to\infty} \frac{2}{\pi} \int_0^{\pi} f(0+t)\, e^{-\frac{1}{2}yt^2} \frac{\sin yt}{t}\, dt$$

(4.21)
$$= \frac{2}{\pi} \sum_{r=1} a_r \int_{\pi/n_r}^{\pi/n_{r-1}} \frac{\sin n_r t\; e^{-\frac{1}{2}yt^2} \sin yt}{t}\, dt \qquad y \to \infty .$$

This limit should be zero, since $f(0) = 0$. For the non-Borel summability of the Fourier series of f(x) at the point zero, it will suffice to show that this sum is divergent for values of $y = n_k$, where k has the values 1,2, ... Putting $\pi/n_r = \lambda_r$, and rewriting the sum,

$$(4.22)\quad B[f(0)] = \frac{2}{\pi}\left(\sum_{r=1}^{k-1} + \sum_{k+1}^{\infty}\right) a_r \int_{\lambda_r}^{\lambda_{r-1}} e^{-\frac{n_k}{2}t^2} \frac{\sin n_r t \sin n_k t}{t}\, dt$$

$$+ a_k \int_{\lambda_k}^{\lambda_{k-1}} e^{-n_k t^2/2} \frac{(\sin n_k t)^2}{t}\, dt.$$

1) when $r < k$,

$$\int_{\lambda_r}^{\lambda_{r-1}} e^{-n_k t^2/2} \frac{\sin n_r t \sin n_k t}{t}\, dt$$

$$= \frac{1}{2}\int_{\lambda_r}^{\lambda_{r-1}} e^{-n_k t^2/2} \left\{\frac{\cos(n_k - n_r)t - \cos(n_k + n_r)t}{t}\right\} dt$$

$$(4.23)\quad = \frac{1}{2}\int_{\lambda_r}^{\lambda_{r-1}} e^{-n_k t^2/2} \{P\}\, dt$$

$$(4.24)\quad = \frac{1}{2} e^{-\frac{n_k}{2}\left(\frac{\pi}{n_r}\right)^2} \int_{\lambda_r}^{\xi} P\, dt + \frac{1}{2} e^{-\frac{n_k}{2}\left(\frac{\pi}{n_{r-1}}\right)^2} \int_{\xi}^{\lambda_{r-1}} P\, dt$$

where ξ is between π/n_r and π/n_{r-1}, by the second mean-value theorem; (e.g., as in Phillips' "Analysis", p. 189):

"If $f(x)$ is monotonic, $f(x)$ and $\phi(x)$ both continuous and $f'(x)$ exists in $a \leq x \leq b$, then

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi} \phi(x)\,dx + f(b)\int_{\xi}^{b} \phi(x)\,dx$$

where ξ is between a and b".

These conditions are obviously satisfied with $f = e^{-n_k t^2/2}$ and ϕ the other factor in (4.23). The exponential factors in (4.24) above, are positive and bounded by 1. The integrals in the same expression are uniformly bounded, e.g., with $\pi/n_r < \xi$,

$$\int_{\pi/n_r}^{\xi} \frac{\cos(n_k - n_r)t}{t}\,dt = \int_{\frac{\pi}{n_r}(n_k - n_r)}^{\xi(n_k - n_r)} \frac{\cos u}{u}\,du .$$

As shown in (2.3), with $r < k$, this lower limit is never zero, in fact is never less than 2π, and the integral is bounded, since

$$\int_1^{\infty} \frac{\cos u}{u}\, du$$

is bounded. Thus the integrals

$$\int_{\lambda_r}^{\lambda_{r-1}} e^{-n_k t^2/2} \,\frac{\sin n_r t \sin n_k t}{t}\, dt$$

are uniformly bounded for all $r < k$.

2) when $r > k$, the integrals in (4.22) can be shown to be uniformly bounded, in the same way. Thus in (4.22) the sums

$$\left(\sum_{r=1}^{k-1} + \sum_{k+1}^{\infty}\right) a_r \int_{\lambda_r}^{\lambda_{r-1}} e^{-n_k t^2/2} \,\frac{\sin n_r t \sin n_k t}{t}\, dt$$

are bounded, since $\sum a_r < \infty$.

