UNIVERSITY OF CINCINNATI

May 29 1948

I hereby recommend that the thesis prepared under my supervision by William Henry Spragens, Jr.

entitled On the Absolute Cesàro Summability of Double Fourier Series

be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy

Approved by:

Charles N. Moore

Form 668—G. S. and T. C.—1M—6-41

ON THE ABSOLUTE CESÀRO SUMMABILITY OF DOUBLE FOURIER SERIES

A dissertation submitted to the

Graduate School of Arts and Sciences
of the University of Cincinnati

in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

1948

by

William Henry Spragens, Jr.
A.B. University of Kentucky 1935
M.S. University of Kentucky 1938

24 Ag '48

1. Introduction

The concept of absolute Cesàro summability of infinite series was introduced by Fekete [8][1] in 1911, using integral orders of summability, and defined more generally by Kogbetliantz [12] in 1925. It is an extension of the concept of absolute convergence, much as summability is an extension of ordinary convergence.

If we think of absolute convergence as convergence of the series of absolute values of terms of the original series, no such generalization would appear to be fruitful, for a series of positive terms which is not convergent is not summable either.

If, however, we think of absolute convergence in terms of the series of absolute values of the differences of successive partial sums, the method of generalization becomes apparent; we use differences of successive Cesàro means. Thus, if s_n^α represents the n-th Cesàro partial sum of order α of a given series, the latter is said to be absolutely summable (C,α), or summable $|C,\alpha|$, if $\sum|s_n^\alpha - s_{n-1}^\alpha|$ converges.

---

[1]Numbers in brackets refer to the bibliography.

The absolute summability of Fourier series has, naturally, received the attention of students of the subject, mainly in the last ten or fifteen years. Moreover, absolute summabilities of types other than Cesàro have been defined and studied; e. g., absolute Abel summability.

The generalization of these matters to double series, however, has been considered only for Bessel summability by Chandrasekharan [6]. The latter investigated the absolute Bessel summability of series of two-dimensional eigenfunctions of a very general type. The trigonometric product functions of double Fourier series appear as a very special case of these eigenfunctions, and in this case the Bessel summability assumes the aspect of Bochner's [3] "spherical summability".

In this paper, then, we take up the generalization of absolute Cesàro summability to double series. It is defined for such series in Section 2, and the definition is shown, in Lemma 1 of that section, to be equivalent to the absolute convergence of a certain related double series. Lemma 2 establishes the "consistency" of this summability, i.e., that summability of certain indices implies summability of higher indices.

We then proceed to study absolute Cesàro summability as applied to double Fourier series. In Section 3 we generalize to double series a sequence of theorems due to Bosanquet [4], [5] which give a sufficient and a necessary condition on a function for the absolute summability of its Fourier series.

In Section 4 is a generalization of a theorem of Hyslop [11] giving a Lipschitz condition on the function as sufficient for the absolute summability of its Fourier series.

Finally, it is shown in Section 5 that absolute Cesàro summability of orders greater than 1 has the cross-neighborhood localization property.

It is a pleasure for the writer to acknowledge the invaluable aid of Professor C. N. Moore, who suggested the investigation of this topic and was generous with counsel and encouragement.

2. Definitions. Preliminary Lemmas

Let $\sum_{m,n=0}^{\infty} a_{mn}$ be a double series, and let $s_{mn}^{\alpha\beta}$ denote the m,n-th Cesàro mean of order α,β of the double sequence $\{s_{\mu\nu}\}=\left\{\sum_{p=0}^{\mu}\sum_{q=0}^{\nu} a_{pq}\right\}$; i. e.,

$$s_{mn}^{\alpha\beta} \equiv \frac{1}{A_m^{\alpha}A_n^{\beta}}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{m-\mu}^{\alpha-1}A_{n-\nu}^{\beta-1}s_{\mu\nu} \equiv \frac{1}{A_m^{\alpha}A_n^{\beta}} S_{mn}^{\alpha\beta},$$

where

$$A_m^{\alpha} \equiv \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)\Gamma(m+1)}, \qquad (\alpha > -1,\ m=0,1,2,\cdots)$$

$$A_0^{-1} \equiv 1,$$

$$A_m^{-1} \equiv 0, \qquad (m=1,2,\cdots).$$

Further, we define $s_{mn}^{\alpha\beta}$ to be zero if either m or n (or both) is negative.

<u>Definition</u> The double series $\sum_{m,n=0}^{\infty} a_{mn}$ is said to be absolutely summable (C,α,β), or summable $|C,\alpha,\beta|$, if the double series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|s_{mn}^{\alpha\beta}-s_{m-1,n}^{\alpha\beta}-s_{m,n-1}^{\alpha\beta}+s_{m-1,n-1}^{\alpha\beta}\right|$$

converges.

Now we let $\tau_{mn}^{\alpha\beta}$ denote the m,n-th Cesàro mean of order α, β of the double sequence $\{\tau_{\mu\nu}\} = \{\mu\nu a_{\mu\nu}\}$, and let τ^{α}_{mo} denote the m-th (C, α) mean of $\{\mu a_{\mu o}\}$, while τ^{β}_{on} denotes the n-th (C, β) mean of $\{\nu a_{o\nu}\}$. Then we have the following fundamental equivalence.

Lemma 1 The summability $|C, \alpha, \beta|$ of $\sum_{m,n=0}^{\infty} a_{mn}$ is equivalent to the convergence of the double series

$$(2.1) \quad \sum_{m=1}^{\infty} m^{-1}|\tau^{\alpha}_{mo}| + \sum_{n=1}^{\infty} n^{-1}|\tau^{\beta}_{on}| + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}|\tau^{\alpha\beta}_{mn}|.$$

Proof We have only to show that

$$(2.2) \quad s^{\alpha\beta}_{mn} - s^{\alpha\beta}_{m-1,n} - s^{\alpha\beta}_{m,n-1} + s^{\alpha\beta}_{m-1,n-1} \begin{cases} = \frac{1}{mn}\tau^{\alpha\beta}_{mn} \quad (m,n=1,2,\cdots) \\ = \frac{1}{m}\tau^{\alpha}_{mo} \quad (n=0, m=1,2,\cdots) \\ = \frac{1}{n}\tau^{\beta}_{on} \quad (m=0, n=1,2,\cdots). \end{cases}$$

We have, for all n and for m = 1, 2, ... ,

$$s^{\alpha\beta}_{mn} - s^{\alpha\beta}_{m-1,n} = \frac{1}{A^{\alpha}_m A^{\beta}_n}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A^{\alpha-1}_{m-\mu}A^{\beta-1}_{n-\nu}s_{\mu\nu} - \frac{1}{A^{\alpha}_{m-1}A^{\beta}_n}\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n} A^{\alpha-1}_{m-\mu-1}A^{\beta-1}_{n-\nu}s_{\mu\nu}$$

$$= \frac{1}{A^{\beta}_n}\sum_{\nu=0}^{n} A^{\beta-1}_{n-\nu}\left(\frac{1}{A^{\alpha}_m}\sum_{\mu=0}^{m} A^{\alpha}_{m-\mu}a_{\mu\nu} - \frac{1}{A^{\alpha}_{m-1}}\sum_{\mu=0}^{m-1} A^{\alpha}_{m-\mu-1}a_{\mu\nu}\right)$$

$$= \frac{1}{A^{\beta}_n}\sum_{\nu=0}^{n} A^{\beta-1}_{n-\nu}\left(\frac{1}{A^{\alpha}_m}\left\{a_{m\nu} + \sum_{\mu=0}^{m-1}\left[A^{\alpha}_{m-\mu} - \frac{A^{\alpha}_m A^{\alpha}_{m-\mu-1}}{A^{\alpha}_{m-1}}\right]a_{\mu\nu}\right\}\right)$$

$$= \frac{1}{A_n^\beta}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1}\left(\frac{1}{A_m^\alpha}\left\{a_{m\nu}+\sum_{\mu=0}^{m-1} A_{m-\mu}^{\alpha-1}\left[\frac{\alpha+m-\mu}{\alpha}-\frac{(\alpha+m)(m-\mu)}{\alpha m}\right]a_{\mu\nu}\right\}\right)$$

$$= \frac{1}{A_n^\beta}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1}\left(\frac{1}{A_m^\alpha}\left\{a_{m\nu}+\sum_{\mu=0}^{m-1}\frac{\mu A_{m-\mu}^{\alpha-1}}{m}\,a_{\mu\nu}\right\}\right)$$

$$= \frac{1}{A_n^\beta}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1}\left(\frac{1}{mA_m^\alpha}\sum_{\mu=0}^{m}\mu A_{m-\mu}^{\alpha-1}\,a_{\mu\nu}\right).$$

In the case $n = 0$, we have (2.2) already. It is proved in similar fashion for $m = 0$. Hence we consider now $m = 1,2,\ldots,$ $n = 1,2,\ldots$. Then by the same reduction as that above, the second pair of terms in the left member of (2.2) yields

$$-s_{m,n-1}^{\alpha\beta}+s_{m-1,n-1}^{\alpha\beta} = -\frac{1}{A_{n-1}^\beta}\sum_{\nu=0}^{n-1} A_{n-\nu-1}^{\beta-1}\left(\frac{1}{mA_m^\alpha}\sum_{\mu=0}^{m}\mu A_{m-\mu}^{\alpha-1}\,a_{\mu\nu}\right).$$

Finally, adding these two equations and employing the same type of reduction with respect to the summations on index ν , we get the left member of (2.2) equal to

$$\frac{1}{nA_n^\beta}\sum_{\nu=0}^{n}\nu A_{n-\nu}^{\beta-1}\left(\frac{1}{mA_m^\alpha}\sum_{\mu=0}^{m}\mu A_{m-\mu}^{\alpha-1}\,a_{\mu\nu}\right)$$

$$= \frac{1}{mn\,A_m^\alpha A_n^\beta}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{m-\mu}^{\alpha-1}A_{n-\nu}^{\beta-1}\,\mu\nu\, a_{\mu\nu}$$

$$= \frac{1}{mn}\,\tau_{mn}^{\alpha\beta} \quad ,$$

which completes the proof of Lemma 1.

