A SET OF NECESSARY AND SUFFICIENT CONDITIONS FOR THE CESÁRO SUMMABILITY OF DOUBLE SERIES, WITH APPLICATION TO THE DOUBLE FOURIER'S SERIES.

by GAYLORD M. MERRIMAN

A Thesis submitted in partial satisfaction of the requirements for the degree of Ph.D.
May, 1926.

A SET OF NECESSARY AND SUFFICIENT CONDITIONS FOR THE CESÁRO SUMMABILITY OF DOUBLE SERIES, WITH APPLICATIONS TO THE DOUBLE FOURIER'S SERIES.*

by GAYLORD M. MERRIMAN

The wide applicability of double Fourier's series to boundary value problems has justified, and even demanded, a close study of the summability, especially that with respect to a pair of Cesáro means, of the series involved.** In view of the importance of this question, it is odd that it has not heretofore been completely solved; that is to say, that conditions of summability (C,r,s) previously obtained have been either necessary or sufficient, but never both.***

The present paper, however, solves this problem, provided a slight modification is made in the question asked. We shall seek here, then, not a necessary and sufficient condition for the Cesaro summability of a double Fourier's series, with respect to two specified means (i.e. summability (Cr,s)),but rather one that the series should be summable with respect to <u>some</u> <u>pair</u> <u>or</u> <u>other</u> of Cesáre means (i.e. summable (C)). Justification for this modification is apparent in that the new problem admits a quite simple solution. In order to derive the result just outlined, it

---

*Presented to the American Mathematical Society, Chicago, April, 1926.

**The preference to be given the process of summability over that of convergence has been pointed out before. Cf.,for example, a paper by Professor C.N.Moore, "Applications of the Theory of Summability to Developments in Orthogonal Functions", Bulletin of the American Mathematical Society, vol 25, (1910), pp, 258-276.

***The same hiatus still exists for the case of convergence, even of a simple Fourier's series. Cf. a paper by Hardy and Littlewood, "Solution of the Cesare Summability Problem for Power Series and Fourier's Series", Mathematische Zeitschrift, December, 1923.

is first necessary to prove a general theorem relative to the summability (C,r,s) of an ordinary double series; this theorem is then applied to the double Fourier's series, with the desired result. The deductions are in all cases generalizations to two variables of those concerning simple series obtained by Hardy and Littlewood in the memoir cited above.

I THE GENERAL PROBLEM

§1. DEFINITIONS AND NOTATION. We have for consideration a double series

(1)
$$A \equiv \sum_0^\infty \sum_0^\infty a_{ij} \; ,$$

where A shall stand for either the series or its sum; by A_c and A_r we shall mean the simple series represented by any column or row of (1), or the sum of such series. We write also, for convenience,

(2)
$$\begin{cases} A_{mn} \equiv A^{00}_{mn} = \sum_0^m \sum_0^n a_{ij} \; , \\ A^{01}_{mn} \equiv \sum_0^n A^{00}_{mj} \; , \qquad A^{10}_{mn} \equiv \sum_0^m A^{00}_{in} \; , \\ A^{11}_{mn} \equiv \sum_0^m \sum_0^n A^{00}_{mn} \\ \text{etc.} \end{cases}$$

The series (1) is said to be summable (C,r,s) to sum A when

$$A^{rs}_{mn} \sim C^{rs}_{mn} A \, ,$$

where

$$C^{rs}_{mn} \equiv \frac{(m+r)(m+r-1)\ldots\ldots(m+1)\,(n+s)(n+s-1)\ldots\ldots(n+1)}{r!\; s!} \, .$$

In these circumstances, we say that $A_{mn} \to A$ (C,r,s). Analogously, we shall use the notations

$$A_{hn} \to A_r \quad (C,s) \qquad (h \text{ fixed})$$

and

$$A_{mk} \to A_c \quad (C,r) \qquad (k \text{ fixed})$$

to indicate Cesaro summability of rows and columns of (1), considered as simple series. When r=s, we shall use the notation

(5) $$\begin{cases} a^{0}_{mk} = a_{mk} & \underline{\text{(k fixed)}} \\ a^{\rho-1}_{mk} = (m+1)\left\{ a^{\rho}_{mk} - a^{\rho}_{m+1\,k} \right\} \end{cases}$$

(6) $$\begin{cases} a^{0}_{hn} = a_{hn} & \underline{\text{(h fixed)}} \\ a^{\rho-1}_{hn} = (n+1)\left\{ a^{\rho}_{hn} - a^{\rho}_{h\,n+1} \right\} \end{cases}$$

<u>such that</u>

(7) $$A^{r+1\,r+1} \equiv \sum_0^\infty \sum_0^\infty a^{r+1\,r+1}_{mn}$$

(8) $$A^{r+1}_{k} \equiv \sum_0^\infty a^{r+1}_{mk}$$

(9) $$A^{r+1}_{h} \equiv \sum_0^\infty a^{r+1}_{hn}$$

<u>are summable (C,-1).</u>

<u>In these circumstances, we have</u>

(10) $$a^{\rho\,\rho}_{mn} = \sum_m^\infty \sum_n^\infty \frac{a^{\rho-1\,\rho-1}_{\mu\nu}}{(\mu+1)(\nu+1)}, \qquad (C, r-\rho)$$

(11) $$a^{\rho}_{mk} = \sum_m^\infty \frac{a^{\rho-1}_{\mu k}}{\mu+1}, \qquad (C, r-\rho)$$

(12) $$a^{\rho}_{hn} = \sum_n^\infty \frac{a^{\rho-1}_{h\nu}}{\nu+1}, \qquad (C, r-\rho)$$

<u>for $\rho = 1,2--- r-1$. The series $A^{\rho\rho}$ is summable $(C,r-\rho)$, and to the same value as A; while A^{ρ}_{k} and A^{ρ}_{h} are summable $(C,r-\rho)$ to the same values as A_r and A_c respectively, for $\rho = 1,2,--- r-1$.</u>

To prove this theorem, we shall need a series of lemmas.

§3. THE AUXILIARY SERIES B, P AND Q. We first introduce three auxiliary double series, B, P and Q, connected with A through the following definitions:

(13) $$p_{mn} = (m+1)\left(b_{mn} - b_{m+1\,n} \right).$$

(13') $$a_{mn} = (n+1)\left(p_{mn} - p_{m\,n+1} \right)$$

(14) $$q_{mn} = (n+1)\left(b_{mn} - b_{m\,n+1} \right)$$

(14') $$a_{mn} = (m+1)\left(q_{mn} - q_{m+1\,n} \right)$$

(15) $$a_{mn} = (m+1)(n+1)\left(b_{mn} - b_{m+1\,n} - b_{m\,n+1} + b_{m+1\,n+1} \right).$$

From the fact that (15) may be obtained through combination either of (13) and (13') or (14) and (14'), we see that these various definitions are consistent.

Lemma 1 : If (i) A_c and A_r are summable (C,r), then P_c and Q_r are summable (C,r-1) to the sums A_c and A_r respectively. If (ii) P_c and Q_r are summable (C,r), then B_c and B_r are summable (C,r-1) to the sums P_c and Q_r respectively.

To prove (i), we have that, in view of (13'), m fixed, and (14'), n fixed, we can, through the medium of Lemma 5 of the Hardy-Littlewood paper, choose the p's and q's so that

(16) $$p_{hn} = \sum_{n}^{\infty} \frac{a_{hv}}{v+1} \qquad (C, r-1)$$

(16') $$q_{mk} = \sum_{m}^{\infty} \frac{a_{\mu k}}{\mu+1} \qquad (C, r-1).$$

On account of this choice, we can further infer from the same source that P_c sums to A_c , and Q_r to A_r. A similar proof holds for (ii), if we make use of (13) and (14).

In particular, if A, A_c and A_r are summable (C,r) to sum zero,* then P_c and Q_r are summable (C,r-1) to zero. We shall take these sums as zero throughout the rest of the paper.

Lemma 2 : If A, A_r and A_c are summable (C,r) to zero, then

(17) $$P_{m\ n+1}^{r\ r-1} = C_m^r\ o(n^{r-1})$$

(17') $$Q_{m+1\ n}^{r-1\ r} = C_n^r\ o(m^{r-1}).$$

*That we can take these sums all zero without loss of generality follows thus: In our given series (1), we can first make the sum of each row zero by altering the elements of the first column to $\bar{a}_{on}$; then the sum of each column can be made zero by changing the elements of the first row to $\bar{a}_{mo}$, and $\bar{a}_{oo}$ to $\bar{\bar{a}}_{oo}$; although this last change might have affected the summability to zero of the first row, we infer that its sum has to be zero because that of the whole series, and those of all the other rows, are zero.

In view of our definition (13'), we can use the process of Lemma 3 of the Hardy-Littlewood paper to show that, for the m^{th} column of (1), we have

(18) $$P^{-1\ r+1}_{m\ n+1} = h_1 C^r_{n+1} + o(n^{r-1}),$$

where $h_1 = (r+1)!\,\psi_1$, ψ_1 being defined as in (2.23) of their paper. But, under the choice of p_{mn}, m fixed, made in Lemma 1, we can, as in Lemma 5 of their paper, show that $h_1 = 0$. Hence, after summation, (18) becomes (17). (17') is obtained in similar manner.

<u>Lemma 3 : If equations (13) - (15) are satisfied, then</u>

(19) $P^{r\ r}_{m\ n} = (r+1)\,B^{r\ r}_{m\ n} - (m+1)\,B^{r-1\ r}_{m+1\ n}$,

(19') $A^{r\ r}_{m\ n} = (r+1)\,P^{r\ r}_{m\ n} - (n+1)\,P^{r\ r-1}_{m\ n+1}$,

(20) $Q^{r\ r}_{m\ n} = (r+1)\,B^{r\ r}_{m\ n} - (n+1)\,B^{r\ r-1}_{m\ n+1}$,

(20') $A^{r\ r}_{m\ n} = (r+1)\,Q^{r\ r}_{m\ n} - (m+1)\,Q^{r-1\ r}_{m+1\ n}$,

(21) $$A^{r\ r}_{m\ n} = (r+1)^2\,B^{r\ r}_{m\ n} + (m+1)(n+1)\,B^{r-1\ r-1}_{m+1\ n+1} - (r+1)\left[(m+1)\,B^{r-1\ r}_{m+1\ n} + (n+1)\,B^{r\ r-1}_{m\ n+1}\right].$$

<u>If r = 0, then</u>

$B^{-1\ -1}_{m+1\ n+1}$	<u>is to be read</u>	$b_{m+1\ n+1}$,
$B^{-1\ 0}_{m+1\ n}$	"	$\sum_{j=0}^{\infty} b_{m+1\ j}$,
$B^{0\ -1}_{m\ n+1}$	"	$\sum_{i=0}^{\infty} b_{i\ n+1}$,

<u>with similar interpretations for the corresponding expressions in terms of P and Q.</u>

We shall indicate only the proofs of (20) and (21); the other results of the lemma will follow in the same way as (20).

In accordance with (14), we have, for the m^{th} column of Q,

(22) $$Q^{-1\ r}_{m\ n} = (r+1)\,B^{-1\ r}_{m\ n} - (n+1)\,B^{-1\ r-1}_{m\ n+1}$$

from (2.12) of the Hardy- Littlewood paper. Hence we obtain (20) by summation; and the other formulae follow analogously.

To prove (21), we first note that we can also obtain by summation of (22),

$$(23)\qquad Q^{r-1\ r}_{m+1\ n} = (r+1)\, B^{r-1\ r}_{m+1\ n} - (n+1)\, B^{r-1\ r-1}_{m+1\ n+1}.$$

Substitution of (20) and (23) in (20') leads to (21).

It should be noted that the above proofs do not depend on the use of the same indices of summability. They will be used later, inconnection with Theorem II, for the case $r \neq s$, in which case their form is substantially the same.

§4. FUNDAMENTAL SUFFICIENCY LEMMA. Lemma 4 : If B is summable (C,r-1) and if each column of P and each row of Q is summable (C,r-1), then A is summable (C,r) to B, and A_c and A_r are summable (C,r) to P_c and Q_r respectively.

We first note that, in view of (13') and (14'), and the results of the Hardy-Littlewood paper, Lemma 2, we caninfer that the summability (C,r-1) of P_c and Q_r imply the summability (C,r) of A_c and A_r respectively, the sums of the corresponding series being the same. Therefore, to complete the proof of Lemma 4, we have only to concern ourselves with the proof of the results regarding the double series A and B.

