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I hereby recommend that the thesis prepared under my supervision by PHILIP SCHWED

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TOPICS IN FIELD THEORY

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TABLE OF CONTENTS

I AUXILIARY CONDITIONS IN GENERALIZED ELECTRODYNAMICS

1. Introduction:

The problem here considered is that of properly generalizing the auxiliary conditions of electrodynamics to fit the generalized electrodynamics proposed by Podolsky[1]. A treatment has previously been given by Pololsky and Kikuchi[2] following somewhat the lines suggested by Stuckelberg[3], but the auxiliary condition given there is unsatisfactory since it does not necessarily subsist forever if satisfied at some particular instant. Our object here is to avoid this difficulty while obtaining a theory in which the main results previously obtained[1,2,4,5,6] are undisturbed. We shall be concerned mainly with the case of charge-free space. The general procedure will be to consider the non-quantum case first and then pass to the quantized theory in the usual way.

2. Need For Auxiliary Conditions in Ordinary Electrodynamics:

In the ordinary electrodynamics, the Lagrangian density* in heaviside units is:

$$L_f = -\frac{1}{4} F_{\alpha\beta}^2 \tag{2.1}$$

where

$$F_{\alpha\beta} = \phi_{\beta,\alpha} - \phi_{\alpha,\beta} \tag{2.2}$$

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*Greek indices run from 1 to 4, Latin from 1 to 3. $x_4 = ict$; $\phi_4 = i\phi$. The summation convention is employed.

$$\phi_{\lambda,\nu} = \frac{\partial \phi_\lambda}{\partial x_\nu} \tag{2.3}$$

This leads to the equation of motion

$$F_{\alpha,\beta\beta} = 0 \tag{2.4}$$

Because ϕ_4 is missing from (2.1), $p_4 = \frac{\partial L}{\partial \phi_4} = 0$ which relation would prove inconsistent with the commutation rules, if we attempted to go to the quantum mechanical form. To avoid this difficulty, we take for the Langrangian density

$$L_f = -\frac{1}{2}\phi_{\alpha,\beta}\,\phi_{\alpha,\beta} \tag{2.5}$$

This leads to the well known wave equation,

$$\Box\,\phi_\alpha = 0 \tag{2.6}$$

Upon writing (2.4) in terms of potentials, we find it to be equivalent to

$$\phi_{\beta,\beta\alpha} - \phi_{\alpha,\beta\beta} = 0 \tag{2.7}$$

Thus (2.4) and (2.6) are equivalent if

$$\phi_{\beta,\beta\alpha} = 0 \tag{2.8}$$

or

$$\phi_{\beta,\beta} = + \text{Const.} \tag{2.9}$$

It is convenient to choose 0 for this constant, thus obtaining the usual Lorentz auxiliary condition

$$\phi_{\beta,\beta} = 0 \tag{2.10}$$

In quantum theory, this equation is in general inconsistent with the commutation rules and it is therefore replaced by

$$\phi_{\beta,\beta}.\Psi = 0 \tag{2.11}$$

From the commutation rules it follows that this condition persists if satisfied at any instant.

3. Need for Auxiliary Conditions in Generalized Electrodynamics:

In generalized electrodynamics, the Lagrangian density analogous to (2.1) is

$$L_f = -\frac{1}{2}\left[\frac{F^2_{\alpha\beta}}{2} + a^2 F^2_{\alpha,\beta,\beta}\right] \qquad (3.1)$$

which leads to the equations of motion

$$(1 - a^2\Box)F_{\alpha\beta,\beta} = 0 \qquad (3.2)$$

To express this theory in Hamiltonian form, it is necessary that $\frac{\partial L_f}{\partial \ddot{\phi}_4}$ not be zero in general[2]. However, for (3.1) we have

$\frac{\partial L_f}{\partial \ddot{\phi}_4} = 0$ and therefore we replace (3.1) by

$$L_f = -\frac{1}{2}\left[\Phi_{\alpha,\beta}\,\Phi_{\alpha,\beta} + a^2\Phi_{\alpha,\beta\beta}\,\Phi_{\alpha,\gamma\gamma}\right] \qquad (3.1')$$

which leads to the equation of motion

$$(1 - a^2\Box)\Box\Phi_\alpha = 0 \qquad (3.3)$$

The general solution of (3.3) may be expressed in terms of Fourier amplitudes as

$$\Phi_\alpha(r,t) = \left(\frac{1}{2\pi}\right)^{3/2}\int\Big\{\Phi_\alpha(\vec{k})e^{-if} + \Phi^*_\alpha(k)e^{if}$$

$$+\tilde{\Phi}_\alpha(\vec{k})e^{-i\tilde{f}} + \tilde{\Phi}^*_\alpha(\vec{k})e^{i\tilde{f}}\Big\}\,d\vec{k} \qquad (3.4)$$

where

$$f = -k_\alpha x_\alpha$$

$$\tilde{f} = -\tilde{k}_\alpha x_\alpha$$

$$\tilde{k}_i = k_i$$
$$k_4 = ik$$
$$\tilde{k}_4 = i\tilde{k}$$
$$\tilde{k} = \left(k^2 + \frac{1}{a^2}\right)^{1/2} \tag{3.5}$$

The Hamiltonian of the system in terms of these amplitudes is then

$$\bar{H}_f = 2\int\left\{\Phi_\alpha^* k_\alpha k^2 - \tilde{\Phi}_\alpha^* \tilde{\Phi}_\alpha \tilde{k}^2\right\} d\vec{k} \tag{3.6}$$

This will prove of importance in quantization.

If now we express (3.2) in terms of potentials, we have

$$(1 - a^2\Box)\left[\Phi_{\beta,\alpha\beta} - \Phi_{\alpha,\beta\beta}\right] = 0 \tag{3.7}$$

We thus see that the necessary and sufficient condition that (3.2) and (3.3) be equivalent is

$$(1 - a^2\Box)\Phi_{\alpha,\beta\alpha} = 0 \tag{3.8}$$

We take, however, the more restrictive condition

$$(1 - a^2\Box)\Phi_{\alpha,\alpha} = 0 \tag{3.9}$$

which is sufficient but not necessary.

4. Possible Auxiliary Conditions:

There are a number of conditions which imply equation (3.9) while reducing to equation (2.10) in the case $a = 0$. We shall consider four of these, which, although only slightly different in appearance, lead to rather different results.

Choice 1

The most obvious choice is equation (3.9) itself:

$$(1 - a^2\Box)\Phi_{\beta,\beta} = 0 \tag{4.1}$$

This condition imposes no restriction on the tilde quantities if we write it in terms of Fourier amplitudes under the assumption that coefficients of different exponentials must vanish. Explicitly it leads to:

$$k_\alpha \Phi_\alpha(\vec{k}) = 0$$
$$k_\alpha \Phi^*_\alpha(\vec{k}) = 0 \qquad (4.2)$$

Since this set places no restriction upon the tilde quantities, it is quite different from that used in reference 2 and fails to remove the infinities from the expression for static interaction. It therefore will not be considered further.

Choice 2

We may take the condition to be the same as (2.9), namely:

$$\Phi_{\beta,\beta} = 0 \qquad (4.3)$$

This choice is considerably more restrictive than (4.1). In terms of Fourier amplitudes it becomes:

$$k_\alpha \Phi_\alpha(\vec{k}) = 0$$
$$k_\alpha I^*_\alpha(\vec{k}) = 0$$
$$\tilde{k}_\alpha \tilde{I}_\alpha(\vec{k}) = 0$$
$$\tilde{k}_\alpha \tilde{\Phi}^*_\alpha(\vec{k}) = 0 \qquad (4.4)$$

It may be remarked that the condition (4.3) implies that the present theory be a generalization of the ordinary meson theory in the following sense. The usual meson equation may be written in the form

$$a^2 F_{\alpha\beta,\beta} + \Phi_\alpha = 0 \tag{4.5}$$

Now let us take as a generalization of (4.5) the equation

$$\Box(a^2 F_{\alpha\beta,\beta} + \Phi_\alpha) = 0 \tag{4.6}$$

If (4.6) is written in terms of potentials, it becomes

$$\Box(a^2 \Phi_{\beta,\alpha\beta} - a^2 \Phi_{\alpha,\beta\beta} + \Phi_\alpha) = 0 \tag{4.7}$$

Application of (4.3) reduces equation (4.7) to equation (3.3) as asserted. If we desired merely to demand the equivalence of (4.5) and (3.3) we would arrive at the condition

$$\Box \Phi_{\beta,\beta,} = 0 \tag{4.8}$$

However, (4.8) and (3.9) together imply (4.3)

<u>Choice 3</u>

It is possible to subject Φ_α to the gage transformation,

$$\Phi_\alpha = \phi_\alpha + a\frac{\partial B}{\partial x_\alpha} \tag{4.9}$$

where B satisfies the equation,

$$(1 - a^2\Box)B = 0 \tag{4.10}$$

and can therefore be written,

$$B(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2}\int\left\{\tilde{B}(\vec{k})e^{-i\tilde{f}} + \tilde{B}^*(\vec{k})e^{i\tilde{f}}\right\}d\vec{k} \tag{4.11}$$

Under the transformation (4.9), equation (4.3) becomes,

$$\phi_{\beta,\beta} + a\Box B = 0 \tag{4.12}$$

which becomes with the aid of (4.10)

$$a\phi_{\beta,\beta} + B = 0 \tag{4.13}$$

In terms of Fourier Amplitudes, (4.13) is

$$\begin{aligned} k_\alpha\phi_\alpha(\vec{k}) &= 0 & \quad ia\tilde{k}_\alpha\tilde{\phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k}) &= 0 \\ k_\alpha\phi^*_\alpha(\vec{k}) &= 0 & \quad -ia\tilde{k}_\alpha\tilde{\phi}^*_\alpha(\vec{k}) + B(\vec{k}) &= 0 \end{aligned} \tag{4.14}$$

It would have been possible, of course, to require merely that B satisfy

$$(1 - a^2\Box)\Box B = 0 \tag{4.15}$$

so that we could write

$$B(r,t) = \left(\frac{1}{2\pi}\right)^{3/2}\int\left\{B(\vec{k})e^{-if} + B^*(\vec{k})e^{if} + \widetilde{B}(\vec{k})e^{-i\widetilde{f}} + \widetilde{B}^*(\vec{k})e^{i\widetilde{f}}\right\}d\vec{k} \tag{4.16}$$

This, however, would merely lead to (4.14) again so that (4.10) and (4.11) are sufficiently general. It should be noted the ϕ_α satisfies the equations

$$(1 - a^2\Box)\Box\phi_\alpha = 0 \tag{4.17}$$

$$F_{\alpha,\beta} = \Phi_{\beta,\alpha} - \Phi_{\alpha,\beta} = \phi_{\beta,\alpha} - \phi_{\alpha,\beta} \tag{4.18}$$

$$(1 - a^2\Box)F_{\alpha\beta,\beta} = 0 \tag{4.19}$$

However, ϕ_α does not satisfy the analogue of (4.5) but rather

$$\Box\left(a^2F_{\alpha\beta,\beta} + \phi_\alpha + a\frac{\partial B}{\partial x_\alpha}\right) = 0 \tag{4.20}$$

Finally it proves convenient to write

$$\phi_\alpha = (\vec{A}, i\phi) \tag{4.21}$$

The use of a gage transformation of this sort corresponds to a proposal of Stuckelberg's[3].