We verify also that the following "lemma of consistency" follows from the corresponding theorem in the case of single series which was proved by Kogbetliantz [12].

<u>Lemma 2</u> A double series which is summable $|C, \alpha_0, \beta_0|$ is also summable $|C, \alpha, \beta|$, $\alpha \geqq \alpha_0$, $\beta \geqq \beta_0$.

<u>Proof</u> Consider first $|C, \alpha, \beta_0|$ summability. We wish to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}\left|\tau_{mn}^{\alpha\beta_0}\right| \leqq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}\left|\tau_{mn}^{\alpha_0\beta_0}\right|,$$

where the series on the right converges by hypothesis. Thus we must show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mnA_m^{\alpha}A_n^{\beta_0}}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{m-\mu}^{\alpha-1}A_{n-\nu}^{\beta_0-1}\mu\nu|a_{\mu\nu}| \leqq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mnA_m^{\alpha_0}A_n^{\beta_0}}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{m-\mu}^{\alpha_0-1}A_{n-\nu}^{\beta_0-1}\mu\nu|a_{\mu\nu}|.$$

For this it suffices that, for each ν,

$$\sum_{m=1}^{\infty}\frac{1}{mA_m^{\alpha}}\sum_{\mu=0}^{m} A_{m-\mu}^{\alpha-1}\mu|a_{\mu\nu}| \leqq \sum_{m=1}^{\infty}\frac{1}{mA_m^{\alpha_0}}\sum_{\mu=0}^{m} A_{m-\mu}^{\alpha_0-1}\mu|a_{\mu\nu}|,$$

which is just the single series theorem already known. The latter also assures us of the convergence of the single series in (2.1). Hence, $|C, \alpha_0, \beta_0|$ summability implies $|C, \alpha, \beta_0|$ summability.

Similarly we see that $|C, \alpha, \beta_0|$ summability implies $|C, \alpha, \beta|$ summability, and the lemma is proved.

3. Absolute Cesàro Summability of Double Fourier Series

In this section we prove a sequence of theorems which are analogues for double series of theorems proved by Bosanquet [4],[5] dealing with absolute Cesàro summability of single Fourier series. Theorems 1 and 2 are special cases of Theorems 3 and 4, respectively, but are stated separately because of their relative simplicity and because they would be proved separately anyhow.

We have first to introduce a number of definitions and notations to be used in this and succeeding sections.

Let f(s,t) be Lebesgue-integrable, with period 2π in each variable. For a fixed point (x,y), let

$$\phi(s,t) = \tfrac{1}{4}\left[f(x+s,y+t) - f(x+s,y-t) - f(x-s,y+t) + f(x-s,y-t)\right]$$

and let its double Fourier series be denoted by

$$\sum_{m,n=0}^{\infty} a_{mn} \cos ms \cos nt.$$

Then the double Fourier series of f(s,t) at the point (x,y) is

$$\sum_{m,n=0}^{\infty} a_{mn}.$$

We observe that the formula for a_{mn} is

$$a_{mn} = \frac{4}{2^{\left[\frac{1}{m+1}\right]+\left[\frac{1}{n+1}\right]}\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(s,t)\cos ms \cos nt \, ds\, dt,$$

in which $[x]$ denotes the largest integer $\leq$ x.

Next we define a sort of "mean value" function associated with $\phi(s,t)$, as follows.

$$\phi_{\alpha\beta}(s,t)\begin{cases} \equiv \phi(s,t), & \alpha = 0,\ \beta = 0; \\ \equiv \alpha s^{-\alpha}\int_0^s (s-u)^{\alpha-1}\phi(u,t)\,du, & \alpha > 0,\ \beta = 0; \\ \equiv \beta t^{-\beta}\int_0^t (t-v)^{\beta-1}\phi(s,v)\,dv, & \alpha = 0,\ \beta > 0; \\ \equiv \alpha\beta\, s^{-\alpha} t^{-\beta}\int_0^s\int_0^t (s-u)^{\alpha-1}(t-v)^{\beta-1}\phi(u,v)\,du\,dv, & \alpha > 0,\ \beta > 0. \end{cases}$$

As a last preliminary, we state the definition of the kind of "bounded variation" we shall use in the ensuing theorems. (For a rather complete study of the various definitions of bounded variation for functions of two variables, we refer to the paper [1] by Adams and Clarkson.)

<u>Definition</u> A function $f(x,y)$ in a region $a \leq x \leq c$, $b \leq y \leq d$, is said to be of <u>bounded variation in the sense of Hardy</u> (<u>B. V. H.</u>) if, in the region,

i) $f(x,y)$ has a finite total variation

$$\int_a^c \int_b^d |d_{x,y} f(x,y)| = \sup_{\sigma} \sum_{p=1}^{j} \sum_{q=1}^{k} |f(x_p,y_q) - f(x_{p-1},y_q) - f(x_p,y_{q-1}) + f(x_{p-1},y_{q-1})|$$

where σ denotes a rectangular net

$a = x_0 < x_1 < x_2 < \cdots < x_j = c, \quad b = y_0 < y_1 < y_2 < \cdots < y_k = d;$

ii) $f(\bar{x},y)$ is of bounded variation with respect to y for at least one $\bar{x}$;

iii) $f(x,\bar{y})$ is of bounded variation with respect to x for at least one $\bar{y}$.

It is known [1] that, as consequences of i), ii) and iii), we have also

iv) property ii) holds for all $\bar{x}$;

v) property iii) holds for all $\bar{y}$;

vi) the total variation of f with respect to y,

$$\int_b^d |d_y f(x,y)| \equiv V_1(x),$$

is dominated by an integrable function $U_1(x)$:

$$V_1(x) \leq U_1(x);$$

vii) a symmetric condition on $V_2(y)$.

<u>Theorem 1</u> If $\phi(s,t)$ is of B.V.H. in $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, then the double Fourier series of f(s,t) is summable $|C, \alpha, \beta|$ at the point (x,y) for $\alpha > 0$, $\beta > 0$.

<u>Proof</u> We begin by setting

$$F_\mu(t) \equiv \frac{2}{\pi}\int_0^\pi \mu \cos \mu s \, \phi(s,t)\, ds, \qquad (\mu = 0,1,\ldots).$$

This defines each function of the sequence for almost all t. For exceptional t, one may define the function by $F_\mu(t+0)$ or $F_\mu(t-0)$.

Let $F_m^\alpha(t)$ represent the m-th (C,α) mean of the sequence $\{F_\mu(t)\}$. Thus we have

$$F_m^\alpha(t) = \int_0^\pi \gamma_m^\alpha(s)\, \phi(s,t)\, ds \qquad (m = 0,1,\ldots),$$

where $\gamma_m^\alpha(s)$ represents the m-th (C,α) mean of the sequence $\left\{\frac{2\mu}{\pi}\cos \mu s\right\}$.

Now we show that, for each m, $F_m^{\alpha}(t)$ is of bounded variation in $(0,\pi)$. Consider any subdivision $0 = t_0 < t_1 < \cdots < t_k = \pi$. We have

$$\sum_{i=1}^{k} \left| F_m^{\alpha}(t_i) - F_m^{\alpha}(t_{i-1}) \right| = \sum_{i=1}^{k} \left| \int_0^{\pi} \gamma_m^{\alpha}(s) \left[\phi(s,t_i) - \phi(s,t_{i-1}) \right] ds \right|$$
$$\leq \int_0^{\pi} \left| \gamma_m^{\alpha}(s) \right| \sum_{i=1}^{k} \left| \phi(s,t_i) - \phi(s,t_{i-1}) \right| ds.$$

Now under the assumption that $\phi(s,t)$ is of B.V.H., we know that $\sum_{i=1}^{k} \left| \phi(s,t_i) - \phi(s,t_{i-1}) \right| \leq V_1(s)$ for any s, and that $V_1(s) \leq U_1(s)$, an integrable function. Hence

$$\sum_{i=1}^{k} \left| F_m^{\alpha}(t_i) - F_m^{\alpha}(t_{i-1}) \right| \leq \int_0^{\pi} \left| \gamma_m^{\alpha}(s) \right| U_1(s)\, ds$$
$$\leq \frac{2m}{\pi} \int_0^{\pi} U_1(s)\, ds.$$

Thus, for each m, the total variation of $F_m^{\alpha}(t)$ is bounded on $(0,\pi)$. In particular, this implies that the derivative of this function exists almost everywhere on $(0,\pi)$.

Returning to the formula defining $F_{\mu}(t)$ and integrating by parts, we have, for almost all t,

$$F_{\mu}(t) = \frac{2}{\pi} \left[\phi(s,t) \sin \mu s \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \sin \mu s \frac{\partial}{\partial s} \phi(s,t)\, ds$$
$$= -\frac{2}{\pi} \int_0^{\pi} \sin \mu s \frac{\partial}{\partial s} \phi(s,t)\, ds ,$$

the partial derivative existing almost everywhere on account of the bounded variation of $\phi(s,t)$ with respect to s.