If $r=0$, (21) becomes

$$(24)\qquad A_{mn} = B_{mn} - (n+1)\sum_0^m b_{i\,n+1} - (m+1)\sum_0^n b_{m+1\,j} + (m+1)(n+1)\, b_{m+1\,n+1},$$

and the result follows at once, since the last three terms tend to zero as m and n become infinite.

If $r>0$, we have the following reduction: by hypothesis,

$$(25)\qquad B^{r-1\ r-1}_{m+1\ n+1} = C^{r-1\ r-1}_{m+1\ n+1}\, B + o(m^{r-1} n^{r-1}).$$

By summing (25) once with respect to both indices, and to each index separately, we obtain

$$(26)\quad \begin{cases} B^{r\ r}_{m\ n} = C^{r\ r}_{m n} B + o(m^r n^r), \\ B^{r-1\ r}_{m+1\ n} = C^{r-1\ r}_{m+1\ n} B + o(m^{r-1} n^r), \\ B^{r\ r-1}_{m\ n+1} = C^{r\ r-1}_{m\ n+1} B + o(m^r n^{r-1}). \end{cases}$$

Substitution of (25) and (26) in (21) gives

$$A^{r r}_{m n} = \left\{ (r+1)^2 C^{r r}_{m n} - (r+1)\left[(m+1) C^{r-1\ r}_{m+1\ n} + (n+1) C^{r\ r-1}_{m\ n+1}\right] + (m+1)(n+1) C^{r-1\ r-1}_{m+1\ n+1} \right\} B + o(m^r n^r),$$

or

$$A^{r r}_{m n} = C^{r r}_{m n} B + o(m^r n^r);$$

and the conclusion follows at once.

§5. FUNDAMENTAL NECESSITY LEMMAS. <u>Lemma 5-: If A, A_r and A_c are summable (C,r), all to sum zero, then</u>

$$(27)\quad B^{r-1\ r-1}_{m\ n} = h\, C^{r\ r}_{m n} + o(m^{r-1} n^{r-1}) + o(m^{r-1} n^r) + o(m^r n^{r-1}),$$

<u>where h is a constant. If r = 0, (27) is to be interpreted as meaning</u>

$$(28)\quad b_{m+1,\ n+1} = h + o\left(\frac{1}{mn}\right) + o\left(\frac{1}{m}\right) + o\left(\frac{1}{n}\right).$$

We prove the lemma first when r = 0. In this case, (21) becomes

$$(29)\quad \begin{aligned} A^{0 0}_{m n} &= o(1) \\ &= (m+2)(n+2) B^{0 0}_{m n} - (m+2)(n+1) B^{0\ 0}_{m\ n+1} \\ &\quad - (m+1)(n+2) B^{0\ 0}_{m+1\ n} + (m+1)(n+1) B^{0\ 0}_{m+1\ n+1} \\ &= B^{0\,0}_{m n} - (m+1) B^{-1\ 0}_{m+1\ n} - (n+1) B^{0\ -1}_{m\ n+1} \\ &\quad + (m+1)(n+1)\, b_{m+1,\ n+1}. \end{aligned}$$

Hence,

$$(30)\quad \begin{aligned} \frac{A^{0 0}_{m n}}{(m+2)(m+1)(n+2)(n+1)} &= o\left(\frac{1}{m^2 n^2}\right) \\ &= \frac{B^{0 0}_{m n}}{(m+1)(n+1)} - \frac{B^{0\ 0}_{m\ n+1}}{(m+1)(n+2)} - \frac{B^{0\ 0}_{m+1\ n}}{(m+2)(n+1)} + \frac{B^{0\ 0}_{m+1\ n+1}}{(m+2)(n+2)}. \end{aligned}$$

Therefore, the double series whose general term is given by (30) is absolutely convergent to some value K_1, and hence,

$$(31)\quad \left.\begin{aligned} & B^{00}_{00} - \frac{B^{00}_{0n}}{n+1} - \frac{B^{00}_{m0}}{m+1} + \frac{B^{00}_{mn}}{(m+1)(n+1)} \\ & = K_1 + o\left(\frac{1}{m}\right) + o\left(\frac{1}{n}\right) + o\left(\frac{1}{mn}\right). \end{aligned}\right.$$

But, from (13) and (14), and application of Lemma 3 of the Hardy-Littlewood paper, to the first column of P and the first row of Q, we can infer that

$$\frac{B^{00}_{m0}}{m+1} = g_1 + o\left(\frac{1}{m}\right) \quad , \quad \frac{B^{00}_{0n}}{n+1} = g_2 + o\left(\frac{1}{n}\right),$$

and (31) becomes

$$(32)\quad \frac{B^{00}_{mn}}{(m+1)(n+1)} = K + o\left(\frac{1}{m}\right) + o\left(\frac{1}{n}\right) + o\left(\frac{1}{mn}\right).$$

Again, from (29),

$$(33)\quad b_{m+1\,n+1} = -\frac{B^{00}_{mn}}{(m+1)(n+1)} + \frac{B^{0\,-1}_{m\,n+1}}{m+1} + \frac{B^{-1\,0}_{m+1\,n}}{n+1} + o\left(\frac{1}{mn}\right).$$

Since P_c and Q_r converge uniformly to zero by Lemma 1, we can conclude from Lemma 3 of the Hardy-Littlewood paper, that the second and third terms on the right-hand side of (33) are of the form $o(1/m)$ and $o(1/n)$ respectively. Hence by substitution of (32) in (33), we obtain (28) as desired.

We have next to prove the result in the general case, $r>0$. We rewrite (21) as

$$(34)\quad \begin{aligned} A^{rr}_{mn} &= o(m^r n^r) \\ &= (m+r+2)(n+r+2)\, B^{rr}_{mn} - (m+r+2)(n+1)\, B^{rr}_{m\,n+1} \\ &\quad - (m+1)(n+r+2)\, B^{rr}_{m+1\,n} + (m+1)(n+1)\, B^{rr}_{m+1\,n+1}. \end{aligned}$$

If we put

$$(35)\quad B^{rr}_{mn} = (m+r+1)(m+r)\cdots(m+1)(n+r+1)(n+r)\cdots(n+1)\,\varphi_{mn},$$

we obtain from (34)

$$\begin{aligned} \varphi_{mn} - \varphi_{m+1\,n} - \varphi_{m\,n+1} + \varphi_{m+1\,n+1} &= o\left(\frac{1}{m^2n^2}\right) \\ &= \frac{A^{rr}_{mn}}{(m+r+2)\cdots(m+1)(n+r+2)\cdots(n+1)}. \end{aligned}$$

Hence we deduce that

$$\varphi_{00} - \varphi_{m0} - \varphi_{0n} + \varphi_{mn}$$

tends to a limit φ', i.e.

$$\varphi_{00} - \varphi_{m0} - \varphi_{0n} + \varphi_{mn} =$$

(36) $$\varphi' - \left\{\sum_{m+1}^{\infty}\sum_{n+1}^{\infty} + \sum_{m+1}^{\infty}\sum_{0}^{n} + \sum_{0}^{m}\sum_{n+1}^{\infty}\right\} \frac{A_{\mu\nu}^{rr}}{(\mu+r+2)\cdots(\mu+1)(\nu+r+2)\cdots(\nu+1)}.$$

But we can infer from (ii) of Lemma 1, and the general analysis in Lemma 3 of the Hardy-Littlewood paper leading to (2.26), that through the substitution of (35), φ_{m0} tends to a limit φ_1, and φ_{0n} tends to a limit φ_2; in fact,

(37) $$\varphi_{m0} = \varphi_1 + \frac{1}{(r+1)!}\sum_{m}^{\infty}\frac{P_{\mu 0}^{r}}{(\mu+r+2)\cdots(\mu+1)} = \varphi_1 + o\left(\frac{1}{m}\right),$$

and

(37') $$\varphi_{0n} = \varphi_2 + \frac{1}{(r+1)!}\sum_{n}^{\infty}\frac{Q_{0\nu}^{r}}{(\nu+r+2)\cdots(\nu+1)} = \varphi_2 + o\left(\frac{1}{n}\right).$$

If we incorporate (37) and (37') in(36), we have

(38) $$\begin{aligned}\varphi_{mn} &= \varphi' - \varphi_{00} + \varphi_1 + \varphi_2 \\ &- \left\{\sum_{m+1}^{\infty}\sum_{n+1}^{\infty} + \sum_{m+1}^{\infty}\sum_{0}^{n} + \sum_{0}^{m}\sum_{n+1}^{\infty}\right\}\frac{A_{\mu\nu}^{rr}}{(\mu+r+2)\cdots(\mu+1)(\nu+r+2)\cdots(\nu+1)} \\ &+ \frac{1}{(r+1)!}\left\{\sum_{m}^{\infty}\frac{P_{\mu 0}^{r}}{(\mu+r+2)\cdots(\mu+1)} + \sum_{n}^{\infty}\frac{Q_{0\nu}^{r}}{(\nu+r+2)\cdots(\nu+1)}\right. \\ &= \varphi + o\left(\frac{1}{mn}\right) + o\left(\frac{1}{m}\right) + o\left(\frac{1}{n}\right).\end{aligned}$$

Thus (35) becomes

(39) $$\begin{aligned}B_{mn}^{rr} &= (m+r+1)\cdots(m+1)(n+r+1)\cdots(n+1)\,\varphi \\ &+ o(m^r n^r) + o(m^r n^{r-1}) + o(m^{r-1} n^r).\end{aligned}$$

We are now in position to deduce our lemma. We first put (19) and (20) in (21) to obtain

(40) $$A_{mn}^{rr} = o(m^r n^r) = -(r+1)^2 B_{mn}^{rr} + (m+1)(n+1)B_{m+1\,n+1}^{r-1\,r-1} + (r+1)\left[P_{mn}^{rr} + Q_{mn}^{rr}\right].$$

We next put (19') and (20') in (40) to get

(41)

$$A_{mn}^{rr} = o(m^r n^r) = (r+1)^2 B_{mn}^{rr} - (n+1) P_{m\ n+1}^{r\ r-1}$$

(41) $$-(m+1) Q_{m+1\ n}^{r-1\ r} - (m+1)(n+1) B_{m+1\ n+1}^{r-1\ r-1},$$

Then, substituting (17) and (17') in (41), we obtain

$$A_{mn}^{rr} = o(m^r n^r) = (r+1)^2 B_{mn}^{rr} - (m+1)(n+1) B_{m+1\ n+1}^{r-1\ r-1} + C_m^r\, o(n^r) + C_n^r\, o(m^r),$$

or

(42) $$(m+1)(n+1) B_{m+1\ n+1}^{r-1\ r-1} = (r+1)^2 B_{mn}^{rr} + o(m^r n^r).$$

Finally, if we put (391 into (42), and divide by $(m+1)(n+1)$, we have

$$B_{m+1\ n+1}^{r-1\ r-1} = h\, C_{m+1\ n+1}^{r\ r} + o(m^{r-1} n^{r-1}) + o(m^{r-1} n^r) + o(m^r n^{r-1})$$

which is the desired result (27), with $m = m+1$, $n = n+1$, and $h = [(r+1)!]^2 \varphi$.

The lemma is thus complete.

Lemma 6 : If A, A_c and A_r are summable (C,r) to sum zero, then (i) there is a solution of (13') and (14') such that P_c and Q_r are summable (C,r-1) to A_c and A_r respectively; and (ii) there is a solution of (13), (14) and (15), such that B is summable (C,r-1), to A.

Part (i) of the lemma follows immediately through an application of Lemma 4 of the Hardy-Littlewood paper to the simple series involved.

To prove (ii), we argue that if b_{mn} is any solution of (13), (14) and (15), all others are of the form

$$b_{mn}^* = b_{mn} - h^*$$

where h* is a constant. If, in particular, h* = h, we have

$$B_{mn}^{*\,r-1\,r-1} = B_{mn}^{r-1\,r-1} - h\, C_{mn}^{rr}$$
$$= o(m^{r-1} n^{r-1}) + o(m^r n^{r-1}) + o(m^{r-1} n^r)$$

by Lemma 5, and hence B* is summable (C,r-1) to sum zero.

§6. FORMS OF THE SOLUTIONS OF (13)-(15). We now wish to determine the forms of the solutions mentioned in Lemma 6 as being necessary for the summability (C,r) of A, A_c and A_r.

