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I hereby recommend that the thesis prepared under my supervision by Lee Lorch

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Approved by:

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SOME PROBLEMS ON THE BOREL SUMMABILITY OF FOURIER SERIES

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by

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CHAPTER I

INTRODUCTION

Borel's method of summability [cf. Borel, 1]* is very powerful. It can, e.g., sum a power series beyond its circle of convergence (in the so-called "Borel polygon", discussed further in chapter VII).

It is far more powerful than Cesáro's method, which cannot sum a power series beyond the circle of convergence. In fact, Cesáro summability requires that the nth term be of order $O(n^{\kappa})$.

On the other hand, Cesáro's method, as Hardy and Littlewood [Hardy and Littlewood, 14] remarked, can sum series which are divergent more delicately than can Borel's. To be more precise, we have:

"If $\sum_0^\infty a_n$ is (C, 1)-summable and $a_n = O(\frac{1}{n})$ as $n \to \infty$, then $\sum_0^\infty a_n$ is convergent." [Hardy, 14]

"If $\sum_{n=0}^\infty a_n$ is Borel summable and if (only) $a_n = O(\frac{1}{\sqrt{n}})$ as $n \to \infty$, then (already) $\sum_0^\infty a_n$ is convergent." [Hardy, and Littlewood, 15].

Relations between Borel and Cesáro summability have been given by Hardy and by Hardy and Littlewood.

"If $\sum_0^\infty a_n$ is Borel summable and $a_n = O(1)$, then $\sum_0^\infty a_n$ is (C, 1)-summable." [Hardy and Littlewood, 15]

and If $\sum_0^\infty a_n$ is (C, 1)-summable, and

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* The numbers in brackets refer to the bibliography at the end.

$\sigma_n = \frac{s_0 + \cdots + s_n}{n+1}$, if $\sigma_n - f = o\left(\frac{1}{\sqrt{n}}\right)$ as $n \to \infty$,

then $\sum_0^\infty a_n$ is Borel summable [Hardy, 10].

In the case of Fourier series, where the nth term is o(1), as n becomes infinite, Borel's method would appear in advance to be less effective than Cesàro's. And this is indeed the case.

C. N. Moore [Moore, 21] pointed out that a certain continuous function is not Borel summable at 0, whereas, of course, as Fejér proved [Fejér, 4] a continuous function is uniformly (C, 1)-summable everywhere.

Moore further remarked that the Lebesgue constants (cf. chapter III) corresponding to Borel's method are of logarithmic magnitude, as are the ordinary Lebesgue constants [Lebesgue, 20, Fejér, 5].

As he stated, this might lead one to think that Borel's method cannot sum divergent Fourier series. That this is actually <u>not</u> the case he showed by constructing a continuous function whose Fourier series diverges at 0 but which is Borel summable there.

Nonetheless, this leaves open problems regarding the connection between convergence and Borel summability for Fourier series. Thus, e.g., given a continuous function whose Fourier series is everywhere Borel summable; would it necessarily be convergent almost everywhere?

As Moore observed, the kernel of the singular integral corresponding to Borel's method when applied to Fourier series is, apart from an exponential factor, essentially the Dirichlet kernel, which oscillates infinitely in the neighborhood of the origin. This stresses the connection between Borel summability and convergence for Fourier series. Such a theorem would be what is called of a Tauberian type.

It is in fact in this Tauberian direction, that Borel summability for Fourier series has lately been studied. Hardy and Littlewood [Hardy and Littlewood, 16, Hardy, 13] have given a criterion for Borel summability of Fourier series (cf. chapter V) and combined it with a Tauberian condition to get convergence criteria.

In this thesis, however, the work is largely in the direction indicated by Moore (loc. cit.). After a preliminary chapter which discusses a lemma of Fejér, I study the Lebesgue constants for Borel summability, and show that

$$L_B(u) = \frac{2}{\pi^2}\log u + O(1) \text{ as } u \to \infty$$

Then chapter IV is devoted principally to proving in detail that the continuous function which -- as Moore stated -- has a non-Borel summable Fourier series at 0, actually does enjoy this property (In so doing, I have had the benefit of C. N. Moore's notes on the subject.).

In chapter V a criterion for the Borel summability of a Fourier series at a point is given.

Chapter VI contains a uniqueness theorem; i.e., a theorem which states that a Borel summable trigonometric series with bounded coefficients is a Fourier series.

Chapter VII contains a brief digression. The object of that chapter is to apply Borel summability to the study of the singularities of functions represented by power series.

Chapter VIII is devoted to strong Borel summability in general together with an application to Fourier series. Also included is a theorem of Tauberian character.

More specific comments on the subject matter are found in the first section of each chapter.

CHAPTER II

ON A LEMMA OF FEJÉR

2.1 Introduction. In the course of calculating the Lebesgue constants, Fejér proved [Fejér, 5, formulas (8) and (9), p. 27] that for a bounded and Riemann integrable function f(x)

$$(2.11)\quad \lim_{n\to\infty}\int_a^b f(x)\,|\sin nx|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx$$

and

$$(2.12)\quad \lim_{n\to\infty}\int_a^b f(x)\,|\cos nx|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx$$

To prove these results, he subdivides the interval (a, b) and writes

$$\int_b^a = \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} + \cdots + \int_{(\ell-1)\frac{\pi}{n}}^{\ell\frac{\pi}{n}} + \epsilon_n,$$

where

$$(k-1)\frac{\pi}{n} < a \leqslant k\frac{\pi}{n} < (k+1)\frac{\pi}{n} < \cdots < \ell\frac{\pi}{n} \leqslant b < (\ell+1)\frac{\pi}{n}$$

and

$$\lim_{n\to\infty}\epsilon_n = 0.$$

Then applying the first mean-value theorem he gets, e.g.,

$$(2.13)\quad \int_a^b f(x)\,|\sin nx|\,dx = \frac{2}{\pi}\left(\frac{\pi}{n} f_1 + \cdots + \frac{\pi}{n} f_{\ell-k}\right) + \epsilon_n,$$

$$m_j \leq f_j \leq M_j, \quad j = 1, \ldots, \ell-k$$

where m_j, M_j are the minimum and maximum, respectively, of $f(x)$ in $(k + j - 1)\frac{\pi}{n} \leq x \leq (k + j)\frac{\pi}{n}$. To get (2.11), we need only let n become infinite in (2.13), since f(x) is assumed bounded and Riemann integrable and $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Adapting this idea, Hobson [Hobson, 7, v. II, p. 519] showed that (2.11) and (2.12) hold even if f(x) is assumed to be only Lebesgue integrable.

Here I derive (2.11) and (2.12) for Lebesgue integrable functions f(x) as a consequence of the Riemann-Lebesgue lemma (section 2.2). The integer n is replaced by the continuous variable u.

In section 2.5, a similar result is established for the case in which f(x) also depends on the parameter u. This particular theorem is needed for the calculation of the Lebesgue constants corresponding to Borel's summability method (cf. chapter III).

Preliminary to this theorem, I require an estimate for

$$(2.14) \quad \int_a^b f(x)\left\{\frac{2}{\pi} - |\sin ux|\right\} dx \quad \text{and} \int_a^b f(x) \left\{\frac{2}{\pi} - |\cos ux|\right\} dx$$

in the special case $f(x) \equiv 1$. This is found in section 2.3. This case is further discussed in section 2.4.

In section 2.6, estimates for the rapidity with which (2.14) tends to zero are given for certain classes of functions f(x).

An extension of (2.11) and (2.12) is given in section 2.7. There $|\sin nx|$ and $|\cos nx|$ are replaced by a more general function $\varphi(nx)$.

2.2. A proof of Fejér's lemma.

Theorem 2.21. If f(x) is Lebesgue integrable in (a, b), then

$$(2.21)\qquad \lim_{u\to\infty}\int_a^b f(x)\,|\sin ux|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx,$$

and

$$(2.22)\qquad \lim_{u\to\infty}\int_a^b f(x)\,|\cos ux|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx.$$

(a) Proof of (2.21):

The Fourier development of $|\sin ux|$ is [de la Vallée Poussin, 29]

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos 2jux}{4j^2-1}$$

and is obviously uniformly convergent.

Hence

$$(2.23)\qquad |\sin ux| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos 2jux}{4j^2-1} \quad *$$

* This expansion was used by G. Szegö [Szegö, 27] to give a representation for the (ordinary) Lebesgue constants ρ_n and, on the basis of this representation, to show that $\operatorname{sgn}\Delta^{\nu}\rho_n = (-1)^{\nu-1}$, $\nu = 1, 2, \ldots\ldots,$

and

$$(2.24)\quad \int_a^b f(x)\left\{\frac{2}{\pi} - |\sin ux|\right\} dx = \int_a^b f(x)\frac{4}{\pi}\sum_{j=1}^{\infty} \frac{\cos 2jux}{4j^2 - 1}\, dx$$

But $\sum_{j=1}^{\infty} \frac{\cos 2jux}{4j^2-1}$ is uniformly (and hence boundedly) convergent. Hence, to prove that termwise integration is permissible, we can appeal to the following standard theorem [cf. Kestelman, 18, theorem 231, p.149]:

"Let g(P) be summable, (i.e., Lebesgue integrable) over B, and let $\{f_r(P)\}$ be a sequence of functions measurable in B whose sum converges boundedly in B; then $g(P)\sum_{r=1}^{\infty} f_r(P)$ is equivalent in B to a summable function, and

$$(2.25)\quad \int_B g(P) \sum_{r=1}^{\infty} f_r(P)\, dP = \sum_{r=1}^{\infty} \int_B g(P)\, f_r(P)\, dP."$$

In this case $g(P) \equiv f(x)$, $B \equiv (a, b)$, $r \equiv j$,

$f_r(P) \equiv \frac{\cos 2jux}{4j^2 - 1}$. Hence

$$(2.26)\quad \int_a^b f(x)\left\{\frac{2}{\pi} - |\sin ux|\right\} dx = \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{4j^2-1} \times \int_a^b f(x) \cos 2jux\, dx.$$

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i.e., complete monotonicity, as Gronwall [Gronwall, 9] had conjectured. The latter established this relation for $\nu = 1$.

But

$$\left|\int_a^b f(x)\cos 2jux\,dx\right| \leq \int_a^b |f(x)|\,|\cos 2jux|\,dx \leq$$

$\int_a^b |f(x)|\,dx$, for all u and j, and $\sum_{j=1}^{\infty} \frac{1}{4j^2-1}$ converges.

The Weierstrass M-test then states the uniform convergence of

$$\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\int_a^b f(x)\cos 2jux\,dx$$

in $0 \leq u$. Its limit at $u \to \infty$ may therefore be taken termwise. Thus

$$(2.27)\quad \lim_{u\to\infty}\int_a^b f(x)\left\{\frac{2}{\pi} - (\sin ux|\right\}dx = \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\lim_{u\to\infty}\int_a^b f(x)\cos 2jux\,dx.$$

From the generalized Riemann-Lebesgue lemma*

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* The generalized Riemann-Lebesgue lemma states: "Let g(x) be summable over (a, b), and let $\varphi(x)$ denote either of the functions cos x and sin x, then

$$\lim_{k\to\infty}\int_a^b g(x)\,\varphi(kx)\,dx = 0$$

[Kestelman,[18] theorem 299, p. 233.]

$\lim_{u \to \infty} \int_a^b f(x) \cos 2jux \, dx$ is zero. This completes the proof of (2.21).

(b) Proof of (2.22):

In (2.23) replace ux by $ux + \frac{\pi}{2}$ [cf. Szász, [6], p. 168], getting

$$(2.28) \qquad |\cos ux| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^{\infty} (-1)^j \frac{\cos 2jux}{4j^2-1}$$

As before, this yields

$$(2.29) \quad \int_a^b f(x) \left\{ \frac{2}{\pi} - |\cos ux| \right\} dx = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{4j^2-1} \int_a^b f(x) \cos 2jux \, dx$$

Again as in part (a), applying the generalized Riemann-Lebesgue lemma gives the desired result: (2.22).

2.3. The case $f(x) \equiv 1$.

Theorem 2.31. $\left| \int_a^b \left\{ \frac{2}{\pi} - |\sin ux| \right\} dx \right| < \frac{2}{\pi} \frac{1}{u}, \quad u > 0.$

Proof: Putting $f(x) \equiv 1$ in (2.24) gives, for $u \neq 0$,

$$(2.31) \quad \begin{cases} \int_a^b \left\{ \frac{2}{\pi} - |\sin ux| \right\} dx = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{4j^2-1} \int_a^b \cos 2jux \, dx \\ = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{4j^2-1} \frac{1}{2ju} (\sin 2jub - \sin 2jua) \end{cases}$$

$$= \frac{2}{\pi}\,\frac{1}{u}\sum_{j=1}^{\infty}\frac{1}{j}\;\frac{1}{4j^2-1}\quad(\sin 2jub - \sin 2jua)$$

Hence, since $|\sin y| \leq 1$,

(2.32) $$\left|\int_a^b \left\{\frac{2}{\pi} - |\sin ux|\right\} dx\right| \leq \frac{4}{\pi}\,\frac{1}{u}\sum_{j=1}^{\infty}\frac{1}{j}\;\frac{1}{4j^2-1} \leq$$

$$\leq \frac{4}{\pi}\,\frac{1}{u}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}.$$

Now put $x = 0$ in (2.23), getting

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j^2-1},$$

whence

(2.33) $$\sum_{j=1}^{\infty}\frac{1}{4j^2-1} = \frac{1}{2}.$$

Substituting this in (2.32) completes the proof.

<u>Corollary 2.31.1.</u> $\int_{k\frac{\pi}{2}}^{l\frac{\pi}{2}} |\sin nx| = l - k$, where l and k are integers,

$$l \geq k,\ n = 1, 2, 3, \ldots\ldots$$

<u>Proof</u>: This follows from (2.31) since the sine of any integral multiple of π is zero.

<u>Theorem 2.32.</u> $\left|\int_a^b \left\{\frac{2}{\pi} - |\cos ux|\right\} dx\right| < \frac{2}{\pi}\;\frac{1}{u},\quad u \to 0.$

Proof: Put $f(x) \equiv 1$ in (2.29). The result then follows as in theorem 2.31.

Corollary 2.32.1: $\int_{k\frac{\pi}{2}}^{\ell\frac{\pi}{2}} |\cos nx| \, dx = \ell - k$, where ℓ and k are integers, $\ell \geq k$, $n = 1, 2, 3, \ldots\ldots$

The proof follows from (2.29).

2.4. Further discussion of the case $f(x) \equiv 1$. Theorems 2.31 and 2.32, respectively, show that

$$\int_a^b \left\{\frac{2}{\pi} - |\sin ux|\right\} dx \quad \text{and} \quad \int_a^b \left\{\frac{2}{\pi} - |\cos ux|\right\} dx$$

approach zero quite rapidly. Consequently one might guess that

$$\int_a^b \left(\frac{2}{\pi} - |\sin ux|\right)^2 dx \quad \text{and} \quad \int_a^b \left(\frac{2}{\pi} - |\cos ux|\right)^2 dx$$

also go to zero as u becomes infinite. Actually this is not the case. More precisely we have

Theorem 2.41. $\lim_{u\to\infty} \int_a^b \left(y - |\sin ux|\right)^2 dx >$

$$\lim_{u\to\infty} \int_a^b \left(\frac{2}{\pi} - |\sin ux|\right)^2 dx = \left(\frac{1}{2} - \frac{4}{\pi^2}\right)\left(b - a\right) > \frac{1}{20}\left(b - a\right),$$

for $y \neq \frac{2}{\pi}$.

Proof:

$$\int_a^b \left(y - |\sin ux|\right)^2 dx = (b-a)y^2 - 2y\int_a^b |\sin ux|\, dx + \int_a^b \sin^2 ux\, dx.$$

But

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

Hence

(2.41) $$\int_a^b (y - |\sin ux|)^2 dx = (b-a)y^2 - 2y\int_a^b |\sin ux|\, dx + \tfrac{1}{2}(b-a) - \tfrac{1}{2}\int_a^b \cos 2ux\, dx.$$

From the Riemann-Lebesgue lemma

$$\lim_{u\to\infty}\int_a^b \cos 2ux\, dx = 0$$

From theorem 2.31 (with $f(x) \equiv 1$)

$$\lim_{u\to\infty}\int_a^b |\sin ux|\, dx = \tfrac{2}{\pi}(b-a).$$

Therefore,

(2.42) $$\lim_{u\to\infty}\int_a^b (y - |\sin ux|)^2 dx = (b-a)\left(y^2 - \tfrac{4}{\pi}y + \tfrac{1}{2}\right)$$

The quadratic on the right has negative discriminant and hence no real zeros. Moreover, it reaches its minimum at $y = \frac{2}{\pi}$ and has there the value $\frac{1}{2} - \frac{4}{\pi^2} = \frac{\pi^2 - 8}{2\pi^2} > \frac{1}{20}$.

q.e.d.

Corollary 2.41.1.

$$\lim_{u\to\infty}\int_a^b \left|\frac{2}{\pi} - |\sin ux|\right| dx > \left(\frac{1}{2} - \frac{4}{\pi^2}\right)(b - a) > \frac{1}{20}\ (b - a).$$

Proof: $\left|\frac{2}{\pi} - |\sin ux|\right| < 1$ whence $\left(\frac{2}{\pi} - |\sin ux|\right)^2 <$

$< \left|\frac{2}{\pi} - |\sin ux|\right|$, making the corollary obvious.

Theorem 2.42:

$$\lim_{u\to\infty}\int_a^b \left(y - |\cos ux|\right)^2 dx > \lim_{u\to\infty}\int_a^b \left(\frac{2}{\pi} - |\cos ux|\right)^2 dx =$$

$$\left(\frac{1}{2} - \frac{4}{\pi^2}\right)(b - a) \qquad \frac{1}{20}\ (b - a), \text{ for } y \neq \frac{2}{\pi}\ .$$

Proof: $$\int_a^b \left(y - |\cos ux|\right)^2 dx = (b - a)y^2 - 2y\int_a^b |\cos ux|\, dx$$

$$+ \int_a^b \cos^2 ux\, dx$$

But

$$\cos^2 x = \frac{1}{2}\ (1 + \cos 2x)$$

Hence

(2.43) $$\int_a^b \left(y - |\cos ux|\right)^2 dx = (b - a)^2 y - 2y\int_a^b |\cos ux|\, dx$$

$$+ \frac{1}{2}\ (b - a) + \int_a^b \cos 2\,ux\, dx.$$

Using the Riemann-Lebesgue lemma and theorem 2.32 $\left(\text{for } f(x) \equiv 1\right)$ we get

(2.44) $$\lim_{u\to\infty}\int_a^b \left(y - |\cos ux|\right)^2 dx = (b - a)\left(y^2 - \frac{4}{\pi}\ y + \frac{1}{2}\right).$$

As before, the quadratic on the right has $y = \frac{2}{\pi}$ for its minimum and has there the value $\frac{1}{2} - \frac{4}{\pi^2} > \frac{1}{20}$.

qed.