Hence, if $\sigma_m^\alpha(s)$ represents the m-th (C,α) mean of the sequence $\left\{\frac{2}{\pi}\sin\mu s\right\}$, we have

$$F_m^\alpha(t) = -\int_0^\pi \sigma_m^\alpha(s)\frac{\partial}{\partial s}\phi(s,t)\,ds$$

for almost all t on $(0,\pi)$.

Now let

$$G_{m\nu}^\alpha = \frac{2}{\pi}\int_0^\pi \nu\cos\nu t\, F_m^\alpha(t)\,dt = -\frac{2}{\pi}\int_0^\pi \sin\nu t\; dF_m^\alpha(t)$$

and we have for the n-th (C,β) mean of the sequence $\left\{G_{m\nu}^\alpha\right\}$ (with ν regarded as the variable index)

$$G_{mn}^{\alpha\beta} = \tau_{mn}^{\alpha\beta} = -\int_0^\pi \sigma_n^\beta(t)\; dF_m^\alpha(t)$$

$$= \int_0^\pi \sigma_n^\beta(t)\;\frac{d}{dt}\left[\int_0^\pi \sigma_m^\alpha(s)\;\frac{\partial}{\partial s}\,\phi(s,t)\,ds\right]dt.$$

Hence we have

$$\sum_{m=1}^\infty\sum_{n=1}^\infty m^{-1}n^{-1}\left|\tau_{mn}^{\alpha\beta}\right| = \sum_{m=1}^\infty\sum_{n=1}^\infty m^{-1}n^{-1}\left|\int_0^\pi \sigma_n^\beta(t)\int_0^\pi \sigma_m^\alpha(s)\,\frac{\partial^2}{\partial s\partial t}\phi(s,t)\,ds\,dt\right|$$

$$\leq \int_0^\pi\int_0^\pi \left|d_{s,t}\phi(s,t)\right|\sum_{m=1}^\infty m^{-1}\left|\sigma_m^\alpha(s)\right|\sum_{n=1}^\infty n^{-1}\left|\sigma_n^\beta(t)\right|$$

$$< \infty,$$

since Bosanquet has shown [4] that $\sum_{m=1}^\infty m^{-1}\left|\sigma_m^\alpha(s)\right|$ is uniformly bounded.

To complete the proof it must be shown also that the two single series in (2.1) converge. Since they are symmetric, it suffices to consider only the first. We have

$$\tau^{\alpha}_{mo} = \frac{1}{\pi}\int_0^{\pi} F^{\alpha}_m(t)\,dt = -\frac{1}{\pi}\int_0^{\pi}\int_0^{\pi}\sigma^{\alpha}_m(s)\frac{\partial}{\partial s}\phi(s,t)\,ds\,dt$$

and consequently

$$\sum_{m=1}^{\infty} m^{-1}\left|\tau^{\alpha}_{mo}\right| \leqq \frac{1}{\pi}\int_0^{\pi}\int_0^{\pi}\sum_{m=1}^{\infty} m^{-1}\left|\sigma^{\alpha}_m(s)\right|\,\left|d_s\phi(s,t)\right|\,dt < \infty,$$

since $\phi(s,t)$ is of bounded variation with respect to s and its total variation function is dominated by an integrable function of t.

<u>Theorem 2</u> If the double Fourier series of f(s,t) is absolutely convergent at the point (x,y), then $\phi_{1+\delta,1+\epsilon}(s,t)$ is of B.V.H. in $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$, for all δ, ϵ such that $\delta > 0$, $\varepsilon > 0$.

<u>Proof</u> By the lemma of consistency, it suffices to prove the theorem for $0 < \delta < 1$, $0 < \varepsilon < 1$.

We start with the definition

$$\phi_{1+\delta,1+\varepsilon}(s,t) = (1+\delta)(1+\varepsilon)\,s^{-1-\delta}t^{-1-\varepsilon}\int_0^{s}\int_0^{t}(s-u)^{\delta}(t-v)^{\varepsilon}\phi(u,v)\,du\,dv.$$

Since $\sum_{m,n=0}^{\infty} a_{mn}$ converges absolutely, so does

$$\sum_{m,n=0}^{\infty} a_{mn}\cos mu\cos nv,$$

and we may integrate term by term if the latter series is substituted for $\phi(u,v)$. Thus

$$\phi_{1+\delta,1+\varepsilon}(s,t) = \sum_{m,n=0}^{\infty} a_{mn}(1+\delta)(1+\varepsilon)s^{-1-\delta}t^{-1-\varepsilon}\int_0^s\int_0^t (s-u)^{\delta}(t-v)^{\varepsilon}\cos mu \cos nv\,du\,dv$$

$$= \sum_{m,n=0}^{\infty} a_{mn}(1+\delta)s^{-1-\delta}\int_0^s (s-u)^{\delta}\cos mu\,du\,(1+\varepsilon)t^{-1-\varepsilon}\int_0^t (t-v)^{\varepsilon}\cos nv\,dv$$

$$= \sum_{m,n=0}^{\infty} a_{mn}\,\sigma_{\delta}(ms)\,\sigma_{\varepsilon}(nt),$$

the last equality defining $\sigma_{\delta}(ms)$ and $\sigma_{\varepsilon}(nt)$;

Then

$$\frac{\partial}{\partial s}\phi_{1+\delta,1+\varepsilon}(s,t) = \sum_{m,n=0}^{\infty} a_{mn} m\,\sigma_{\delta}'(ms)\,\sigma_{\varepsilon}(nt),$$

$$\frac{\partial}{\partial t}\phi_{1+\delta,1+\varepsilon}(s,t) = \sum_{m,n=0}^{\infty} a_{mn}\,\sigma_{\delta}(ms)\,n\,\sigma_{\varepsilon}'(nt),$$

$$\frac{\partial^2}{\partial s\,\partial t}\phi_{1+\delta,1+\varepsilon}(s,t) = \sum_{m,n=0}^{\infty} a_{mn} m\,\sigma_{\delta}'(ms)\,n\,\sigma_{\varepsilon}'(nt),$$

provided the differentiated series converge uniformly. That is seen to be true by using the inequalities ([4], p. 14):

$$|\sigma_{\alpha}'(x)| \begin{cases} \leq A \\ \leq Ax^{-1-\alpha} \end{cases} \qquad (0<\alpha<1,\ x>0),$$

where we use the letter A to denote a constant, not necessarily the same at each occurrence.

From its definition, $\phi_{1+\delta,1+\varepsilon}(s,t)$ is seen to be absolutely continuous in any region excluding a cross-neighborhood of the origin, and consequently of B.V.H. in any such region. Hence the finiteness of the total variation over the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, and that of the functions $V_1(s)$ and $V_2(t)$, may be obtained by integrating the absolute values of the above derivatives.

For the total variation we have

$$\int_0^\pi\int_0^\pi |d_{s,t}\phi_{1+\delta,1+\varepsilon}(s,t)| \leq \int_0^\pi\int_0^\pi \sum_{m,n=0}^{\infty} |a_{mn}|\, m|\sigma_\delta'(ms)|\, n\,|\sigma_\varepsilon'(nt)|\, ds\, dt$$

$$= \sum_{m,n=0}^{\infty} |a_{mn}|\, m\int_0^\pi |\sigma_\delta'(ms)|\, ds\; n\int_0^\pi |\sigma_\varepsilon'(nt)|\, dt$$

$$\leq A\sum_{m,n=0}^{\infty} |a_{mn}|$$

$$< \infty,$$

since Bosanquet ([4], p. 14) has shown that the integrals like $m\int_0^\pi |\sigma_\delta'(ms)|\, ds$ are uniformly bounded, and since $\sum_{m,n=0}^{\infty} |a_{mn}|$ is convergent.

For $V_1(s)$ a similar argument applies, using a single integration of $|d_s\phi_{1+\delta,1+\varepsilon}(s,t)|$, and the uniform boundedness of $\sigma_\varepsilon(nt)$. Then from symmetry we conclude also that $V_2(t)$ is finite. Thus we see that the conditions in the definition of B.V.H. on page 11 are satisfied by $\phi_{1+\delta,1+\varepsilon}(s,t)$ in $0 \leq s \leq \pi$, $0 \leq t \leq \pi$.

Theorem 3 If $\phi_{\alpha\beta}(s,t)$ is of B.V.H. in $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$, then the double Fourier series of $f(s,t)$ is summable $|C, \gamma, \delta|$ at the point (x,y), when $\gamma > \alpha \geqq 0$, $\delta > \beta \geqq 0$.

Proof Theorem 1 is the case $\alpha = 0$, $\beta = 0$. We consider here the case where the indices are positive. The case in which one of the indices is positive and the other zero can be treated by a combination of the methods used in these proofs. By the lemma of consistency, we may suppose that

$$h \leqq \alpha < \gamma < h+1, \quad k \leqq \beta < \delta < k+1,$$

where $h = [\alpha]$ (the largest integer $\leqq \alpha$) and $k = [\beta]$.