Lemma 7 -: If A, A_c and A_r are summable (C,r) to sum zero, then

(i)

(43) $$A'' = \sum_0^\infty \sum_0^\infty \frac{a_{\mu\nu}}{(\mu+1)(\nu+1)}$$

is summable (C,r-1), where

(44) $$A''_{\ } = [(r+1)!]^2 \sum_0^\infty \sum_0^\infty \frac{A_{\mu\nu}^{rr}}{(\mu+r+2)\dots(\mu+1)(\nu+r+2)\dots(\nu+1)};$$

and if

(45) $$b_{mn} = \sum_m^\infty \sum_n^\infty \frac{a_{\mu\nu}}{(\mu+1)(\nu+1)} \qquad (C, r-1),$$

then B is summable (C,r-1) to sum A.

(ii) Also

(46) $$A_h^1 = \sum_{\nu=0}^\infty \frac{a_{h\nu}}{\nu+1}$$

is summable (C,r-1), where

(47) $$A_h' = (r+1)! \sum_0^\infty \frac{A_{h\nu}^{-1\,r}}{(\nu+r+2)\dots(\nu+1)};$$

and if

(48) $$p_{hn} = \sum_n^\infty \frac{a_{h\nu}}{\nu+1} \qquad (C, r-1),$$

then P_c is summable (C,r-1) to sum A_c.

(iii) Similar results to those in (ii) follow for A_k' and Q_r, if we replace A_h' by A_k', and P_c by Q_r, and n by m.

Parts (ii) and (iii) follow immediately from the analysis in our Lemma 6 and in Lemma 5 of the Hardy-Littlewood paper; in fact, they are in no wise dependent on our work and were used freely in the deduction of our Lemma 2; they are included here merely for the sake of completeness.

Before proceeding to the proof of (i), we note that if we consider the simple series formed from the first column of P and the first row of Q, then we have, by the Lemma 5 of the Hardy-Littlewood paper, that

(49) $$b_{00} = (r+1)! \sum_0^\infty \frac{P_{\mu 0}^{r}}{(\mu+r+2)\cdots(\mu+1)},$$

and

(49') $$b_{00} = (r+1)! \sum_0^\infty \frac{Q_{0\nu}^{r}}{(\nu+r+2)\cdots(\nu+1)}.$$

For the proof of (i), we infer from Lemma 6 that B* is summable (C,r-1) to zero; thus

$$\sum_0^\infty \sum_0^\infty \left\{ b^*_{mn} - b^*_{m+1\,n} - b^*_{m\,n+1} + b^*_{m+1\,n+1} \right\}$$

is summable (C,r-1) to b^*_{00}, or A'' is summable (C,r-1) to b^*_{00}. Thus there exists a $b_{mn} = b^*_{mn} + h^*$, which is plainly a solution of (13), (14) and (15); but $b^*_{mn} \to 0$ (C,r-1) as m and n become infinite, since B* is summable; hence $h^* = 0$. Therefore $b_{mn} = b^*_{mn}$, and B is summable (C,r-1) to b^*_{00} or to b_{00}.

We now can infer that, since h is zero, the φ of (38) is zero.

Hence

$$\begin{aligned} A'' &= b^*_{00} = b_{00} = -b_{00} + b_{00} + b_{00} \\ &= B^{rr}_{00} = [(r+1)!]^2 \varphi_{00} \\ &= -[(r+1)!]^2 \sum_0^\infty \sum_0^\infty \frac{A_{\mu\nu}^{rr}}{(\mu+r+2)\cdots(\mu+1)(\nu+r+2)\cdots(\nu+1)} \\ &\quad + (r+1)! \sum_0^\infty \frac{P_{\mu 0}^{r}}{(\mu+r+2)\cdots(\mu+1)} + (r+1)! \sum_0^\infty \frac{Q_{0\nu}^{r}}{(\nu+r+2)\cdots(\nu+1)}, \end{aligned}$$

by (35) and (37). Using (49) and (49'), we obtain (44) as desired.

§7, TWO ADDITIONAL LEMMAS. Lemmas 4,6 and 7 furnish us with all the desired results for the proof of Theorem I; however we insert two more lemmas for the sake of completeness. Their origin is at once apparent.

<u>Lemma 8 -: If (i) A'' is summable (C,r-1) and A'_k and $A^{\prime}_{k}$ are also</u>

summable (C,r-1), (ii) b_{mn} is defined through (45), p_{kn} through (48) and q_{mk} through a similar expression, and (iii) B is summable (C,r-1), while P_c and Q_r are summable (C,r-1); then A, A_c and A_r are summable (C,r).

Lemma 9 -: A necessary and sufficient condition that A, A_c and A_r should be summable (C,r), is that, if b_{mn}, p_{kn} and q_{mk} are defined as in Lemma 7, B, P_c and Q_r should be summable (C,r-1).

§8, PROOF OF THEOREM I. We are now ready to prove Theorem I.

For the sufficiency of the conditions outlined in the statement of the theorem, we suppose that $A^{r+1\,r+1}$, A_K^{r+1} and A_k^{r+1} are summable (C,-1). Then, by successive applications of Lemma 4, we conclude that $A^{\rho\rho}$, A_K^{ρ} and A_k^{ρ} are summable (C,r-ρ), for $\rho = r$, r-1,----0; also $A^{rr} = A^{r-1\,r-1} = \cdots = A^{\rho\rho}$, with corresponding equalities for the sums of the simple series involved.

For the necessity of the conditions of the theorem, we suppose that A, A_c and A_r are summable (C,r). Then, by Lemmas 6 and 7, there exist numbers

$$a'^{\,\prime}_{mn} = b_{mn} = \sum_{m}^{\infty}\sum_{n}^{\infty} \frac{a_{\mu\nu}}{(\mu+1)(\nu+1)},$$

$$a'_{mK} = q_{mK} = \sum_{m}^{\infty} \frac{a_{\mu K}}{\mu+1},$$

$$a'_{kn} = p_{kn} = \sum_{n}^{\infty} \frac{a_{k\nu}}{\nu+1},$$

such that A'', A'_K and A'_k are summable (C,r-1). Successive applications of this reasoning lead to the conditions as enunciated. Theorem I is thus complete.

§9. THE GENERAL CASE, $r \neq s$. We now wish to consider the case when $r \neq s$. We shall discuss the results only when $s > r$, say $s = r + s'$, $s' > 0$; it will of course be understood that the results when $r > s$ are entirely analogous to these.

Theorem II -: The necessary and sufficient conditions that A should be summable (C,r,s), where $s = r + s'$, and that A_c and A_r should be summable (C,s) and (C,r) respectively, are that there should exist systems of numbers

$$(50)\qquad \begin{cases} a^{0\,0}_{mn} = a_{mn} \\ a^{0\,\sigma-1}_{mn} = (n+1)\left(a^{0\,\sigma}_{mn} - a^{0\,\sigma}_{m\,n+1}\right) & (\sigma = 1, 2 \dots s') \end{cases}$$

$$(51)\qquad \begin{cases} a^{0}_{kn} = a_{kn} \\ a^{\sigma-1}_{kn} = (n+1)\left(a^{\sigma}_{kn} - a^{\sigma}_{k\,n+1}\right) & (\sigma = 1, 2 \dots s') \end{cases}$$

(52)

$$(52)\qquad a^{0}_{mk} = a_{mk}.$$

such that

$$(53)\qquad A^{0\,s'}_{mn} = \sum_0^\infty \sum_0^\infty a^{0\,s'}_{mn}$$

is summable (C,r,s-s'), while

$$(54)\qquad A^{s'}_{k} = \sum_0^\infty a^{s'}_{kn}$$

is summable (C,s-s'), and A_r is summable (C,r). In these circumstances, we have

$$(55)\qquad a^{0\,\sigma}_{mn} = \sum_m^\infty \sum_n^\infty \frac{a^{0\,\sigma-1}_{m\nu}}{\nu+1} \qquad (C, r, s-\sigma)$$

for $\sigma = 1, 2, \dots s'$, and

$$(56)\qquad a^{\sigma}_{kn} = \sum_n^\infty \frac{a^{\sigma-1}_{k\nu}}{\nu+1} \qquad (C, s-\sigma).$$

The sums of all the corresponding series are the same.

To prove Theorem II, we shall need two additional lemmas, corresponding to Lemma 4 and Lemmas 6 and 7 of the analysis leading to Theorem I. The

Proofs of these lemmas, 10 and 11, follow reasoning in all cases analogous to that of their prototypes, so that most of the details will be omitted here. They are based on the two equations

$$a_{mn} = (n+1)\left(p_{mn} - p_{m\,n+1}\right) \tag{13'}$$

$$A^{rs}_{mn} = (s+1)\,P^{rs}_{mn} - (n+1)\,P^{r\,s-1}_{m\,n+1}. \tag{57}$$

The latter is (19') generalized to the present case; its proof differs in no way from that of (19'), as was indicated in Lemma 3.

Lemma 10 -: If P is summable (C,r,s-1), and P_c is summable (C,s-1) then A is summable (C,r,s) and A_c is summable (C,s); the sums of the corresponding series are the same.

Lemma 11 -: If A is summable (C,r,s) and A_c is summable (C,s), both to zero, then

(i) there exists a solution of (13') such that P is summable (C,r,s-1), and P_c is summable (C,s-1), to zero;

(ii) also

$$A^{01} = \sum_{0}^{\infty}\sum_{0}^{\infty} \frac{a_{\mu\nu}}{\nu+1} \tag{58}$$

is summable (C,r,s-1); and if

$$p_{mn} = \sum_{m}^{\infty}\sum_{n}^{\infty} \frac{a_{\mu\nu}}{(\nu+1)} \qquad (C,r,s-1), \tag{59}$$

then P is summable (C,r,s-1) to zero;

(iii) also

$$A^{1}_{h} = \sum_{0}^{\infty} \frac{a_{h\nu}}{\nu+1} \tag{46}$$

is summable (C,s-1), and if

$$p_{hn} = \sum_{n}^{\infty} \frac{a_{h\nu}}{\nu+1} \qquad (C,s-1) \tag{48}$$

then P_c is summable (C,s-1) to zero.

The proof of (i) is exactly analogous to that of Lemma 6, and is based on the formula

$$(60) \qquad P^{r\,s-1}_{m\,n} = h\,C^{r\,s}_{m\,n} + o(m^r n^{s-1}),$$

which is proved along the lines of Lemma 5, through use of (13') and (57). The proof of (ii) is similar to that of Lemma 7; while (iii) is merely a restatement of (iii) of that lemma.

We can now establish Theorem II. For the sufficiency of the conditions of the theorem, we suppose that $A^{0\,s'}$ is summable (C,r,s-s') to zero, and $A^{s'}_k$ is summable (C,s-s') to zero. Then, by repeated application of Lemma 10, we have in succession that $A^{0\,s'-1}$ is summable (C,r,s-s'+1), --- A^{00} is summable (C,r,s); while $A^{s'-1}_k$ is summable (C,s-s'+1),----A^0_k is summable (C,s). For the necessity of the conditions, we suppose A summable (C,r,s), A_c summable (C,s), to zero. By Lemma 11, there exist numbers

$$a^{0\,1}_{m\,n} = p_{mn} = \sum_m^\infty \sum_n^\infty \frac{a_{\mu\nu}}{\nu+1},$$

$$a^1_{kn} = p_{kn} = \sum_n^\infty \frac{a_{k\nu}}{\nu+1},$$

such that A^{01} is summable (C,r,s-1), and A^1_k is summable (C,s-1), to zero. Repeated applications, s' times, of this reasoning establish the necessity of the condition of the theorem. Theorem II is thus complete.

THE MOST GENERAL THEOREM

§10. It is interesting to view Theorem II in conjunction with Theorem I. The former gives as condition that A should be summable (C,r,s) that A should be summable (C,r,r), where s = r+s'. But Theorem I states that the condition for this latter property is that $A^{r+1\;r+s'+1}$ should be summable (C,-1,-1). The conditions regarding the rows and columns follow similarly.