3) when $r = k$, the last integral in (4.22)

$$\int_{\lambda_k}^{\lambda_{k-1}} e^{-n_k t^2/2} \,\frac{(\sin n_k t)^2}{t}\, dt \geq e^{-\frac{n_k}{2}\left(\frac{\pi}{n_{k-1}}\right)^2} \int_{\lambda_k}^{\lambda_{k-1}} \frac{1 - \cos 2 n_k t}{2t}\, dt$$

by the generalized mean value theorem.

As in (2.34) this last integral is equal to

$$\tfrac{1}{2} \log N_k - \tfrac{1}{2} \int_{2\pi}^{2\pi N_k} \frac{\cos u}{u}\, du$$

the second term being bounded, and the first increasing indefinitely as $k \to \infty$. The factor $e^{-\frac{n_k}{2}\left(\frac{\pi}{n_{k-1}}\right)^2}$ before the integral approaches 1 as $k \to \infty$, since the exponent is $o(1)$, by the assumption in (4.11).

Thus $B\left[f(0)\right]$, (4.22), is made up of three bounded terms added to $\frac{1}{2} a_k \log N_k$, which increases indefinitely as $k \to \infty$, by the assumption in (4.1), and so the Fourier series of $f(x)$ at $x = 0$ is not Borel summable.

CHAPTER V.

F(x), A SIMPLE FUNCTION WITH DIVERGENT FOURIER SERIES AT A POINT OF CONTINUITY.

5.1. Introduction. As it stands the example of Schwarz has a Fourier series that is divergent and not Borel summable at $x = 0$, a point of continuity. It can be modified in a simple manner so as to be still continuous and possess a Fourier series that is divergent, but summable (B), at $x = 0$, a point of continuity. The only change made is in defining $F(x)$ as $a_r \sin n_r^3 x$ in $(\pi/n_r, \pi/n_{r-1})$ where $f(x)$ was $a_r \sin n_r x$. It has an advantage over Fejer's example * by being defined more simply, less artificially, and especially by being defined "graphically".

5.2. Definition of F(x).

(5.21) $F(x)$ is defined in the closed interval $(0, \pi)$, by

$$F(0) = 0$$

$$F(x) = a_r \sin n_r^3 x \qquad \pi/n_r \leq x \leq \pi/n_{r-1}$$

and by evenness and periodicity elsewhere. $F(x)$ is continuous in the closed interval $(-\pi, \pi)$.

As before (2.1), N_r is an odd integer, at least 3,

$$n_0 = 1, \quad n_r = N_1 N_2 \dots N_r, \quad N_r \to \infty,$$

$$a_r > 0, \quad \sum a_r < \infty, \quad a_r \log N_r \to \infty.$$

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* Fejer's example is $\theta(x) = \sum_{n=1}^{\infty} \dfrac{\sin 2^{n^3} x}{n^2} \qquad (0 \leq x \leq \pi).$

As mentioned in the Introduction (I (1)), $\theta(x)$ has a divergent Fourier series.

<u>5.3. F(x) has a divergent Fourier series at x = 0</u>.   As seen in (2.2)

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x+t)\,\frac{\sin nt}{t}\,dt \qquad n \to \infty$$

must converge to $F(x)$, in order that the Fourier series of $F(x)$, a periodic and even function, be convergent. To show divergence of $s_n(x)$, it will be sufficient to show divergence for values of $n = n_k^3$, (k = 1,2, ...) as $k \to \infty$. At the point $x = 0$,

$$s_n(0) = \frac{2}{\pi}\int_0^{\pi} F(0+t)\,\frac{\sin n_k^3 t}{t}\,dt$$

(5.31) $$= \frac{2}{\pi}\sum_{r=1}^{\infty} a_r \int_{\pi/n_r}^{\pi/n_{r-1}} \frac{\sin n_r^3 t \sin n_k^3 t}{t}\,dt \qquad k \to \infty \;.$$