We shall employ the notation

$$\Phi_{\alpha\beta}(s,t) = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} s^{\alpha} t^{\beta} \phi_{\alpha\beta}(s,t),$$

where α and β may be positive or zero. Then, in analogous fashion to Bosanquet's proof [5], we get

$$\begin{aligned}\tau_{mn}^{\gamma\delta} = &\left\{ \left[\sum_{\mu=1}^{h}\sum_{\nu=1}^{k}(-1)^{\mu+\nu} \Phi_{\mu\nu}(s,t)\left(\frac{d}{ds}\right)^{\mu}\sigma_m^{\gamma}(s)\left(\frac{d}{dt}\right)^{\nu}\sigma_n^{\delta}(t)\right]_{s=0}^{s=\pi}\right\}_{t=0}^{t=\pi} \\ &+ \left[\sum_{\nu=1}^{k}(-1)^{\nu+h-1}\left(\frac{d}{dt}\right)^{\nu}\sigma_n^{\delta}(t)\int_0^{\pi}\Phi_{h\nu}(s,t)\left(\frac{d}{ds}\right)^{h+1}\sigma_m^{\gamma}(s)\,ds\right]_{t=0}^{t=\pi} \\ &+ \left[\sum_{\mu=1}^{h}(-1)^{\mu+k-1}\left(\frac{d}{ds}\right)^{\mu}\sigma_m^{\gamma}(s)\int_0^{\pi}\Phi_{\mu k}(s,t)\left(\frac{d}{dt}\right)^{k+1}\sigma_n^{\delta}(t)\,dt\right]_{s=0}^{s=\pi} \\ &+ (-1)^{h+k}\int_0^{\pi}\!\!\int_0^{\pi}\Phi_{hk}(s,t)\left(\frac{d}{ds}\right)^{h+1}\sigma_m^{\gamma}(s)\left(\frac{d}{dt}\right)^{k+1}\sigma_n^{\delta}(t)\,ds\,dt.\end{aligned}$$

Now Bosanquet's estimates ([5], p. 522) show that the magnitude of the first term on the right in the preceding formula is $O(m^{\ell-\gamma})O(n^{k-\delta})$, while the others become, respectively,

$$O(m^{\ell-\gamma})\left[O(n^{\beta-\delta}) + \int_0^{\pi} V(n,v)\,d_v\phi_{\alpha\beta}(u,v)\right],$$

$$O(n^{k-\delta})\left[O(m^{\alpha-\gamma}) + \int_0^{\pi} V(m,u)\,d_u\phi_{\alpha\beta}(u,v)\right],$$

and

$$O(m^{\alpha-\gamma})O(n^{\beta-\delta}) + \int_0^{\pi}\int_0^{\pi} V(m,u)\;V(n,v)\,d_{u,v}\,\phi_{\alpha\beta}(u,v),$$

in which $V(m,u)$ is a function satisfying the inequalities

$$(3.1) \qquad |V(m,u)| \begin{cases} \leqq A m^{\alpha} u^{\alpha} \\ \leqq A m^{\alpha-\gamma} u^{\alpha-\gamma}. \end{cases}$$

Accordingly, we have, by combining certain terms, and writing only one of a symmetric pair of middle terms,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}\left|\tau_{mn}^{\gamma\delta}\right|$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big|O(m^{\alpha-\gamma-1})O(n^{\beta-\delta-1}) + O(m^{\alpha-\gamma-1})\int_0^{\pi} n^{-1}V(n,v)\,d_v\phi_{\alpha\beta}(u,v)$$
$$+ \int_0^{\pi}\int_0^{\pi} m^{-1}V(m,u)\,n^{-1}V(n,v)\,d_{uv}\phi_{\alpha\beta}(u,v)\Big|$$

which is not greater than

$$A\sum_{m=1}^{\infty} m^{\alpha-\gamma-1}\sum_{n=1}^{\infty} n^{\beta-\delta-1} + A\sum_{m=1}^{\infty} m^{\alpha-\gamma-1}\int_0^{\pi}|d_v\phi_{\alpha\beta}(u,v)|\sum_{n=1}^{\infty} n^{-1}|V(n,v)|$$
$$+\int_0^{\pi}\int_0^{\pi}|d_{uv}\phi_{\alpha\beta}(u,v)|\sum_{m=1}^{\infty} m^{-1}|V(m,u)|\sum_{n=1}^{\infty} n^{-1}|V(n,v)|.$$

But since $0<\alpha<\gamma$, $0<\beta<\delta$, the first double series above is convergent. Considering also (3.1), we have

$$\begin{aligned}\sum_{m=1}^{\infty} m^{-1}|V(m,u)| &= \sum_{m=1}^{[u^{-1}]} m^{-1}O(m^{\alpha}u^{\alpha}) + \sum_{m=[u^{-1}]+1}^{\infty} m^{-1}O(m^{\alpha-\gamma}u^{\alpha-\gamma})\\ &\leqq A\,u^{\alpha}\sum_{m=1}^{[u^{-1}]} m^{\alpha-1} + A\,u^{\alpha-\gamma}\sum_{m=[u^{-1}]+1}^{\infty} m^{\alpha-\gamma-1}\\ &\leqq A + A,\end{aligned}$$

the sums like that on the left thus converging uniformly. Hence

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}|\tau_{mn}^{\gamma\delta}| \leqq A + A\int_0^{\pi}|d_v\phi_{\alpha\beta}(u,v)| + A\int_0^{\pi}\int_0^{\pi}|d_{uv}\phi_{\alpha\beta}(u,v)|$$
$$<\infty,$$

because of the bounded variation of $\phi_{\alpha\beta}(u,v)$.

The observation that the convergence of the single series in (2.2) follows from the single series theorem concludes the proof of this theorem.

<u>Theorem 4</u> If the double Fourier series of f(s,t) is summable $|C,\alpha,\beta|$ at the point (x,y), then $\phi_{\gamma\delta}(s,t)$ is of B.V.H. in $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, for $\gamma > \alpha + 1$, $\delta > \beta + 1$ $(\alpha \geq 0, \beta \geq 0)$.

<u>Proof</u> The case $\alpha = 0$, $\beta = 0$ is covered by Theorem 2. As in the preceding theorem, we exhibit the proof for positive indices here; and we may suppose, without loss of generality, that

$$0 < \alpha < \gamma - 1 < h + 1, \qquad 0 < \beta < \delta - 1 < k + 1,$$

where $h = [\alpha]$, $k = [\beta]$.

As in Theorem 2, we observe that $\phi_{\gamma\delta}(s,t)$ is absolutely continuous in any region excluding an arbitrarily small cross-neighborhood of the origin, and consequently we can prove that it is of B.V.H. on the whole square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$ by integrating the absolute values of its mixed partial derivative over that square, and integrating the absolute values of its first partials from 0 to π for some value of the fixed variable. The finiteness of these three integrals will be enough to establish our theorem.

We shall write, for $\alpha > 0$,

$$\gamma_\alpha(x) \equiv \int_0^1 (1-u)^{\alpha-1} \cos xu \, du.$$

Furthermore, we use the notation

$$\Delta f(n) \equiv f(n) - f(n+1).$$

Now, starting with the definition of $\phi_{\gamma\delta}(s,t)$, we get

$$\begin{aligned} \gamma^{-1}\delta^{-1}\phi_{\gamma\delta}(s,t) &= s^{-\gamma} t^{-\delta} \int_0^s \int_0^t (s-u)^{\gamma-1} (t-v)^{\delta-1} \phi(u,v)\,du\,dv \\ &= \sum_{m,n=0}^{\infty} a_{mn}\, \gamma_\gamma(ms)\, \gamma_\delta(nt) \\ &= \sum_{m,n=0}^{\infty} S_{mn}^{hk}\, \Delta^{k+1}\gamma_\gamma(ms)\, \Delta^{k+1}\gamma_\delta(nt) \end{aligned}$$

by repeated application of Abel's transformation, since the Fourier coefficients a_{mn} are $o(1)$. Then by the same argumants as those used by Bosanquet ([5], pp. 524–525) this reduces to

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \Delta^{k+1}\gamma_\gamma(ms)\, \Delta^{k+1}\gamma_\delta(nt) \sum_{p,q=0}^{p=m,\,q=n} A_{m-p}^{k-\alpha}\, A_{n-q}^{k-\beta}\, S_{pq}^{\alpha-1,\beta-1} \\ &= \sum_{p,q=0}^{\infty} S_{pq}^{\alpha-1,\beta-1} \sum_{m=p}^{\infty} A_{m-p}\, \Delta^{k+1}\gamma_\gamma(ms) \sum_{n=q}^{\infty} A_{n-q}^{k-\beta}\, \Delta^{k+1}\gamma_\delta(nt) \\ &= \sum_{p,q=0}^{\infty} S_{pq}^{\alpha-1,\beta-1}\, J_p(s)\, J_q(t) \end{aligned}$$

where we use the notation

$$J_p(s) = \sum_{m=p}^{\infty} A_{m-p}^{k-\alpha} \Delta^{k+1} \gamma_\gamma(ms) = O(p^{-\gamma}).$$

We also introduce the symbol, to be used directly,

$$V_p(s) = \sum_{n=p}^{\infty} A_r^{\alpha} \Delta J_r(s) = O(p^{\alpha-\gamma}).$$

The estimates on the last two formulas are due to Bosanquet ([5], pp. 526-527).