Choice 4

We can take instead of (3.9)

$$\Phi_{\beta,\beta} = -\frac{B}{a} \tag{4.22}$$

where B is a solution of

$$(1 - a^2\Box)B = 0 \tag{4.23}$$

In terms of Fourier amplitudes, these may be written

$$k_\alpha \Phi_\alpha(\vec{k}) = 0$$
$$k_\alpha \Phi^*_\alpha(\vec{k}) = 0$$
$$ia\tilde{k}_\alpha \quad \tilde{\Phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k}) = 0$$
$$ia\tilde{k}_\alpha \Phi^*_\alpha(\vec{k}) - \tilde{B}(\vec{k}) = 0 \qquad (4.24)$$

(4.23) insures that (4.22) implies (3.9). This is essentially the choice of Podolsky and Kikuchi[1,2]. Although the difference between this choice and choice 3 is not now apparent, the further development should make this clear.

5. Choice 2:

We go now immediately to the quantized case. Here we have instead of (4.4),

$$k_\alpha \Phi_\alpha(\vec{k})\Psi = 0$$
$$k_\alpha I^*_\alpha(\vec{k})\Psi = 0$$
$$\tilde{k}_\alpha \hat{\Phi}_\alpha(\vec{k})\Psi = 0$$
$$\tilde{k}_\alpha \hat{\Phi}^*_\alpha(\vec{k})\Psi = 0 \qquad (5.1)$$

Our use of the generalization of (4.4) instead of that of (4.3)

$$\Phi_{\beta,\beta}\Psi = 0 \qquad (5.2)$$

may be considered as an extra assumption. We still have to determine the commutation rules. Since the operators are time dependent, we determine the commutation rules by the requirement

$$[\bar{H}_f F(\vec{r},t)]\Psi = \frac{h}{i}\dot{F}(r,t)\Psi \qquad (5.3)$$

$\bar{H}_f$, of course, is given by (3.6). We assume that the commutation rules for the ordinary potentials are

$$\left[\Phi^*_\alpha(\vec{k}), \Phi_\beta(\vec{k}')\right] = \frac{c\hbar}{2k}\delta_{\alpha\beta}\,\delta(\vec{k} - \vec{k}') \tag{5.4}$$

with all other commutators involving ordinary potentials vanishing. It may be easily verified that (5.4) is consistent with (5.3). We write the rules for the extraordinary potentials in the rather general form

$$\left[\tilde{\Phi}^*_\beta(\vec{k}),\ \tilde{\Phi}_\alpha(\vec{k}')\right] = g_{\beta\alpha}(\vec{k},\ \vec{k}') \tag{5.5}$$

with all other commutators involving extraordinary potentials vanishing. Upon applying (5.3) and equating coefficients of exponentials, we get as one condition:

$$-2\int \tilde{k}^2 \Phi_\beta(\vec{k})\, g_{\beta\alpha}(\vec{k},\vec{k}')\, d\vec{k}\,\Psi = -c\hbar\int \tilde{k}\,\Phi_\alpha(\vec{k})\,\delta(\vec{k} - \vec{k}')\, d\vec{k} \tag{5.6}$$

This will be satisfied if the following holds:

$$\Phi_\beta(\vec{k})\, g_{\beta\alpha}\Psi = \frac{c\hbar}{2\tilde{k}}\,\Phi_\alpha(\vec{k})\,\delta(\vec{k} - \vec{k}') \tag{5.7}$$

of which a particular solution is

$$G_{\beta\alpha}(\vec{k},\vec{k}') = \frac{\delta_{\alpha\beta}\, c\hbar}{2\tilde{k}}\ \delta(\vec{k} - \vec{k}') \tag{5.8}$$

to which may be added any solution of

$$\Phi_\beta(\vec{k})\,\bar{g}_{\beta\alpha}\Psi = 0 \tag{5.9}$$

In view of (5.1), one solution of (5.9) is

$$\bar{g}_{\beta,\alpha}(\vec{k},\vec{k}') = c(\vec{k},\vec{k}')\tilde{k}_\beta \tag{5.10}$$

If we assume $g_{\beta,\alpha}$ to be symmetric in $\vec{k}$ and $\vec{k}'$ and α and β, we can put

$$\bar{g}_{\beta\alpha}(\vec{k},\vec{k}') = c(\vec{k},\vec{k}')\tilde{k}_\alpha \check{k}_\beta \tag{5.11}$$

where

$$c(\vec{k},\vec{k}') = c(\vec{k}',\vec{k}) \tag{5.12}$$

The commutation rules are thus:

$$\left[\tilde{\Phi}^*_\beta(\vec{k}),\tilde{\Phi}_\alpha(\vec{k}')\right] = \frac{\delta_{\alpha\beta}c\hbar}{2\tilde{k}}\,\delta(\vec{k}-\vec{k}') + c(\vec{k},\vec{k}')\tilde{k}'_\alpha\tilde{k}_\beta \qquad (5.13)$$

We now seek to determine $c(\vec{k},\vec{k}')$ so that the conditions (5.1) are consistent with each other. This requires, among others, that

$$\left[\tilde{k}_\beta\Phi^*_\beta(\vec{k}),\ \tilde{k}'_\alpha\Phi_\alpha(\vec{k}')\right] = 0 \qquad (5.14)$$

This can be written

$$\frac{\tilde{k}^2_\alpha}{2\tilde{k}}\,c\hbar\,\delta(\vec{k}-\vec{k}') + \tilde{k}^{2\prime}_\alpha\tilde{k}^2_\beta\,c(\vec{k},\vec{k}') = 0 \qquad (5.15)$$

However, $\tilde{k}^2_\alpha = -\frac{1}{a^2}$ so that we have

$$c(\vec{k},\vec{k}') = \frac{a^2c\hbar}{2\tilde{k}}\,\delta(\vec{k}-\vec{k}') \qquad (5.16)$$

This is the revision of the commutation rules suggested by Pauli[7].

With this revision the condition for consistency of the commutation rules is met. However, the results are considerably different from those obtained in the usual manner. For one thing, we have

$$\left[\tilde{\Phi}^*_\beta(\vec{k}'),\tilde{k}'_\alpha\tilde{\Phi}_\alpha(\vec{k}')\right] = \frac{\delta_{\alpha\beta}\tilde{k}'_\beta}{2\tilde{k}}\,\delta(\vec{k}-\vec{k}') + \frac{a^2\tilde{k}^{2\prime}_\alpha\tilde{k}_\beta}{2\tilde{k}}\,\delta(\vec{k}-\vec{k}') = 0 \qquad (5.17)$$

This, of course, guarantees that

$$\left[H_f,\tilde{k}_\alpha\tilde{\Phi}_\alpha(\vec{k}')\right] = 0 \qquad (5.18)$$

as is required. However, it also implies that

$$\left[R_s,\tilde{k}_\alpha\tilde{\Phi}_\alpha(\vec{k}')\right] = 0 \qquad (5.19)$$

Thus the auxiliary conditions do not need modification and the interaction energy is to be expected to be considerably different from that obtained by Podolsky and Kikuchi.

6. Choice 3

We consider first the case in which the field quantities are not quantized. In terms of Fourier amplitudes the gage transformation employed is

$$\tilde{\Phi}_\alpha(\vec{k}) = \tilde{\phi}_\alpha(\vec{k}) + ia\tilde{k}_\alpha\tilde{B}(\vec{k})$$
$$\tilde{\Phi}^*_\alpha(\vec{k}) = \tilde{\phi}^*_\alpha(\vec{k}) - ia\tilde{k}_\alpha\tilde{B}^*(\vec{k}) \qquad (6.1)$$

To compute $\bar{H}_f$ we need $\tilde{\Phi}^*_\alpha(k)\tilde{\Phi}_\alpha(\vec{k})$ which is given by

To compute $\bar{H}_f$ we need $\tilde{\Phi}^*_\alpha(k)\tilde{\Phi}_\alpha(\vec{k})$ which is given by,

$$\tilde{\Phi}^*_\alpha(\vec{k})\tilde{\Phi}_\alpha(\vec{k}) = \tilde{\phi}^*_\alpha(\vec{k})\tilde{\phi}_\alpha(\vec{k}) + a^2\tilde{k}^2_\alpha\tilde{B}^*(\vec{k})\tilde{B}(\vec{k})$$
$$- ia\tilde{k}_\alpha\tilde{B}^*(\vec{k})\tilde{\phi}_\alpha(\vec{k}) + ia\tilde{k}_\alpha\tilde{\phi}^*_\alpha(\vec{k})\tilde{B}(\vec{k}) \qquad (6.2)$$