Corollary 2.42.1.

$$\lim_{u \to \infty} \int_a^b \left| \frac{2}{\pi} - |\cos ux| \right| dx > \left(\frac{1}{2} - \frac{4}{\pi^2} \right)(b - a) > \frac{1}{20} (b - a).$$

Proof: $\left| \frac{2}{\pi} - |\cos ux| \right| < 1$ whence $\left(\frac{2}{\pi} - |\cos ux| \right)^2 <$
$< \left| \frac{2}{\pi} - |\cos ux| \right|$, making the corollary obvious.

25. f(x) dependent on the parameter u. The theorem of this section is needed for the calculation of the Lebesgue constants corresponding to Borel summability.

Theorem 2.51. Give $f_u(x)$, $u > 0$. Let $f_u(x)$ be differentiable, $f_u'(x)$ integrable and let further

(2.51) $$\int_a^b |f_u'(x)| \, dx = o(u^\beta) \text{ and } f_u(\ ) = o(u^\beta) \text{ as } u \to \infty,$$

where $0 < \beta \leq 1$.

Then

(2.52) $$\int_a^b f_u(x) \left\{ \frac{2}{\pi} - |\sin ux| \right\} dx = o(u^{\beta-1}) \text{ as } u \to \infty .$$

and

(2.53) $$\int_a^b f_u(x) \left\{ \frac{2}{\pi} - |\cos ux| \right\} dx = o(u^{\beta-1}) \text{ as } u \to \infty .$$

In the proof we require an integration by parts. The general formula reads:

(25.4) $$\int_a^b g(x)\ F(x)\ dx + \int_a^b f(x)\ G(x)\ dx = F(b)\ G(b) - F(a)\ G(a),$$

where f(x) and g(x) are summable over (a, b) and

(2.55) $$F(x) = \int_a^x f(t)\ dt \quad \text{and} \quad G(x) = \int_a^x g(t)\ dt.$$

<u>(a) Proof of (2.52):</u>

Integrate by parts, putting

$$f(x) = f_u^1\ (x)$$

and

$$g(x) = \tfrac{2}{\pi} - |\sin ux.|$$

This gives

(2.56) $$\int_a^b f_u(x) \left\{\tfrac{2}{\pi} - |\sin ux|\right\} dx = f_u(b) \int_a^b \left\{\tfrac{2}{\pi} - |\sin ux|\right\} dx - \int_a^b f_u^1(x) \left[\int_a^x \left\{\tfrac{2}{\pi} - |\sin ut|\right\} dt\right] dx.$$

From theorem 2.31 and the second part of hypothesis (2.51) it follows that

(2.57) $$f_u(b) \int_a^b \left\{\tfrac{2}{\pi} - |\sin ux|\right\} dx = o\ (u^{\beta-1})\ \text{as}\ u \to \infty .$$

Also

$$\left| \int_a^b f_u^1(x) \left[\int_a^x \left\{\tfrac{2}{\pi} - |\sin ut|\right\} dt\right] dx \right| <$$

$$< \int_a^b |f_u'(x)| \left| \int_a^x \left\{ \frac{2}{\pi} - |\sin ut| \right\} dt \right| dx.$$

Applying theorem 2.31 (for $b = x$) and (2.51) yields

$$(2.58) \quad \int_a^b f_u'(x) \left[\int_a^x \left\{ \frac{2}{\pi} - |\sin ut| \right\} dt \right] dx \;=\; o\,(u^{\beta-1}) \text{ as } u \to \infty .$$

Taking (2.57) and (2.58) together with (2.56) completes the proof of (2.52).

<u>(b) Proof of (2.53):</u>

This follows the lines of part (a). Here take $f(x) = f_u'(x)$ and $g(x) = \frac{2}{\pi} - |\cos ux|$ in (2.54). Simply using theorem 2.32 instead of theorem 2.31, each step in the proof is the same.

qed.

Taking $\beta = 1$, we have

<u>Corollary 2.51.1:</u> If

$$(2.59) \quad \int_a^b |f_u'(x)|\, dx \;=\; o(u) \quad \text{and} \quad f_u(b) = o \quad \text{as } u \to \infty ,$$

then

$$(2.510) \quad \int_a^b f_u(x) \left\{ \frac{2}{\pi} - |\sin ux| \right\} dx \;=\; o(1) \text{ as } u \to \infty .$$

and

$$(2.511) \quad \int_a^b f_u(x) \left\{ \frac{2}{\pi} - |\cos ux| \right\} dx \;=\; o(1) \text{ as } u \to \infty .$$

Next, note that in theorem 2.51 and its proof <u>o</u> can be replaced by <u>O</u>, throughout, without requiring a single alteration in the proof. This result reads in full as follows:

Theorem 2.52. Given $f_u(x)$, $u > 0$. Let $f_u(x)$ be differentiable, $f_u'(x)$ be integrable and let further (

(2.512) $$\int_a^b |f_u'(x)|\,dx = O(u^\beta) \text{ and } f_u(b) = O(u^\beta) \quad \text{as } u \to \infty ,$$

where $0 < \beta \leq 1$.

Then

(2.513) $$\int_a^b f_u(x)\left\{\frac{2}{\pi} - |\sin ux|\right\} dx = O(u^{\beta-1}) \text{ as } u \to \infty$$

and

(2.514) $$\int_a^b f_u(x)\left\{\frac{2}{\pi} - |\cos ux|\right\} dx = O(u^{\beta-1}) \text{ as } u \to \infty .$$

Again taking $\beta = 1$, gives

Corollary 2.52.1. If

(2.515) $$\int_a^b |f_u'(x)|\,dx = O(u) \text{ and } f_u(b) = O(u) \text{ as } u \to \infty ,$$

then

(2.516) $$\int_a^b f_u(x)\left\{\frac{2}{\pi} - |\sin ux|\right\} dx = O(1) \text{ as } u \to \infty$$

and

(2.517) $$\int_a^b f_u(x)\left\{\frac{2}{\pi} - |\cos ux|\right\} dx = O(1) \text{ as } \to \infty .$$

2.6. Estimates. Now let us find estimates for

(2.61) $$\int_a^b f(x)\left\{\frac{2}{\pi} - |\sin ux|\right\} dx$$

and

(2.62) $$\int_a^b f(x)\left\{\frac{2}{\pi} - |\cos ux|\right\} dx.$$

The point of departure in the case of (2.61) is (2.26); of (2.62), (2.29) -- the places where the Riemann-Lebesgue lemma was applied to obtain theorem 2.21. For f(x) $\equiv$ 1, (2.26) and (2.29) have already been used to obtain estimates, given by theorems 2.31 and 2.32 with their respective corollaries.

One obvious deduction from (2.24) and (2.29) is

<u>Theorem 2.61.</u> If

(2.63) $$\int_a^b f(x) \cos 2nx \, dx = 0, \quad n = 1, 2, \ldots,$$

then

(2.64) $$\int_a^b f(x) |\sin nx| \, dx = \int_a^b f(x) |\cos nx| \, dx = \frac{2}{\pi} \int_a^b f(x) dx,$$

$n = 1, 2, \ldots.$

<u>Theorem 2.62.</u> If f(x) is of bounded variation in (a, b), then there exists a positive constant <u>c</u>, such that

(2.66) $$\int_a^b f(x) \left\{ \frac{2}{\pi} - |\sin ux| \right\} dx < \frac{c}{u}, \quad u > 0.$$

and

(2.67) $$\left| \int_a^b f(x) \left\{ \frac{2}{\pi} - |\cos ux| \right\} dx \right| < \frac{c}{u}, \quad u > 0.$$

<u>Proof:</u>

For such f(x) there exists a positive constant c, such that

(2.68) $$\left| \int_a^b f(x) \cos ux \, dx \right| < \frac{c}{u}$$

[Tonelli, 28, p. 75]. Together with (2.26) and (2.29), this establishes the desired result.

Theorem 2.63. If f(x) is absolutely continuous in $(0,2\pi)$, then

$$(2.69)\quad \lim_{u\to\infty}\left[u\int_0^{2\pi} f(x)\left\{\frac{2}{\pi} - |\sin ux|\right\}dx - \frac{2}{\pi}f(2\pi)\sum_{j=1}^{\infty}\frac{1}{j}\,\frac{1}{4j^2-1}\sin 4ju\pi\right] = 0$$

and

$$(2.610)\quad \lim_{u\to\infty}\left[u\int_0^{2\pi} f(x)\left\{\frac{2}{\pi} - |\cos ux|\right\}dx - \frac{2}{\pi}f(2\pi)\sum_{j=1}^{\infty}\frac{(-1)^j}{j(4j^2-1)}\sin 4ju\pi\right] = 0.$$

(a) Proof of (2.69).

From (2.26) it follows that

$$(2.611)\quad u\int_0^{2\pi} f(x)\left\{\frac{2}{\pi} - |\sin ux|\right\}dx = \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\,u\int_0^{2\pi} f(x)\cos 2jux\,dx.$$

Now, in the formula for integration by parts, (2.54), take f(x) to be cos ux and G(x) to be the present f(x). Then, multiplying through by u,

$$\int_0^{2\pi} f'(x)\sin ux\,dx + u\int_0^{2\pi} f(x)\cos ux\,dx = f(2\pi)\sin 2u\pi.$$

Instead of u, consider 2ju. This equation then becomes

$$(2.612)\quad u\int_0^{2\pi} f(x)\cos 2jux\,dx = \frac{1}{2}\frac{1}{j}f(2\pi)\sin 4ju\pi - \frac{1}{2}\frac{1}{j}\int_0^{2\pi} f'(x)\sin 2jux\,dx.$$

Substituting in (2.611) and transposing, we get

$$(2.613)\quad u\int_0^{2\pi} f(x)\left\{\frac{2}{\pi} - |\sin ux|\right\}dx - \frac{2}{\pi}f(2\pi)\sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{4j^2-1}\times$$

$$\times\sin 4ju\pi = \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{4j^2-1}\int_0^{2\pi} f'(x)\times$$

$$\times\sin 2jux\,dx.$$

The series on the right is uniformly convergent for $u \geqq 0$, since

$$\left|\int_0^{2\pi} f'(x)\sin 2jux\,dx\right| \leq \int_0^{2\pi} |f'(x)|\,dx \text{ and } \sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{4j^2-1}$$

converges. Hence its limit as $u\to\infty$ may be computed termwise. But, according to the Riemann-Lebesgue lemma, each term goes to zero. This establishes (2.69).

<u>(b) Proof of (2.610):</u> This proceeds as part (a). We get

$$(2.614)\quad u\int_0^{2\pi} f(x)\left\{\frac{2}{\pi} - |\cos ux|\right\}dx - \frac{2}{\pi}f(2\pi)\sum_{j=1}^{\infty}\frac{(-1)^j}{j(4j^2-1)}\times$$

$$\times\sin 4ju\pi$$

$$= \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{j(4j^2-1)}\int_0^{2\pi} f'(x)\sin 2jux\,dx.$$

The Riemann-Lebesgue lemma then implies (2.610).

qed.

Corollary 2.63.1. If f(x) is absolutely continuous in $(0,2\pi)$, and if $f(2\pi) = 0$, then

(2.615) $$\lim_{u\to\infty} u\int_0^{2\pi} f(x)\left\{\tfrac{2}{\pi} - |\sin ux|\right\}dx = 0$$

and

(2.616) $$\lim_{u\to\infty} u\int_0^{2\pi} f(x)\left\{\tfrac{2}{\pi} - |\cos ux|\right\}dx = 0$$

Proof: Obvious from the theorem.

Corollary 2.63.2: If f(x) is absolutely continuous in (0,2), then for y of the form $\frac{n}{4}$, n integral,

(2.617) $$\lim_{y\to\infty} y\int_0^{2\pi} f(x)\left\{\tfrac{2}{\pi} - |\sin yx|\right\}dx = 0$$

and

(2.618) $$\lim_{y\to\infty} y\int_0^{2\pi} f(x)\left\{\tfrac{2}{\pi} - |\cos yx|\right\}dx = 0$$

Proof:

c (2.617) and (2.618) follow from (2.69) and (2.610) respectively, since the sine of an integral multiple of is zero.

Corollary 2.63.3. If f(x) is absolutely continuous in $(0,2\pi)$, then

(2.619) $$\lim_{u\to\infty} u\int_0^{2\pi} \{f(x) - f(2\pi)\}\left\{\tfrac{2}{\pi} - |\sin ux|\right\}dx = 0$$

and

(2.620) $$\lim_{u\to\infty} u\int_0^{2\pi} \{f(x) - f(2\pi)\}\left\{\tfrac{2}{\pi} - |\cos ux|\right\}dx = 0$$

(a) Proof of (2.619):

In (2.31) take $a = 0$, $b = 2\pi$. Then

(2.621) $$u\int_0^{2\pi} \left\{\frac{2}{\pi} - |\sin ux|\right\} dx = \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{1}{j} \frac{1}{4j^2-1} \sin 4 j u\pi$$

Substituting this in (2.69) gives the result.

(b) Proof of (2.620):

This result follows directly from the cosine analogue of (2.31), gotten from (2.29).

Theorem 2.64. If $f(0) = f(2\pi)$ and if $f(x)$ satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, then there is a constant $k > 0$, such that

(2.622) $$\left|\int_0^{2\pi} f(x) \left\{\frac{2}{\pi} - |\sin nx|\right\} dx\right| < \frac{K}{n^\alpha}, \quad n = 1, 2, \ldots.$$

and

(2.623) $$\left|\int_0^{2\pi} f(x) \left\{\frac{2}{\pi} - |\cos nx|\right\} dx\right| < \frac{K}{n^\alpha}, \quad n = 1, 2, \ldots$$

Proof: This is a consequence of the known estimate of the Fourier coefficients of such functions [Tonelli, 28, p. 225.]

2.7. Another extension of Fejér's lemma.

Theorem 2.71. Let $f(x)$, $\varphi(x)$ be Legesgue integrable over (a, b) and $(0, 2\pi)$ respectively. Let $\varphi(x)$ have an absolutely convergent Fourier series

(2.71) $$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Then

(2.72) $$\lim_{u\to\infty} \int_a^b f(x)\varphi(ux)dx = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x)\,dx \cdot \int_a^b f(x)dx.$$

Proof: We have, since $\frac{1}{2}a_0 = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,dx$,

$$\varphi(ux) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,dx + \sum_{k=1}^{\infty}(a_k\cos kux + b_k\sin kux)$$

and

$$f(x)\varphi(ux) = \frac{1}{2\pi}f(x)\int_0^{2\pi}\varphi(x)\,dx + \sum_{k=1}^{\infty} f(x)(a_k\cos kux + b_k\sin kux)$$

Hence

$$(2.73)\quad \begin{cases}\int_a^b f(x)\,\varphi(ux)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,dx\int_a^b f(x)\,dx \\ \qquad + \int_a^b \sum_{k=1}^{\infty} f(x)(a_k\cos kux + b_k\sin kux)\,dx.\end{cases}$$

Now, since $\sum_{k=1}^{\infty}(a_k\cos kux + b_k\sin kux)$ is absolutely convergent, $\sum_{k=1}^{\infty}(|a_k| + |b_k|)$ converges [Zygmund, 33, p. 131.] Hence, since $|\cos y| \leq 1$ and $|\sin y| \leq 1$, $\sum_1^{\infty}(a_k\cos kux + b_k\sin kux)$ is uniformly convergent. The second term on the right side of (2.73) may then be integrated termwise and we get

$$(2.74)\quad \begin{cases}\int_a^b f(x)\,\varphi(ux)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,dx\cdot\int_a^b f(x)\,dx \\ \qquad + \sum_{k=1}^{\infty}\left\{a_k\int_a^b f(x)\cos kux + b_k\int_a^b f(x)\sin kux\,dx\right\},\end{cases}$$

Now, since $|\cos kux| \leq 1$, $|\sin kux| \leq 1$

$$(2.75)\quad \left|\int_a^b f(x)\cos kux\,dx\right| \leq \int_a^b |f(x)|\,dx \text{ and } \left|\int_a^b f(x)\sin kux\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Since $\sum_{k=1}^{\infty}(|a_k| + |b_k|) < \infty$, we have from the Wierstrass M-test that (2.74) is uniformly convergent in u. Hence we may pass to the limit termwise, i.e.,

$$(2.76)\quad \begin{cases}\lim_{u\to\infty}\int_a^b f(x)\varphi(ux) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)\,dx\int_a^b f(x)\,dx \\ \qquad + \sum_{k=1}^{\infty}\left\{a_k\lim_{u\to\infty}\int_a^b f(x)\cos kux\,dx + b_k\lim_{u\to\infty}\int_a^b f(x)\sin kux\,dx\right\}.\end{cases}$$

From the Riemann-Lebesgue lemma,

$$\lim_{n\to\infty}\int_a^b f(x)\cos knx\,dx = \lim_{n\to\infty}\int_a^b f(x)\sin knx\,dx = 0,$$

Thus (2.76) reduces to (2.72).

qed.

<u>Remark:</u>

The functions $|\cos x|$ and $|\sin x|$ used by Fejér both have absolutely convergent Fourier series. His lemma is therefore included in theorem 2.71.

CHAPTER III

ON THE LEBESGUE CONSTANTS CORRESPONDING TO BOREL SUMMABILITY

3.1. Introduction. It is known that there exist continuous functions whose Fourier series diverge at a point. DuBois Reymond constructed such a function as far back as 1876. [DuBois Reymond, 3]. Substantially simpler examples have been given by Fejér [Fejér, 6, 5].

Another approach to this problem, far more general in outlook, is due to Lebesgue [Lebesgue, 10]. What he did was to consider the partial sums of the Fourier series of the totality of continuous (in fact merely bounded) functions at a point,

$$\text{(3.11)} \qquad \underset{f \in E}{\text{l.u.b.}} \, |s_n(f; x)| \equiv \rho_n(x)$$

where E is the class of Lebesgue integrable functions bounded by 1. Translating (3.11) into other terms he found that $\rho_n(x)$ is independent of x and

$$\text{(3.12)} \qquad \rho_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{|\sin(2n+1)t|}{\sin t}\, dt.$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} |D_n(t)|\, dt$$

where $D_n(t)$ is the Dirichlet kernel.

