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I hereby recommend that the thesis prepared under my supervision by JAMES MASSEY SHAHEEN

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ON THE THEORY OF NÖRLUND MEANS AND THEIR APPLICATION TO POWER SERIES AND FOURIER SERIES

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To them, and to the other members of the Mathematics Department, this thesis is dedicated, in appreciation for the invaluable mathematical training I have received.

James M. Shaheen

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Introduction

In the general theory of summability, two well-known methods for summing divergent series are the Cesàro method and the Hölder method, defined as follows:

Definition 1 A divergent series, whose nth partial sum is s_n, is said to be Cesàro summable, or (C,1) summable, to the value s if $\lim_{n\to\infty} \frac{s_0+s_1+\cdots+s_n}{n+1} = s$.

The same defining statement holds for the Hölder method. It is in their scale developments that the two methods differ. By a scale development we mean an iteration process which yields successively more and more powerful methods of summation in the sense that a higher order of the scale will sum divergent series that a lower order will not. Iteration processes, or scales, have been developed for several other methods of summing divergent series, as well as for the Cesàro and Hölder methods.

To obtain the Cesàro scale, let $s_n^{(0)} = s_n$, and let

$s_n^{(r)} = s_0^{(r-1)} + s_1^{(r-1)} + \cdots + s_n^{(r-1)}$, for r a positive integer.

The symbol γ_n^r is used to denote the binomial coefficient: $\gamma_n^r = \frac{(n+1)\cdots(n+r)}{1\cdots r}$.

Definition 2 A divergent series, whose nth partial sum is s_n, is said to be Cesàro summable of order r (r a positive integer), or (C,r) summable, to the value s if $\lim_{n\to\infty} \frac{s_n^{(r)}}{\gamma_n^{(r)}} = s$.

By using the properties of the binomial coefficients, the nth Cesàro mean of order r, that is, $\frac{S_n^{(r)}}{Y_n^{(r)}}$, may be expressed in terms of the original partial sums: $\frac{S_n^{(r)}}{Y_n^{(r)}} = \frac{\sum_{k=0}^{n} Y_{n-k}^{(r-1)} s_k}{Y_n^{(r)}}$. It is easily seen that the process of obtaining the rth order of the Cesàro scale involves r-1 summations followed by a single division. This scale has been extended to all integral and non-integral values of r greater than -1.

To obtain the Hölder scale, let $H_n^{(1)} = \frac{s_0+s_1+\cdots+s_n}{n+1}$, and let $H_n^{(r)} = \frac{H_0^{(r-1)}+H_1^{(r-1)}+\cdots+H_n^{(r-1)}}{n+1}$, for r a positive integer.

Definition 3 A divergent series, whose nth partial sum is s_n, is said to be Hölder summable of order r (r a positive integer), or (H,r) summable, to the value s if $\lim_{n\to\infty} \frac{H_n^{(r)}}{n+1} = s$.

This process of obtaining the rth order of the Hölder scale involves summation and division repeated r times. Unlike the Cesàro scale, we cannot obtain simple formulas which express the Hölder means of order r in terms of the original partial sums s_n. We can see from the construction of $H_n^{(r)}$ that such formulas are very complicated.

Another important method of summability is the Nörlund method defined as follows:

Definition 4 Let $p_0>0$, $p_n \geq 0$, $P_n = p_0+p_1+\cdots+p_n$, $\frac{p_n}{P_n} \to 0$.

A divergent series, whose nth partial sum is s_n, is said to be Nörlund summable, or (N, p_n) summable, to the value s if

$$\lim_{n \to \infty} \frac{p_0 s_n + p_1 s_{n-1} + \cdots + p_n s_0}{P_n} = s . ^{*}$$

We can readily see that definition 1 is a special case of definition 4 in which the arbitrary positive weights, p_n, are all taken to be 1. The literature on the Nörlund method of summability, however, contains no discussion of scales.

One of the accomplishments of this thesis is the development of two Nörlund scales, the first of which is analogous to, and includes as a special case, the Cesàro scale, and the second of which is analogous to, and includes as a special case, the Hölder scale. Although each order of the Nörlund scale transformation is itself a Nörlund transformation, a single sequence of positive numbers $\{p_n\}$ and a scale now sums a wider class of divergent series than was previously possible with this single sequence.

For these Nörlund scales, theorems about regularity, consistency, inclusion, a Tauberian theorem, etc., are developed in this thesis. A very interesting problem, suggested by Professor

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*The definition of summation was first given by G.F. Woronoi (22). (Numbers in parentheses are references to the bibliography.) Woronoi's paper was scarcely observed at the time of its appearance and was at any rate soon forgotten. It is customary to attach this definition to the name of N.E. Nörlund (17).

C.N. Moore, is the following: If $\{p_n\}$ and $\{q_n\}$ are two regular Nörlund sequences, considered as coefficients of two power series $\sum_{n=0}^{\infty} p_n z^n$, $\sum_{n=0}^{\infty} q_n z^n$, and if the functions associated with these series, $p(z)$, $q(z)$, have poles of order r and s respectively at $z=1$ and no others on the unit circle, then this should imply, if s $>$ r, that the second scale, with $\{q_n\}$, is stronger than the first, with $\{p_n\}$. With additional hypotheses, this problem is discussed in two theorems.

In the application of Nörlund means to power series, several theorems are developed which concern Nörlund summability of the series on the circle of convergence. One such theorem, important in itself, shows that the Nörlund limit and the functional limit are the same at a point on the circle provided we approach the point from inside the circle. This is a generalization of Hölder's theorem (12) for the Cesàro means of order r and requires only an additional hypothesis on the Nörlund function $p(z)$.

Another problem suggested by Professor C.N. Moore and discussed in this thesis is the summability of a series associated with a certain class of functions which have a singularity of a definite type on the circle of convergence.

In the field of Fourier series, the double conjugate Fourier series of a function $f(x,y)$, continuous throughout the fundamental square, is shown in this thesis to be summable, with regular Nörlund weights, to the conjugate function. Nörlund summability of the double conjugate Fourier series at a point of discontinuity of the function $f(x,y)$ is also discussed in a manner similar to Professor C.N. Moore's treatment (15) of the Cèsaro summability of the double Fourier series of discontinuous functions.

1. A Nörlund Scale Development

We begin with a sequence of positive numbers $\{p_n\}$ such that $p_0 > 0$, $p_n \geq 0$, $P_n = p_0 + p_1 + \cdots + p_n$, $\frac{p_n}{P_n} \to 0$. Let

[1.1] $$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad , \quad P(x) = \sum_{n=0}^{\infty} P_n x^n .$$

Since $\frac{P_{n-1}}{P_n} \to 1$, $P(x)$ converges for $|x| < 1$. Since $p(x) = (1-x)\sum_{n=0}^{\infty} P_n x^n$, $p(x)$ converges, for $|x| < 1$, to $(1-x)\, P(x)$.

Let $\sum_{n=0}^{\infty} a_n$ be a given series with partial sums $s_n = s_n^{(0)}$, and consider

[1.2] $$f(x) = \sum_{\nu=0}^{\infty} s_\nu^{(0)} x^\nu = \frac{1}{1-x} \sum_{\nu=0}^{\infty} a_\nu x^\nu .$$

Let $s_n^{(1)} = p_0 s_n^{(0)} + p_1 s_{n-1}^{(0)} + \cdots + p_n s_0^{(0)}$; then

[1.3] $$\sum_{\nu=0}^{\infty} p_\nu x^\nu \cdot \sum_{\nu=0}^{\infty} s_\nu^{(0)} x^\nu = \sum_{\nu=0}^{\infty} s_\nu^{(1)} x^\nu .$$

In general, let $s_n^{(r)} = p_0 s_n^{(r-1)} + p_1 s_{n-1}^{(r-1)} + \cdots + p_n s_0^{(r-1)}$, for r a positive integer. Let

[1.4] $$\left(p(x)\right)^r = \left(\sum_{\nu=0}^{\infty} p_\nu x^\nu\right)^r = \sum_{\nu=0}^{\infty} p_\nu^{(r)} x^\nu ,^{*}$$

where $\sum_{\nu=0}^{\infty} p_\nu^{(r)} x^\nu$ converges absolutely as long as $\sum_{\nu=0}^{\infty} p_\nu x^\nu$ itself does (13a).

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*Recurrence formulas for the evaluation of $p_n^{(r)}$ have been developed by J.W.L. Glaisher (7).

Then

[1.5] $$\sum_{\nu=0}^{\infty} p_\nu x^\nu \cdot \sum_{\nu=0}^{\infty} s_\nu^{(r-1)} x^\nu = \sum_{\nu=0}^{\infty} s_\nu^{(r)} x^\nu \quad .$$

The coefficients in the power series on the right side of [1.5] may be expressed in terms of the original partial sums, s_n , by making use of [1.4]:

[1.6] $$\sum_{\nu=0}^{\infty} s_\nu^{(r)} x^\nu = \sum_{\nu=0}^{\infty} p_\nu x^\nu \cdot \sum_{\nu=0}^{\infty} s_\nu^{(r-1)} x^\nu = \sum_{\nu=0}^{\infty} p_\nu^{(r)} x^\nu \cdot \sum_{\nu=0}^{\infty} s_\nu^{(0)} x^\nu$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\nu=0}^{n} p_\nu^{(r)} s_{n-\nu}^{(0)} \right) x^n \quad .$$

Consider $t_n^{(r)} = \dfrac{s_n^{(r)}}{\sum_{\nu=0}^{n} p_\nu^{(r)}}$, the Nörlund mean of order r in this scale development. If the limit of $t_n^{(r)}$ exists, as $n \to \infty$, the given series is said to be Nörlund summable of order r, or (N_r, p_n) summable,to this limit.

The Cesàro scale of integral order is a special case of this Nörlund scale (set $p_n = 1$ for all n).

2. Regularity

In this and later articles, we shall make use of several results that can readily be proved by induction. Noting that $p_n^{(k)} = p_0 p_n^{(k-1)} + \cdots + p_n p_0^{(k-1)}$, $p_0^{(k)} + \cdots + p_n^{(k)} = P_n p_0^{(k-1)} + \cdots + P_0 p_n^{(k-1)}$, and assuming that p_n and P_n tend to limits, finite or infinite, we **have**

Theorem 1 (a) $\dfrac{p_n^{(k)}}{\sum_{v=0}^{n} p_v^{(k)}} \to 0$ may be deduced from $\dfrac{p_n}{P_n} \to 0$.

(b) $\dfrac{\sum_{v=0}^{n-r} p_v^{(k)}}{\sum_{v=0}^{n} p_v^{(k)}} \to 1$, for each fixed r, follows from $\dfrac{P_{n-r}}{P_n} \to 1$.

(c) $\sum_{v=0}^{n} p_v^{(k)} \to \infty$ if $P_n \to \infty$.

The proofs are omitted. Henceforward, whenever reference is made to this theorem, the assumption is implicit that p_n and P_n tend to limits, finite or infinite.

Theorem 2 The (N_k, p_n) method is regular.