The sum at the bottom of page 22 becomes

$$\sum_{p,q=0}^{\infty} S_{pq}^{\alpha\beta} \Delta J_p(s) \Delta J_q(t)$$

$$= \sum_{p,q=0}^{\infty} s_{pq}^{\alpha\beta} A_p^{\alpha} \Delta J_p(s) A_q^{\beta} \Delta J_q(t)$$

$$= \sum_{p,q=0}^{\infty} s_{pq}^{\alpha\beta} \Delta V_p(s) \Delta V_q(t)$$

$$= \sum_{p,q=0}^{\infty} (s_{pq}^{\alpha\beta} - s_{p-1,q}^{\alpha\beta} - s_{p,q-1}^{\alpha\beta} + s_{p-1,q-1}^{\alpha\beta}) V_p(s) V_q(t).$$

Now by differentiating we obtain, using $\Delta_{11} s_{pq}^{\alpha\beta}$ as an abbreviation for the quantity in parentheses above,

$$(3.2) \quad \gamma^{-1}\delta^{-1} \frac{\partial}{\partial s} \phi_{\gamma\delta}(s,t) = \sum_{p,q=0}^{\infty} \Delta_{11} s_{pq}^{\alpha\beta} V_p'(s) V_q(t),$$

$$(3.3)\quad \gamma^{-1}\delta^{-1}\frac{\partial}{\partial t}\phi_{\gamma\delta}(s,t) = \sum_{p,q=0}^{\infty} \Delta_{11}s_{pq}^{\alpha\beta}\, V_p(s)\, V_q'(t),$$

$$(3.4)\quad \gamma^{-1}\delta^{-1}\frac{\partial^2}{\partial s\partial t}\phi_{\gamma\delta}(s,t) = \sum_{p,q=0}^{\infty} \Delta_{11}s_{pq}^{\alpha\beta}\, V_p'(s)\, V_q'(t),$$

provided differentiation term by term is allowable, and hence we get

$$(3.5)\quad \gamma^{-1}\delta^{-1}\int_0^{\pi}\left|\frac{\partial}{\partial s}\phi_{\gamma\delta}(s,t)\right| ds \leq \sum_{p,q=0}^{\infty}\left|\Delta_{11}s_{pq}^{\alpha\beta}\right|\, |V_q(t)| \int_0^{\pi}|V_p'(s)|\, ds,$$

a symmetric inequality from (3.3), and

$$(3.6)\quad \gamma^{-1}\delta^{-1}\int_0^{\pi}\!\!\int_0^{\pi}\left|\frac{\partial^2}{\partial s\partial t}\phi_{\gamma\delta}(s,t)\right| ds\, dt \leq \sum_{p,q=0}^{\infty}\left|\Delta_{11}s_{pq}^{\alpha\beta}\right| \int_0^{\pi}|V_p'(s)|\, ds \int_0^{\pi}|V_q'(t)|\, dt.$$

But Bosanquet (loc. cit.) has shown that the integrals in the right members of (3.5) and (3.6) are uniformly bounded. Since, by hypothesis,

$$\sum_{p,q=0}^{\infty}\left|\Delta_{11}s_{pq}^{\alpha\beta}\right| < \infty ,$$

we have the convergence of the right members of (3.5) and (3.6), which means that $\phi_{\gamma\delta}(s,t)$ is of B.V.H.

It remains only for us to justify the differentiations leading to (3.2), (3.3) and (3.4). Bosanquet (loc. cit.) has shown that

$$\left|V_p'(s)\right| \begin{cases} \leqq A\ p^{\alpha}\ s^{\alpha-1} \\ \leqq A\ p^{1+\alpha-\gamma}\ s^{\alpha-\gamma}, \end{cases}$$

which enables us to see that the differentiated series converge uniformly with respect to s and t outside an arbitrarily small cross-neighborhood of the origin. This completes the proof.

4. Lipschitz Conditions Sufficient for Absolute Summability

The following brief statements of known results indicate our object in this section:

Theorem (Bernstein [2])

If $f(x) \in \text{Lip}\,\alpha$, $\alpha > \frac{1}{2}$, then its Fourier series is absolutely convergent.

Theorem (Hyslop [11])

If $f(x) \in \text{Lip}\,\alpha$, $0 < \alpha \leq \frac{1}{2}$, then its Fourier series is summable $|C,\gamma|$, $\gamma > \frac{1}{2} - \alpha$.

Theorem (Reves [14])

If $f(x,y) \in \text{Lip}\, S_{11}(\alpha,\beta)$, Lip α in x, and Lip β in y, $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, then its double Fourier series is absolutely convergent.

We propose to complete this array by proving the two-variable analogue of Hyslop's theorem. The result may also be regarded as an extension of Reves' theorem, inasmuch as absolute summability may be considered an extension of absolute convergence.

The theorem attributed to Reves is in his dissertation [14], and is included in a more general result in a joint paper of Reves and Szász [15]. Practically the same theorem was published recently by Chelidze [7].

We first state the definitions of the Lipschitz conditions we shall impose on f(x,y).

<u>Definition</u> Lip $S_{11}(\alpha,\beta)$ denotes the class of functions f(x,y) which are such that, for $-\pi \leq x \leq \pi$, $-\pi \leq y \leq \pi$,

$$|\Delta_{11} f(x,y;h,k)| \equiv |f(x+h,y+k)-f(x,y+k)-f(x+h,y)+f(x,y)| \leq K|h|^{\alpha}|k|^{\beta}.$$

<u>Definition</u> Lip α in x denotes the class of functions f(x,y) which are such that, for $-\pi \leq x \leq \pi$, $-\pi \leq y \leq \pi$,

$$|\Delta_{10} f(x,y;h)| \equiv |f(x+h,y)-f(x,y)| \leq K|h|^{\alpha}.$$

<u>Definition</u> Lip β in y denotes the class of functions f(x,y) which are such that, for $-\pi \leq x \leq \pi$, $-\pi \leq y \leq \pi$,

$$|\Delta_{01} f(x,y;k)| \equiv |f(x,y+k)-f(x,y)| \leq K|k|^{\beta}.$$

It is to be understood that the above conditions hold uniformly for all x and y.

As in the preceding section, we consider f(x,y) to be L-integrable and periodic. Then our theorem may be stated as follows.

<u>Theorem 5</u> If $f(x,y)$ belongs to each of the three classes Lip $S_{11}(\alpha,\beta)$, Lip α in x, and Lip β in y, $0 < \alpha \leq \frac{1}{2}$, $0 < \beta \leq \frac{1}{2}$, then its double Fourier series is summable $|C,\gamma,\delta|$ for all (x,y) when $\gamma > \frac{1}{2}-\alpha$, $\delta > \frac{1}{2}-\beta$.

<u>Proof</u> We define the function $\psi(x,y;s,t) \equiv \psi(s,t)$ to be

$$\frac{1}{4}\left[f(x+s,y+t)+f(x+s,y-t)+f(x-s,y+t)+f(x-s,y-t)\right] - $$
$$- \frac{1}{2}\left[f(x+s,y)+f(x-s,y)\right] - \frac{1}{2}\left[f(x,y+t)+f(x,y-t)\right] + f(x,y)$$

$$= \frac{1}{4}\Delta_{11}f(x,y;s,t)+\frac{1}{4}\Delta_{11}f(x,y;s,-t)+\frac{1}{4}\Delta_{11}f(x,y;-s,t)+\frac{1}{4}\Delta_{11}f(x,y;-s,-t).$$

We observe that $\psi(s,t)$ is an even-even function. If the double Fourier series of $f(x,y)$ is

$$\frac{1}{4}a_{oo} + \frac{1}{2}\sum_{m=1}^{\infty}(a_{mo}\cos mx + b_{mo}\sin mx) + \frac{1}{2}\sum_{n=1}^{\infty}(a_{on}\cos ny + c_{on}\sin ny)$$
$$+ \sum_{m,n=1}^{\infty}(a_{mn}\cos mx\cos ny + b_{mn}\sin mx\cos ny +$$
$$+ c_{mn}\cos mx\sin ny + d_{mn}\sin mx\sin ny)$$

$$\equiv A_{oo} + \sum_{m=1}^{\infty}A_{mo}(x) + \sum_{n=1}^{\infty}A_{on}(y) + \sum_{m,n=1}^{\infty}A_{mn}(x,y) \equiv \sum_{m,n=0}^{\infty}A_{mn},$$

then the double Fourier series of $\psi(s,t)$ is

$$\sum_{m,n=0}^{\infty}A_{mn}(1 - \cos ms - \cos nt + \cos ms\cos nt).$$

Hence,

$$A_{mn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \psi(s,t)\cos ms\cos nt\,ds\,dt, \quad (m,n = 1,2,\dots)$$

$$A_{mo} = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi \left\{\psi(s,t) + \tfrac{1}{2}\left[f(x+s,y) + f(x-s,y)\right]\right\}\cos ms\,ds\,dt,$$
$$(m = 1,2,\dots)$$

$$A_{on} = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi \left\{\psi(s,t) + \tfrac{1}{2}\left[f(x,y+t) + f(x,y-t)\right]\right\}\cos nt\,ds\,dt,$$
$$(n = 1,2,\dots).$$

Now using, as before, $\tau_{mn}^{\gamma\delta}$ to denote the m,n-th Cesàro mean of order γ,δ of the double sequence $\{\mu\nu A_{\mu\nu}\}$, and $C_m^{\gamma}(s)$ to denote the m-th Cesaro mean of order γ of the sequence $\{\frac{2}{\pi}\mu\cos\mu s\}$, we have

$$\tau_{mn}^{\gamma\delta} = \int_0^\pi\int_0^\pi \psi(s,t)\; C_m^{\gamma}(s)\; C_n^{\delta}(t)\,ds\,dt$$

for positive integral m and n, and similarly

$$\tau_{mo}^{\gamma} = \frac{1}{\pi}\int_0^\pi\int_0^\pi \left\{\psi(s,t) + \tfrac{1}{2}\left[f(x+s,y) + f(x-s,y)\right]\right\}C_m(s)\,ds\,dt,$$

$$\tau_{on}^{\delta} = \frac{1}{\pi}\int_0^\pi\int_0^\pi \left\{\psi(s,t) + \tfrac{1}{2}\left[f(x,y+t) + f(x,y-t)\right]\right\}C_n(t)\,ds\,dt.$$

To prove summability $|C,\gamma,\delta|$ we must show that

$$(4.1)\quad \sum_{m=1}^{\infty} m^{-1}|\tau_{mo}^{\gamma}| + \sum_{n=1}^{\infty} n^{-1}|\tau_{on}^{\delta}| + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}|\tau_{mn}^{\gamma\delta}| < \infty .$$

The convergence of the two single series in (4.1) can be deduced by means of Hyslop's theorem quoted at the beginning of this section. Consider

$$A_{mo} = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi \psi(s,t)\cos ms\, ds\, dt + \frac{2}{\pi^2}\int_0^\pi\int_0^\pi \tfrac{1}{2}\left[f(x+s,y)+f(x-s,y)\right]\cos ms\, ds\, dt$$
$$= \frac{2}{\pi^2}\int_0^\pi \cos ms\left(\int_0^\pi \psi(s,t)\,dt\right)ds + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+s,y)\cos ms\, ds.$$

the inner integral in the first term of the last line is a function belonging to Lip α in x, and hence this term is the general term of the Fourier series of such a function. The second term is likewise the m-th term of the single Fourier series of a function in Lip α in x, and hence the single series theorem assures us of the $|C,\gamma|$ summability of $\sum_{m=1}^{\infty} A_{mo}$. Similarly, $\sum_{n=1}^{\infty} A_{on}$ is summable $|C,\delta|$.