He then showed that

(3.13) $\rho_n \to +\infty \text{ as } n \to \infty.$

Following upon this, he proved a general theorem on functionals from which it followed that (3.13) implies the existence of a continuous function having a divergent Fourier series at a point.

An asymptotic formula for ρ_n was given by Fejér. In particular he showed that ρ_n becomes logarithmically infinite.

The broader scope of Lebesgue's method is at once apparent. Replacing $s_n(f; x)$ in (3.11) by its transform by some summability method, one gets immediately a practical approach to the following problem:

Given a summability method. Does there exist a continuous function whose Fourier series is not everywhere summable by this method?

For the case of Borel summability, this approach was first taken by C. N. Moore [Moore, 21]. He pointed out, among other things, that the constants which correspond to the ρ_n in the case of Borel summability again become logarithmically infinite and thence deduced the existence of a Continuous function whose Fourier series is not Borel summable at 0.

It is the objective of the present chapter to analyze the structure of the constants in this case more closely than was necessary for Moore's purposes.

2. Further remarks. For the constants ρ_n, Fejer [Fejér, 5] introduced the term "Lebesgue constants", by which they are now universally known. The importance of these constants led him to make a precise study of the manner in which they become infinite. He showed

$$(3.21)\quad \rho_n = \frac{4}{\pi^2}\log n + c_0 + o(1) \text{ as } n \to \infty$$

where

$$(3.22)\quad c_0 = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin t} - \frac{1}{t}\right)dt + 2\int_0^1 \log\Gamma(t)\cos\pi t\,dt\,.$$

and conjectured that ρ_n is monotonic, n = 1, 2,..... This guess was confirmed by Gronwall [Gronwall, 9] who then pointed out that the sequence might also be completely monotonic, a fact later verified by Szegö [Szegö, 27].

In keeping with C. N. Moore's notation, I shall hereafter use the designation L(n) instead of ρ_n for the Lebesgue constants.

After finding the integral expression for the Borel means, I obtain an integral representation for the Lebesgue constants $L_B(u)$ in this case. This integral is quite complicated. I therefore give a simpler integral whose differences from $L_B(u)$ vanishes as u becomes infinite. Then I estimate $L(u) - L_B(u)$

and, after doing this, refer to Fejer's estimate of L (n) to give the estimate I seek for $L_B(u)$.

I then close the chapter with several conjectures relative to the behavior of $L_B(u)$.

3.3. The Borel means of Fourier series.

Definition 3.31. Let a series $\sum_{n=0}^{\infty} a_n$ be given and let

$s_n = \sum_{k=0}^{\infty} a_k.$

Let

$$(3.31) \qquad B(u) \equiv \sum_{n=0}^{\infty} s_n \frac{u^n}{n!}.$$

If

$$(3.32) \qquad \lim_{u \to \infty} e^{-u} B(u)$$

exists and equals s, then the series $\sum_{n=0}^{\infty} a_n$ is said to be Borel summable to the value s, or, more exactly, summable to the value s by Borel's exponential method.

The Borel method is regular - i.e., it sums every convergent series and the Borel sum is the value to which the series converges.

Definition 3.32. $e^{-u}B(u)$ is the uth Borel mean of $\sum_{n=0}^{\infty} a_n$.

In the case of Fourier series, we have [Zygmund, 33, p. 20], for a Lebesgue integrable f(t),

$$(3.33)\quad s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin\{(n+\frac{1}{2})(t-x)\}}{\sin\frac{1}{2}(t-x)}\,dt.$$

Let us compute $e^{-u}B(u)$ in this case. Here B(u) depends on f and x; so we write $B_f(u;x)$ instead of simply B(u).

<u>Theorem 3.31</u>. If f(t) is Lebesgue integrable, then

$$(3.34)\quad \begin{cases} e^{-u}B_f(u;x) = e^{-u}\displaystyle\sum_{n=0}^{\infty}\frac{s_n(x)}{n!}u^n \\ \quad = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\,e^{u\cos(t-x)-u}\,\frac{\sin\{u\sin(t-x)+\frac{1}{2}(t-x)\}}{\sin\frac{1}{2}(t-x)}\,dt. \end{cases}$$

<u>Proof</u>: From (3.33), for $u > 0$, we get

$$(3.35)\quad \sum_{n=0}^{\infty}\frac{s_n(x)}{n!}u^n = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{u^n}{n!}\int_{-\pi}^{\pi} f(t)\,\frac{\sin\{(2n+1)\frac{1}{2}(t-x)\}}{\sin\frac{1}{2}(t-x)}\,dt.$$

It would be helpful to be able to interchange the order of summation and integration. To show that this is possible, we appeal to a standard result [Kestelman, 18, p. 144]:

"Let $\{f_r(P)\}$ be a sequence of functions measurable in B,* and such that

$$(3.36)\quad \sum_{r=1}^{\infty}\int_B |f_r(P)|\,dP < \infty;$$

*Here B denotes a point-set of finite measure.

then $\sum_{r=1}^{\infty} f_r P$ is equivalent in B to a summable (i.e., Lebesgue integrable) function, and

$$(3.37) \qquad \int_B \sum_{r=1}^{\infty} f_r(P)dP = \sum_{r=1}^{\infty} \int_B f_r(P)dP."$$

In view of this theorem what we have to show is

$$(3.38) \quad \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{u^n}{n!} \int_{-\pi}^{\pi} |f(t)| \frac{|\sin(2n+1)\frac{1}{2}(t-x)|}{|\sin\frac{1}{2}(t-x)|} dt < \infty, \quad u > 0.$$

Now, it is known (and may readily be proved by induction) that

$$\left|\sin(2n+1)\frac{t-x}{2}\right| \leq (2n+1)\left|\sin\frac{t-x}{2}\right|.$$

Hence

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{u^n}{n!} \int_{-\pi}^{\pi} |f(t)| \frac{|\sin(2n+1)\frac{t-x}{2}|}{|\sin\frac{t-x}{2}|} dt$$

$$\leq \sum_{n=0}^{\infty} \frac{(2n+1)u^n}{n!} = 2u \sum_{n=0}^{\infty} \frac{u^{n-1}}{(n-1)!} + e^{-u} < \infty, \quad u > 0.$$

The interchange of sum and integral is thus permissible and we get

$$(3.39) \quad \beta f(u;x) = \sum_{n=0}^{\infty} \frac{s_n(x)}{n!} u^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{\sin\frac{t-x}{2}} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\frac{t-x}{2}}{n!} u^n dt.$$

Now,

$$(3.310)\quad \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{\sin(2n+1)\frac{1}{2}(t-x)}{n!} u^n \\ \displaystyle = \sum_{n=0}^{\infty} \frac{\sin n(t-x)\cos\frac{1}{2}(t-x) + \sin\frac{1}{2}(t-x)\cos n(t-x)}{n!} u^n \\ \displaystyle = \cos\tfrac{1}{2}(t-x)\sum_{n=0}^{\infty} \frac{\sin n(t-x)}{n!} u^n + \sin\tfrac{1}{2}(t-x)\sum_{n=0}^{\infty} \frac{\cos n(t-x)}{n!} u^n. \end{cases}$$

To evaluate this, we use the exponential function:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Take $z = ue^{i(t-x)}$. Then

$$e^{ue^{i(t-x)}} = \sum_{n=0}^{\infty} \frac{u^n e^{in(t-x)}}{n!} = \sum_{n=0}^{\infty} \frac{\cos n(t-x)}{n!} u^n + i\sum_{n=0}^{\infty} \frac{\sin n(t-x)}{n!} u^n.$$

Moreover,

$$e^{ue^{i(t-x)}} = e^{u\{\cos(t-x) + i\sin(t-x)\}} = e^{u\cos(t-x)} \cdot e^{iu\sin(t-x)}$$

$$= e^{u\cos(t-x)}\cos\{u\sin(t-x)\} + ie^{u\cos(t-x)}\sin\{u\sin(t-x)\}$$

Equating real and imaginary parts yields

$$(3.311)\quad e^{u\cos(t-x)}\cos\{u\sin(t-x)\} = \sum_{n=0}^{\infty} \frac{\cos n(t-x)}{n!} u^n$$

and

$$(3.312)\quad e^{u\cos(t-x)}\sin\{u\sin(t-x)\} = \sum_{n=0}^{\infty} \frac{\sin n(t-x)}{n!} u^n.$$

Hence, from (3.310),

$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)\frac{t-x}{2}}{n!} u^n = \cos\tfrac{1}{2}(t-x)\, e^{u\cos(t-x)} \sin\{u\sin(t-x)\}$$

$$+ \sin\tfrac{1}{2}(t-x)\, e^{u\cos(t-x)} \cos\{u\sin(t-x)\}$$

$$= e^{u\cos(t-x)} \sin\{u\sin(t-x)+\tfrac{1}{2}(t-x)\}$$

Substituting this in (3.39) and multiplying both sides by e^{-u} completes the proof.

3.4. The Lebesgue constants corresponding to Borel's method. The objective of this section is to define the Lebesgue constants corresponding to Borel's method and to express them in terms of an integral. The procedure is the same as in the case of the ordinary Lebesgue constants. First we define the appropriate class of functions.

Definition 3.41. If $|f(t)| \leq 1$ for all t in (a, b) and Lebesgue integrable, then f(t) is said to belong to the class E. (a, b).

The actual definition of the Borel Lebesgue constants can now be given:

Definition 3.42. The u^{th} Lebesgue constant corresponding to Borel's summability method is denoted by $L_B(u; x)$ and defined to be

(3.41) $$L_B(u;x) \equiv \underset{f\in E}{\text{l.u.b.}} \left|e^{-u} B_f(u;x)\right|,$$

where $E = E(-\pi, \pi)$ and where u and x are kept fixed.

In order to transform (3.41) into a form more adapted to calculation, the following theorem is established [Note that part of the theorem states that $L_\beta(u, x)$ is independent of x]:

Theorem 3.41:

$$(3.42)\quad L_\beta(u) \equiv L_\beta(u;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-u(1-\cos t)} \frac{|\sin(u \sin t + \frac{1}{2}t)|}{|\sin \frac{1}{2}t|}\, dt$$

Proof: From (3.34) it follows that

$$|e^{-u}\beta_f(u;x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\, e^{u\cos(t-x)-u} \frac{|\sin\{u \sin(t-x)+\frac{1}{2}(t-x)\}|}{|\sin\frac{1}{2}(t-x)|}\, dt$$

Hence

$$(3.43)\quad L_\beta(u;x) = \underset{f \in E}{\text{l.u.b.}}\, |e^{-u}\beta_f(u;x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{u\cos(t-x)-u} \frac{|\sin\{u\sin(t-x)+\frac{1}{2}(t-x)\}|}{|\sin\frac{1}{2}(t-x)|}\, dt$$

On the other hand, also in E is the function

$$(3.44)\quad f^*(t) = \operatorname{sgn} \frac{\sin\{u \sin(t-x)+\frac{1}{2}(t-x)\}}{\sin\frac{1}{2}(t-x)}, \quad -\pi < t < \pi$$

where, as usual,

$$(3.45) \qquad \operatorname{sgn} a = \begin{cases} 1 & \text{for } a > 0 \\ 0 & \text{for } a = 0 \\ -1 & \text{for } a < 0 \end{cases}$$

Hence

$$L_B(u;x) \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(t)\ \frac{\sin\{u\sin(t-x)+\frac{1}{2}(t-x)\}}{\sin\frac{1}{2}(t-x)}\ e^{u\cos(t-x)-u}\,dt.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{u\cos(t-x)-u}\ \frac{|\sin\{u\sin(t-x)+\frac{1}{2}(t-x)\}|}{|\sin\frac{1}{2}(t-x)|}\,dt.$$

Comparing this with (3.43), we see that

$$(3.47) \quad L_B(u;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{u\cos(t-x)-u}\ \frac{|\sin\{u\sin(t-x)+\frac{1}{2}(t-x)\}|}{|\sin\frac{1}{2}(t-x)|}\,dt.$$

✓ The integrand is periodic of period 2π in the variable x, the integral is therefore independent of x. Taking $x = 0$ completes the proof of the theorem.

3.5. Asymptotic simplification of the Borel Lebesgue-constants. The representation thus far obtained for the Lebesgue constants corresponding to Borel's summability method is

$$(3.51) \quad L_B(u) = \frac{1}{\pi}\int_{0}^{\pi} e^{-u(1-\cos t)}\ \frac{|\sin(u\sin t+\frac{1}{2}t)|}{\sin\frac{1}{2}t}\,dt.$$

This is a rather complicated integral. Its integrand contains an iterated sine and a trigonometric exponential. In order to find the "asymptotic value" of $L_\beta(u)$ as u becomes infinite, a somewhat "simpler asymptotic representation" of $L_\beta(u)$ is necessary. That is, I shall seek to replace (3.51) by an asymptotically equal integral which is easier to handle. This is what is meant by "asymptotic simplification" and this is the object of the present section. Its achievement requires several stages. The first is

Theorem 3.51. Let

(3.52) $$L_{\beta,1}(u) \equiv \frac{1}{\pi}\int_0^{\pi} e^{-u(1-\cos t)}\frac{\sin(u+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

Then

(3.53) $$L_\beta(u) = L_{\beta,1}(u) + o(1) \quad \text{as } u \longrightarrow \infty$$

Before proceding to the proof proper, several trigonometric inequalities are required. These constitute the content of the lemmas below.

Lemma 3.51.1. For $0 \leq x \leq \frac{\pi}{2}$, we have

(3.54) $$x - \frac{x^3}{3!} \leq \sin x \leq x$$

and

(3.55) $$0 \le \sin x - x + \frac{x^3}{3!} \le \frac{x^5}{5!}$$

Proof of lemma 3.51.1. Be it recalled that

(3.56) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdot$$

The series is a convergent series of alternating signs whose terms decrease in absolute value in $(0, \frac{\pi}{2})$. Hence, the remainder is of the same sign as and less in absolute value than the first term neglected. Neglecting the first term gives the right-hand inequality in (3.54); neglecting the second, the left. Neglecting the third term gives the right-hand inequality in (3.55). The left-hand inequality in (3.55) is simply a rewrite of the left-hand inequality in (3.54).

Lemma 3.51.2. If $|x - y| \le \pi$, then

(3.57) $$|\sin x - \sin y| \le |x - y|.$$

Proof of lemma 3.51.2. Be it recalled that

(3.58) $$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y).$$

Using $|\cos a| \le 1$ and the right-hand side of (3.54), with x replaced by $\frac{1}{2}(x - y)$, gives (3.57).

Lemma 3.51.3.

(3.59) $$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1, \quad 0 \leq x \leq \frac{\pi}{2}.$$

Proof of lemma 3.51.3. Dividing each side of (3.56) by x yields

(3.510) $$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

To establish (3.59), it is of course sufficient to show that the function in (3.510) decreases as x increases from 0 to And to show this, it is sufficient to prove that the derivative with respect to x of that series is negative in $(0,\frac{\pi}{2})$. If termwise differentiation is permissible, this will not be difficult. The following standard theorem will be used [Hobson,[7],v.II, p. 334]:

"If the series $\sum_{n=1}^{\infty} u_n(x)$ converges everywhere in the finite interval (a, b), and the terms of the series $\sum_{n=1}^{\infty} u'_n(x)$ be all continuous in (a, b), and this latter series is uniformly convergent in (a, b), then $s'(x)$ exists, and is the sum of the series $\sum_{n=1}^{\infty} u'_n(x)$, at all points of (a,b)."

Here $s(x) = \sum_{n=1}^{\infty} u_n(x)$.

In our case, we have

$$s(x) = \frac{\sin x}{x}, \; a = 0, \; b = \frac{\pi}{2}, \; u_{n+1}(x) = \frac{(-1)^n x^{2n}}{(2n+1)!}, \; u'_{n+1}(x) = \frac{2n}{(2n+1)!}(-1)^n x^{2n-1}$$

According to the Weierstrass M-test, the series

$$(3.511) \quad \sum_{n=1}^{\infty} u'_n(x) \equiv \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{(2n+1)!} x^{2n-1}$$

will be uniformly convergent in $(0,\frac{\pi}{2})$ if

$$\max_{0\le x\le \frac{\pi}{2}} \left| \frac{(-1)^n 2n}{(2n+1)!} x^{2n-1} \right| \le M_n$$

and

$$\sum_{n=1}^{\infty} M_n \qquad \text{converges.}$$

But

$$\max_{0\le x\le \frac{\pi}{2}} \left| \frac{(-1)^n 2n}{(2n+1)!} x^{2n-1} \right| = \frac{2n}{(2n+1)!} \max_{0\le x\le \frac{\pi}{2}} |x|^{2n-1} = \frac{2n}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n-1}$$

And, from the test-ratio test,

$$\sum_{n=1}^{\infty} \frac{2n}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n-1} \qquad \text{converges.}$$

Hence (3.511) is uniformly convergent in $(0,\frac{\pi}{2})$.

Thus, all the conditions of the theorem quoted are satisfied. Therefore

$$(3.512) \quad \frac{d}{dx} \frac{\sin x}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{(2n+1)!} x^{2n-1},$$

where the series is uniformly convergent in $(0,\frac{\pi}{2})$.

In that interval, the series in (3.512) is a convergent series of alternating signs the absolute values of whose terms forms a decreasing sequence. Hence it has the same sign as its first term. But this term is $-\frac{1}{3}x$ and is negative throughout the entire open interval. Hence

(3.513) $$\frac{d}{dx}\frac{\sin x}{x} < 0, \quad 0 < x \le \frac{\pi}{2}.$$

$\frac{\sin x}{x}$ is thus a decreasing function of x as x increases from 0 to $\frac{\pi}{2}$. Since $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and since $\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi}$, this completes the proof of the lemma.

<u>Lemma 3.51.4.</u> If $a \geqslant 1$, $0 \le x \le \frac{\pi}{2}$, $0 \le ax \le \frac{\pi}{2}$, then

(3.51.4) $$0 < \frac{\sin ax}{\sin x} \le a.$$

<u>Proof of lemma 3.51.4</u>:

Since $0 \le ax \le \frac{\pi}{2}$ and $a \geqslant 1$, we have

$$0 \le \cos ax \le \cos x$$

inasmuch as $\cos y$ decreases in $(0, \frac{\pi}{2})$. Hence

$$\int_0^{\pi} \cos at\,dt \le \int_0^{\pi} \cos t\,dt.$$

But this inequality is simply

$$\frac{\sin ax}{a} \le \sin x,$$

a rewriting of (3.51.4).

