UNIVERSITY OF CINCINNATI

April 26 1951

I hereby recommend that the thesis prepared under my supervision by Cohn L Morrison *entitled* A Differential Analogue

be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy

Approved by:

Carl A. Ludeke

FORM 668—G. S. & T. C.—500—5-48

A DIFFERENTIAL ANALOGUE

A dissertation submitted to the
Graduate School of Arts and Sciences
of the University of Cincinnati
in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

1951

by

Cohn L. Morrison

A. B. DePauw University, 1931
M. A. Indiana University, 1936
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AN ELECTRO-DYNAMICAL ANALOGUE

1. Introduction

A differential analogue which could represent linear equations with variable coefficients was discussed in my thesis for the degree of Master of Science. The present dissertation is concerned with several additions and changes which have been made to the apparatus. In particular, interest centers around representing differential equations of systems with non-linear restoring forces. This class of equations consists of some very interesting equations which cannot be linearized, or, if so, only for very small displacements.

By proper selection it was possible to find equations whose solutions can be approximated by analytical methods. In addition to these, I have shown a few solutions of equations which have not been solved by analytical methods.

2. Theory of Representing Non-linear Spring Forces

A coil suspended so that it can rotate freely in a uniform field will act as a pendulum.

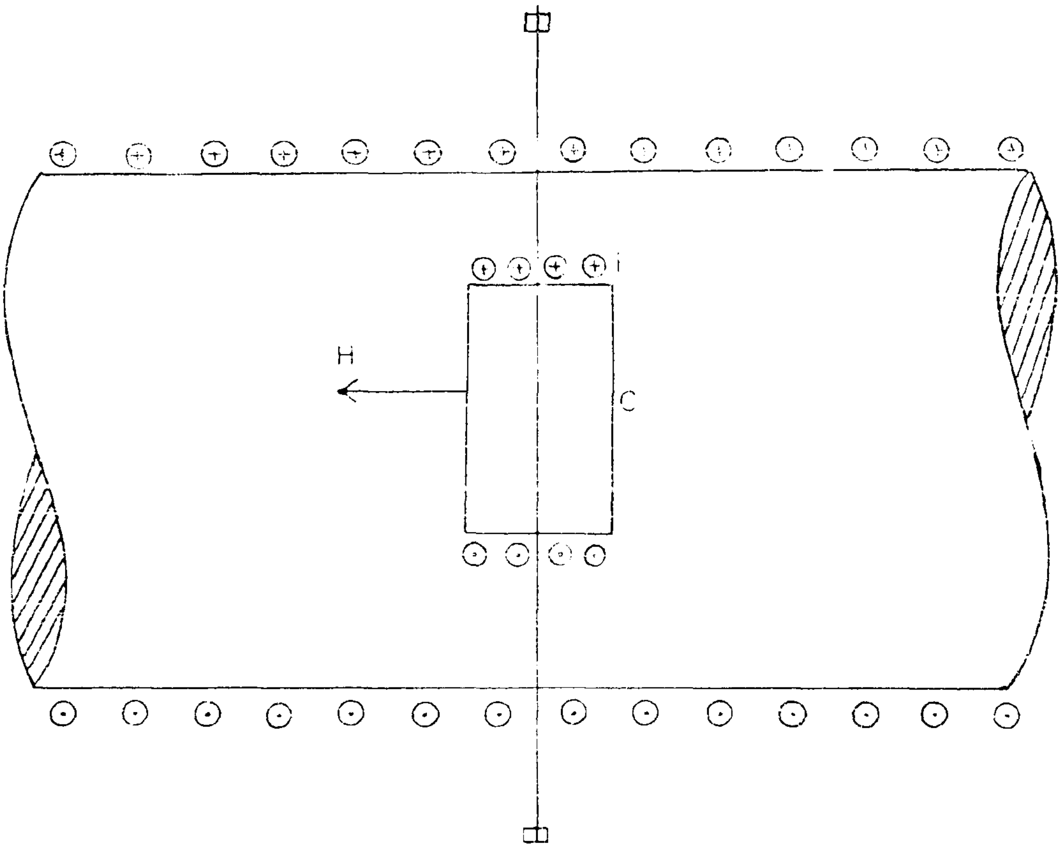


figure no. 1

If H is kept constant, then the restoring couple M will be proportional to the current i in the rotating coil and to the displacement.

$$M = k\,i\,\sin\theta \cong k\,i \qquad (2.1)$$

k depends on the geometry and number of turns of wire on the small coil. Replacement of $\sin\theta$ by θ is justified if we stay within a maximum displacement less than say 15°. Now if i is made a function of θ, then the torque,

$$M = k\,i\,\theta \cong k\,\theta\, f(\theta), \qquad i = f(\theta) \qquad (2.2)$$

In order to make $i = f(\theta)$ a photo-integrating sphere is used. A sphere with a light-diffusing inside surface has a small opening through which passes a beam of light having a uniform density per unit area. A photo-tube signal is placed so that it picks up only indirect light. The signal from the photo-tube is amplified so that current flows which is proportional to the amount of light flux entering the sphere. The opening consists of a moveable shutter so that the area of light which enters the sphere can be changed with displacement. The purpose of the integrating sphere is to eliminate the directional effect of the photo-electric cell which receives the main beam of light. The direction from which a beam strikes a photocell has considerable effect on the response.

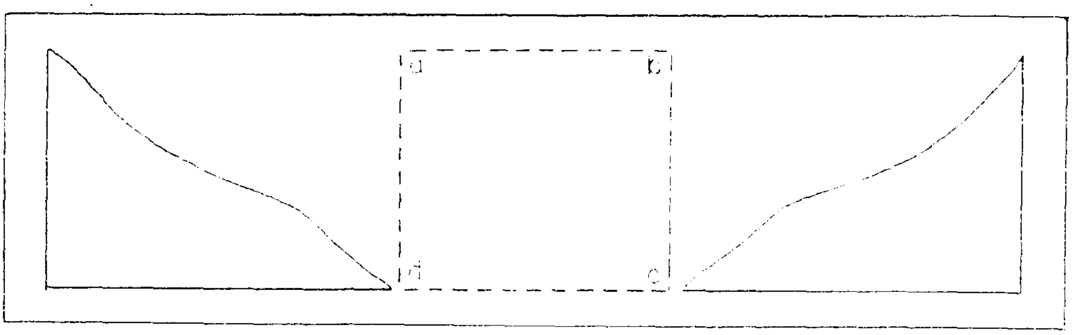


figure no. 2

Now if the shutter has an opening cut into it which is shaped by the θ axis and $y = f(\theta)$, then the quantity of light which passes through the opening is proportional to $\phi(\theta) = \int_0^\theta f(\theta)\, d\theta$. Hence, if we want a spring force of $k\theta\phi(\theta)$, the first step is to differentiate $\phi(\theta)$ so that we have $f(\theta)$. The shutter then must have a top perimeter described by $f(\theta)$.

Suppose we want to represent

$$\ddot{\theta} + \psi(\theta) = 0 \qquad (2.3)$$

The first step is to factor $k\theta$ out of $\psi(\theta)$ so that

$$\ddot{\theta} + k\theta\phi(\theta) = 0, \qquad \psi(\theta) = k\theta\phi(\theta) \qquad (2.4)$$

As an example, suppose we want to represent

$$\ddot{\theta} + k\theta^5 = 0 \qquad (2.5)$$

Here $\psi(\theta)$ in (2.3) is $k\theta^5 = k\theta(\theta^4)$. $\phi(\theta)$ in (2.4) is θ^4.

$$f(\theta) = \frac{d\phi(\theta)}{d\theta} = 4\theta^3 \qquad (2.6)$$

When cutting the actual curve on the shutter we need not be concerned with the constant 4 since we can control the amplification by other means.

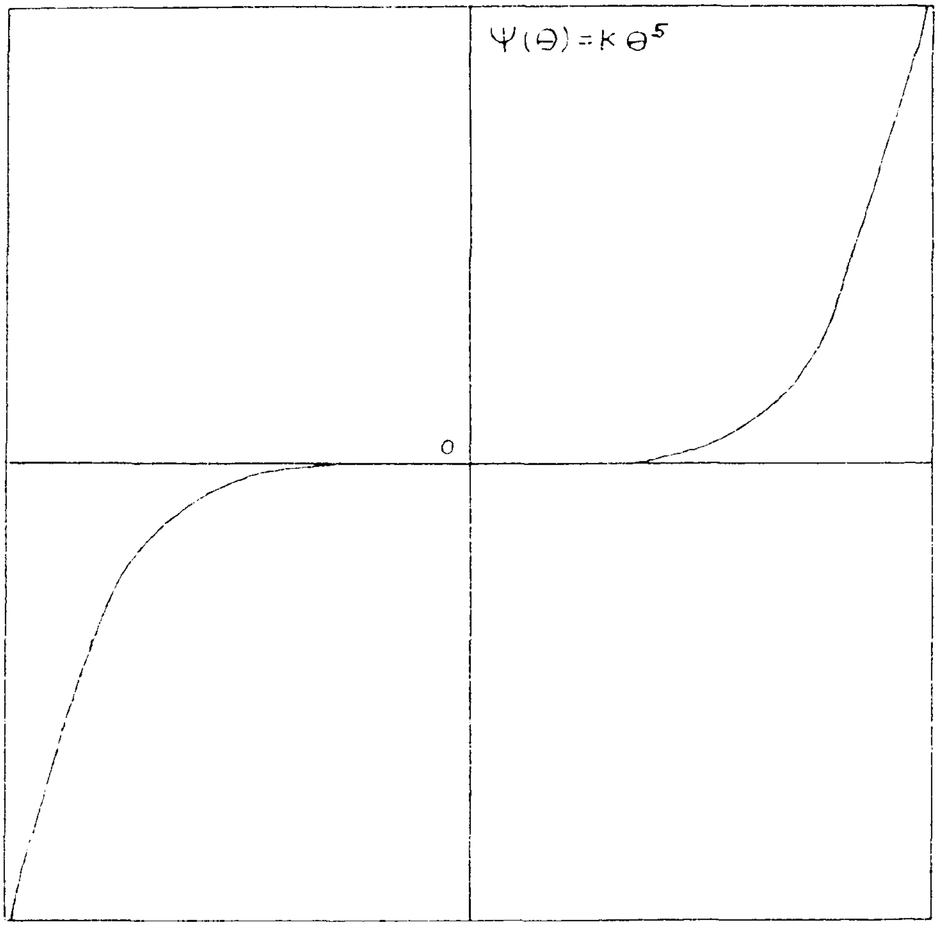


figure no. 3

Figure no. 3 above shows how the negative of the spring force varies with the displacement.

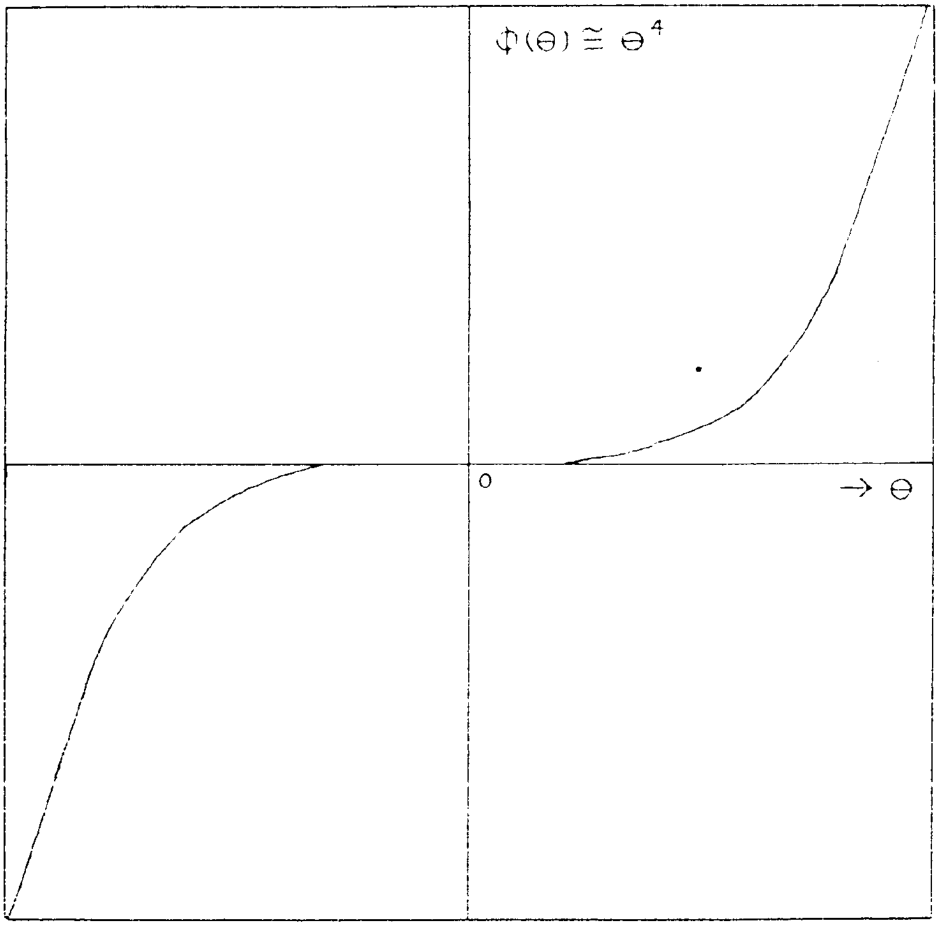


figure no. 4

Figure no. 4 above shows how $\phi(\theta)$ varies with displacement. Of course, the current which flows in the oscillating coil must be proportional to $\phi(\theta)$. Actually the current must be of such a value that it will produce a magnetic field in the oscillating coil which, when acted upon by the uniform field in which it is located, produces a torque equal to $k\,\theta\,\phi(\theta)$.

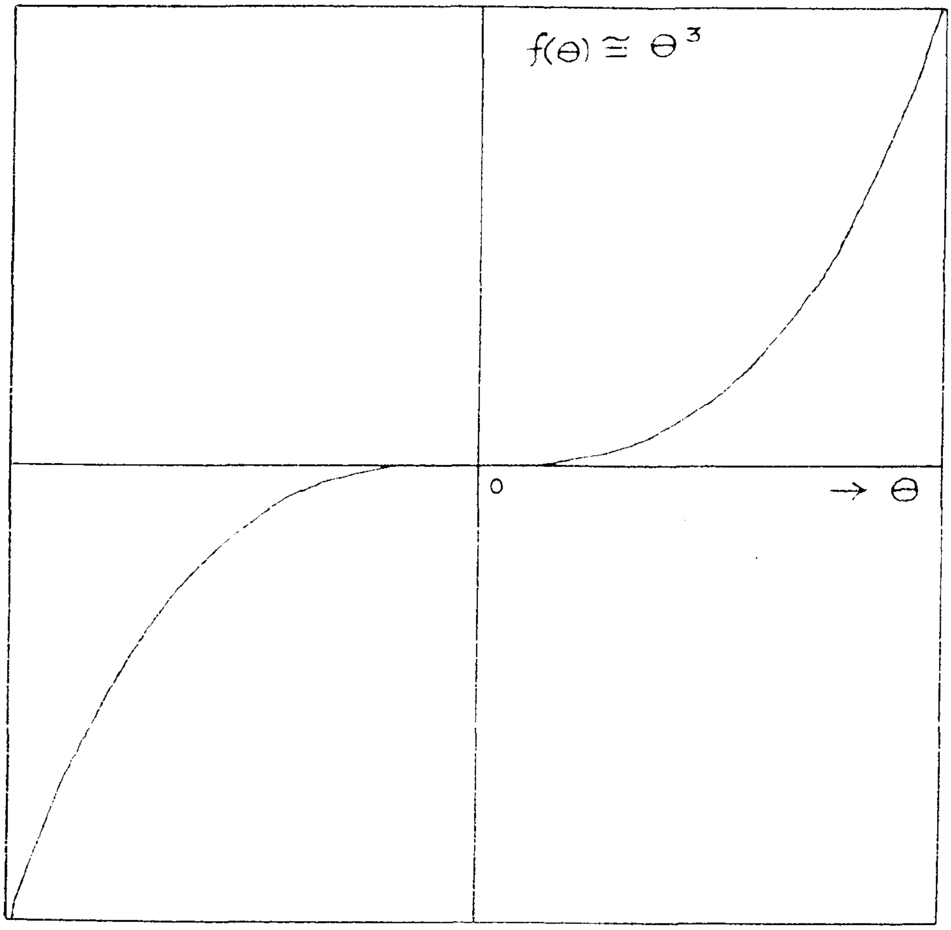


figure no. 5

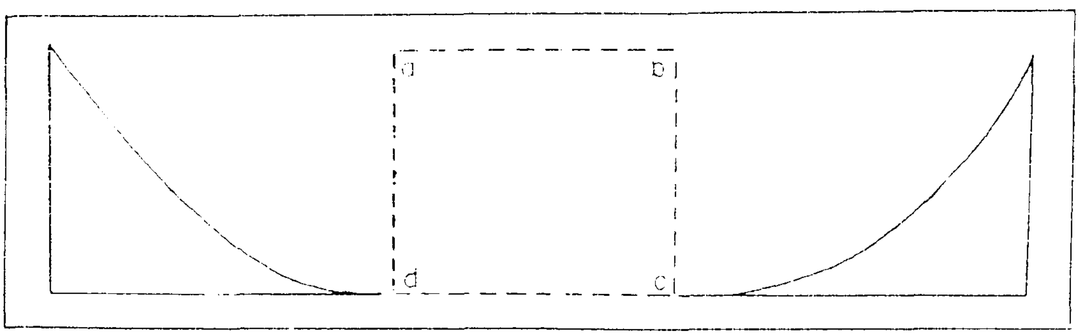


figure no. 6

The shutter which gives a negative spring force of $k\theta^3$ is shown in figure no. 3. The shutter is placed at the position shown when the oscillating coil is at its position of equilibrium. The upper portion of the curve which in this case is proportional to θ^3 is such that on a $2\frac{1}{4}$" x $3\frac{1}{4}$" negative the clear portion of the shutter comes to within about $\frac{1}{4}$" of each edge. The amplifier control is used to control the constant of proportionality.