Thus, to prove the theorem, it remains for us to prove the convergence of the double deries

$$(4.2) \qquad \sum_{m,n=1}^{\infty} m^{-1}n^{-1}\left|\tau_{mn}^{\gamma\delta}\right|$$
$$= \sum_{m,n=1}^{\infty} m^{-1}n^{-1}\left|\int_0^\pi\int_0^\pi \psi(s,t)\frac{1}{A_m^\gamma A_n^\delta}\sum_{k=0}^{m}\sum_{l=0}^{n}(m-k)A_k^{\gamma-1}\cos(m-k)s\,(n-l)A_l^{\gamma-1}\cos(n-l)t\, ds\, dt\right|$$

Since $A_m^\gamma \sim \dfrac{m^\gamma}{\Gamma(\gamma+1)}$, (4.2) is not greater than some absolute constant times the sum

$$\sum_{m,n=1}^{\infty} m^{-1-\gamma} n^{-1-\delta} \left| \int_0^{\pi}\int_0^{\pi} \psi(s,t) \sum_{k=0}^{m} (m-k) A_k^{\gamma-1} \cos(m-k)s \sum_{\ell=0}^{n} (n-\ell) A_\ell^{\delta-1} \cos(n-\ell)t \, ds\, dt \right|$$

$$\leqq \sum_{m,n=1}^{\infty} S_1(m,n) + \sum_{m,n=1}^{\infty} S_2(m,n) + \sum_{m,n=1}^{\infty} S_3(m,n) + \sum_{m,n=1}^{\infty} S_4(m,n),$$

where

$$S_1 = m^{-\gamma} n^{-\delta} \left| \int_0^{\pi}\int_0^{\pi} \psi(s,t) \sum_{k=0}^{\infty} A_k^{\gamma-1} \cos(m-k)s \sum_{\ell=0}^{\infty} A_\ell^{\delta-1} \cos(n-\ell)t \, ds\, dt \right|,$$

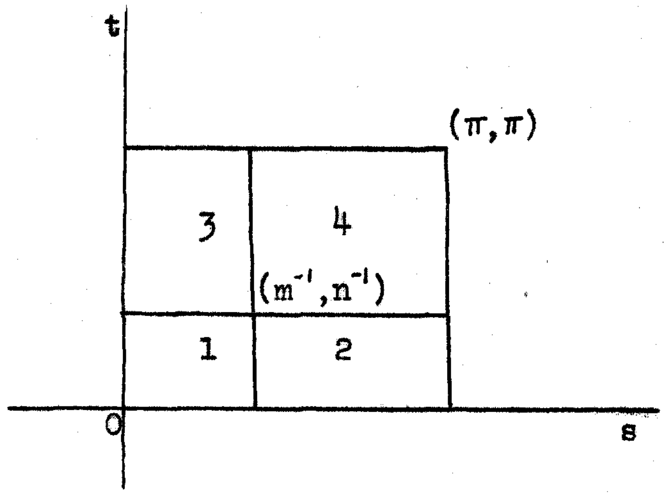
$$S_2 = m^{-\gamma} n^{-1-\delta} \left| \int_0^{\pi}\int_0^{\pi} \psi(s,t) \sum_{k=0}^{\infty} A_k^{\gamma-1} \cos(m-k)s \left\{ \sum_{\ell=n+1}^{\infty} n A_\ell^{\delta-1} \cos(n-\ell)t + \sum_{\ell=0}^{n} \ell A_\ell^{\delta-1} \cos(n-\ell)t \right\} ds dt \right|,$$

$$S_3 = m^{-1-\gamma} n^{-\delta} \left| \int_0^{\pi}\int_0^{\pi} \psi(s,t) \left\{ \sum_{k=m+1}^{\infty} m A_k^{\gamma-1} \cos(m-k)s + \sum_{k=0}^{m} k A_k^{\gamma-1} \cos(m-k)s \right\} \sum_{\ell=0}^{\infty} A_\ell^{\delta-1} \cos(n-\ell)t \, ds dt \right|,$$

$$S_4 = m^{-1-\gamma} n^{-1-\delta} \left| \int_0^{\pi}\int_0^{\pi} \psi(s,t) \left\{ \sum_{k=m+1}^{\infty} m A_k^{\gamma-1} \cos(m-k)s + \sum_{k=0}^{m} k A_k^{\gamma-1} \cos(m-k)s \right\} \times \right.$$
$$\left. \times \left\{ \sum_{\ell=n+1}^{\infty} n A_\ell^{\delta-1} \cos(n-\ell)t + \sum_{\ell=0}^{n} \ell A_\ell^{\delta-1} \cos(n-\ell)t \right\} ds\, dt \right|.$$

We shall show here proof of the convergence of the first and fourth of the above series; the other two can be treated by a combination of the methods to be used.

Now considering $S_4(m,n)$ first, we break up the square of integration into four rectangular subintervals having a common corner at the point (m^{-1},n^{-1}), as shown in the figure on page 32, and get



$$S_4(m,n) \leq S_{4,1}(m,n) + S_{4,2}(m,n) + S_{4,3}(m,n) + S_{4,4}(m,n)$$

the second subscript indicating the sub-rectangle over which the integral is taken.

Since, as k increases, $A_k^{\gamma-1}$ decreases (by the lemma of consistency we may consider $\gamma < \frac{1}{2}$) and $kA_k^{\gamma-1}$ increases, the absolute value of each of the sums in the first pair of braces in $S_{4,1}$ is less than $A m^{\gamma} s^{-1}$. Similarly for the second pair. Hence

$$\sum_{m,n=1}^{\infty} S_{4,1}(m,n) < A \sum_{m,n=1}^{\infty} m^{-1-\gamma} n^{-1-\delta} \int_0^{m^{-1}}\int_0^{n^{-1}} m^{\gamma} n^{\delta} s^{-1} t^{-1} |\psi(s,t)| \, ds\, dt$$

$$< A \sum_{m,n=1}^{\infty} m^{-1} n^{-1} \int_0^{m^{-1}}\int_0^{n^{-1}} s^{\alpha-1} t^{\beta-1} \, ds\, dt$$

$$< \infty .$$

On the other terms we shall require the estimate obtained by Hyslop ([11], pp. 478-479)

$$\sum_{k=m+1}^{\infty} mA_k^{\gamma-1}\cos(m-k)s + \sum_{k=0}^{m} kA_k^{\gamma-1}\cos(m-k)s = O(s^{-1-\gamma}), \quad m^{-1} < s < \pi,$$

by means of which we obtain

$$\begin{aligned}
\sum_{m,n=1}^{\infty} S_{4,2}(m,n) &< A\sum_{m,n=1}^{\infty} m^{-1-\gamma} n^{-1-\delta}\int_{m^{-1}}^{\pi}\int_{0}^{n^{-1}} n^{\delta} s^{-1-\gamma} t^{-1}\,|\psi(s,t)|\,ds\,dt \\
&< A\sum_{m,n=1}^{\infty} m^{-1-\gamma} n^{-1}\int_{m^{-1}}^{\pi} s^{\alpha-1-\gamma}\,ds\int_{0}^{n^{-1}} t^{\beta-1}\,dt \\
&< A\sum_{m,n=1}^{\infty} m^{-1-\gamma} n^{-1}\left[s^{\alpha-\gamma}\right]_{m^{-1}}^{\pi}\left[t^{\beta}\right]_{0}^{n^{-1}} \\
&< A\sum_{m,n=1}^{\infty} n^{-1-\beta}(m^{-1-\gamma} + m^{-1-\alpha}) \\
&< A\sum_{n=1}^{\infty} n^{-1-\beta}\sum_{m=1}^{\infty}(m^{-1-\gamma} + m^{-1-\alpha}) < \infty .
\end{aligned}$$

Similarly for $S_{4,3}$ and $S_{4,4}$. Hence the double series $\sum_{m,n=1}^{\infty} S_4(m,n)$ is convergent.