Proof of theorem 3.51:

We have

(3.515) $$L_{\beta,1}(u) - L_\beta(u) = \frac{1}{\pi}\int_0^\pi \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}\left\{|\sin(u+\tfrac{1}{2})t| - |\sin(u\sin t + \tfrac{1}{2}t)|\right\}dt$$

Using the fact that

$$\big||a| - |b|\big| \leq |a-b|$$

for any numbers a and b, we have

$$|L_{\beta,1}(u) - L_\beta(u)| \leq \frac{1}{\pi}\int_0^\pi \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}\big||\sin(u+\tfrac{1}{2})t| - |\sin(u\sin t + \tfrac{1}{2}t)|\big|dt$$

$$\leq \frac{1}{\pi}\int_0^\pi \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}|\sin(u+\tfrac{1}{2})t - \sin(u\sin t + \tfrac{1}{2}t)|dt.$$

From (3.58) and $|\cos a| \leq 1$,

$$|L_{\beta,1}(u) - L_\beta(u)| \leq \frac{2}{\pi}\int_0^\pi \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}|\cos\tfrac{1}{2}(ut + u\sin t + t)\sin\tfrac{1}{2}(t-\sin t)u|dt$$

$$\leq \frac{2}{\pi}\int_0^\pi \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}|\sin\tfrac{1}{2}(t-\sin t)u|\,dt$$

$$\equiv \frac{2}{\pi}\int_0^{\frac{1}{\sqrt[3]{u}}} + \frac{2}{\pi}\int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \equiv \mathrm{I} + \mathrm{II}.$$

Consider II. From $|\sin a| \leq 1$ and $2\sin^2\frac{1}{2}t = 1-\cos t$, we have

$$\frac{2}{\pi}\int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}\left|\sin\tfrac{1}{2}(t-\sin t)u\right|dt \leq \frac{2}{\pi}\int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{2}{\pi}\int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \frac{e^{-2u\sin^2\frac{1}{2}t}}{\sin\frac{1}{2}t}\,dt \leq \frac{2}{\pi}\left(\pi-\frac{1}{\sqrt[3]{u}}\right)\max_{\frac{1}{\sqrt[3]{u}}\leq t\leq\pi} \frac{e^{-2u\sin^2\frac{1}{2}t}}{\sin\frac{1}{2}t}$$

$$\leq 2\max_{\frac{1}{\sqrt[3]{u}}\leq t\leq\pi} \frac{e^{-2u\sin^2\frac{1}{2}t}}{\sin\frac{1}{2}t} = 2\max_{\frac{1}{2\sqrt[3]{u}}\leq t\leq\frac{\pi}{2}} \frac{e^{-2u\sin^2 t}}{\sin t}$$

Noting that $\frac{e^{-2ux^2}}{x}$ decreases as x increases, and recalling (3.54), we have

$$\max_{\frac{1}{2\sqrt[3]{u}}\leq t\leq\frac{\pi}{2}} \frac{e^{-2u\sin^2 t}}{\sin t} \leq \max_{\frac{1}{2\sqrt[3]{u}}\leq t\leq\frac{\pi}{2}} \frac{\exp.\left\{-2u\left(t^2-\frac{t^4}{3}+\frac{t^6}{36}\right)\right\}}{t-\frac{t^3}{6}}$$

$$= \frac{\exp.\left\{-2u\left(\frac{1}{4}\frac{1}{u^{2/3}}-\frac{1}{2^4}\frac{1}{3}\frac{1}{u^{4/3}}+\frac{1}{2^6}\frac{1}{36}\frac{1}{u^2}\right)\right\}}{\frac{1}{2\sqrt[3]{u}}\left(1-\frac{1}{4}\frac{1}{6}\frac{1}{u^{2/3}}\right)}$$

$$= \frac{\operatorname{Exp}\left\{-\frac{1}{2}\sqrt[3]{u}+\frac{1}{2^3}\frac{1}{3}\frac{1}{\sqrt[3]{u}}-\frac{1}{2^5}\frac{1}{36}\frac{1}{u}\right\}}{\frac{1}{2\sqrt[3]{u}}}\;\frac{1}{1-\frac{1}{24}\frac{1}{u^{2/3}}}$$

$$= o(1) \quad \text{as } u\to\infty$$

Together with (3.517) this states

$$(3.518)\quad \mathrm{II} \equiv \frac{2}{\pi}\int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}\left|\sin\tfrac{1}{2}(t-\sin t)u\right|dt = o(1) \text{ as } u\to\infty.$$

The other integral, I, requires a bit more manipulation. First, I shall show that it differs by an additive o (1) as $u \to \infty$ from

$$(3.519) \quad \ell(u) \equiv \frac{2}{\pi} \int_0^{\frac{1}{\sqrt[3]{u}}} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left| \sin \frac{1}{2} \frac{t^3}{3!} u \right| dt .$$

We have

$$\left| I - \ell \right| \equiv \frac{2}{\pi} \left| \int_0^{\frac{1}{\sqrt[3]{u}}} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left\{ \left| \sin\frac{1}{2}(t - \sin t) u \right| - \left| \sin\frac{1}{2} \frac{t^3}{3!} u \right| \right\} dt \right|$$

$$\leq \frac{2}{\pi} \int_0^{\frac{1}{\sqrt[3]{u}}} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left| \left| \sin\frac{1}{2}(t-\sin t) u \right| - \left| \sin\frac{1}{2} \frac{t^3}{3!} \right| \right| dt$$

$$\leq \frac{2}{\pi} \int_0^{\frac{1}{\sqrt[3]{u}}} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left| \sin\frac{1}{2}(t - \sin t) u - \sin\frac{1}{2} \frac{t^3}{3!} \right| dt$$

From (3.57) we get that this last integral is

$$\leq \frac{2}{\pi} \int_0^{\frac{1}{\sqrt[3]{u}}} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \frac{u}{2} \left| \frac{t^3}{3!} - t + \sin t \right| dt.$$

Now, (3.55), with x replaced by t, states that

$$0 \leq \sin t - t + \frac{t^3}{3!} \leq \frac{t^5}{5!}, \quad 0 \leq t \leq \frac{\pi}{2} .$$

Hence

$$(3.520)\begin{cases} |I-\ell| \le \frac{u}{5!\pi}\int_0^{\frac{1}{\sqrt[3]{u}}} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t}\, t^5\,dt \\ \quad = \frac{2u}{5!\pi}\int_0^{\frac{1}{\sqrt[3]{u}}} e^{-u(1-\cos t)}\frac{\frac{1}{2}t}{\sin\frac{1}{2}t}\, t^4\,dt. \end{cases}$$

From (3.59), with x replaced by $\frac{1}{2}t$, we get

$$(3.521) \qquad 1 \le \frac{\frac{1}{2}t}{\sin\frac{1}{2}t} \le \frac{\pi}{2}, \quad 0 \le \frac{1}{2}t \le \frac{\pi}{2}.$$

Together with (3.520), this gives

$$(3.522) \qquad |I-\ell| \le \frac{u}{5!}\int_0^{\frac{1}{\sqrt[3]{u}}} e^{-u(1-\cos t)}\, t^4\,dt$$

Since $|\cos t| \le 1$, $e^{-u(1-\cos t)} \le 1$, and we have

$$(3.523) \qquad |I-\ell| \le \frac{u}{5!}\int_0^{\frac{1}{\sqrt[3]{u}}} t^4\,dt = \frac{1}{600}\,\frac{1}{u^{2/3}} = o(1) \text{ as } u \to \infty.$$

Now, in the interval $(0, \frac{1}{\sqrt[3]{u}})$, $ut^3 \le \frac{u}{u} = 1$. Hence, in that interval $\sin\frac{1}{2}\frac{t^3}{3!}u$ is non-negative. The integral ℓ, (3.519), may therefore be written

$$(3.524) \qquad \ell(u) = \frac{2}{\pi}\int_0^{\frac{1}{\sqrt[3]{u}}} e^{-u(1-\cos t)}\frac{\sin\frac{1}{2}t\cdot\frac{ut^2}{6}}{\sin\frac{1}{2}t}\,dt.$$

We now can handle $\ell(u)$ conveniently by considering the values of t for which $\frac{ut^2}{6} \geq 1$, separately from those values for which $\frac{ut^2}{6} < 1$. That is, consider $t \geq \sqrt{\frac{6}{u}}$ separately from $t < \sqrt{\frac{6}{u}}$. In order that $t \geq \sqrt{\frac{6}{u}}$ shall fall within the interval of integration, u must be such that

$$\frac{1}{\sqrt[3]{u}} \geq \sqrt{\frac{6}{u}}$$

This inequality is valid for $u \geq 6^3 = 216$. So, henceforth consider only values of $u \geq 216$. We can now write (3.524) as

(3.525) $$\ell(u) \equiv \frac{2}{\pi}\int_0^{\sqrt{\frac{6}{u}}} + \frac{2}{\pi}\int_{\sqrt{\frac{6}{u}}}^{\frac{1}{\sqrt[3]{u}}}, \quad u \geq 216.$$

Now, consider

$$\frac{2}{\pi}\int_0^{\sqrt{\frac{6}{u}}} e^{-u(1-\cos t)} \frac{\sin\frac{1}{2}t\cdot\frac{ut^2}{6}}{\sin\frac{1}{2}t}\, dt.$$

Since $0 \leq t \leq \sqrt{\frac{6}{u}}$, we have $\frac{ut^2}{6} \leq 1$. The sine is an increasing function in the first quadrant. Consequently

$$0 \leq \frac{\sin\frac{1}{2}t\cdot\frac{ut^2}{6}}{\sin\frac{1}{2}t} \leq 1, \text{ for } 0 \leq t \leq \sqrt{\frac{6}{u}} .$$

Hence

$$\frac{2}{\pi}\int_0^{\sqrt{\frac{6}{u}}} e^{-u(1-\cos t)} \frac{\sin\frac{1}{2}t\cdot\frac{ut^2}{6}}{\sin\frac{1}{2}t}\, dt$$

$$\leq \frac{2}{\pi}\int_0^{\sqrt{\frac{6}{u}}} e^{-u(1-\cos t)}\,dt \leq \frac{2}{\pi}\sqrt{\frac{6}{u}} = o(1) \text{ as } u \to \infty.$$

Now consider

$$\frac{2}{\pi}\int_{\sqrt{\frac{6}{w}}}^{\frac{1}{\sqrt[3]{w}}} e^{-w(1-\cos t)} \frac{\sin\frac{1}{2}t\,\frac{wt^2}{6}}{\sin\frac{1}{2}t}\,dt$$

Here $\frac{wt^2}{6} \geqslant 1$. We use lemma 3.51.4 with

$a = \frac{wt^2}{6}$, $x = \frac{1}{2}t$

$a \geqslant 1$ and $0 \leq ax = \frac{1}{2}\,\frac{wt^2}{6} \leq \frac{1}{12} < \frac{\pi}{2}$ Hence, from (3.514)

$$\frac{\sin\frac{1}{2}t\cdot\frac{wt^2}{6}}{\sin\frac{1}{2}t} \leq \frac{wt^2}{6}$$

Therefore

(3.527) $$\frac{2}{\pi}\int_{\sqrt{\frac{6}{w}}}^{\frac{1}{\sqrt[3]{w}}} e^{-w(1-\cos t)} \frac{\sin\frac{1}{2}t\,\frac{wt^2}{6}}{\sin\frac{1}{2}t}\,dt \leq \frac{w}{3\pi}\int_{\sqrt{\frac{6}{w}}}^{\frac{1}{\sqrt[3]{w}}} t^2 e^{-w(1-\cos t)}\,dt$$

Since $1-\cos t = 2\sin^2\frac{1}{2}t$, the last integral may be written

$$\frac{w}{3\pi}\int_{\sqrt{\frac{6}{w}}}^{\frac{1}{\sqrt[3]{w}}} t^2 e^{-2w\sin^2\frac{1}{2}t}\,dt$$

From (3.59), with $x = \frac{1}{2}t$,

$$\frac{\sin\frac{1}{2}t}{\frac{1}{2}t} \geqslant \frac{2}{\pi}$$

Hence

$$\sin\tfrac{1}{2}t \geqslant \frac{t}{\pi}$$

and

$$\sin^2\tfrac{1}{2}t \geqslant \frac{1}{\pi^2}\,t^2$$

Therefore,

$$(3.528)\quad \begin{cases} \frac{u}{3\pi}\int_{\sqrt{\frac{6}{u}}}^{\frac{1}{\sqrt[3]{u}}} t^2 e^{-u(1-\cos t)}dt = \frac{u}{3\pi}\int_{\sqrt{\frac{6}{u}}}^{\frac{1}{\sqrt[3]{u}}} t^2 e^{-2u\sin^2\frac{1}{2}t}dt \\ \leq \frac{u}{3\pi}\int_{\sqrt{\frac{6}{u}}}^{\frac{1}{\sqrt[3]{u}}} t^2 e^{-\frac{2}{\pi^2}ut^2}dt \leq \frac{u}{3\pi}\int_{0}^{\infty} t^2 e^{-\frac{2}{\pi^2}\cdot ut^2}dt. \end{cases}$$

Now, formula 494 in Pierce's Short Table of Integrals, (p. 63), states

$$\int_0^\infty x^{2n}e^{-ax^2}dx = \frac{1.3.5\ldots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}.$$

Taking $n = 1$, $x = t$, $a = \frac{2}{\pi^2}u$, this becomes

$$(3.529)\quad \int_0^\infty t^2 e^{-\frac{2}{\pi^2}ut^2}dt = \frac{1}{16}\frac{\pi^3\sqrt{2\pi}}{u\sqrt{u}}$$

Substituting in (3.528) yields

$$(3.530)\quad \frac{u}{3\pi}\int_{\sqrt{\frac{6}{u}}}^{\frac{1}{\sqrt[3]{u}}} t^2 e^{-u(1-\cos t)}dt \leq \frac{1}{48}\pi^2\sqrt{2\pi}\frac{1}{\sqrt{u}}$$

$$= o(1) \text{ as } u \to \infty.$$

This, together with (3.527), gives

$$(3.531)\quad \frac{2}{\pi}\int_{\sqrt{\frac{6}{u}}}^{\frac{1}{\sqrt[3]{u}}} e^{-u(1-\cos t)}\frac{\sin\frac{1}{2}t\cdot\frac{ut^2}{6}}{\sin\frac{1}{2}t}dt = o(1) \text{ as } u \to \infty$$

And this together with (3.526) establishes

(3.532) $\quad \ell(u) = o(1)$ as $u \to \infty$.

Hence, combining this with (3.523), we have

(3.533) $\quad I(u) = o(1)$ as $u \to \infty$

This and (3.518) establishes

(3.534) $\quad I + II = o(1)$ as $u \to \infty$.

The conclusion of theorem 3.51 then follows from (3.516).

qed.

The next simplification is given by

<u>Theorem 3.52.</u> Let

(3.535) $\quad L_{\beta,2}(u) = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{1}{2}ut^2} \frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt.$

Then

(3.536) $\quad L_\beta(u) = L_{\beta,2}(u) + o(1)$ as $u \to \infty$.

Before proceding to the proof of theorem 3.52, another elementary inequality is needed:

<u>Lemma 3.52.1:</u>

(3.537) $\quad 0 \leq e^{-x} - e^{-y} \leq y - x,$ for $0 \leq x \leq y$.

Proof of lemma 3.52.1:

$$\frac{d}{dx}\left(x+e^{-x}\right) = 1-e^{-x}.$$

For $x \geq 0$, therefore,

$$\frac{d}{dx}\left(x+e^{-x}\right) = 0 .$$

Hence $x+e^{-x}$ increases as x increases from 0 on, and we may write $x+e^{-x} \leq y+e^{-y}$ for $0 \leq x \leq y$.

But this is just a rewriting of (3.537).

qed.

Proof of theorem 3.52:

Having already shown (theorem 3.51) that

(3.53) $L_{\beta}(u) = L_{\beta,1}(u) + o(1)$ as $u \longrightarrow \infty$,

to prove (3.536) it will be sufficient to establish

(3.538) $L_{\beta,1}(u) = L_{\beta,2}(u) + o(1)$ as $u \longrightarrow \infty$.