$$= \frac{2}{\pi}\sum_{1}^{\infty} a_r I_r .$$

1) when $r = k$, I_r becomes infinite with k; writing $\pi/n_r = \lambda_r$,

(5.32) $$I_r = \frac{1}{2}\int_{\lambda_k}^{\lambda_{k-1}} \frac{(1 - \cos 2n_k^3 t)}{t}\,dt$$

$$= \frac{1}{2}\log(n_k/n_{k-1}) - \frac{1}{2}\int_{\lambda_k}^{\lambda_{k-1}} \frac{\cos 2n_k^3 t}{t}\,dt$$

$$= \frac{1}{2}\log N_k - \frac{1}{2}\int_{2\pi n_k^2}^{2\pi N_k^3 n_{k-1}^2} \cos u / u \;\; du.$$

This last integral is convergent for all k . All the limits are greater than 1, and the integral $\int_1^{\infty} \cos u / u \;du$ is convergent. The term $\frac{1}{2}\log N_k$ becomes infinite with k .

2) when $r \neq k$, it can be shown that I_r in (5.31) is uniformly bounded for all r and k, and so the sums $\left(\sum_1^{k-1} + \sum_{k+1}^{\infty}\right) a_r I_r$, will be bounded. If $r \neq k$,

(5.34)
$$I_r = \int_{\lambda_r}^{\lambda_{r-1}} \frac{1}{t} (\sin n_r^3 t)(\sin n_k^3 t)\, dt$$

$$= \frac{1}{2} \int_{\lambda_r}^{\lambda_{r-1}} \frac{\cos |n_r^3 - n_k^3|\, t - \cos (n_r^3 + n_k^3) t}{t}\, dt$$

$$= \frac{1}{2} \int_{\frac{\pi}{n_r}|n_r^3 - n_k^3|}^{\frac{\pi}{n_{r-1}}|n_r^3 - n_k^3|} \frac{\cos u}{u}\, du - \frac{1}{2} \int_{\frac{\pi}{n_r}(n_r^3 + n_k^3)}^{\frac{\pi}{n_{r-1}}(n_r^3 + n_k^3)} \frac{\cos u}{u}\, du .$$

None of these limits is less than 1, since the smallest,

$$\frac{\pi}{n_r} |n_r^3 - n_k^3| = \pi / n_r\, |n_r - n_k| \cdot (n_r^2 + n_r \cdot n_k + n_k^2),$$

and it was seen before (2.3) that $\frac{\pi}{n_r} |n_r - n_k| \geq 2\pi/3$,

and that $\cos u / u$ is integrable from 1 to any upper limit.

Since $\sum_1^{\infty} a_r I_r$ (5.31) is not bounded as $k \to \infty$, the Fourier series of the continuous function $F(x)$ is divergent at the point $x = 0$.

CHAPTER VI.

THE FOURIER SERIES OF F(x) IS BOREL SUMMABLE AT A POINT OF CONTINUITY.

6.1. Introduction. F(x) was defined (5.2) in the closed interval $(0, \pi)$ as

$$F(0) = 0$$

$$F(x) = a_r \sin n_r^3 x \qquad \pi/n_r \leq x \leq \pi/n_{r-1}$$

and by evenness and periodicity elsewhere. F(x) is continuous at $x = 0$, but its Fourier series is divergent at that point (5.3). It will be shown that this series is Borel summable at $x = 0$.

6.2. Moore's condition for summability (B). As seen before (3.34), the necessary and sufficient condition that the Fourier series of a function $f(x)$* be summable (B) at a point x is that

$$B\left[f(x_0)\right] = \frac{1}{\pi}\int_0^{\pi}\left\{f(x_0+t)+f(x_0-t)\right\}\frac{e^{-\frac{1}{4}yt^2}}{t}\sin yt\,dt$$

approach the finite value $f(x_0)$, as continuous $y \to \infty$.