Hence, amalgamating the two theorems, we have

Theorem III -: The necessary and sufficient conditions that A should be summable (C,r,s), s = r + s', while A_c and A_r are summable (C,s) and (C,r) respectively, are that there should exist numbers

$$a_{mn}^{\rho\sigma}, \quad a_{m\kappa}^{\rho}, \quad a_{kn}^{\rho},$$

$\rho = 1,2---r+1$, $\sigma = 1,2,--s+1$, such that

(61) $$\sum_0^\infty \sum_0^\infty a_{m\,n}^{r+1\;s+1} = A_{m\,n}^{r+1\;s+1}$$

is summable (C,-1,-1), while (8) and

(62) $$A_k^{s+1} = \sum_0^\infty a_{kn}^{s+1}$$

are summable (C,-1); provided that, in building the auxiliary series (61) and (62), we interpret the numbers $a_{mn}^{\rho\sigma}$ and a_{kn}^{r} as those defined in (50) and (51) as long as r ≠ s , and as those defined in (4) and (6) when r = s.

§11. THEOREM III'. For our application to the double Fourier's series, we shall need a slightly different form of Theorem III, which amounts to nothing but a change of notation. If we put

$$a_{mn}^{\rho\sigma} = \alpha_{m-1\;n-1}^{\rho\;\;\sigma}$$

A^{rs} is replaced by

$$A^{*rs} = \left\{ \begin{array}{l} 0 + 0 + \cdots \\ +0 + \alpha_{00}^{\rho\sigma} + \alpha_{01}^{\rho\sigma} + \cdots \\ +0 + \alpha_{10}^{\rho\sigma} + \alpha_{11}^{\rho\sigma} + \cdots \\ + \cdots \end{array} \right\}.$$

But the addition of the zeroes does not affeft the summability or the sum of the series, and hence we have

Theorem III' -: Theorem III still holds true if we replace the factors m + 1 and n + 1 in (5),(6),(7) (50) and (51) by m and n, and the factors $1/\mu+1$ and $1/\nu+1$ in (10), (11), (12), (55) and (56) by $1/\mu$ and $1/\nu$.

II APPLICATION TO THE DOUBLE FOURIER'S SERIES

We now use Theorem III' as the basis of the derivation of necessary and sufficient conditions for the summability (C) of the double Fourier's series of a function $f(\alpha\beta)$, which is , in a certain region, integrable in the Lebesgue sense and doubly periodic, the summation to be taken at the point $\alpha = x, \beta = y$.

§12 We write, for convenience,

$$\varphi(\alpha\beta) \equiv f(x+\alpha, y+\beta) + f(x+\alpha, y-\beta) + f(x-\alpha, y+\beta) + f(x-\alpha, y-\beta) - 4A, \tag{63}$$

$$\begin{cases} \chi(\alpha) \equiv \int_{-\pi}^{\pi} [f(x+\alpha,\beta) - f(x-\alpha,\beta)]\, d\beta - 2A_r, \\ \zeta(\beta) \equiv \int_{-\pi}^{\pi} [f(x, y+\beta) - f(x, y-\beta)]\, d\alpha - 2A_c, \end{cases} \tag{64}$$

where A, A_r and A_c are constants to be determined later. We introduce also

$$\begin{cases} \varphi_{11} \equiv \frac{1}{\alpha\beta} \int_0^\alpha \int_0^\beta \varphi(\alpha_1, \beta_1)\, d\alpha_1\, d\beta_1 \equiv \varphi_1(\alpha\beta), \\ \varphi_{22} \equiv \frac{1}{\alpha\beta} \int_0^\alpha \int_0^\beta \varphi_1(\alpha_1, \beta_1)\, d\alpha_1\, d\beta_1 \equiv \varphi_2(\alpha\beta) \\ \text{etc.} \end{cases} \tag{65}$$

$$\begin{cases} \varphi_{10} \equiv \frac{1}{\alpha} \int_0^\alpha \varphi(\alpha_1, \beta)\, d\alpha_1 \\ \varphi_{01} \equiv \frac{1}{\beta} \int_0^\beta \varphi(\alpha\, \beta_1)\, d\beta_1 \\ \varphi_{21} \equiv \frac{1}{\alpha} \int_0^\alpha \varphi_1(\alpha_1, \beta)\, d\alpha_1 \qquad \text{etc.} \end{cases} \tag{66}$$

$$(67)\quad \begin{cases} \chi_1(\alpha) = \frac{1}{\alpha}\int_0^\alpha \chi(\alpha_1)\,d\alpha_1 \\ \xi_1(\beta) = \frac{1}{\beta}\int_0^\beta \xi(\beta_1)\,d\beta_1 \end{cases}$$

etc.

If we except a constant term and a factor $1/4$, the double Fourier's series for $f(\alpha\beta)$ at $\alpha = x,\ \beta = y$ is the same as that of $\varphi(\alpha\beta)$ for $\alpha = \beta = 0$. Hence, if we take $A = 0$, A_r and $A_c = 0$, $a_{00} = 0$, we have

$$(68)\quad \begin{cases} \varphi(\alpha\beta) \sim \sum_1^\infty \sum_1^\infty a_{ij} \cos i\alpha \cos j\beta \\ \chi(\alpha) \sim \sum_1^\infty a_{ik} \cos i\alpha \\ \xi(\beta) \sim \sum_1^\infty a'_{kj} \cos j\alpha \end{cases},$$

since now we may consider only even functions. Thus, ~~if~~ we replace $f(\alpha\beta)$ by $\varphi(\alpha\beta)$ in the discussions which follow.

The theorem we wish to prove is as follows:

Theorem IV -: A set of necessary and sufficient conditions that the double Fourier's series of $\varphi(\alpha\beta)$ should be summable (C) to sum A, for $\alpha=\beta=0$, and that each row and column should be summable (C) as a simple series to sums A_r and A_c respectively, is that there should exist numbers h and k, such that

$$(69)\quad \begin{cases} (a) & \varphi_{hk} \longrightarrow 0 \\ (b) & \chi_h \longrightarrow 0 \\ (c) & \xi_k \longrightarrow 0 \end{cases}$$

as α and $\beta \longrightarrow 0$.

§13. THE AUXILIARY FUNCTIONS $\psi_{ij}(\alpha\beta)$. We introduce next a set of auxiliary functions, defined a s follows:

$$
(70)\quad\begin{cases}
\psi_0(\alpha\beta) \equiv \varphi_0(\alpha\beta) = \varphi(\alpha\beta) \\
\psi_1(\alpha\beta) \equiv \frac{1}{4}\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\int_0^{\alpha}\int_0^{\beta}\psi_0(\alpha_1\beta_1)\,d\alpha_1\,d\beta_1 + \gamma_1 \sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} \\
\psi_2(\alpha\beta) \equiv \frac{1}{4}\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\int_0^{\alpha}\int_0^{\beta}\psi_1(\alpha_1\beta_1)\,d\alpha_1\,d\beta_1 + \gamma_2 \sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} \\
\psi_{01}(\alpha\beta) \equiv \frac{1}{2}\cot\frac{\beta}{2}\int_0^{\beta}\psi_0(\alpha\beta_1)\,d\beta_1 + \gamma_{01}\sin^2\frac{\beta}{2} \\
\text{etc.}
\end{cases}
$$

where the γ's are constants to be determined later.

<u>Lemma 12 -: The existence of any ψ_{rs} (r, s = 0, 1, ---) as a continuous function of its upper limits implies that of the corresponding φ_{rs}, and conversely; their difference is $o(\alpha\beta)$.</u>

The result is obvious for r = s = 0; andfollows readily from the corresponding analysis in IV. 1 of the Hardy-Littlewood paper if either r or s is zero, since in such event there is in the definition of ψ, integration with respect to only one variable.

Supposing it true for $r = m,\ s = n$, we have

$$
\begin{aligned}
\psi_{m+1\,n+1}(\alpha\beta) &= \frac{1}{4}\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\int_0^{\alpha}\int_0^{\beta}\left[\varphi_{mn} + o(\alpha_1\beta_1)\right]d\alpha_1\,d\beta_1 \\
&\quad + \gamma_{m+1\,n+1}\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} \\
&= \left\{\frac{1}{\alpha\beta} + O\left(\frac{\beta}{\alpha}\right) + O\left(\frac{\alpha}{\beta}\right) + O(\alpha\beta)\right\}\int_0^{\alpha}\int_0^{\beta}\left[\varphi_{mn} + o(\alpha_1\beta_1)\right]d\alpha_1\,d\beta_1 \\
&\quad + o(\alpha\beta) \\
&= \varphi_{m+1\,n+1} + o(\alpha\beta),
\end{aligned}
$$

and the lemma follows by induction. We may thus replace each φ by its corresponding ψ, and conversely.

<u>Lemma 13 -: Every $\psi_{rs}(\alpha\beta)$, (r, s = 0, 1, ---), is an even-even period-</u>

ic function, and its rows and columns are even periodic series:

(71) $$\psi_{rs}(\alpha\beta) \sim \sum_1^\infty \sum_1^\infty a_{ij}^{(rs)} \cos i\alpha \cos j\beta$$

(72) $$\begin{cases} \psi_{rk}(\alpha\beta) \sim \sum_1^\infty a_{ik}^{(r)} \cos i\alpha \\ \psi_{ks}(\alpha\beta) \sim \sum_1^\infty a_{kj}^{(s)} \cos j\beta . \end{cases}$$

The proof of (72) follows directly from (4.311) of the Hardy-Littlewood paper, while that of (71) is entirely analogous to the reasoning leading to (4.311).

§14. THE SPECIAL HYPOTHESIS φ^2 INTEGRABLE (L). From this point on, we shall divide the discussion into two parts, one being the special case when φ^2 is integrable (L), the other being the general case where φ is integrable (L). The reason for this division is fairly obvious In the first case, it can be shown as in Lemma 14 below, that φ^2_{rs} and ψ^2_{rs} are integrable (L), whence φ_{rs} and ψ_{rs} are also. But this is not the immediate result in the second case*, and to prove our theorem we are led to more special considerations invalving generalized integrals.

We proceed, then, under the hypothesis that φ^2 is integrable (L).

<u>Lemma (14) -: If φ^2 is integrable (L), so are φ^2_{rs} and ψ^2_{rs}, and consequently φ_{rs} and ψ_{rs} $(r,s=0,1,--)$.</u>

We shall indicate the proofs of the lemma in the cases of φ^2_{01}, φ^2_{10} and φ^2_{1} ; sucessive applications of the methods used will then serve to complete the lemma.

*For a more complete exposition of this point, and for an example, the reader is referred to IV.5 of the Hardy-Littlewood paper.

We first show that

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}\varphi_{01}^2(\alpha_1\beta_1)\,d\alpha_1\,d\beta_1 < M;$$

the proof of the integrability of φ_{10}^2 is analogous. We note that the part of φ_{01} depending on α is the same as that of φ, and therefore φ_{01}^2 is integrable as far as the α-integration is concerned. For the β- integration, then, we can use the results of Lemma 10 of the Hardy-Littlewood paper, to infer our result, since the definitions in (66) are comparable to those in (1.272) of that paper.

To show that

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}\varphi_1^2(\alpha_1\beta_1)\,d\alpha_1\,d\beta_1 < M,$$

we introduce

$$(72)\qquad \Phi_1(\alpha\beta) \equiv \int_0^{\alpha}\int_0^{\beta}\varphi(\alpha_1\beta_1)\,d\alpha_1\,d\beta_1,$$

whence, by Schwarz's inequality*,

$$(73)\qquad \Phi_1^2(\alpha\beta) \leqq \alpha\beta\int_0^{\alpha}\int_0^{\beta}\varphi^2(\alpha_1\beta_1)\,d\alpha_1\,d\beta_1 = o(\alpha\beta).$$

We have then by an integration by parts**,

$$\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}\varphi_1^2\,d\alpha_1\,d\beta_1 = \int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}\frac{\Phi_1^2(\alpha_1\beta_1)}{\alpha_1^2\beta_1^2}\,d\alpha_1\,d\beta_1$$

$$= \frac{\Phi_1^2(\alpha\beta)}{\alpha\beta}\Bigg|_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2} + 2\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}(\varphi_1\varphi_{10}+\varphi_1\varphi_{01})\,d\alpha_1\,d\beta_1$$

$$-4\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}\varphi_1\varphi_{10}\,d\alpha_1\,d\beta_1 - 8\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}(\varphi_1\varphi+\varphi_{10}\varphi_{01})\,d\alpha_1\,d\beta_1$$

$$= o(1) + O(1) + O\left(\sqrt{\int_{\varepsilon_2}^{\delta_2}\varphi_{01}^2\,d\beta}\right)$$

$$+ O\left(\sqrt{\int_{\varepsilon_1}^{\delta_1}\varphi_{10}^2\,d\alpha}\right) + O\left(\sqrt{\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_2}^{\delta_2}\varphi_1^2\,d\alpha\,d\beta}\right).$$

* Cf. W.R.Lovitt, "Integral Equations", p. 125.
**Cf. Hilda Geiringer, "Trigonometrische Doppelreihen", Monatshefte fur Mathematik und Physik, 1918, p. 82.

