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A METHOD FOR OBTAINING

THE FREQUENCY-RESPONSE FUNCTION

FOR THE BALLISTOCARDIOGRAPHIC SYSTEM

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Preface

Possible uses of the ballistocardiograph include the determination of cardiac output, heart work per cycle, and the establishment of limits for the normal pattern. In order to arrive at a method of determining these quantities, the forces which cause the ballistocardiogram must be known.

If the frequency-response function for the system consisting of the heart, body, and ballisto-table were known it should be possible to determine these forces since the output is known and we could work backward to determine the input. Thus, the object of this thesis was to find a method for determining the overall frequency-response function for the system.

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Cecil E. Oelker

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I Introduction

The input and output of the system consisting of the heart, body, and ballisto-table are related by a differential equation, the solution of which would give the output for any given input. This equation provides a complete description or characterization of the system but one that cannot be conveniently obtained or used. Other methods of characterizing a system are related to the outputs produced by special types of inputs:

a) The frequency-response function relates a sinusoidal input to the output it produces.

b) The weighting function is the output produced by an impulse input.

c) The response to a step-function input.

These modes of description are simply related. Thus knowledge of any one characterizes the system and any of the others may be determined from it (1).

In determining the forces which produce the ballistocardiogram, the frequency-response function probably would be the most convenient. The output would be analyzed into its harmonic components and the sinusoidal force to produce each component determined. The sum of the input components would thus be the input function.

However, to determine the frequency-response function any of the types of inputs may be used and the method chosen depends upon the relative simplicity, accuracy and convenience of obtaining each of the inputs.

II Theory

A. Use of Cadavers

It seems apparent, if a known input is to be introduced at the heart of a man and the output determined, that use must be made of a cadaver. Assuming that tests have been made on a cadaver, one must be prepared to answer the question of how the data may be interpreted for a live body.

In anticipation of this question it was decided to devise a method of comparing the response function of a body before and after death. The method chosen was the frequency-response method. A sinusoidal force generator could be devised which could be fastened to the chest of a person and would be coupled to the body in much the same way as the heart. In this way the frequency-response function could be obtained for the body before and after death. There are three possibilities for the results:

a) The frequency-response function for the body may not change appreciably after death and the results obtained on cadavers can be generalized for use on live bodies.

b) The frequency-response function may change but the response function for the sine generator on the chest after death may agree well with the response function obtained from the input at the heart. In this case the sinusoidal generator response obtained from live persons may be used to give the frequency-response function.

c) None of the frequency-response functions may agree. In this case different mounting of the sine generator to closer approximate the heart may be tried or an approximation made to relate the response of cadavers to live bodies.

B. Selection of Input at Heart

Next let us consider the choice of input to be used at the heart. The frequency-response function could of course be obtained most easily by the application of a sinusoidal force at the heart and recording the response directly. The most apparent means for obtaining a sinusoidal force seems to be the use of a rotating unbalance. This requires a motor to drive it and appropriate gearing, etc. This would be rather bulky in comparison with the mass (about 400 grams) and size of the heart and would seriously alter the characteristics of the heart as a vibrating system. This seems to eliminate the use of the sinusoidal response method.

Thus we were led to the use of either a step function input or an impulse function input. An impulse force can be obtained by producing a sudden change of momentum in the heart. A particle of mass m could be ejected from or delivered to the heart with a velocity v and would produce a step function momentum, mv, or an impulse force function since $F = \frac{d}{dt} mv$.

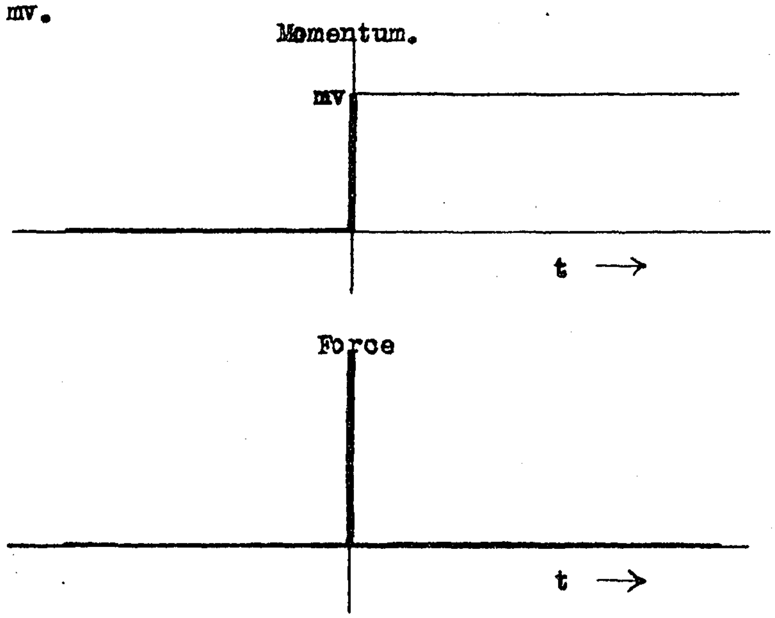


Figure 1. Impulse Force

The sudden ejection of a particle from the heart with a known velocity would require a mechanism of springs, a release, etc. which would also be fairly bulky. However, a particle delivered to the heart could be stopped with little or no bounce by simply allowing it to fall into the heart or to strike the end of a tube placed solidly in the heart muscle. The tube could be made of aluminum and the total mass of the tube and particle made small compared with that of the heart. This was the method finally chosen.

C. Effect of a Finite Width of Impulse

In as much as the particle cannot be stopped instantaneously in a practical experiment, the effect of the use of an impulse with a finite time width was investigated*. Consider a single degree of freedom with mass M, damping coefficient C, and spring constant K. Since it can be shown (1) that the response to an impulse function is the derivative of the response to a step function input, we shall first determine the response to a step function P(t), ie., P(t) = 0 t < 0, P(t) = P t > 0. Then

$$M \frac{d^2x}{dt^2} + C \frac{dx}{dt} + Kx = P$$

or

$$\frac{d^2x}{dt^2} + \frac{C}{M}\frac{dx}{dt} + \frac{K}{M}x = \frac{P}{M}$$

or

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = \frac{P}{M}$$

where $C_c = 2\sqrt{KM}$, $\frac{C}{C_c} = \zeta$, and $\omega_n = \sqrt{\frac{K}{M}}$

The general solution of this equation is:

$$(1) \quad x = e^{-\zeta\omega_n t}\left(C_1 \cos\omega_d t + C_2 \sin\omega_d t\right) + \frac{P}{K}$$

Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

*This was done with the help of Dr. Podolsky.

Using the initial conditions that at $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$, we find

$$C_1 = -\frac{P}{K} \quad , \quad C_2 = -\frac{\xi P}{K\sqrt{1-\xi^2}}$$

or

$$x = e^{-\xi\omega_n t}\left(-\frac{P}{K}\cos\omega_d t - \frac{P}{K}\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right) + \frac{P}{K}$$

(2) $$= \frac{P}{K}\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \delta)\right] \qquad t > 0$$

where $$\delta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\sqrt{\frac{1}{\xi^2} - 1}$$

Now the response to an impulse function is given by the derivative of the response to the step function. Thus, if X is the response to an impulse,

$$X = \frac{dx}{dt} = \frac{P\xi\omega_n}{K} e^{-\xi\omega_n t}\left(\frac{1}{\sqrt{1-\xi^2}}\sin(\omega_d t + \delta)\right)$$

$$- \frac{P\omega_d}{K\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\cos(\omega_d t + \delta)$$

$$= \frac{P\omega_n}{K\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin(\omega_d t + \delta + \phi)$$

where $$\phi = \tan^{-1}\frac{-\sqrt{1-\xi^2}}{\xi} = -\delta$$

Therefore (3) $$X = \frac{P\omega_n}{K\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\omega_d t$$

Since at t = 0, the initial momentum = $M\left(\frac{dX}{dt}\right)_{t=0}$ = P, we see that P is the initial momentum.

Consider now the response to an impulse force, F, of time width λ as shown in Figure 2.

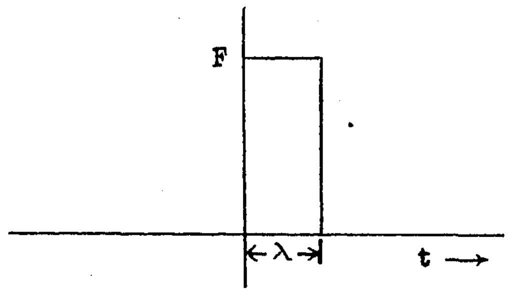


Figure 2. Impulse Force of Finite Time Width

Now $P = \int F\,dt$ is the total change of momentum.