Proof: Noting that $s_n^{(k)} = \sum_{v=0}^{n} p_v^{(k)} s_{n-v}^{(0)}$ and that

$$t_n^{(k)} = \frac{s_n^{(k)}}{\sum_{v=0}^{n} p_v^{(k)}} = \sum_{v=0}^{n} \frac{p_{n-v}^{(k)}}{\sum_{\mu=0}^{n} p_\mu^{(k)}} s_v \,, \tag{2.1}$$

this transform can be shown to be a regular transform by an application of the Silverman-Toeplitz theorem (20) and part (a) of Theorem 1. Details are omitted.

The next theorem establishes the relative regularity of the (N_k, p_n) method when $P_n \to \infty$.

Theorem 3 If $\sum_{v=0}^{\infty} a_n = s(N_k, p_n)$ and $P_n \to \infty$, then $\sum_{v=0}^{\infty} a_v = s(N_{k+1}, p_n)$.

Proof: We have

$$t_n^{(k+1)} = \frac{s_n^{(k+1)}}{\sum_{v=0}^{n} p_v^{(k+1)}} = \frac{p_0 s_n^{(k)} + \cdots + p_n s_0^{(k)}}{p_0^{(k+1)} + \cdots + p_n^{(k+1)}}$$

$$= \frac{p_0 s_n^{(k)} + \cdots + p_n s_0^{(k)}}{p_0[p_0^{(k)} + \cdots + p_n^{(k)}] + \cdots + p_n p_0^{(k)}} = \frac{p_0 s_n^{(k)} + \cdots + p_n s_0^{(k)}}{p_0\left(\sum_{v=0}^{n} p_v^{(k)}\right) + \cdots + p_n p_0^{(k)}} \to s$$

since $\dfrac{s_n^{(k)}}{\sum_{v=0}^{n} p_v^{(k)}} \to s$ by hypothesis. We make use of the result

$\dfrac{y_0 + \cdots + y_n}{\alpha_0 + \cdots + \alpha_n} \to \xi$ provided $\dfrac{y_n}{\alpha_n} \to \xi$, and $\alpha_0, \alpha_1, \cdots, \alpha_n$ are any

positive numbers for which $\alpha_0 + \alpha_1 + \cdots + \alpha_n \to \infty$ (13b). Here, $P_n \to \infty$

implies $\alpha_n \equiv \sum_{v=0}^{n} p_v^{(k)} \to \infty$ by Theorem 1 (a).

We next prove a Tauberian theorem.

Theorem 4 If $\sum_{v=0}^{\infty} a_v = s\,(N_k, p_n)$ and $\dfrac{a_{n-r}}{p_{n-r}^{(k)}} \sum_{v=0}^{n-r} p_{n-v}^{(k)} = o(1)$

for each $r = 0, 1, 2, \cdots$, then $\sum_{v=0}^{\infty} a_v = s$, i.e., the series converges.

Proof: We have

$$s_n - t_n^{(k)} = s_n - \frac{s_n^{(k)}}{\sum_{v=0}^{n} p_v^{(k)}} = \frac{s_n\left(\sum_{v=0}^{n} p_v^{(k)}\right) - \left(p_0^{(k)} s_n + \cdots + p_n^{(k)} s_0\right)}{\sum_{v=0}^{n} p_v^{(k)}}$$

$$= \frac{s_n\left(p_0^{(k)} + \cdots + p_n^{(k)}\right) - \left(p_0^{(k)} s_n + \cdots + p_n^{(k)} s_0\right)}{\sum_{v=0}^{n} p_v^{(k)}} = \frac{p_1^{(k)}(s_n - s_{n-1}) + p_2^{(k)}(s_n - s_{n-2}) + \cdots + p_n^{(k)}(s_n - s_0)}{\sum_{v=0}^{n} p_v^{(k)}}$$

$$= \frac{p_1^{(k)} a_n + p_2^{(k)}(a_n + a_{n-1}) + \cdots + p_n^{(k)}(a_n + \cdots + a_1)}{\sum_{v=0}^{n} p_v^{(k)}} = \frac{a_n\left(p_1^{(k)} + \cdots + p_n^{(k)}\right) + a_{n-1}\left(p_2^{(k)} + \cdots + p_n^{(k)}\right) + \cdots + a_1 p_n^{(k)}}{p_0^{(k)} + \cdots + p_n^{(k)}}.$$

Since $t_n^{(k)} \to s$, a necessary condition for convergence is

$$\frac{a_n(p_1^{(k)}+\cdots+p_n^{(k)})+a_{n-1}(p_2^{(k)}+\cdots+p_n^{(k)})+\cdots+a_1\, p_n^{(k)}}{p_0^{(k)}+\cdots+p_n^{(k)}} = o(1)$$

. This condition is satisfied if $\dfrac{a_{n-r}(p_{r+1}^{(k)}+\cdots+p_n^{(k)})}{p_{n-r}^{(k)}} = o(1)$ for each $r = 0, 1, 2, \cdots$

since the $(N, p_n^{(k)})$ method is regular and since $\dfrac{\sum_{\nu=0}^{n-r} p_\nu^{(k)}}{\sum_{\nu=0}^{n} p_\nu^{(k)}} \to 1$,

for each fixed r, by Theorem 1 (b).

<u>Theorem 5</u> A necessary condition that $\sum_{\nu=0}^{\infty} a_\nu$ be (N_k, p_n) summable is

$$\frac{p_0^{(k)}a_n + p_1^{(k)}a_{n-1} + \cdots + p_n^{(k)}a_0}{\sum_{\nu=0}^{n} p_\nu^{(k)}} = o(1) .$$

<u>Proof</u>: We have

$$\left(\sum_{\nu=0}^{n} p_\nu^{(k)}\right)t_n^{(k)} - \left(\sum_{\nu=0}^{n-1} p_\nu^{(k)}\right)t_{n-1}^{(k)} = S_n^{(k)} - S_{n-1}^{(k)} = (p_0^{(k)}s_n+\cdots+p_n^{(k)}s_0) - (p_0^{(k)}s_{n-1}+\cdots+p_{n-1}^{(k)}s_0)$$

$$= p_0^{(k)}(s_n - s_{n-1}) + \cdots + p_{n-1}^{(k)}(s_1 - s_0) + p_n^{(k)}s_0 = p_0^{(k)}a_n + \cdots + p_n^{(k)}a_0 .$$

Therefore,

$$t_n^{(k)} - \frac{\sum_{\nu=0}^{n-1} p_\nu^{(k)}}{\sum_{\nu=0}^{n} p_\nu^{(k)}}\, t_{n-1}^{(k)} = \frac{p_0^{(k)}a_n + \cdots + p_n^{(k)}a_0}{\sum_{\nu=0}^{n} p_\nu^{(k)}} . \tag{2.2}$$

Since $t_n^{(k)} \to s$ and $\dfrac{\sum_{\nu=0}^{n-1} p_\nu^{(k)}}{\sum_{\nu=0}^{n} p_\nu^{(k)}} \to 1$ by Theorem 1(b), the left side of [2.2] tends to zero and the result follows.

<u>Theorem 6</u>: (a) If $\sum_{\nu=0}^{\infty} a_\nu = s\ (N_k, p_n)$, then $\sum_{\nu=1}^{\infty} a_\nu = s - a_0\ (N_k, p_n)$.

(b) If $\sum_{\nu=0}^{\infty} a_\nu = s\ (N_k, p_n)$ and $\sum_{\nu=0}^{\infty} b_\nu = t\ (N_k, p_n)$, then

$\sum_{\nu=0}^{\infty} (a_\nu \pm b_\nu) = s \pm t\ (N_k, p_n)$.

(c) If $\sum_{\nu=0}^{\infty} a_\nu = s\ (N_k, p_n)$, then $\sum_{\nu=0}^{\infty} c a_\nu = cs\ (N_k, p_n)$.

The proofs are omitted.

<u>Theorem 7</u> If $P_n \to \infty$, the condition $\frac{p_n}{P_n} \to 0$ is necessary and sufficient for the regularity of the (N_k, p_n) method.

<u>Proof</u>: By Theorem 1(a), $\frac{p_n^{(k)}}{\sum_{\nu=0}^{n} p_\nu^{(k)}} \to 0$ may be deduced from $\frac{p_n}{P_n} \to 0$ and the converse is also true when $P_n \to \infty$. If we replace the condition $\frac{p_n}{P_n} \to 0$ by the equivalent condition $\frac{p_n^{(k)}}{\sum_{\nu=0}^{n} p_\nu^{(k)}} \to 0$, then the remainder of the proof is similar to the proof of a special case ($k = 1$) of this theorem given in Hardy's Divergent Series, p.64-5, which makes use of the theorem of Silverman and Toeplitz (20).

3. <u>Consistency, Inclusion, Equivalence</u>

We say that two methods A and B are <u>consistent</u> if $s_n \to s\ (A)$, $s_n \to s'\ (B)$ imply $s' = s$, i.e., if they cannot sum the same series to different sums.

Theorem 8   Any two regular Nörlund methods are consistent: if $s_n \to s\ (N_k, p_n)$ and $s_n \to s'\ (N_k, q_n)$, then $s' = s$.

Proof: The proof is similar to the proof of a special case (k = 1) of this theorem given by Nörlund (17). Nörlund's proof also appears in Hardy's Divergent Series, p.65.

We say that the method Q includes the method P if $s_n \to s\,(P)$ implies $s_n \to s\ (Q)$ and that the methods are equivalent if each includes the other. If Q includes P but is not equivalent to P, we say Q is stronger than P.

We shall make use of the following results which are readily established.

(i) If (N_1, p_n) and (N_1, q_n) are regular, then $\frac{p_n}{P_n} \to 0$, $\frac{q_n}{Q_n} \to 0$, and $\frac{p_n^{(k)}}{\sum_{\nu=0}^{n} p_\nu^{(k)}} \to 0$, $\frac{q_n^{(k)}}{\sum_{\nu=0}^{n} q_\nu^{(k)}} \to 0$, by Theorem 1(a).

(ii) The series $p(x) = \sum_{\nu=0}^{\infty} p_\nu x^\nu$, $P(x) = \sum_{\nu=0}^{\infty} P_\nu x^\nu$, $q(x) = \sum_{\nu=0}^{\infty} q_\nu x^\nu$, $Q(x) = \sum_{\nu=0}^{\infty} Q_\nu x^\nu$, are convergent for $|x| < 1$.

(iii) The series $(p(x))^k = \sum_{\nu=0}^{\infty} p_\nu^{(k)} x^\nu$, $(P(x))^k = \sum_{\nu=0}^{\infty} P_\nu^{(k)} x^\nu$, $(q(x))^k = \sum_{\nu=0}^{\infty} q_\nu x^\nu$, $(Q(x))^k = \sum_{\nu=0}^{\infty} Q_\nu^{(k)} x^\nu$, are also convergent for $|x| < 1$ (13a).

(iv) The series $(k(x))^k = \sum_{v=0}^{\infty} k_v^{(k)} x^v = \frac{(q(x))^k}{(p(x))^k} = \frac{(Q(x))^k}{(P(x))^k}$ and

$(l(x))^k = \sum_{v=0}^{\infty} l_v^{(k)} x^v = \frac{(p(x))^k}{(q(x))^k} = \frac{(P(x))^k}{(Q(x))^k}$ are convergent for small x (3).