However, from (4.14) it follows that

$$- ia\tilde{k}_\alpha\tilde{B}^*(\vec{k})\tilde{\phi}_\alpha(\vec{k}) + ia\tilde{k}_\alpha\tilde{\phi}^*_\alpha(\vec{k})\tilde{B}(\vec{k}) = 2\tilde{B}^*(\vec{k})\tilde{B}(\vec{k}) \qquad (6.3)$$

while $a^2\tilde{k}^2_\alpha = -1$, so that we have

$$\tilde{\Phi}^*_\alpha(\vec{k})\tilde{\Phi}_\alpha(\vec{k}) = \tilde{\phi}^*_\alpha(\vec{k})\tilde{\phi}_\alpha(\vec{k}) + \tilde{B}^*(\vec{k})\tilde{B}(\vec{k}) \qquad (6.4)$$

and

$$\bar{H}_f = 2\int\left\{k^2\phi^*_\alpha(\vec{k})\phi_\alpha(\vec{k}) - \tilde{k}^2\left[\tilde{\phi}^*_\alpha(\vec{k})\tilde{\phi}_\alpha(\vec{k}) + \tilde{B}^*(\vec{k})\tilde{B}(\vec{k})\right]\right\}d\vec{k} \qquad (6.5)$$

In the theory in which the particles as well as the field are treated classically, the interaction between the particles and field depends only upon $F_{\alpha\beta}$. Thus the equation of motion of any particle in the field is

$$\frac{d^2x_\alpha}{ds^2} = \frac{e}{mc^2}F_{\alpha\beta}\frac{dx_\beta}{ds} \qquad (6.6)$$

It may be noted that the potentials occuring in $F_{\alpha\beta}$ no longer need satisfy (3.3) since we are no longer dealing with charge-free space. Now by equation (4.18) it is immaterial whether we use ϕ's or Φ's in computing $F_{\alpha\beta}$. On the other hand, when the particles are treated by the methods of quantum mechanics, their wave equation is given by

$$\left[\sum_{s=1}^{n} R_s + \frac{\bar{h}}{i}\frac{\partial}{\partial t}\right]\Omega = 0 \qquad (6.7)$$

where we assume n particles and where we may take for R_s either

$$R_s = c\alpha_{is}\left[p_{is} - \frac{e_s\phi_i(\vec{r}_s,t)}{c}\right] + m_s c^2\beta_s + e_s\phi(\vec{r}_s,t) \qquad (6.8)$$

or

$$R_s = c\alpha_{is}\left[p_{is} - \frac{e_s\Phi_i(\vec{r}_s,t)}{c}\right] + m_s c^2\beta_s + e_s\Phi(\vec{r}_s,t) \qquad (6.9)$$

We are free to use either (6.8) or (6.9) in (6.7) at our convenience. However, it can easily be shown they are equivalent when the fields are not quantized. To do this assume R_s given by (6.9) and apply to (6.7) the canonical transformation,

$$\Omega = \exp\left[\frac{ai}{ch}\sum_{s=1}^{n} e_s\ (\vec{r}_s,t)\right]\Omega' \qquad (6.10)$$

the wave equation thus obtained for Ω' is the same as that obtained for Ω when (6.8) is substituted in (6.7).

We are now ready to quantize the field quantities. We start as usual with the case of charge-free space. We can then apply the requirement (5.3) with $\bar{H}_f$ given by (6.4) to obtain the following commutation rules:

$$\left[\phi^*_\alpha(\vec{k}), \phi_\beta(\vec{k}')\right] = -\frac{c\hbar}{2k}\delta_{\alpha\beta}\delta(\vec{k} - \vec{k}')$$

$$\left[\tilde{\phi}^*_\alpha(\vec{k}), \tilde{\phi}_\beta(\vec{k}')\right] = \frac{c\hbar}{2k}\delta_{\alpha\beta}\delta(\vec{k} - \vec{k}') \tag{6.11}$$

$$\left[\tilde{B}^*(\vec{k}), \tilde{B}(\vec{k}')\right] = \frac{c\hbar}{2k}\delta\ (\vec{k} - \vec{k}')$$

The auxiliary conditions are:

$$k_\alpha\phi_\alpha(\vec{k})\Omega = 0$$

$$k_\alpha\phi^*_\alpha(\vec{k})\Omega = 0$$

$$\left[ia\tilde{k}_\alpha\tilde{\phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k})\right]\Omega = 0 \tag{6.12}$$

$$\left[-ia\tilde{k}_\alpha\tilde{\phi}^*_\alpha(\vec{k}) + \tilde{B}^*(\vec{k})\right]\Omega = 0$$

These are equations (5.8) of 2. To show that the second pair are consistent with each other we compute

$$\left[-ia\tilde{k}_\alpha\tilde{\phi}^*_\alpha(\vec{k}) + \tilde{B}^*(\vec{k}), ia\tilde{k}_\beta\phi_\beta(\vec{k}') + \tilde{B}(\vec{k})\right] =$$

$$a^2\tilde{k}^2_\alpha\frac{c\hbar}{2k}\delta(\vec{k} - \vec{k}') + \frac{c\hbar}{2k}\delta(\vec{k} - \vec{k}') = 0 \tag{6.13}$$

In the above we made use of $\tilde{k}^2_\alpha = -\frac{1}{a^2}$. Further, once these conditions are imposed, they will continue to hold, for we have

$$\left[H_f, ia\tilde{k}_\alpha\tilde{\phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k})\right]\Omega = -c\hbar k^2\left[ia\tilde{k}_\alpha\tilde{\phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k})\right]\Omega = 0 \tag{6.14}$$

and similar equations for the commutators between H_f and the other operators in (6.12).

We can now pass to the case where charged particles are present. First, however, we must decide between the two expressions (6.8) and (6.9) for R_s since, because of the commutation rules (6.11), these are no longer equivalent. If we use (6.9) the present approach becomes like that of choice 2, since we would then everywhere employ the potentials Φ_α. If, however, take for R_s (6.8), we obtain for the equations used in deriving the interaction energy among the charged particle the expressions given by Podolsky and Kikuchi[4]. More specifically, the wave equation for the whole system may be written,

$$\left[\sum_{s=1}^{n} R_s + \bar{H}_f + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\Omega = 0 \qquad (6.15)$$

from which $\bar{H}_f$ may be eliminated by the transformation

$$\Omega = \exp\left[\frac{-i}{k}\bar{H}_f t\right]\Psi \qquad (6.16)$$

As the result of this transformation, (6.16) becomes,

$$\left[\sum_{s=1}^{n} R_s + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\Psi = 0 \qquad (6.17)$$

where now the ϕ's occuring in R_s now satisfy (3.3). Finally, it is easy to see that the set

$$s = 1,\ldots,n \left(R_s + \frac{\hbar}{i}\frac{\partial}{\partial t_s}\right)\Psi = 0 \qquad (6.18)$$

becomes identical with (6.17) if we add its members together and set in them $t = t_1 = \ldots t_s = t_n$. When we use (6.9) for R_s with the potentials required to satisfy (3.3), (6.18) is identical with (1.2) of Podolsky and Kikuchi[3]. The usual procedure may now be followed, that is, (6.12) may be appropriately modified to be consistent with (6.18) and the interaction energy calculated. It may be noted that (6.14) makes if possible to consider (6.12) as holding for either Ω or Ψ.

There is one more remark of interest which may be made about the present choice. We noted in discussing the case in which the particles but not the field were treated according to the methods of quantum mechanics, that the equivalence of (6.8) and (6.9) might be demonstrated by means of the transformation (6.10). We may consider this transformation to have been performed before the fields were quantized so that equation (6.7) with R_s given by (6.9) was obtained in this way. There is then no reason why in the case of quantized fields we could not take for the auxiliary conditions to be associated with (6.17):

$$\begin{aligned}
k_\alpha \phi_\alpha(\vec{k}) \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s,t)\right]\Psi &= 0 \\
k_\alpha^* \phi_\alpha^*(\vec{k}) \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s,t)\right]\Psi &= 0 \\
\left[ia\tilde{k}_\alpha \tilde{\phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k})\right] \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s,t)\right]\Psi &= 0 \\
\left[-ia\tilde{k}_\alpha \tilde{\phi}_\alpha^*(\vec{k}) + \tilde{B}^*(\vec{k})\right] \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s,t)\right]\Psi &= 0
\end{aligned} \tag{6.19}$$

Here $B(\vec{r}_s,t)$ satisfies the wave equation because we consider the transformation (6.16) as having been performed. Similarly we can associate with equations (6.18) the set:

$$
\begin{aligned}
&k_\alpha \phi_\alpha(\vec{k}) \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s, t_s)\right]\Psi = 0 \\
&k_\alpha \phi_\alpha^*(\vec{k}) \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s, t_s)\right]\Psi = 0 \\
&\left[ia\tilde{k}_\alpha \tilde{\phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k})\right] \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s, t_s)\right]\Psi = 0 \\
&\left[-ia\tilde{k}_\alpha \tilde{\phi}_\alpha^*(\vec{k}) + \tilde{B}^*(\vec{k})\right] \exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s, t_s)\right]\Psi = 0
\end{aligned}
\tag{6.20}
$$