3. Additions Made to the Machine Since the Last Paper

The basic parts of the machine were described in a previous paper. Only the additions to the apparatus will be described here. The entire apparatus has been rearranged so that operation of the machine can be controlled from a position directly in front of the coils. An instrument panel (photograph no. 6) was placed to the right of the operator when facing the coils. On the panel there is a switch for the following: safe lights; recording light; timing light; galvanometer recording light; light for integrating spheres; motor on recording drum; room lights; actuating current for solenoid-oil-switch for large coils; rheostat for adjusting the speed of the rotating drum; and a rheostat and ammeter for adjusting the amount of current which flows through the large coils. It is necessary for these to be placed in a position accessible to the operator while he is adjusting the position of the moving coil. This can be understood when one realizes that all of the above switches must be operated each time a record is made. Since a record requires about only one minute after the paper is placed on the drum, one can see that ease of operation is essential if confusion with the inevitable resulting mistakes are to be avoided.

The point sources of light for the recording, the timing, and the galvanometer now are arranged for operation with A.C. The bulbs are 8-10 volt, automobile headlight types. These were changed from D. C. to avoid fluctuations in the voltage which occur with battery operation. The voltage now is regulated by reducing it by a step-down transformer to 15 volts. Further regulation is obtained by rheostats in the output side of the transformers. These are shown in photograph no. 6, page 20.

A scale, shown in photograph no. 1, is fastened onto the wall so that the number of degrees or the number of radians can be read directly. Readings can be made up to 15° on either side of the equilibrium position. A point source projector and lens system, placed as shown in photograph no. 1, is used to project a dot on the wall by means of a mirror attached to the shaft of the oscillating coil. The scale is essential for the calibration of the system and for the operator's use in finding the correct original displacement when running problems with free vibrations.

An oil immersion solenoid switch now is used to turn the current on and off in the coil used to produce a uniform field. An ordinary switch was not satisfactory because the spark caused fogging of the photographic recording paper. When an iron core is used in the coils, an ordinary switch even fails to extinguish itself.

The gear ratio of the recording drum has been changed by replacing change gears with worm gears so that the time of one complete revolution occurs in about thirty seconds when the driving motor is operating at full speed. Speed regulation of the motor is increased to the extent that only slight variations in motor speed over one revolution can be detected.

Leveling screws now are located under each leg of the supporting table. Exact leveling of the coils now is achieved in a short time.

A record processing section, shown in photograph no. 3, is located in one corner of the room. An aerial film type developing unit is installed and arranged so that the film carrier can be loaded while it is sitting on the drain board, located at one end of the table, and then can be progressively placed into the developing tank, the washing tank, the fixing bath, and back into the washing tank. The developer and fixing bath rest in a galvanized tray into which the overflow from the washing bath flows. The tray, in addition to preventing messiness, keeps the developer and fixer at a constant temperature.

The oscillating coils are now changed to a pair of coils, each wound of 1500 turns of number 36 varnished copper wire. A new type pressed brass coil holder was designed so that better centering of the coils could be achieved. The leads now are .0015 inch thick gold strips instead of number 40 copper wire. A maximum amount of current now can be carried to the coils with a minimum amount of restoring torque. Although soldering the gold leads onto the coils presents some difficulty, the durability of the leads is good despite their fragile nature.

An iron yoke was constructed so that the uniform field of the large coils could be increased in intensity. The yoke is made from Armco Magnetic Ingot Iron 3 inch outside diameter. The design, as shown in photograph no. 7, is such as to allow the pole pieces to be removed without disturbing the remainder of the apparatus in any way. No bolts need be removed in order to remove the pole pieces. When the iron is in place, the field is increased several times.

In order to represent non-linear spring forces, it is necessary to use some method of varying the current in the coils as a function of displacement. As explained previously in Section 2, a photointegraph can be used to do this. Two hemispheres of spun copper are fitted together to make a sphere 12 inches in diameter. In the hemisphere suspended from the base of the coils an opening is cut 2½ inches wide and 10 inches long. The edges of the opening are covered by brass strips in which slots 1/16 inch wide are milled. These permit any part of the slot to be masked off by sliding stencil board in and out of the slots. In most cases it is found convenient to leave an opening 2¼ inches by 2¼ inches in the sphere for the shutter to cover. The inside of the sphere is painted with a coating of flat zinc oxide paint. While the binder in the paint is still tacky, magnesium oxide is dusted over the interior so that a highly reflective matte surface is obtained. The sphere is attached to the bottom of the aluminum plate which holds the coils at such a position that the shutter, when fastened to the shaft of the oscillating coils, just misses the sphere at all points. The shutter is made of brass, balsa wood, and photographic film. The arm and collar which are attached to the shaft are of brass. A curved holder made of balsa wood is at the end of the brass arm. The photographic film is then attached to the balsa wood holder.

Some increase in the moment of inertia caused by the shutter is desirable. However, in order to keep the increase in the moment of inertia from becoming excessive the balsa wood is used in the construction to reduce the mass of the rotating parts.

The uniform flux for the integrating sphere is from a mazda lamp at a distance of about three feet from the shutter. The lamp is in a light tight box, shown in photograph no. 1. Within the box the lamp can be moved from a distance of about three feet to a distance of about one foot so that the intensity of the light beam can be varied as needed. Tests show that a uniform beam is obtained when the light is as close as one foot to the integrating sphere. The light source and the integrating sphere are enclosed in a light tight enclosure below the coils. On either side of the light tight box a panel can be removed to gain access to the shutter and sphere, and the end of the box in which the source light is located can be removed to gain access to the mazda lamp.

Inside the sphere is located a 919 type vacuum photo-tube. The orientation of the tube is such that no direct light falls on it. The target faces the top of the sphere in such a manner that only after multiple reflections can any light reach the target. This guarantees that the response of the photo-cell will depend only on the quantity of light entering the sphere.

The production of the film for the shutter was reduced to a routine procedure which was as follows. The curve of $\phi'(\theta)$ when $\theta\phi(\theta)$ is the spring force, is first drawn on a large sheet of white cardboard so that the base line, or θ-axis, is about 14 inches long and the vertical depth at the maximum is about 10 inches. A cutout of the opening is then made with sharp knives to give a sharp edge to the contour. A photograph of this is taken with an Eastman Precision enlarger. The camera head is run to the extreme top of the enlarger post. The dimensions of 14 inches by 10 inches for the cutout are correct to fill the negative to within a quarter of an inch margin. Beneath the cutout a piece of black velvet is placed to produce the maximum contrast between the cutout and the cardboard. A two-second exposure at f-22 on contrast process film was found to give a negative transparent enough in the space under the curve and yet opaque at all other points. The negatives are developed together in pairs so that each has exactly the same development.

The block wiring diagram for the photo-tube, the control

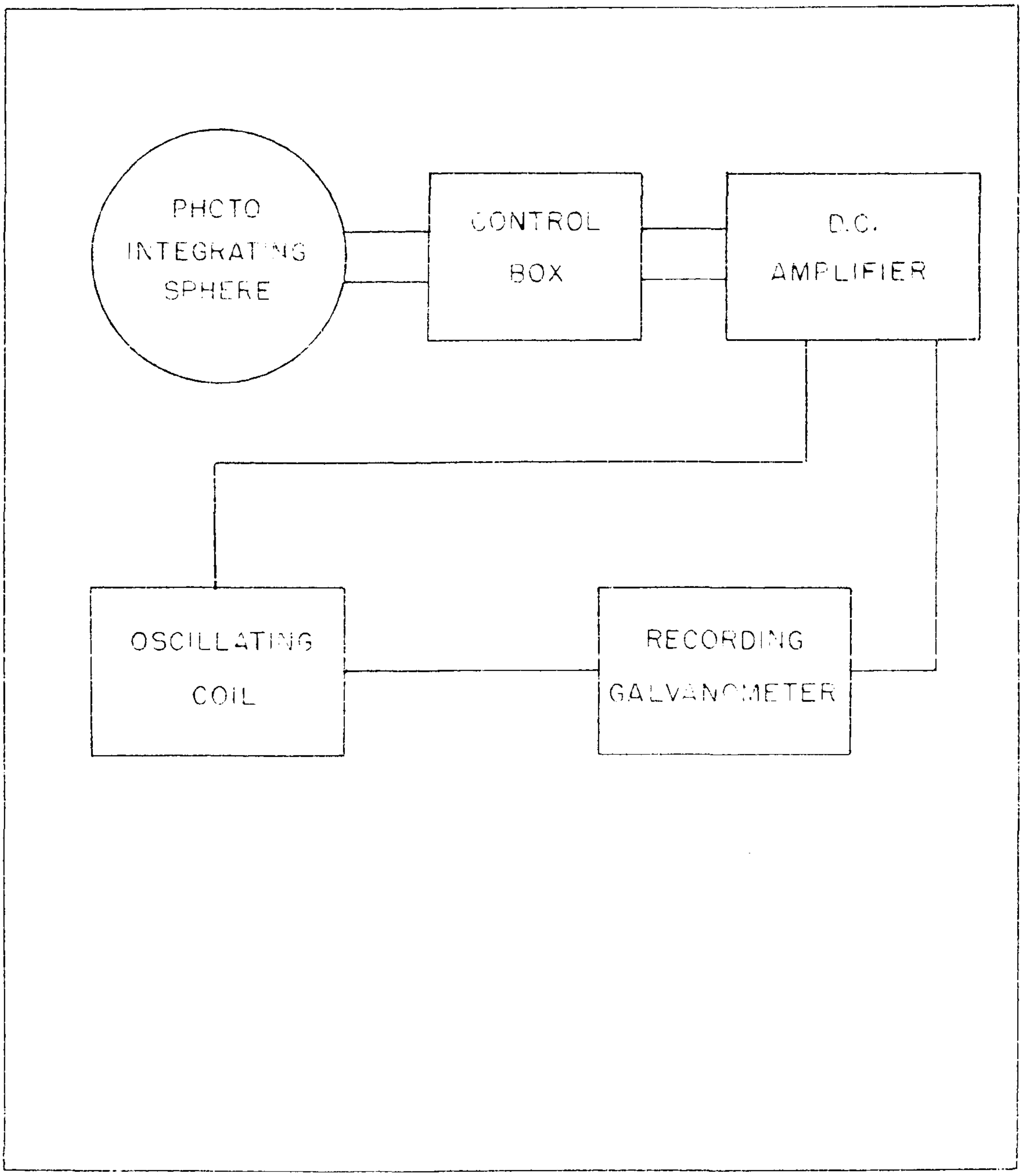
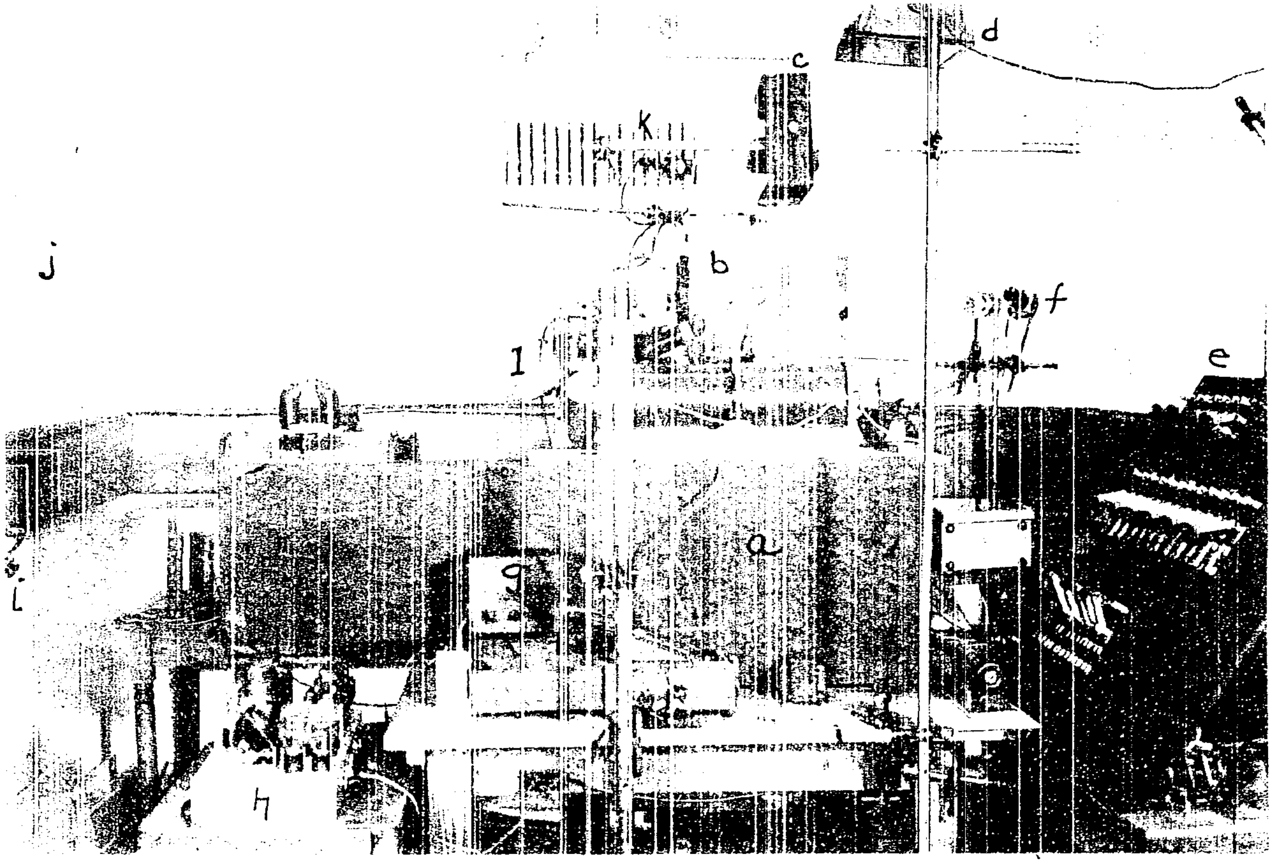


PHOTO
INTEGRATING
SPHERE
CONTROL
BOX
D.C.
AMPLIFIER
OSCILLATING
COIL
RECORDING
GALVANOMETER

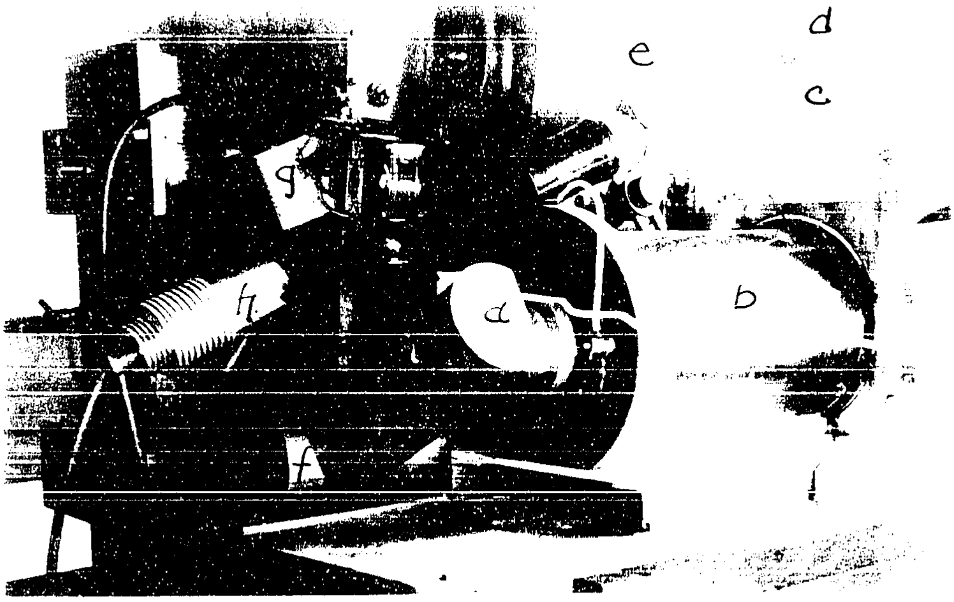
diagram no. 1

box, the recording string galvanometer, and the D. C. amplifier is diagram no. 1. The maximum amount of current used was about 25 milliamperes. Over the range from no light to enough light in the integrating sphere to cause the amplifier to give an output of 25 milliamperes the output was found to be essentially proportional to the amount of light which entered the sphere. For any given curve a corrected curve on the shutter can always be drawn so that the spring force-deflection curves are what one desires. When recording, the input, as well as the displacement, is always recorded. If the input for a given displacement is not correct, a slight adjustment in the shutter opening can be made to bring the current to the correct value. Over the range used in the following examples no corrections were needed. Only in exceptional cases, when large currents are needed, will there ever arise the necessity for correcting the curves to bring the spring force to the correct value.