We proceed to $\sum_{m,n=1}^{\infty} S_1(m,n)$. We write

$$\begin{aligned}
&\int_0^{\pi}\int_0^{\pi}\psi(s,t)\sum_{k=0}^{\infty}A_k^{\gamma-1}\cos(m-k)s\sum_{\ell=0}^{\infty}A_\ell^{\delta-1}\cos(n-\ell)t\,ds\,dt \\
&= \int_0^{\pi}\int_0^{\pi}\psi(s,t)p^{\gamma}(s)p^{\delta}(t)\cos ms\cos nt\,ds\,dt \\
&\quad + \int_0^{\pi}\int_0^{\pi}\psi(s,t)p^{\gamma}(s)q^{\delta}(t)\cos ms\sin nt\,ds\,dt \\
&\quad + \int_0^{\pi}\int_0^{\pi}\psi(s,t)q^{\gamma}(s)p^{\delta}(t)\sin ms\cos nt\,ds\,dt \\
&\quad + \int_0^{\pi}\int_0^{\pi}\psi(s,t)q^{\gamma}(s)q^{\delta}(t)\sin ms\sin nt\,ds\,dt \\
&= a_{mn} + b_{mn} + c_{mn} + d_{mn},
\end{aligned}$$

where $p^{\gamma}(s) \equiv \sum_{k=0}^{\infty} A_k^{\gamma-1} \cos ks$ and $q^{\gamma}(s) \equiv \sum_{k=0}^{\infty} A_k^{\gamma-1} \sin ks$. These functions are continuous for $\eta \leq s \leq \pi$, and their absolute values when $0 < s \leq \eta$ are less than $As^{-\gamma}$. Hence, remembering that $\gamma < \frac{1}{2}$, $\delta < \frac{1}{2}$, the function $p^{\gamma}(s)p^{\delta}(t)$ is in Lebesgue class L^2 over the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$. Furthermore, since $\psi(s,t)$ is continuous, the function $\psi(s,t)p^{\gamma}(s)p^{\delta}(t)$ is in L^2. Thus a_{mn} is the Fourier coefficient of an even-even function in L^2. Similarly, b_{mn} is the Fourier coefficient of an even-odd function in L^2, and so on. If we put

$$G(s,t) \equiv \psi(s,t)\, p^{\gamma}(s)\, p^{\delta}(t)$$

we have

$$\begin{aligned}
a_{mn} \sin mh \sin nk &= \int_0^{\pi}\int_0^{\pi} G(s,t) \cos ms \sin mh \cos nt \sin nk \, ds\, dt \\
&= \tfrac{1}{4}\int_0^{\pi}\int_0^{\pi} G(s,t)\left[\sin m(s+h) - \sin m(s-h)\right]\left[\sin n(t+k) - \sin n(t-k)\right] ds\,dt \\
&= \tfrac{1}{4}\int_h^{\pi+h}\int_k^{\pi+k} G(s-h,t-k) \sin ms \sin nt \, ds\, dt \\
&\quad - \tfrac{1}{4}\int_h^{\pi+h}\int_{-k}^{\pi-k} G(s-h,t+k) \sin ms \sin nt \, ds\, dt \\
&\quad - \tfrac{1}{4}\int_{-h}^{\pi-h}\int_k^{\pi+k} G(s+h,t-k) \sin ms \sin nt \, ds\, dt \qquad (4.3) \\
&\quad + \tfrac{1}{4}\int_{-h}^{\pi-h}\int_{-k}^{\pi-k} G(s+h,t+k) \sin ms \sin nt \, ds\, dt
\end{aligned}$$

The four terms in (4.3) are equivalent to

$$(4.4)\quad \frac{1}{4}\int_0^{\pi}\int_0^{\pi}\Big[G(s-h,t-k)-G(s-h,t+k)-G(s+h,t-k)+G(s+h,t+k)\Big]\sin ms\,\sin nt\,ds\,dt.$$

This may be seen by considering, for example,

$$\int_{\pi}^{\pi+h}\int_{k}^{\pi+k} G(s-h,t-k)\sin ms\,\sin nt\,ds\,dt,$$

an "excess" part of the first integral in (4.3). We make the substitution $s' = 2\pi - s$, and, bearing in mind that G is an even function of its first argument and periodic with period 2π, while $\sin ms$ is odd and periodic, we obtain for the above integral

$$\int_{\pi}^{\pi-h}\int_{k}^{\pi+k} G(s'+h,t-k)\sin ms'\,\sin nt\,ds'dt$$

$$= -\int_{\pi-h}^{\pi}\int_{k}^{\pi+k} G(s+h,t-k)\sin ms\,\sin nt\,ds\,dt,$$

which is a "missing" part of the third integral in (4.3). In such fashion the various integrals in (4.3) complement each other over the peripheral strips of the area of integration, combining to give the expression (4.4).

Now by Bessel's inequality

$$\sum_{m,n=1}^{\infty} a_{mn}^2 \sin^2 mh\,\sin^2 nk$$

$$\leq A\int_0^{\pi}\int_0^{\pi}\Big[G(s+h,t+k)-G(s+h,t-k)-G(s-h,t+k)+G(s-h,t-k)\Big]^2 ds\,dt,$$

and the expression at the bottom of page 35 is certainly not greater than

$$2A\int_0^\pi\int_0^\pi\left\{\left[G(s+h,t+k)-G(s+h,t-k)\right]^2+\left[G(s-h,t+k)-G(s-h,t-k)\right]^2\right\}ds\,dt$$

$$\begin{aligned}&= 2A\int_0^\pi\int_0^\pi\left[\psi(s+h,t+k)p^\gamma(s+h)p^\delta(t+k)-\psi(s+h,t-k)p^\gamma(s+h)p^\delta(t-k)\right]^2 ds\,dt\\&+ 2A\int_0^\pi\int_0^\pi\left[\psi(s-h,t+k)p^\gamma(s-h)p^\delta(t+k)-\psi(s-h,t-k)p^\gamma(s-h)p^\delta(t-k)\right]^2 ds\,dt.\end{aligned} \tag{4.5}$$

Considering the first integral in (4.5), we see that it is not greater than the sum

$$4A\left\{I_1(h,k)+I_2(h,k)\right\}$$

where

$$I_1(h,k)\equiv\int_0^\pi\int_0^\pi\left[p^\gamma(s+h)p^\delta(t+k)\right]^2\left[\psi(s+h,t+k)-\psi(s+h,t-k)\right]^2 ds\,dt,$$

$$I_2(h,k)\equiv\int_0^\pi\int_0^\pi\left[p^\gamma(s+h)\,\psi(s+h,t-k)\right]^2\left[p^\delta(t+k)-p^\delta(t-k)\right]^2 ds\,dt.$$

We get first an estimate of $I_1(h,k)$. Taking h and k as positive, we have

$\psi(s+h,t+k) - \psi(s+h,t-k)$

$$= \tfrac{1}{4}[f(x+s+h,y+t+k)-f(x+s+h,y+t-k)-f(x,y+t+k)+f(x,y+t-k)]$$
$$+ \tfrac{1}{4}[f(x+s+h,y-t-k)-f(x+s+h,y-t+k)-f(x,y-t-k)+f(x,y-t+k)]$$
$$+ \tfrac{1}{4}[f(x-s-h,y+t+k)-f(x-s-h,y+t-k)-f(x,y+t+k)+f(x,y+t-k)]$$
$$+ \tfrac{1}{4}[f(x-s-h,y-t-k)-f(x-s-h,y-t+k)-f(x,y-t-k)+f(x,y-t+k)]$$

and since f is in Lip β in y, this is $\leqq Ak^{\beta}$. Hence

$$I_1(h,k) \leqq Ak^{2\beta}\int_0^{\pi}\int_0^{\pi} s^{-2\gamma}t^{-2\delta}\,ds\,dt = O(k^{2\beta})$$

since $\gamma < \frac{1}{2}$, $\delta < \frac{1}{2}$.

Turning to $I_2(h,k)$, we write it as a sum of two parts:

$$I_2(h,k) = \int_0^{\pi}\int_{-k}^{k}[p^{\gamma}(s+h)\,\psi(s+h,t)]^2[p^{\delta}(t+2k)-p^{\delta}(t)]^2 ds\,dt$$
$$+ \int_0^{\pi}\int_k^{\pi-k}[p^{\gamma}(s+h)\,\psi(s+h,t)]^2[p^{\delta}(t+2k)-p^{\delta}(t)]^2 ds\,dt$$
$$= I_{2,1}(h,k) + I_{2,2}(h,k).$$

Then we have

$$I_{2,1}(h,k) \leqq 2\int_0^{\pi}\int_{-k}^{k}[p^{\gamma}(s+h)\psi(s+h,t)]^2\left[\{p^{\delta}(t+2k)\}^2+\{p^{\delta}(t)\}^2\right]ds\,dt$$
$$= O\left(\int_0^{\pi}\int_{-k}^{k}(s+h)^{2\alpha-2\gamma}t^{2\beta}(t+2k)^{-2\delta}\,ds\,dt\right) + O\left(\int_0^{\pi}\int_{-k}^{k}(s+h)^{2\alpha-2\gamma}t^{2\beta-2\delta}\,dsdt\right)$$
$$= O(h^{2\alpha-2\gamma+1})\cdot O(k^{2\beta-2\delta+1}).$$

For the second part we have

$$I_{2,2}(h,k) = 4k^2\int_0^{\pi}\int_k^{\pi-k}\left[p^{\gamma}(s+h)\psi(s+h,t)\right]^2\left[p^{\delta'}(t+2\theta k)\right]^2 ds\,dt$$
$$(0<\theta<1)$$
$$= O\left(k^2\int_0^{\pi}\int_k^{\pi}(s+h)^{2\alpha-2\gamma}t^{2\beta}(\sin\tfrac{1}{2}t)^{-2\delta-2}ds\,dt\right)$$
$$= O\left(k^2\int_0^{\pi}\int_k^{\pi}(s+h)^{2\alpha-2\gamma}\,t^{2\beta-2\delta-2}\,ds\,dt\right)$$
$$= O(h^{2\alpha-2\gamma+1})\cdot O(k^{2\beta-2\delta+1}).$$

The second integral in (4.5) yields the same estimate. It follows that, when h and k tend to zero,

$$\sum_{m,n=1}^{\infty} a_{mn}^2 \sin^2 mh \sin^2 nk = O(h^{2\alpha}k^{2\beta}),$$

and the same may be proved in like manner for b_{mn}, c_{mn} and d_{mn}.