The response for $0 < t < \lambda$ will be the same as the response to a step function input as found previously in Equation 2.

$$x = \frac{F}{K}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \delta\right)\right]$$

Then

$$(4)\quad x(\lambda^-) = \frac{F}{K}\left[1 - \frac{e^{-\zeta\omega_n \lambda}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d \lambda + \delta\right)\right]$$

And from Equation 3

$$(5)\quad \dot{x}(\lambda^-) = \frac{F\omega_n}{K\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n\lambda}\sin\omega_d\lambda$$

From Equation 1 we get

$$(6) \quad x(\lambda+) = e^{-\zeta\omega_n\lambda}\left(C_3 \cos \omega_d\lambda + C_4 \sin \omega_d\lambda\right)$$

$$\dot{x}(\lambda+) = -\zeta\omega_n e^{-\zeta\omega_n\lambda}\left(C_3 \cos \omega_d\lambda + C_4 \sin \omega_d\lambda\right)$$
$$+\omega_d e^{-\zeta\omega_n\lambda}\left(-C_3 \sin \omega_d\lambda + C_4 \cos \omega_d\lambda\right)$$

$$(7) \quad = -C_3\omega_n e^{-\zeta\omega_n\lambda}\cos(\omega_d\lambda-\delta) - C_4\omega_n e^{-\zeta\omega_n\lambda}\sin(\omega_d\lambda-\delta)$$

Rearranging Equations 6 and 7 gives

$$C_3 \cos \omega_d\lambda + C_4 \sin \omega_d\lambda = x(\lambda+)\, e^{\zeta\omega_n\lambda} = \alpha$$
$$C_3 \cos(\omega_d\lambda-\delta) + C_4 \sin(\omega_d\lambda-\delta) = -\frac{\dot{x}(\lambda+)}{\omega_n} e^{\zeta\omega_n\lambda} = \beta$$

Using the method of determinants to solve for C_3 and C_4 and

letting

$$\Delta = \sin(\omega_d\lambda-\delta)\cos\omega_d\lambda - \cos(\omega_d\lambda-\delta)\sin\omega_d\lambda$$
$$= \sin(-\delta) = -\sin\delta$$

$$C_3 = \frac{1}{\Delta}\begin{vmatrix} \alpha & \sin\omega_d\lambda \\ \beta & \sin(\omega_d\lambda-\delta) \end{vmatrix} = \frac{\beta\sin\omega_d\lambda - \alpha\sin(\omega_d\lambda-\delta)}{\sin\delta}$$

$$C_4 = \frac{1}{\Delta}\begin{vmatrix} \cos\omega_d\lambda & \alpha \\ \cos(\omega_d\lambda-\delta) & \beta \end{vmatrix} = \frac{\alpha\cos(\omega_d\lambda-\delta) - \beta\cos\omega_d\lambda}{\sin\delta}$$

Putting in now the values of α and β leads to

$$C_3 = \frac{-F}{K(\sin\delta)\sqrt{1-\zeta^2}}\Big[\sqrt{1-\zeta^2}\, e^{\zeta\omega_n\lambda}\sin(\omega_d\lambda-\delta) + \sin^2\omega_d\lambda$$
$$- \sin(\omega_d\lambda+\delta)\sin(\omega_d\lambda-\delta)\Big]$$

$$C_4 = \frac{F}{K \sin\delta}\left\{ e^{\zeta\omega_n\lambda}\cos(\omega_d\lambda - \delta) - \frac{1}{\sqrt{1-\zeta^2}}\left[\sin(\omega_d\lambda+\delta)\cos(\omega_d\lambda - \delta) - \sin\omega_d\lambda\cos\omega_d\lambda\right]\right\}$$

Reducing by use of trigonometric identities, C_3 and C_4 reduce to

$$C_3 = -\frac{F}{K}\left[\frac{e^{\zeta\omega_n\lambda}\sin(\omega_d\lambda - \delta)}{\sin\delta} + \frac{\sin\delta}{\sqrt{1-\zeta^2}}\right]$$

$$C_4 = \frac{F}{K}\left|\frac{e^{\zeta\omega_n\lambda}\cos(\omega_d\lambda - \delta)}{\sin\delta} - \frac{\cos\delta}{\sqrt{1-\zeta^2}}\right|$$

Putting these back into Equation 6 and recalling that

$$\delta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

or $\quad \sin\delta = \sqrt{1-\zeta^2} \quad$ and $\quad \cos\delta = \zeta$

$$x = \frac{F e^{-\zeta\omega_n t}}{K\sqrt{1-\zeta^2}}\left\{\left[e^{\zeta\omega_n\lambda}\cos(\omega_d\lambda - \delta) - \zeta\right]\sin\omega_d t - \left[\sqrt{1-\zeta^2} + e^{\zeta\omega_n\lambda}\sin(\omega_d\lambda - \delta)\right]\cos\omega_d t\right\}$$

We now assume that the response can be represented by the response to an impulse function, Equation 3, plus λ times a correction term, Q, plus negligible correction terms of higher orders of λ. That is

$$(9) \qquad x = \frac{p\omega_n}{K\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t + \lambda Q$$

Now $$P = \int F\,dt = F\lambda$$

or $$F = \frac{P}{\lambda}$$

Putting this in Equation 8 and determining the contribution of the C_3 term to $Q_0 = Q(\lambda = 0)$ we get:

$$\text{Contribution of } C_3 \text{ term to } Q_0 = \lim_{\lambda \to 0} \frac{-Pe^{-\zeta\omega_n t}}{K\sqrt{1-\zeta^2}} \left\{ \frac{\left[\sqrt{1-\zeta^2} + e^{\zeta\omega_n\lambda} \sin(\omega_d\lambda - \delta)\right] \cos\omega_d t}{\lambda^2} \right\}$$

Using L'Hospital's Rule
$$= \frac{-Pe^{-\zeta\omega_n t}\cos\omega_d t}{K\sqrt{1-\zeta^2}} \left[\frac{\zeta^2\omega_n^2(-\sqrt{1-\zeta^2})}{2} + \frac{\zeta^2\omega_n\omega_d + \omega_d^2\sqrt{1-\zeta^2}}{2} \right]$$

$$= \frac{-Pe^{-\zeta\omega_n t}\cos\omega_d t}{K\sqrt{1-\zeta^2}} \; \frac{\omega_n^2\sqrt{1-\zeta^2}}{2}$$

$$= \frac{-P\omega_n^2 e^{-\zeta\omega_n t}\cos\omega_d t}{2K}$$

Similarly, the contribution of the C_4 term to Q_0

$$= \lim_{\lambda \to 0} \frac{Pe^{-\zeta\omega_n t}\sin\omega_d t}{K\sqrt{1-\zeta^2}} \; \frac{\left[e^{-\zeta\omega_n\lambda}\cos(\omega_d\lambda - \delta) - \zeta\right]}{\lambda^2}$$

$$= \frac{Pe^{-\zeta\omega_n t}\sin\omega_d t}{K\sqrt{1-\zeta^2}} \; \frac{\omega_n^2\zeta^2}{2}$$

$$= \frac{P\zeta\omega_n^2}{2K\sqrt{1-\zeta^2}}$$

Hence $$Q_0 = \frac{-P\omega_n^2 e^{-\zeta\omega_n t}\cos\omega_d t}{2K} + \frac{P\zeta\omega_n^2 e^{-\zeta\omega_n t}\sin\omega_d t}{2K\sqrt{1-\zeta^2}}$$

$$= \frac{P\omega_n^2 e^{-\zeta\omega_n t}}{2K\sqrt{1-\zeta^2}}\left[\zeta\sin\omega_d t - \sqrt{1-\zeta^2}\cos\omega_d t\right]$$

$$= \frac{P\omega_n^2 e^{-\zeta\omega_n t}}{2K\sqrt{1-\zeta^2}}\sin(\omega_d t - \delta)$$

We want λ chosen so that the magnitude of the λQ_0 term in Equation 9 is small in comparison with the constant term. That is

$$\frac{\text{Magnitude } \lambda Q_0}{\frac{P\omega_n}{K\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}} = \frac{\frac{\lambda P\omega_n^2 e^{-\zeta\omega_n t}}{2K\sqrt{1-\zeta^2}}}{\frac{P\omega_n e^{-\zeta\omega_n t}}{K\sqrt{1-\zeta^2}}} < 1$$

$$\frac{\lambda\omega_n}{2} < 1$$

or $$\lambda < \frac{2}{\omega_n} = \frac{2}{2\pi f_n} = \frac{1}{\pi f_n} = \frac{T}{\pi}$$ where T is the period.

Thus, if the impulse is very short in comparison with $\frac{T}{\pi}$ no serious error is introduced.

D. Fitting the Weighting Function Curve

The weighting function, that is the response to an impulse input, for a single degree of freedom was shown in section C. to be of the form

$$W(t) = Ae^{-Bt}\sin Ct$$

For a system having several degrees of freedom the weighting function will be a sum of such terms with the addition of a phase factor. Thus, for two degrees of freedom, we have,

$$W(t) = Ae^{-Bt}\sin Ct + De^{-Et}\sin(Ft+G) \qquad (10)$$

Since the weighting function will be obtained experimentally as a curve, it will be necessary to fit it as accurately as possible with an equation of the form of Equation 10. Thus, if we assume two such terms, there are 7 constants to be determined to give the best fit. This may be done in a systematic way by the method of least squares.

A number, n, of ordinates of the curve to be fitted are selected corresponding to certain values of t. An approximate value for each of the constants in the equation is also assumed (A_0, B_0, ...G_0).

Then

$$A = A_0 + \alpha \qquad D = D_0 + \delta$$

$$B = B_0 + \beta \qquad E = E_0 + \epsilon$$

$$C = C_0 + \gamma \qquad F = F_0 + \psi$$

$$G = G_0 + \eta$$

$$y = Ae^{-Bt}\sin Ct + De^{-Et}\sin(Ft+G)$$

Then

$$\frac{\partial y}{\partial A} = d = e^{-Bt}\sin Ct$$

$$\frac{\partial y}{\partial B} = f = -Ate^{-Bt}\sin Ct$$

$$\frac{\partial y}{\partial C} = g = Ate^{-Bt}\cos Ct$$

$$\frac{\partial y}{\partial D} = h = e^{-Et}\sin(Ft+G)$$

$$\frac{\partial y}{\partial E} = j = -Dte^{-Et}\sin(Ft+G)$$

$$\frac{\partial y}{\partial F} = k = Dte^{-Et}\cos(Ft+G)$$

$$\frac{\partial y}{\partial G} = l = De^{-Et}\cos(Ft+G)$$

Then

$$y = A_0e^{-B_0t}\sin C_0t + D_0e^{-E_0t}\sin(F_0t+G_0) + \alpha\left(\frac{\partial y}{\partial A}\right)_0 + \beta\left(\frac{\partial y}{\partial B}\right)_0 + \gamma\left(\frac{\partial y}{\partial C}\right)_0 + \delta\left(\frac{\partial y}{\partial D}\right)_0 + \epsilon\left(\frac{\partial y}{\partial E}\right)_0 + \psi\left(\frac{\partial y}{\partial F}\right)_0 + \eta\left(\frac{\partial y}{\partial G}\right)_0$$

If we call $y_0 = A_0e^{-B_0t}\sin C_0t + D_0e^{-E_0t}\sin(F_0t+G_0)$, then S, the sum of the square of the deviations of the calculated ordinates from the measured ordinates, is given by

$$S = \sum_1^m (y - y_0 - \alpha d - \beta f - \gamma g - \delta h - \epsilon j - \psi k - \eta l)^2$$

We want $\alpha, \beta, \ldots \eta$ to be chosen so S is a minimum.