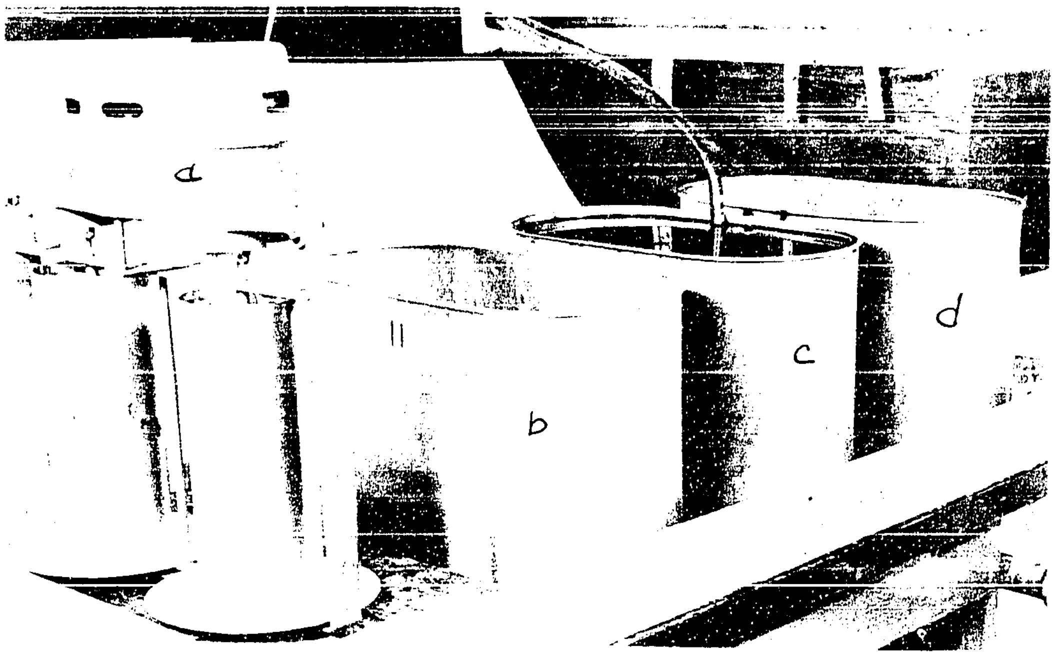
Photograph No. 1

a. housing for photo-integrating sphere and light
b. drum
c. recording string galvanometer
d. safe light
e. control panel
f. point source for record
g. control for gain
h. D. C. amplifier
i. point source for scale
j. scale
k. point source for galvanometer
l. point source for timing



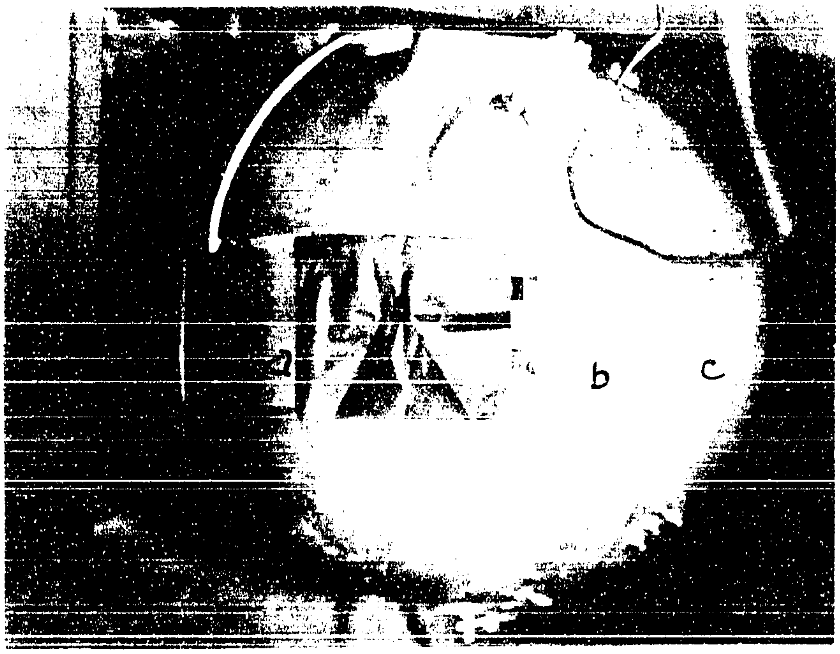
Photograph No. 2

a. pole piece
b. coil
c. scale mirror
d. recording mirror
e. drum
f. iron yoke
g. timing disc
h. point source for timing



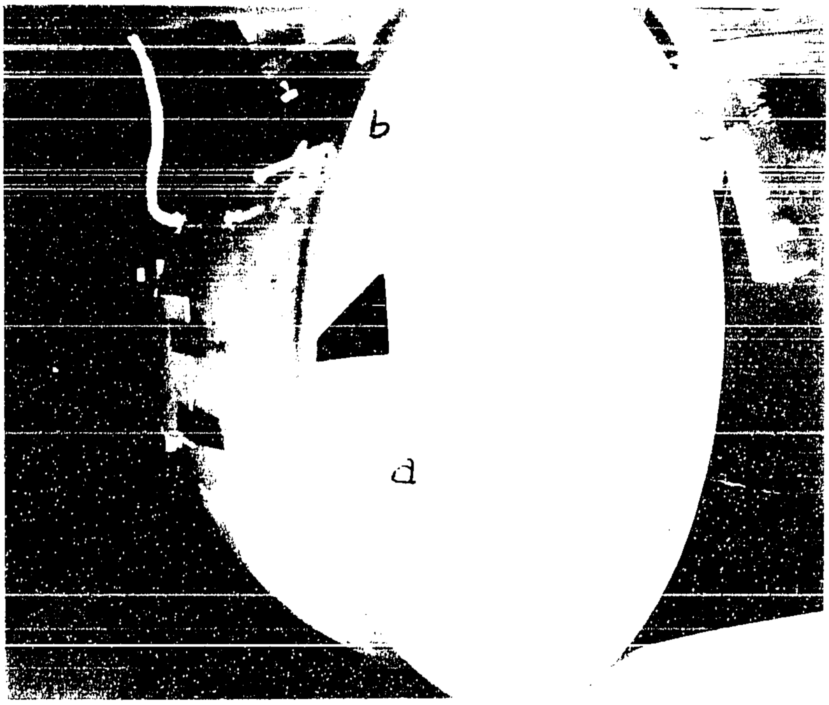
Photograph No. 3

a. agitator for linagraph paper
b. developer
c. washing tank
d. fixing bath



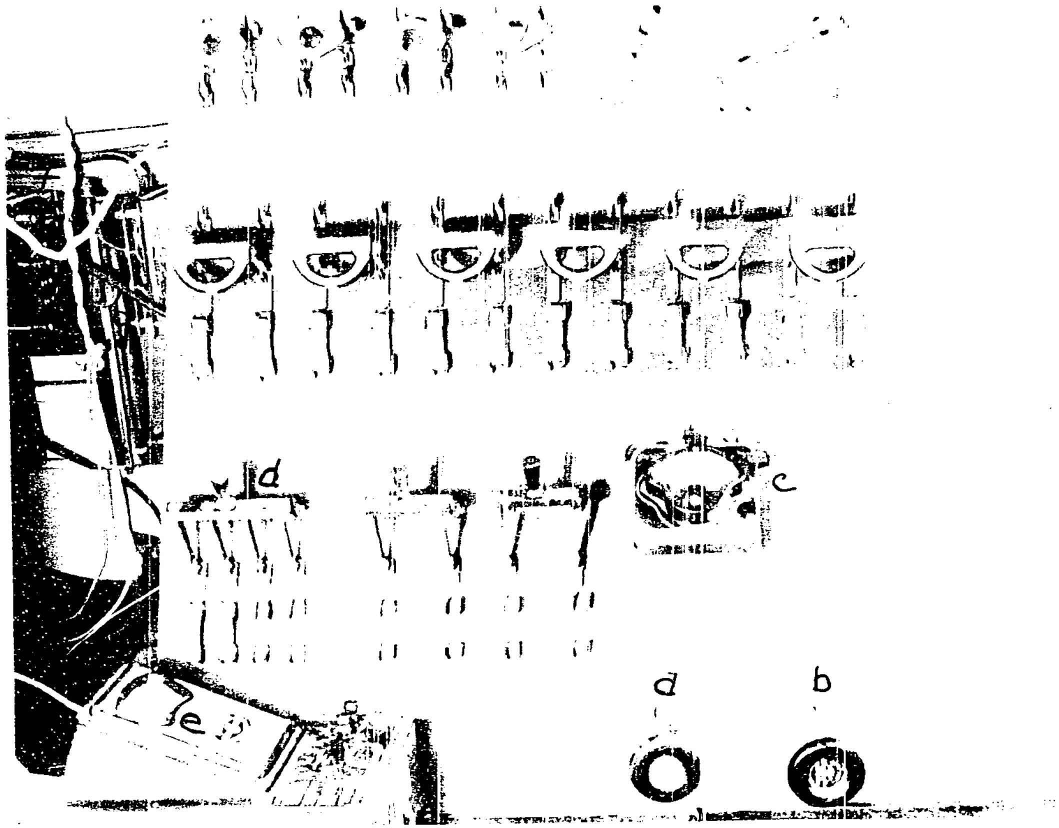
Photograph No. 4

a. shutter b. hemisphere c. removable back



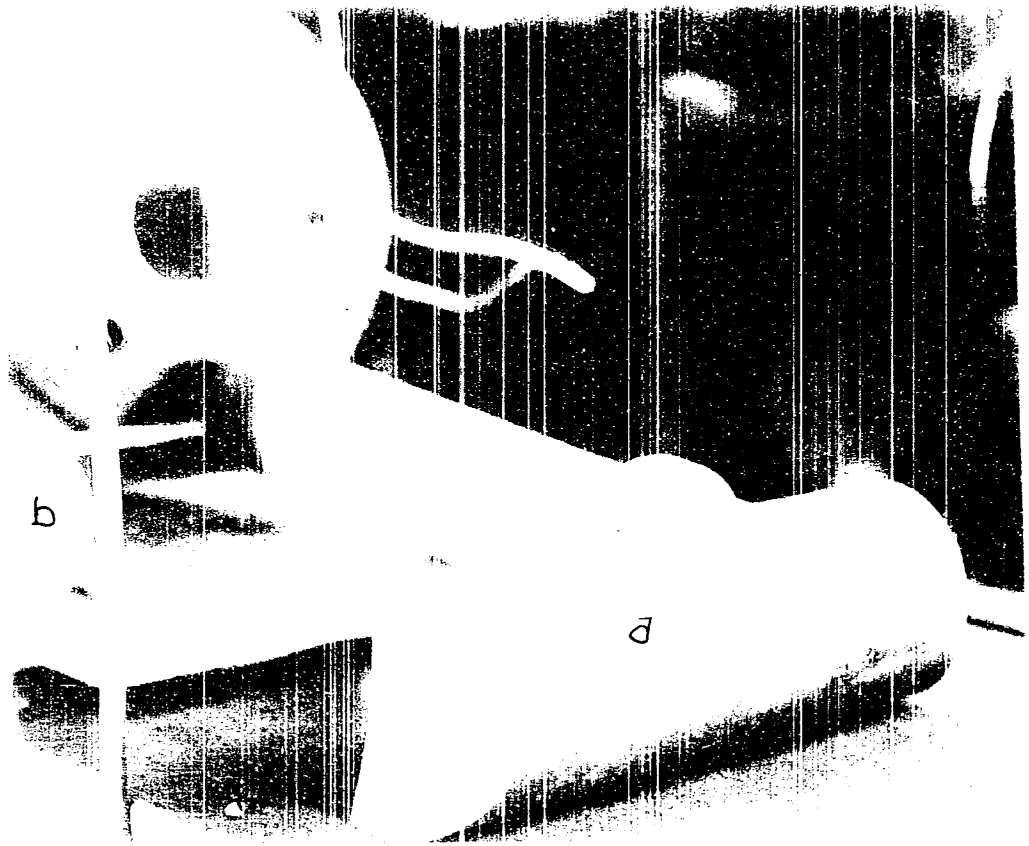
Photograph No. 5

a. matte coating inside sphere b. photo-cell



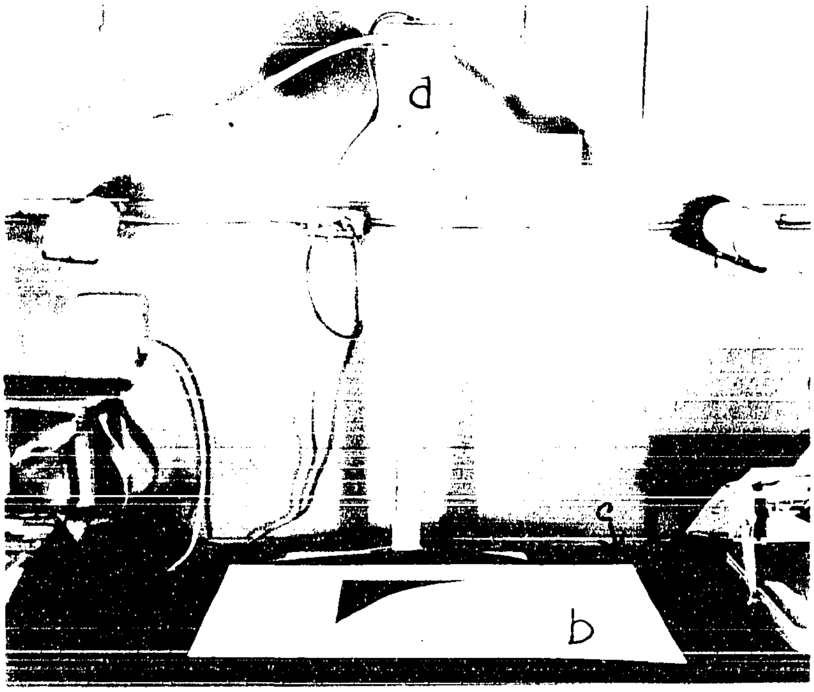
Photograph No. 6

a. rheostat control for field coil
b. rheostat control for drum motor
c. ammeter for field coil
d. switch for light for photo-integrating sphere
e. milliammeter for calibrating recording galvanometer
f. transformer for recording light system
g. rheostat for controlling intensity of recording light



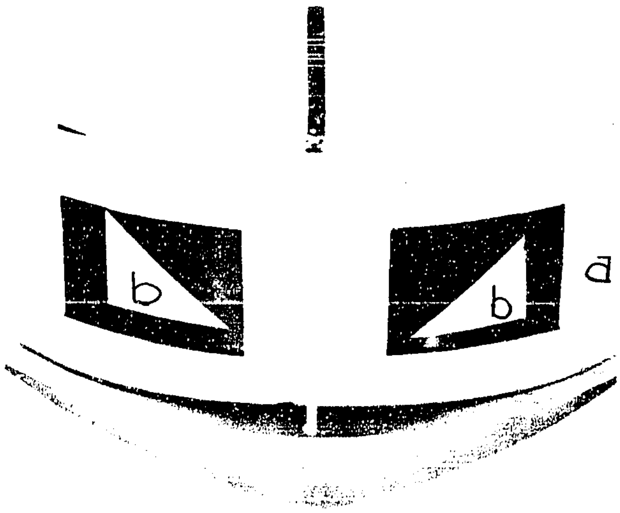
Photograph No. 7

a. pole piece removed to show construction
b. cross piece of yoke



Photograph No. 8

a. Eastman precision enlarger used as a camera
b. white cardboard
c. black velvet



Photograph No. 9

a. balsa wood shutter
b. 2-1/4" × 3-1/4" film for mask

4. Spring Forces and Damping

a. **Spring Forces**. There are two methods available for measuring the spring force produced by any given current in the coils. Load deflection curves could be determined by placing known amounts of current through the coils and then measuring the torque necessary to displace to various values of displacement. Use of this method necessitates measuring the moment of inertia of the moving system in addition to the spring force. A second method, requiring the measurement of but one quantity, is used in this paper. No measuring devices other than those with which the analogue is equipped are needed. This method consists of measuring the frequency of a free vibration for various values of current.