We conclude then that since the right-hand side of the above equation is bounded, so must be the left; the lemma is thus established.

§15. FOURIER FORM OF THE NUMBERS EXISTING IN THEOREM III'. Lemma 15. If φ^2 is integrable (L), the a_{mn}^{rs} $(r, s = 0, 1, ---)$ in the notation of Theorem III', differ from the $a_{mn}^{(rs)}$, in the notation of Lemma 13, by the general term of a double series summable (C, -1) to zero. Also, a_{mk}^{r} differs from $a_{mk}^{(r)}$, and $a_{kn}^{(s)}$ from a_{kn}^{s} by the general terms of series summable (C, -1) to zero.

The second part of the lemma follows from an application of Lemma 11 of the Hardy-Littlewood paper to the rows and columns of the double series concerned.

We shall consider two cases in the proof of the first part of the lemma. The first is a reduction to be used when $r = s = 1$, the second when either r or s is zero, and the other unity. The lemma in general follows from a combination of successive applications of these two cases.

(1) We have, for $r = s = 1$,

$$\sum_m^M \sum_n^N \frac{a_{\mu\nu}}{\mu\nu} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi(\alpha\beta) \sum_m^M \sum_n^N \frac{\cos\mu\alpha}{\mu}\,\frac{\cos\nu\beta}{\nu}\, d\alpha\, d\beta$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta) \sum_m^M \sum_n^N \sin\mu\alpha \sin\nu\beta \, d\alpha\, d\beta$$

(74)

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta) \frac{\cos(m-\frac{1}{2})\alpha - \cos(M-\frac{1}{2})\alpha}{\sin\frac{\alpha}{2}}\,\frac{\cos(n-\frac{1}{2})\beta - \cos(N-\frac{1}{2})\beta}{\sin\frac{\beta}{2}}\, d\alpha\, d\beta$$

$$\to \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\left[\cos m\alpha \cot\frac{\alpha}{2} + \sin m\alpha\right]\left[\cos n\beta \cot\frac{\beta}{2} + \sin n\beta\right] d\alpha\, d\beta$$

as $M, N \to \infty$, by the generalized Riemann-Lebesgue theorem. This becomes, on expansion,

$$
(75)\quad
\begin{aligned}
a''_{mn} = {} & \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\cos m\alpha \cot\frac{\alpha}{2}\cos n\beta \cot\frac{\beta}{2}\, d\alpha\, d\beta \\
& + \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\sin m\alpha \sin n\beta\, d\alpha\, d\beta \\
& + \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\cos m\alpha \cot\frac{\alpha}{2}\sin n\beta\, d\alpha\, d\beta \\
& + \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\sin m\alpha \cos n\beta \cot\frac{\beta}{2}\, d\alpha\, d\beta .
\end{aligned}
$$

If we use (70), we can rewrite the first term of (75) as

$$
a^{(11)}_{mn} - \frac{\gamma_1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sin^2\frac{\alpha}{2}\cos m\alpha \sin^2\frac{\beta}{2}\cos n\beta\, d\alpha\, d\beta .
$$

Thereupon, we have

$$
(76)\quad
\begin{aligned}
a''_{mn} &= a^{(11)}_{mn} + b''_{mn} \\
&= a^{(11)}_{mn} + b'^{\,11}_{mn} + b''^{\,11}_{mn} + b'''^{\,11}_{mn} + b''''^{\,11}_{mn},
\end{aligned}
$$

where, from (75),

$$
(77)\quad
\left\{
\begin{aligned}
b'^{\,11}_{mn} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\sin m\alpha \sin n\beta\, d\alpha\, d\beta \\
b''^{\,11}_{mn} &= \frac{-\gamma_1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sin^2\frac{\alpha}{2}\cos m\alpha \sin^2\frac{\beta}{2}\cos n\beta\, d\alpha\, d\beta \\
b'''^{\,11}_{mn} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\cos m\alpha \cot\frac{\alpha}{2}\sin n\beta\, d\alpha\, d\beta \\
b''''^{\,11}_{mn} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\sin m\alpha \cos n\beta \cot\frac{\beta}{2}\, d\alpha\, d\beta .
\end{aligned}
\right.
$$

We next proceed to sum the series $B^{11} = \sum_1^\infty \sum_1^\infty b'^{\,11}_{mn}$ in the ordinary manner. We have

$$
(78)\quad
\begin{aligned}
\sum_1^M \sum_1^N b'^{\,11}_{mn} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\,\frac{\cos\frac{\alpha}{2} - \cos(M-\frac{1}{2})\alpha}{2\sin\frac{\alpha}{2}}\;\frac{\cos\frac{\beta}{2} - \cos(N-\frac{1}{2})\beta}{2\sin\frac{\beta}{2}}\, d\alpha\, d\beta \\
&\to \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta)\cdot\frac{1}{4}\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\, d\alpha\, d\beta
\end{aligned}
$$

as M and N $\to\infty$.

For the summation of $\sum\sum b''^{\,11}_{mn}$, we infer that if $m=n=1$, $b''^{\,11}_{mn} = -\frac{\gamma_1}{4}$, while in any other case, $b''^{\,11}_{mn}$ is zero. Hence

$$(79) \qquad \sum_1^\infty \sum_1^\infty b''^{\,11}_{mn} = -\frac{\gamma_1}{4}.$$

We can sum $\sum\sum b'''^{\,11}_{mn}$ and $\sum\sum b''''^{\,11}_{mn}$ to the same value, viz.

$$(80) \qquad \sum_1^\infty \sum_1^\infty b'''^{\,11}_{mn} = \sum_1^\infty \sum_1^\infty b''''^{\,11}_{mn} = -\sum_1^\infty \sum_1^\infty b'^{\,11}_{mn}.$$

Hence, combining (78),(79) and (80), we conclude that

$$\sum_1^\infty \sum_1^\infty b^{11}_{mn} = -\frac{\gamma_1}{4} - \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{4}\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\,\Phi_1(\alpha\beta)\,d\alpha\,d\beta$$
$$= -\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \psi_1(\alpha\beta)\,d\alpha\,d\beta \qquad = 0$$

We have then the following facts concerning the series B^{11}:

a) The series B^{11} converges to zero.

b) Since $b'^{\,11}_{mn}$ is the Fourier sine-sine coefficient of a double integral, it is $o\left(\frac{1}{mn}\right)$, and hence b^{11}_{mn} is of that form.

c) If we form $\sum_{m=1}^\infty b'^{\,11}_{mn}$, n fixed, we have

$$\sum_{m=1}^\infty b'^{\,11}_{mn} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_1(\alpha\beta) \cdot \frac{1}{2}\cot\frac{\alpha}{2}\,\frac{\sin n\beta}{n}\,d\alpha\,d\beta,$$

which is the Fourier sine coefficient of an integral, and so of the form $o\left(\frac{1}{n}\right)$; the same form is therefore possessed by $\sum_{m=1}^\infty b^{11}_{mn}$. Similarly, $\sum_{n=1}^\infty b^{11}_{mn}$ is $o\left(\frac{1}{m}\right)$.

The above three properties make B^{11} summable (C,-1,-1) to zero.

(ii) We now consider the second case of our lemma, i.e. that where, for example, $r=0$, $s=1$; similar analysis to that to be employed here will hold in case $r=1$, $s=0$. We now have

$$a^{01}_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi(\alpha\beta)\sum_m^\infty \sum_n^\infty \cos\mu\alpha\,\frac{\cos\nu\beta}{\nu}\,d\alpha\,d\beta$$

$$(81) \qquad = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_{01}(\alpha\beta)\sum_m^\infty \sum_n^\infty \cos\mu\alpha\,\sin\nu\beta\,d\alpha\,d\beta$$

where

$$\Phi_{01}(\alpha\beta) = \int_0^\beta \varphi(\alpha\,\beta_1)\,d\beta_1 .$$

By a process similar to the one outlined in case (i), we find that (81) reduces to

$$(82)\qquad \begin{aligned} a^{01}_{mn} &= a^{(01)}_{mn} + b^{01}_{mn} \\ &= a^{(01)}_{mn} + b'^{\,01}_{mn} + b''^{\,01}_{mn} + b'''^{\,01}_{mn} + b''''^{\,01}_{mn}, \end{aligned}$$

where

$$(83)\qquad \begin{aligned} b'^{\,01}_{mn} &= -\frac{\gamma_{01}}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos m\alpha \cos n\beta \sin^2\tfrac{\beta}{2}\,d\alpha\,d\beta \\ b''^{\,01}_{mn} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_{01}(\alpha\beta)\cos m\alpha \sin n\beta\,d\alpha\,d\beta \\ b'''^{\,01}_{mn} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_{01}(\alpha\beta)\sin m\alpha \cot\tfrac{\alpha}{2}\cos n\beta \cot\tfrac{\beta}{2}\,d\alpha\,d\beta \\ b''''^{\,01}_{mn} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_{01}(\alpha\beta)\sin m\alpha \sin n\beta \cot\tfrac{\alpha}{2}\,d\alpha\,d\beta . \end{aligned}$$

When we sum $B^{01} = \sum_1^\infty\sum_1^\infty b^{01}_{mn}$, we find that the last two terms of (83) sum to the same value with opposite signs, while the first two add together to produce zero, with the help of (70). As before, in the case of B^{11}, we also can conclude that B^{01} is summable (C,-1,-1) to zero. Similarly, $B^{10} = \sum\sum b^{10}_{mn}$ where $b^{10}_{mn} = a^{10}_{mn} - a^{(10)}_{mn}$, is summable (C,-1-1) to zero.

Our lemma is completed by the remark that, since all ψ's possess the relevant properties of ψ_0, we can carry out these processes in (i) and (ii) any number of times and in any combinations.

§16. A CONVERGENCE THEOREM. <u>Lemma 16-: If a_{mn} is $o(\frac{1}{mn})$, a_{hn} is $o(\frac{1}{n})$, and a_{mk} is $o(\frac{1}{m})$, then a necessary and sufficient condition that $A^{(rs)}$ should be convergent to sum A, while $A^{(r)}_h$ and $A^{(s)}_k$ are convergent to A_h and A_k respectively, is that</u>

(a)

$$(84)\quad \begin{cases} \text{(a)} & \varphi_{r+1,\, s+1} \longrightarrow A \\ \text{(b)} & \chi_{r+1} \longrightarrow A_r \\ \text{(c)} & \zeta_{s+1} \longrightarrow A_c \end{cases}$$

when α and $\beta \longrightarrow 0$.

This lemma is a restatement, in our present notation, of the corollary to the generalization of a theorem due to Fatou*, which generalization we append to the present paper. (84b) and (84c) follow from Lemma 12 of the Hardy-Littlewood paper.

§17. PROOF OF THEOREM IV, φ^2 INTEGRABLE. We are now ready to prove Theorem IV, in the case φ^2 is integrable (L). If A is summable (C,r,s) A_r is summable (C,r) and A_c is summable (C,s), all to zero, then, by Theorem III', $A^{r+1\, s+1}$, A^{r+1}_{κ} and A^{s+1}_{λ} are summable (C,-1) to zero; hence, by Lemma 15, $A^{(r+1,s+1)}$, $A^{(r+1)}_{\kappa}$ and $A^{(s+1)}_{\lambda}$ are summable (C,-1) to zero, and therefore by Lemma 16, $\varphi_{r+1,\,s+1}$, χ_{r+1} and $\zeta_{s+1} \longrightarrow 0$. The condition is thus necessary.

For the sufficiency of the condition, we have that, if $\varphi_{r+1,\,s+1}$, χ_{r+1} and $\zeta_{s+1} \longrightarrow 0$, then $A^{(r+1\; s+1)}$ is summable (C,1,1) to zero by the generalized Fejér theorem**, while $A^{(r+1)}_{\kappa}$ and $A^{(s+1)}_{\lambda}$ are summable (C,1) to zero by Fejér's theorem itself; therefore $A^{r+1\, s+1}$, A^{r+1}_{κ} and A^{s+1}_{λ} are summable (C,1) to zero by Lemma 15; and hence A is summable (C,r+1,s+1) to zero, and A_r and A_c are summable (C,r+1) and (C,s+1) to zero, all by the reductions used in the derivation of Theorem III'. Theorem IV is thus complete for the case in which φ^2 is integrable.