Because of the commutation rules, there is a possible ambiguity in the expression $\exp\left[\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s B(\vec{r}_s, t_s)\right]$ and we therefore define it to mean:

$$
\begin{aligned}
&\exp\left\{\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s \left(\frac{1}{2\pi}\right)^{3/2}\int d\vec{k}\left[\tilde{B}(\vec{k})\exp(i\vec{k}\cdot\vec{r}_s - i\tilde{k}ct_s)\right]\right\} \\
\times\ &\exp\left\{\frac{ai}{c\hbar}\sum_{s=1}^{n} e_s \left(\frac{1}{2\pi}\right)^{3/2}\int d\vec{k}\left[\tilde{B}^*(\vec{k})\exp(-i\vec{k}\cdot\vec{r}_s + i\tilde{k}ct_s)\right]\right\}
\end{aligned}
\tag{6.21}
$$

if now we multiply (6.20) on the left by

$$
\begin{aligned}
&\exp\left[\frac{-ai}{c\hbar}\sum_{s=1}^{n} e_s \left(\frac{1}{2\pi}\right)^{3/2}\int d\vec{k}\ \tilde{B}^*(\vec{k})\exp(i\vec{k}\cdot\vec{r}_s + i\tilde{k}ct_s)\right] \\
\times\ &\exp\left[\frac{-ai}{c\hbar}\sum_{s=1}^{n} e_s \left(\frac{1}{2\pi}\right)^{3/2}\int d\vec{k}\ \tilde{B}(\vec{k})\exp(i\vec{k}\cdot\vec{r}_s - i\tilde{k}ct_s)\right]
\end{aligned}
$$

and make use of the commutation rules (6.12) we obtain

$$
\begin{aligned}
&k_\alpha \phi_\alpha(\vec{k})\Psi = 0 \\
&k_\alpha \phi_\alpha^*(\vec{k})\Psi = 0 \\
&\left\{ia\tilde{k}_\alpha \tilde{\phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k}) - \frac{ia}{2\tilde{k}}\left(\frac{1}{2\pi}\right)^{3/2}\sum_{s=1}^{n} e_s \exp(ickt_s - ik\cdot r_s)\right\}\Psi = 0 \\
&\left\{-ia\tilde{k}_\alpha \tilde{\phi}_\alpha^*(\vec{k}) + \tilde{B}^*(\vec{k}) + \frac{ia}{2\tilde{k}}\left(\frac{1}{2\pi}\right)^{3/2}\sum_{s=1}^{n} e_s \exp(-ickt_s + ik\cdot r_s)\right\}\Psi = 0
\end{aligned}
\tag{6.22}
$$

The third member of (6.22) follows quite simply if we make the association:

$$\tilde{B}(\vec{k}) \longrightarrow \frac{-c\hbar}{2k}\frac{\delta}{\delta B^*(\vec{k})} \tag{6.23}$$

While to obtain the fourth we let

$$\tilde{B}^*(\vec{k}) \longrightarrow \frac{c\hbar}{2k}\frac{\delta}{\delta B(\vec{k})} \tag{6.24}$$

The last pair of (6.22) is, however, consistent with (6.17) and in fact, is just the modification of (6.12) which is usually employed in the case when charges are present. The first pair of (6.22), however, still requires modification to make it consistent with (6.17). It may be noted that the desired modification can be obtained by letting a go to infinity in the second pair.

7. Choice 4

We consider the case of charge-free space and quantized fields. $\bar{H}_f$ is given by (3.6). The commutation rules for the Φ's are:

$$\left[\Phi^*_\alpha(\vec{k}),\ \Phi_\beta(\vec{k}')\right] = -\frac{c\hbar}{2k}\delta_{\alpha\beta}\,\delta(\vec{k}-\vec{k}')$$

$$\left[\tilde{\Phi}^*_\alpha(\vec{k}),\ \tilde{\Phi}_\beta(\vec{k}')\right] = \frac{c\hbar}{2k}\delta_{\alpha\beta}\,\delta(\vec{k}-\vec{k}') \tag{7.1}$$

In order that $B(\vec{r},t)$ satisfy principle (5.3) it is necessary that it be expressible in terms of Φ_α's. Assume that such a B exists which makes the auxiliary conditions consistent with one another. These conditions are:

$$k_\alpha \Phi_\alpha(\vec{k})\Psi = 0$$

$$k_\alpha I^*_\alpha(\vec{k})\Psi = 0$$
$$\left[ia\tilde{k}_\alpha \tilde{\Phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k})\right]\Psi = 0$$
$$\left[-ia\tilde{k}_\alpha \tilde{I}^*_\alpha(\vec{k}) + \tilde{B}^*(\vec{k})\right]\Psi = 0 \qquad (7.2)$$

Now let us pass to the case where particles are present. According to the many times theory, the following set of equations must be satisfied by Ψ

$$s = 1, \ldots n\left(R_s + \frac{\hbar}{i}\frac{\partial}{\partial t_s}\right)\Psi = 0 \qquad (7.3)$$

In the present case, R_s is given by:

$$R_s = c\alpha_{si}\left[p_{si} - \frac{e_s}{c}\Phi_i(\vec{r}_s, t_s)\right] + m_s c^2 \beta_s + e_s \Phi(\vec{r}_s, t_s) \qquad (7.4)$$

In order that (7.3) and (7.2) be consistent, it is necessary that the operators occuring in each member of (7.2) commute with those occuring in each member of (7.3). To accomplish this, it is customary to alter the operators of (7.2). Let us therefore try to modify the operator in the third member of (7.2) so that it commutes with

$$R_s + \frac{\hbar}{i}\frac{\partial}{\partial t_s} \qquad (7.5)$$

To do this we let it be replaced by

$$ia\tilde{k}_\alpha \tilde{\Phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k}) + g(\vec{r}_s, t_s) \qquad (7.6)$$

where $g(\vec{r}_s, t_s)$ is simply some undetermined function of $\vec{r}_s$ and t_s. If we assume the commutation rules between $\tilde{B}(\vec{k})$ and $\tilde{\Phi}_\alpha(\vec{k})$, which are as yet undefined, to be given by:

$$\left[\tilde{\Phi}_\alpha(\vec{k}),\ \tilde{B}(\vec{k})\right] = 0$$
$$\left[\Phi_i^*(\vec{k}),\ \tilde{B}(\vec{k}')\right] = \Delta_i\,\delta(\vec{k}-\vec{k}') \qquad (7.7)$$
$$\left[\tilde{\Phi}^*(\vec{k}),\ \tilde{B}(\vec{k}')\right] = -\Delta\ \delta(\vec{k}-\vec{k}')$$

where Δ_i and Δ represent arbitrary functions of $\vec{k}$ and $\vec{k}'$, the condition that the commutator between (7.5) and (7.6) vanish becomes

$$\left[R_s + \frac{\hbar}{i}\frac{\partial}{\partial t_s},\ i\tilde{k}_\alpha\tilde{\Phi}_\alpha(\vec{k}) + \tilde{B}(\vec{k}) + g(\vec{r}_s,t_s)\right] =$$
$$c\vec{\alpha}_s\left\{\frac{\hbar}{i}\nabla_s g - \frac{e_s}{c}\left(\frac{1}{2\pi}\right)^{3/2}\left[\frac{\vec{k}c\hbar}{2\tilde{k}} + \vec{\Delta}\right] \times\right.$$
$$\left.\exp\left[i(\tilde{k}ct_s - \vec{k}\cdot\vec{r}_s)\right]\right\} + \frac{\hbar}{i}\frac{\partial g}{\partial t_s} +$$
$$e_s\left(\frac{1}{2\pi}\right)^{3/2}\left[\frac{-c\hbar}{2} - \Delta\right]\exp\left[i(\tilde{k}ct_s - \vec{k}\cdot\vec{r}_s)\right] = 0 \qquad (7.8)$$

If we take α_{sx}, α_{sy}, α_{sz}, and 1 as independent quantities, this leads to the set of conditions

$$\frac{h}{i}\nabla_s g - \frac{e_s}{c}\left(\frac{1}{2\pi}\right)^{3/2}\left[\frac{\vec{k}c\hbar}{2\tilde{k}} + \vec{\Delta}\right]\exp\left[i(\tilde{k}ct_s - \vec{k}\cdot\vec{r}_s)\right] = 0$$

$$\frac{h}{i}\frac{\partial g}{\partial t_s} + e_s\left(\frac{1}{2\pi}\right)^{3/2}\left[-\frac{c\hbar}{2} - \Delta\right]\exp\left[i(\tilde{k}ct_s - \vec{k}\cdot\vec{r}_s)\right] = 0 \qquad (7.9)$$

In order that the equations of (7.9) be consistent with one another, it is necessary that say $\frac{\partial g}{\partial x\,\partial y} = \frac{\partial g}{\partial y\,\partial x}$ and hence that

$$\nabla_s g = G(\vec{r}_s,t_s)\vec{k}$$

$$\frac{\partial g_s}{\partial t_s} = -\,G(\vec{r}_s,t_s)c\tilde{k} \qquad (7.10)$$

This implies that

$$\vec{\Delta} = C\vec{k}$$
$$\Delta = C\tilde{k} \qquad (7.11)$$

Reference to the commutation rules (7.1) shows that for (7.11) to hold, $\tilde{B}(\vec{k})$ must be given by

$$\tilde{B}(\vec{k}) \quad \text{const.}\ \tilde{k}_\alpha \tilde{\Phi}_\alpha(\vec{k}) \qquad (7.12)$$

Now let us again consider the case where no charges are present. The presence of $\tilde{B}^*(\vec{k})$ and $\tilde{B}(\vec{k})$ in the equations (7.2) does not cause them to differ essentially from the equations:

$$\begin{aligned} k_\alpha \Phi_\alpha(\vec{k}) \Psi &= 0 \\ k_\alpha \Phi^*_\alpha(\vec{k}) \Psi &= 0 \\ \tilde{k}_\alpha \tilde{\Phi}_\alpha(\vec{k}) \Psi &= 0 \\ \tilde{k}_\alpha \tilde{\Phi}^*_\alpha(\vec{k}) \Psi &= 0 \end{aligned} \qquad (7.13)$$

Indeed, on account of the relation (7.12), equations (7.13) and (7.2) are the same except for a possible constant factor. However, because of the commutation rules, (7.1), the last two members of (7.13) are not consistent with one another. Therefore, choice 4 is inadmissible.

8. Conclusions:

The treatment of auxiliary conditions by the method of choice 3 may be considered satisfactory in that it fulfills the requirements stated in the introduction. The auxiliary conditions are consistent with each other; if they are once imposed in charge-free space, they continue to hold; they can be modified to be consistent with the wave equation for the charged particles; and they lead to the same equation basic for the determination of the interaction energy as were found by Podolsky and Kikuchi[2]. Furthermore, a procedure is given in which special modification of the auxiliary conditions containing tilde quantities is unnecessary. The approach of choice 3 is essentially that used by Podolsky and Schwed[8].