If we consider

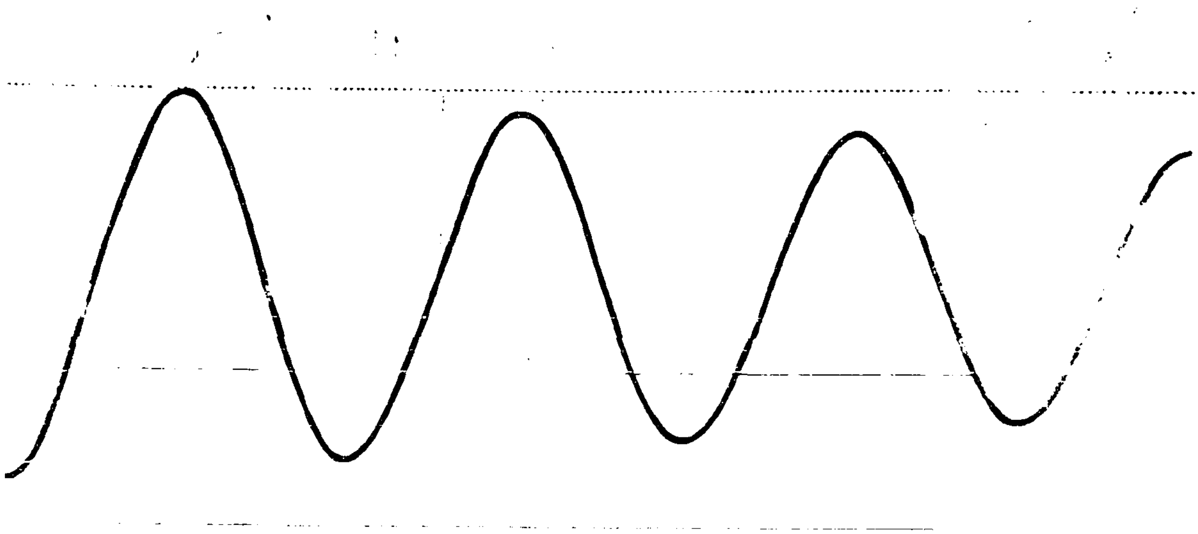
$$J\ddot{\theta} + k\theta = 0 \tag{4.1}$$

we see that we can divide the equation by J and write

$$\ddot{\theta} + \frac{k}{J}\theta = 0 \tag{4.2}$$

But the spring constant, $\frac{k}{J}$, as now written is ω_n^2, when ω_n is the frequency in radians per second of a free vibration.

In this discussion we are interested in a constant current in the field coil so that a constant field will be maintained. A value of 4 amperes with the large coils in place was selected. With 4 amperes of current in the large coil, the current in the oscillating coil was set at several values and a free oscillation was recorded.



photograph no. 10

In photograph no. 10 the curve "a" is the record of the oscillating coil with a current of 15 milliamperes. Line "b" is a row of timing dots at 1/30 second apart. ω_n^2 from this record is 16.65 $(\text{radian/second})^2$. Line "c" is a record made with zero current in the oscillating coil. Line "d" is produced by the recording galvanometer when 15 milliamperes flow.

Now we can convert any given value of current into force from the curve. Since it is a straight line, we see that 1.24 1, with 1 measured in milliamperes.

b. Damping. The discussion in this paper assumes that the amount of damping is very small.

From the theory of free vibration in a harmonic oscillator with viscous damping we know that the equation

$\frac{d^2\theta}{dt} + c\frac{d\theta}{dt} + \omega_n^2\theta = 0$, when ω_n = natural angular frequency and c = coefficient of viscous damping, can be rewritten as $\frac{d^2\theta}{dt^2} + 2b\omega_n\frac{d\theta}{dt} + \omega_n^2\theta = 0$ where ω_n is the natural frequency of the system and b is the damping factor. The damping factor b is the ratio of c to c_{cR} when c_{cR} is the critical damping.

δ, the log decrement, is approximately the logarithm of the ratio of successive peaks of the damped record.

In terms of b

$$\delta = \frac{2\pi b}{(1 - b^2)^{1/2}} \qquad (4.3)$$

A typical record gave δ as ln 1.1 or δ = .095 and ω_n=4.76 radians per second.

Then $.095 = \frac{b}{(1 - b^2)^{1/2}}$ or b = .015

This value of b is considered well below a value of b that would have to be considered when studying the effect of non-linearity in the spring force on the solutions of differential equations.

Figure no. 7 shows a graph of ω_n^2 plotted against current in the oscillating coil when 4 amperes are in the large coil producing the uniform magnetic field.

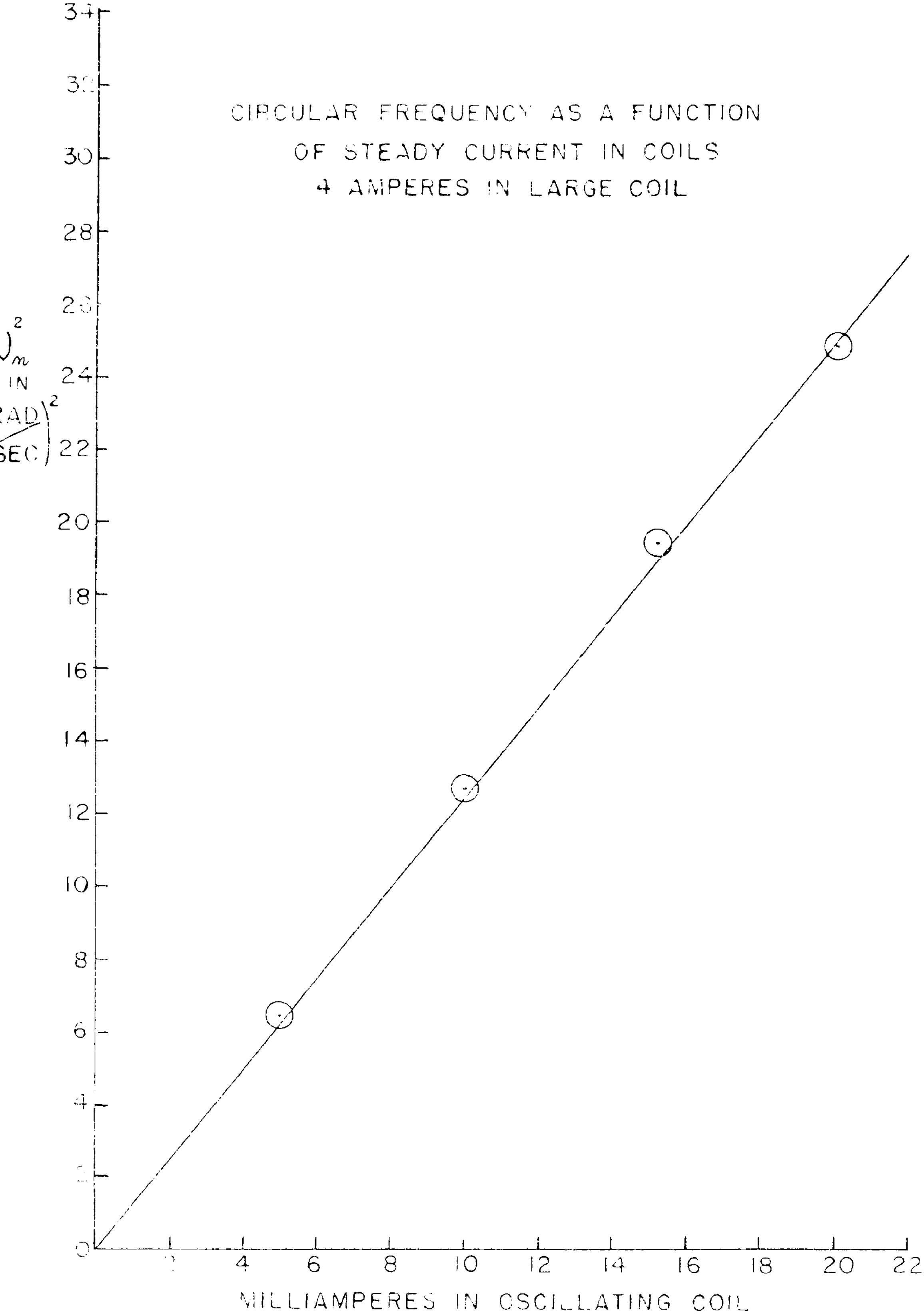

CIRCULAR FREQUENCY AS A FUNCTION
OF STEADY CURRENT IN COILS
4 AMPERES IN LARGE COIL
34
30
28
24
22
20
18
16
14
12
10
8
6
4
0
IN
4
6
8
10
12
14
16
18
20
22
MILLIAMPERES IN OSCILLATING COIL

figure no. 7

5. Sample Solutions

a. First let us consider

$$\frac{d^2x}{dt^2} + kx^3 = 0 \qquad (5.1)$$

an equation which has a periodic solution, the period of which can be found in terms of the amplitude.[1]

This equation can be written as

$$v\frac{dv}{dx} + kx^3 = 0, \qquad v = \frac{dx}{dt} \qquad (5.2)$$

$$\frac{dv}{dx} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

Integrating, this becomes

$$\frac{v^2}{2} + \frac{kx^4}{4} = h \qquad (5.3)$$

Now at $x = a$, $v = 0$ and, therefore,

$$h = \frac{ka^4}{4} \qquad (5.4)$$

From (5.2) we see that the phase trajectory is symmetric with respect to the origin. Therefore, the period can be written as

$$T = 4\int_0^a \frac{dx}{v} = 4\int_0^a \frac{dx}{\sqrt{2h - \frac{k}{2}x^4}}$$

$$= 4\int_0^a \frac{dx}{\sqrt{\frac{k}{2}a^4 - \frac{k}{2}x^4}} \quad \frac{4\sqrt{2}}{\sqrt{k}}\int_0^a \frac{dx}{\sqrt{a^4 - x^4}} \qquad (5.5)$$

Now substitute in (5.4) $x = a \sin\theta$.

[1]Stoker, J. J. Non-linear Vibrations in Mechanical and Electrical Systems. New York: Interscience Publishers, Inc., 1950. P. 21.

Then

$$T = \frac{4\sqrt{2}}{\sqrt{k}} \int_0^{\frac{\pi}{2}} \frac{a \cos\theta \; d\theta}{\sqrt{(a^2 + x^2)(a^2 - x^2)}}$$

$$= \frac{4\sqrt{2}}{\sqrt{k}} \int_0^{\frac{\pi}{2}} \frac{a \cos\theta \; d\theta}{\sqrt{(a^2 + a^2 \sin^2\theta)(a^2 - a^2 \sin^2\theta)}}$$

$$= \frac{4\sqrt{2}}{a\sqrt{k}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + \sin^2\theta}} \qquad (5.6)$$

From tables the definite integral $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + \sin^2\theta}}$ is found to have the value of 1.301.

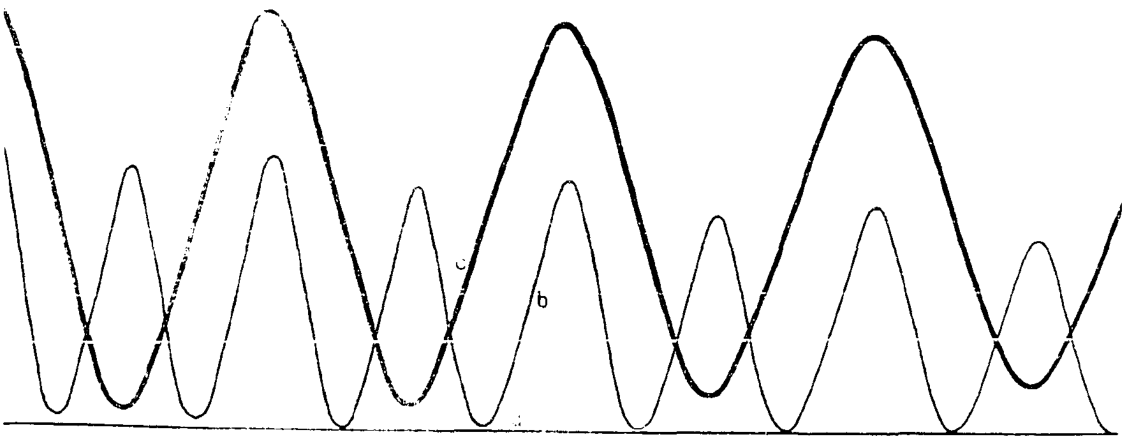
Hence,

$$T = \frac{1.301 \times 4\sqrt{2}}{\sqrt{k}} \cdot \frac{1}{a} \qquad (5.7)$$

or we have

$$\frac{T_1}{T_2} = \frac{a_2}{a_1} \qquad (5.8)$$

as the relation between the period and the amplitude.



photograph no. 11

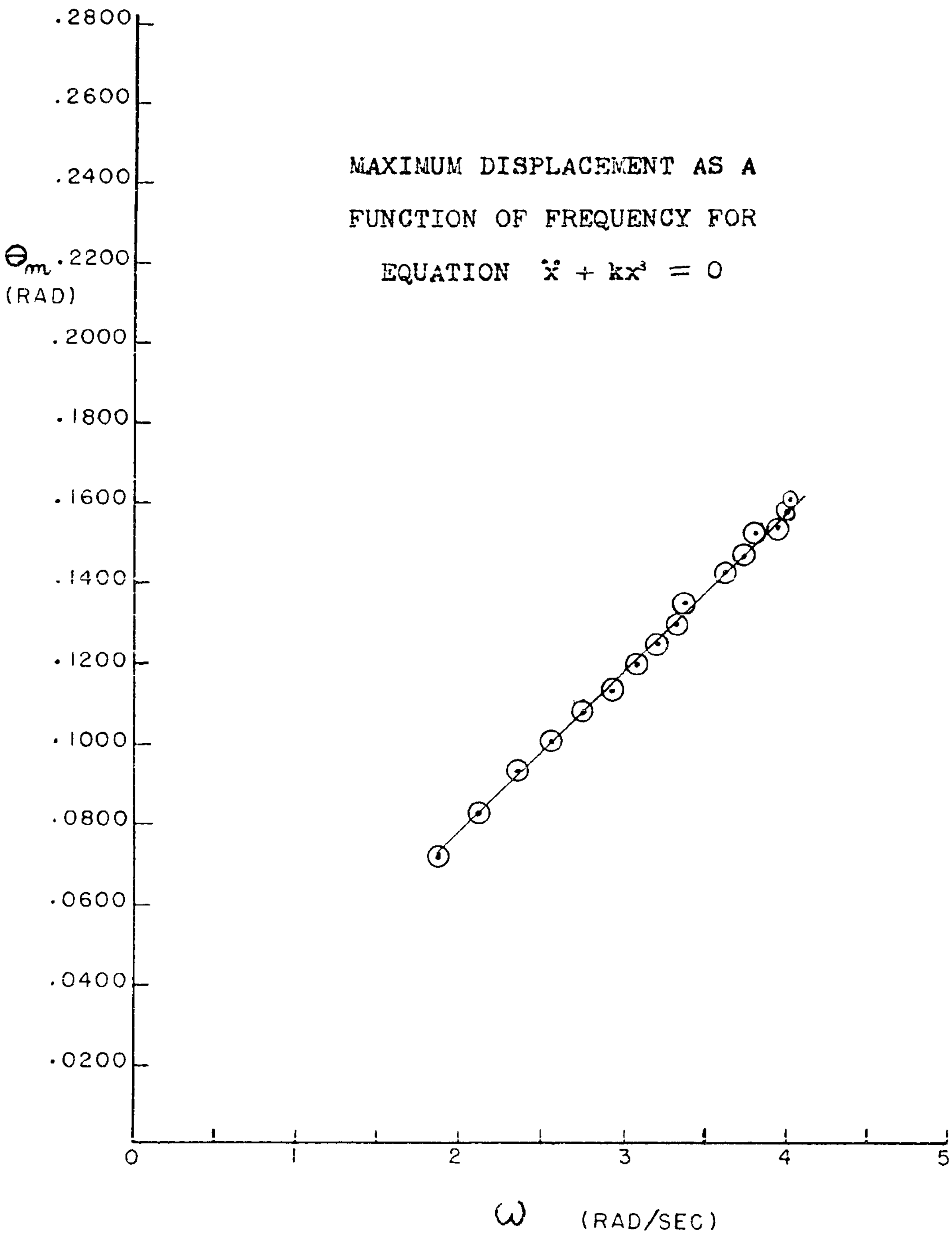

MAXIMUM DISPLACEMENT AS A
FUNCTION OF FREQUENCY FOR
EQUATION $\ddot{x} + kx^3 = 0$
Θ_m (RAD)
.2800
.2600
.2400
.2200
.2000
.1800
.1600
.1400
.1200
.1000
.0800
.0600
.0400
.0200
0
1
2
3
4
5
ω (RAD/SEC)

figure no. 8

Photograph no. 11 of a photographic record for $\frac{d^2\theta}{dt^2} + k\theta^3 = 0$ shows the various graphs from which can be obtained the solution and a check on the input. At the top of the photograph we see the series of dots which are placed there by a light intercepted in such a manner as to make the dots 1/30 of a second apart. The bottom curve on the photograph is the record of the input into the small coil, while the curve just below the timing dots is the displacement record of the coil. We see, therefore, that a plot of displacement against time and a plot of input current against time are obtained on the Linagraph recording paper.