Equating coefficients in the power series identities in (iv), we get

$$\text{(v)} \qquad k_0^{(k)} p_n^{(k)} + \dots + k_n^{(k)} p_0^{(k)} = q_n^{(k)}, \qquad k_0^{(k)} P_n^{(k)} + \dots + k_n^{(k)} P_0^{(k)} = Q_n^{(k)},$$

$$l_0^{(k)} q_n^{(k)} + \dots + l_n^{(k)} q_0^{(k)} = p_n^{(k)}, \qquad l_0^{(k)} Q_n^{(k)} + \dots + l_n^{(k)} Q_0^{(k)} = P_n^{(k)}.$$

Theorem 8 If (N_k, p_n) and (N_k, q_n) are regular, then in order that (N_k, q_n) should include (N_k, p_n) it is necessary and sufficient that both

$$\text{(a)} \qquad |k_0^{(k)}| \sum_{v=0}^{n} p_v^{(k)} + |k_1^{(k)}| \sum_{v=0}^{n-1} p_v^{(k)} + \dots + |k_n^{(k)}| p_0^{(k)} \leq H \sum_{v=0}^{n} q_v^{(k)},$$

where H is independent of n, and

$$\text{(b)} \qquad \frac{k_n^{(k)}}{\sum_{v=0}^{n} q_v^{(k)}} \to 0 \quad .$$

If $\sum_{n=0}^{\infty} p_v^{(k)} \to \infty$, the second condition may be omitted.

Proof; The proof is a corollary of the theorem given by Marcel Riesz (18c) for any two regular Nörlund methods.

Theorem 9 In order that two regular Nörlund methods (N_k, p_n) and (N_k, q_n) should be equivalent, it is necessary and sufficient that $\sum_{\nu=0}^{\infty} |K_\nu^{(\kappa)}| < \infty$ and $\sum_{\nu=0}^{\infty} |\lambda_\nu^{(\kappa)}| < \infty$.

Proof: The proof is similar to the proof of a special case (k = 1) of this theorem given by M. Riesz (18c).

4. Inclusion Determined by the Poles of the Function

In this article we shall discuss the problem suggested by Professor C.N. Moore and stated in the Introduction. In order to do so we shall make use of Wiegert's Theorem (9)* which we now state and prove.

Wiegert's Theorem If $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ has the unit circle as the circle of convergence and if z = 1, a pole of order p (an integer) is the only singularity on $|z| = 1$, then $a_n = O(n^{p-1})$.

Proof: Write

$$f(z) = \frac{c_p}{(1-z)^p} + \frac{c_{p-1}}{(1-z)^{p-1}} + \cdots + \frac{c_1}{1-z} + g(z),$$

where $g(z)$ is regular for $|z| < 1 + \delta$, $\delta > 0$. Hence $g(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu$, where $b_n = o(1)$. Since $\frac{1}{(1-z)^p} = \sum_{n=0}^{\infty} \gamma_n^{p-1} z^n$, we compute

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*Wiegert's Theorem is stated but not proved either in this reference or in Titchmarsh's Theory of Functions, p214. So far as the writer knows , there is no proof anywhere in the literature.

$a_n = c_p \gamma_n^{p-1} + c_{p-1}\gamma_{n-1}^{p-1} + \cdots + c_1\gamma_n^0 + b_n$. Since $\gamma_n^\alpha \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$, $\alpha \neq -1, -2, \cdots$

we have $a_n = O(n^{p-1})$, which proves the theorem.

Let $p(x) = \sum_{\nu=0}^{\infty} p_\nu x^\nu$, $q(x) = \sum_{\nu=0}^{\infty} q_\nu x^\nu$, $P(x) = \sum_{\nu=0}^{\infty} P_\nu x^\nu$,

$Q(x) = \sum_{\nu=0}^{\infty} Q_\nu x^\nu$, each convergent for $|x| < 1$. Since $p(x) = (1-x)P(x)$,

and $q(x) = (1-x)\,Q(x)$, then $K(x) = \sum_{\nu=0}^{\infty} K_\nu x^\nu = \frac{q(x)}{p(x)} = \frac{Q(x)}{P(x)}$ is

convergent for small x (3). The coefficients K_ν satisfy the

equations $K_0 p_n + \cdots + K_n p_0 = q_n$, $K_0 P_n + \cdots + K_n P_0 = Q_n$.

Assume that $p(z)$ and $q(z)$ are rational and that $p(z)$ has no zeros inside or on the unit circle, or both $p(z)$ and $q(z)$ have the same zeros of the same order. Suppose $q(z)$ has a pole of order s and $p(z)$ has a pole of order r . Then we can write

$$K(z) = \frac{q(z)}{p(z)} = \frac{q_1(z)}{p_1(z)}\frac{(1-z)^r}{(1-z)^s} = \sum_{\nu=0}^{\infty} b_\nu z^\nu \cdot \sum_{\nu=0} \gamma_\nu^{s-r-1} z^\nu ,$$

where $\frac{q_1(z)}{p_1(z)}$ is analytic at z = 1 and its power series development,

$\sum_{\nu=0}^{\infty} b_\nu z^\nu$, converges for all z in and on $|z| = 1$. If we write

$K(z) = \sum_{\nu=0}^{\infty} K_\nu z^\nu = \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} b_\nu \gamma_{n-\nu}^{s-r-1}\right) z^\nu$, then $K_n = \sum_{\nu=0}^{n} b_\nu \gamma_{n-\nu}^{s-r-1}$.

<u>Theorem 10</u> If both $p(z) = \sum_{\nu=0}^{\infty} p_\nu z^\nu$ and $q(z) = \sum_{\nu=0}^{\infty} q_\nu z^\nu$ satisfy the hypotheses of Wiegert's Theorem, with $p(z)$ having a pole of order r at z = 1 and $q(z)$ having a pole of order s at z = 1, and if s > r, and $n|b_n| < M$, then (N_1, q_n) includes (N_1, p_n).

Proof: For inclusion, we must show (by Theorem 8, with k = 1) that $|K_0|P_n + |K_1|P_{n-1} + \cdots + |K_n|P_0 \leq H\, Q_n$. By Wiegert's Theorem, $p_n = O(n^{r-1})$, $q_n = O(n^{s-1})$, so that $P_n = O(1^{r-1} + \cdots + n^{r-1}) = O(n^r)$,

$Q_n = O(1^{s-1} + \cdots + n^{s-1}) = O(n^s)$. Furthermore, $K_n = b_0 \gamma_n^{s-r-1} + b_1 \gamma_{n-1}^{s-r-1} + \cdots + b_n \gamma_0^{s-r-1}$.

We must now show that $|K_0|P_n + |K_1|P_{n-1} + \cdots + |K_n|P_0 = O(n^s)$. We have

$$|K_0|P_n + \cdots + |K_n|P_0 \leq |b_0\gamma_0^{s-r-1}|P_n + |b_0\gamma_1^{s-r-1} + b_1\gamma_0^{s-r-1}|P_{n-1} + \cdots + |b_0\gamma_n^{s-r-1} + \cdots + b_n\gamma_0^{s-r-1}|P_0$$

$$\leq |b_0|\gamma_0^{s-r-1}P_n + (|b_0|\gamma_1^{s-r-1} + |b_1|\gamma_0^{s-r-1})P_{n-1} + \cdots + (|b_0|\gamma_n^{s-r-1} + \cdots + |b_n|\gamma_0^{s-r-1})P_0,$$

since $\gamma_n^{s-r-1} > 0$ for $s-r > 0$. Making use of the inequalities $\gamma_n^{s-r-1} < M_1 n^{s-r-1}$ and $P_n < M_2 n^r$, we have

$$|K_0|P_n + \cdots + |K_n|P_0 \leq M_3\left[|b_0|n^r + (|b_0|1^{s-r-1} + |b_1|)(n-1)^r + \cdots + (|b_0|n^{s-r-1} + |b_1|(n-1)^{s-r-1} + \cdots + |b_n|)\right]$$

$$\leq M_3\left[|b_0|(n^r + 1^{s-r-1}(n-1)^r + \cdots + n^{s-r-1}) + |b_1|((n-1)^r + 1^{s-r-1}(n-2)^r + \cdots + (n-1)^{s-r-1}) + \cdots + |b_n|\right].$$

From properties of the Beta Function, we can establish the inequality

[4.1] $$1^{s-r-1}(n-1)^r + \cdots + (n-1)^{s-r-1}1^r < M_4 n^s.$$

On making use of this result, we have

$$|K_0|P_n + \cdots + |K_n|P_0 \leq M_5\left[|b_0|n^s + |b_1|(n-1)^s + \cdots + |b_n|\right].$$

Since $|b_n| < M_6 n^{-1}$, then

$$|K_0|P_n + \cdots + |K_n|P_0 \leq M_7\left[n^s + 1^{-1}(n-1)^s + 2^{-1}(n-2)^s + \cdots + n^{-1}\right] = O(n^s),$$

on making use of [4.1] again. This completes the proof.

Next we show that the hypotheses of Theorem 10 imply that (N_k, q_n) includes (N_k, p_n). We first show that the hypotheses imply that (N_2, q_n) includes (N_2, p_n), and then we repeat the argument k times.

In view of Theorem 8, with k = 2, we must show that

$$|K_0^{(2)}| \sum_{v=0}^{n} p_v^{(2)} + |K_1^{(2)}| \sum_{v=0}^{n-1} p_v^{(2)} + \cdots + |K_n^{(2)}| p_0^{(2)} \leq H \sum_{v=0}^{n} g_v^{(2)} ,$$

where $g_n^{(2)} = g_0 g_n + \cdots + g_n g_0$, $p_n^{(2)} = p_0 p_n + \cdots + p_n p_0$, $K_n^{(2)} = K_0 K_n + \cdots + K_n K_0$.

Since $p(z) = \sum_{v=0}^{\infty} p_v z^v$ has a pole of order r at z = 1,

$(p(z))^2 = \sum_{v=0}^{\infty} p_v^{(2)} z^v$ has a pole of order 2r at z = 1. Similarly,

$(g(z))^2 = \sum_{v=0}^{\infty} g_v^{(2)} z^v$ has a pole of order 2s at z = 1, and 2s > 2r

since s > r. Both $(p(z))^2 = \sum_{v=0}^{\infty} p_v^{(2)} z^v$ and $(g(z))^2 = \sum_{v=0}^{\infty} g_v^{(2)} z^v$ satisfy

the hypotheses of Wiegert's Theorem, so that $p_n^{(2)} = O(n^{2r-1})$,

$g_n^{(2)} = O(n^{2s-1})$, $\sum_{v=0}^{n} g_v^{(2)} = O(1^{2s-1} + \cdots + n^{2s-1}) = O(n^{2s})$,

$\sum_{v=0}^{n} p_v^{(2)} = O(1^{2r-1} + \cdots + n^{2r-1}) = O(n^{2r})$. From the definition of $K(z)$,

$$(K(z))^2 = \left(\sum_{v=0}^{\infty} K_v z^v\right)^2 = \sum_{v=0}^{\infty} K_v^{(2)} z^v = \left(\frac{g(z)}{p(z)}\right)^2 = \frac{\sum_{v=0}^{\infty} g_v^{(2)} z^v}{\sum_{v=0}^{\infty} p_v^{(2)} z^v}$$

is convergent for small z (3). We assumed $p(z)$ and $g(z)$ rational and $p(z)$ has no zeros inside or on $|z| = 1$, or $p(z)$ and $g(z)$ have the same zeros of the same order. This assumption still holds for $(p(z))^2$ and $(g(z))^2$. Then

$$\sum_{v=0}^{\infty} K_v^{(2)} z^v = \left(\frac{g(z)}{p(z)}\right)^2 = \frac{g_1(z)}{p_1(z)} \frac{(1-z)^{2r}}{(1-z)^{2s}} = \left(\frac{g_1(z)}{p_1(z)}\right)^2 \frac{1}{(1-z)^{2s-2r}} = \sum_{v=0}^{\infty} b_v^{(2)} z^v \cdot \sum_{n=0}^{\infty} \gamma_n^{2s-2r-1} z^n ,$$

where $\left(\frac{g_1(z)}{p_1(z)}\right)^2$ is analytic at z = 1 since $\frac{g_1(z)}{p_1(z)}$ is analytic there.