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II CHARGE DEPENDENCE IN GENERALIZED MESON THEORY

1. Introduction:

Green[1] has recently generalized the neutral meson theory in much the same manner that electrodynamics was previously generalized[2-7]. The purpose of the present paper is to investigate the extension of generalized meson theory to the charged meson case. In doing this, it proves necessary, however, to alter the Hamiltonian of the neucleons in the presence of a field in the manner suggested by Stuckelberg[8].

2. Classical Theory:

In introducing charge dependence, it is convenient to consider the case in which only charged mesons are present inasmuch as the neutral mesons are represented by an essentially independent set of potentials. In adapting the procedure previously used to the present case, it is necessary to introduce complex potentials. A suitable Lagrangian density for the meson field in neucleon free space is

$$L_f = -\frac{1}{1-\mu^2 a^2}\left[\mu^2 \Phi_\alpha^* \Phi_\alpha + (1+\mu^2 a^2)\Phi_{\alpha,\beta}^* \Phi_{\alpha,\beta} + a^2(\Box \Phi_\alpha^*)(\Box \Phi_\alpha)\right] \tag{2.1}$$

Here Φ_α and Φ_α^* are considered independent quantities and we have for the equations of motion,

$$\begin{aligned}(a^2\Box - 1)(\Box - \mu^2)\Phi_\alpha &= 0\\ (a^2\Box - 1)(\Box - \mu^2)\Phi_\alpha^* &= 0\end{aligned} \tag{2.2}$$

Following the notation of Pauli[9], the solution of the first set can be written

$$\Phi_\alpha(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int \Big\{ \Phi^{0}_{\alpha+}(\vec{k})e^{-if_0} + \Phi^{0*}_{\alpha-}(\vec{k})e^{if_0} + \Phi^{1}_{\alpha+}(\vec{k})e^{-if_1} + \Phi^{1*}_{\alpha-}(\vec{k})e^{if_1} \Big\} d\vec{k} \qquad (2.3)$$

with

$$f_0 = ck_0t - \vec{k}\cdot\vec{r} \qquad f_1 = ck_1t - \vec{k}\cdot\vec{r}$$
$$k_0 = \sqrt{k^2 + \mu^2} \qquad k_1 = \sqrt{k^2 + \frac{1}{a^2}} \qquad (2.4)$$

That of the second set is the formal conjugate. The Hamiltonian in terms of Fourier amplitudes is then

$$\bar{H}_f = 2\int \Big\{ \left[\Phi^{0*}_{\alpha+}(\vec{k})\Phi^{0}_{\alpha+}(\vec{k}) + \Phi^{0*}_{\alpha-}(\vec{k})\Phi^{0}_{\alpha-}(\vec{k})\right]k_0^2 - \left[\Phi^{1*}_{\alpha+}(\vec{k})\Phi^{1}_{\alpha+}(\vec{k}) + \Phi^{1*}_{\alpha-}(\vec{k})\Phi^{1}_{\alpha-}(\vec{k})\right]k_1^2 \Big\} d\vec{k} \qquad (2.5)$$

It should be noted that $\Phi_\alpha(r,t)$ and $\bar{H}_f$ both can be decomposed into two parts according to

$$\Phi_\alpha(\vec{r},t) = \Phi^0_\alpha(\vec{r},t) + \Phi^1_\alpha(\vec{r},t) \qquad \bar{H}_f = \bar{H}^0_f + \bar{H}^1_f \qquad (2.6)$$

where

$$\Phi^0_\alpha(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int \left[\Phi^{0}_{\alpha+}(\vec{k})e^{-if_0} + \Phi^{0*}_{\alpha-}(\vec{k})e^{if_0}\right] d\vec{k}$$
$$\Phi^1_\alpha(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int \left[\Phi^{1}_{\alpha+}(\vec{k})e^{-if_1} + \Phi^{1*}_{\alpha-}(\vec{k})e^{-if_1}\right] d\vec{k} \qquad (2.7)$$
$$\bar{H}^0_f = 2\int \left[\Phi^{0*}_{\alpha+}(\vec{k})\Phi^{0}_{\alpha+}(\vec{k}) + \Phi^{0*}_{\alpha-}(\vec{k})\Phi^{0}_{\alpha-}(\vec{k})\right]k_0^2 d\vec{k}$$
$$\bar{H}^1_f = -2\int \left[\Phi^{1*}_{\alpha+}(\vec{k})\Phi^{1}_{\alpha+}(\vec{k}) + \Phi^{1*}_{\alpha-}(\vec{k})\Phi^{1}_{\alpha-}(\vec{k})\right]k_1^2 d\vec{k}$$

Quantities with the superscript, o, will be called μ quantities and those with superscript, 1, will be called <u>a</u> quantities. Finally, we take as auxiliary conditions

$$\Phi_{\alpha,\alpha} = 0 \tag{2.8}$$

and their conjugates. This implies that the field equations may be written

$$\begin{aligned}(a\square -1)(F_{\alpha\beta,\beta} + \mu^2\Phi_\alpha) &= 0\\ (\square - \mu^2)(a^2F_{\alpha\beta,\beta} + \Phi_\alpha) &= 0\end{aligned} \tag{2.9}$$

and their conjugates.

3. Quantized Theory:

To pass to the Quantized theory, we introduce the gage transformation

$$\Phi_\alpha(\vec{r},t) = \phi_\alpha(\vec{r},t) + a\frac{\partial B^\circ}{\partial x_\alpha} + \frac{1}{\mu}\frac{\partial B^{\prime}}{\partial x_\alpha} \tag{3.1}$$

where the B's satisfy the equations

$$\begin{aligned}(a^2\square -1)B^{\prime} &= 0 & (a^2\square -1)B^{\prime *} &= 0\\ (\square - \mu^2)B^\circ &= 0 & (\square - \mu^2)B^{\circ *} &= 0\end{aligned} \tag{3.2}$$

and therefore may be written as

$$\begin{aligned}B^\circ(\vec{r},t) &= \left(\frac{1}{2\pi}\right)^{3/2}\int\left\{B^\circ_+(\vec{k})e^{-if_\circ} + B^{\circ *}_-(\vec{k})e^{if_\circ}\right\}d\vec{k}\\ B^{\prime}(\vec{r},t) &= \left(\frac{1}{2\pi}\right)^{3/2}\int\left\{B^{\prime}_+(\vec{k})e^{-if_1} + B^{\prime *}_-(\vec{k})e^{if_1}\right\}d\vec{k}\end{aligned} \tag{3.3}$$

The Hamiltonian may be expressed in terms of the new potentials as

$$\begin{aligned}\bar{H}_f = 2\int\Big\{k_\circ^2\big[&\phi^{\circ *}_{\alpha +}(\vec{k})\phi^\circ_{\alpha +}(\vec{k}) + \phi^{\circ *}_{\alpha -}(\vec{k})\phi^\circ_{\alpha -}(\vec{k}) + B^{\circ *}_+(\vec{k})B^\circ_+(\vec{k})\\ &+ B^{\circ *}_-(\vec{k})B^\circ_-(\vec{k})\big] - k_1^2\big[\phi^{\prime *}_{\alpha +}(\vec{k})\phi^{\prime}_{\alpha +}(\vec{k}) + \phi^{\prime *}_{\alpha -}(\vec{k})\phi^{\prime}_{\alpha -}(\vec{k})\\ &+ B^{\prime *}_+(\vec{k})B^{\prime}_+(\vec{k}) + B^{\prime *}_-(\vec{k})B^{\prime}_-(\vec{k})\big]\Big\}d\vec{k}\end{aligned} \tag{3.4}$$

It is now possible to obtain the commutation rules which are as follows. Each quantity commutes with all other quantities except its complex conjugate. The rules for the non-commuting quantities are:

$$\left[\phi^{o*}_{\alpha\pm}(\vec{k}),\ \phi_{\beta\pm}(\vec{k}')\right] = -\frac{c\hbar}{2k}\delta_{\alpha\beta}\,\delta(\vec{k}-\vec{k}')$$

$$\left[B^{o*}_{\pm}(\vec{k}),\ B^{o}_{\pm}(\vec{k}')\right] = -\frac{c\hbar}{2k_o}\,\delta(\vec{k}-\vec{k}')$$

$$\left[\phi^{1*}_{\alpha\pm}(\vec{k}),\ \phi^{1}_{\beta\pm}(\vec{k}')\right] = \frac{c\hbar}{2k_1}\delta_{\alpha\beta}\delta(\vec{k}-\vec{k}') \tag{3.5}$$

$$\left[B^{1*}_{\pm}(\vec{k}),\ B^{1*}_{\pm}(\vec{k}')\right] = \frac{c\hbar}{2k_1}\,\delta(\vec{k}-\vec{k}')$$

In terms of Fourier amplitudes the auxiliary conditions, taken now as conditions on Ψ, are:

$$\left[\vec{k}\cdot\vec{A}^{o}_{\pm}(\vec{k}) - k_o\phi^{o}_{\pm}(\vec{k}) - i\mu B^{o}_{\pm}(\vec{k})\right]\Psi = 0$$

$$\left[\vec{k}\cdot\vec{A}^{o*}_{\pm}(\vec{k}) - k_o\phi^{o*}_{\pm}(\vec{k}) + i\mu B^{o*}_{\pm}(\vec{k})\right]\Psi = 0$$

$$\left\{a\left[\vec{k}\cdot\vec{A}^{1}_{\pm}(\vec{k}) - k_1\phi^{1}_{\pm}(k)\right] + iB^{1}_{\pm}(\vec{k})\right\}\Psi = 0 \tag{3.6}$$

$$\left\{a\left[\vec{k}\cdot\vec{A}^{1*}_{\pm}(\vec{k}) - k_1\phi^{1}_{\pm}(\vec{k})\right] - iB^{1*}_{\pm}(\vec{k})\right\}\Psi = 0$$

These are consistent with one another and once imposed (in neucleon free space) continue to subsist.