The theory for $\frac{d^2\theta}{dt^2} + k\theta^3 = 0$ developed at the beginning of section 5 does not take damping into account. It can be shown that the influence of damping is of the second order when a relationship between the period and amplitude is sought. It will be assumed here that over a given period it is possible to ignore the damping. Now determining the maximum amplitude $a_m = \Theta_m$ from the record and the corresponding frequency ω_m, it follows from the theory that a_m plotted against ω_m will give a straight line. From figure no. 8 we see that "a" plotted against ω_m is a straight line as predicted by the theory.

The method used to obtain the term $k\theta^3$ in

$$\frac{d^2\theta}{dt^2} + k\theta^3 = 0 \tag{5.9}$$

is as follows. The equation can be written

$$\frac{d^2\Theta}{dt^2} + (k\,\Theta^2)\Theta = 0 \qquad (5.10)$$

From the theory of the apparatus in section 2 we know that the curve on the shutter should be $\phi'(\Theta) = 2k\Theta$. The completed shutter is shown in photograph no. 9. The constant k in the equation can be varied by changing the output of the amplifier. A new shutter does not have to be produced for various values of k.

On page 32 are three graphs which show the negative of the spring force plotted against displacement, $\phi(\Theta)$ against displacement and $\phi'(\Theta)$ against displacement.

b. Now consider $\frac{d^2\Theta}{dt^2} + k_1\Theta - k_3\Theta^3 = 0$, k_1 and $k_3 > 0$, another equation for which an expression for frequency as a function of time can be obtained.

If for $\frac{d^2x}{dt^2} + k_1 x - k_3 x^3 = 0$ we write $v\frac{dv}{dx} + k_1 x - k_3 x^3 = 0$ we can integrate and obtain

$$\frac{v^2}{2} + \frac{k_1 x^2}{2} - \frac{k_3 x^4}{4} = p \qquad (5.11)$$

At $x = a$, $v = 0$, therefore

$$k_1 a^2 - \frac{k_3 a^4}{2} = h \qquad (5.12)$$

Then, since the curve in the phase plane is symmetric with the origin

$$T = 4\int_0^a \frac{dx}{v} = 4\int_0^a \frac{dx}{\sqrt{h - (k_1 x^2 - \frac{k_3 x^4}{2})}} \qquad (5.13)$$

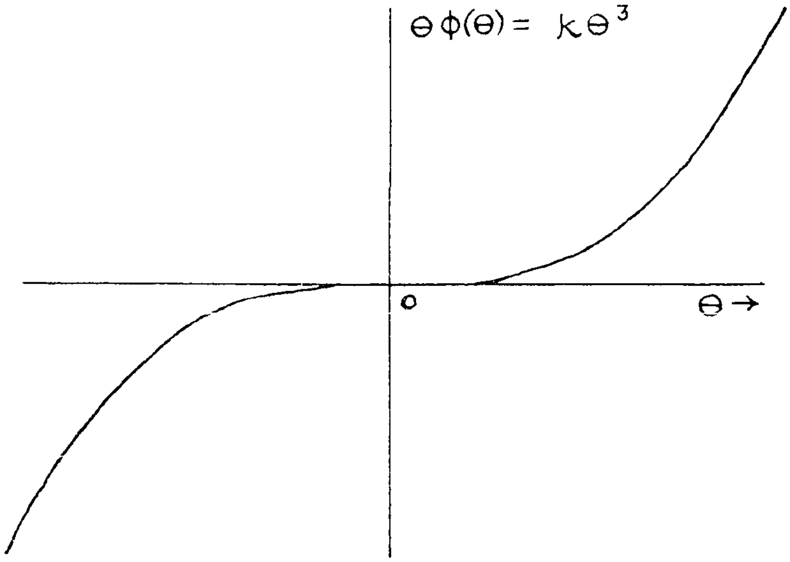


figure no. 9

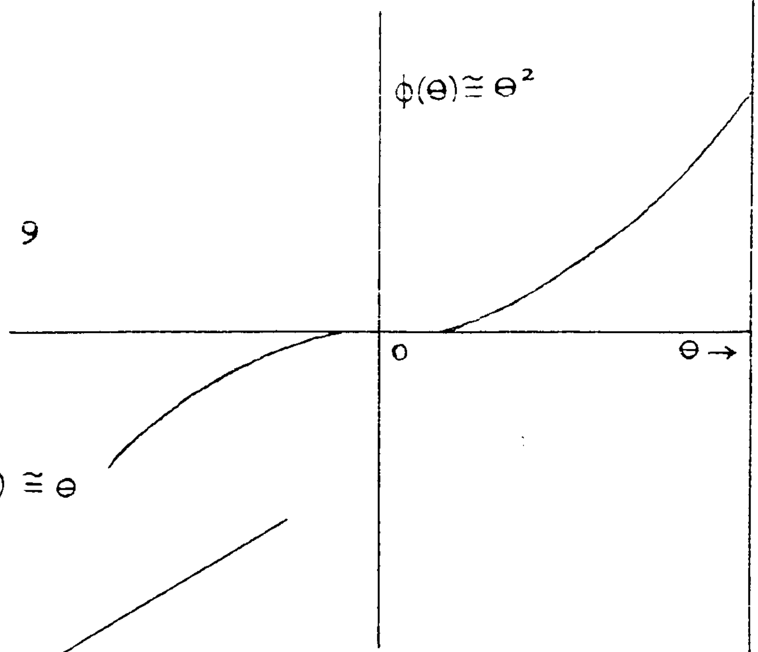


figure no. 10

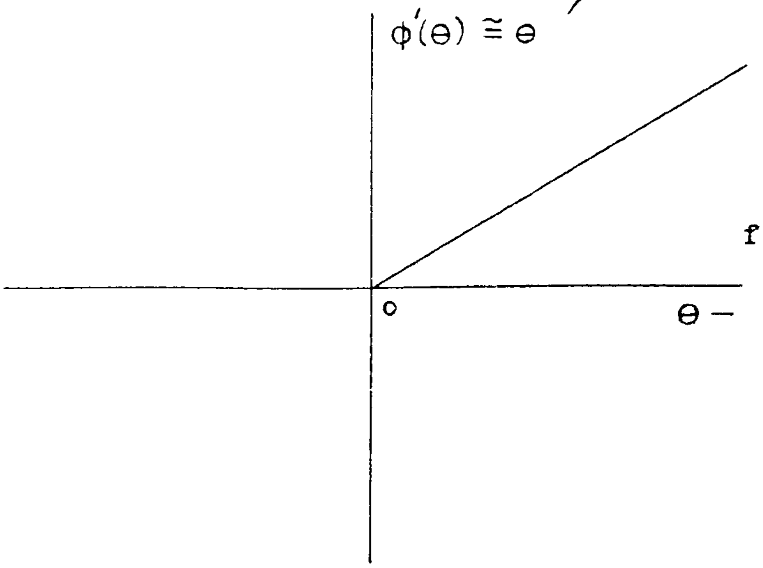


figure no. 11

Since a^2 is a root of

$$h - (\alpha z + \beta \frac{z^2}{2}) = 0 \qquad (5.14)$$

we may write

$$h - (k_1 x^2 - \frac{k_2 x^4}{2}) = \frac{\beta}{2}(a^2 - x^2)(b^2 + x^2) \qquad (5.15)$$

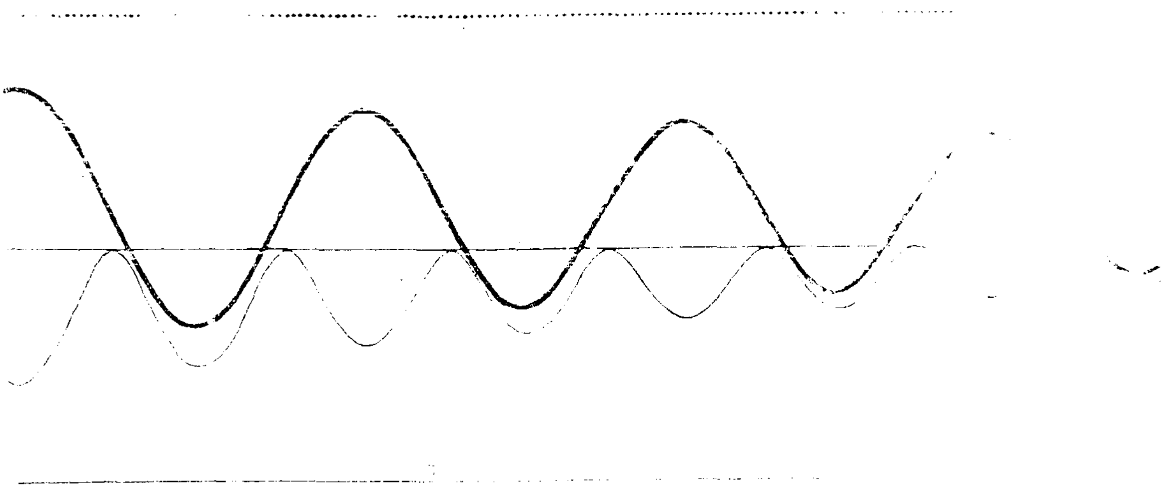
and

$$-\frac{k_2}{2}(-b + a^2) = -\alpha \quad , \text{ or } -k_2 b^2 = -k_2 a^2 + 2k_1 \qquad (5.16)$$

Now if we let $x = a \sin\theta$ and eliminate b by (5.13),

$$T = 4\sqrt{2} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2k_1 - k_2 a^2 - k_2 a^2 \sin\theta}} \qquad (5.17)$$

Equation (5.14) can then be used to calculate the period for given values of a, k_1, and k_2.



photograph no. 12

The above photograph is a record taken of

$$\frac{d^2\theta}{d\tau^2} + \theta - \frac{k_2 \theta^3}{k_1} = 0 \qquad \frac{k_2}{k_1} = \frac{1}{6} \qquad (5.18)$$

On this record curve "a" was produced by turning the drum one complete revolution with no current flowing in the coil. Curve "c" is a curve drawn at the peak current flowing. This is the current flowing at no displacement. Curve "b" represents the actual current flowing in the coil while oscillating. Curve "d" is the displacement record. Line of dots "e" is the time record. Since we have simultaneously recorded the current input and the displacement, we can check on the accuracy of the spring force at any time.

Spring force is obtained in the following way. Suppose that we wish $\frac{k_3}{k_1}$ to equal $\frac{1}{6}$. A plot of the spring force is shown in figure no. 12.

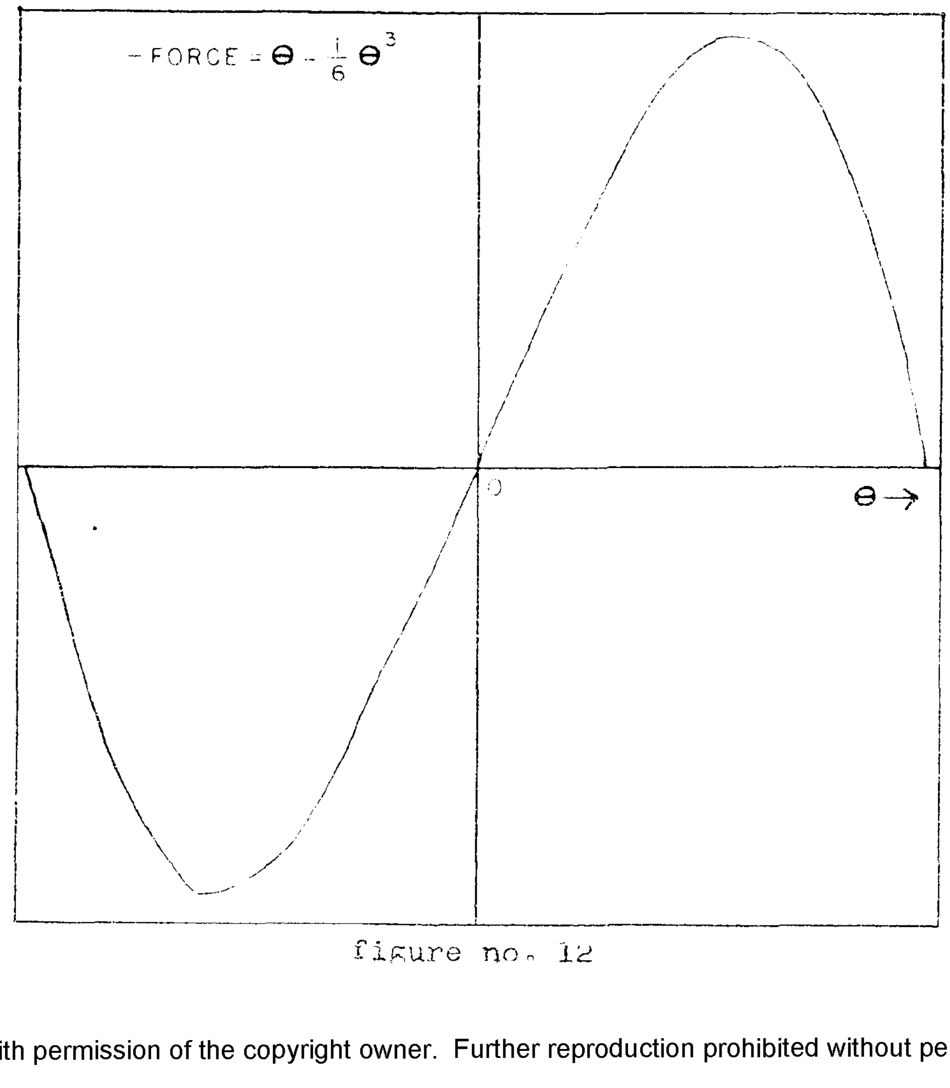


figure no. 12

Now we need

$$\theta\,\phi(\theta) = \theta(1 - \tfrac{1}{6}\theta^2) \tag{5.19}$$

The function of force goes to zero over $\sqrt{6}$ radians. In order to represent this motion on the analogue, a displacement of $\frac{1}{\sqrt{365}}$ of a radian will represent a displacement of 1 radian in the original equation.

Suppose we represent the equation

$$\frac{d^2\theta}{dt^2} + k_1\theta - k_2\theta^3 = 0 \tag{5.20}$$

by the equation

$$\frac{d^2\Psi}{dt^2} + k_1'\Psi - k_3'\Psi^3 = 0, \quad \theta = \lambda\Psi \tag{5.21}$$

Then

$$\frac{d^2\Psi}{dt^2} + k_1\Psi - \lambda^2 k_3\Psi^3 = 0 \tag{5.22}$$

Now, if

$$\tau = \sqrt{k_1}\cdot t \tag{5.23}$$

we have

$$\frac{d^2\Psi}{d\tau^2} + \Psi - \lambda^2\frac{k_3}{k_1}\Psi^3 = 0 \tag{5.24}$$

Since we chose to have $\frac{k_3}{k_1} = \frac{1}{6}$, we will have an effective constant of $\lambda^2\frac{1}{6}$ when λ is equal to the ratio of θ to Ψ. λ was chosen to equal $\sqrt{365}$ in this problem. Therefore, $\lambda^2\frac{1}{6} = \frac{365}{6} = 60.8$. Our equation is now

$$\frac{d^2\Psi}{d\tau^2} + \Psi - 60.8\,\Psi^3 = 0 \tag{5.25}$$

The spring force in terms of ψ is shown in figure no. 13.

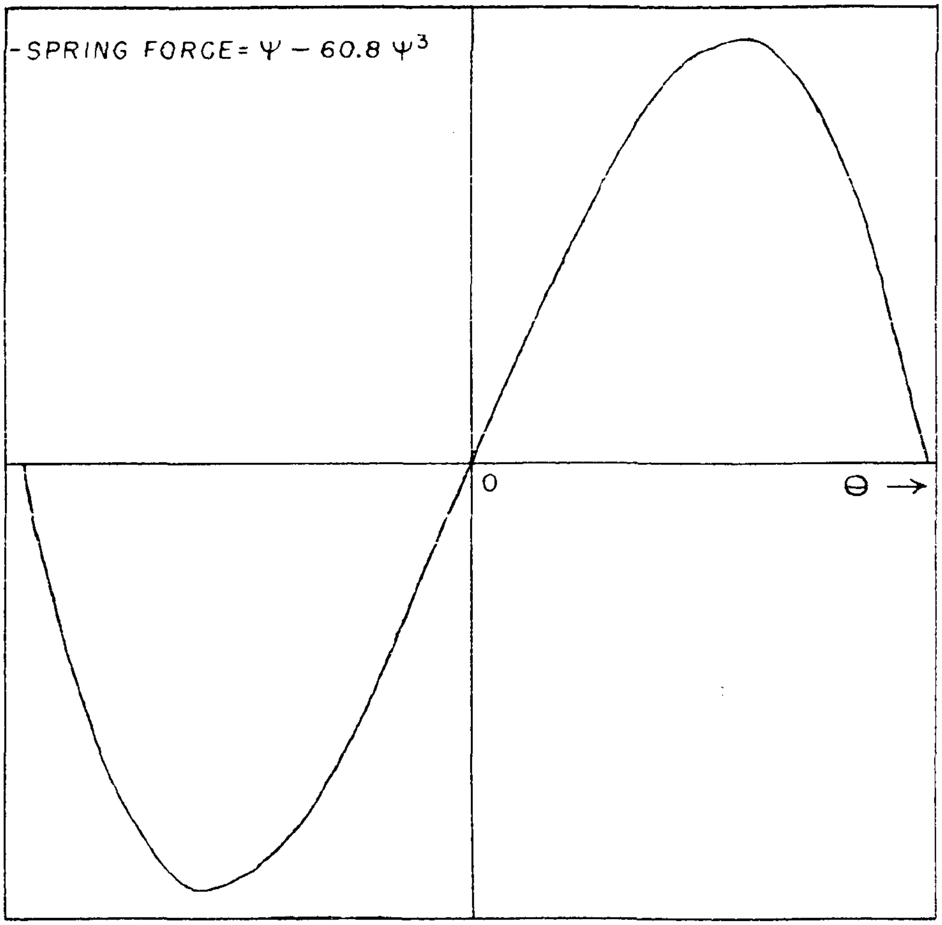


figure no. 13

$\psi\,\phi(\psi) = \psi\,(1 - 60.8\,\psi^2)$; therefore, $\phi(\psi)$ is made up of two terms as illustrated in figure no. 14.