6.3. F(x) is Borel summable at x = 0. Since F(x) is an even function, this condition becomes at $x = 0$,

$$B\left[F(0)\right] = \frac{2}{\pi}\int_0^{\pi}F(0+t)\,\frac{e^{-\frac{1}{4}yt^2}}{t}\sin yt\,dt$$

$$(6.31) \qquad = \frac{2}{\pi}\sum_{r=1}^{\infty}a_r\int_{\pi/n_r}^{\pi/n_{r-1}}\frac{(\sin n_r^3 t)\,e^{-\frac{1}{4}yt^2}\sin yt}{t}\,dt,\quad y\to\infty.$$

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* f(x) is Lebesgue integrable and periodic.

(6.31) might be written as

$$B\left[F(0)\right] = \frac{2}{\pi}\sum_{r=1}^{\infty} a_r L_r .$$

Since $F(0) = 0$, this sum must approach zero, for the Fourier series of $F(x)$ to be summable (B) at $x = 0$.

We shall show that the integral, L_r, is $o(1)$ for values of $r \leq k$, and has a finite upper bound for $r > k$, where k is determined so that $\sum_{k+1}^{\infty} a_r < \epsilon$; ϵ, being a small positive number. This choice of k is possible since $\sum a_r$ is convergent. The value of y is taken in the range $n_k^3 \leq y < n_{k+1}^3$, and thus $y \to \infty$ continuously as $k \to \infty$. In the y-interval, y is first taken equal to n_k^3, and then $n_k^3 < y < n_{k+1}^3$.

Writing $\pi/n_r = \lambda_r$,

$$L_r = \int_{\lambda_r}^{\lambda_{r-1}} \frac{e^{-\frac{1}{2}yt^2}}{t} (\sin n_r^3 t)(\sin y t)\, dt$$

1) When $y = n_k^3$:

a) with $k = r$,

$$L_r = \int_{\lambda_k}^{\lambda_{k-1}} \frac{e^{-\frac{1}{2}n_k^3 t^2}}{t} (\sin n_k^3 t)^2\, dt$$

$$\leq \frac{e^{-\frac{n_k^3}{2}\left(\frac{\pi}{n_k}\right)^2}}{\pi/n_k} \int_{\lambda_k}^{\lambda_{k-1}} (\sin n_k^3 t)^2\, dt$$

$$\leq e^{-n_k}(n_k)\left[\pi/n_{k-1} - \pi/n_k\right]$$

$$= o(1).$$

b) with $k \neq r$,

$$L_r = \tfrac{1}{2} \int_{\lambda_r}^{\lambda_{r-1}} e^{-\frac{1}{2}yt^2} \left\{ \frac{\cos |n_k^3 - n_r^3|\, t - \cos (n_k^3 + n_r^3) t}{t} \right\} dt$$

$$= \tfrac{1}{2} e^{-\frac{n_k^3}{2}\left(\frac{\pi}{n_r}\right)^2} \int_{\lambda_r}^{\xi} \left\{ \frac{\cos |n_k^3 - n_r^3|\, t - \cos (n_k^3 + n_r^3) t}{t} \right\} dt$$

$$+ \tfrac{1}{2} e^{-\frac{n_k^3}{2}\left(\frac{\pi}{n_{r-1}}\right)^2} \int_{\xi}^{\lambda_{r-1}} \left\{ \frac{\cos |n_k^3 - n_r^3|\, t - \cos (n_k^3 + n_r^3) t}{t} \right\} dt$$

by the mean value theorem, where $\lambda_r < \xi < \lambda_{r-1}$. The integrals in this last expression are uniformly bounded for all $r < k$ and $r > k$ (5.34). The exponential factors are $o(1)$ for $r < k$, and $O(1)$ for $r > k$, (actually $o(1)$ whenever $n_k^3 > n_r^2$ or $n_k^3 > n_{r-1}^2$). Thus L_r itself is $o(1)$ for $r < k$ and at most $O(1)$ for $r > k$, as $k \to \infty$.