Therefore,

$$\frac{\partial S}{\partial \alpha} = -2\sum_1^m d(y - y_0 - \alpha d - \beta f - \gamma g - \delta h - \epsilon j - \psi k - \eta l) = 0$$

$$\frac{\partial S}{\partial \beta} = -2\sum_1^m f(\qquad\qquad) = 0$$

$$\vdots$$

$$\frac{\partial S}{\partial \eta} = -2\sum_1^m l(\qquad\qquad) = 0$$

or

$$\alpha \sum_1^m d^2 + \beta \sum_1^m df + \gamma \sum_1^m dg + \delta \sum_1^m dh + \epsilon \sum_1^m dj + \psi \sum_1^m dk + \eta \sum_1^m dl = \sum_1^m d(y-y_0)$$

$$\alpha \sum_1^m df + \beta \sum_1^m f^2 + \gamma \sum_1^m fg + \delta \sum_1^m fh + \epsilon \sum_1^m fj + \psi \sum_1^m fk + \eta \sum_1^m fl = \sum_1^m f(y-y_0)$$

$$\alpha \sum_1^m dg + \beta \sum_1^m fg + \text{-----------------} = \sum_1^m g(y-y_0)$$

$$\vdots$$

$$\alpha \sum_1^m dl + \beta \sum_1^m fl + \gamma \sum_1^m gl + \delta \sum_1^m hl + \epsilon \sum_1^m jl + \psi \sum_1^m kl + \eta \sum_1^m l^2 = \sum_1^m l(y-y_0)$$

These may be written in summation notation as

$$\sum a_{ij} \alpha_j = f_i$$

where $\alpha_1 = \alpha \quad , \quad \alpha_2 = \beta \quad ,$ etc.

$a_{11} = \sum d^2, \quad a_{12} = \sum df \quad ,$ etc.

and $a_{ij} = a_{ji}$

This is a set of 7 equations in 7 unknowns $(\alpha, \beta, \cdots, \eta)$ which may be solved simultaneously to yield a set of corrections to the originally chosen values of A, B,.....G.

E. Transformation of Weighting Function to Frequency-Response Function

After obtaining the weighting function from the recorded curve as explained in the last section, we must next transform it to give the desired frequency-response function. It can be shown (2) that the output, $E_0(t)$, for any input, $E_i(t)$, is given by

$$E_0(t) = \int_{0-}^{\infty} E_i(t-T)\, W(T)\, dT$$

where $W(T)$ is the weighting function. The response to a sinusoidal input, $E_i(t) = A\, e^{j\omega t}$, is then given by

$$\begin{aligned} E_0(t) &= A\int_{0-}^{\infty} e^{j\omega(t-T)} W(T)\, dT \\ &= A\, e^{j\omega t} \int_{0-}^{\infty} e^{-j\omega T}\, W(T)\, dT \\ &= E_i(t) \int_{0-}^{\infty} e^{-j\omega T}\, W(T)\, dT \end{aligned}$$

Calling the frequency-response function $Y(j\omega)$, then

$$Y(j\omega) = \frac{E_0(t)}{E_i(t)} = \int_{0-}^{\infty} e^{-j\omega T}\, W(T)\, dT$$

If we assume $W(T)$ is of the form of Equation 10, then

$$\begin{aligned} Y(j\omega) &= \int_0^{\infty} e^{-j\omega T}\left[A e^{-BT} \sin CT + D\, e^{-ET} \sin(FT+G)\right] dT \\ &= \int_0^{\infty} e^{-j\omega T}\left[\frac{Ae^{-BT}}{2j}\left(e^{jCT} - e^{-jCT}\right) + \frac{De^{-ET}}{2j}\left(e^{j(FT+G)} - e^{-j(FT+G)}\right)\right] dT \\ &= \frac{A}{2j}\int_0^{\infty}\left[e^{-j\omega T - BT + jCT} - e^{-j\omega T - BT - jCT}\right] dT \\ &\quad + \frac{D}{2j}\int_0^{\infty}\left[e^{-j\omega T - ET + j(FT+G)} - e^{-j\omega T - ET - j(FT+G)}\right] dT \end{aligned}$$

$$Y(j\omega) = \frac{A}{2j}\left[\frac{e^{(-j\omega-B+jC)T}}{-j(\omega-C)-B} - \frac{e^{(-j\omega-B-jC)T}}{-B-j(\omega+C)}\right]_0^\infty + \frac{D}{2j}\left[\frac{e^{jG}e^{(-j\omega-E+jF)T}}{-E-j(\omega-F)} - \frac{e^{-jG}e^{(-j\omega-E-jF)T}}{-E-j(\omega+F)}\right]_0^\infty$$

$$Y(j\omega) = \frac{A}{2j}\left[\frac{B+j(\omega+C)-B-j(\omega-C)}{B^2+C^2-\omega^2+2jB\omega}\right] + \frac{D}{2j}\left[\frac{(\cos G+j\sin G)[E+j(\omega+F)]-(\cos G-\sin G)[E+j(\omega-F)]}{E^2+F^2-\omega^2+2j\omega E}\right]$$

$$Y(j\omega) = \frac{A\ C}{B^2+C^2-\omega^2+2jB\omega} + D\left[\frac{F\cos G + E\sin G + j\omega\sin G}{E^2+F^2-\omega^2+2j\omega E}\right]$$

$$= M\, e^{j\phi} + P\, e^{j\theta} = (M\cos\phi + P\cos\theta) + j(M\sin\phi + P\sin\theta)$$

Thus, the magnitude of the frequency-response is given by

$$|Y| = \sqrt{(M\cos\phi + P\cos\theta)^2 + (M\sin\phi + P\sin\theta)^2}$$

and the phase by

$$\text{Arg } Y(j\omega) = \tan^{-1}\frac{M\sin\phi + P\sin\theta}{M\cos\phi + P\cos\theta}$$

This is the desired frequency-response function.

III Experimental

A. Sinusoidal Force Generator

To fulfill the need of a sinusoidal force generator that could be fastened to the chest of a person, the equipment shown in Figure 3 was designed and constructed. Two rotating unbalances 180° out of phase provide a sinusoidal force along the head to foot direction but cancel out in the side to side direction in order to arouse only one mode of oscillation at a time.

A sets of weights was devised to be used at different speed ranges in order to provide approximately a constant amplitude force of 400 gr. over the frequency range used. Table I shows the sets of weights, the appropriate radius used, the speed range and the force constant which, when

Table I

Set of Weights	Speed Cycles /Sec.	Radius To Be Used	Force Constant To Get Force in Grams Multiply by f^2
Lead	1-3 Cycles	Medium	69.25
Brass & Lead	3-6	Medium	25.25
Brass	6-10	Large	6.695
Aluminum	10-20	Medium	1.822
Screws	20-30	Small	.4535