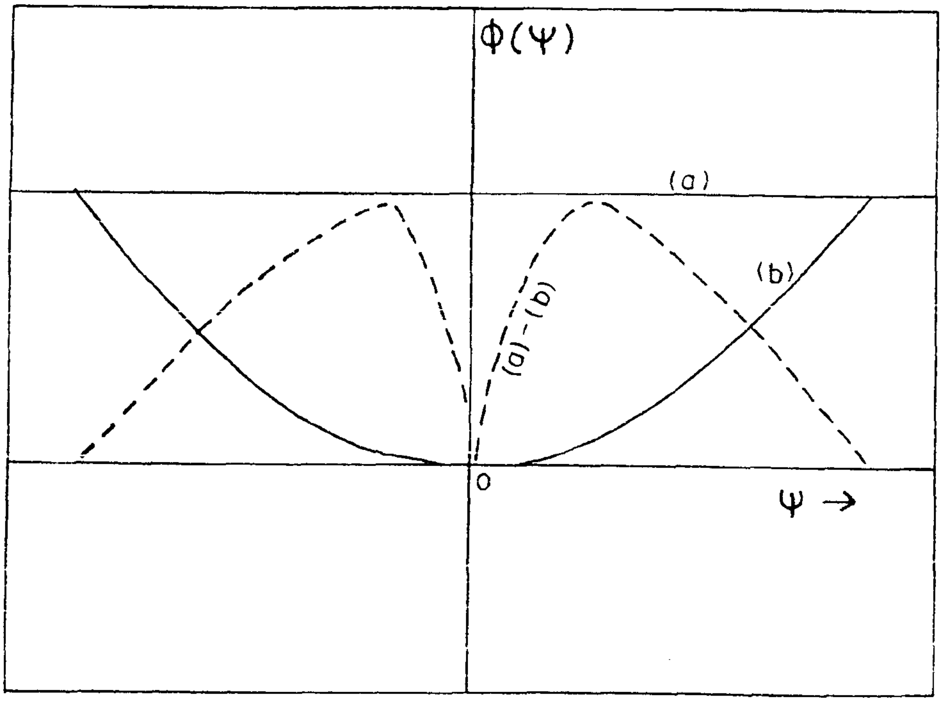


figure no. 14

Curve "a" in figure no. 14 represents unit force; curve "b" represents 60.8 ψ^2 ; and curve "c" represents $\phi(\psi)$, which is found by subtracting curve "b" from curve "a". In order to achieve this constant on the machine, the amplifier is adjusted to cause 19.8 milliamperes to flow in the oscillating coils. This means that $k_1 = 23.8$.

The shutter is a triangular shaped opening whose area increases with displacement squared or ψ^2 .

Since $\lambda^2 \frac{k_3}{k_1} = 60.8$, $\lambda^2 = 365$ and $k_1 = 23.8$, we have

$$(\lambda^2 k_3) = 60.8 \times 23.8 = 1450 \qquad (5.26)$$

$$i = (\lambda^2 k_3)\,\Theta_m^2 \cdot 1.24 = 1450 \times \Theta_m^2 = 1450\,\Theta_m^2\,\text{mA} \qquad (5.27)$$

1.24 = conversion factor between force and current.

Now if a value of Θ_m is chosen and the coil is displaced by that amount, the control can be adjusted so that the correct amount of current flows. A value of $\Theta_m = .06$ radians was chosen. Then $i = 1170 \times (.06)^2 = 1170 \times .0036 =$ 4.21 mA is the value of the current which must pass through the coil at a displacement of .06 radians. To be sure that the correct settings have been chosen a trial record can be run and the input checked against the displacement.

On page 33 is photograph no. 12 which is the record taken of $\frac{d^2\Theta}{dt^2} + \Theta - \frac{1}{6}\Theta^3 = 0$.

"a" is a line recorded with zero current through the oscillating coils. "b" is the line produced by the recording string galvanometer when the record was made. "c" is an envelope of the maximum currents. Notice this is a straight line. The distance between "a" and "c" is proportional to the linear force term in the differential equation. "d" is the displacement curve. "e" is the line of timing dots.

The formula (5.17) for calculating values for t at various amplitudes and the photographic record for finding t for that amplitude have been used to prepare the table of values in table no. 1.

	Amplitude Radians	Period From Record	Period Calculated	Diff	% Diff.
0	1.845	1.710	1.705	.005	.73%
1	1.48	1.532	1.521	.011	.84%
2	1.25	1.452	1.440	.012	.5 %
3	1.10	1.412	1.405	.007	.15%
4	.923	1.373	1.375	.002	.07%
5	.775	1.340	1.339	.001	0 %
6	.656	1.325	1.325	.000	.03%
7	.559	1.319	1.320	.001	.00%

table no. 1

c. Now consider the equation

$$\frac{d^2\theta}{dt^2} + k_1\theta + k_3\theta^3 = 0, \qquad k_1, k_3 > 0 \tag{5.28}$$

This equation is selected here because it can be solved by several methods for approximate relationships between amplitude and frequency when $\frac{k_3}{k_1}$ is small[1] and a considerable amount of data is available for cases in which $\frac{k_3}{k_1}$ is large[2].

Suppose we first make the transformation $\tau = \sqrt{k_1}\, t$ on equation (5.28). Then we have

$$\frac{d^2\theta}{d\tau^2} + \theta + \mu\theta^3 = 0, \qquad \mu = \frac{k_3}{k_1} \tag{5.29}$$

Using the method of slowly varying parameters of Kryloff and Bogoliuboff, we can show that, given the equation

[1]Minorsky, N. Introduction to Non-linear Mechanics. Ann Arbor: J. W. Edwards Co., 1947. p. 190.

[2]Ludeke, C. A. Journal of Applied Physics, Vol. 20, No. 6 pp. 600-607, June, 1949; Vol. , No. pp. 694-699, July, 1949

$$\ddot{x} + \omega^2 x + \mu f(x,\dot{x}) = 0 \qquad (5.30)$$

to the first approximation when μ is small,

$$x = a \sin \psi \qquad (5.31)$$

with "a" considered a variable given by the differential equation

$$\frac{da}{dt} = -\frac{1}{2\pi}\frac{\mu}{\omega}\int_0^{2\pi} f(a \sin\phi, a\omega \cos\phi)\cos\phi\, d\phi = \Phi(a) \qquad (5.32)$$

and "Ψ" considered a variable given by the differential equation

$$\frac{d\Psi}{dt} = \omega + \frac{\mu}{a\omega}\frac{1}{2\pi}\int_0^{2\pi} f(a \sin\phi, a\omega \cos\phi)\sin\phi\, d\phi = \Omega(a) \qquad (5.33)$$

$f(a \sin\phi, a\omega \cos\phi) = f(x,\dot{x})$ with x replaced by $a \sin\phi$ and $\dot{x}$ by $a\omega \cos\phi$.

Now if we consider equations in which the damping can be ignored, as

$$\ddot{x} + \omega^2 x + \mu f(x) = 0 \qquad (5.34)$$

we have for equation (5.32)

$$\frac{da}{dt} = -\frac{\mu}{2\pi\omega}\int_0^{2\pi} f(a \sin\phi)\cos\phi\, d\phi \qquad (5.35)$$

and for equation (5.33)

$$\frac{d\Psi}{dt} = \omega + \frac{\mu}{2\pi a\omega}\int_0^{2\pi} f(a \sin\phi)\sin\phi\, d\phi \qquad (5.36)$$

Since

$$\int_0^{2\pi} f(a \sin\phi)\cos\phi\, d\phi = \frac{1}{a}\psi(a \sin\phi) = 0 \qquad (5.37)$$

we see that the amplitude does not change with time in the solution.

From (5.33) it follows that the solution has the form

$$\Psi = \Omega(a)\, t + \Psi_0 \qquad (5.38)$$

Squaring (5.33) and neglecting the term of the second order in μ, one has

$$\Omega^2(a) = \omega^2 + \frac{\mu}{\pi a}\int_0^{2\pi} f(a \sin\phi)\sin\phi \, d\phi \tag{5.39}$$

Now using this equation to find an approximate relation between frequency and amplitude for the equation under consideration, we find that

$$\frac{d^2\theta}{d\tau^2} + \theta + \mu\theta^3 = 0 \tag{5.40}$$

Using (5.39) when $f(a \sin\phi) = a^3 \sin^3\phi$ and $\omega^2 = 1$, we have

$$\Omega^2(a) = 1 + \frac{\mu}{\pi a}\int_0^{2\pi} a^3 \sin^3\phi \, d\phi = 1 + \frac{\mu a^2}{\pi} \quad \sin^3\phi \, d\phi$$

$$= 1 + \frac{\mu a^2}{\pi} \cdot \frac{3\pi}{4} = 1 + \frac{3}{4}\mu a^2 \tag{5.41}$$

It can be shown by methods of higher approximation that

$$\Omega^2(a) = 1 + \frac{3}{4}\mu a^2 + \frac{3}{128}\mu^2 a^4 \tag{5.42}$$

We should keep in mind that the above development was for μa^2 small. If μa^2 is large, then we will resort to a comparison of our results with those obtained by Dr. Ludeke on a different type of analogue.

A plot of the spring force with displacement is given in figure no. 15.

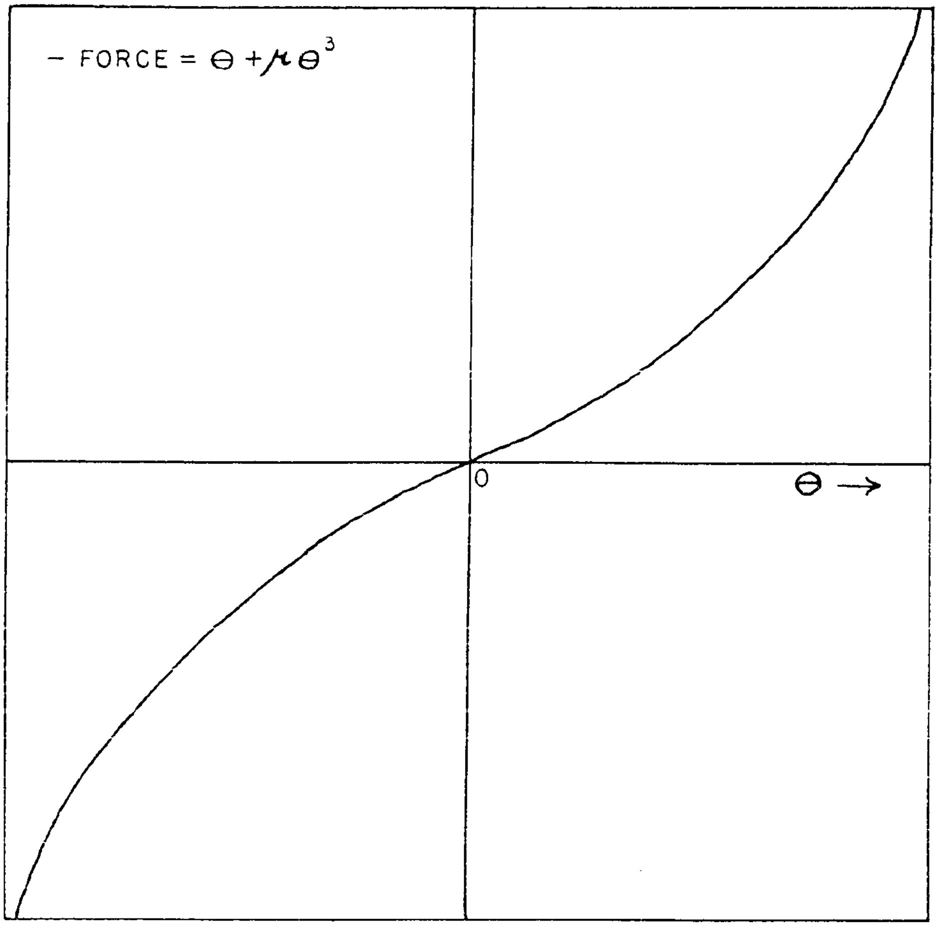
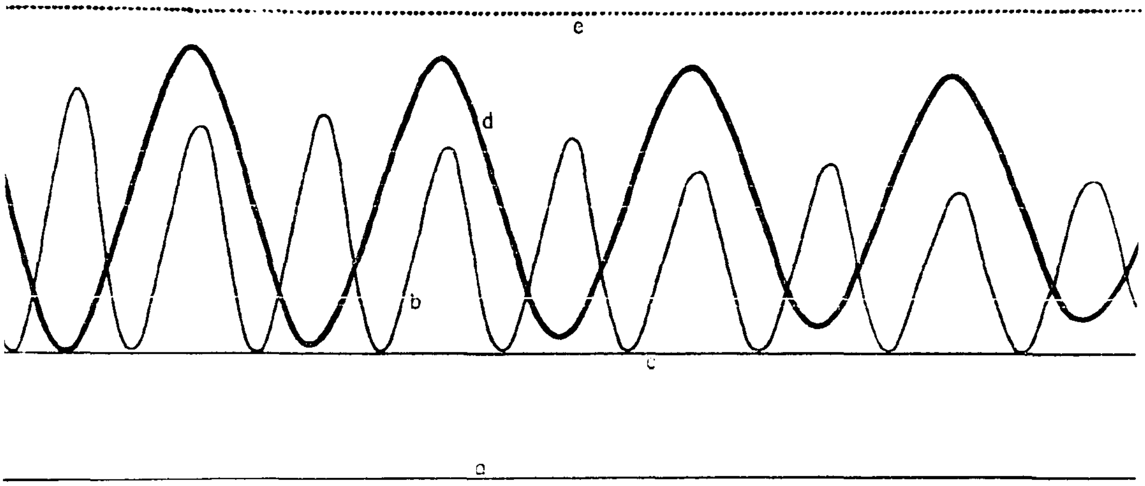


figure no. 15

For our representation it is then necessary to write $\Theta\phi(\Theta) = \Theta(1 \;+\mu\Theta^3)$. $\phi(\Theta)$ is then made up of two parts, a constant plus $\mu\Theta^3$. $\mu\Theta^2$ can be represented by a current which varies as the square of the displacement. As in section 5.2, we use triangular openings in the shutter, but this time the amplifier is adjusted so that a constant current flows and a current proportional to Θ^2 is added to it.

Photograph no. 13 is a record of $\ddot{\theta} + \theta + \mu\theta^3 = 0$.



photograph no. 13

Line "a" was recorded by running the drum one complete revolution with no current flowing in the small coils. "b" is the record of the galvanometer while the record was made. "c" is a line drawn so that it touches the bottom of the loops of "b". "d" is the solution with respect to time. "e" is a line of timing dots. On the photograph we see that the current represented by the galvanometer displacement between "a" and "c" is a constant and that the variable current is superimposed onto the constant current.

Figure no. 16 shows a plot of a^2 with $\left(\frac{\omega}{\omega_0}\right)^2$. Notice that the plot is a straight line despite the value of $\mu =$ 89.9.