*P.Fatou: "Series Trigonometriques et Series de Taylor", Acta Mathematica, vol. 30, 1906.
**C.N.Moore: "On Convergence Factors in Double Series and the Double Fourier's Series", Transactions of the American Mathematical Society, vol. 14, #1, Jan. 1913.

§18. THE GENERAL CASE, φ INTEGRABLE (L). GENERALIZED INTEGRALS. We have still to prove Theorem IV in the case when φ is integrable (L), which, as was pointed out before, requires more delicate considerations. The difficulty lies inthe fact that the integrability (L) of φ does not necessarily imply that of φ_{rs} ; that is to say, the analogue of Lemma 14 for this general case does not hold true if we consider only our ordinary integrals. To obviate this difficulty, we are led to consider a kind of "generalized integral". We say that $F(\alpha\beta)$ is integrable (Cauchy) provided

$$\int_0^{\alpha}\int_0^{\beta} F(\alpha\beta)\,d\alpha\,d\beta = \lim_{\varepsilon_1,\varepsilon_2\to 0}\int_{\varepsilon_1}^{\gamma}\int_{\varepsilon_2}^{\beta} F(\alpha\beta)\,d\alpha\,d\beta,$$

where the integral on the right is a Lebesgue integral, but the other is not. Single Cauchy integrals are definéd in analogous fashoin*. Our integrals are understood to be "generalized" here only in reference to the origin.

With respect to these new integrals, our argument requires modification. In the first place, we want an analogue to Lemma 14; and secondly, if our reductions in Lemma 15 are to remain valid, we need to prove an analogue to the generalized Riemann-Lebesgue theorem. We note, however, that in the case of the sufficiency of the condition, the new integrals were not altogether needed in all cases, for our condition asserts that some φ_{hk} is continuous for $\gamma = \beta = 0$, which certainly presupposes the existence in some form or other of the φ_{ij} $(i = 1,2\text{--}h-1, j = 1,2\text{--}k-1)$. If these integrals exist in the Lebesgue sense, then the preceding proof of the sufficiency of the condition holds without change. If, however, they are Cauchy integrals, we require an additional lemma in order to prove the sufficiency case.

§19. INTEGRABILITY (L) OF φ_{ij}, χ_i, AND ξ_j. <u>Lemma 17-: If φ,</u>

* Cf. the Hardy-Littlewood paper, IV.5.

χ and ζ are integrable (L), while φ_{10}, φ_{01}, φ_1 ------ $\varphi_{r+1\,s-1}$, χ_1 --- χ_{r-1} and ζ_1 ---- ζ_{s-1} are integrable (C), and if $\varphi_{r-1\,s}$, $\varphi_{r\,s-1}$ and φ_{rs}, χ_r and ζ_s are continuous for $\alpha=\beta=0$, then φ_{ij}, χ_i and ζ_j, $(i=0,1---r,\ j=0,1--s)$, are integrable (L).

It is convenient to make a change of variables; let us put

$$(85)\quad \left\{\begin{aligned} &\alpha = e^{-x} \qquad \beta = e^{-y}\\ &F(xy) = e^{-(x+y)}\,\varphi(\alpha\beta)\\ &F_{01}(xy) = e^{-y}\,\varphi_{01}(\alpha\beta) \qquad\qquad F_{10}(xy) = e^{-x}\,\varphi_{10}(\alpha\beta)\\ &F_1(xy) = e^{-(x+y)}\,\varphi_1(\alpha\beta)\\ &F_{21}(xy) = e^{-(x+y)}\,\varphi_{10}(\alpha\beta)\\ &etc.\\ &G(x) = e^{-x}\,\chi(\alpha)\\ &G_1(x) = e^{-x}\,\chi_1(\alpha)\\ &etc.\\ &H(y) = e^{-y}\,\zeta(y)\\ &H_1(y) = e^{-y}\,\zeta_1(y)\\ &etc.\end{aligned}\right.$$

Our hypothesis is then that

(a) the integrals

$$\int_x^\infty\int_y^\infty |F(xy)|\,dx\,dy,\qquad \int_x^\infty |G(x)|\,dx,\qquad \int_y^\infty |H(y)|\,dy,$$

$$F_{01}(xy) = \int_y^\infty F(xy)\,dy,\quad G_1(x) = \int_x^\infty G(x)\,dx,\quad H_1(y) = \int_y^\infty H(y)\,dy,$$

$$F_1(xy) = \int_x^\infty\int_y^\infty F(xy)\,dx\,dy,\quad etc.$$

exist as Cauchy integrals up to ∞;

(b) $$F_{rs}(xy) = o(e^{-(x+y)}) \qquad\qquad G_r(x) = o(e^{-x})$$

$$F_{r\,s-1}(xy) = o(e^{-(x+y)}) \qquad\qquad H_s(y) = o(e^{-y})$$

$$F_{r-1\,s}(xy) = o(e^{-(x+y)})$$

except when either r-1 or s-1 is zero, inwhich case $F_{r0} = o(e^{-x})$, etc. We wish to prove that

$$(86)\quad \left\{\begin{array}{l} \int_x^\infty \int_y^\infty |F_{\rho\sigma}(xy)|\,dx\,dy, \\ \int_x^\infty |G_\rho(x)|\,dx \quad \text{and} \quad \int_y^\infty |H_\sigma(y)|\,dy \end{array}\right. \quad \begin{pmatrix} \rho = 0,1\ldots r \\ \sigma = 0,1,\ldots s \end{pmatrix}$$

exist in the Lebesgue sense.

We shall use only a part of (D); i.e. that $F_{rs}(xy) = O(e^{-a(x+y)})$, where a is a constant, $G_r(x) = O(e^{-ax})$, etc. Also since the various functions F, G and H are continuous and tend to zero as x and y become infinite, we may assume that $|F_{\rho\sigma}(xy)|$, $|G_\rho(x)|$ and $|H_\sigma(y)|$ are all less than unity.

In the first place, on account of the definitions of G and H in (85), we can easily follow the method of Lemma 13 of the Hardy-Littlewood paper to show that our results as to χ and ζ are true.

We shall divide the proof of the rest of the lemma into three parts: (i) the proof that F_{11} is integrable (L); (ii) the proof that F_{01} and F_{10} are integrable (L); (iii) the combination of these to prove the general result.

(i) Consider the surface

$$z = F_1(xy) \qquad \begin{pmatrix} \xi < x < 2\xi \\ \eta < y < 2\eta \end{pmatrix},$$

where ξ and η are large and positive. The set of points for which

$$|z| > \Lambda = e^{-\frac{1}{2}a(\xi+\eta)}$$

consists of a ser of open nets (XX_1, YY_1). Also, we have

$$\int_X^{X_1}\int_Y^{Y_1} \Lambda\,dx\,dy \leqq \left| \int_X^{X_1}\int_Y^{Y_1} F_1(xy)\,dx\,dy \right|$$
$$\leqq O(e^{-a(X+Y)}) \leqq O(e^{-a(\xi+\eta)}).$$

Thence we obtain

$$(87) \qquad (X_1 - X)(Y_1 - Y) < \frac{A e^{-a(\xi+\eta)}}{\Lambda} < A_1 \Lambda .$$

Suppose now that $(X \leqq x \leqq X_1, Y \leqq y \leqq Y_1)$; then we have that*

$$(88) \qquad |F_1(xy)| \leqq \Lambda + \int_X^{X_1}\int_Y^{Y_1} |F(xy)|\, dx\, dy .$$

Hence it follows that

$$\begin{aligned}
\int_\xi^{2\xi}\int_\eta^{2\eta} |F_1(xy)|\, dx\, dy &\leqq \Lambda\xi\eta + \sum_\xi^{2\xi}\sum_\eta^{2\eta}\left\{ \int_X^{X_1}\int_Y^{Y_1} |F(xy)|\, dx\, dy \right\} \\
&\leqq 4\Lambda\xi\eta + \sum\sum \left\{ (X_1 - X)(Y_1 - Y)\int\int |F(xy)|\, dx\, dy \right. \\
&\leqq 4\Lambda\xi\eta + \sum\sum \left\{ A_1\Lambda \int\int |F|\, dx\, dy \right. \\
&< 4\Lambda\xi\eta + A_2\Lambda\xi\eta \\
&< A_3\Lambda\xi\eta \\
&\leqq A_3\,\xi\eta\, e^{-\frac{1}{2}a(\xi+\eta)} .
\end{aligned}$$

It can be inferred at once from the last inequality that F_1 is integrable (L)

(ii) To show that F_{01} is integrable (L) (the discussion for F_{10} is similar), we have only to show that F_{01} is integrable (L) with respect to y, as it is already so with regard to x. This result can be obtained as in Lemma 15 of the Hardy-Littlewood paper.

(iii) Successive applications of the processes used in (i) and (ii) on the basis of the results obtained in those sections, show that F_{ij}, F_{i0} and F_{0j} ($i = 0,1,$---r, $j = 0,1,$--s) are integrable (L), and the lemma is complete.

§20. PROOF OF SUFFICIENCY, φ INTEGRABLE (L). Lemma 17 enables us to infer the sufficiency of the condition in Theorem IV when φ is integrable

*This is the special case, $r = 2$, of the development of $F_{r-1,r-1}(xy)$ into a Taylor's series with remainder expressed as an iterated r-1 fold double integral.

(L), for in this event, even if the $\varphi_{p\sigma}$ exist as functions integrable (C), they are also integrable (L). Hence the previous reasoning applied when was integrable (§§15-16), can now be applied to the general case, and the sufficiency of the condition is established.

We have yet to prove the necessity of the condition when φ is integrable (L); for this purpose we require a series of further lemmas.

§21. Lemma 18-: The Fourier cosine-cosine coefficient of a function integrable (C) is $o(mn)$.

Consider

$$a_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_1(\alpha\beta)\cos m\alpha \cos n\beta \, d\alpha\, d\beta$$

where $f_1(\alpha\beta)$ is integrable (C). Let us put

$$F(\alpha\beta) = \int_0^{\alpha}\int_0^{\beta} f_1(\alpha_1\beta_1)\, d\alpha_1\, d\beta_1 \;;$$

then

$$a_{mn} = \frac{mn}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(\alpha\beta)\sin m\alpha \sin n\beta \, d\alpha\, d\beta + \frac{1}{\pi^2}\Big[F(\alpha\beta)\cos m\alpha \cos n\beta\Big]_{-\pi}^{\pi}$$

$$= O(1) + o(mn) = o(mn)$$

since F is continuous.

Lemma 19-: If $\sum\sum \frac{a_{mn}}{mn}$ is convergent, then

$$\Omega(\varepsilon_1\varepsilon_2) = \sum\sum \frac{a_{mn}}{mn}\int_{m\varepsilon_1}^{m\alpha}\int_{n\varepsilon_2}^{n\beta}\left[\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\;\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right]^2 d\alpha\, d\beta$$

is continuous for $\varepsilon_1, \varepsilon_2 = 0$.

The proof of this result is exactly analogous to that of a similar one in Lemma 16 of the Hardy-Littlewood paper.

§22. ANALOGUE OF THE RIEMANN-LEBESGUE THEOREM FOR CAUCHY INTEGRALS. Lemma 20-: If $f(\alpha\beta)$ and $\frac{\mathfrak{F}(\alpha\beta)}{\alpha\beta}$, where

$$\mathfrak{F}(\alpha\beta) = \int_0^\alpha \int_0^\beta f(\alpha\beta)\,d\alpha\,d\beta,$$

are integrable (C), then the four integrals

$$(89) \qquad \int_0^\alpha \int_0^\beta \frac{\mathfrak{F}(\alpha\beta)}{\alpha\beta} \begin{matrix}\cos m\alpha & \cos n\beta \\ \sin m\alpha & \sin n\beta\end{matrix} \, d\alpha\, d\beta$$

approach zero (C,1) as m and n $\to \infty$ together or singly.

The result is obvious if the sine-sine integral is chosen. We shall indicate the proof of the result for the cosine-cosine integral, and the proofs of the other two will be dismissed with the remark that they can be argued in similar fashion.