To do this consider

(3.539) $0 \leq L_{\beta,1}(u) - L_{\beta,2}(u) = \frac{1}{\pi}\int_0^{\pi}\left(e^{-2u\sin^2\frac{1}{2}t - 2u\frac{t^2}{4}}\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt.$

Here again it helps to separate the integral into two parts:

$$(3.540)\ \frac{1}{\pi}\int_0^{\pi}\left(e^{-2u\sin^2\frac{1}{2}t}-e^{-2u\frac{t^2}{4}}\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt=\frac{1}{\pi}\int_0^{\frac{1}{\sqrt[4]{u}}}+\frac{1}{\pi}\int_{\frac{1}{\sqrt[4]{u}}}^{\pi}=\frac{1}{\pi}I_1(u)+\frac{1}{\pi}II(u).$$

√

Consider first $I_1(u)$. From (3.537) we get

$$I_1(u)=\int_0^{\frac{1}{\sqrt[4]{u}}}\left(e^{-2u\sin^2\frac{1}{2}t}-e^{-2u\frac{t^2}{4}}\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt$$

$$\leq 2u\int_0^{\frac{1}{\sqrt[4]{u}}}\left(\frac{t^2}{4}-\sin^2\tfrac{1}{2}t\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt$$

$$=2u\int_0^{\frac{1}{\sqrt[4]{u}}}\left(\frac{t}{2}+\sin\tfrac{1}{2}t\right)\left(\frac{t}{2}-\sin\tfrac{1}{2}t\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt.$$

Since $\sin x \leq x$, this last integral is

$$\leq 2u\int_0^{\frac{1}{\sqrt[4]{u}}} t\left(\frac{t}{2}-\sin\tfrac{1}{2}t\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt.$$

Hence

$$(3.541)\quad 0\leq I_1(u)\leq 2u\int_0^{\frac{1}{\sqrt[4]{u}}} t\left(\frac{t}{2}-\sin\tfrac{1}{2}t\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt.$$

But from (3.54), with $x=\frac{1}{2}t$, it follows that

$$(3.542)\quad 0\leq \tfrac{1}{2}t-\sin\tfrac{1}{2}t\leq\frac{(\frac{1}{2}t)^3}{3!}=\frac{t^3}{48}.$$

Applying this inequality to (3.541) gives

$$0\leq I_1(u)\leq\frac{u}{24}\int_0^{\frac{1}{\sqrt[4]{u}}} t^4\,\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt$$

$$=\frac{u}{12}\int_0^{\frac{1}{\sqrt[4]{u}}} t^3\,\frac{|\sin(u+\frac{1}{2})t|}{\frac{\sin\frac{1}{2}t}{\frac{1}{2}t}}\,dt.$$

From (3.59) with $x = \frac{1}{2}t$, we get

$$\frac{\sin \frac{1}{2}t}{\frac{1}{2}t} \geq \frac{2}{\pi}, \quad 0 \leq \tfrac{1}{2}t \leq \tfrac{1}{2}\pi.$$

Hence

$$\int_0^{\frac{1}{\sqrt[3]{u}}} t^3 \frac{|\sin(u+\frac{1}{2})t|}{\frac{\sin\frac{1}{2}t}{\frac{1}{2}t}}\,dt \leq \frac{\pi}{2}\int_0^{\frac{1}{\sqrt[3]{u}}} t^3 |\sin(u+\tfrac{1}{2})t|\,dt$$

$$\leq \frac{\pi}{2}\int_0^{\frac{1}{\sqrt[3]{u}}} t^3\,dt = \frac{\pi}{8}\,\frac{1}{u}\,\frac{1}{\sqrt[3]{u}}$$

Substituting in (3.543) yields

(3.544) $$0 \leq I_1(u) \leq \frac{\pi}{96}\,\frac{1}{\sqrt[3]{u}} = o(1) \text{ as } u \to \infty.$$

Next $II_1(u)$ is considered

$$0 \leq II_1(u) = \int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \left(e^{-2u\sin^2\frac{1}{2}t} - e^{-2u\frac{t^2}{4}}\right) \frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt$$

$$\leq \int_{\frac{1}{\sqrt[3]{u}}}^{\pi} e^{-2u\sin^2\frac{1}{2}t}\,\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt$$

From (3.59) with $x = \frac{1}{2}t$, it follows that

$$\sin^2\tfrac{1}{2}t \geq \frac{1}{\pi^2}\,t^2$$

Hence

$$0 \leq \text{II}_1(u) \leq \int_{\frac{1}{\sqrt[3]{u}}}^{\pi} e^{-\frac{2}{\pi^2} u t^2} \frac{|\sin(u+\frac{1}{2})t|}{\sin \frac{1}{2}t} dt$$

$$\leq e^{-\frac{2}{\pi^2} \sqrt[3]{u}} \int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \frac{|\sin(u+\frac{1}{2})t|}{\sin \frac{1}{2}t} dt$$

$$\leq e^{-\frac{2}{\pi^2} \sqrt[3]{u}} \int_0^{\pi} \frac{|\sin(u+\frac{1}{2})t|}{\sin \frac{1}{2}t} dt$$

$$\leq (2u+1)\pi \, e^{-\frac{2}{\pi^2} \sqrt[3]{u}} = o(1) \text{ as } u \longrightarrow \infty.$$

This, together with (3.544) and (3.540), when applied to (3.539) establishes (3.538). And (3.538) in conjunction with (3.53) gives (3.536). This proves theorem 3.52.

qed.

For future convenience let us make a simple transformation of the variable of integration in

(3.535) $$L_{\beta,2}(u) \equiv \frac{1}{\pi} \int_0^{\pi} e^{-\frac{1}{2} u t^2} \frac{|\sin(u+\frac{1}{2})t|}{\sin \frac{1}{2}t} dt$$

by replacing t by 2t. Then

(3.546) $$L_{\beta,2}(u) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-2ut^2} \frac{|\sin(2u+1)t|}{\sin t} dt.$$

Now we simplify (3.546) asymptotically.

<u>Theorem 3.53</u>: Let

(3.547) $$L_{\beta,3}(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-2ut^2}\frac{|\sin(2u+1)t|}{t}\,dt.$$

Then

(3.548) $$L_\beta(u) = L_{\beta,3}(u) + o(1) \text{ as } u\to\infty.$$

<u>Proof</u>: By virtue of (3.536) it will be sufficient to demonstrate

(3.549) $$L_{\beta,2}(u) = L_{\beta,3}(u) + o(1) \text{ as } u\to\infty.$$

Now,

$$L_{\beta,2}(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-2ut^2}\frac{|\sin(2u+1)t|}{\sin t}\,dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-2ut^2}\frac{|\sin(2u+1)t|}{t}\,dt + \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin t}-\frac{1}{t}\right)e^{-2ut^2}|\sin(2u+1)t|\,dt.$$

Hence

(3.550) $$L_{\beta,2}(u) = L_{\beta,3}(u) + \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin t}-\frac{1}{t}\right)e^{-2ut^2}|\sin(2u+1)t|\,dt$$

Consider

$$J(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin t}-\frac{1}{t}\right)e^{-2ut^2}|\sin(2u+1)t|\,dt$$

First we obtain an estimate for

$$\frac{1}{\sin t}-\frac{1}{t} = \frac{t-\sin t}{t\sin t},\qquad 0\le t\le\frac{\pi}{2}.$$

From (3.54) with $x = t$ it follows that

$$0 \leq t - \sin t \leq \frac{t^3}{3!}, \quad 0 \leq t \leq \frac{\pi}{2}$$

Hence

(3.551) $$0 \leq \frac{t - \sin t}{t \sin t} \leq \frac{t^2}{6 \sin t} = \frac{1}{6} t \cdot \frac{t}{\sin t}.$$

From (3.59) it follows that

$$\frac{t}{\sin t} = \frac{\pi}{2}, \quad 0 \leq t \leq \frac{\pi}{2} .$$

Substituting this in (3.55) gives

(3.552) $$0 \leq \frac{1}{\sin t} - \frac{1}{t} = \frac{t - \sin t}{t \sin t} \leq \frac{\pi}{12} t, \quad 0 \leq t \leq \frac{\pi}{2}$$

Hence, from (3.552) and $|\sin(2u+1)t| \leq 1$,

$$J(u) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin t} - \frac{1}{t}\right) e^{-2ut^2} |\sin(2u+1)t| \, dt$$

$$\leq \frac{1}{6} \int_0^{\frac{\pi}{2}} t e^{-2ut^2} |\sin(2u+1)t| \, dt \leq \frac{1}{6} \int_0^{\frac{\pi}{2}} t e^{-2ut^2} \, dt$$

Now,

(3.554) $$0 = \frac{1}{6} \int_0^{\frac{\pi}{2}} t e^{-2ut^2} dt \leq \frac{\pi}{12} \int_0^{\frac{\pi}{2}} e^{-2u\tau^2} d\tau \leq \frac{\pi}{12} \int_0^{\infty} e^{-2u\tau^2} d\tau .$$

Referring again to Pierce's Short Table of Integrals, we find formula 492 (p. 63):

$$\int_0^\infty e^{-a^2x^2}dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma\left(\tfrac{1}{2}\right).$$

Here $a = \sqrt{2u}$, $x = t$. Thus.

(3.555) $$\frac{\pi}{2}\int_0^\infty e^{-2ut^2}dt = \frac{\pi\sqrt{\pi}}{24}\frac{1}{\sqrt{2u}} = o(1) \text{ as } u \longrightarrow \infty.$$

Consequently, from (3.554)

$$1/6\int_0^{\frac{\pi}{2}} t\,e^{-2ut^2}dt = o(1) \text{ as } u \longrightarrow \infty,$$

and, together with (3.553),

(3.556) $$J(u) \equiv \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin t} - \frac{1}{t}\right)e^{-2ut^2}|\sin(2n+1)t|\,dt = o(1) \text{ as } u \longrightarrow \infty.$$

Together with (3.550) this establishes (3.549) and, consequently, with (3.536), theorem 3.53.

qed.

3.6. On a function considered by Fejér. In the course of calculating the ordinary Lebesgue constants, Fejér [Fejér, 5, p. 29, formula (10)] introduced

(3.61) $$\lambda_n \equiv \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{|\sin(2n+1)t|}{t}dt$$

and wrote

$$\lambda_n = \frac{2}{\pi}\int_0^{\frac{\pi(n+1)}{2n+1}} \frac{|\sin(2n+1)t|}{t}\,dt - \frac{2}{\pi}\int_0^{\frac{\pi(n+1)}{2n+1}} \frac{|\sin(2n+1)t|}{t}\,dt$$

$$\equiv \nu_n \qquad\qquad + \epsilon_n$$

In order to prepare for the computation of the Lebesgue constants corresponding to Borel summability, knowledge of λ_n, ν_n, ϵ_n is needed for continuous and not just integral values of n. We define

$$(3.63)\quad L_1(u) \equiv \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{|\sin(2u+1)t|}{2t}\,dt$$

and write

$$L_1(u) = \frac{2}{\pi}\int_0^{(u+1)\frac{\pi}{2n+1}} \frac{|\sin(2u+1)t|}{t}\,dt - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{(u+1)\frac{\pi}{2n+1}} \frac{|\sin(2u+1)t|}{t}\,dt$$

$$\equiv \nu(u) \qquad\qquad + \epsilon(u)$$

Adapting the device used by Fejér in his study of ν_m as well as his result (infra), we prove

Theorem 3.61

$$(3.65)\quad L_1(u) = \frac{4}{\pi^2}\log u + 2\int_0^1 \log\Gamma(t)\cos\pi t\,dt + o(1) \quad \text{as } u \longrightarrow \infty.$$

Proof: Recall (3.64)

$$L_1(u) = \nu(u) + \epsilon(u).$$

We have

$$|\epsilon(u)| = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{(u+1)\frac{\pi}{2u+1}} \frac{|\sin(2u+1)t|}{t}\,dt \le \frac{2}{\pi}\int_{\frac{\pi}{2}}^{(u+1)\frac{\pi}{2u+1}} \frac{1}{t}\,dt$$

$$= \frac{2}{\pi}\left(\log \pi\frac{u+1}{2u+1} - \log\frac{\pi}{2}\right) = \frac{2}{\pi}\log 2\cdot\frac{u+1}{2u+1} \to \frac{2}{\pi}\log 2\cdot\frac{1}{2}$$
$$= 0 \text{ as } u\to\infty.$$

Hence

(3.66) $\epsilon(u) = o(1)$ as $u \to \infty$,

and

(3.67) $L_1(u) = \nu(u) + o(1)$ as $u \to \infty$.

(3.68) Now,
$$\nu(u) = \frac{2}{\pi}\int_0^{(u+1)\frac{\pi}{2u+1}} \frac{|\sin(2u+1)t|}{t}\,dt = \frac{2}{\pi}\int_0^{(u+1)} \frac{|\sin t|}{t}\,dt$$
$$= \frac{2}{\pi}\int_0^{([u]+1)\pi} + \frac{2}{\pi}\int_{([u]+1)\pi}^{(u+1)\pi}\ ,$$

where, as usual, $[u]$ denotes the largest integer in u.

Note that
$$0 \le \frac{2}{\pi}\int_{([u]+1)\pi}^{(u+1)\pi} \frac{|\sin t|}{t}\,dt \le \frac{2}{\pi}\int_{([u]+1)\pi}^{(u+1)\pi} \frac{1}{t}\,dt = \frac{2}{\pi}\left\{\log(u+1)\pi - \log([u]+1)\pi\right\}$$

$$= \frac{2}{\pi}\log\frac{u+1}{[u]+1} \to \frac{2}{\pi}\log 1 = 0 \text{ as } u\to\infty.$$

Hence, from (3.68)

(3.610) $$\nu(u) = \frac{2}{\pi}\int_0^{([u]+1)\pi} \frac{|\sin t|}{t}\,dt + o(1) \text{ as } u \to \infty$$

Fejér showed [Fejér, 5], formula(15), p. 30] that

(3.611) $$\nu_n = \frac{4}{\pi^2}\log n - \frac{2}{\pi^2}\int_0^{\pi} \frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}\sin t\,dt + \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Now,

$$\nu_n = \frac{2}{\pi}\int_0^{(n+1)\frac{\pi}{2n+1}} \frac{|\sin(2n+1)t|}{t}\,dt = \frac{2}{\pi}\int_0^{(n+1)\pi} \frac{|\sin t|}{t}\,dt.$$

Hence

(3.612) $$\nu(u) = \nu_{[u]} + o(1) \text{ as } u \to \infty.$$

From (3.611) it then follows that

$$\nu(u) = \frac{4}{\pi^2}\log[u] - \frac{2}{\pi^2}\int_0^{\pi} \frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}\sin t\,dt + o(1) \text{ as } u \to \infty.$$

But

$$0 \le \log u - \log[u] = \log\frac{u}{[u]} = \log\frac{[u] + (-[u]+u)}{[u]}$$
$$= \log\left(1 + \frac{-[u]+u}{[u]}\right) \le \log\left(1 + \frac{1}{[u]}\right) \to 0 \text{ as } u \to \infty.$$

Therefore we may write

(3.613) $$\nu(u) = \frac{4}{\pi^2}\log u - \frac{2}{\pi^2}\int_0^{\pi} \frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}\sin t\,dt + o(1) \text{ as } u \to \infty.$$

and, from (3.67),

(3.614) $$L_1(u) = \frac{4}{\pi^2}\log u - \frac{2}{\pi^2}\int_0^{\pi}\frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}\sin t\,dt + o(1) \text{ as } u\to\infty.$$

Now with Fejér we find another representation for

$\int_0^{\pi}\frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}\sin t\,dt$, namely: $-\pi^2\int_0^1 \log\Gamma(t)\cos\pi t\,dt$.

To do so, refer again to the general formula for integration by parts (2.54). There put

$x = t$, $g(x) = \frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}$; $F(x) = \sin t$.

This gives

$$\int_0^{\pi}\frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}\sin t\,dt + \pi\int_0^{\pi}\log\Gamma(\tfrac{t}{\pi})\cos t\,dt = 0$$

or, writing $t = \frac{t}{\pi}$,

(3.615) $$\int_0^{\pi}\frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}\sin t\,dt = -\pi^2\int_0^1\log\Gamma(t)\cos\pi t\,dt.$$

Substituting this in (3.614) completes the proof of the formula stated above, and thus theorem 3.61.

3.7. An asymptotic value of $L_\beta(u)$. In order to estimate $L_\beta(u)$ we consider now

(3.71) $$L_1(u) - L_\beta(u)$$

and estimate this. Since $L_{\beta,3}(u) = L_\beta(u) + o(1)$ as u becomes infinite, we may consider instead

$$(3.72)\quad d(u) \equiv h_1(u) - h_{\beta,3}(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1-e^{-2ut^2}}{t}\,|\sin(2u+1)t|\,dt$$

To simplify this, we prove

Theorem 3.71. Let

$$(3.73)\quad d_1(u) \equiv \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\,\frac{2ut^2}{1+2ut^2}\,|\sin(2u+1)t|\,dt\,.$$

Then

$$(3.74)\quad d(u) = d_1(u) + O(1) \text{ as } u \longrightarrow \infty\,.$$

Proof:

$$(3.75)\quad \begin{cases} d(u) - d_1(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(1 - \frac{2ut^2}{1+2ut^2} - e^{-2ut^2}\right)|\sin(2u+1)t|\,dt \\ \qquad = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right)|\sin(2u+1)t|\,dt \end{cases}$$

Now,

$$e^x = 1 + x + \sum_2^\infty \frac{x^n}{n!} \geqslant 1 + x,\quad x \geqslant 0,$$

whence

$$e^{-x} \leqslant \frac{1}{1+x},\quad x \geqslant 0\,.$$

The integrand in (3.75) is therefore positive (taking $x = 2ut^2$). Since $|\sin(2u+1)t| \leqslant 1$, we then have

(3.76) $0 \leq d(u) - d_1(u) \leq \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right) dt.$

Making the transformation

$$x = 2ut^2$$

we get

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right)dt = \frac{1}{\pi}\int_0^{\frac{\pi^2}{2}u} \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right)dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi^2}{2}u} \frac{e^x - 1 - x}{xe^x(1+x)}\, dx$$

Or

(3.77) $\lim_{u\to\infty} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right) dt = \frac{1}{\pi}\int_0^{\infty} \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right)dx,$

if the right-hand integral exists.

To prove that it does exist, write

(3.78) $$\begin{cases} \frac{1}{\pi}\int_0^{\infty} \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right)dx = \frac{1}{\pi}\int_0^{\infty}\frac{e^x - 1 - x}{e^x(1+x)x}\, dx \\ \qquad = \frac{1}{\pi}\int_0^1 + \frac{1}{\pi}\int_1^{\infty}. \end{cases}$$

Now

(3.79) $\frac{1}{\pi}\int_0^1 \frac{e^x - 1 - x}{xe^x(1+x)}\, dx = \frac{1}{\pi}\int_0^1 \frac{e^x - 1 - x}{x} \frac{1}{e^x(1+x)}\, dx.$

Note now that for $0 \leq x \leq 1$

$$e^x - 2x \leq 1,$$

whence

$$e^x - 1 - x \leq x,$$

or

$$\frac{e^x - 1 - x}{x} \leq 1$$

And, since e^{-x} $(1\ x)^{-1}$ is integrable in (0, 1), this shows that (3.79) exists.

On the other hand,

$$(3.710)\quad \frac{1}{\pi}\int_1^\infty \frac{e^x-1-x}{e^x x(1+x)}\,dx < \frac{1}{\pi}\int_1^\infty \frac{1}{x(1+x)}\,dx < \frac{1}{\pi}\int_1^\infty \frac{dx}{x^2}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{dx}{(1+x)^2} = \frac{1}{\pi}$$

Combining this with the fact that (3.79) exists, we see that the right-hand integral in (3.77), and hence the limit on the left, exists.

Together with (3.76) this completes the proof.

qed.

Further to expedite the study of d(u), we prove

Theorem 3.72. Let

(3.711) $d_2(u) \equiv \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} \frac{1}{t}\,\frac{2ut^2}{1+2ut^2}\,dt.$

Then

(3.712) $d_1(u) = d_2(u) + o(1)$ as $u \longrightarrow \infty$.

Proof: Write

$$d_2(u) - d_1(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\,\frac{2ut^2}{1+2ut^2}\left\{\frac{2}{\pi} - |\sin(2u+1)t|\right\} dt.$$

Now we seek to apply corollary 2.51.1 to $d_2(u) - d_1(u)$. The $f_u(x)$ of the corollary is

$$\frac{1}{t}\,,\quad \frac{2ut^2}{1+2ut^2}$$

We have, $a = 0$, $b = \frac{\pi}{2}$ and

(3.713) $f_u(b) = f_u(\frac{\pi}{2}) = \frac{2}{\pi}\,\frac{2u\frac{\pi^2}{4}}{1+2u\frac{\pi^2}{4}} = O(1) = o(u)$ as $u \longrightarrow \infty$.