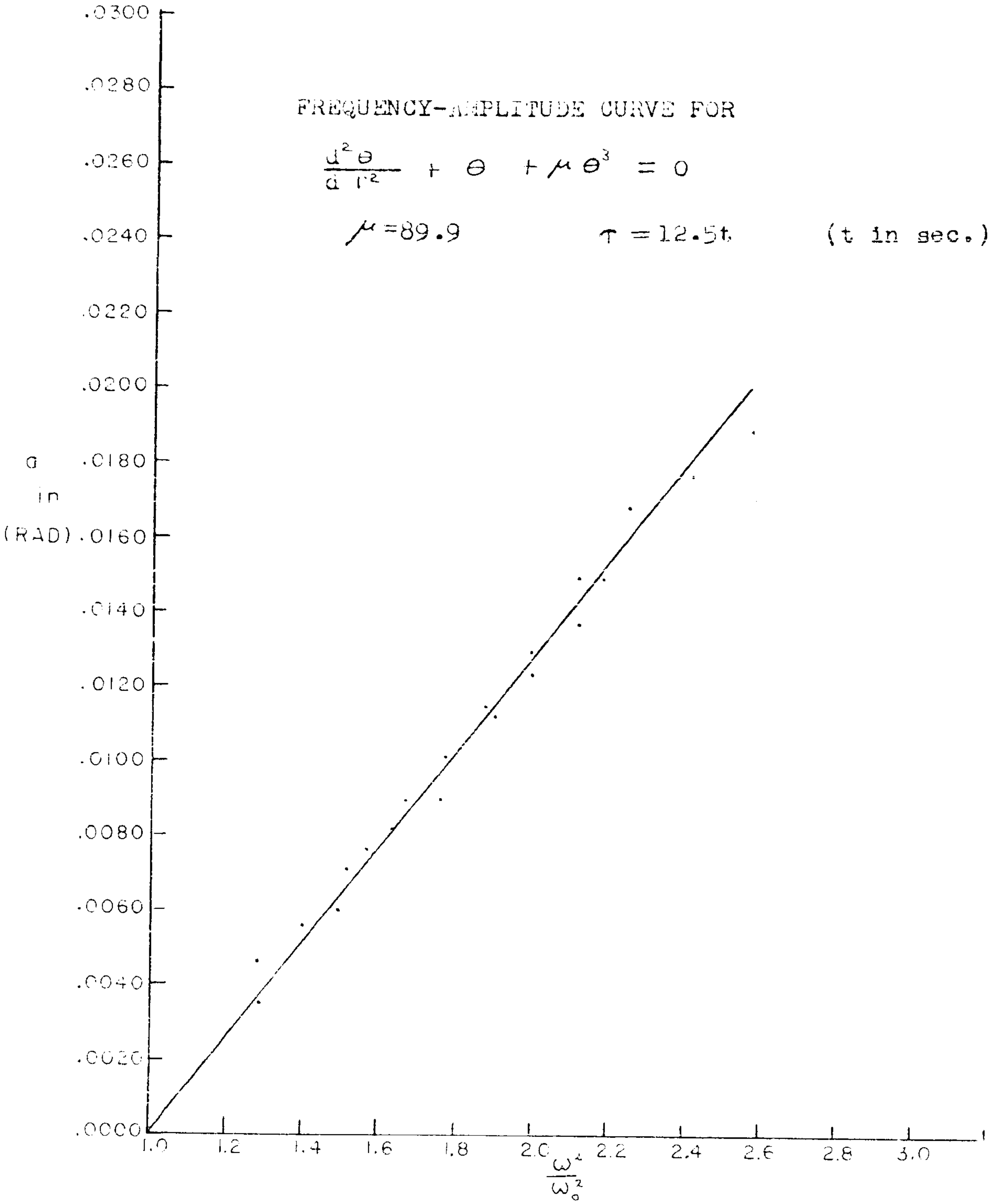

FREQUENCY-AMPLITUDE CURVE FOR
$\frac{d^2\theta}{d\tau^2} + \theta + \mu\theta^3 = 0$
μ =89.9
τ = 12.5t
(t in sec.)
a in (RAD)
.0300
.0280
.0260
.0240
.0220
.0200
.0180
.0160
.0140
.0120
.0100
.0080
.0060
.0040
.0020
.0000
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
$\frac{\omega^2}{\omega_0^2}$

figure no. 16

d. Consider[1]

$$\frac{d^2y}{dt^2} - \frac{1}{4}\, y(1 - y^7) = 0 \qquad (5.43)$$

The Lane-Emden equation

$$\frac{d^2\Theta}{d\xi^2} + \frac{2}{\xi}\frac{d\Theta}{d\xi} + \Theta^{\mu} = 0 \qquad (5.44)$$

arises from a study of gravitational equilibrium of a gaseous configuration in stellar space. The substitution $\frac{d}{d\xi} = -x^2 \frac{d}{dx}$ transforms the equation to

$$x^4 \frac{d^2\Theta}{dx^2} + \Theta^{\mu} = 0 \qquad (5.45)$$

If $\mu = 5$, the equation is integrable. Substituting $\Theta = (\frac{1}{4})^{\frac{1}{4}} x^{\frac{1}{2}} y$ into (5.45) gives

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{1}{4}\, y(1 - y^4) = 0 \qquad (5.46)$$

With $x = e^t$, (5.46) becomes

$$\frac{d^2y}{dt^2} - \frac{1}{4}\, y(1 - y^4) = 0 \qquad (5.47)$$

If we write (5.47) as

$$v\frac{dv}{dy} = \frac{1}{4}\, y(1 - y^4) \qquad (5.48)$$

we can integrate and get

$$v^2 = \frac{1}{4}\, y^2(1 - \frac{1}{3}\, y^4) + 2A \qquad (5.49)$$

If we use the initial conditions $y = v = 0$, which are

[1] McLachlan, N. W. Ordinary Non-linear Differential Equations in Engineering and Physical Sciences. Oxford: Clarendon Press, 1950. p. 23.

useful in the Lane-Emden equation

$$v = \frac{dy}{dt} = \pm \frac{1}{2} y(1 - \frac{1}{3} y^4)^{\frac{1}{2}} \qquad (5.50)$$

If we choose the negative sign in (5.50), then

$$t = -2 \int \frac{dy}{y(1 - \frac{1}{3} y^4)^{\frac{1}{2}}} + B' \qquad (5.51)$$

Making the substitution $\sin^2 w = \frac{1}{3} y^4$

$$t = - \int \frac{dw}{\sin w} + B' = -\ln \tan \frac{1}{2} w + B' \qquad (5.52)$$

or

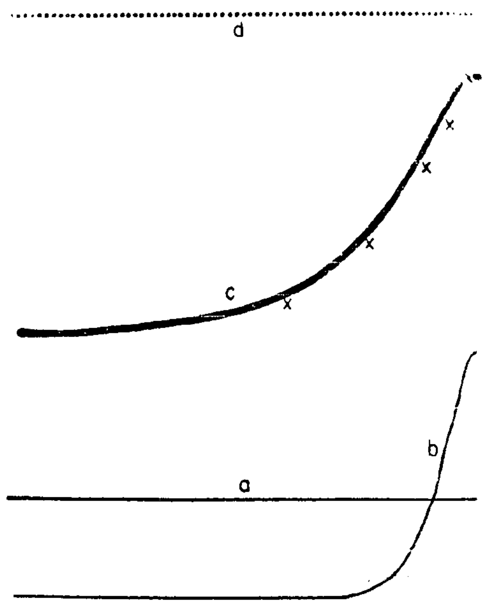
$$\tan \frac{1}{2} w = B e^{-\mu} \qquad (5.53)$$

Since

$$\sin^2 w = \frac{4 \tan^2 \frac{1}{2} w}{(1 + \tan^2 \frac{1}{2} w)^2} = \frac{1}{3} y^4 \qquad (5.54)$$

$$y = \frac{12B^2 e^{-2t}}{(1 + B^2 e^{-2t})} \qquad (5.55)$$

The record shown in photograph no. 14 shows the record of displacement and current. The record was made by releasing the moving coil at O and allowing it to move away from its point of unstable equilibrium at y O. Line "a" is the line of zero current. Curve "b" is a graph of the current which flowed during the record. Curve "c" is the displacement curve. The line of dots "d" indicates timing. Curve "c" is the record as given by the machine, and the crosses are points calculated from equation (5.55)



photograph no. 14

The restoring force is represented in the apparatus by setting the amplifier in such a way that a constant current is directed so that the oscillating coil is driven away from its center position and then a mask with $f(y) \cong y^3$ is used to generate a current in the opposite direction.

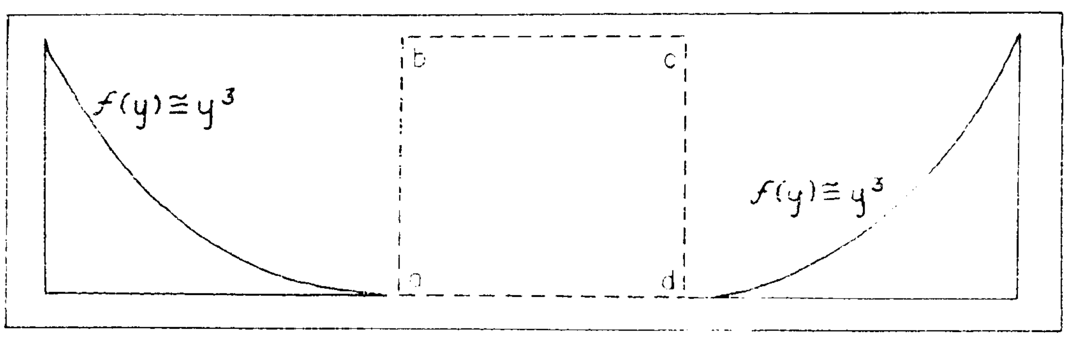


figure no. 17

Below is a figure showing the currents in the oscillating coil.

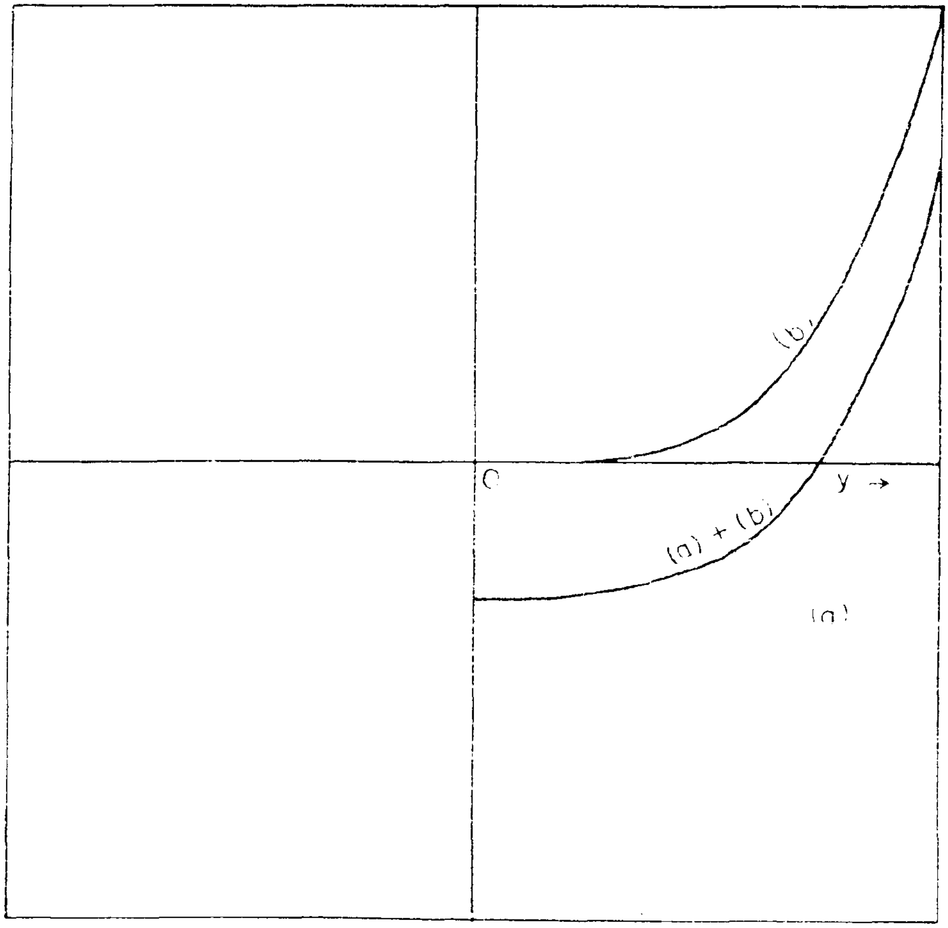


figure no. 18

When "a" is added to "b" a current proportional to $k_1 - k_2 y^4$ is the result. When this is multiplied by y, we have $y(k_1 - k_4 y^4)$ as the spring force.

(e) Let us consider the equation

$$\ddot{x} + k_1 x + \begin{Bmatrix} k_3' x^3 \\ k_3 x^3 \end{Bmatrix} = 0 \qquad \begin{matrix} x > 0 \\ x < 0 \end{matrix} \qquad (5.56)$$

That is, when x is positive $\ddot{x} + k_1 x + k_3' x^3 = 0$ is the differential equation governing the motion, and when x is negative $\ddot{x} + k_1 x + k_3 x^3 = 0$ is the differential equation governing the motion. This can be achieved by setting the amplifier to deliver a constant current plus the current generated by using a mask with two triangular openings but with different slopes as shown in figure no. 19.

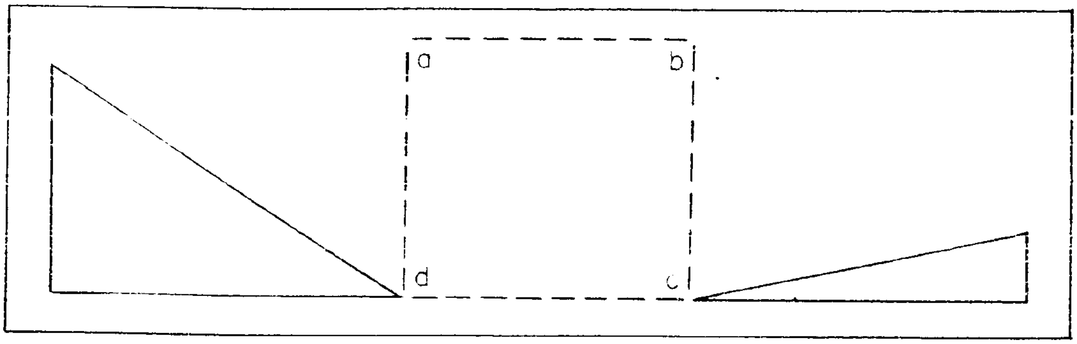
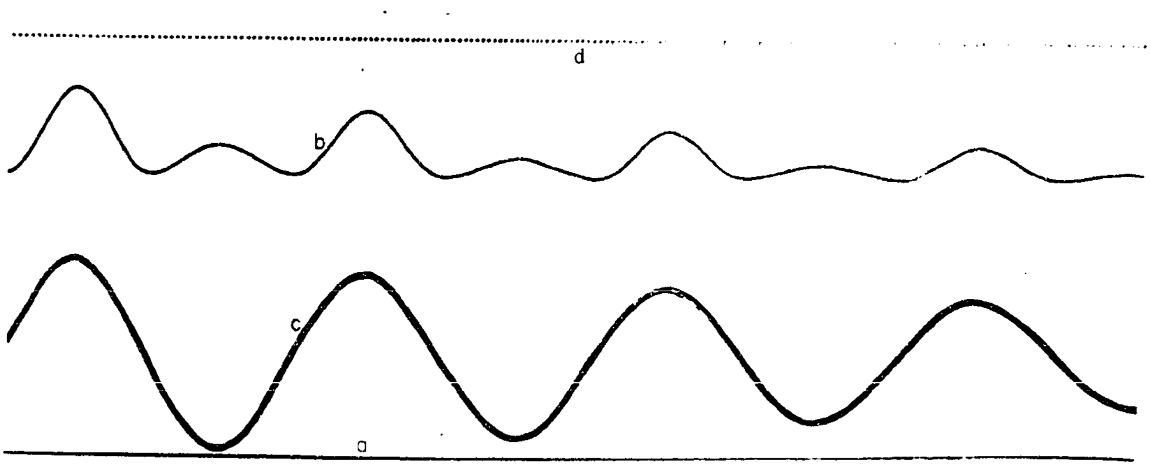


figure no. 19

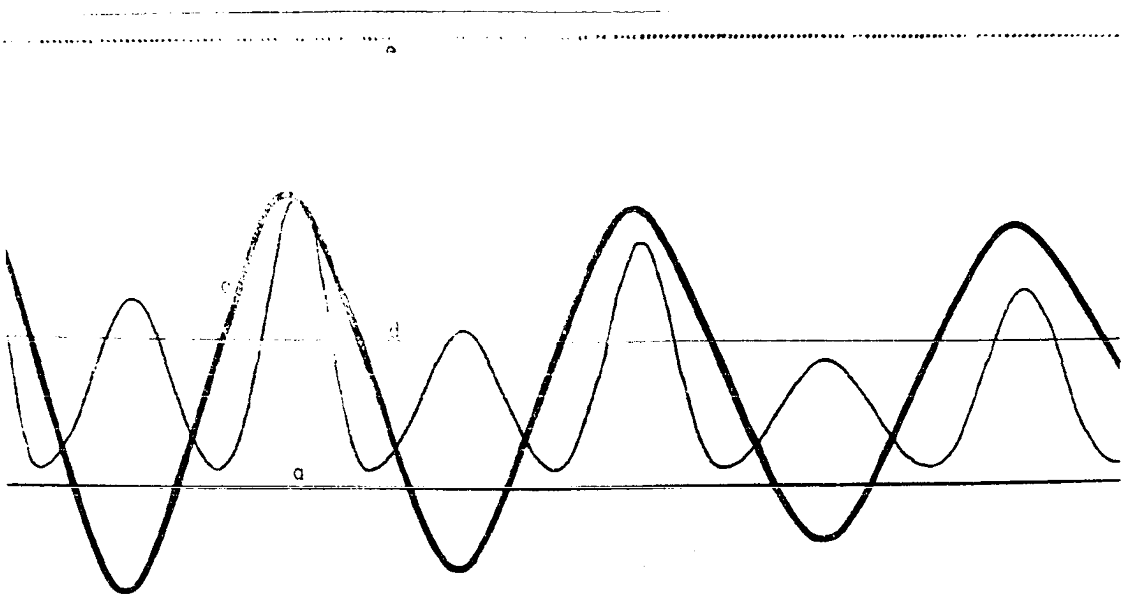
In the example shown in photograph no. 15 the value of $k_1 = 15.5$, $k_3' = 735$, and $k_3 = 275$. In the photograph "a" is the zero current line; "b" is the recording of the galvanometer while the record was being run; "c" is the displacement curve; and "d" is the line of timing dots.