Let $h = \frac{\pi}{2M}$, $k = \frac{\pi}{2N}$. Then we get

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}^2 \sin^2\frac{m\pi}{2M}\sin^2\frac{n\pi}{2N} = O(M^{-2\alpha}N^{-2\beta}),$$

and, writing $M = 2^{\mu}$, $N = 2^{\nu}$, we see that

$$\sum_{m=2^{\mu-1}+1}^{2^{\mu}}\ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} a_{mn}^2 = O(2^{-2\mu\alpha-2\nu\beta}).$$

Now, by application of Schwarz's inequality,

$$\sum_{m=2^{\mu-1}+1}^{2^{\mu}} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} m^{-\gamma} n^{-\delta} |a_{mn}|$$

$$\leq \left\{ \sum_{m=2^{\mu-1}+1}^{2^{\mu}} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} a_{mn}^2 \right\}^{\frac{1}{2}} \left\{ \sum_{m=2^{\mu-1}+1}^{2^{\mu}} m^{-2\gamma} \right\}^{\frac{1}{2}} \left\{ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} n^{-2\delta} \right\}^{\frac{1}{2}}$$

$$= O(2^{-\mu\alpha-\nu\beta}) \cdot O(2^{\mu(\frac{1}{2}-\gamma)}) \cdot O(2^{\nu(\frac{1}{2}-\delta)})$$

$$= O(2^{-\mu(\alpha+\gamma-\frac{1}{2})}) \cdot O(2^{-\nu(\beta+\delta-\frac{1}{2})}).$$

Similar relations hold for b_{mn}, c_{mn} and d_{mn}.

It follows that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_1(m,n) \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-\gamma} n^{-\delta} (|a_{mn}| + |b_{mn}| + |c_{mn}| + |d_{mn}|)$$

$$\leq A \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} 2^{-\mu(\alpha+\gamma-\frac{1}{2})} \, 2^{-\nu(\beta+\delta-\frac{1}{2})}$$

$$< A,$$

since $\gamma > \frac{1}{2} - \alpha$, $\delta > \frac{1}{2} - \beta$.

We conclude that (4.2) is convergent, and the theorem follows.

5. Localization Properties

A summability method may be said to possess a "localization property" if it sums to zero at a point the Fourier series of any integrable function vanishing throughout a neighborhood of the point. Thus, in single Fourier series, Riemann's principle of localization asserts that convergence itself has this property.

When we consider summability methods for multiple Fourier series, we have to distinguish between two sorts of localization properties, corresponding to two ways of generalizing the concept of "neighborhood". The first is the "cross-neighborhood", in which at least one of the variables is within a prescribed distance of its value at the fixed point. The other, which we shall call simply the "neighborhood", is the set of points <u>all</u> of whose coordinates are within prescribed distances of their respective values at the fixed point.

It is well known that, in extending the theory of single Fourier series to double or multiple

Fourier series, such properties as Riemann's localization principle find their counterparts in connection with the cross-neighborhood, rather than the neighborhood.

It has become, indeed, a matter of peculiar concern to students of multiple series to investigate what sorts of summability methods have the neighborhood localization property. The only methods found to have it are those in which the indices of the partial sums do not become infinite independently, for example, the restricted summability of C. N. Moore ([13], p. 567), or the spherical summability of Bochner [3].

Since the <u>absolute</u> convergence of a multiple series does not depend upon the relative manner in which the different indices become infinite, we should not expect to find <u>neighborhood</u> localization properties for absolute summability. Instead, it will be the <u>cross-neighborhood</u> which will figure in our generalization to double series of the single series localization properties.

For single series, such properties of absolute Cesàro summability have been investigated by Bosanquet [5] and Foà [9]. The former pointed out that, while $|C,0|$ summability (absolute convergence) does not possess the localization property, $|C,\alpha|$, for $\alpha > 1$, does, since his sufficient condition for $|C,\alpha|$ summability is, for $\alpha > 1$, merely a neighborhood condition on the function.

Foà showed that for $0 < \alpha \leq 1$, $|C,\alpha|$ does <u>not</u> possess the localization property. He did this by exhibiting two functions coinciding in a neighborhood of the origin, at which point the Fourier series of the first is summable $|C,\alpha|$ for $\alpha \geq 0$, but that of the second is not summable $|C,1|$, and hence not $|C,\alpha|$, $\alpha \leq 1$.

The extension of these results to double Fourier series is embodied in the following theorem.

<u>Theorem 6</u> $|C,\alpha,\beta|$ summability possesses the cross-neighborhood localization property for $\alpha > 1$, $\beta > 1$, and only for such values of α and β.

<u>Proof</u> The first part of this theorem follows from Theorem 3 (page 18). For $\phi_{11}(s,t)$ is clearly of B.V.H. outside a cross-neighborhood of the origin,

whence by Theorem 3 we get $|C,\alpha,\beta|$ summability of the Fourier series for $\alpha > 1$, $\beta > 1$, provided only that the function behave suitably in the cross-neighborhood. But that is just another way of saying that such summability possesses the cross-neighborhood localization property.

To prove the second part of the theorem, we construct by use of Foà's examples two functions $F_1(s,t)$ and $F_2(s,t)$, such that F_1 has an absolutely convergent double Fourier series at the origin, while F_2 coincides with F_1 in a cross-neighborhood of the origin, but the Fourier series of F_2 is not even summable $|C,\alpha,1|$ at the origin, for any α, or $|C,1,\beta|$ for any β.

The functions constructed by Foà are:

$$f_1(s) = \begin{cases} -1, & -\pi \leq s < -\varepsilon, \\ 0, & -\varepsilon \leq s \leq \varepsilon, \qquad \varepsilon < \pi - e^{-2}; \\ 1, & \varepsilon < s < \pi, \end{cases}$$

$$f_2(s) = \begin{cases} 0, & -\pi \leq s < \pi - e^{-2}, \\ \dfrac{\sin \frac{1}{2} s}{(\pi - s)\log^2(\pi - s)}, & \pi - e^{-2} \leq s < \pi. \end{cases}$$

Foà showed [9] that the Fourier series of $f_1(s)$ is absolutely convergent at the origin, but that of $f_2(s)$ is not summable $|C,1|$.

Now consider $F_1(s,t) \equiv f_1(s)f_1(t)$. Its double Fourier series is absolutely convergent, since each term is a product of terms from absolutely convergent single series.

On the other hand, $F_2(s,t) \equiv f_2(s)f_2(t)$ has a double Fourier series which is not summable $|C,1,1|$ since, again, each term is a product of a term from a single series in s and a term from a single series in t, and neither single series is summable $|C,1|$. (The origin is understood to be the point under consideration in all these statements.) As a matter of fact, it is necessary that only <u>one</u> of the indices α,β be as small as 1 in order to thwart the $|C,\alpha,\beta|$ summability of the double Fourier series of $F_2(s,t)$ at the origin.

Now since F_1 and F_2 each equals zero throughout a cross-neighborhood of the origin, it is their behavior outside this region that makes the difference in their summability $|C,\alpha,\beta|$ with $\min(\alpha,\beta) \leq 1$, and so such summability does not have the cross-neighborhood localization property. This completes the proof.

6. References

[1] Adams, C. R., and Clarkson, J. A., Definitions of bounded variation for functions of two variables. Trans. Am. Math. Soc. 35: 824-854 (1933)

[2] Bernstein, S., Sur le convergence absolue des séries trigonométriques. Comptes Rendus 158: 1661-1664 (1914)

[3] Bochner, Salomon, Summation of multiple Fourier series by spherical means. Trans. Am. Math. Soc. 40: 175-207 (1936)

[4] Bosanquet, L. S., On the absolute summability (C) of a Fourier series. Journal Lond. Math. Soc. 11: 11-15 (1936)

[5] Bosanquet, L. S., The absolute Cesàro summability of a Fourier series. Proc. Lond. Math. Soc. (2) 41: 517-528 (1936)

[6] Chandrasekharan, K., The absolute summability of series of eigenfunctions. Journal Indian Math. Soc. (N.S.) 7: 25-30 (1943)

[7] Chelidze, V. G., On the absolute convergence of double Fourier series. C. R. (Doklady) Acad. Sci. U.R.S.S. (N.S.) 54: 117-120 (1946). Math. Reviews 8: 376 (1947)

[8] Fekete, Mihály, A széttartó végetelen sorok elméletéhez. Mathematikai és Természettudományi Értesitö 29: 719-726 (1911)

[9] Foà, Alberto, Il fenomeno di Riemann per la somma $|C,\alpha|$ di una serie di Fourier con $0 < \alpha \leqq 1$. Rend. Inst. Lomb. Sc. e Lett. 71: 359-366 (1938)

[10] Hobson, E. W., The theory of functions of a real variable. Cambridge. Vol. I (3rd ed.) 1927, Vol. II (2nd ed.) 1926.

[11] Hyslop, J. M., On the absolute summability of Fourier series. Proc. Lond. Math. Soc. (2) 43: 475-483 (1937)

[12] Kogbetliantz, E. G., Sur les séries absolument sommables. Bull. des Math. (2) 49: 234-256 (1925)

[13] Moore, C. N., On the summability of the double Fourier's series of discontinuous functions. Math. Annalen 74: 555-572 (1913)

[14] Reves, G. E., On the absolute convergence of double Fourier series. Thesis, University of Cincinnati, 1941.

[15] Reves, G. E., and Szász, Otto, Some theorems on double trigonometric series. Duke Math. Journal 9: 693-705 (1942)