4. Choice of R_s:

Following the many times theory , the equations for the particles are:

$$R_s\Psi = \hbar i\,\frac{\partial\Psi}{\partial t_s} \qquad s = 1\ldots n \tag{4.1}$$

R_s may be given either by

$$R_s = c\vec{\alpha}_s\cdot\vec{p}_s + g_s\Big\{\tau_{-s}\left[\phi(\vec{r}_s,t_s) - \alpha_{s,j}\phi_j(\vec{r}_s,t_s)\right]$$
$$+ \tau_{+s}\left[\phi^*(\vec{r}_s,t_s) - \alpha_{s,j}\phi_j^*(\vec{r}_s,t_s)\right]\Big\} + m_s c^2\beta_s \tag{4.2}$$

or by

$$R_s = c\vec{\alpha}_s \cdot \vec{p}_s + g_s\left\{\mathcal{T}_{-s}\left[\Phi(\vec{r}_s, t_s) - \alpha_{s,j}\Phi_j(\vec{r}_s, t_s)\right] + \mathcal{T}_{+s}\left[\Phi^*(\vec{r}_s, t_s) - \alpha_{s,j}\Phi^*(\vec{r}_s, t_s)\right]\right\} + m_s c^2 \beta_s \qquad (4.3)$$

The first choice corresponds to that made by Green[1] while the second is of Stuckelberg[8]. It is now necessary to modify the auxiliary conditions so that they are consistent with equation (4.1). Because of the presence of the $\mathcal{T}$'s, this proves impossible for the choice (4.2) while the choice (4.3) is consistent with the auxiliary conditions without modification. In the following, the Φ's will be expressed in terms of ϕ's and B's.

At this point it is convenient to make use of the fact that the $\underline{\mu}$ and $\underline{a}$ fields can be treated separately. Thus, we consider the case where only the $\underline{\mu}$ field is present and obtain the interaction energy for the case where both fields are present by subtracting from the expression for the energy of the $\underline{\mu}$ field the same expression with $\underline{\mu}$ replaced by $1/\underline{a}$. As usual, we introduce

$$Q^\circ_\pm(\vec{k}) = \frac{\vec{k}\cdot\vec{A}^\circ_\pm(\vec{k}) - i\mu B_\pm(\vec{k})}{k_o} \qquad (4.4)$$

and its conjugate. The auxiliary conditions are then

$$\left[Q^\circ_\pm(\vec{k}) - \phi^\circ_\pm(\vec{k})\right]\Psi = 0 \qquad (4.5)$$

and their conjugates.

In view of the commutation rules it is possible to make the associations:

$$\begin{aligned} \phi_{\pm}^{o*}(\vec{k}) &\longrightarrow \frac{c\hbar}{2k_o}\frac{\delta}{\delta\phi_{\pm}^{o}(\vec{k})} \\ A_{\pm|}^{o}(\vec{k}) &\longrightarrow \frac{c\hbar}{2k_o}\frac{\delta}{\delta A_{\pm|}^{o*}(\vec{k})} \\ B_{\pm}^{o}(\vec{k}) &\longrightarrow \frac{c\hbar}{2k_o}\frac{\delta}{\delta B_{\pm}^{o*}(\vec{k})} \\ Q_{\pm}^{o}(\vec{k}) &\longrightarrow \frac{c\hbar}{2k_o}\frac{\delta}{\delta Q_{\pm}^{o*}(\vec{k})} \end{aligned} \tag{4.6}$$

The equation (4.5) can be thus written:

$$\begin{aligned} \left[\frac{c\hbar}{2k_o}\frac{\delta}{\delta Q_{\pm}^{o*}(\vec{k})} - \phi_{+}^{o}(\vec{k})\right]\Psi &= 0 \\ \left[Q_{\pm}^{o*}(\vec{k}) - \frac{c\hbar}{2k_o}\frac{\delta}{\delta\phi_{\pm}^{o}(\vec{k})}\right]\Psi &= 0 \end{aligned} \tag{4.7}$$

5. Elimination of Auxiliary Conditions:

We may take as the solution of the equations (4.7) the expression:

$$\Psi = \exp(X_{+}^{o} + X_{-}^{o})\Omega \tag{5.1}$$

where

$$X_{\pm}^{o} = \frac{2}{c\hbar}\int\phi_{\pm}^{o*}(\vec{k})\phi_{\pm}^{o}(\vec{k})k_o d\vec{k} \tag{5.2}$$

and Ω is independent of $Q_{\pm}^{o}(\vec{k})$ and $\phi_{\pm}^{o*}(\vec{k})$. To obtain the wave equation[3] in terms of Ω we need merely replace each operator F in R_s by $\exp[-(X_{+}^{o}+X_{-}^{o})]\,F\exp(X_{+}^{o}+X_{-}^{o})$. We first consider how the Fourier amplitudes of the potentials transform. With the use of (4.6) it follows at once that we have:

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] \phi_\pm^{\circ *}(\vec{k}) \exp(X_+^\circ + X_-^\circ) = \phi_\pm^{\circ *}(\vec{k}) + Q_\pm^{\circ *}(\vec{k})$$

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] B_\pm^\circ(\vec{k}) \exp(X_+ + X_-) = B_+^\circ(\vec{k}) + \frac{i\mu\phi_\pm^\circ(\vec{k})}{k_\circ}$$

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] \vec{A}_\pm^\circ(\vec{k}) \exp(X_+^\circ + X_-^\circ) = \vec{A}_+^\circ(\vec{k}) + \frac{\vec{k}}{k_\circ}\phi_\pm^\circ(\vec{k}) \quad (5.3)$$

The others are unchanged. These lead at once to the following expressions for the transformed potentials as functions of space and time.

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] \vec{A}^\circ(\vec{r}_s, t_s) \exp(X_+^\circ + X_-^\circ) = \vec{A}^\circ(\vec{r}_s, t_s) + \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{\vec{k}}{k_\circ} \phi_+^\circ(\vec{k}) \exp\left[i(\vec{k}\cdot\vec{r}_s - k_\circ c t_s)\right] d\vec{k}$$

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] \phi^\circ(\vec{r}_s, t_s) \exp(X_+^\circ + X_-^\circ) = \phi^\circ(\vec{r}_s, t_s) + \left(\frac{1}{2\pi}\right)^{3/2} \int Q_-^{\circ *}(\vec{k}) \exp\left[-(\vec{k}\cdot\vec{r}_s - k_\circ c t_s)\right] d\vec{k}$$

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] \frac{\partial B^\circ(\vec{r}_s, t_s)}{\partial x_{sj}} \exp(X_+^\circ + X_-^\circ) = \frac{\partial B^\circ(\vec{r}_s, t_s)}{\partial x_{sj}} - \mu\left(\frac{1}{2\pi}\right)^{3/2} \int \frac{k_j \phi_+^\circ}{\hat{k}}(\vec{k}) \exp\left[i(\vec{k}\cdot\vec{r}_s - k_\circ c t_s)\right] d\vec{k}$$

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] \frac{\partial B^\circ(\vec{r}_s, t_s)}{\partial x_{s\circ}} \exp(X_+^\circ + X_-^\circ) = \frac{\partial B^\circ(\vec{r}_s, t_s)}{\partial x_s} + \mu\left(\frac{1}{2\pi}\right)^{3/2} \int \phi_+^\circ(\vec{k}) \exp\left[i(\vec{k}\cdot\vec{r}_s - k_\circ c t_s)\right] d\vec{k} \quad (5.4)$$

A typical expression for the transformation of the complex conjugate potentials is:

$$\exp\left[-(X_+^\circ + X_-^\circ)\right] \vec{A}^{\circ *}(\vec{r}_s, t_s) \exp(X_+^\circ + X_-^\circ) = \vec{A}^{\circ *}(\vec{r}_s, t_s) + \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{\vec{k}}{k} \phi_-^\circ(\vec{k}) \exp\left[i(\vec{k}\cdot\vec{r}_s - k_\circ c t_s)\right] d\vec{k} \quad (5.5)$$

The others are related to (5.4) in a similiar manner.

Reference to equation (4.3) shows that the quantities in which we are interested are of the sort:

$$\exp\left[-(X_+ + X_-)\right]\left[\phi^{\circ}(\vec{r}_s, t_s) - \frac{1}{\mu}\frac{\partial B^{\circ}(\vec{r}_s, t_s)}{\partial x_{s0}}\right]\exp(X_+ + X_-)\Omega \qquad (5.6)$$

and

$$\exp\left[-(X_+ + X_-)\right]\left[A_j^{\circ}(\vec{r}_s, t_s) + \frac{1}{\mu}\frac{\partial B^{\circ}(\vec{r}_s, t_s)}{\partial x_{sj}}\right]\exp(X_+ + X_-)\Omega \qquad (5.7)$$

By the use of (5.4) and of the equations:

$$Q_{\pm}^{*\circ}(\vec{k})\Omega = 0$$
$$Q_{\pm}^{\circ}(\vec{k})\Omega = 0 \qquad (5.8)$$

we obtain for (5.6)

$$\left\{-\frac{1}{\mu}\frac{\partial B^{\circ}(r_s, t_s)}{\partial x_{s0}} + \left(\frac{1}{2\pi}\right)^{3/2}\int Q_-^{\circ *}(\vec{k})\exp\left[-i(\vec{k}\cdot\vec{r}_s - k_0 c t_s)\right] d\vec{k}\right\}\Omega \qquad (5.9)$$

and for (5.7)

$$\left\{A_j^{\circ}(\vec{r}_s, t_s) + \frac{1}{\mu}\frac{\partial B^{\circ}(\vec{r}_s, t_s)}{\partial x_{sj}}\right\}\Omega \qquad (5.10)$$