2) When $n_k^3 < y < n_{k+1}^3$.

$$(6.32) \qquad L_r = \tfrac{1}{2} \int_{\lambda_r}^{\lambda_{r-1}} \frac{e^{-\frac{1}{2}yt^2}}{t} \left[\cos |n_r^3 - y|\, t - \cos (n_r^3 + y) t \right] dt$$

Let us call the expression in brackets C_r. Then

a) with $r \leq k$,

$$L_r = \tfrac{1}{2} e^{-\frac{y}{2}\left(\frac{\pi}{n_r}\right)^2} \int_{\pi/n_r}^{\xi} C_r \, dt + \tfrac{1}{2} e^{-\frac{y}{2}\left(\frac{\pi}{n_{r-1}}\right)^2} \int_{\xi}^{\pi/n_{r-1}} C_r \, dt$$

where $\pi/n_r < \xi < \pi/n_{r-1}$. Each of these integrals is less in absolute value than $2(\pi/n_{r-1} - \pi/n_r) = o(1)$.

Each of the exponential factors is $o(1)$, e.g.,

$$\frac{e^{-\frac{y}{2}\left(\frac{\pi}{n_r}\right)^2}}{\pi/n_r} \leq e^{-n_k^3/n_r^2}(n_r) \leq e^{-n_k}(n_k) = o(1).$$

Thus L_r is $o(1)$ as $k \to \infty$, for $r \leq k$, and y in the interval $n_k^3 < y < n_{k+1}^3$.

b) with $r > k$, (and $n_k^3 < y < n_{k+1}^3$),

$$L_r = \tfrac{1}{2}\int_{\lambda_r}^{\lambda_{r-1}} e^{-\frac{1}{2}yt^2}\frac{\cos(n_r^3-y)t}{t}\,dt - \tfrac{1}{2}\int_{\lambda_r}^{\lambda_{r-1}} e^{-\frac{1}{2}yt^2}\frac{\cos(n_r^3+y)t}{t}\,dt$$

$$(6.33) \qquad = \frac{1}{2}(L_{r_1} + L_{r_2}).$$

Using the second mean-value theorem this last integral is seen to be uniformly bounded for all $r > k$, so that all we need consider is the first integral, L_{r_1}. Because of the difficulties that may arise when $(n_r^3 - y)$ is small, the interval for y is divided into two parts.

b_1) when $r = k+1$, and $n_k^3 < y < n_{k+1}^3 - n_{k+1}$.

$$L_{r_1} = \int_{\lambda_r}^{\lambda_{r-1}} e^{-\frac{1}{2}yt^2}\,\frac{\cos(n_r^3-y)t}{t}\,dt$$

$$= e^{-\frac{y}{2}\left(\frac{\pi}{n_r}\right)^2}\int_{\pi/n_r}^{\xi}\frac{\cos(n_r^3-y)t}{t}\,dt + e^{-\frac{y}{2}\left(\frac{\pi}{n_{r-1}}\right)^2}\int_{\xi}^{\pi/n_{r-1}}\frac{\cos(n_r^3-y)t}{t}\,dt$$

where $\pi/n_r < \xi < \pi/n_{r-1}$. This first integral is seen to be convergent, if written as

$$\int_{\frac{\pi}{n_r}(n_r^3-y)}^{\xi(n_r^3-y)} \frac{\cos u}{u}\,du$$

since the lower limit $\frac{\pi}{n_r}(n_r^3-y) \geq \frac{\pi}{n_{k+1}}(n_{k+1}) = \pi$, since $r = k+1$. Thus L_{r_1} is uniformly convergent in this y-interval.