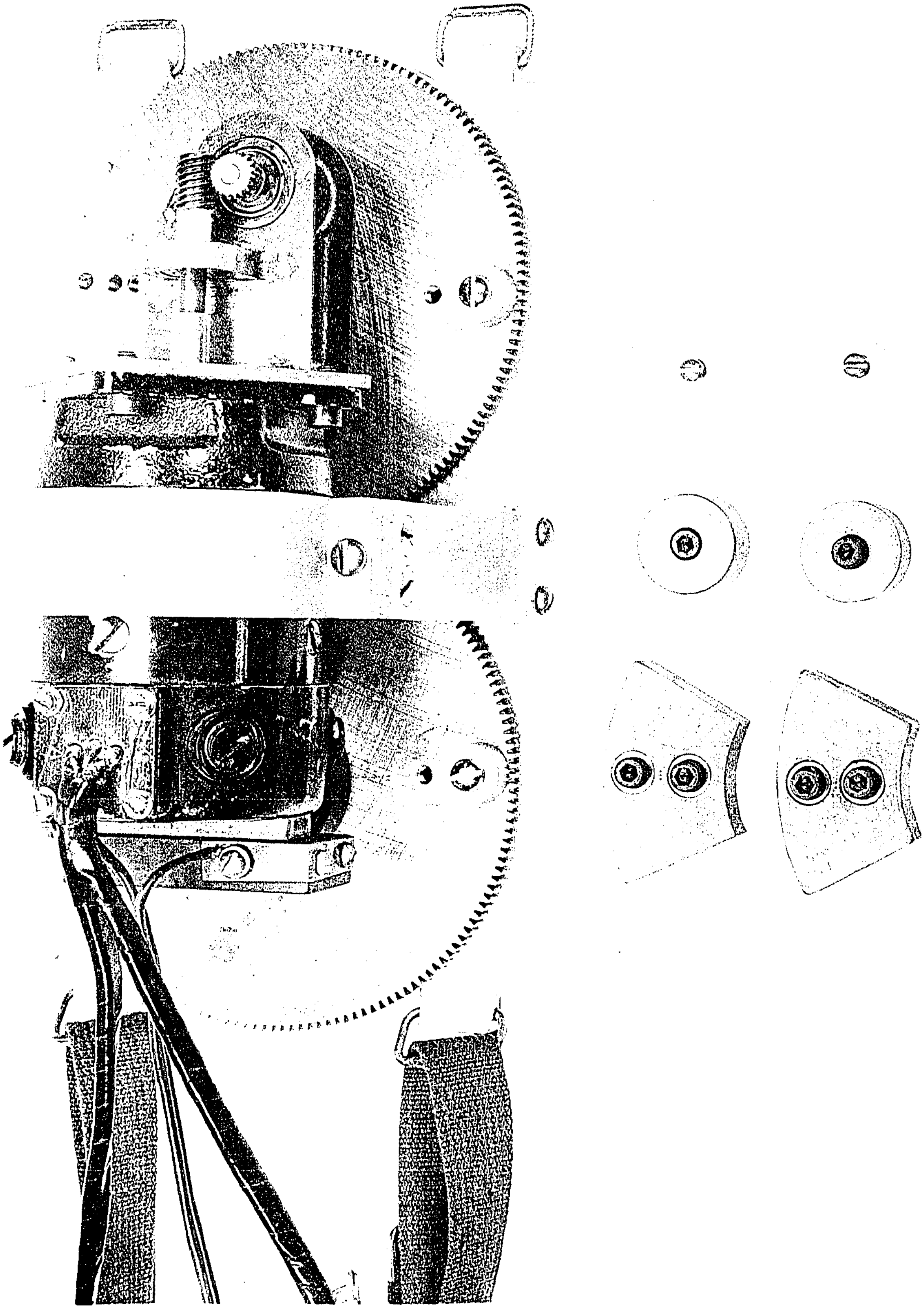


Figure 3. Sinusoidal Force Generator

multiplied by the frequency squared, gives the amplitude of the force.

The generator is strapped to the chest by two webb belts. It is driven by a small 12 volt D.C. motor supplied by two lead storage batteries. The motor is mounted above the rotating gears which act as the unbalance supports. The motor speed is controled by rheostats in the armature circuit and the speed is adjusted by setting the armature voltage according to Table II. Two of the gears in the gear chain from the motor to the rotating gears may be interchanged. This changes the ratio of the motor speed to the unbalance speed from $12\frac{1}{2}$:1 to 2:1. The low speed range is from 1 to 6 cycles per second and the

Table II

Speed Cycles/Sec.	Armature Voltage	Motor Speed in R.P.M.	Gear Ratio
1	2.03	750	Low Speed Range
2	3.65	1500	
4	6.90	3600	$12\frac{1}{2}$:1
8	2.80	960	High Speed Range
15	4.65	1800	
20	5.95	2400	2:1
25	7.30	3000	
30	8.65	3600	

high speed range from 4 to 30 cycles per second. The control panel may be seen in Figure 4.

In order to be able to determine the speed exactly a contact is mounted on one of the rotating unbalance gears so that it is closed for an instant each revolution of the gear. This shorts out an element in a bridge circuit on a Brush amplifier and provides a pulse on the record each cycle. Thus by counting the pulses and knowing the recorder paper speed, the frequency of the sinusoidal force generator can be determined quite accurately. In addition the pulse enables one to measure the phase shift of the output since the pulse occurs at the zero of the sine input.

B. Impulse Equipment

The particle used to produce the impulse input is a steel bearing ball 7/8 in. in diameter. The mass of the ball is 44.658 grams. It obtains its horizontal velocity by rolling down an inclined plane and coming off horizontally with a known velocity. It then falls through a tube bent in the form of a parabola, so the ball does not touch the tube, and into an aluminum tube placed in the heart. The parabolic tube serves only to locate the aluminum tube properly so the ball is sure to strike only the end of the aluminum tube. The arrangement of the apparatus for an experiment is shown in Figure 5.