The power series development $\left(\frac{g_1(z)}{p_1(z)}\right)^2 = \sum_{v=0}^{\infty} b_v^{(2)} z^v$ converges for all z in and on $|z| = 1$. From the series for $(k(z))^2$, we get

$$k_n^{(2)} = b_0^{(2)} \gamma_n^{2s-2r-1} + b_1^{(2)} \gamma_{n-1}^{2s-2r-1} + \cdots + b_n^{(2)} \gamma_0^{2s-2r-1} .$$

We assume $n|b_n^{(2)}| < M$ and apply Theorem 10 with r replaced by 2r and s replaced by 2s. Then this result, which is a necessary and sufficient condition that (N_2, g_n) include (N_2, p_n), follows:

$$|k_0^{(2)}| \sum_{v=0}^{n} p_v^{(2)} + |k_1^{(2)}| \sum_{v=0}^{n-1} p_v^{(2)} + \cdots + |k_n^{(2)}| p_0^{(2)} \leq H \sum_{v=0}^{n} g_v^{(2)} .$$

We repeat the above argument k times to obtain

<u>Theorem 11</u> If $p(z) = \sum_{v=0}^{\infty} p_v z^v$ and $g(z) = \sum_{v=0}^{\infty} g_v z^v$ have poles of order r and s, respectively, at z = 1, and no other singularities in and on $|z| = 1$, if s > r and $n|b_n^{(k)}| < M$, then (N_k, g_n) includes (N_k, p_n) provided $p(z)$ has no zeros in or on $|z| = 1$, or $p(z)$ and $g(z)$ have the same zeros of the same order.

Here, $p(z) = \frac{p_1(z)}{(1-z)^r}$, $g(z) = \frac{g_1(z)}{(1-z)^s}$ and the coefficients, $b_n^{(k)}$, are those in the power series development of the function $\left(\frac{g_1(z)}{p_1(z)}\right)^k = \sum_{v=0}^{\infty} b_v^{(k)} z^v$; the function is analytic at z = 1; the series converges for all z in and on $|z| = 1$.

We note that the second condition of Theorem 8 is easily seen to be satisfied. The condition is $k_n^{(k)} = o\left(\sum_{v=0}^{n} g_v^{(k)}\right)$. To prove this, we have, for k = 1, $K_n = O(n^{s-r})$, $Q_n = O(n^s)$. Thus $K_n = o(Q_n)$ since $s > r$ and $|K_n| < M n^{s-r} = o(n^s)$.

For any integral k > 1, we repeat the argument k times.

To illustrate Theorem 10, let us consider the case in which the Nörlund means are the Cesàro means. We have

$$p(z) = \frac{1}{(1-z)^r} = \sum_{v=0}^{\infty} \gamma_v^{r-1} z^v \quad , \quad q(z) = \frac{1}{(1-z)^s} = \sum_{v=0}^{\infty} \gamma_v^{s-1} z^v ,$$

$$p_n = \gamma_n^{r-1} = O(n^{r-1}) \quad , \quad q_n = \gamma_n^{s-1} = O(n^{s-1}) \quad ,$$

$$P_n = \sum_{v=0}^{n} \gamma_v^{r-1} = \gamma_n^r = O(n^r) \quad , \quad Q_n = \sum_{v=0}^{n} \gamma_v^{s-1} = \gamma_v^s = O(n^s),$$

$$K(z) = \sum_{v=0}^{\infty} k_v z^v = \frac{q(z)}{p(z)} = \frac{(1-z)^r}{(1-z)^s} = \frac{1}{(1-z)^{s-r}} = \frac{q_1(z)}{p_1(z)} \frac{1}{(1-z)^{s-r}} .$$

Here, $\frac{q_1(z)}{p_1(z)} = 1$, which in a power series $\sum_{v=0}^{\infty} b_v z^v$ has $b_0 = 1$ $b_n = 0$ for all $n \geq 1$. The hypothesis $n|b_n| < M$ is therefore satisfied. Thus

$$\sum_{v=0}^{\infty} k_v z^v = \sum_{v=0}^{\infty} b_v z^v \sum_{v=0}^{\infty} \gamma_v^{s-r-1} z^v = \sum_{v=0}^{\infty} \gamma_v^{s-r-1} z^v$$ and therefore $K_n = \gamma_n^{s-r-1}$;

$$|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0 = \gamma_0^{s-r-1}P_n + \gamma_1^{s-r-1}P_{n-1} + \cdots + \gamma_n^{s-r-1}P_0$$

$$\leq M_1\left(n^r + 1^{s-r-1}(n-1)^r + 2^{s-r-1}(n-2)^r + \cdots + n^{s-r-1}\right).$$

On making use of [4.1] , we have

$$|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0 \leq M_2 n^s = O(n^s) .$$

Therefore $|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0 \leq H Q_n$ since $Q_n = O(n^s)$.

Thus for integral r and s ,(C,s) summability includes (C,r) summability if $s > r$.

This example can be modified to illustrate Theorem 11.

Both Theorems 10 and 11 are restricted to integral r and s by Wiegert's Theorem. We can partially remove this restriction by the following consideration, and, specializing as above, show that (C,s) summability includes (C,r) summability for non-integral s and r, $s > r$.

<u>The Non-integral Case</u>

Consider λ_1, λ_2, positive, non-integral. Let $(1-x)^{-\lambda_1}(a_0+a_1x+a_2x^2+\cdots)=p_0+p_1x+p_2x^2+\cdots$, where $a_0>0$, $a_n\geq 0$. Since $\gamma_n^{\lambda_1-1}>0$, then $p_0>0$, $p_n\geq 0$. Suppose $a_n < M$ and assume $p_1(x)=\sum_{\nu=0}^{\infty} a_\nu x^\nu$ is convergent for $|x|<1+\delta$, $\delta>0$.

Since $\sum_{\nu=0}^{\infty}\gamma_\nu^{\lambda_1-1}x^\nu$ converges absolutely for $|x|\leq 1$, the product series $\sum_{\nu=0}^{\infty} p_\nu x^\nu$ also converges, for $|x|<1$. Introduce

$$\sum_{\nu=0}^{\infty} p_\nu x^\nu=\sum_{\nu=0}^{\infty}\gamma_\nu^{\lambda_1-1}x^\nu\cdot\sum_{\nu=0}^{\infty}a_\nu x^\nu=\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n}\gamma_\nu^{\lambda_1-1}a_{n-\nu}\right)x^n \text{ so that } p_n=\sum_{\nu=0}^{n}\gamma_\nu^{\lambda_1-1}a_{n-\nu}.$$

Since $a_n<M$, then $p_n=O(n^{\lambda_1})$, $P_n=O(1^{\lambda_1}+\cdots+n^{\lambda_1})=O(n^{\lambda_1+1})$.

Let $(1-x)^{-\lambda_2}(b_0+b_1x+b_2x^2+\cdots)=g_0+g_1x+g_2x^2+\cdots$, where $b_0>0$, $b_n\geq 0$. Since $\gamma_n^{\lambda_2-1}>0$, then $g_0>0$, $g_n\geq 0$. Assume $g_1(x)=\sum_{\nu=0}^{\infty}b_\nu x^\nu$ converges for $|x|<1+\delta$, $\delta>0$, and $|b_n|<M$.

Since $\sum_{\nu=0}^{\infty}\gamma_\nu^{\lambda_2-1}x^\nu$ converges absolutely for $|x|\leq 1$, the product series, $\sum_{\nu=0}^{\infty} g_\nu x^\nu$, converges absolutely for $|x| < 1$. We have

$$\sum_{\nu=0}^{\infty} g_\nu x^\nu = \sum_{\nu=0}^{\infty} b_\nu x^\nu \cdot \sum_{\nu=0}^{\infty} \gamma_\nu^{\lambda_2 - 1} x^\nu = \sum_{n=0}^{\infty} \Big(\sum_{\nu=0}^{n} b_\nu \gamma_{n-\nu}^{\lambda_2-1} \Big) x^\nu \;;\; g_n = \sum_{\nu=0}^{n} b_\nu \gamma_{n-\nu}^{\lambda_2 - 1}.$$

Since $b_n < M$, for all n, then $g_n = O(n^{\lambda_2})$ and $Q_n = O(1^{\lambda_2} + \cdots + n^{\lambda_2}) = O(n^{\lambda_2+1})$.

Now $p_1(z)$ and $g_1(z)$ are analytic in and on $|z| = 1$. Assuming $p_1(z)$ has no zeros in and on $|z| = 1$, or both $p_1(z)$ and $g_1(z)$ have the same zeros of the same order, we have

$$\frac{g_1(z)}{p_1(z)} = \frac{\sum_{\nu=0}^{\infty} b_\nu z^\nu}{\sum_{\nu=0}^{\infty} a_\nu z^\nu} = \sum_{\nu=0}^{\infty} c_\nu z^\nu,$$ where the power series $\sum_{\nu=0}^{\infty} c_\nu z^\nu$ converges

for all z in and on $|z| = 1$.

<u>Theorem 12</u> (a) If $p(x) = \sum_{\nu=0}^{\infty} p_\nu x^\nu$ and $g(x) = \sum_{\nu=0}^{\infty} g_\nu x^\nu$ are given as above, with $\lambda_2 > \lambda_1 > 0$, and if $n|c_n| < M$, then (N, g_n) includes (N, p_n).

(b) To show (N_k, g_n) includes (N_k, p_n), we require, in place of $n|c_n| < M$, that $n|c_n^{(k)}| < M$, where $\left(\frac{g_1(x)}{p_1(x)}\right)^k = \left(\frac{\sum_{\nu=0}^{\infty} b_\nu x^\nu}{\sum_{\nu=0}^{\infty} a_\nu x^\nu}\right)^k = \sum_{\nu=0}^{\infty} c_\nu^{(k)} x^\nu$.