b_2) when $r = k+1$, and $n_{k+1}^3 - n_{k+1} \leq y < n_{k+1}^3$,

$$|L_{r_1}| = \left| \int_{\pi/n_r}^{\pi/n_{r-1}} \frac{e^{-\frac{1}{2}yt^2}}{t} \cos (n_{k+1}^3 - y)t \; dt \right|$$

$$\leq \int \frac{e^{-\frac{1}{2}yt^2}}{t} \; dt = \int o(1) \; dt = o(1)$$

since, as $k \to \infty$,

$$\frac{e^{-yt^2/2}}{t} \leq \frac{e^{-(n_{k+1}^3 - n_{k+1}) \cdot \frac{1}{(n_{k+1})^2}}}{\pi/n_{k+1}} = \frac{e^{-n_{k+1}}}{\pi/n_{k+1}} [O(1)] = o(1).$$

b_3) when $r > k+1$, and $n_k^3 < y < n_{k+1}^3$,

$$L_{r_1} = \int_{\lambda_r}^{\lambda_{r-1}} \frac{e^{-\frac{1}{2}yt^2} \cos (n_r^3 - y)t}{t} \; dt$$

is uniformly bounded for all $r > k+1$, as is clearly seen by using the second mean-value theorem as in the preceding section ($2b_1$).

Thus when $n_k^3 < y < n_{k+1}^3$, L_r, the integral in (6.32), is $o(1)$ for $r \leq k$, and uniformly bounded for $r > k$, as $k \to \infty$.

<u>6.4. Summary</u>. Summarizing our results, we have shown that with $\sum_{k+1}^{\infty} a_r < \epsilon$,

$$\frac{2}{\pi} \left(\sum_{r=1}^{k} + \sum_{r=k+1}^{\infty} \right) a_r \int_{\pi/n_r}^{\pi/n_{r-1}} (\sin n_k^3 t) \frac{e^{-\frac{1}{2}yt^2}}{t} \sin yt \; dt$$

of (6.31) approaches zero as continuous y becomes infinite, by showing that for $n_k^3 \leq y < n_{k+1}^3$, the integral is $o(1)$ for

$r \leq k$, and uniformly bounded for all $r > k$, as k ,(and therefore y), becomes infinite.

Since this sum approaches zero, $F(x)$ as defined above (5.2) has its Fourier series Borel summable to $F(0)$ at $x = 0$, a point of continuity, where the same series is divergent.

6.5. Comparison of Hardy and Moore conditions. In (3.2) we saw that the integral definition of Borel summability was:

A sum $\sum_0^\infty u_n$ is said to be Borel summable it

$$\lim \int_0^y e^{-y} \sum_0^\infty \frac{u_n y^n}{n!} dy \rightarrow s.$$

Since the converse of this is true, the condition for summability (B) of Fourier series (3.34)

$$(6.51) \qquad \frac{1}{\pi}\int_0^\pi \left\{f(x+t) + f(x-t)\right\} e^{-\frac{1}{4}yt^2} \frac{\sin yt}{t} dt \qquad y \rightarrow \infty$$

can be stated as a necessary and sufficient one.

At first sight this may seem to be entirely different from Hardy's necessary and sufficient condition (4.41) that

$$(6.52) \qquad \int_0^{\xi^{-1/2}} \varphi(t) \frac{\sin \xi t}{t} dt$$

must approach zero, where $\varphi(t) = \frac{1}{2}\left\{f(x+t) + f(x-t) - 2s\right\}$ approaches zero as $t \rightarrow 0$. Here $s = 0$. This condition seems to eliminate the need of the exponential factor which was necessary in the method we used to work out the problem.

Actually we know that (6.51) for Fourier series is essentially equivalent to its counterpart for the exponential definitiion of Borel (3.2). This condition arising from the exponential definition is shown to differ by $o(1)$ from (6.52), [Hardy, 3, pp. 179- 181]. Thus these two conditions, (6.51) and (6.52), will be essentially equivalent.