Since the horizontal velocity does not change after

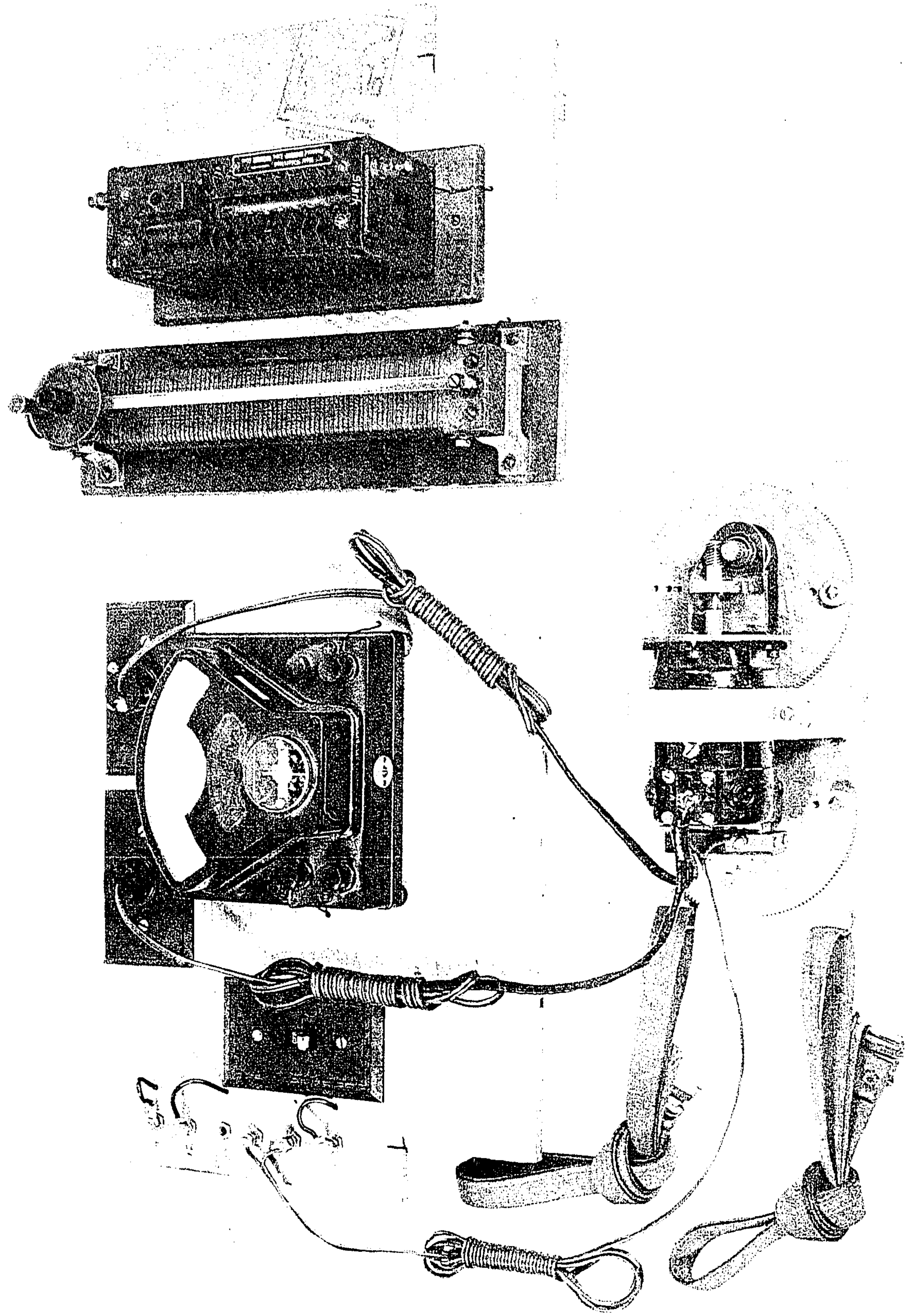


Figure 4. Sinusoidal Force Generator and Control Panel

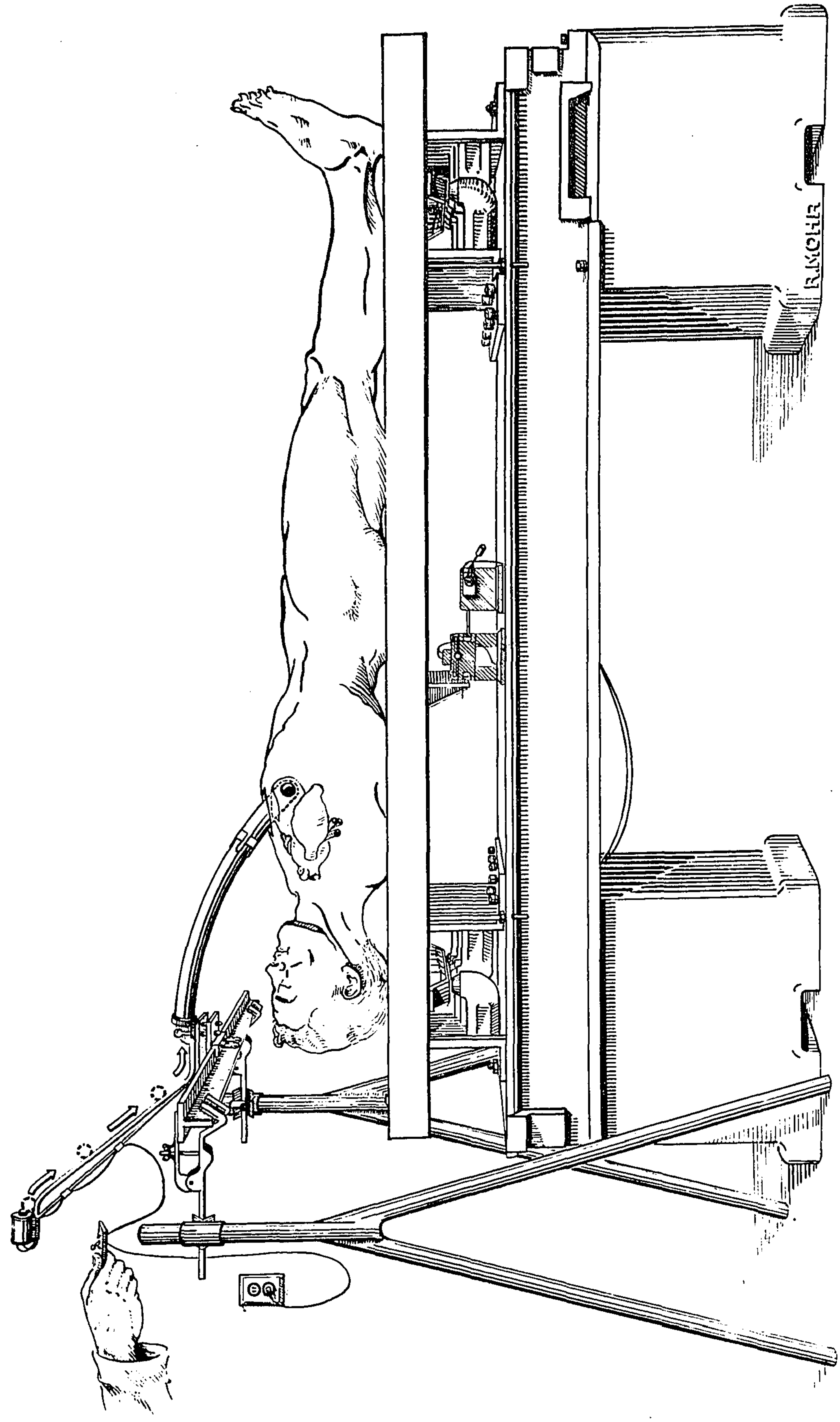


Figure 5. Impulse Equipment Arranged for Experiment

leaving the horizontal part of the inclined plane, the horizontal momentum of the ball is known. The vertical component of momentum produces a force in the heart perpendicular to the surface of the table and hence produces no resulting motion of the table.

The horizontal velocity of the ball was determined by allowing it to fall from the inclined plane through a known vertical distance and measuring the horizontal distance the ball travels. The velocity can then be calculated by use of elementary physics. The measured value of the horizontal velocity was 176 ± 2 cm./sec. The horizontal momentum of the ball is then 7860 ± 89 gr. cm./sec.

To prevent the ball from bouncing when it strikes the end of the tube, a chunk of paraffin was placed in the end of the tube. The indentation of the ball in the paraffin is somewhat less than $\frac{1}{2}$ mm. deep, so the ball comes to rest in less than $\frac{1}{2}$ mm. Using an average velocity of 176/2 cm./sec., this corresponds to about 0.0005 sec. for the impulse time width. From Sec. II C. we recall that the time width λ should be small compared with $\frac{T}{\pi}$. Taking $\frac{T}{\pi} = 10\lambda$ we see that T can be of the order .02 sec. or f = 50 c.p.s. which is about 3 times the highest frequency expected in our system. In other words, the arrangement gives $\lambda = \frac{1}{30} \frac{T}{\pi}$.

The ball is held in place at the top of the inclined plane by a small electro-magnet supplied by one cell of the lead storage batteries. The inclined plane is leveled by means of adjustable thumb screws until an "L" level mounted on the base shows it to be horizontal.

The aluminum tube which is placed in the heart has a mass of less than 24 grams. Thus, the total mass added to the heart by the tube and ball is only about 68 grams.

C. Frequency-Response Curve for Live Persons

Using the sinusoidal generator described previously strapped to the chest, response curves were obtained on two persons. Typical records for two different frequencies are shown in Figure 6. By subtracting the normal ballistocardiograph pattern from these curves, the amplitude and phase shift of the response is obtained.

The frequency is determined by counting the pulses, corresponding to the cycles, occurring in one second (25 lines on the curves in Figure 6). Multiplying the force constant given in Table I by f^2 gives the amplitude of the impressed sinusoidal force in grams. The desired amplitude, X_d, (3) is given by $\frac{F}{K}$ where K is the spring constant of the ballistocardiograph. The K for our table at standard amplification is 3.135 gr./mm. The ratio of the amplitude of response, X, to the desired amplitude, X_d, is then determined and plotted against the impressed frequency to give the frequency response function. The frequency-response functions obtained for the two subjects

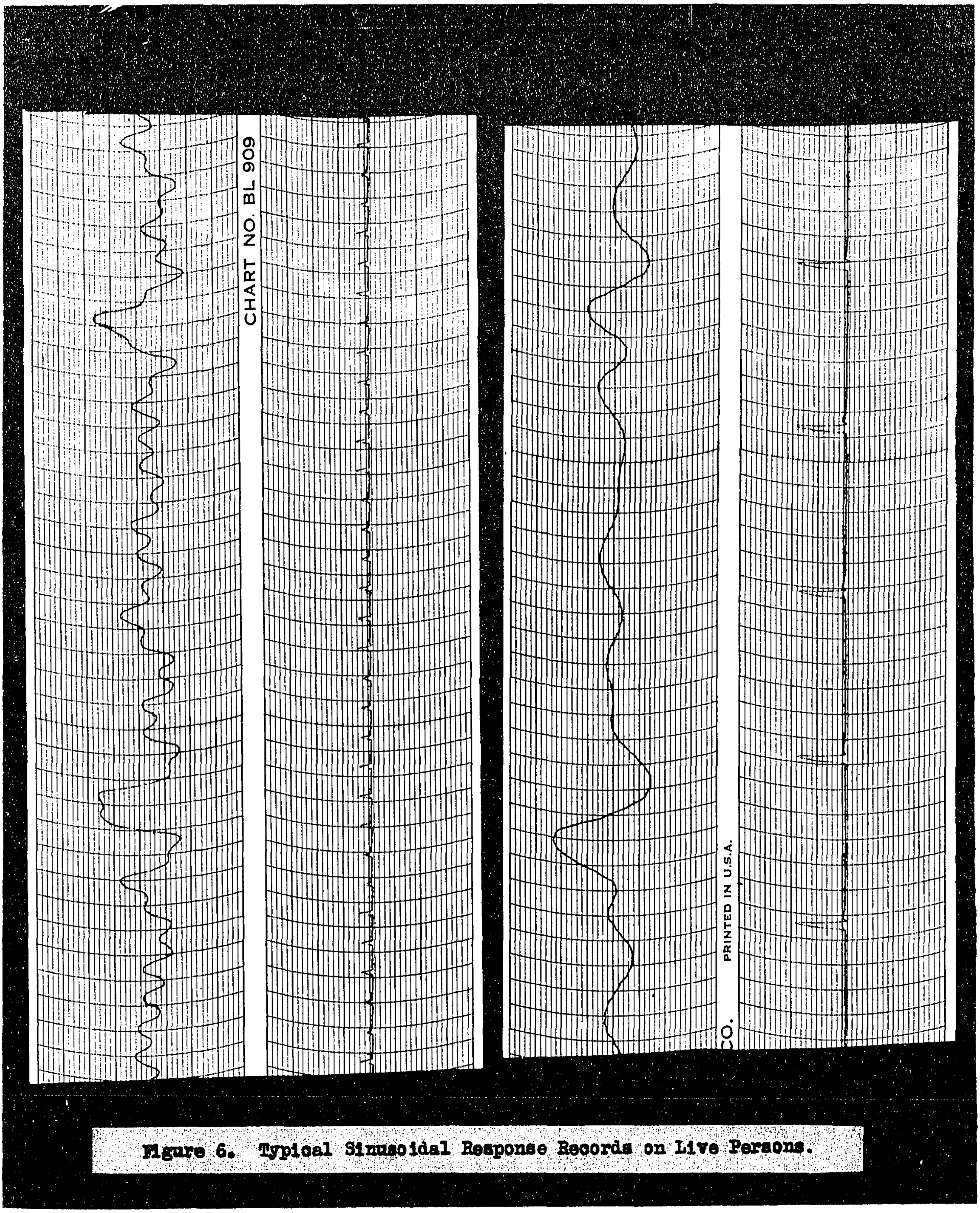


Figure 6. Typical Sinusoidal Response Records on Live Persons.