If the operator in braces in equation (5.9) is written as a Fourier integral, we have for the coefficient of $\left(\frac{1}{2\pi}\right)^{3/2} \times$ $\exp\left[-i(\vec{k}\cdot\vec{r}_s - k_0 c t_s)\right]$ in this integral:

$$\frac{-ik_0 B_-^{\circ *}(\vec{k})}{\mu} + \frac{\vec{k}\cdot\vec{A}_-^{\circ *}(\vec{k}) + i\mu B_-^{\circ *}(\vec{k})}{k_0}$$
$$= \frac{i\mu B_-^{\circ *}(\vec{k})}{k_0}\left(1 - \frac{k_0^2}{\mu^2}\right) + \frac{\vec{k}\cdot\vec{A}_-^{\circ *}(\vec{k})}{k_0}$$
$$= \frac{i\mu B_-^{\circ *}(\vec{k})}{k_0}\left(-\frac{k^2}{\mu^2}\right) + \frac{\vec{k}\cdot\vec{A}_-^{\circ *}(\vec{k})}{k_0}$$

$$= \frac{\vec{k}}{k_0} \cdot \left[A_-^{\circ *}(\vec{k}) - \frac{i\vec{k}B_-^{\circ *}(\vec{k})}{\mu} \right] \tag{5.11}$$

Similiarly the Fourier amplitude corresponding to the quantity in braces in (5.10) is

$$A_j^\circ(\vec{k}) + \frac{i}{\mu} B$$

The field quantities occurring in (5.8) and (5.9) and hence in R_s may thus all be expressed in terms of the quantity $\vec{C}_\pm^\circ(\vec{k})$ which we define as:

$$\vec{C}_\pm^\circ(\vec{k}) = \vec{A}_\pm^\circ(\vec{k}) + \frac{i\vec{k}B_\pm^\circ(\vec{k})}{\mu} \tag{5.12}$$

In addition it is possible to express $\vec{A}_\pm^\circ(\vec{k})$ and $B_\pm^\circ(\vec{k})$ in terms of $C_\pm^\circ(\vec{k})$ and $Q_\pm^\circ(\vec{k})$. We can thus express Ω in terms of Q's and C's. However, since $Q_\pm^\circ(\vec{k})$ is an operator and, therefore, not present in Ω and, since Ω has been assumed independent of $Q_\pm^{\circ *}(\vec{k})$, Ω depends only on the C's. The commutation rules for the C's are:

$$\left[C_{i\pm}^{\circ *}(\vec{k}), C_{j\pm}^\circ(\vec{k})\right] = \left[A_\pm^{\circ *}{}_i(\vec{k}), A_{\pm j}^\circ(\vec{k}')\right]$$

$$+ \frac{k_i k_j}{\mu^2}\left[B_\pm^{\circ *}(\vec{k}), B_\pm^\circ(\vec{k}')\right]$$

$$= -\frac{c\hbar}{2k_0}\left[\delta_{ij} + \frac{k_i k_j}{\mu^2}\right]\delta(\vec{k} - \vec{k}') \tag{5.13}$$

Upon adding together the equations (4.1) and setting $t_1 = t_s = t_n = t$, we get

$$\sum_{s=1}^{n}\Big\{c\vec{\alpha}_s\cdot\vec{p}_s + \varepsilon_s\tau_{-s}\left(\frac{1}{2\pi}\right)^{3/2}\int\left(\frac{\vec{k}}{k_o}+\vec{\alpha}_s\right)\cdot\Big(\vec{C}^{\circ}_{+}(\vec{k})\times$$

$$\exp\left[i(\vec{k}\cdot\vec{r}_s - k_o ct)\right] + \vec{C}^{\circ *}_{-}(\vec{k})\exp\left[-i(\vec{k}\cdot\vec{r}_s - k_o ct)\right]\Big)d\vec{k}$$

$$+\varepsilon_s\tau_{+s}\left(\frac{1}{2\pi}\right)^{3/2}\int\left(\frac{\vec{k}}{k_o}+\vec{\alpha}_s\right)\cdot\Big(\vec{C}^{\circ}_{-}(\vec{k})\exp\left[i(\vec{k}\cdot\vec{r}_s - k_o ct)\right]$$

$$+\vec{C}^{\circ *}_{+}(\vec{k})\exp\left[-i(\vec{k}\cdot\vec{r}_s - k_o ct)\right]\Big)\,d\vec{k} + m_s c^2\beta_s\Big\}\Omega = i\hbar\frac{\partial\Omega}{\partial t} \qquad (5.14)$$

This result can be obtained in a somewhat simpler manner. We assume the field quantities upon which Ψ depends are:

$$\phi^{\circ}_{\pm}(\vec{k}) + \frac{ik_{\ell}B^{\circ}_{\pm}(\vec{k})}{\mu} = \Phi^{\circ}_{\pm}(\vec{k})$$

and

$$A^{\circ}_{\pm\ell}(\vec{k}) + \frac{ik_{\ell}B^{\circ}_{\pm}(\vec{k})}{\mu} = \Phi^{\circ}_{\pm\ell}(\vec{k}) \qquad (5.15)$$

Further, we take the auxiliary conditions to be the operational relations:

$$k_o\left[\phi^{\circ}_{\pm}(\vec{k}) + \frac{ik_o B^{\circ}_{\pm}(\vec{k})}{\mu}\right] - \vec{k}\cdot\vec{C}^{\circ}_{\pm}(\vec{k}) = 0$$

$$k_o\left[\phi^{\circ *}_{\pm}(\vec{k}) - \frac{ik_o B^{\circ *}_{\pm}(\vec{k})}{\mu}\right] - \vec{k}\cdot C^{\circ *}_{\pm}(\vec{k}) = 0 \qquad (5.16)$$

Since each of these commutes with the quantities (5.15) and their conjugates, these auxiliary conditions are consistent with the commutation rules. We can use (5.16) to eliminate from R_s and Ψ the quantities $\phi^{\circ}_{\pm}(\vec{k}) - ik_o B^{\circ}_{\pm}(\vec{k})/\mu$ and their conjugates; if we call Ψ when expressed in this manner Ω, we obtain (5.14)

6. Computation of Interaction Energy:

It is convenient to write $\vec{C}^\circ_\pm(\vec{k})$ as

$$\vec{C}^\circ_\pm(\vec{k}) = \sqrt{\frac{c\hbar}{2k_0}}\left[\vec{b}^\circ_\pm(\vec{k}) - N\vec{k}\vec{k}\cdot\vec{b}^\circ_\pm(\vec{k})\right] \tag{6.1}$$

with

$$N = \frac{1 + k_0/\mu}{k^2} \tag{6.2}$$

This serves to determine the commutation rules of the b's as:

$$\left[b^{\circ *}_{\pm j}(\vec{k}), b^\circ_{\pm i}(\vec{k}')\right] = -\delta_{ji}\,\delta(\vec{k} - \vec{k}') \tag{6.3}$$

To confirm that the commutation rules for the C's are preserved by this representation, we compute

$$\left[C^{\circ *}_{\pm i}(\vec{k}), C^\circ_{\pm j}(\vec{k}')\right] = \frac{c\hbar}{2k_0}\left\{\left[b^{\circ *}_{\pm i}(\vec{k}), b^\circ_{\pm j}(\vec{k}')\right] - \left(\frac{1 + k_0/\mu}{k^2}\right) \times\right.$$

$$\left(\left[b^{\circ *}_{\pm i}(\vec{k}), \vec{k}'\cdot\vec{b}^\circ_\pm(\vec{k}')\right]k_j' + k_i\left[\vec{k}\cdot b^{\circ *}_\pm(\vec{k}), b^\circ_{\pm j}(\vec{k}')\right]k_j\right)$$

$$\left. + \left(\frac{1 + k_0/\mu}{k^2}\right)^2 \left[\vec{k}\cdot\vec{b}^{\circ *}_\pm(\vec{k}), \vec{k}'\cdot b^\circ_\pm(\vec{k}')\right]k_i k_j'\right\}$$

$$= -\frac{c\hbar}{2k_0}\left\{\delta_{ij} - 2\left(\frac{1 + k_0/\mu}{k^2}\right)k_i k_j + \frac{1 + 2k_0/\mu + k_0^2/\mu^2}{k^2}\, k_i k_j\right\} \times$$

$$\delta(\vec{k} - \vec{k}') = -\frac{c\hbar}{2k_0}\left\{\delta_{ij} + k_i k_j/\mu^2\right\}\delta(\vec{k} - \vec{k}') \tag{6.4}$$

The time dependence of the potentials in (5.13) may be eliminated by the transformation

$$\Omega = e^{iwt}\,\Omega'$$

with

$$w = c\int k_0\left[\vec{b}^{\circ *}_+(\vec{k})\cdot\vec{b}^\circ_+(\vec{k}) + b^{\circ *}_-(\vec{k})\cdot\vec{b}^\circ_-(\vec{k})\right]d\vec{k} \tag{6.5}$$

It is now possible to write the transformed equation (5.13) as;

$$\left\{H_o + \frac{\hbar}{i}\frac{\partial}{\partial t} + c\hbar w\right\}\Omega' = \int d\vec{k}\left[\vec{G}_+^{o*}(\vec{k})\cdot\vec{b}_+^{o}(\vec{k}) + \vec{G}_+^{o}(\vec{k})\cdot\vec{b}_+^{o*}(\vec{k})\right.$$

$$\left. + \vec{G}_-^{o*}(\vec{k})\cdot\vec{b}_-^{o}(\vec{k}) + \vec{G}_-^{o}(\vec{k})\cdot\vec{b}_-^{o*}(\vec{k})\right]\Omega' \qquad (6.6)$$

where

$$H_o = \sum_{s=1}^{n}\left[c\vec{\alpha}_s\cdot\vec{p}_s + m_s c^2\beta_s\right] \qquad (6.7)$$

and

$$\vec{G}_\pm^*(\vec{k}) = \left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{c\hbar}{2k_o}\right)^{1/2}\sum_{s=1}^{n} g_s \mathcal{T}_{\mp s}\left[\vec{\alpha}_s - N\vec{\alpha}_s\cdot\vec{k}\vec{k} - \frac{\vec{k}}{k_o}\right.$$

$$\left. + \frac{Nk^2\vec{k}}{k_o}\right] \qquad (6.8)$$

and conjugate.