<u>Proof</u>: The proof of (a) is similar to the proof of Theorem 10 with

$$\sum_{\nu=0}^{\infty} k_\nu x^\nu = \frac{g(x)}{p(x)} = \frac{g_1(x)}{p_1(x)} \frac{(1-x)^{\lambda_1}}{(1-x)^{\lambda_2}} = \sum_{n=0}^{\infty} c_n x^n \cdot \sum_{n=0}^{\infty} \gamma_n^{\lambda_2 - \lambda_1 - 1} x^n \;;\quad k_n = \sum_{\nu=0}^{n} c_\nu \gamma_{n-\nu}^{\lambda_2 - \lambda_1 - 1}.$$

The proof of (b) is similar to the proof of Theorem 11.

This concludes the discussion of the problem.

Some of the open questions concerning this Nörlund scale are now listed. First, an extension of this scale to non-integral orders would be valuable, and this should carry in its wake most, if not all, of the theorems developed here. Second, a comparison of the Nörlund scale with other summability scales would be interesting, and so would relief from the restriction that $\{p_n\}$ be a sequence of positive numbers. Perhaps more can be done with the above problem suggested by Professor Moore. Also, a collection of examples to illustrate the theory is always welcome.

An interesting question which involves the Nörlund scale is the determination of the orderof summability of the Cauchy product series of two given series, each known to be summable by some order of the scale. In this connection, much work has been done on multiplication theorems for Nörlund means (of order one) by Mears (16) and Silverman and Szasz (19). There is sufficient generality in these theorems to answer the question for the scale, by making the necessary changes in points of detail, in fact, to have many of their theorems carry over.

5. The Iterative Scale

In this article we shall develop a scale, analogous to the Hölder scale, in which the means are defined by a process of summation and division, summation and division, etc. This development differs from the first Cesàro-type scale where the means are defined by a process of k summations followed by a single division. We shall first state the definition for summability of order one and then for summability of order k. The superscript notation, $N^{(k)}$, will be used for this iterative scale.

Definition 5 Suppose $p_0 > 0$, $p_n \geq 0$, $P_n = p_0 + p_1 + \cdots + p_n$, $\frac{p_n}{P_n} \to 0$.

If $t_n^{(1)} = \dfrac{p_0 s_n + p_1 s_{n-1} + \cdots + p_n s_0}{p_0 + p_1 + \cdots + p_n} \to s$ when $n \to \infty$, we shall write

$\sum_{\nu=0}^{\infty} a_\nu = s(N^{(1)}, p_n)$, that is, $\sum_{\nu=0}^{\infty} a_\nu$ is $N^{(1)}$ summable withweights p_n.

Definition 6 We assume k is a positive integer. If

$t_n^{(k)} = \dfrac{p_0 t_n^{(k-1)} + \cdots + p_n t_0^{(k-1)}}{p_0 + \cdots + p_n} \to s$ when $n \to \infty$, we shall write

$\sum_{\nu=0}^{\infty} a_\nu = s(N^{(k)}, p_n)$, that is, $\sum_{\nu=0}^{\infty} a_\nu$ is $N^{(k)}$ summable with weights p_n.

The Hölder scale is a special case of this iterative scale (set $p_n = 1$ for all n).

We now list some examples of divergent series which are summable by this iterative scale.

Example 1 The series $\sum_{\nu=0}^{\infty} a_\nu$ for which $s_n = (-2)^n$, that is, $a_n = s_n - s_{n-1} = -3(-2)^{n-1}$, can be shown to be $(N^{(2)}, p_n)$ summable with weights $p_n = 2^{n+1}$. More generally, we have

Example 2 The series $\sum_{\nu=0}^{\infty} a_\nu$ for which $s_n = (-k)^n$, $k \neq 1, k > 0$, that is, $a_n = s_n - s_{n-1} = (-1)^n (k+1) k^{n+1}$ can be shown to be $(N^{(2)}, p_n)$ summable with weights $p_n = k^{n+1}$. For $k = 1$, the series for which $s_n = (-1)^n$ is $(N^{(1)}, p_n)$ summable to $\frac{1}{2}$ with weights $p_n = n+1$.

The series given in these examples are also summable $N^{(1)}$, but the weights are difficult to determine. The scale is very useful in eliminating this difficulty.

6. Regularity and Consistency

Theorem 13 The iterative method of summation is regular if and only if $\frac{p_n}{P_n} \to 0$.

Proof: We use the proof of the regularity of the $(N^{(1)}, p_n)$ method (8) and we repeat it k times to establish the regularity of the $(N^{(k)}, p_n)$ method.

Regularity assures us that $\sum_{\nu=0}^{\infty} a_\nu = s(N^{(k)}, p_n)$ implies $\sum_{\nu=0}^{\infty} a_\nu = s(N^{(k+1)}, p_n)$.

Theorem 14 (a) If $\sum_{\nu=0}^{\infty} a_\nu = s(N^{(k)}, p_n)$, then $\sum_{\nu=1}^{\infty} a_\nu = s - a_0(N^{(k)}, p_n)$.

(b) If $\sum_{\nu=0}^{\infty} a_\nu = s\ (N^{(k)}, p_n)$ and $\sum_{\nu=0}^{\infty} b_\nu = t\ (N^{(k)}, p_n)$, then $\sum_{\nu=0}^{\infty} (a_\nu \pm b_\nu) = s \pm t\ (N^{(k)}, p_n)$.

(c) If $\sum_{\nu=0}^{\infty} a_\nu = s\,(N^{(k)}, p_n)$, then $\sum_{\nu=0}^{\infty} c a_\nu = c s\ (N^{(k)}, p_n)$.

The proofs are omitted .

<u>Theorem 15</u> Any two regular iterative methods are consistent: if $\sum_{\nu=0}^{\infty} a_\nu = s\ (N^{(k)}, p_n)$ and $\sum_{\nu=0}^{\infty} a_\nu = s'\ (N^{(k)}, q_n)$, then $s' = s$.

<u>Proof:</u> On using the Silverman-Toeplitz Theorem (20), the proof is similar to the proof of consistency for the Cesàro-type scale (Theorem 8, page 12) or to the proof given in Hardy's Divergent Series, page 65, with s_n replaced by $t_n^{(k-1)}$.

Due to the manner in which the means are formed, the expression for $t_n^{(k)}$ in terms of $s_n (\equiv t_n^{(0)})$ is quite complicated:

$$t_n^{(k)} = \sum_{i_1=0}^{n} \frac{p_{i_1}}{P_n} t_{n-i_1}^{(k-1)} = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \frac{p_{i_1} p_{i_2}}{P_n P_{n-i_1}} t_{n-i_1-i_2}^{(k-2)} = \cdots\cdots$$

$$= \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \cdots\cdots \sum_{i_k=0}^{n-i_1-\cdots-i_{k-1}} \frac{p_{i_1} p_{i_2} \cdots p_{i_k}}{P_n P_{n-i_1} \cdots P_{n-i_1-\cdots-i_{k-1}}} t_{n-i_1-\cdots-i_k}^{(0)} .$$

This greatly restricts its use.

7. Application of Nörlund Means to Power Series

We shall develop, in this article, several theorems about Nörlund summablity of a power series on its circle of convergence. If the radius of convergence of the series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ is r, the transformation $z' = rz$ leads to a series with radius of convergence 1. Therefore we shall suppose without loss of generality that the radius of convergence is 1.

The first theorem to be developed states the conditions under which the Nörlund limit and the functional limit are the same at a point on the circle of convergence. An important theorem in itself, this result will be used in a later theorem.

<u>Theorem 16</u> If $e^{i\theta}$ is any fixed point on $|z| = 1$ and if $t_n^{(r)}(e^{i\theta}) \to L$ as $n \to \infty$ and $\frac{p(|z|)}{|p(z)|} < M$ as $z \to e^{i\theta}$ inside an angle $< \pi$, vertex at $e^{i\theta}$, then $\lim_{z \to e^{i\theta}} \sum_{\nu=0}^{\infty} a_\nu z^\nu = L$, provided $p(z)$ has no zeros in or on $|z| = 1$.

<u>Proof</u>: With no loss of generality, set $\theta = 0$. By hypothesis,

$$t_n^{(r)} = \frac{s_n^{(r)}}{\sum_{\nu=0}^{n} p_\nu^{(r)}} \to L .$$

This can also be expressed as

$$s_n^{(r)} = L \sum_{\nu=0}^{n} p_\nu^{(r)} + \epsilon_n \sum_{\nu=0}^{n} p_\nu^{(r)} ,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Therefore

[7.1] $$\frac{f(z)}{1-z}\left(p(z)\right)^r = \sum_{\nu=0}^{\infty} s_\nu^{(r)} z^\nu = L\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)z^n + \sum_{n=0}^{\infty}\epsilon_n\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)z^n.$$

If η_p is the greatest of $|\epsilon_{p+1}|$, $|\epsilon_{p+2}|$, ... , we have

[7.2] $$\left|\sum_{n=0}^{\infty}\epsilon_n\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)z^n\right| \le \sum_{n=0}^{p}|\epsilon_n|\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)|z|^n + \eta_p\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)|z|^n,$$

and

$$\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)|z|^n = \frac{1}{1-|z|}\left(p(|z|)\right)^r .$$

Multiplying both sides of [7.1] by $\frac{|1-z|}{(p(z))^r}$, we have

$$\frac{f(z)|1-z|(p(z))^r}{(1-z)\,(p(z))^r} = \frac{L\,|1-z|}{(p(z))^r}\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)z^n + \frac{|1-z|}{(p(z))^r}\sum_{n=0}^{\infty}\epsilon_n\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)z^n,$$

$$\frac{f(z)\,|1-z|}{(1-z)} = \frac{L\,|1-z|}{(p(z))^r}\,\frac{(p(z))^r}{(1-z)} + \frac{|1-z|}{(p(z))^r}\sum_{n=0}^{\infty}\epsilon_n\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)z^n ,$$

$$\left(f(z)-L\right)\frac{|1-z|}{(1-z)} = \frac{|1-z|}{(p(z))^r}\sum_{n=0}^{\infty}\epsilon_n\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)z^n .$$

Making use of [7.2],we have

[7.3] $$|f(z)-L| \le \frac{|1-z|}{|p(z)|^r}\sum_{n=0}^{p}|\epsilon_n|\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)|z|^n + \frac{|1-z|}{|p(z)|^r}\eta_p\sum_{n=0}^{\infty}\left(\sum_{0}^{n} p_\nu^{(r)}\right)|z|^n.$$

When $z\to 1$ in any way, the first term on the right tends to zero for a fixed p. Setting $z = 1-\rho e^{i\alpha}$, we have $\cos\alpha > \delta > 0$ inside an angle $< \pi$, vertex at 1. Hence, for $\rho<\delta$,

$$|z|^2 = 1-2\rho\cos\alpha+\rho^2 < 1-2\rho\delta+\rho\delta = 1-\rho\delta < 1-\rho\delta+\frac{\rho^2\delta^2}{4} = \left(1-\frac{\rho\delta}{2}\right)^2,$$

that is, $|z| < 1-\frac{\rho\delta}{2} = 1-\frac{\delta\,|1-z|}{2}$ and thus $\frac{|1-z|}{1-|z|} < \frac{2}{\delta}$.