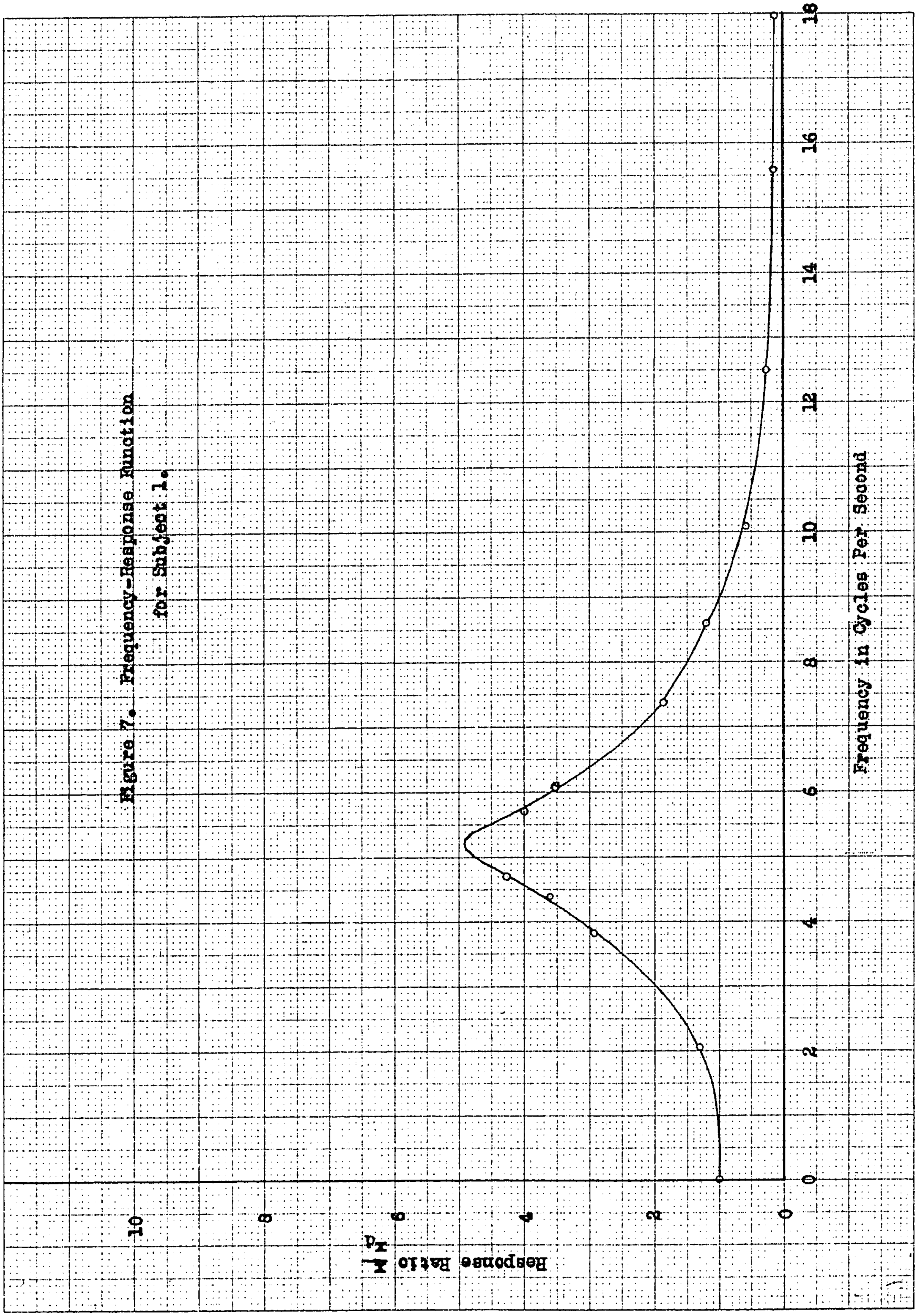


Figure 7. Frequency-Response Function for Subject 1.

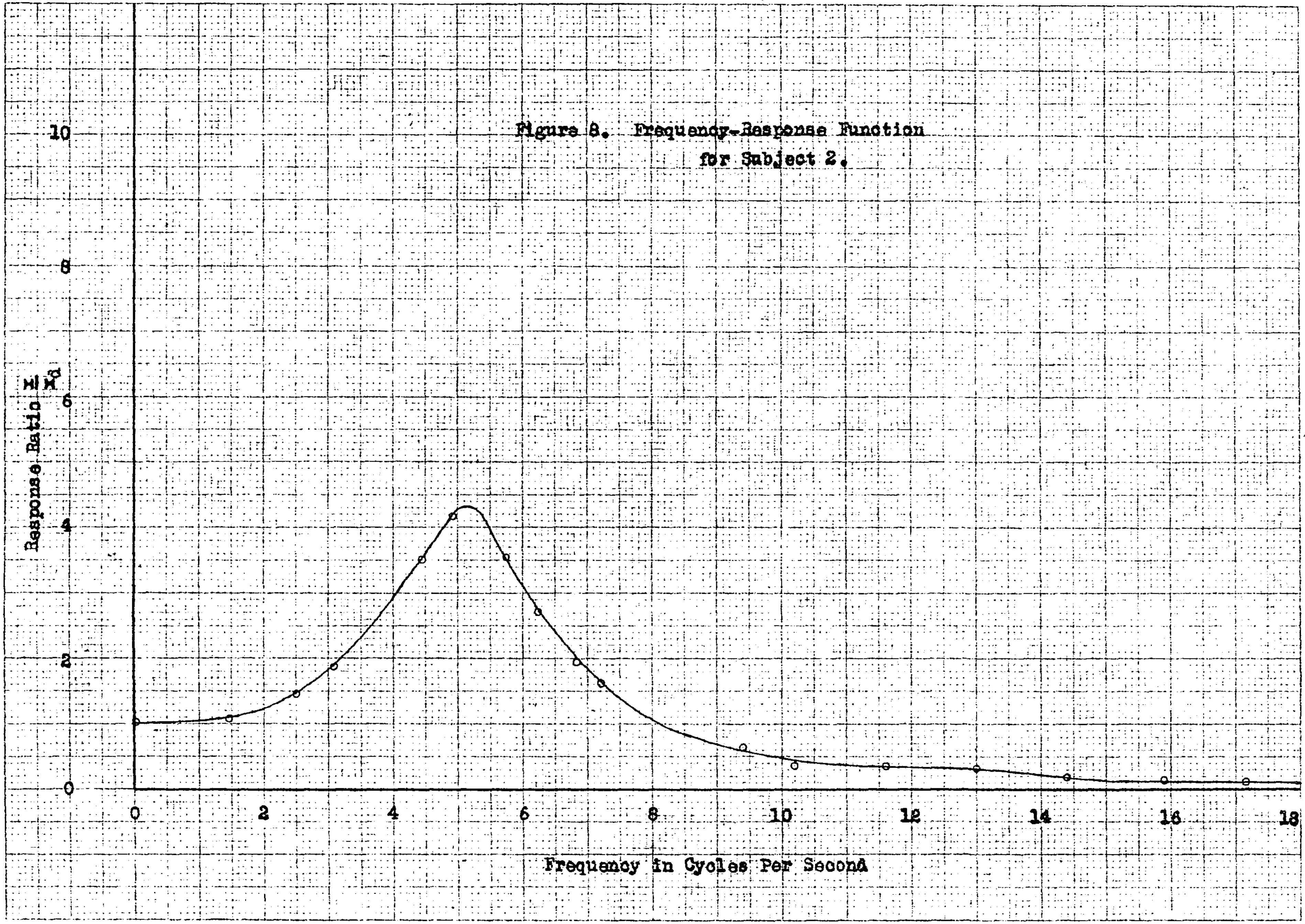


Figure 8. Frequency-Response Function for Subject 2.

tested are shown in Figure 7 and 8.

D. Frequency-Response Function for Cadavers

In a similar way the frequency-response function was obtained on the second of the above subjects after death. This curve is shown in Figure 9.

It is easily seen that there has been a distinct change after death. The peak occurs at a higher frequency and is about twice the value obtained on living persons. This indicates an increase in the spring constant and a decrease in the damping coefficient with death. This may be accounted for by the loss of muscle tone. Thus, we must conclude that the data which we may obtain from the use of cadavers is not directly applicable for use on living persons.

E. Weighting Function for a Cadaver

After the frequency-response test was made using the sinusoidal generator on the chest of the cadaver, an opening was made in the chest by a surgeon and the aluminum tube was placed firmly in the heart muscle at the apex of the right ventricle. The opening in the chest was made as small as possible and with the aim of disturbing the heart and body coupling as little as possible.

The equipment was then arranged as shown in Figure 5 and the ball was released into the tube in the heart. The test was repeated four times and the resulting outputs were found to