photograph no. 15

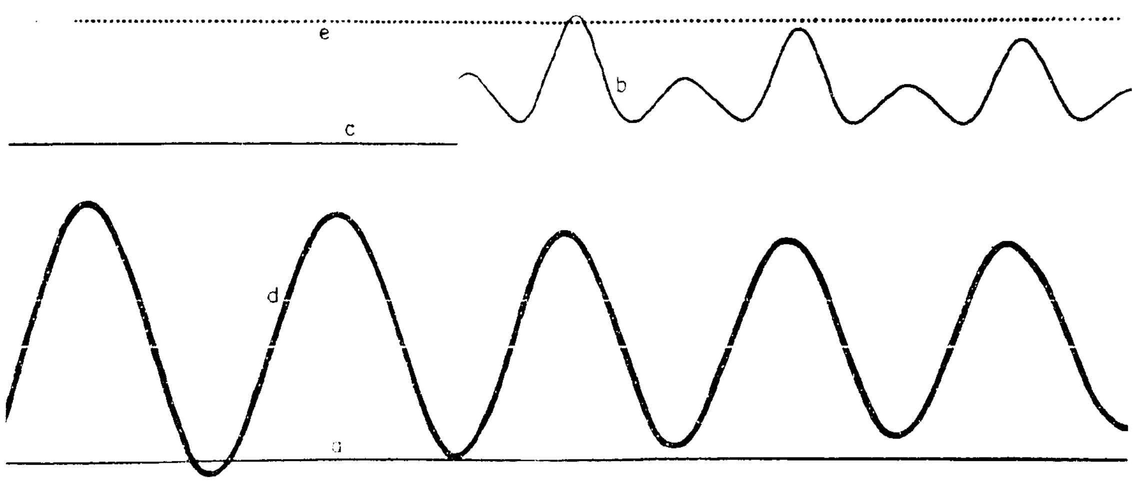
In photograph no. 16 is a similar example. Here the additional line of zero displacement, "d", is shown to emphasize the fact that the displacement curve is not symmetrical. "a" is zero current line; "b" is the record of current while the record was being run; "c" is the displacement curve; and "e" is a line of timing dots. The equation is

$$\ddot{x} + 1.14x + \begin{Bmatrix} 1581x^3 \\ 174x^3 \end{Bmatrix} = 0 \qquad \begin{matrix} x > 0 \\ x < 0 \end{matrix} \qquad (5.57)$$



photograph no. 16

Photograph no. 17 illustrates that a ripple on a spring force does not greatly change the displacement curves. "a" is the record of zero current in the galvanometer; "b" is the record of ripple current on constant current; "c" is the record of constant current; and "e" is timing dots.



photograph no. 17

(f) If the shutter is arranged as shown in figure no. 20, a spring force such as shown in figure no. 21 is the result.

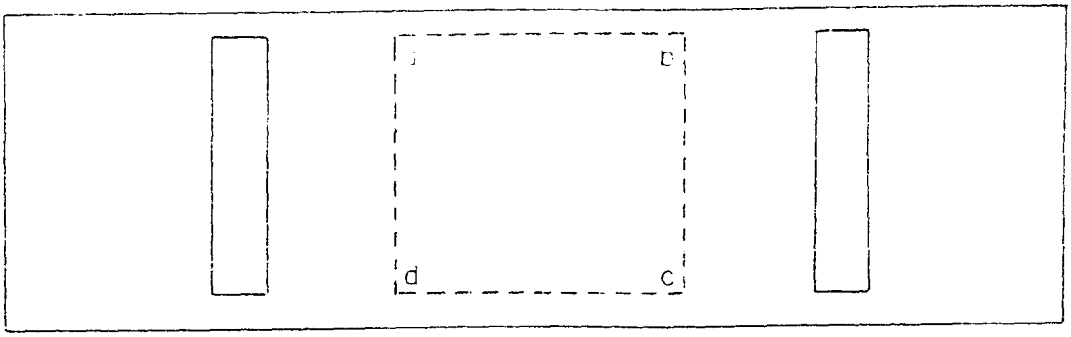


figure no. 20

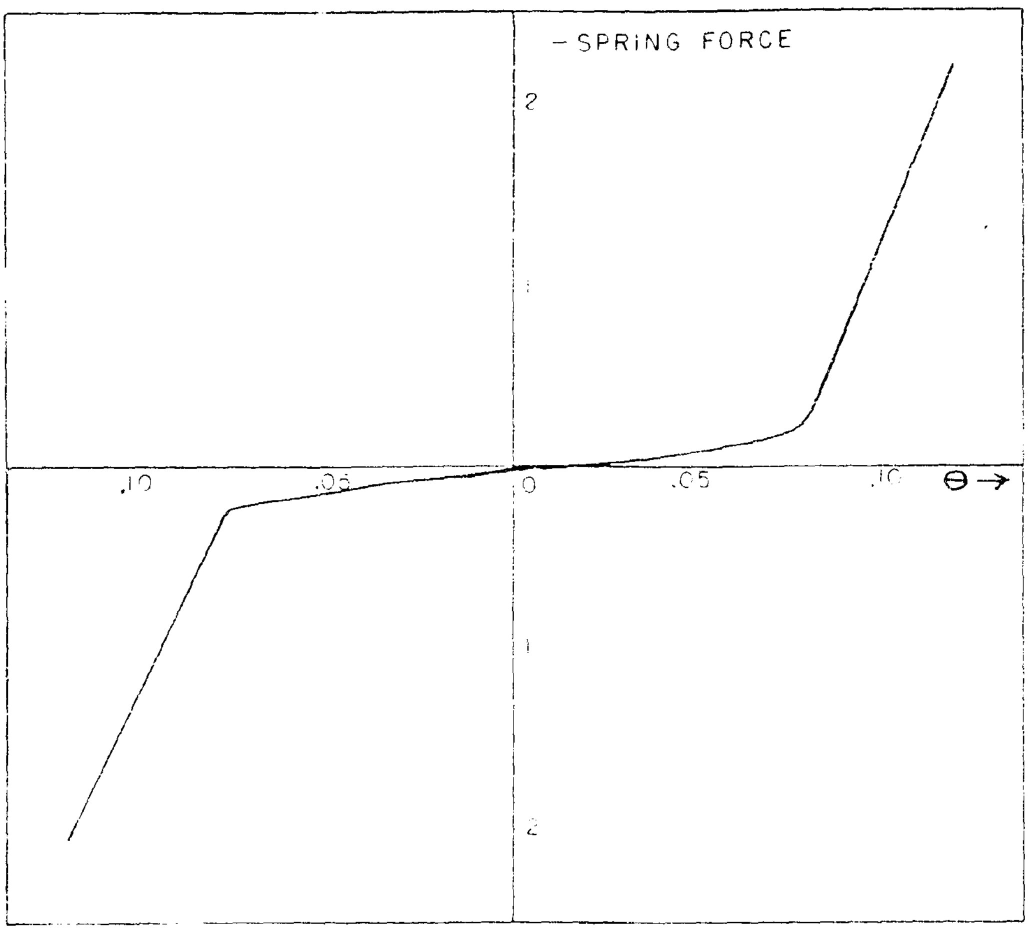
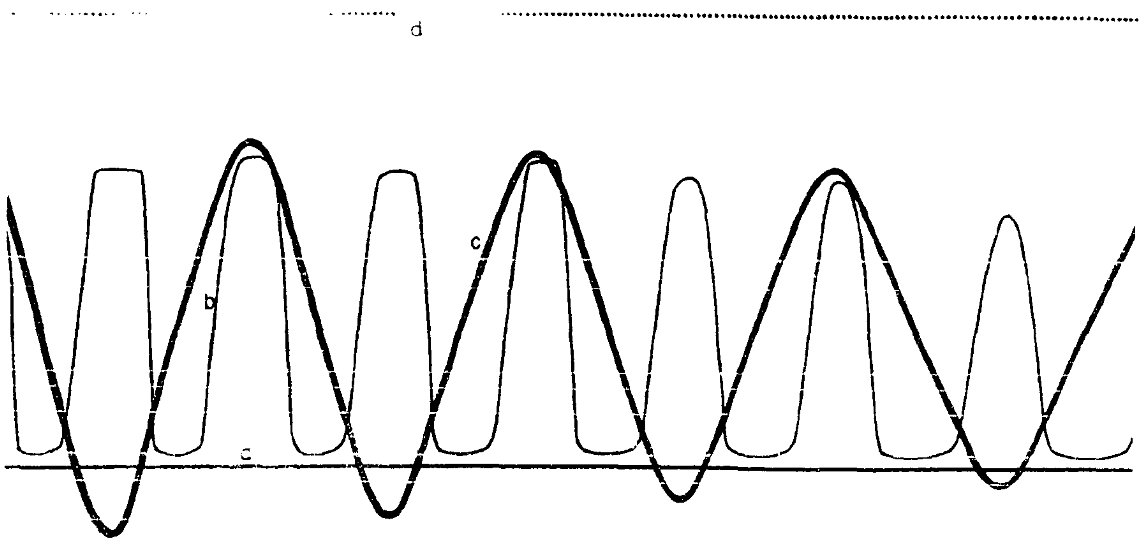


figure no. 21

The resulting record is shown in photograph no. 18 below. "a" is the record made with zero current through the galvanometer; "b" is the record of the current; "c" is the record of displacement; and "d" is a line of timing dots.



photograph no. 18

(g) A shutter as shown below in figure no. 22 gives a spring force as shown in figure no. 23.

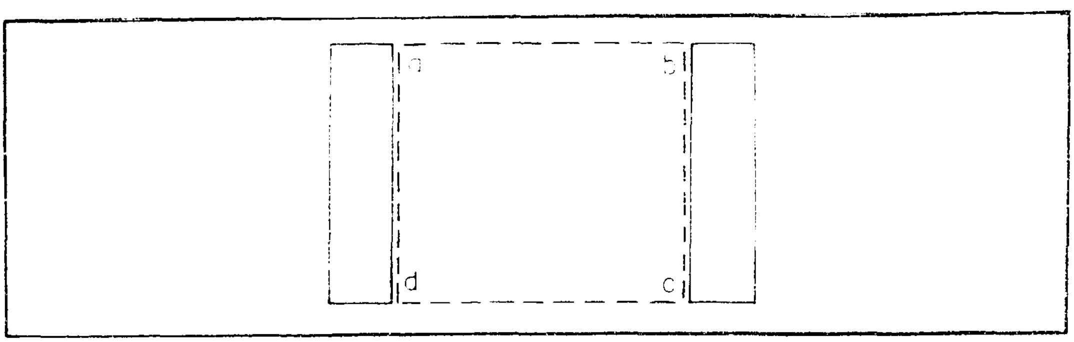


figure no. 22

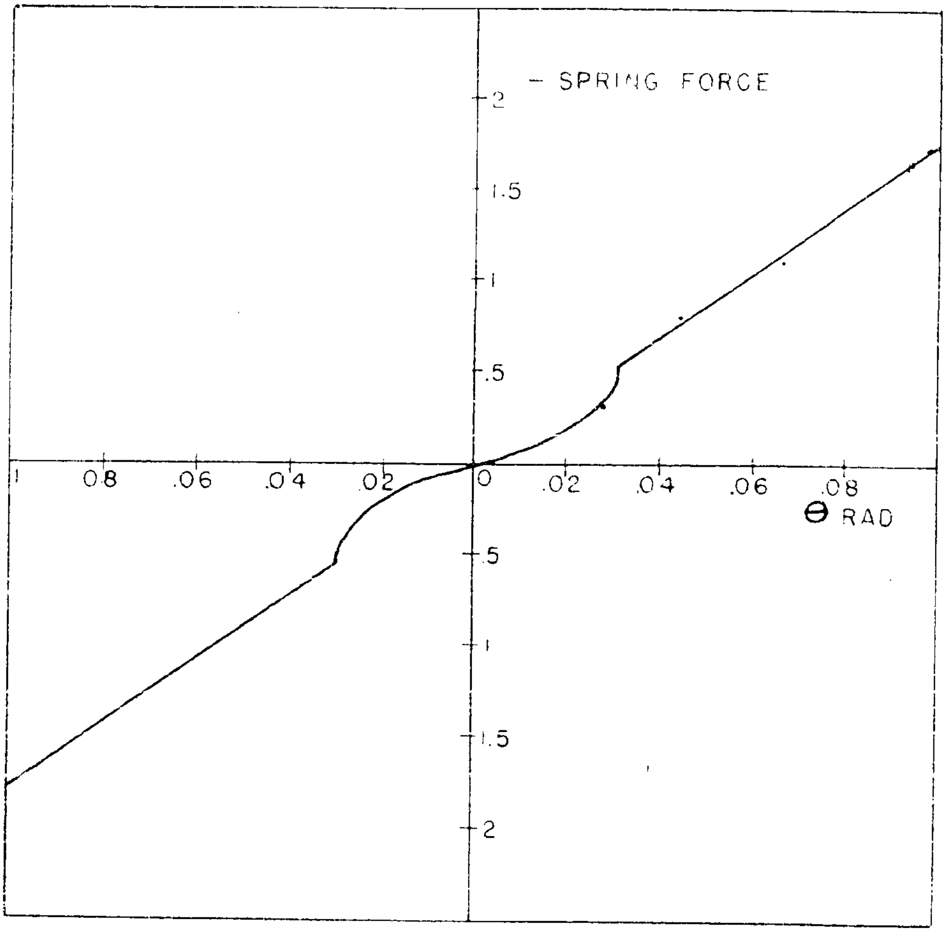
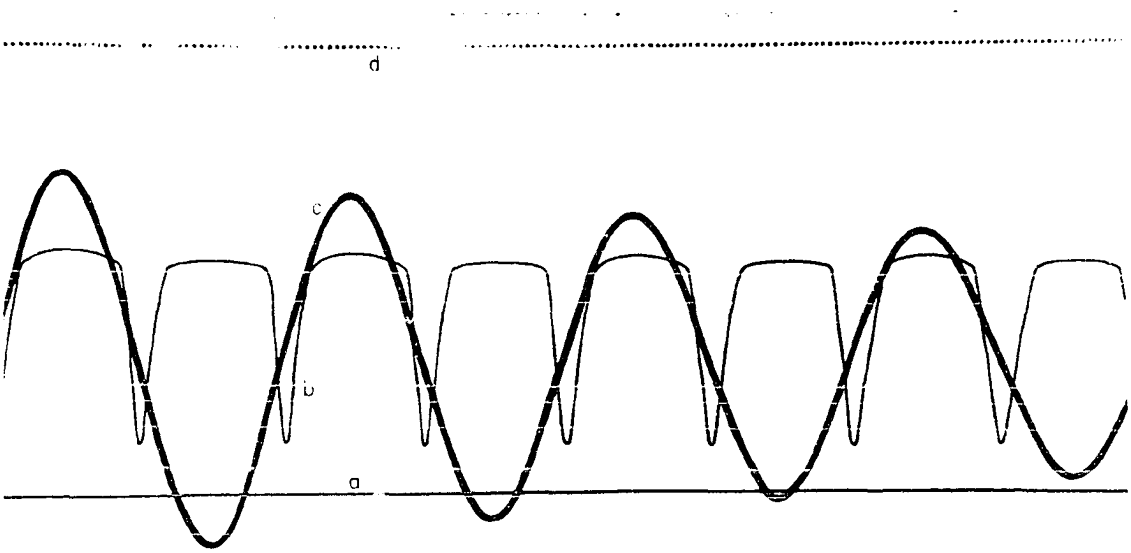


figure no. 23

Photograph no. 19 shows a record with this type shutter. "a" is the record of zero current in the galvanometer; "b" is the record of the current; "c" is the displacement record; and "d" is a line of timing dots.



photograph no. 19

ACKNOWLEDGEMENT

I appreciate the valuable help given to me by various members of the Physics Department and the student body. In particular, I wish to thank Dr. Carl A. Ludeke for his suggestion of the apparatus and his continued interest and help. Much valuable assistance was given by Mr. Allen Chase in the construction of the apparatus.

The funds for the purchase of material were supplied by the Frederick Cottrell Grant in Aid to Dr. Carl A. Ludeke from the Research Corporation, New York.