To consider the cosine-cosine integral, let us put

$$v_{mn} = \int\int \cos m\alpha \cos n\beta \, \frac{\mathfrak{F}(\alpha\beta)}{\alpha\beta}\, d\alpha\, d\beta,$$

whence

$$\begin{aligned}(90) \qquad \sum_1^m \sum_1^n v_{ij} = {} & \frac{1}{4}\int\int \frac{\mathfrak{F}(\alpha\beta)}{\alpha\beta}\,\frac{\sin(m+\frac{1}{2})\alpha\,\sin(n+\frac{1}{2})\beta}{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}\,d\alpha\,d\beta \\ & - \frac{1}{4}\int\int \frac{\mathfrak{F}(\alpha\beta)}{\alpha\beta}\,\frac{\sin(n+\frac{1}{2})\beta}{\sin\frac{1}{2}\beta}\,d\alpha\,d\beta \\ & - \frac{1}{4}\int\int \frac{\mathfrak{F}(\alpha\beta)}{\alpha\beta}\,\frac{\sin(m+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\,d\alpha\,d\beta + O(1).\end{aligned}$$

By considering the β integration of the second term alone, and reducing it as in Lemma 15 of the Hardy-Littlewood paper, we easily show that the integral is $o(mn)$; a similar remark dismisses the third term of (90), and we have now only to show that the first term is $o(mn)$.

We have that

$$\frac{1}{4}\int\int \frac{\mathcal{F}(\alpha\beta)}{\alpha\beta}\,\frac{\sin(m+\frac{1}{2})\alpha\,\sin(n+\frac{1}{2})\beta}{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}\,d\alpha\,d\beta$$

$$= \frac{1}{4}\int\int \sin m\alpha\cot\tfrac{1}{2}\alpha\,\sin n\beta\cot\tfrac{1}{2}\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha\beta}\,d\alpha\,d\beta$$
$$+\frac{1}{4}\int\int \cos m\alpha\,\sin n\beta\cot\tfrac{1}{2}\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha\beta}\,d\alpha\,d\beta$$
$$+\frac{1}{4}\int\int \sin m\alpha\cot\tfrac{1}{2}\alpha\,\cos n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha\beta}\,d\alpha\,d\beta$$
$$+\frac{1}{4}\int\int \cos m\alpha\,\cos n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha\beta}\,d\alpha\,d\beta$$
$$= \int\int \sin m\alpha\,\sin n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha^2\beta^2}\,d\alpha\,d\beta$$
$$+\frac{1}{2}\int\int \cos m\alpha\,\sin n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha\beta^2}\,d\alpha\,d\beta$$
$$+\frac{1}{2}\int\int \sin m\alpha\,\cos n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha^2\beta}\,d\alpha\,d\beta$$
$$+\ o(mn)$$

by Lemma 18. What we have to prove, then, is that

$$\text{(91)}\quad \begin{aligned} v'_{mn} &= \int\int \sin m\alpha\,\sin n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha^2\beta^2}\,d\alpha\,d\beta = o(mn)\\ v''_{mn} &= \int\int \cos m\alpha\,\sin n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha\beta^2}\,d\alpha\,d\beta = o(mn)\\ v'''_{mn} &= \int\int \sin m\alpha\,\cos n\beta\,\frac{\mathcal{F}(\alpha\beta)}{\alpha^2\beta}\,d\alpha\,d\beta = o(mn).\end{aligned}$$

In considering v'_{mn}, we break up the region of integration as follows:

$$\text{(92)}\quad \int_0^a\int_0^b = \int_0^{1/m}\int_0^{1/n} + \int_0^{1/m}\int_{1/n}^{\varepsilon_2} + \int_{1/m}^{\varepsilon_1}\int_0^{1/n} + \int_{1/m}^{\varepsilon_1}\int_{1/n}^{\varepsilon_2} + \int_{\varepsilon_1}^{a}\int_{\varepsilon_2}^{b} + \int_{\varepsilon_1}^{a}\int_0^{\varepsilon_2} + \int_0^{\varepsilon_1}\int_{\varepsilon_2}^{b},$$

each integral having the integrand of v'_{mn} in (91).

For the first integral in (92), we have

$$\text{(93)}\quad \int_0^{1/m}\int_0^{1/n} = mn\int_0^{1/m}\int_0^{1/n}\frac{\sin m\alpha}{m\alpha}\,\frac{\sin n\beta}{n\beta}\,\frac{\mathcal{F}(\alpha\beta)}{\alpha\beta}\,d\alpha\,d\beta.$$

$$= mn \int_0^{t_1}\int_0^{t_2} \frac{\mathcal{F}(\alpha\beta)}{\alpha\beta}\, d\alpha\, d\beta \qquad \left(\begin{matrix} 0 < t_1 < 1/m \\ 0 < t_2 < 1/n \end{matrix}\right),$$

since $\frac{\sin m\alpha}{m\alpha}$ decreases steadily from unity in the region of integration. The second term of (92) becomes

$$(94) \qquad \left| \int_0^{1/m}\int_{1/n}^{\varepsilon_2} \right| \leqq mn \int_0^{1/m}\int_{1/n}^{\varepsilon_2} \frac{|\mathcal{F}(\alpha\beta)|}{\alpha\beta}\, d\alpha\, d\beta$$

by treating the α-integration as before, and putting in the greatest value of $\sin n\beta$ and the least value of β. The third term reduces analogously to

$$(95) \qquad \left| \int_{1/m}^{\varepsilon_1}\int_0^{1/n} \right| \leqq mn \int_{1/m}^{\varepsilon_1}\int_0^{1/n} \frac{|\mathcal{F}(\alpha\beta)|}{\alpha\beta}\, d\alpha\, d\beta.$$

For the fourth term of (92) we have

$$(96) \qquad \left| \int_{1/m}^{\varepsilon_1}\int_{1/n}^{\varepsilon_2} \right| \leqq M \int_{1/m}^{\varepsilon_1}\int_{1/n}^{\varepsilon_2} \frac{d\alpha\, d\beta}{\alpha^2\beta^2} \leqq M\, mn,$$

where M is the maximum value of $|\mathcal{F}(\alpha\beta)|$ in the region $(0,0 — \varepsilon_1\varepsilon_2)$.

If now we choose, first ε_1 and ε_2, then m and n, we can make (93), (94), (95) and (96) each $o(mn)$. Also, since ε_1 and ε_2 are now fixed, we have the following data concerning the remaining terms of (92):

The fifth term is obviously $o(mn)$ by the generalized Riemann-Lebesgue theorem, since this integral is now a Lebesgue integral. For the sixth term of (92), (and the seventh reduces analogously), we have

$$(97) \qquad \left| \int_{\varepsilon_1}^{a}\int_0^{\varepsilon_2} \right| \leqq \frac{2n}{\pi\varepsilon_1} \int_{\varepsilon_1}^{a}\int_0^{\varepsilon_2} \frac{|\mathcal{F}(\alpha\beta)|}{\alpha\beta}\, d\alpha\, d\beta,$$

which is $o(mn)$. Hence we obtain the desired result in the case of v'_{mn}.

There is no difficulty in reducing the v''_{mn} and v'''_{mn} in analogous fashion. In fact, since α occurs only to the first power in v''_{mn}, we find that the reduced integrals will have no coefficient m, but will at worst have only a coefficient n; hence they are obviously $o(mn)$. The lemma is thus complete.

It is to Lemma 20 that we appeal in dealing with Cauchy integrals, rather than to the generalized Riemann-Lebesgue theorem.

§23. NECESSITY OF CAUCHY INTEGRABILITY OF $\varphi_{rs}(\alpha\beta)$, $\chi_r(\alpha)$ AND $\zeta_s(\beta)$.

Lemma 21-: If (i) $\varphi(\alpha\beta)$ is integrable (C), $\chi(\alpha)$ and $\zeta(\beta)$ are integrable (C); (ii) a_{mn}, $a_{\mu n}$, and $a_{m\kappa}$ are $o(1)$; and (iii) A is summable (C,r,s), A_c and A_ρ being summable (C,s) and (C,r) respectively; then $\varphi_{rs}(\alpha\beta)$ or $\psi_{rs}(\alpha\beta)$ are integrable (C), and $\chi_r(\alpha)$ and $\zeta_s(\beta)$ are integrable (C).

We first note that $\chi_r(\alpha)$ and $\zeta_s(\beta)$ are integrable (C) by Lemma 17 of the Hardy-Littlewood paper. For the proof of the lemma in the case of $\varphi_{rs}(\alpha\beta)$, we shall, as before, indicate the proof only in the cases of φ_{10}, φ_{01} and φ_1. Successive applications of these processes, or proper combinations of them, will serve to prove the lemma in its general aspect.

We put, for convenience,

$$(98)\qquad \begin{cases} \Phi_{11}(\alpha\beta) \equiv \int_0^\alpha\int_0^\beta \varphi(\alpha_1,\beta_1)\,d\alpha_1\,d\beta_1 \\ \Phi_{10}(\alpha\beta) \equiv \int_0^\alpha \varphi(\alpha_1\beta)\,d\alpha_1, \qquad \Phi_{01}(\alpha\beta) = \int_0^\beta \varphi(\alpha\beta_1)\,d\beta_1 \\ \Phi_{22}(\alpha\beta) \equiv \int_0^\alpha\int_0^\beta \varphi_1(\alpha_1\beta_1)\,d\alpha_1\,d\beta_1. \end{cases}$$

We shall prove first the integrability (C) of $\varphi_1(\alpha\beta)$. By an integration by parts,

$$(99)\qquad \begin{aligned} \int_0^a\int_0^b \varphi_1(\alpha\beta)\,d\alpha\,d\beta &= \int_{\varepsilon_1}^a\int_{\varepsilon_2}^b \frac{\Phi_{11}(\alpha\beta)}{\alpha\beta}\,d\alpha\,d\beta \\ &= \frac{\Phi_{22}(\alpha\beta)}{\alpha\beta}\Big]_{\varepsilon_1,\varepsilon_2}^{a,b} + \int_{\varepsilon_1}^a\int_{\varepsilon_2}^b \frac{\Phi_{22}(\alpha\beta)}{\alpha^2\beta^2}\,d\alpha\,d\beta \\ &\quad - \int_{\varepsilon_1}^a \left[\frac{\Phi_{22}(\alpha\beta)}{\alpha^2\beta}\right]_{\varepsilon_2}^b d\alpha - \int_{\varepsilon_2}^b \left[\frac{\Phi_{22}(\alpha\beta)}{\alpha\beta^2}\right]_{\varepsilon_1}^a d\beta \end{aligned}$$

The last two terms of (99), being single integrals, can be shown to exist in the limit by the processes of Lemma 17 of the Hardy-Littlewood paper, and therefore we have only to show that

(100) $$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \int_{\varepsilon_1}^{a} \int_{\varepsilon_2}^{b} \frac{\Phi_{22}(\alpha\beta)}{\alpha^2\beta^2}\, d\alpha\, d\beta$$

exists.

Since $\Phi_{11}(\alpha\beta)$ is continuous, it possesses a double Fourier's series which is uniformly summable (C,1); also, since it is periodic, we have that

(101) $$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_{11}(\alpha\beta) \sin m\alpha \sin n\beta \, d\alpha\, d\beta = \frac{1}{mn\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi(\alpha\beta) \cos m\alpha \cos n\beta \, d\alpha\, d\beta = \frac{a_{mn}}{mn}.$$

Hence we conclude that

(102) $$\Phi_{11}(\alpha\beta) = \sum\sum \frac{a_{mn}}{mn} \sin m\alpha \sin n\beta.$$

Since (102) is uniformly summable and its general term is $o\left(\frac{1}{mn}\right)$, it is uniformly convergent.

If we integrate (102) doubly, once with respect to each variable, divide by $\alpha^2\beta^2$ and integrate again in the same way, we have

(103) $$\int_{\varepsilon_1}^{a}\int_{\varepsilon_2}^{b} \frac{\Phi_{22}(\alpha\beta)}{\alpha^2\beta^2}\, d\alpha\, d\beta = \sum\sum \frac{a_{mn}}{m^2n^2} \int_{\varepsilon_1}^{a}\int_{\varepsilon_2}^{b} \frac{(1-\cos m\alpha)(1-\cos n\beta)}{\alpha^2\,\beta^2}\, d\alpha\, d\beta.$$

If we make a change of variable, putting $m\alpha = \alpha'$, $n\beta = \beta'$, we obtain

(103') $$\int_{\varepsilon_1}^{a}\int_{\varepsilon_2}^{b} \frac{\Phi_{22}(\alpha\beta)}{\alpha^2\beta^2} = \frac{1}{4}\sum\sum \frac{a_{mn}}{mn} \int_{m\varepsilon_1}^{ma}\int_{n\varepsilon_2}^{nb} \left(\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}} \frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^2 d\alpha\, d\beta.$$

Since A is summable, $\sum\sum \frac{a_{mn}}{mn}$ is summable, and it is convergent a s noted above; Hence, by Lemma 19, (103') is continuous for $\varepsilon_1, \varepsilon_2 = 0$. Therefore φ_1 is integrable (C).