Therefore the second term on the right side of [7.3] becomes

$$\frac{|1-z|}{|p(z)|^r}\eta_p \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} p_\nu^{(r)}\right)|z|^n = \frac{|1-z|}{|p(z)|^r}\eta_p \frac{1}{1-|z|}\sum_{n=0}^{\infty} p_n^{(r)}|z|^n = \frac{|1-z|}{|p(z)|^r}\eta_p \frac{1}{1-|z|}\left(p(|z|)\right)^r.$$

Since $\left(\frac{p(|z|)}{|p(z)|}\right)^r$ is bounded, by hypothesis, and η_p is arbitrarily small, the right side of [7.3] is arbitrarily small, and the theorem is established.

If we consider Cesàro means of order r in this theorem, in place of Nörlund means of order r, the second hypothesis becomes $\frac{p(|z|)}{|p(z)|} = \left(\frac{|1-z|}{1-|z|}\right)^{r+1} < \left(\frac{2}{\delta}\right)^{r+1}$, and can be omitted. The theorem for Cesàro means of order r was first proved by Abel (1) in 1826 for r = 0 (convergence) at least for radial approach. Frobenius (6), in 1880, proved the case r = 1, and Hölder (12), in 1882, proved the general result. Theorem 16 is important as a generalization of the Cesàro result, even in cases in which the Nörlund means are weaker than the Cesàro means.

<u>Theorem 17</u>. If $|t_n^{(k)}(e^{i\theta})| < M$ independently of θ, then $|f(z)| \leq M$ for $|z| < 1$.

<u>Proof</u>: By definition,

$$\frac{f(re^{i\theta})}{1-r}(p(r))^k = \sum_{n=0}^{\infty} s_n^{(k)}(e^{i\theta})r^n = \sum_{n=0}^{\infty}\frac{s_n^{(k)}(e^{i\theta})}{\sum_{\nu=0}^{n} p_\nu^{(k)}}\left(\sum_{\nu=0}^{n} p_\nu^k\right)r^n.$$

By hypothesis,

$$\left|\frac{s_n^{(k)}(e^{i\theta})}{\sum_{\nu=0}^{n} p_\nu^{(k)}}\right| \leq M .$$

Hence, $$\frac{|f(re^{i\theta})|}{|1-r|}(p(r))^k \le M\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} p_\nu^{(k)}\right)r^n \quad .$$

Simplifying, we get
$$\frac{f(re^{i\theta})}{|1-r|}(p(r))^k \le M\frac{1}{1-r}\sum_{\nu=0}^{\infty} p_\nu^{(k)} r^n = \frac{M}{1-r}\left(p(r)\right)^k.$$

Finally, $|f(re^{i\theta})| \le M$ and the theorem is established.

Theorem 18 If $|f(z)|<M_1$ in $|z|<1$, then $|t_n^{(1)}(e^{i\theta})|<M_2$.

Proof: The proof is similar to Landau's proof (14) for (C,1) summability. From
$$\frac{f(z)p(z)}{1-z} = \sum_{n=0}^{\infty} z^n\sum_{n=0}^{\infty} a_n z^n\sum_{n=0}^{\infty} p_n z^n = \sum_{n=0}^{\infty} s_n z^n\sum_{n=0}^{\infty} p_n z^n = \sum_{n=0}^{\infty} P_n t_n^{(1)}(1) z^n,$$
we have
$$P_n t_n^{(1)}(1) = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)\,p(z)}{(1-z)z^{n+1}}\,dz \quad .$$

But we can add $f(z)P(z)(p_0+p_1z+\cdots+p_nz^n)$ to the integrand without altering the value of the integral since this term is regular at the origin and hence in $|z|=1$. Thus we can write
$$P_n t_n^{(1)}(1) = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)P(z)}{z^{n+1}}\left[(p_0+p_1z+\cdots+p_nz^n)z^{n+1}+1\right]dz \quad ,$$
and since $|f(z)|<M_1$, $|P(z)|<M_2$, in $|z|\le r<1$, we have, on setting $z=re^{i\theta}$,
$$\begin{aligned} P_n t_n^{(1)}(1) &\le \frac{M}{2\pi r^n}\int_0^{2\pi}\left\{|p_0+p_1z+\cdots+p_nz^n|r^{n+1}+1\right\}d\theta \\ &\le \frac{M}{2\pi r^n}\int_0^{2\pi}\left\{p_0+\cdots+p_nr^n\right\}d\theta + \frac{M}{r^n} \\ &\le \frac{M}{2\pi r^n}\cdot 2\pi\left(p_0+p_1r+p_2r^2+\cdots+p_nr^n\right) + \frac{M}{r^n} \quad . \end{aligned}$$

Since the left side is independent of r < 1, we have

$$P_n t_n^{(1)}(1) \le \lim_{r\to 1}\left[\frac{M}{r^n}(p_0+p_1r+\cdots+p_nr^n)+\frac{M}{r^n}\right] = MP_n + M = M(P_n+1).$$

Therefore $|t_n^{(1)}(1)| \le M_1$. Applying this result to the function $f_1(z) = f(e^{i\theta}z)$, we have $|t_n^{(1)}(e^{i\theta})| \le M_1$.

We now generalize a result due to M. Riesz (18a),(18b). Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1. Let

$$T(z) = \sum_{n=0}^{\infty}(t_n^{(1)}-t_{n-1}^{(1)})z^n = (1-z)\sum_{n=0}^{\infty} t_n^{(1)}z^n = (1-z)\sum_{n=0}^{\infty}\frac{P_n t_n^{(1)} z^n}{P_n} = (1-z)\sum_{n=0}^{\infty} P_n t_n^{(1)} z^n \sum_{n=0}^{\infty} b_n z^n,$$

where the b_ν's are defined by $\sum_{\nu=0}^{n} P_\nu t_\nu^{(1)} b_{n-\nu} = t_n^{(1)}$, $b_0 = \frac{1}{p_0} \ne 0$.

At the origin, $b(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu \ne 0$.

<u>Theorem 19</u> If $\frac{s_n}{P_n} \to 0$, the Nörlund means, $t_n^{(1)}(e^{i\theta})$, tend to a limit at every regular point of $p(z)f(z)$ which is not a singular point of $b(z)$ or $T(z)$.

<u>Proof</u>: With no loss of generality, set $\theta = 0$. We have

$$|t_n^{(1)} - t_{n-1}^{(1)}| = \left|\frac{p_0 s_n + \cdots + p_n s_0}{P_n} - \frac{p_0 s_{n-1} + \cdots + p_{n-1}s_0}{P_{n-1}}\right|$$

$$= \left|\frac{p_0 s_n}{P_n} + \frac{(p_1P_{n-1}-p_0P_n)s_{n-1} + (p_2P_{n-1}-p_1P_n)s_{n-2} + \cdots + (p_nP_{n-1}-p_{n-1}P_n)s_0}{P_nP_{n-1}}\right|$$

$$= \left|-\frac{\left(p_0-p_1\frac{P_{n-1}}{P_n}\right)s_{n-1} + \left(p_1-p_2\frac{P_{n-1}}{P_n}\right)s_{n-2} + \cdots + \left(p_{n-1}-p_n\frac{P_{n-1}}{P_n}\right)s_0}{P_n} + \frac{p_0 s_n}{P_n}\right| .$$

Therefore

$$\left| t_n^{(1)} - t_{n-1}^{(1)} \right| \leq \frac{p_0 \frac{|s_{n-1}|}{P_{n-1}} + p_1 \frac{|s_{n-2}|}{P_{n-2}} + \cdots + p_{n-1} \frac{|s_0|}{P_n}}{P_{n-1}} + p_0 \frac{|s_n|}{P_n} .$$

Since $\frac{|s_n|}{P_n} \to 0$, its Nörlund mean also tends to zero, and consequently $t_n^{(1)} - t_{n-1}^{(1)} \to 0$. Since

$$T(z) = \sum_{n=0}^{\infty} (t_n^{(1)} - t_{n-1}^{(1)}) z^n = (1-z) \sum_{n=0}^{\infty} P_n t_n^{(1)} z^n \sum_{n=0}^{\infty} b_n z^n = p(z) f(z) b(z) ,$$

and since the radii of convergence of $\sum_{\nu=0}^{\infty} (t_n^{(1)} - t_{n-1}^{(1)}) z^n$ and $\sum_{\nu=0}^{\infty} P_n t_n^{(1)} z^n$ are both 1, the radius of convergence of $\sum_{\nu=0}^{\infty} b_\nu z^\nu$ is at least 1. For if it were less than 1, we may be led to a possible contradiction: the radius of convergence of the Cauchy product series of two given series is at least as great as the smaller radius of convergence. Since $\frac{T(z)}{b(z)} = p(z) f(z)$, then $\frac{T(z)}{b(z)}$ is analytic whenever $p(z) f(z)$ is analytic so that $T(z)$ and $b(z)$ must have the same singularities of the same order on the unit circle, if they have any at all. Assuming $z = 1$ is a regular point of $p(z) f(z)$ and not a singular point of $T(z)$ or $b(z)$, then, by Fatou's theorem (5)*, we have $\sum_{\nu=0}^{\infty} (t_n^{(1)} - t_{n-1}^{(1)}) < \infty$. The partial sums are precisely $t_n^{(1)}$ which must tend to a limit. This completes the proof.

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*Fatou's Theorem: If $a_n \to 0$, then the series $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ converges at every regular point of $|z| = 1$.

We next generalize a result due to Dienes (4d),(4e), and this generalization may be considered a partial converse of Theorem 16. We shall make use of the following lemmas (references to their proofs are given):

<u>Lemma 1</u> (4e) If, along every path inside $|z|=1$ leading to $e^{i\phi}$, $\lim_{z\to e^{i\phi}} |f(z)| \leq M$, then, to any given η, there is an arc (α, β) containing ϕ such that when z tends inside $|z|=1$ to any point of (α, β), we have $\lim_{z\to e^{i\phi}} |f(z)| \leq M+\eta$.

<u>Lemma 2</u> (4e) If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is bounded in a sector (0;a,b), then $f(z)=\sum_{\nu=0}^{\infty} b_\nu z^\nu + \sum_{\nu=0}^{\infty} c_\nu z^\nu = f_1(z)+f_2(z)$, where $f_2(z)$ is regular on the open arc (a,b) of $|z|=1$, and $b_n \to 0$.

<u>Lemma 3</u> (4e) If $a_n \to 0$ and along every path inside $|z|=1$ leading to $e^{i\phi}$, $\lim_{z\to e^{i\phi}} f(z)=A$, then $s_n^{(1)}(e^{i\phi})\to A$, where $s_n^{(1)}$ are the arithmetic means.

<u>Lemma 4</u> A regular Nörlund method (N_1, p_n) includes (C,1) if

$$n p_0 + \sum_{k=0}^{n-1} (n-k)|p_k - p_{k-1}| < A_1 P_n .$$

The proof is omitted.