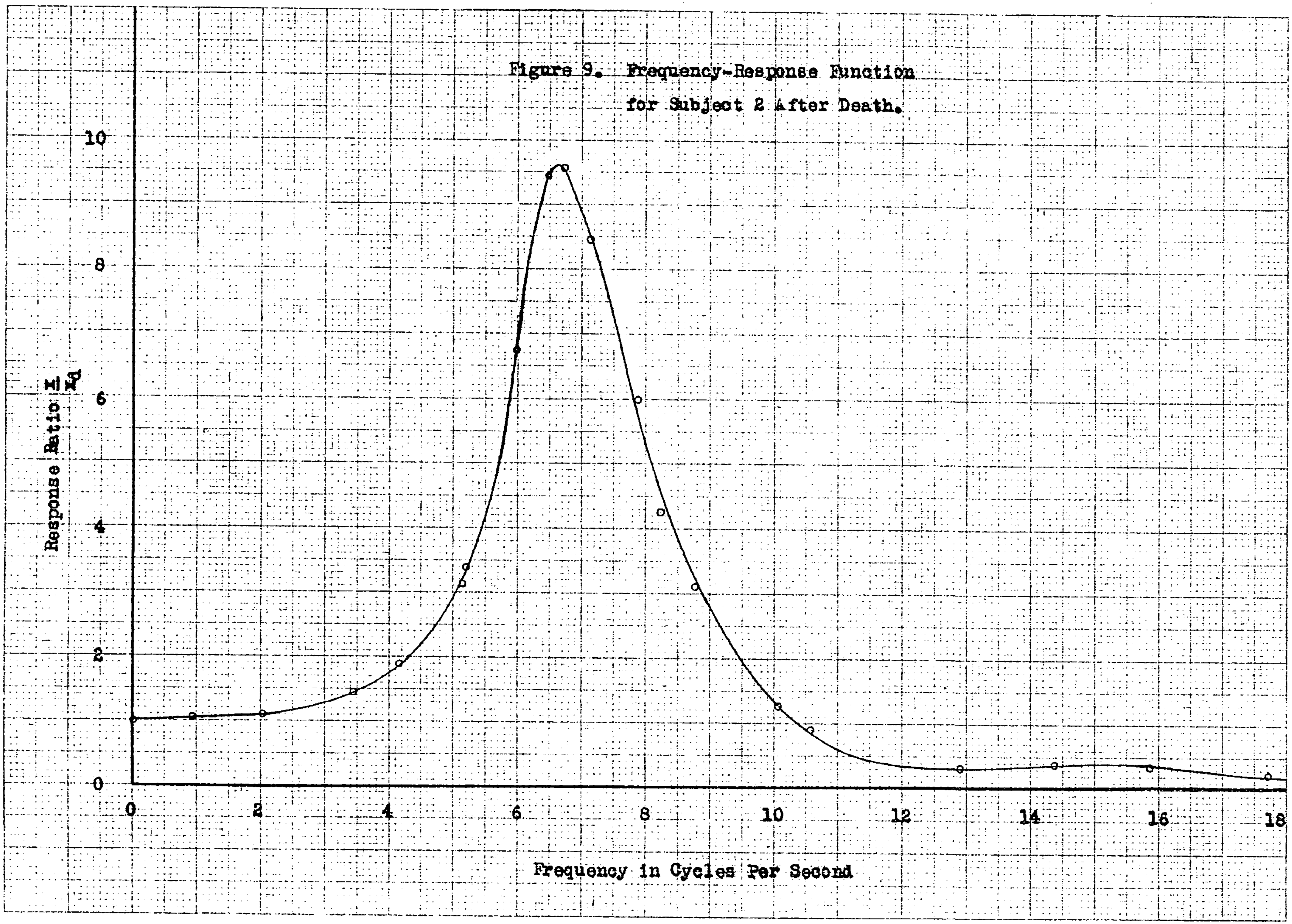


Figure 9. Frequency-Response Function for Subject 2 After Death.

be practically identical. The response to the impulse input, that is the weighting function, was photographically enlarged and is shown in Figure 10.

The weighting function was found to contain three distinct components. The three components, their sum, and the original curve are shown in Figure 11. The highest frequency component was found to have a frequency of 16.6 cycles per second which is easily identified as that of the table with a load of 130 lbs. The other two components have frequencies of 7.14 and 7.74 cps. Dr. Starr (4) has reported the natural frequencies for cadavers of from 5 to 7 cps. in other tests made on the ballistocardiograph. This, along with the fact that the 7.74 cps. component leads the 7.14 cps. one, allows us to identify the 7.14 cps. component as that contributed by the body and the 7.74 cps. component as due to the heart. The relative amplitude, damping exponent, phase angle, and angular frequency for each component are listed in Table III.

Table III

Component	Relative Amplitude	Damping Exponent	Angular Frequency	Phase
Heart	18.50	5.34	48.60	0
Body	12.65	3.12	44.85	-59
Table	5.69	12.04	104.40	-109

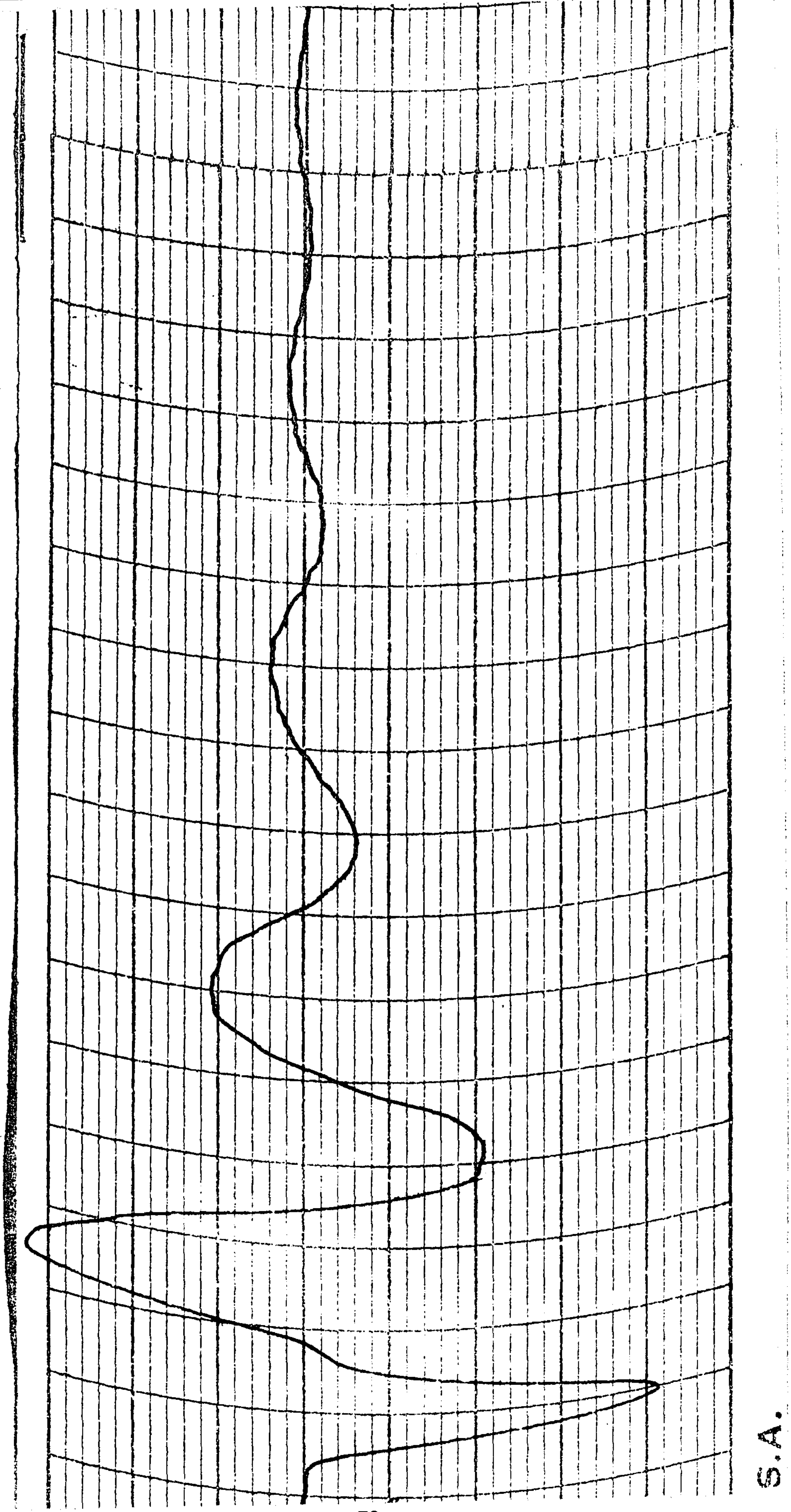
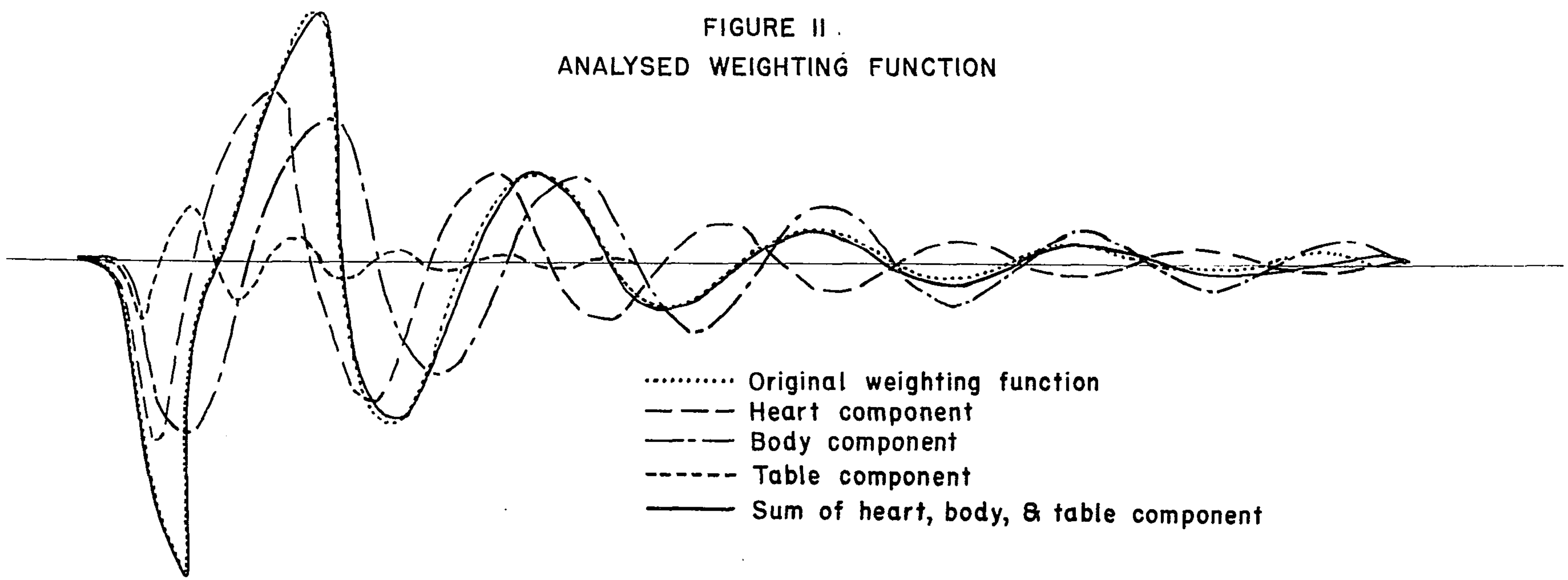


Figure 10. Weighting Function

FIGURE II

ANALYSED WEIGHTING FUNCTION



F. Frequency-Response Function Obtained From Weighting Function

Using the method outlined in Section II E. and the constants given in Table III, the frequency-response function was determined. The result is shown in Figure 12. The corresponding phase shift diagram is shown in Figure 13.