In the conjugate of (6.8) $\mathcal{T}_\pm$ is to be replaced by $\mathcal{T}_\mp$.

We are now ready to determine the interaction energy by the perturbation method. To do this we make the association:

$$b^{o}_{\pm j}(\vec{k}) \rightarrow \frac{\delta}{\delta b^{o\prime}_{\pm j}(\vec{k})}$$

$$b^{o*}_{\pm j}(\vec{k}) \rightarrow b^{o\prime}_{\pm j}(\vec{k}) \qquad (6.9)$$

We assume Ω' to be a functional of $\vec{b}'_\pm(\vec{k})$ and expand it in the series

$$\Omega' = \Omega_{oo} + \Omega_{o1} + \Omega_{1o} \qquad (6.10)$$

where

$$\Omega_{oo} = \Psi_{oo}$$

$$\Omega_{1o} = \int\vec{\Psi}_{1o}(k)\cdot\vec{b}_+^{o\prime}(\vec{k})d\vec{k}$$

$$\Omega_{o1} = \int\vec{\Psi}_{o1}(k)\cdot\vec{b}_-^{o\prime}(\vec{k})d\vec{k} \qquad (6.11)$$

If we substitute (6.10) in (6.6) and assume

$$\left|\left[H_0 + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\vec{\Psi}_{01}(\vec{k})\right| << \left|\hbar c k_0 \vec{\Psi}_{01}(\vec{k})\right| \tag{6.12}$$

we obtain the following equation for Ψ_{00} :

$$\left[H_0 + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\Psi_{00} = \int\frac{d\vec{k}}{\hbar c k_0}\left[\vec{G}_+^{0*}(\vec{k})\cdot\vec{G}_+^{0}(\vec{k}) + \vec{G}_-^{0*}(\vec{k})\cdot\vec{G}_-^{0}(\vec{k})\right]\Psi_{00} \tag{6.13}$$

Thus the interaction energy is:

$$U^0 = -\int\frac{d\vec{k}}{\hbar c k_0}\left[\vec{G}_+^{0*}(\vec{k})\cdot\vec{G}_+^{0}(\vec{k}) + \vec{G}_-^{0*}(\vec{k})\cdot\vec{G}_-^{0}(\vec{k})\right] \tag{6.14}$$

By making use of the definitions (6.7) and after performing some simplifications, we find:

$$\vec{G}_+^{0*}(\vec{k})\cdot\vec{G}_+^{0}(\vec{k}) = \left(\frac{1}{2\pi}\right)^3 \frac{c\hbar}{2k}\sum_{u,v=1}^{n} \tau_{-u}\tau_{+v} g_u g_v \left\{ \vec{\alpha}_u\cdot\vec{\alpha}_v + \frac{\vec{\alpha}_u\cdot k\,\vec{\alpha}_v\cdot k}{\mu^2} + \frac{k^2}{\mu^2} + \frac{\vec{\alpha}_u\cdot\vec{k}k_0}{\mu^2} + \frac{\vec{\alpha}_v\cdot\vec{k}k_0}{\mu^2}\right\}\exp\left(i\vec{k}\cdot\vec{R}_{uv}\right) \tag{6.15}$$

where

$$\vec{R}_{uv} = \vec{r}_u - \vec{r}_v \tag{6.16}$$

We are now ready to evaluate U_1^0. To do this we write it as

$$U^0 = U_0^0 + U_1^0 \tag{6.17}$$

where U_0^0 corresponds to the summation over $u = v$ in (6.15), and U_1^0, to that over $u \neq v$ in (6.15). Upon expressing the integrals in terms of spherical coordinates and integrating over the angles, we have for U_1^0,

$$U_1^0 = -\frac{1}{16\pi^2}\sum_{u\neq v} g_u g_v (\tau_{-u}\tau_{+v} + \tau_{+u}\tau_{-v}) \left\{\frac{\vec{\alpha}_u\cdot\vec{\alpha}_v}{R_{uv}} \times\right.$$

$$\left(2I_1 + \frac{I_2}{\mu^2} - \frac{I_3}{\mu^2}\right) + \frac{\vec{\alpha}_u \cdot \vec{R}_{uv} \vec{\alpha}_v \cdot \vec{R}_{uv}}{R^3_{uv}} \left(3I_3 - I_2\right) + \frac{2I_2}{\mu^2 R_{uv}} \Big\} \qquad (6.18)$$

Where

$$I_1 = \int_{-\infty}^{\infty} \frac{k\exp(ik\ R_{uv})}{ik_o^2}\ dk$$

$$I_2 = \int_{-\infty}^{\infty} \frac{k^3\exp(ik\ R_{uv})}{ik_o^2}\ dk$$

$$I_3 = \int_{-\infty}^{\infty} \left[1 + \frac{2i}{kR_{uv}} - \frac{2}{k^2R^2_{uv}}\right] \frac{k^3\exp(ik\ R_{uv})}{ik_o^2}\ dk \qquad (6.19)$$

Because the integrands of I_2 and I_3 diverge in absolute value for $k \to \infty$ it is convenient to rewrite I_2 and I_3 as follows:

$$I_2 = \int_{-\infty}^{\infty} \left[k - \frac{k\mu^2}{k_o^2}\right] \frac{\exp(ik\ R_{uv})}{i}\ dk$$

$$I_3 = \int_{-\infty}^{\infty} \left\{k - \frac{k\mu^2}{k_o^2} + \frac{2i}{R_{uv}} - \frac{2\mu^2 i}{k_o^2 R_{uv}} - \frac{2k}{k_o^2 R^2_{uv}}\right\} \frac{\exp(ik\ R_{uv})}{i}\ dk$$

$$(6.20)$$

If we use the formulas

$$\delta(R_{uv}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ik\ R_{uv})\ dk$$

and

$$\delta'(R_{uv}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ik\exp(ik\ R_{uv})\ dk \qquad (6.21)$$

and apply the method of residues we can now evaluate the integrals. Since our present interest is in the case, $u \neq v$, all δ functions vanish and we omit them. We find in this manner:

$$I_1 = \pi \exp(-\mu R_{uv})$$

$$I_2 = -\pi\mu^2 \exp(-\mu R_{uv})$$

$$I_3 = -\pi\left[\mu^2 + \frac{2\mu}{R_{uv}} + \frac{2}{R^2_{uv}}\right]\exp(-\mu R_{uv}) \qquad (6.22)$$

We thus have:

$$U_1^\circ = -\frac{1}{8\pi}\sum_{u\neq v} g_u g_v (\tau_{-u}\tau_{+v} + \tau_{+u}\tau_{-v})\exp(-\mu R_{uv})\left\{\frac{\vec{\alpha}_u\cdot\vec{\alpha}_v}{R_{uv}} \times \right.$$

$$\left.\left(1+\frac{1}{\mu R_{uv}}+\frac{1}{\mu^2 R_{uv}}\right) - \frac{\vec{\alpha}_u\cdot\vec{R}_{uv}\vec{\alpha}_v\cdot\vec{R}_{uv}}{R^3_{uv}}\left(1+\frac{3}{\mu R_{uv}}+\frac{3}{\mu^2 R^2_{uv}}\right) - \frac{1}{R_{uv}}\right\}$$

(6.23)

Similarly we find for the interaction enery of the $\underline{a}$ field:

$$U_1' = \frac{1}{8\pi}\sum_{u\neq v} g_u g_v(\tau_{-u}\tau_{+v} + \tau_{+u}\tau_{-v})\,\mathrm{esp}\left[-\frac{R_{uv}}{a}\right]\left\{\frac{\vec{\alpha}_u\cdot\vec{\alpha}_v}{R_{uv}} \times\right.$$

$$\exp\left(-\frac{R_{uv}}{a}\right)\left(1+\frac{a}{R_{uv}}+\frac{a^2}{R^2_{uv}}\right) - \frac{\vec{\alpha}_u\cdot\vec{R}_{uv}\vec{\alpha}_v\ \vec{R}_{uv}}{R^3_{uv}} \times$$

$$\left.\left(1+\frac{3a}{R_{uv}}+\frac{3a^2}{R^2_{uv}}\right) - \frac{1}{R_{uv}}\right\} \qquad (6.24)$$

Finally we find for the self energy upon expressing the integrals in spherical coordinates and integrating over the angles:

$$U = U_0^\circ + U_0' = \frac{1}{4\pi}\sum_u g^2{}_u(J_1 + J_2)$$

$$J_1 = \left(\mu^2 - \frac{1}{d^2}\right)\int_0^\infty \frac{k^2dk}{k^2k_1^2} = \mu - \frac{1}{a}$$

$$J_2 = 2\left(a^2 - \frac{1}{\mu^2}\right)\int_0^\infty k^2dk \qquad (6.26)$$

7. Discussion:

The result obtained is disappointing in two ways. First, because of the presence of J_2 in (6.26), the self energy remains infinite. Secondly, it was to be hoped that, aside from the presence of the Γ's and a possible constant factor, the interaction energy would be that found by Green[1]. This is indeed the case for the terms not involving $\vec{\alpha}$'s. However, because of the different treatment of auxiliary conditions, the rest of the expression is quite different from his. Furthermore, it is unsuited for application since, on account of the high powers of $\frac{1}{R_{uv}}$ occurring in our expression, any attempt to solve the wave equation containing this potential would require the artificial device of "cutting off."

It should be remarked that the difficulty is not due to the generalization but rather to the treatment of the auxiliary conditions necessitated by the presence of Γ's. Indeed, (6.23) is just the expression one would obtain in the ungeneralized theory by this approach. On the other hand, (6.24) (except for the presence of the Γ's and a possible constant factor) shows the result which would have been obtained for the tilde contribution of the interaction energy had choice 2 of Part I been employed. It is for this reason that (6.23) and (6.24) were obtained separately.

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