<u>Theorem 20</u> If $\frac{a_n}{P_n} \to 0$, and along every inner path leading

to $e^{i\theta}$, $\lim_{z \to e^{i\theta}} f(z) = A$, then $t_n^{(1)}(e^{i\theta}) \to A$, provided

$\sum_{k=1}^{n-1} (n-k)|p_k - p_{k-1}| = O(P_n)$, and $e^{i\theta}$ is a regular point of $p(z)f(z)$

satisfying the hypotheses of Theorems 16 and 19.

Proof: By Lemma 1, $f(z)$ is bounded in a sector (0;a,b), the arc (a,b) containing θ, and hence, by Lemma 2, we can decompose it into the sum $f_1(z)+f_2(z)$. Since $f_2(z)$ is regular on the open arc (a,b), and $b_n \to 0$, i.e., $\frac{c_n}{P_n} \to 0$, then Theorem 19 applies since $\frac{C_n}{P_n} \to 0$ ($C_n = c_0 + c_1 + \cdots + c_n$) implies $\frac{c_n}{P_n} \to 0$. Thus the Nörlund means of the partial sums of $f_2(z)$ at $e^{i\theta}$ converge to $f_2(e^{i\theta})$. On the other hand, the hypothesis on the p_ν's implies the hypothesis of Lemma 4, so that from Lemmas 3 and 4 together, we get the Nörlund means of the partial sums converging at $e^{i\theta}$. Putting these results together, and noting that the functional limit and the limit of the Nörlund means are the same, we get the theorem.

Singularities on the Circle of Convergence (4b),(4c)

We consider a function $f(z)$ and its associated series, $\sum_{\nu=0}^{\infty} a_\nu z^\nu$, on the circle of convergence. Suppose in the neighborhood of $e^{i\theta}$,

$$f(z) = \frac{P_g\left(\log \frac{1}{1-e^{-i\theta}z}\right)}{(1-e^{-i\theta}z)^{\rho}} + f_1(z) \quad ,$$

where $P_q(z) = A_q z^q + A_{q-1} z^{q-1} + \cdots + A_0$, and the order ρ' of $f_1(z)$ at $e^{i\theta}$ is $<\rho$ ($\rho > 0$). We shall discuss the summability of the series associated with this class of functions at points on the circle of convergence. This problem was suggested by Professor C.N. Moore and is discussed completely.

Case 1: $0 < \rho < 1$ The coefficients, a_n, of the series tend to zero since the function is of order $\rho < 1$ on the entire circle of convergence ($|z| = 1$): a_n behaves like $\gamma_n^{\rho-1}$, the dominant contributor, (since log z is weaker than any power of z), which tends to zero for $0 < \rho < 1$. In 1911, Dienes (4c) proved that, for the case $0 < \rho < 1$, $\lim_{n \to \infty} \frac{s_n(e^{i\theta})}{n^\rho \log^q n} = \frac{A_q}{\Gamma(q+1)}$. At regular points of the function on the circle of convergence, if $0 < \rho < 1$, i.e., $a_n \to 0$, the series is convergent by Fatou's Theorem (5). At the singular point, $e^{i\theta}$, the series is not Nörlund summable since $s_n(e^{i\theta})$ becomes infinite like $n^\rho \log^q n$, and the Nörlund method is totally regular.

Case 2: $\rho \geq 1$ The coefficients, a_n, behave like $\gamma_n^{\rho-1}$, which becomes infinite with n. The circle of convergence of $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ is again the unit circle. However, $\frac{a_n}{n^r} \to 0$ for $r > \rho - 1$, i.e., $r+1 > \rho \geq 1$. A theorem due to M. Riesz (18a) is the basis for the next remark. Riesz has shown that if $\frac{b_n}{n^r} \to 0$ ($r > 0$), the arithmetic means of order r have a well determined limit at each regular point on the circle of convergence and this limit is the value of the function, $\phi(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu$, at this point.

In view of this result, the series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ for $f(z) = \dfrac{P_\delta\left(\log \frac{1}{1-e^{-i\theta}z}\right)}{(1-e^{-i\theta}z)^\rho} + f_1(z)$ is not (C, ρ) summable at regular points on the circle of convergence, since $\frac{a_n}{n^\rho}$ does not tend to zero, but it is (C, $\rho+\alpha$) summable for $\alpha > 0$. However we can find a Nörlund sequence $\{p_n\}$ which will interpolate precisely and still sum the series. Let

$$p_n = n^\rho \log^\delta n \quad , \quad p(z) = \sum_{n=0}^{\infty} n^\rho \log^\delta n \, z^n \, , \quad P(z) = \sum_{n=0}^{\infty} P_n z^n \, .$$

Consider a regular point z_0 $(\neq e^{i\theta}, \neq 1)$. The relation between the coefficients b_n and the order of the function , $\phi(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu$, on the entire circle of convergence is given by $\Omega = 1 + \overline{\lim_{n\to\infty}} \frac{\log |b_n|}{\log n}$, the order being Ω (4e). Thus the order of $f(z)$ on the circle of convergence is $\leq \rho+1$. We can write $f(zz_0) = f(z_0) + (1-z)\,\varphi_1(z)$, where $\varphi_1(z)$ is analytic at all regular points of $f(z)$ in and on the circle, except at $e^{i\theta}$ and 1. The function $\varphi_1(z)$ is also of order $\leq \rho+1$ on the entire circle. On multiplying the equation $f(zz_0) = f(z_0) + (1-z)\varphi_1(z)$ through by $P(z) = \sum_{\nu=0}^{\infty} P_\nu z^\nu$, we have

$$\sum_{\nu=0}^{\infty} P_\nu z^\nu \sum_{n=0}^{\infty} a_n z_0^n z^n \equiv \sum_{n=0}^{\infty} S_n^{(1)}(z_0) z^n = f(z_0) \sum_{n=0}^{\infty} P_n z^n + (1-z)\varphi_1(z) \sum_{n=0}^{\infty} P_n z^n .$$

Equating coefficients, we get $S_n^{(1)}(z_0) = f(z_0) P_n (1+\epsilon_n)$, since the order of $(1-z)\varphi_1(z)\, P(z) = p(z)\varphi_1(z)$ is one less than the order of $P(z)$ on the entire circle except at the point 1, which we avoid. The function $\varphi_1(z)$ is analytic at z_0 and does not influence the order. The ratio of the Taylor coefficients of the second member to the first therefore tend to zero. Hence $t_n^{(1)}(z_0) = \frac{S_n^{(1)}(z_0)}{P_n} \to f(z_0)$, and we have proved

Theorem 21 At a regular point z_0 of $f(z)$, on the circle of convergence, the series $\sum_{n=0}^{\infty} a_n z^n$ is summable (N, $n^\rho \log^\delta n$) to $f(z_0)$.

In 1913, Dienes (4e) proved that if $\frac{a_n}{n^r} \to 0$, $r \geq 1$, and $f(z) = \frac{P_\delta\left(\log \frac{1}{1-e^{-i\theta}z}\right)}{(1-e^{-i\theta}z)^\rho} + f_1(z)$, and the order of $f_1(z)$ at $e^{i\theta}$ is $\rho' < \rho$ $(0 < \rho < r+1)$, then $\frac{S_n^{(r-\rho)}(e^{i\theta})}{n^\rho \log^\delta n} \to \frac{\Gamma(r-\rho+1)}{\Gamma(r+1)} p_\delta$.

For $\rho \geq 1$ and $r > \rho$, we have $r-\rho > 0$ and the Cesàro means of positive order become infinite like $n^\rho \log^\delta n$ at the singular point $e^{i\theta}$. Since the Cesàro and Nörlund methods are totally regular, the Cesàro means of any higher order become infinite, and so will any Nörlund means which are more general.

This completes the discussion of the problem.

8. Application of Nörlund Means to Fourier Series

In this article we shall develop several theorems about Nörlund summability of the double conjugate Fourier series, hereafter abbreviated D.C.F.S. The following preliminary material can be found in any paper or text which treats D.C.F.S. (2), (11a), (21).

The function $f(u,v)$ is assumed to be integrable over the fundamental square Q$[-\pi,-\pi;\ \pi,\pi]$ and to have period 2π in each variable. The double Fourier series (abbreviated D.F.S. hereafter) is denoted by $\sigma(f)$ and the rectangular partial sums by $S_{mn}(x,y;f)$.

We have

[8.1] $f(u,v) \sim$

$$\sum_{\substack{m \\ n}\}=0}^{\infty} \lambda_{mn}\left[a'_{mn}\cos mu \cos nv + b'_{mn}\sin mu \cos nv + c'_{mn}\cos mu \sin nv + d'_{mn}\sin mu \sin nv\right],$$

where

[8.2]
$$\lambda_{mn} = \begin{cases} \frac{1}{4} & \text{if} \quad m = n = 0 \\ \frac{1}{2} & \text{if} \quad m=0,\ n>0 \ \text{or} \ m>0,\ n=0 \\ 1 & \text{if} \quad m>0, \quad n>0 \end{cases}$$

and

[8.3]
$$\begin{cases} a'_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\cos mu \cos nv \, du\, dv \\ b'_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sin mu \cos nv \, du\, dv \\ c'_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\cos mu \sin nv \, du\, dv \\ d'_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sin mu \sin nv \, du\, dv \end{cases}.$$

We may set $\lambda_{mn} a'_{mn} = a_{mn}$, $\lambda_{mn} b'_{mn} = b_{mn}$,
$\lambda_{mn} c'_{mn} = c_{mn}$, $\lambda_{mn} d'_{mn} = d_{mn}$,

and use the unprimed coefficients. This we do for the D.C.F.S., which we take to be

[8.4] $\displaystyle\sum_{m,n=0}^{\infty} \lambda_{mn}\left[a'_{mn}\sin mu \sin nv - b'_{mn}\sin mu \cos nv - c'_{mn}\cos mu \sin nv + d'_{mn}\cos mu \cos nv\right],$

where the coefficients are given by [8.3] . With unprimed coefficients, the rectangular partial sums of the D.C.F.S. are

[8.5] $\qquad \bar{S}_{mn}(x,y;f) =$

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\left[a_{\mu\nu}\sin\mu x \sin\nu y - b_{\mu\nu}\cos\mu x \sin\nu y - c_{\mu\nu}\sin\mu x \cos\nu y + d_{\mu\nu}\cos\mu x \cos\nu y\right].$$

On changing to the integral form for the coefficients, we have

$$\bar{S}_{mn}(u,v;f) = \frac{1}{\pi^2}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\cdot$$

$$\left[\cos\mu u \cos\nu v \sin\mu x \sin\nu y - \sin\mu u \cos\nu v \cos\mu x \sin\nu y - \cos\mu u \sin\nu v \sin\mu x \cos\nu y + \sin\mu u \sin\nu v \cos\mu x \cos\nu y\right] du\,dv$$

$$\bar{S}_{mn}(x,y;f) = \frac{1}{\pi^2}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\left[\sin\mu x \cos\mu u - \cos\mu x \sin\mu u\right]\cdot\left[\sin\nu y \cos\nu v - \cos\nu y \sin\nu v\right] du\,dv$$

$$\bar{S}_{mn}(x,y;f) = \frac{1}{\pi^2}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sin\mu(x-u)\sin\nu(y-v)\,du\,dv$$

$$\bar{S}_{mn}(x,y;f) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\left(\sum_{\mu=0}^{m}\sin\mu(x-u)\right)\left(\sum_{\nu=0}^{n}\sin\nu(y-v)\right) du\,dv$$

$$\bar{S}_{mn}(x,y;f) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\,\bar{D}_m(x-u)\bar{D}_n(y-v)\,du\,dv$$

[8.6] $\qquad \bar{S}_{mn}(x,y;f) = \dfrac{1}{\pi^2}\displaystyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,\,y+v)\,\bar{D}_m(u)\bar{D}_n(v)\,du\,dv \qquad .$

If we note that $\bar{D}_m(u) = \sum_{\mu=0}^{m} \sin \mu u$ is an odd function of u, it is not difficult to show that we can write [8.6] in the form

[8.7] $$\bar{S}_{mn}(x,y;f) = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} \psi_{xy}\, \bar{D}_m(u)\, \bar{D}_n(v)\, du\, dv ,$$

where $\psi_{xy} = f(x+u, y+v) - f(x-u, y+v) - f(x+u, y-v) + f(x-u, y-v)$.