Also

$$f'_u(t) = \frac{d}{dt}\,\frac{2ut}{1+2ut^2} = 2u\frac{2ut^2-t+1}{(1+2ut^2)^2} \geq 0 \text{ for } u \geq \frac{1}{8}.$$

Thus for $u > 1$,

$$(3.714)\begin{cases}\int_a^b |f_u'(t)|\,dt = \int_a^b f_u'(t)\,dt = f_u(b) - f_u(a) \\ = f_u(b) = O(1) = o(u) \text{ as } u \to \infty\end{cases}$$

From (3.713) and (3.714) it follows that the hypotheses of corollary 2.51.1 are satisfied. Hence the conclusion follows. But the conclusion of that corollary in this case simply states that

$$d_2(u) - d_1(u) = o(1) \text{ as } u \to \infty,$$

and the theorem follows.

qed.

Now combining theorem 3.72 with theorem 3.71 yields without further ado

<u>Theorem 3.73.</u>

(3.715) $d(u) = d_2(u) + O(1)$ as $u \to \infty$.

The next step is the approximate calculation of

(3.711) $d_2(u) = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} \frac{1}{t}\,\frac{2ut^2}{1+2ut^2}\,dt$

This integral is

$$\frac{2}{\pi^2} \log \left(1 + \frac{\pi^2}{2} u\right) = \frac{2}{\pi^2} \log \frac{\pi^2}{2} u + o(1)$$

$$\doteq \frac{2}{\pi^2} \log u + \frac{2}{\pi^2} \log \frac{\pi^2}{2} + o(1) \text{ as } u \to \infty .$$

Hence

(3.716) $d_2(u) = \frac{2}{\pi^2} \log u + \frac{2}{\pi^2} \log \frac{\pi^2}{2} + o(1)$ as $u \to \infty$.

From (3.712) it then follows that

(3.717) $d_1(u) = \frac{2}{\pi^2} \log u + \frac{2}{\pi^2} \log \frac{\pi^2}{2} + o(1)$ as $u \to \infty$,

and from (3.74) we get

(3.718) $d(u) = \frac{2}{\pi^2} \log u + O(1)$ as $u \to \infty$.

Together with (3.72) this states

(3.719) $L_{B,3}(u) = L_1(u) - \frac{2}{\pi^2} \log u + O(1)$ as $u \to \infty$,

Since $L_{B,3}(u) = L_B(u) + o(1)$ as $u \to \infty$ we have

(3.720) $L_B(u) = L_1(u) - \frac{2}{\pi^2} \log u + O(1)$ as $u \to \infty$,

And, finally, from theorem 3.65 and (3.720) we get the principal result of this chapter:

Theorem 3.74:

(3.721) $L_B(u) = \frac{2}{\pi^2}\log u + O(1)$ as $u \to \infty$

3.8. Existence of a continuous function whose Fourier series is not Borel summable everywhere. From theorem 3.74 we can again deduce what C. N. Moore [Moore, 21] similarly remarked, namely, that there exists a continuous function whose Fourier series is not everywhere Borel summable.

The theorem on which this deduction rests is

"If the sequence

(3.81) $$u_n(x) = \int_a^b x(t) y_n(t)\, dt$$

in bounded* for every bounded, or even only continuous, function x, then $M[y_n; a, b] = O(1)$." [Zygmund, 33, p. 99].

Here, as usual

(3.82) $$M[y_n; a, b] = \int_a^b |y_n(t)|\, dt.$$

We recall now (3.34):

(3.34) $$e^{-u} B_f(u, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{u\cos(t-x) - u}\, \frac{\sin\{u\sin(t-x) + \frac{1}{2}(t-x)\}}{\sin\frac{1}{2}(t-x)}\, dt$$

For x = 0 and integral values of u, this is of

---

* i.e., with respect to n for any given (x(t).

the form (3.81). Let now f(t) run through the set of continuous functions. If

$$e^{-u}\beta_f(u;0)$$

is bounded for f(t) in the set of continuous functions, it would then follow that

$$M[y_n;-\pi,\pi] = \frac{2}{\pi}\int_{-\pi}^{\pi} e^{-u(1-\cos t)}\frac{|\sin(u\sin t+\frac{1}{2}t)|}{|\sin\frac{1}{2}t|}\,dt$$

is likewise bounded. But the integral on the right is $L_B(u)$, which, from theorem 3.74, is not bounded, but becomes infinite.

Hence there is a continuous function f(t) for which $e^{-u}\beta_f(u;0)$ is not bounded; that is, there is a continuous function whose Fourier series is not Borel summable at 0.

3.9. Remarks and conjectures. Let us recall

(3.721) $L_B(u) = \frac{2}{\pi^2}\log u + O(1)$ as $u \to \infty$.

The additive factor O(1) could stand simply for a constant plus o(1) or it could cover some such oscillating function as sin u plus o(1).

If we look back upon the entire asymptotic derivation of $L_B(u)$ we see that it is in only one place that we have had to introduce a factor (O(1), in theorem

3.71. Everywhere else where an estimate was given the estimate took the form of a constant plus o(1).

What is the estimate which thus weakened the final result? It is

(3.91) $$d(u) - d_1(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right)\left|\sin(2u+1)t\right| dt.$$

All we were able to show in theorem 3.71 was

(3.92) $$d(u) - d_1(u) = O(1) \text{ as } u \to \infty.$$

A more precise result would sharpen our final conclusion. Could, e.g., O(1) be replaced by o(1) in (3.92)? This is hardly likely in view of

(3.77) $$\lim_{u\to\infty} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right) dt = \frac{1}{\pi}\int_0^{\infty} \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right) dx.$$

Looking back to theorem 2.51 and, more particularly to corollary 2.51.1, a more likely guess would appear to be

(3.93) $$\lim_{u\to\infty} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right)\left(\frac{2}{\pi} - \left|\sin(2u+1)t\right|\right) dt = 0.$$

If this were true, we could improve theorem 3.74 as follows:

Conjecture 3.91:

(3.94)
$$\begin{cases} L_B(u) = \frac{2}{\pi^2}\log u - \frac{2}{\pi^2}\int_0^\infty \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right)dx \\ + 2\int_0^1 \log\Gamma(t)\cos\pi t\,dt - \frac{2}{\pi^2}\log\frac{\pi^2}{2} + o(1) \end{cases}$$

~~as u → ∞~~.

as u → ∞ .

Another possibility is of course that (3.93) does not hold but that we nevertheless have the existence of

(3.95)
$$\lim_{u\to\infty} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right)|\sin(2u+1)t|\,dt = c .$$

In this case (which obviously includes the previous) we would have

Conjecture 3.92:

(3.96)
$$\begin{cases} L_B(u) = \frac{2}{\pi^2}\log u - c + 2\int_0^1 \log\Gamma(t)\cos\pi t\,dt \\ \qquad - \frac{2}{\pi^2}\log\frac{\pi^2}{2} + o(1) \quad \text{as } u\to\infty, \end{cases}$$

Thus, it is only the behavior of ~~(3.97)~~

(3.97)
$$\lim_{u\to\infty} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{t}\left(\frac{1}{1+2ut^2} - e^{-2ut^2}\right)|\sin(2u+1)t|\,dt$$

which is the doubtful part in this situation. If (3.97) should oscillate, then O(1) would oscillate too.

Another type of conjecture deals with the behavior of

$$(3.98)\qquad L_\beta(u+\delta) - L_\beta(u)$$

For what δ and what range of u would (3.98) be positive? A related question is connected with

$$(3.99)\qquad L'_\beta(u)$$

For what range of u is (3.99) positive?

In view of the structure of $L_\beta(u)$ and in view of the fact that L(n) is monotonic (in fact completely monotonic, as Szegö [Szegö, 27] showed) it would appear likely that the following guess is fairly safe:

Conjecture 3.93.

$$(3.910)\qquad L'_\beta(u) > 0, \quad u \geq 1.$$

If this is true, it would of course imply that (3.98) is positive for any δ, in the range $u \geq 1$. If not, the question surrounding (3.98) would still be open. And open too, would be the question: Is (3.910) valid if u is more restricted?

CHAPTER IV

AN EXAMPLE OF A CONTINUOUS FUNCTION WHOSE FOURIER SERIES IS NOT BOREL SUMMABLE AT 0.

4.1. Introduction. In his analysis of the (ordinary) Lebesgue constants, Fejér [Fejér, 5] considered the functions

(4.11) $\varphi_n(x) = \operatorname{sgn} \dfrac{\sin(2n+1)\frac{1}{2}x}{\sin\frac{1}{2}x}, \quad 0 \le x \le 2\pi, \; n = 0, 1, \ldots$,

all of which are in E and proved that the n^{th} Fourier(cosine) partial sum of $\varphi_n(x)$ has for $x = 0$, a value which becomes infinite with n (the n^{th} Lebesgue constant).

He then remarked that

(4.12) $\varphi_n(x) \equiv \operatorname{sgn} \sin(2n+1)\frac{1}{2}x, \quad 0 \le x < 2\pi, \; n = 0, 1, \ldots$

and pointed out that each $\varphi_n(x)$, save $\varphi_0(x)$, is discontinuous, the number of discontinuities becoming infinite with n.

Pointing to the desirability of having a specific set of continuous functions $\psi_n(x)$ of E having the same property that the n^{th} Fourier partial sum of $\psi_n(x)$ becomes infinite with n at $x = 0$, he defined

(4.13) $$\begin{cases} \psi_n(x) = \sin(2n+1)\frac{1}{2}x, & 0 \le x \le \pi \\ \psi_n(x) \equiv \psi_n(-x), & -\pi \le x \le 0 \end{cases} \qquad n = 0, 1, \ldots$$

For $\psi_n(x)$ he had

(4.14) $s_n(0) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin\frac{1}{2}(2n+1)t}{\sin t}\,dt \equiv \sigma_n$

and proved [Fejér, 5 , p. 36]

(4.15) $\sigma_n = \frac{1}{\pi}\log n + \gamma_0 + o(1)$ as $n \to \infty$,

where

(4.16) $\gamma_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin t} - \frac{1}{t}\right)dt + 2\int_0^1 \log\Gamma(t)\sin 2\pi t\,dt.$

He showed that the functions (4.13) have the fol-
lowing properties:

1. All are in E.
2. Each Fourier partial sums of any of these functions are positive for $x = 0$.
3. The nth partial sum of $\Psi_m(x)$ at $x = 0$ has a value larger than $\frac{1}{2\pi}\log(2n+1)$ which, consequently, becomes infinite with n.

On the basis of this work, he proved [Fejér, 5 , p. 44]

"The infinite series

(4.17) $f_1(x) = \sum_{k=1}^{\infty} \frac{\sin(2k^3+1)\frac{1}{2}x}{k^2}$

represent an everywhere continuous function in $0 \leq x \leq 2\pi$, such that its Fourier (cosine) series diverges at $x = 0$."

Here we are faced with a similar situation. In chapter III, in studying the Borel Lebesgue constants, the family of functions corresponding to (4.11) was

$$(4.18)\qquad g_u(t) = \operatorname{sgn} \frac{\sin(u\sin t + \frac{1}{2}t)}{\sin\frac{1}{2}t}, \quad 0 \leq t < \pi, \ u \geq 0$$
$$g_u(-t) = g_u(t), \quad -\pi \leq t \leq 0$$

Obviously,

$$(4.19)\qquad g_u(t) = \operatorname{sgn} \sin(u\sin t + \tfrac{1}{2}t), \quad 0 \leq t < 2\pi, \ u \geq 0,$$

again with all $g_u(t)$ save $g_0(t)$ discontinuous.

In place of $g_u(t)$ we should like to consider a specific set of continuous functions of E, $h_u(t)$ having the property that the u^{th} Borel mean of the Fourier (cosine) series of $h_u(x)$ becomes infinite with u at $x = 0$.

In the next section we show that $\psi_u(x)$, considered for $u \geqslant 0$, satisfies this requirement and devote this chapter to discussing the behavior of the constants which for Borel summability correspond to Fejér's σ_n. An asymptotic lower bound is given, derived along the lines of the work of chapter III. Also it is shown, as C. N. Moore has pointed out [Moore, 21] that $f_1(x)$ is not Borel summable at $x = 0$.

4.2. The analogues of the σ_n. To provide the set of continuous functions in E $h_u(t)$, $u \geq 0$, having the property that the u^{th} Borel mean of the Fourier (cosine) series of the u^{th} function becomes infinite with u at $t = 0$, we define

$$(4.21) \quad \begin{aligned} \psi_u(t) &\equiv \sin(u+\tfrac{1}{2})t, \quad 0 \leq t \leq \pi \\ \psi_u(t) &= \psi_u(-t), \quad -\pi \leq t \leq 0, \quad u \geq 0 \end{aligned}$$

Each is obviously continuous; each is obviously in E.

From (3.34) we get that the u^{th} Borel mean of $h_u(t)$ at 0 is

$$(4.22) \quad F_B(u) \equiv e^{-u}\beta_{\psi_u}(u;0) = \frac{1}{\pi}\int_0^{\pi} e^{-u(1-\cos t)} \frac{\sin(u+\frac{1}{2})t \sin(u\sin t + \frac{1}{2})t}{\sin\frac{1}{2}t}\, dt.$$

<u>4.3. A simpler integral whose difference from $F_B(u)$ is o(1).</u>

As in the discussion of $L_B(u)$, it is convenient to replace $F_B(u)$ by an integral simpler in character and yet whose difference from $F_B(u)$ is o(1) as u becomes infinite. This will make it easier to calculate the asymptotic value of $F_B(u)$.

The method is exactly the same as for $L_B(u)$. Indeed, many of the specific steps are here taken over.

First we have

<u>Theorem 4.31.</u> Let

$$(4.31) \quad F_{B,1}(u) \equiv \frac{1}{\pi}\int_0^{\pi} e^{-u(1-\cos t)} \frac{\sin^2(u+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt$$

Then

(4.32) $$\mathcal{F}_{B}(u) = \mathcal{F}_{B,1}(u) + o(1) \quad \text{as } u \to \infty.$$

Proof: Write

$$\mathcal{F}_{B,1}(u) - \mathcal{F}_{B}(u) = \frac{1}{\pi}\int_0^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left\{\sin^2(u+\tfrac{1}{2})t - \sin^2(u\sin t + \tfrac{1}{2}t)\right\} dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \sin(u+\tfrac{1}{2}t)\left\{\sin(u+\tfrac{1}{2})t - \sin(u\sin t + \tfrac{1}{2}t)\right\} dt.$$

But $|\sin x| \le 1$. Hence

(4.33) $$\left|\mathcal{F}_{B,1}(u) - \mathcal{F}_{B}(u)\right| \le \frac{1}{\pi}\int_0^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left|\sin(u+\tfrac{1}{2})t - \sin(u\sin t + \tfrac{1}{2}t)\right| dt.$$

From (3.58) and $|\cos x| \le 1$, it follows that

(4.34) $$\left\{\begin{aligned} \left|\mathcal{F}_{B,1}(u) - \mathcal{F}_{B}(u)\right| &\le \frac{2}{\pi}\int_0^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left|\cos\tfrac{1}{2}(ut + u\sin t + t)\sin\tfrac{1}{2}(t - \sin t)u\right| dt \\ &\le \frac{2}{\pi}\int_0^{\pi} \frac{e^{-u(1-\cos t)}}{\sin\frac{1}{2}t} \left|\sin\tfrac{1}{2}(t-\sin t)u\right| dt \\ &\equiv \frac{2}{\pi}\int_0^{\frac{1}{\sqrt[3]{u}}} + \frac{2}{\pi}\int_{\frac{1}{\sqrt[3]{u}}}^{\pi} \equiv I + II, \end{aligned}\right.$$

where I and II are the same as in (3.516).

But (3.534) states

(3.534) $\text{I} + \text{II} = o(1)$ as $u \to \infty$.

Combining this with (4.34) completes the proof of the theorem.

qed.

The next simplification is found in

Theorem 4.32. Let

(4.35) $$F_{B,2}(u) \equiv \frac{2}{\pi}\int_0^{\pi} e^{-\frac{1}{2}ut^2} \frac{\sin^2(u+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

Then

(4.36) $$F_B(u) = F_{B,2}(u) + o(1) \quad \text{as } u \to \infty$$

Proof: In view of (4.32), it is sufficient to prove

(4.37) $$F_{B,1}(u) = F_{B,2}(u) + o(1) \quad \text{as } u \to \infty$$

Now, from

(4.38) $$1 - \cos t = 2\sin^2 \tfrac{1}{2}t$$

we may write

(4.39) $$F_{B,1}(u) = \frac{2}{\pi}\int_0^{\pi} e^{-2u\sin^2\frac{1}{2}t} \frac{\sin^2(u+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

Now we consider

$$(4.310)\quad 0 \le F_{\beta,1}(u) - F_{\beta,2}(u) = \frac{1}{\pi}\int_0^{\pi}\left(e^{-2u\sin^2\frac{1}{2}t} - e^{-2u\frac{t^2}{4}}\right)\frac{\sin^2(u+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

But

$$(4.311)\quad \sin^2(u+\tfrac{1}{2})t \le |\sin(u+\tfrac{1}{2})t|$$

since $|\sin(u+\frac{1}{2})t| \le 1$.

Hence

$$(4.312)\quad 0 \le F_{\beta,1}(u) - F_{\beta,2}(u) \le \frac{1}{\pi}\int_0^{\pi}\left(e^{-2u\sin^2\frac{1}{2}t} - e^{-2u\frac{t^2}{4}}\right)\frac{|\sin(u+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt$$

From (3.539) this states

$$(4.313)\quad 0 \le F_{\beta,1}(u) - F_{\beta,2}(u) \le L_{\beta,1}(u) - L_{\beta,2}(u).$$

But, as theorem 3.52 established, $L_{\beta,1}(u) - L_{\beta,2}(u)$ is o(1) as u becomes infinite. This, applied to (4.313), proves theorem 4.32.

qed.

In $F_{\beta,2}(u)$, replace t by 2t; then (4.35) becomes

$$(4.314)\quad F_{\beta,2}(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-2ut^2}\frac{\sin^2(2u+1)t}{\sin t}\,dt.$$

We prove now

<u>Theorem 4.33.</u> Let

$$(4.315)\quad F_{\beta,3}(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-2ut^2}\frac{\sin^2(2u+1)t}{t}\,dt$$

Then

(4.316) $F_{\beta}(u) = F_{\beta,3}(u) + o(1)$ as $u \to \infty$.