It can be seen by comparing Figures 9 and 12 that the sinusoidal generator and the impulse at the heart do not give comparable results. We must conclude then, that the last of our three original possibilities (Section II A.) is correct. That is, none of the frequency-response functions agree and thus further assumptions must be made to give a frequency-response function for a living body.

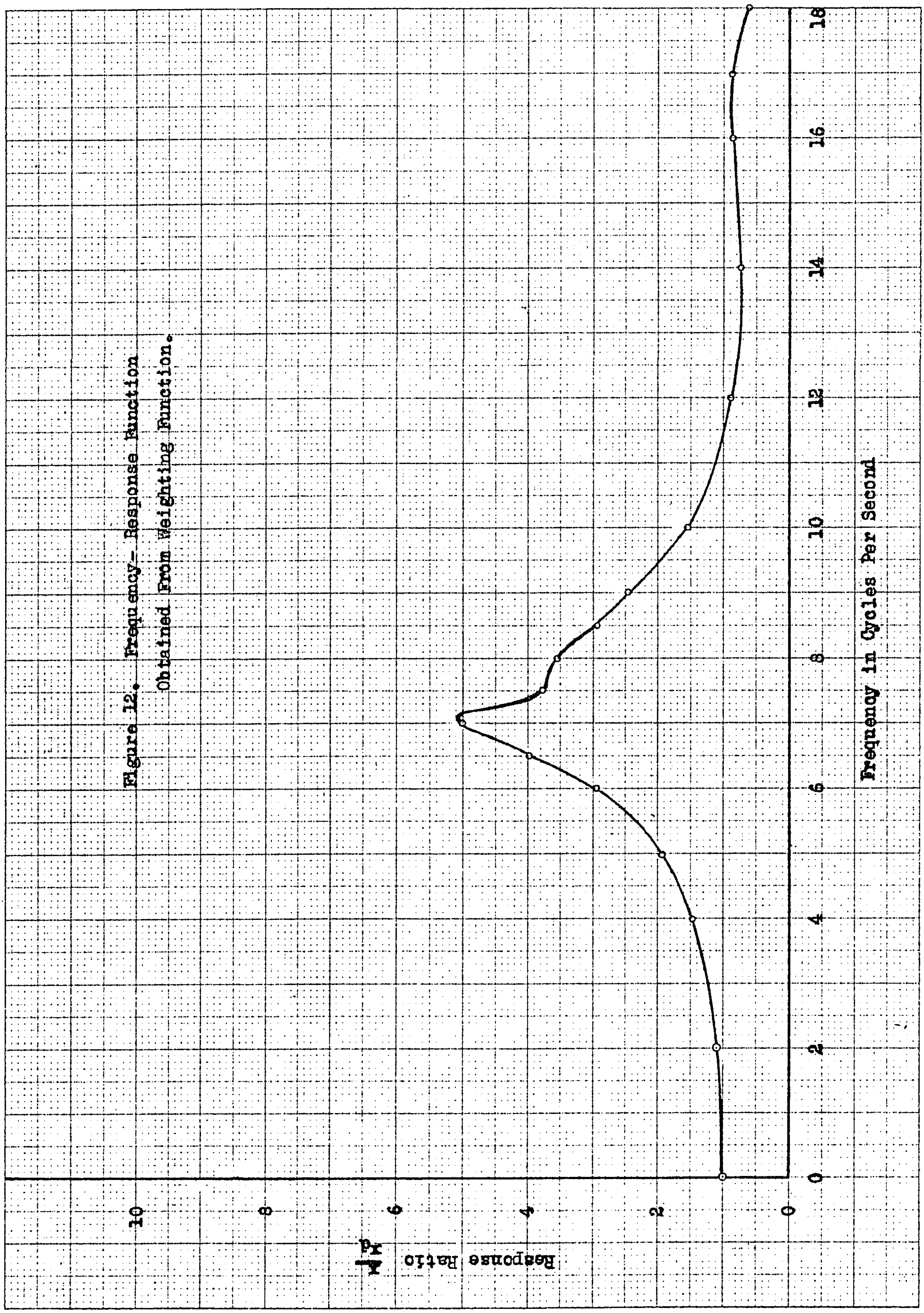


Figure 12. Frequency- Response Function
Obtained From Weighting Function.

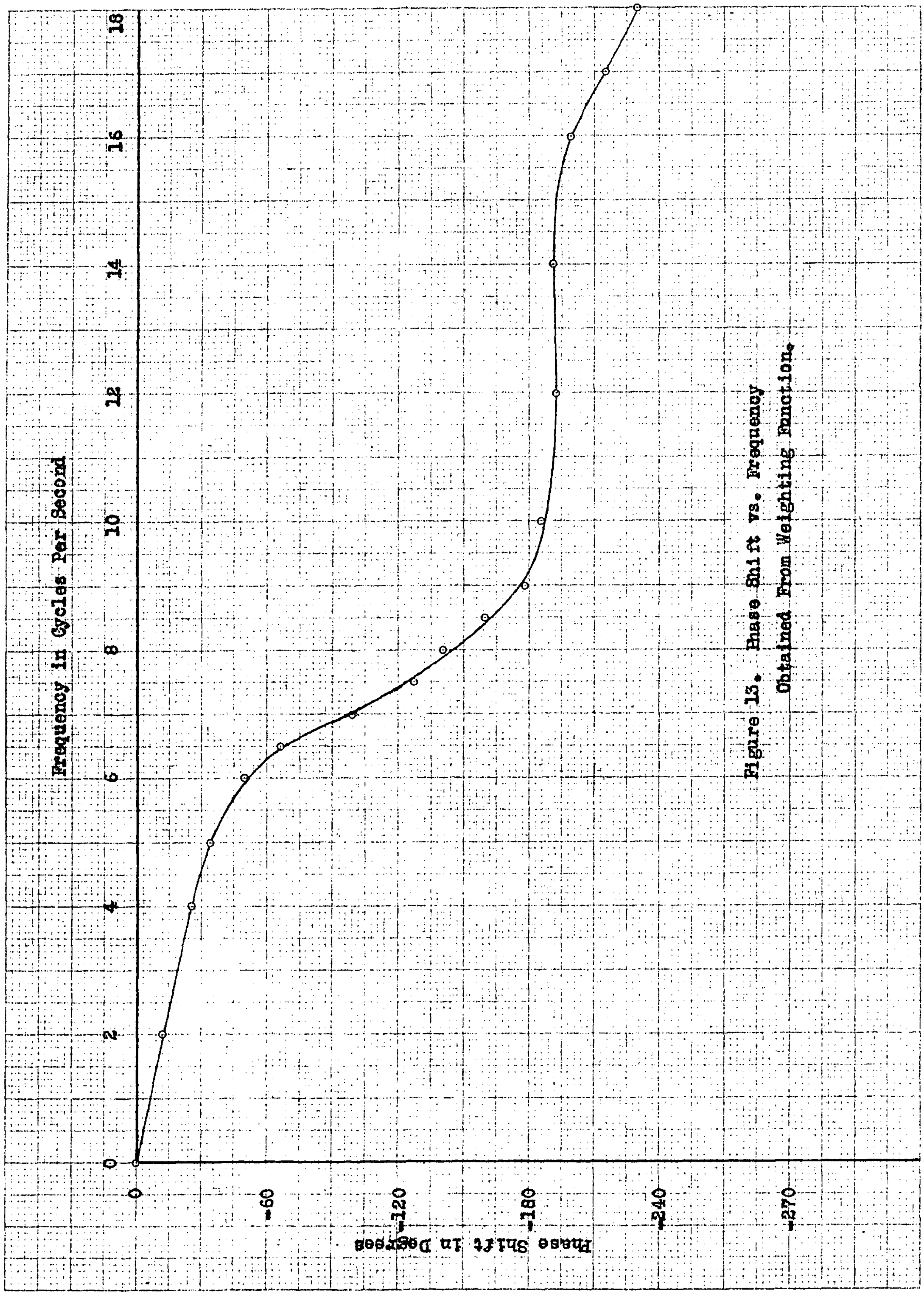


Figure 13. Phase Shift vs. Frequency
Obtained From Weighting Function.

IV Conclusion

From the foregoing, one may be inclined to believe that the likelihood of finding a frequency-response function for a living being is an extremely remote possibility, and that nothing can be gained from the information obtained. This, however, is not a fair evaluation of the facts.

First, the data has been obtained from a single experiment using one cadaver. For obvious reasons the tests cannot be rerun if the data seems suspicious. In addition, results obtained from one subject are not necessarily the same as would be found if a series of subjects were used which would bring in the variations due to the variety of body builds, normal deviations, etc.

Secondly, assuming that the data obtained is valid and would be similar for other subjects, it may still be possible to obtain a frequency-response function for living persons, upon making certain assumptions. For example, since the heart is an organ composed of striped muscle, a logical assumption might be that the damping and spring constant change in the same manner as the body damping and spring constant change due to loss of muscle tone with death. These changes could be determined from the frequency-response functions obtained with the sinusoidal force generator before and after death.

Another possibility is to attempt to change the method of mounting the sine generator on the chest so that its natural frequency and the coupling to the body give a better approximation of the heart and body coupling. The frequency-response function for living persons could then be obtained directly. Since no information was formerly available on the natural frequency or damping coefficient for a human heart, this could not be done previously.

In any event, it would seem that the methods developed in this thesis are practical and applicable to the problem at hand. Time does not permit investigation of the possibilities given above but work along this line will continue.