The functions φ_{01} and φ_{10} are already integrable (C) as far as the α-and β-integrations respectively are concerned. Since they are defined interms of a single integration of φ, we can then use Lemma 17 of the Hardy-

Littlewood paper to conclude that they are integrable (C) with respect to their β and α integrations respectively. The lemma is thus complete.

§24. ANALOGUE OF LEMMA 15 FOR φ INTEGRABLE (C). Lemma 22-:If φ is integrable (C), the hypoheses of Lemma 21 being satisfied, then all the results of Lemma 15 hold without change.

In fact, we find that the reductions of Lemma 15 are all valid. In this case, the statement of (74) is still true if we use Lemma 20 instead of the Riemann-Lebesgue theorem; thus, for example,

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_{11}(\alpha\beta)\ \frac{\cos(M-\frac{1}{2})\alpha}{\alpha}\ \frac{\cos(N-\frac{1}{2})\beta}{\beta}\,d\alpha\,d\beta \rightarrow 0 \quad (C,1)$$

by Lemma 20, and hence

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_{11}(\alpha\beta)\ \frac{\cos(M-\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\ \frac{\cos(N-\frac{1}{2})\beta}{\sin\frac{1}{2}\beta}\,d\alpha\,d\beta \rightarrow 0 \quad (C,1);$$

therefore (74) is true (C,1), and we may even omit the (C,1) since $\frac{a_{\mu\nu}}{\mu\nu}$ is $o\left(\frac{1}{\mu\nu}\right)$. Moreover, Lemma 21 assures us that the γ's can be determined again so that the φ_{rs} satisfy all the conditions of the φ, so that we can make successive applications of the reasoning of Lemma 15, (i) and (ii), as before, to prove Lemma 22 in general.

§25. PROOF OF THEOREM IV IN GENERAL CASE, φ INTEGRABLE (C). We are now in position to prove Theorem IV in the general case where φ is integrable (C). The sufficiency of the conditions has already been disposed of, in §20. For the necessity, we simply replace Lemma 14 by Lemma 21, and Lemma 15 by Lemma 22, and carry out the argument of §17 precisely as before. Theorem IV is, then, completed.

APPENDIX

§26. GENERALIZATION OF A THEOREM OF FATOU. In the article cited in connection with Lemma 16, Fatou obtained a necessary and sufficient condition fot the convergence of a Fourier's series of a certain type. We base our Lemma 16 on a generalization to two variables of his result.

Let the double Fourier's series representing $f(\alpha\beta)$ be given by

$$(104)\qquad f(\alpha\beta) = \frac{a_{00}}{4} + \sum_1^\infty\sum_1^\infty \left\{ \begin{array}{l} a_{ij}\cos i\alpha\cos j\beta + b_{ij}\cos i\alpha\sin j\beta \\ + c_{ij}\sin i\alpha\cos j\beta + d_{ij}\sin i\alpha\sin j\beta \end{array} \right\}$$

$$= \frac{a_{00}}{4} + \sum_1^\infty\sum_1^\infty A_{ij}.$$

We let $g(\alpha\beta)$ represent the integral of (104) obtained by integration of (104) once with respect to each variable; thus

$$(105)\qquad g(\alpha\beta) = \frac{a_{00}\alpha\beta}{4} + \sum_1^\infty\sum_1^\infty \frac{1}{ij} \left\{ \begin{array}{l} a_{ij}\sin i\alpha\sin j\beta - b_{ij}\sin i\alpha\cos j\beta \\ - c_{ij}\cos i\alpha\sin j\beta + d_{ij}\cos i\alpha\cos j\beta \end{array} \right\}$$

The theorem we wish to prove is, then,

<u>Theorem V-: If a_{mn}, b_{mn}, c_{mn} and d_{mn} are each $o\left(\frac{1}{mn}\right)$ and if a_{hn} etc. (h fixed) and a_{mk} etc. (k fixed) are $o\left(\frac{1}{n}\right)$ and $o\left(\frac{1}{m}\right)$ respectively, then a necessary and sufficient condition that (104) should be convergent is that there should exist</u>

$$(106)\qquad L \equiv \lim_{x,y\to 0} \frac{g(\alpha+x,\beta+y) - g(\alpha+x,\beta) - g(\alpha,\beta+y) + g(\alpha\beta)}{xy} = f(\alpha\beta).$$

We first prove two lemmas:

<u>Lemma 23-: If the series</u>

(107) $$\varphi(x\,y) = \sum_1^{\infty}\sum_1^{\infty} c_{ij}\, x^i y^j$$

is convergent for $x < 1$, $y < 1$, and if $c_{mn} = o\left(\frac{1}{mn}\right)$, $c_{mk} = o\left(\frac{1}{m}\right)$, k fixed, and $c_{hn} = o\left(\frac{1}{n}\right)$, h fixed, then a necessary and sufficient condition for the convergence of (107) when $x = y = 1$ is that the value of the series should approach a finite limit as $x \to 1$, $y \to 1$ through real, proper-fractional values.*

We break up the sum in (107) into four parts:

(108) $$\sum_1^{\infty}\sum_1^{\infty} c_{ij}\, x^i y^j = \left\{\sum_1^{\mu}\sum_1^{\nu} + \sum_{\mu}^{\infty}\sum_{\nu}^{\infty} + \sum_1^{\mu}\sum_{\nu}^{\infty} + \sum_{\mu}^{\infty}\sum_1^{\nu}\right\} c_{ij}\, x^i y^j,$$

where, from the point (μ, ν) on, we have

(109) $$\begin{cases} i\,j\,|c_{ij}| < \varepsilon_1 \\ i\,|c_{ik}| < \varepsilon_2 \\ j\,|c_{hj}| < \varepsilon_3 . \end{cases}$$

We then have, for the second summation in (108), that

(110) $$\left|\sum_{\mu}^{\infty}\sum_{\nu}^{\infty} c_{ij}\, x^i y^j\right| < \frac{\varepsilon_1}{\mu\nu}\,\frac{x^{i-1}\, y^{j-1}}{(1-x)(1-y)} < \frac{\varepsilon_1}{\mu\nu(1-x)(1-y)} < \varepsilon_1'$$

if we make the substitution

$$x = 1 - \frac{1}{\mu}, \quad y = 1 - \frac{1}{\nu}.$$

With the same substitutions,

(111) $$\sum_1^{\mu}\sum_1^{\nu} c_{ij}\left(1 - \frac{1}{\mu}\right)^i\left(1 - \frac{1}{\nu}\right)^j = \sum_1^{\mu}\sum_1^{\nu} c_{ij} - \sum_1^{\mu} i\, c_{i1}\,\frac{\delta_{i1}}{\mu} - \sum_1 j\, c_{ij}\,\frac{\delta_{ij}}{\nu},$$

since μ and ν are large; the δ's are positive numbers less than unity. Hence we have

(112) $$\left|\sum_1^{\mu}\sum_1^{\nu}\right| < \left|\sum_1^{\mu}\sum_1^{\nu} c_{ij}\right| + \delta\left\{\sum_1^{\mu}\frac{i}{\mu}|c_{i1}| + \sum_1 \frac{j}{\nu}|c_{ij}|\right\},$$

---

* This lemma is a generalization to two variables of a theorem due to Pringsheim: cf. Fatou, loc. cit., p. 381.

where δ is the largest of the δ's occurring in (111). But since $i|c_{i1}|$ and $j|c_{1j}|$ approach zero with $1/\mu$ and $1/\nu$ respectively, the same is true of their mean values; hence the last term on the right-hand side of (112) can be made as small as we please:

$$(112') \qquad \left|\sum_1^{\mu}\sum_1^{\nu} c_{ij}\, x^i y^j\right| < \left|\sum_1^{\mu}\sum_1^{\nu} c_{ij}\right| + \varepsilon''.$$

For the third term of (108), we have

$$(113) \qquad \left|\sum_1^{\mu}\sum_{\nu}^{\infty} c_{ij}\left(1-\frac{1}{\mu}\right)^i\left(1-\frac{1}{\nu}\right)^j\right| < \left|\sum_1^{\mu}\sum_{\nu}^{\infty} c_{ij}\right| + \theta_1(\mu) + \theta_2(\nu),$$

where θ_1 and θ_2 are sums such as occurred in (111) and approach zero by our hypothesis. A similar reduction is valid for the fourth term of (108).

The net result of the foregoing reductions is that

$$(114) \qquad \left|\varphi\left(1-\frac{1}{\mu},\ 1-\frac{1}{\nu}\right) - \left\{\sum_1^{\mu}\sum_1^{\nu} + \sum_1^{\mu}\sum_{\nu}^{\infty} + \sum_{\mu}^{\infty}\sum_1^{\nu}\right\} c_{ij}\right|$$

can be made as small as we please; hence if one of the expressions in (114) approaches a limit as $\mu, \nu \to \infty$, the other does also. But as μ and $\nu \to \infty$, the last two sums in the second term of (114) approach zero, since the series converges by hypothesis; hence the lemma is complete.

<u>Lemma 24 -: If (104) is convergent and its coefficients satisfy the conditions of Theorem V, then there exists</u>

$$(115) \qquad \lim_{x,y\to 0} \frac{g(\alpha+x,\beta+y) - g(\alpha-x,\beta+y) - g(\alpha+x,\beta-y) + g(\alpha-x,\beta-y)}{4xy} = f(\alpha\beta).$$

By an easy reduction, we have that the fractional expression in (115) can be replaced by

$$(116) \qquad \sum_1^{\infty}\sum_1^{\infty} A_{ij}\ \frac{\sin ix}{ix}\ \frac{\sin jy}{jy}.$$

We consider this series as made up of the terms of (104) each multiplied by a factor

$$\frac{\sin ix}{ix}\;\frac{\sin jy}{jy} \qquad (i, j = 1, 2 \ldots).$$

Since (104) is convergent, i.e., summable (C,0), by hypothesis, and since the factors satisfy all conditions required of them to make them "convergence factors" *, we can infer that (104) and (116) have the same limits, and hence (115) is established.

The proof of Theorem V can now be completed. For the sufficiency, we use Lemma 23. Let α and β be arguments for which L exists; then, by hypothesis,

$$\lim_{r,s\to 1} \sum_1^\infty \sum_1^\infty A_{ij}\, r^i s^j = L,$$

and thence, by Lemma 23, (104) is convergent. The necessity of the condition (106) is a consequence of the combination of (115) with

$$(117)\quad \lim_{x,y\to 0} \frac{\left\{\begin{array}{l} g(\alpha+x,\beta+y)+g(\alpha+x,\beta-y)+g(\alpha-x,\beta+y)+g(\alpha-x,\beta-y) \\ +4g(\alpha\beta)-2g(\alpha+x,\beta)-2g(\alpha-x,\beta)-2g(\alpha,\beta+y)-2g(\alpha,\beta-y)\end{array}\right.}{xy}=0$$

(117) being a generalization to two variables of a result due to Riemann.** This combination results immediately in the necessity of (106), and Theorem V is complete.

Instead of requiring the existence of the limit in (106), we can replace that condition by the existence of (115), because of the validity of (117). But (115) immediately translates into

$$\frac{1}{4xy}\int_{\alpha-x}^{\alpha+x}\int_{\beta-y}^{\beta+y} f(\alpha\beta)\,d\alpha\,d\beta \to f(\alpha\beta) \quad \text{as } x,y\to 0,$$

*Theseconditions are given by C.N.Moore, "Convergence Factors in Multiple Series", as yet unpublished.
**H.Geiringer, loc. cit. p. 73.

which in turn is equivalent to

$$(118) \qquad \frac{1}{xy}\int_0^{\alpha+x}\int_0^{\beta+y} f(\alpha\beta)\,d\alpha\,d\beta \to f(\alpha\beta) \quad \text{as } x,y \to 0.$$

We thus have the following useful corollary to Theorem V :

Corollary : If the conditions of Theorem V are satisfied, then a necessary and sufficient condition that (104) should be convergent is that (118) should be true.

It is on this corollary that Lemma 16 is based.

Cincinnati, Ohio,
May 28, 1926.