Proof: In view of theorems 4.31 and 4.32, it is sufficient to show

(4.317) $F_{\beta,2}(u) \leq F_{\beta,3}(u) + o(1)$ as $u \to \infty$.

Now,

$$F_{\beta,2}(u) \equiv \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-2ut^2} \frac{\sin^2(2u+1)t}{t} dt$$

$$+ \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin t} - \frac{1}{t}\right) e^{-2ut^2} \sin^2(2u+1)t \, dt.$$

Noting that $|\sin(2u+1)t| \leq 1$, we have

$$\sin^2(2u+1)t \leq |\sin(2u+1)t|.$$

Hence, from $0 \leq \frac{1}{\sin t} - \frac{1}{t}$, $0 \leq t \leq \frac{\pi}{2}$,

(4.318) $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin t} - \frac{1}{t}\right) e^{-2ut^2} \sin^2(2u+1)t\,dt \leq \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin t} - \frac{1}{t}\right) e^{-2ut^2} |\sin(2u+1)t|\,dt$

The latter integral is o(1) as $u \to \infty$, from (3.556). Hence (4.317) and consequently also (4.316) are established.

qed.

4.4. On a function considered by Fejér. In his study of σ_n Fejér used

(4.41) $$\mathcal{J}_n \equiv \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2(2n+1)t}{t}\,dt.$$

Here we need

(4.42) $$\mathcal{J}(u) \equiv \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2(2u+1)t}{t}\,dt.$$

We have

$$\mathcal{J}(u) = \frac{2}{\pi}\int_0^{(u+1)\frac{\pi}{2u+1}} \frac{\sin^2(2u+1)t}{t}\,dt - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{(u+1)\frac{\pi}{2u+1}} \frac{\sin^2(2u+1)t}{t}\,dt$$

$$\equiv \mathcal{J}(u) \qquad\qquad + \delta(u).$$

We prove now

Theorem 4.41.

(4.44) $$\mathcal{J}(u) = \frac{4}{\pi}\log u + 2\int_0^1 \log\Gamma(t)\sin\frac{2}{\pi}t\,dt + o(1) \text{ as } u \to \infty,$$

Proof: We have

$$|\delta(u)| = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{(u+1)\frac{\pi}{2u+1}} \frac{\sin^2(2u+1)t}{t}\,dt \le \frac{2}{\pi}\int_{\frac{\pi}{2}}^{(u+1)\frac{\pi}{2u+1}} \frac{1}{t}\,dt$$

$$= \frac{2}{\pi}\left(\log \pi\frac{u+1}{2u+1} - \log\frac{\pi}{2}\right) = \frac{2}{\pi}\log 2\,\frac{u+1}{2u+1} = o(1) \text{ as } u \to \infty.$$

Hence

(4.45) $$\delta(u) = o(1) \text{ as } u \to \infty.$$

and

(4.46) $\mathcal{J}(u) = \mathcal{S}(u) + o(1)$ as $u \to \infty$.

Now

(4.47) $$\begin{cases} \mathcal{J}(u) = \frac{2}{\pi}\int_0^{(u+1)\frac{\pi}{2u+1}} \frac{\sin^2(2u+1)t}{t}\,dt \\ = \frac{2}{\pi}\int_0^{(u+1)\pi} \frac{\sin^2 t}{t}\,dt \end{cases}$$

Recalling that Fejér [Fejér, 5 , p. 36] showed

(4.48) $$\mathcal{J}_n = \frac{1}{\pi}\log n + 2\int_0^1 \log\Gamma(t)\sin 2\pi t\,dt + o(1)$$ as $u \to \infty$

we have, for $n = [u]$,

(4.49) $$\begin{cases} \mathcal{J}([u]) = \frac{1}{\pi}\log[u] + 2\int_0^1 \log\Gamma(t)\sin 2\pi t\,dt + o(1) \\ = \frac{1}{\pi}\log u + 2\int_0^1 \log\Gamma(t)\sin 2\pi t\,dt + o(1) \quad \text{as } u \to \infty \end{cases}$$

From (4.46) and (4.47) we therefore get

(4.410) $$\mathcal{J}([u]) = \frac{2}{\pi}\int_0^{([u]+1)\pi} \frac{\sin^2 t}{t}\,dt = \frac{1}{\pi}\log u + 2\int_0^1 \log\Gamma(t)\sin 2\pi t\,dt + o(1)$$

as $u \to \infty$.

But

(4.411) $$0 \le \mathcal{S}(u) - \mathcal{S}([u]) = \frac{2}{\pi}\int_{([u]+1)\pi}^{(u+1)\pi} \frac{\sin^2 t}{t}\,dt$$

$$\le \frac{2}{\pi}\int_{([u]+1)\pi}^{(u+1)\pi} \frac{|\sin t|}{t}\,dt$$

From (3.69) the latter integral is $o(1)$ as $u \to \infty$.

Hence

(4.412) $j(u) = j([u]) + o(1)$ as $u \to \infty$.

Combining this with (4.46) and (4.41) now gives (4.44).

qed.

4.5. Magnitude of $F_\beta(u)$. As we compared $L_\beta(u)$ with $L(u)$, so do we compare $F_\beta(u)$ with $\nu(u)$ in order to find an asymptotic estimate for $F_\beta(u)$.

Instead of studying

(4.51) $\nu(u) - F_\beta(u)$ as $u \to \infty$

we can, in view of theorem 4.33 (which states that $F_\beta(u)$ $= F_{\beta,3}(u) + o(1)$ as $u \to \infty$), consider

(4.52) $b(u) \equiv \nu_2(u) - F_\beta(u) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1-e^{-2ut^2}}{t} \sin^2(2u+1)t\,dt.$

A detailed study of $b(u)$ along lines similar to the study of $d(u)$ made in chapter III should result in obtaining at least the first term in the asymptotic representation of $F_\beta(u)$.

For our present purpose all that is needed, however, is the rough behavior of $F_\beta(u)$ as $u \to \infty$, its magnitude.

First we prove

Theorem 4.51: b -

(4.53) $b(u) \leq \frac{2}{\pi^2} \log u + O(1)$ as $u \to \infty$.

Proof: From $\sin^2(2u+1)t \leq |\sin(2u+1)t|$ and (3.72)

(4.54) $$b(u) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1 - e^{-2ut^2}}{t} \sin^2(2u+1)t \, dt$$

$$\leq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1 - e^{-2ut^2}}{t} |\sin(2u+1)t| \, dt = d(u)$$

But (3.718) states

(3.718) $d(u) = \frac{2}{\pi^2} \log u + O(1)$ as $u \to \infty$.

qed.

Now we give an asymptotic lower bound for $F_\beta(u)$ (defined in (4.22)).

Theorem 4.52:

(4.55) $F_\beta(u) > \frac{\pi - 2}{\pi^2} \log u + O(1)$ as $u \to \infty$.

Proof: From theorem 4.51 and (4.51) we have

(4.56) $b(u) = g(u) - F_\beta(u) \leq \frac{2}{\pi^2} \log u + O(1)$ as $u \to \infty$.

or

(4.57) $F_B(u) \geq \phi(u) - \frac{2}{\pi^2}\log u + O(1)$ as $u \to \infty$

But, from (4.44)

(4.58) $\phi(u) = \frac{1}{\pi}\log u + O(1)$ as $u \to \infty$.

Hence, substituting in (4.57) we get (4.55).

qed.

4.6. Proof that the Fourier series of $f_1(t)$ is not Borel summable at $t = 0$. C. N. Moore has remarked [C. N. Moore, 21] that $f_1(t)$ is not only an example of a continuous function having a divergent Fourier series at $t=0$, but also that it is an example of a continuous function whose Fourier series is not Borel summable at $t=0$. Here I fill in the details of the proof.

We recall that

(4.61) $$f_1(t) = \sum_{k=1}^{\infty} \frac{\sin(2^{k^3}+1)\frac{1}{2}t}{k^2} \equiv \sum_{k=1}^{\infty} \frac{\psi_{2^{k^3}}(t)}{k^2}$$

$f_1(t)$ is continuous, since it is the sum of a uniformly convergent series of continuous functions.

What we wish to show is that the Fourier cosine series of $f_1(t)$ is not Borel summable at $t = 0$; i.e., that the limit of

(4.62) $$e^{-u}\beta_{f_1}(u;0) = \frac{1}{2\pi}\int_0^{\pi} f_1(t)\, e^{-2u\sin^2\frac{1}{2}t}\, \frac{\sin(u\sin t + \frac{1}{2}t)}{\sin\frac{1}{2}t}\, dt.$$

as u becomes infinite, does not exist. Actually we shall show that the limit superior is $+\infty$.

In (4.62) after replacing $f_1(t)$ by (4.61), termwise integration is permissible, since $f_1(t)$ is represented by a uniformly convergent series and $e^{-2u\sin^2\frac{1}{2}t}\frac{\sin(u\sin t+\frac{1}{2}t)}{\sin\frac{1}{2}t}$ is integrable in $(0,\pi)$.

Hence we have

Theorem 4.61:

(4.63) $$e^{-u}\beta_{f_1}(u;0)=\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\pi}e^{-2u\sin^2\frac{1}{2}t}\sin(2^{k^3}+1)\tfrac{1}{2}t\,\frac{\sin(u\sin t+\frac{1}{2}t)}{\sin\frac{1}{2}t}\,dt$$

Next we need

Theorem 4.62: For all u,

(4.64) $$\int_0^{\pi}e^{-2u\sin^2\frac{1}{2}t}\sin(2^{k^3}+1)\tfrac{1}{2}t\,\frac{\sin(u\sin t+\frac{1}{2}t)}{\sin\frac{1}{2}t}\,dt>0,\ k=0,1,\ldots$$

i.e., each term in (4.63) is positive.

Proof: From (4.22) we have

(4.65) $$\pi e^{-u}\beta_{\psi_{2^{k^3}}}(u;0)=\int_0^{\pi}e^{-2u\sin^2\frac{1}{2}t}\sin(2^{k^3}+1)\tfrac{1}{2}t\,\frac{\sin(u\sin t+\frac{1}{2}t)}{\sin\frac{1}{2}t}\,dt$$

Also

(4.66) $$\pi e^{-u}\beta_{\psi_{2^{k^3}}}(u;0)=\pi e^{-u}\sum_{n=0}^{\infty}s_n(0)\frac{u^n}{n!},$$

where $s_n(0)$ is the n^{th} Fourier partial sum of $\Psi_{2^{k^3}}(t)$ at $t = 0$.

Now, Fejér proved (in the result quoted in §4.1) that each $s_n(0) > 0$, $n = 0, 1, \ldots$

Hence, from (4.66) and (4.65) we get (4.64).

qed.

Finally we have

Theorem 4.63:

(4.67) $\overline{\lim_{u \to \infty}} \; e^{-u} \beta_{f_1}(u; 0) = +\infty$

Proof: Consider the sequence

(4.68) $\frac{1}{2} \cdot 2^{1^3}, \frac{1}{2} \cdot 2^{2^3}, \ldots \frac{1}{2} \cdot 2^{m^3}, \ldots$

We shall prove that as y takes on the values in (4.68)

(4.69) $\lim_{y \to \infty} e^{-y} \beta_{f_1}(y; 0) = +\infty$

This will establish (4.67)

Let $m > 0$ be given, and take

(4.61) $y_1 = \frac{1}{2} \, 2^{([M]+1)^3}, \; y_j = \frac{1}{2} \, 2^{([M]+j)^3}, \ldots$

We now have, from (4.64), taking that term in the infinite series in (4.63) for which $k = [M]+j$,

(4.611) $e^{-y_j} \beta_{f_1}(y_j; 0) > \frac{1}{2\pi} \frac{1}{\{[M]+j\}^2} \int_0^{\pi} e^{-2y_j \sin^2 \frac{1}{2}t} \sin(y_j+1)\tfrac{1}{2}t \, \frac{\sin(y_j \sin t + \frac{1}{2}t)}{\sin \frac{1}{2}t} \, dt$

From (4.55), for all large j say, $j > j_0$, we have for the integral

$$\int_0^{\pi} e^{-2y_j \sin^2 \frac{1}{2}t} \sin (y_j+1)\tfrac{1}{2}t \, \frac{\sin(y_j \sin t + \frac{1}{2}t)}{\sin \frac{1}{2}t} \, dt$$

$$> \frac{1}{2} \, \frac{\pi - 2}{\pi^2} \log y_j = \frac{1}{2} \frac{\pi-2}{\pi^2} \log \frac{1}{2} + \left(\frac{1}{2} \, \frac{\pi-2}{\pi^2} \log 2\right)([M]+j)^3$$

From (4.611) we thus get, for $j \geq j_0$

$$e^{-y_j} \beta_{f_1}(y_j; 0) > \left(\frac{1}{4\pi} \, \frac{\pi-2}{\pi^2} \log \frac{1}{2}\right)\left(\frac{1}{([M]+j)}\right)^2$$

$$+ \left(\frac{1}{4\pi} \, \frac{\pi-2}{\pi^2} \log 2\right)([M]+j) > M$$

Hence, for $y \geq y_{j_0}$,

$$(4.613) \qquad e^{-y} \beta_{f_1}(y; 0) > M .$$

This proves (4.69) and hence also (4.67).

qed.

CHAPTER V

THE BOREL SUMMABILITY OF FOURIER SERIES AT A POINT.

5.1. Introduction.

The objective of this chapter is to establish a criterion for the Borel summability of Fourier series at a point. One such criterion is due to Hardy and Littlewood [G. H. Hardy and J. E. Littlewood, 16], who used it, together with a Tauberian theorem, to obtain convergence criteria. Their summability criterion states:

If $\varphi_x(t) \equiv f(x+2t) + f(x-2t) - 2f(x) = o\left[\left(\log \frac{1}{t}\right)^{-1}\right]$ as $t \to 0$, then the Fourier series of $f(x)$ is Borel summable to $f(x)$ at x.

We recall here that C. N. Moore [C. N. Moore, 21] showed that even for continuous functions Borel summability need not imply convergence.

5.2. The Criterion.

Theorem 5.21: If, for a fixed x, and a corresponding $S(x)$,

$$(5.21) \quad \frac{1}{h}\int_0^h |f(x+2t)+f(x-2t)-2S(x)|\,dt = o(\sqrt{h}) \text{ as } h \to 0,$$

then the Fourier series of $f(x)$ is Borel summable to $S(x)$ at the point x.

Proof: Denote, as usual, the $(C, 1)$ means of the Fourier series of $f(x)$ by $\sigma_m(x)$ and the partial sums of the Fourier series by $s_m(x)$.

With this notation in mind, we appeal to a theorem of O. Szász [O. Szász, 24, v. 48(1926), pp. 353-362] in order to make use of (5.21). His theorem states:

"If, corresponding to a positive $\alpha < 1$ and to an x-value, there exist two quantities $s = s(x)$, $g_\alpha = g_\alpha(x)$, such that

$$\lim_{h \to +0} \frac{1}{h^{1+\alpha}} \int_0^h \left| f(x+2t) + f(x-2t) - 2s - g_\alpha \sin^\alpha t \right| dt = 0$$

then

$$\lim_{n\to\infty} n^\alpha \left[\sigma_n(x) - s(x)\right] = \frac{\gamma_\alpha}{\pi} g_\alpha(x);$$

where

$$\gamma_\alpha = \lim_{n\to\infty} n^{\alpha-1} \int_0^{\frac{\pi}{2}} \left(\frac{\sin nt}{\sin t}\right)^2 dt = \int_0^\infty \frac{\sin^n t}{t^{2-\alpha}} dt$$
$$= \frac{\Gamma(\alpha) \sin \frac{\alpha\pi}{2}}{(1-\alpha) 2^\alpha}$$

In this case $\alpha = \frac{1}{2}$, $g_\alpha(x) \equiv 0$. Hence, from Szász's theorem, we get

$$(5.22) \quad \lim_{n\to\infty} \sqrt{n} \left[\sigma_n(x) - S(x)\right] = 0$$

A theorem of Hardy now permits us to conclude our theorem from (5.22). His theorem [G. H. Hardy, 10, v.35 (1904), pp. 22-67; p. 40] reads as follows:

"If

$$\left(\sigma_n =\right) \quad \frac{S_0 + S_1 + \cdots\cdots + S_n}{n+1} = s + \rho_n,$$

and

$$\lim_{n=\infty} \rho_n \sqrt{n} = 0,$$

then

$$\lim_{x=\infty} e^{-x} \sum_{(?)}^{\infty} \frac{s_n x^n}{n!} = s.$$

Now, (5.22) is just another way of writing the hypothesis of Hardy's theorem. Hence, the conclusion follows.

5.3. Some further problems. Several problems, or rather types of problems, suggest themselves immediately. In the first place, it is desirable to exhibit a specific function satisfying (5.21) with $S(x) \neq f(x)$, whose Fourier series is divergent. This would then furnish an example of a Borel summable Fourier series which is not Borel summable to the function to which the Fourier series corresponds.

Another type of problem is this: We know(as C. N. Moore first showed) that a continuous function need not have an everywhere Borel-summable Fourier series. Perhaps, however, the Fourier series of a continuous function would have to be Borel summable almost everywhere. If a continuous function satisfies (5.21) almost everywhere, then by theorem 5.21 the Fourier series is Borel summable almost everywhere. Is (5.21) satisfied almost everywhere? And, if not almost everywhere, would a continuous function satisfy (5.21) in a set of positive measure? What classes of functions would have such properties?

CHAPTER VI

A UNIQUENESS THEOREM

6.1 Introduction. Not every trigonometric series is a Fourier series. Conditions under which they are, have been extensively investigated, and are described in Hobson's and Zygmund's books.

The object here is to give another such theorem.

6.2. The Theorem.

Theorem 6.21. If

(6.21) $$a_n = O(1) \quad \text{and} \quad b_n = O(1) \quad \text{as } n \to \infty$$

and if $\frac{1}{2} a_0 + \sum$

(6.22) $$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is Borel summable to a finite Lebesgue integrable function f(x) everywhere in $(-\pi, \pi)$, then (6.22) is a Fourier series.

Proof: According to Hardy and Littlewood [Hardy and Littlewood, 15, v. 41(1916), pp. 36-54; p.54] we have:

"If $\sum a_n$ is summable (B) and $a_n = O(1)$, then $\sum a_n$ is summable (C,1)."

Applying that result here, as assumption (6.21) together with the assumption of Borel summability permits, we learn that (6.22) is (C,1) summable everywhere in $(-\pi, \pi)$. Moreover, it is known that whenever a series is summable both by Borel's method and by (C,1) means, then the sums are the same. (This is what is called the consistency of the two methods with one another.)