In the Schwarz example (4.1) the diffioult term was (4.22) the term with $r = k$,

$$a_k \int_{\pi/n_k}^{\pi/n_{k-1}} e^{-\frac{1}{2}n_k t^2} \frac{(\sin n_k t)^2}{t} \, dt \quad .$$

We can suppose the integrands to be written in the following integrals:

$$\sum_{k+1}^{\infty} \int_{\pi/n_R}^{\pi/n_{R-1}} + \int_{\pi/n_k}^{\pi/n_k} \qquad \int_{\pi/\sqrt{n_k}}^{\pi/\sqrt{n_{k-1}}} \qquad + \sum_{1}^{k-1} \int_{\pi/n_r}^{\pi/n_{r-1}}$$

the first representing all the terms with $R > k$, the last, those with $r < k$, the dotted integral inserted to indicate the limit of Hardy's condition. This condition would say that the sum of the integrals from $(\pi/\sqrt{n_k}\, , \pi/\sqrt{n_{k-1}})$ to the left must be $o(1)$ for summability (B). We have shown that from $(\pi/n_k\, , \pi/n_{k-1})$ to the left the sum of the integrals increased indefinitely, thus showing non-summability (B). Hence the two conditions agree on this problem. The exponential factor above is not effective to the left of $(\pi/\sqrt{n_k}\, , \pi/\sqrt{n_{k-1}})$.

With reference to the problem in this chapter (6.31) :

$$\cdots \int_{\pi(n_k)^{-3/2}}^{\pi(n_{k-1})^{-3/2}} \cdots + \int_{\pi/n_k}^{\pi/n_{k-1}} + \cdots + \int_{\pi/n_r}^{\pi/n_{r-1}} a_r \frac{e^{-\frac{1}{2}yt^2}}{t} \sin n_r^3 t \sin n_k^3 t \, dt \cdots$$

we have shown that the sum of these integrals is $o(1)$. The exponential factor is only effective (in the left direction) up to the interval $(\pi/(n_k)^{3/2}, \pi/(n_{k-1})^{3/2})$. Hardy's condition would say that all the integrals from

$$\int_{\pi(n_k)^{-3/2}}^{\pi(n_{k-1})^{-3/2}} a_r \frac{\sin n_r^3 t \sin n_k^3 t}{t} \, dt$$

to the left should be $o(1)$, for summability (B).

Thus the two conditions are essentially equivalent. Hardy's emphasizes the absolute minimum that is necessary. Our use of Moore's condition shows what happened to the term, with $r = k$, that kept Schwarz' original example from being Borel summable. Our method also emphasized the need of the exponential factor in the term where $r = k$, and in the terms where $r < k$.

BIBLIOGRAPHY.

1) T.J.Bromwich - Introduction to the Theory of Infinite Series. (1908 edition).

2) L. Fejer - Beispiele stetiger Functionen mit divergenter Fourier-reihe, Journal fur die r. u. a. Mathematik, (Crelle) v. 137 (1910), pp. 1 - 5.

3) G.H.Hardy - Remarks on some points in the theory of divergent series, Annals of Mathematics, (2), v.36 (1935) pp. 167 - 181.

4) G.H.Hardy and W.W.Rogosinski - Fourier Series, 1944.

5) E.W.Hobson - The Theory of Functions of a Real Variable, (1926), v.2.

6) L.Lorch - The Lebesgue constants for Borel summability, Duke Mathematical Journal, v.11, (1944), pp. 459 - 467.

7) C.N.Moore - On the application of Borel's method to the summation of Fourier series, Proceedings National Acad. of Sciences, v.11, (1925), pp. 284 - 288.

8) E.Picard - Traite D'Analyse, (1922), v.1.

9) A. Sachse - Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variablen durch trigonometrische Reihen, Schlomilch's Zeitschrift, supplement v.25 (1880), pp. 231 - 274. or Abhandlungen zur Geschichte der Mathematik, v.3, (1880)pp. 231 - 274.

10) O.Szasz - Introduction to the Theory of Divergent Series, (lithoprinted), (1944), University of Cincinnati.

11) E.C.Titchmarsh - The Theory of Functions, 2nd. edition, (1929).

12) A.Zygmund - Trigonometrical Series, (1935).