Hence (6.22) is (C,1) - summable to f(x) everywhere in $(-\pi,\pi)$.

To complete the proof, the following theorem is needed [Hobson, 17 , v. II, p. 681]:

"If $\frac{a_n}{n^k}, \frac{b_n}{n^k}$ are bounded, where k is some number such that $0 \leq k < 1$, and if the series $\sum(a_n \cos nx + b_n \sin nx)$ is such that its upper and lower Cesàro sums of integral order r are finite at each point of $(-\pi,\pi)$, and summable (i.e., Lebesgue integrable) in that interval, then the series is a Fourier's series."

In our case $k = 0$, $r = 1$ and the upper and lower (C, 1) sums are the same and finite. Therefore, since f(x) is Lebesgue integrable, the conclusion that (6.22) is a ~~Four~~ Fourier series follows.

qed.

CHAPTER VII

AN APPLICATION OF BOREL SUMMABILITY TO POWER SERIES.

7.1 Introduction and generalities.

Borel summability has been extensively applied to the study of power series. It is particularly useful as a method of analytic continuation, since its sums power series beyond their circle of convergence except where the circle of convergence is a natural boundary. For a complete discussion of this situation, the read will do well to consult Borel's own exposition [Borel, 1 , second edition, 1928].

Here let us recall a few results. Given a power series

$$(7.11) \qquad \sum_{n=0}^{\infty} a_n z^n$$

having 0 as the center of its circle of convergence. The circle itself may of course consist solely of that point, as is the case if $a_n = n!$; it may have a finite radius of convergence, as is the case if $a_n = 1$; it may converge everywhere, as is the case if $a_n = \frac{1}{n!}$

Here we are concerned with the case in which (7.11) has a finite radius of convergence, not zero.

What, then, is the domain of Borel summability of (7.11)? Referring to chapter IV of Borel's book, we learn that it is the inside of the so-called <u>polygon of summability</u>

(sometimes called the Borel polygon), constructed as follows:

In the present case (7.11) defines an analytic function f(z), which is regular inside the circle of convergence of (7.11) and has at least one singularity on that circle.

Now, connect 0 with each singularity of f(z) by a straight line. Then erect a perpendicular to each of these lines at the respective singularities. To each singularity corresponds a half-plane containing the circle of convergence. The common-part (or product or cross-section) of all these half-planes is called the "polygon of summability" or the "Borel polygon", (although it obviously need not be a polygon in the strict sense of the term: if f(z) has only one singularity, it would be a half-plane; if f(z) has the circle of convergence as a natural boundary, the "Borel polygon" would be simply the circle of convergence.)

From this discussion one clear fact emerges: The power series (7.11) cannot be summed by Borel's means at points beyond a singular point on the radius vector connecting that singular point with the origin. The converse is also true:

Theorem 7.11. If the power series (7.11) cannot be summed by Borel's means at any point lying on the prolongation of a radius vector beyond a given point A on the circle of convergence, then A is a singular point of the function defined by the power series (7.11).

Proof:

Assume that A is a regular point. Then there is a neighborhood of A (a complete circle around A) containing only regular points of the function. At each such point, the power series is Borel summable. But among these points are points on the prolongation of the radius vector to A beyond A. At these points (7.11) was assumed to be not Borel summable. This is a contradiction. Hence A is a singular point.

qed.

Thus, one way of proving that a power series is singular at a given point on the boundary of the circle of convergence is to prove that it is nowhere Borel summable along the radius vector beyond that point. (Note that nothing is said regarding Borel summability at the point. A power series could even converge at a singular point on the circle fof convergence, as does

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} \text{ at } z = 1)$$

This idea was used by G. N. Watson [G. N. Watson, 31, v. 42 (1911), pp. 41-54] to prove that a certain function has its circle of convergence as a natural boundary.

As a matter of fact, Watson used what is known as LeRoy's extension of Borel's method. But, since theorem 7.11 is also valid for that extension [Cf. Watson, 31 ,] the idea is essentially the same.

Here the possibilities of theorem 7.11 are illustrated by giving a slight extension of the Pringsheim-Vivanti theorem (infra).

7.2. On the Pringsheim-Vivanti theorem. The object of this section is to give a new proof of the Pringsheim-Vivanti theorem (using theorem 7.11) and, in so doing, to extend that theorem slightly.

Theorem 7.21. Suppose (i) that the power series (7.11) has the unit circle for its circle of convergence. Let

(7.21) a_n be real $n=0,1,2,\ldots$ (ii) $\epsilon > 0$ and

(7.22) $$s_n(\epsilon) \equiv \sum_{k=0}^{n} a_k (1+\epsilon)$$

Assume (iii)

(7.23) $$\lim_{n\to\infty} s_n(\epsilon) = +\infty \text{ for each } \epsilon > 0.$$

Then the point 1 is a singular point of the power series (7.11).

Proof:

Let $m > 0$ be given. From (7.23) we learn the existence of $N > 0$, such that for a given ϵ

(7.24) $$s_n(\epsilon) > M \quad \text{for } M \geq N = N(M,\epsilon).$$

Hence $$e^{-x}\sum_{n=0}^{\infty}\frac{s_n(\epsilon)}{n!}x^n = e^{-x}\sum_{n=0}^{}\frac{s_n(\epsilon)}{n!}x^n + e^{-x}\sum_{n=N}^{\infty}\frac{s_n(\epsilon)}{n!}x^n$$

$$> e^{-x}\sum_{n=0}^{N-1}\frac{s_n(\epsilon)}{n!}x^n + e^{-x}M\sum_{n=N}^{\infty}\frac{x^n}{n!}$$

$$= e^{-x}\sum_{n=0}^{n-1}\frac{s_n(\epsilon)}{n!}x^n + e^{-x}M\sum_{n=0}^{\infty}\frac{x^n}{n!} - e^{-x}M\sum_{n=0}^{N-1}\frac{x^n}{n!}$$

$$= M + e^{-x}\sum_{n=0}^{N-1}\frac{s_n(\epsilon)}{n!}x^n - e^{-x}M\sum_{n=0}^{N-1}\frac{x^n}{n!}$$

Now, both $\sum_{n=0}^{N-1}\frac{s_n(\epsilon)}{n!}$ and $\sum_{n=0}^{N-1}\frac{x^n}{n!}$ are polynomials in x and, hence, are $o(e^x)$ as x becomes infinite. Therefore

(7.25) $$\lim_{x\to\infty} e^{-x}\sum_{n=0}^{\infty}\frac{s_n(\epsilon)}{n!}x^n \geq M \text{ for any } \epsilon > 0.$$

Appealing now to theorem 7.11 completes the proof that 1 is a singularity.

As a consequence of this, we get a new proof of the Pringsheim-Vivanti theorem [A. Pringsheim, 21, v. 44 (1894), pp. 41.56]:

<u>Theorem 7.22.</u> Let (i) the power series (7.11) have radius of convergence 1; let (ii) a be real and non-negative. Then 1 is a singular point of (7.11).

<u>**Proof:**</u> For any $\epsilon > 0$, $s_n(\epsilon)$ is divergent as n becomes in-

finite, since (7.11) has 1 for its radius of convergence. From assumption (ii) it follows that $s_n(\epsilon)$ is real and positive. Hence (7.22) is satisfied, in view of the divergence of $s_n(\epsilon)$ as n becomes infinite.

Theorem 7.21 consequently applies and the point 1 is therefore a singularity of (7.11).

qed.

CHAPTER VIII

ON STRONG BOREL SUMMABILITY

8.1. Introduction. If $\{s_n\}$ is Borel summable to s, it need not follow that $\{s_{n+1}\}$ is Borel summable. This was shown by Hardy [cf. Knapp, 19 , pp. 488 and 494]. V. Garten proved that if, in addition, it be assumed that $s_n = o$ as $n \to \infty$ for some $k > 0$, that it will then follow that $\{s_{n+1}\}$ will also be Borel summable to s [Garten, 8]

Here I prove that strong Borel summability of $\{s_n\}$ to s implies that of $\{s_{n+1}\}$ without further restriction (sections 8.2, 8.3, 8.4).

Section 8.5 is devoted to showing that convergence in the mean implies that $\{s_n(x)$ satisfies a necessary condition for strong Borel summability almost everywhere and in applying this result to Fourier series.

In section 8.6, a Tauberian-type theorem is proved.

8.2. Definition of strong Borel summability.

Definition 8.21. If $p > 0$ and if $\lim_{t \to \infty} e^{-t} \sum_{n=0}^{\infty} |s_n - s|^p \frac{t^n}{n!} = 0$ then $\{s_n\}$ is said to be strongly Borel summable to s with index p. [cf. Szász, 25 , p. 113].

8.3. Two lemmas. Here are proved two lemmas, of which the second is a corollary of the first. Actually it is only the second which is needed for the proof of the main theorem, but it is so special a case of the first that it appears worthwhile to give both.

Lemma 8.31. If

(8.31) $$G'(t) > 0;$$

if

(8.32) $G'(t)$ is monotonically increasing with t

and if

(8.33) $$G(t+1) - G(t) = o(e^t) \text{ as } t \to \infty,$$

then

(8.34) $$G'(t) = o(e^t) \text{ as } t \to \infty.$$

Proof: From the mean-value theorem of differential calculus we learn the existence of a number y, $t \leq y \leq t+1$, such that

(8.35) $$G'(y) = G(t+1) - G(t).$$

Hence

(8.36) $$\frac{G'(y)}{e^t} = \frac{G(t+1) - G(t)}{e^t}$$

From (8.31) and (8.32) it now follows that

(8.37) $$0 < \frac{G'(t)}{e^t} \leq \frac{G'(y)}{e^t} = \frac{G(t+1) - G(t)}{e^t}$$

since

Letting t become infinite, (8.34) now follows from (8.37) and (8.33).

qed.

Lemma 8.32. If (8.31) and (8.32) are satisfied, and if

(8.38) $G(t) = o(e^t)$ as $t \to \infty$

then (8.34) holds.

Proof: We have

$$\frac{G(t+1)-G(t)}{e^t} = e\,\frac{G(t+1)}{e^{t+1}} - \frac{G(t)}{e^t}$$

Hence (8.33) follows from (8.38). All the conditions of lemma 8.31 are therefore satisfied (the others having been assumed) and (8.34) holds.

qed.

8.4. On shifting the index. Lemma 8.32 is now used to prove

Theorem 8.41. If $\{s_n\}$ is strongly Borel summable to s, with index p > 0, then so too is $\{s_{n+1}\}$.

Proof: Define

(8.41) $G(t) \equiv \sum_{n=0}^{\infty} |s_n - s|^p \frac{t^n}{n!}$,

Then, from the definition of strong Borel summability, (8.38) follows. Moreover,

(8.42) $G'(t) = \sum_{n=1}^{\infty} |s_n - s|^p \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} |s_{n+1} - s|^p \frac{t^n}{n!}$.

Obviously $G'(t) > 0$ and is monotonically increasing with t. These are conditions (8.31) and (8.32). Lemma 8.32 now applies, since (8.38) has been shown to hold. (8.34) then follows. But, from (8.42), this is simply the definition of strong Borel summability of $\{s_{n+1}\}$ to s, with index p.

qed.

8.5. On a necessary condition for strong Borel summability.

Here I show that certain types of sequences of functions and some classes of Fourier series satisfy a necessary condition for strong Borel summability. It is not excluded that the sequences and series in question may actually be strongly Borel summable or at least Borel summable. It would be very interesting to establish either result or to give examples to the corollary.

First we have

Theorem 8.51. A necessary condition for the strong Borel summability of $\{s_n\}$ to s with index p, is that

(8.51) $\sum_{n=0}^{\infty} |s_n - s|^p \frac{t^n}{n!}$ is an entire function of t.

Proof: Obvious from definition 8.21.

We now recall

Definition 8.51. If $p > 0$, if the functions $s_n(x)$, $n = 0,1,\ldots$ $s(x)$ are in $L_p(a, b)$, and if

(8.52) $$\lim_{n\to\infty} \int_a^b |s_n(x) - s(x)|^p dx = 0$$

then $s_n(x)$ is said to converge in the mean of order p to

$s(x)$.

Before coming to the theorem referred to in §8.1, (theorem 8.52) we need an important result due to Fubini [cf. Hobson, 17, v.II, p. 335] is used:

"If all the functions of the convergent series $\sum_{n=1} u_n(x)$ are monotone non-diminishing, or all are monotone non-increasing, and the series converges in (a, b) to $s(x)$, then $s'(x)$ exists and is the sum-function of $\sum_{n=1} u'_n(x)$ almost everywhere* in (a, b)."

We proceed now to

Theorem 8.52. If $s(x)$, $s_n(x)$, n = 0, 1, 2,..., are in $L_p(a,b)$, $p > 0$, and if $s_n(x)$ converges to $s(x)$ in the mean of order p, then $\{s_n(x)\}$ satisfies the necessary condition (8.51) for strong Borel summability to $s(x)$ with index p, for almost all x in (a, b).

Proof: From the assumption on convergence in the mean, it follows that

(8.53) $$\int_a^x |s_n(y) - s(y)|^p dy = o(1) \text{ as } n \to \infty.$$

Hence, from the regularity of Borel's method,

(8.54) $$\sum_{n=0}^{\infty} \int_a^x |s_n(y) - s(y)|^p dy \frac{t^n}{n!} = o(e^t) \text{ as } t \to \infty.$$

---

* The "almost everywhere" refers also to the domain of existence of (x).

It now follows from theorem 8.51 that

$$(8.55)\quad \sum_{n=0}^{\infty} \int_a^x |s_n(y) - s(y)|^p dy \frac{t^n}{n!}$$

is an entire function of t for all x in (a, b).

But

$$\int_a^x |s_n(y) - s(y)|^p dy$$

is obviously an increasing function of x in (a, b). Theorem 8.52 now follows from Fubini's theorem.

qed.

Applying this theorem to Fourier series, we shall prove the corollaries below.

Corollary 8.52.1. If f(x) is in L_p, $p > 1$, and if $s_n(x)$ denotes the n^{th} partial sum of the Fourier series of f(x), then

(8.56) $\sum_{n=0}^{\infty} |s_n(x) - f(x)|^p \frac{t^n}{n!}$ is an entire function of t for almost all x in $(-\pi, \pi)$.

Proof: Under the hypotheses of the corollary, $s_n(x)$ converges to f(x) in the mean of order p. [cf. Zygmund, 33, p.153]. The result then follows from theorem 8.52.

Corollary 8.52.2. If f(x) is Lebesgue integrable, then (8.56) is satisfied for every $0 \le p \le 1$.

Proof: Under the hypotheses of the corollary, $s_n(x)$ converges in the mean of order p for every $0 < p < 1$ to $f(x)$ [cf. Zygmund, 31 , p. 153]. The result then follows for theorem 8.52.

Corollary 8.52.3. If $|f(x)| \log^+ |f(x)|$ is Lebesgue integrable, then

(8.57) $\sum_{n=0}^{\infty} |s_n(x) - f(x)| \frac{t^n}{n!}$ is an entire function of t for almost all x in $(-\pi, \pi)$.

Proof: Under the hypotheses of the corollary (x) converges in the mean of order 1 to f(x) [cf. Zygmund, 31 , p. 153]. The result then follows from theorem 8.52.

8.6. On a Tauberian theorem. Here I derive a Tauberian result by combining the results of the previous section with a Tauberian theorem due to Hardy. First we need

Definition 8.61. A series

$$(8.61) \quad \sum_{n=0}^{\infty} a_n$$

is said to be summable (B, α) to sum s, if (i) the series

$$(8.62) \quad A_\alpha(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)}$$

is convergent for all t, and (ii) the integral

(8.63) $\int_0^\infty e^{-t} A_\alpha(t)dt = \lim_{T\to\infty} \int_0^T e^{-t} A_\alpha(t)\,dt$

converges to A.

[cf. Hardy, 12]

First, what is the connection between (B,α) summability and the type of Borel summability we have thus far considered?

For $\alpha = 1$ we have (B, 1) summability which is

$$\int_0^\infty e^{-t} \sum_{n=0}^\infty a_n \frac{t^n}{n!}\,dt$$

which we call Borel's integral method. If $a_n \to 0$ and in particular for Fourier series, then it is equivalent to Borel's exponential method [cf. Hardy, 13].

Now let us state the theorem of Hardy in question [Hardy, 12]:

"If (i) $\beta > \alpha > 0$, (ii) the series (8.61) is summable (B,β), and (iii) the series (8.62) is convergent for all t, then the series (8.61) is summable (B,α)."

Restricting ourselves to Fourier series, we may write Hardy's theorem for the case $\alpha \geq 1$ as follows:

<u>Theorem 8.61</u>: If (i) $\beta > \alpha > 1$, (ii) the Fourier series

(8.64) $\frac{1}{2}a_0 + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx)$

is summable (B, β) at the point x, and (iii) the series

(8.65) $A(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \frac{t^n}{n!}$

is convergent for all t, then the series (8.64) is summable at the point x by Borel's exponential method.

Before proceeding to the proof of the next theorem, we state a standard lemma [cf. Szász, 25, p. 110]:

"If $\sum_0^{\infty} u_\nu \frac{t^\nu}{\nu!}$ converges for all $t > 0$, then $\sum_0^{\infty} s_\nu \frac{t^\nu}{\nu!}$ converges for all $t > 0$, and conversely."

We now prove the following Tauberian result:

Theorem 8.62: If $|f(x)| \log^+ |f(x)|$ is Lebesgue integrable, f(x) has the Fourier series (8.64), if (8.64) is summable (B, β), $\beta > 1$, for all x, then (8.64) is Borel summable (by exponential means) for almost all x.

Proof: Under the assumptions of the theorem we learn from corollary 8.52.3 that, for almost all x,

$$\sum_{n=0}^{\infty} |s_n(x) - f(x)| \frac{t^n}{n!}$$

is an entire function of t. Hence, for these values of x,

$$\sum_{n=0}^{\infty} \{s_n(x) - f(x)\} \frac{t^n}{n!}$$

and consequently also $\sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}$

converges for all t.

From the lemma quoted just prior to the theorem, it now follows that (8.65) is convergent for all $t > 0$, for almost all x.

Theorem 8.61 is now applicable, since it was assumed that (8.64) is summable (B,β) $\beta > 1$. Hence for all x for which (8.65) is convergent for all t we have that (8.64) is Borel summable. But, as we have just seen, this is true for almost all x.

qed.

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