We can form either the single or the double Nörlund mean* of the double sequence of partial sums [8.7]. We shall work with the single Nörlund means first. This Nörlund mean of the square partial sums, $s_{kk}(x,y;f)$, is

[8.8] $$t_n(x,y;f) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, s_{kk}(x,y;f) \quad .$$

Assuming (N, p_n) is a regular method, if $t_n(x,y;f)$ tends to a limit, as $n \to \infty$, the sequence of square partial sums is said to be (N, p_n) summable to this limit.

In integral form, the Nörlund mean [8.8] becomes

[8.9] $$t_n(x,y;f) = \int_0^{\pi}\int_0^{\pi} \psi_{xy}\, \bar{K}_n(u,v)\, du\, dv ,$$

where $\bar{K}_n(u,v) = \frac{1}{4\pi^2 P_n}\sum_{k=0}^{n} p_{n-k}\, \bar{D}_k(u)\, \bar{D}_k(v)$, and ψ_{xy} is given by [8.7].

Let

[8.10] $$s(u) = \lim_{k\to\infty} S_k(u) = \lim_{k\to\infty} \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2} \quad .$$

---

*Herriot (10a), (10b), considers both. The double Nörlund means are defined later, where they are first used.

The function , $S(u)$, is defined for all u since the series is convergent ($< \frac{\pi^2}{6}$) for all u. Let

$$F_K(x,y) = \int_0^{\pi}\int_0^{\pi} S_K(u)\, S_K(v)\, \psi_{xy}^{\prime * \prime *}\, du\, dv \quad ,$$

where $'$ and $*$ denote differentiation with respect to u and v , respectively. We now have the following theorem:

<u>Theorem 22</u> If, for each point (x_0, y_0) within $Q[-\pi,-\pi\,;\,\pi,\pi]$, the sequence $\{F_K(x_0,y_0)\}$ is Nörlund summable with weights p_n, then the Nörlund means of the D.C.F.S. tend to a limit which is the limit of the Nörlund mean of $\{F_K(x_0,y_0)\}$, provided $f^{\prime * \prime *}(u,v)$ exists and is integrable. If $\lim_{K\to\infty} F_K(x_0,y_0)$ exists, the D.C.F.S. converges to this limit.

<u>Proof</u>: We integrate $\int_0^{\pi}\int_0^{\pi} \psi_{xy}\, \bar{K}_n(u,v)\,du\,dv$ by parts four times.

We first integrate $\int_0^{\pi} \psi_{xy}\, \bar{K}_n(u,v)\,du$ by parts with $U = \psi_{xy}$,

$dV = K_n(u,v)\,du$; $dU = \psi'_{xy}\,du$, $V = \int \bar{K}_n(u,v)\,du$.

Then

[8.11] $$\int_0^{\pi}\int_0^{\pi} \psi_{xy}\bar{K}_n(u,v)\,du\,dv = \int_0^{\pi}\left[\left\{\psi_{xy}\int \bar{K}_n(u,v)\,du\right\}_0^{\pi} - \int_0^{\pi}\left\{\int \bar{K}_n(u,v)\,du\right\}\cdot \psi'_{xy}\,du\right]dv$$

$$= \int_0^{\pi}\left\{\psi_{xy}\Big|_0^{\pi}\int_0^{\pi} \bar{K}_n(u,v)\,du - \int_0^{\pi}\left[\int \bar{K}_n(u,v)\,du\right]\cdot \psi'_{xy}\,du\right\}dv .$$

On making use of the periodicity of $f(u,v)$ and the definition of ψ_{xy} , we can readily show that $\psi_{xy}\Big|_{u=0} = \psi_{xy}\Big|_{u=\pi} = 0$ so that $\psi_{xy}\Big|_0^{\pi} = 0$.

Moreover,

$$\int \bar{K}_n(u,v)\,du = \frac{1}{4\pi^2 P_n}\int \sum_{k=0}^{n} p_{n-k}\,\bar{D}_k(u)\bar{D}_k(v)\,du =$$

$$\frac{1}{4\pi^2 P_n}\sum_{k=0}^{n} p_{n-k}\,\bar{D}_{n-k}(v)\int \sum_{\nu=0}^{k} \sin \nu u\,du = -\frac{1}{4\pi^2 P_n}\sum_{k=0}^{n} p_{n-k}\,\bar{D}_k(u)\sum_{\nu=0}^{k}\frac{\cos \nu u}{\nu},$$

so that [8.11] becomes

[8.12] $$\int_0^\pi\int_0^\pi \psi_{xy}\,\bar{K}_n(u,v)\,du\,dv = \frac{1}{4\pi^2 P_n}\sum_{k=0}^{n} p_{n-k}\int_0^\pi \bar{D}_k(v)\left(\int_0^\pi \sum_{\nu=0}^{k}\frac{\cos \nu u}{\nu}\,\psi'_{xy}\right)dv .$$

To integrate the right side of [8.12] by parts, we set $dV = \bar{D}_k(v)dv$,

$$dU = \left\{\int_0^\pi \sum_{\nu=0}^{k}\frac{\cos \nu u}{\nu}\,\psi'^{*}_{xy}\,du\right\}dv\ ; \qquad V = \int \sum_{\nu=0}^{k}\sin \nu v\,dv = -\sum_{\nu=0}^{k}\frac{\cos \nu v}{\nu}\ , \quad U = \int_0^\pi \sum_{\nu=0}^{k}\frac{\cos \nu u}{\nu}\,\psi'_{xy}\,du .$$

Then [8.12] becomes

[8.13] $$\int_0^\pi\int_0^\pi \psi_{xy}\,\bar{K}_n(u,v)\,du\,dv =$$

$$\frac{1}{4\pi^2 P_n}\sum_{k=0}^{n} p_{n-k}\left\{\left[\left[\int_0^\pi \sum_{\nu=0}^{k}\frac{\cos \nu u}{\nu}\,\psi'_{xy}\,du\right]\left\{-\sum_{\nu=0}^{k}\frac{\cos \nu v}{\nu}\right\}\right]_{v=0}^{v=\pi} + \int_0^\pi\left[\left(\sum_{\nu=0}^{k}\frac{\cos \nu v}{\nu}\right)\int_0^\pi \frac{\cos \nu u}{\nu}\,\psi'^{*}_{xy}\,du\right]dv\right\}.$$

Since $f(u,v)$ is periodic, $f'(u,v)$ and $f^{*}(u,v)$ are also periodic, and we can readily show that $\psi'_{xy}\big|_{v=0} = \psi'_{xy}\big|_{v=\pi} = 0$, if we note that

$$\psi'_{xy} = f'(x+u,\,y+v) + f'(x-u,\,y+v) - f'(x+u,\,y-v) - f'(x-u,\,y-v) .$$

Then [8.13] becomes

[8.14] $$\int_0^\pi\int_0^\pi \psi_{xy}\,\bar{K}_n(u,v)\,du\,dv = \frac{1}{4\pi^2 P_n}\sum_{k=0}^{n} p_{n-k}\left\{\int_0^\pi \sum_{\nu=0}^{k}\frac{\cos \nu u}{\nu}\left(\int_0^\pi \sum_{\nu=0}^{k}\frac{\cos \nu u}{\nu}\,\psi'^{*}_{xy}\,du\right)dv\right\}.$$

To integrate the inner integral of [8.14] by parts, we set

$$U = \psi_{xy}^{1*}, \quad dV = \sum_{\nu=0}^{k} \frac{\cos \nu u}{\nu}\, du \; ; \quad dU = \psi_{xy}^{1*1} du, \quad V = \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2} .$$

Then

[8.15] $$\int_0^{\pi}\int_0^{\pi} \psi_{xy}\, \bar{K}_n(u,v)\, du\, dv =$$

$$\frac{1}{4\pi^2 P_n} \sum_{k=0}^{n} p_{n-k} \left[\int_0^{\pi} \sum_{\nu=0}^{k} \frac{\cos \nu v}{\nu} \left\{ \left[\psi_{xy}^{1*} \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2} \right]_{u=0}^{u=\pi} - \int_0^{\pi} \psi_{xy}^{1*1} \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2}\, du \right\} dv \right]$$

$$= -\frac{1}{4\pi^2 P_n} \sum_{k=0}^{n} p_{n-k} \int_0^{\pi} \sum_{\nu=0}^{k} \frac{\cos \nu v}{\nu} \left\{ \int_0^{\pi} \psi_{xy}^{1*1} \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2}\, du \right\} dv .$$

Finally, we integrate the right side of [8.15] by parts with

$$U = \int_0^{\pi} \psi_{xy}^{1*1} \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2}\, du, \quad dV = \sum_{\nu=0}^{k} \frac{\cos \nu v}{\nu}\, dv \; ; \quad V = \sum_{\nu=0}^{k} \frac{\sin \nu v}{\nu^2}, \quad dU = \left\{ \int_0^{\pi} \psi_{xy}^{1*1*} \sum_{0}^{k} \frac{\sin \nu u}{\nu^2} \right\} dv.$$

Then [8.15] becomes

$$\int_0^{\pi}\int_0^{\pi} \psi_{xy}\, \bar{K}_n(u,v)\, du\, dv = -\frac{1}{4\pi^2 P_n} \sum_{k=0}^{n} p_{n-k} \left\{ \left[\left(\int_0^{\pi} \psi_{xy}^{1*1} \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2}\, du \right) \sum_{\nu=0}^{k} \frac{\sin \nu v}{\nu^2} \right]_{v=0}^{v=\pi} - \int_0^{\pi} \left(\sum_{\nu=0}^{k} \frac{\sin \nu v}{\nu^2} \right) \left(\int_0^{\pi} \psi_{xy}^{1*1*} \sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2}\, du \right) dv \right\}$$

$$= \frac{1}{4\pi^2 P_n} \sum_{k=0}^{n} p_{n-k} \int_0^{\pi}\int_0^{\pi} \left(\sum_{\nu=0}^{k} \frac{\sin \nu u}{\nu^2} \right) \left(\sum_{\nu=0}^{k} \frac{\sin \nu v}{\nu^2} \right) \psi_{xy}^{1*1*}\, du\, dv$$

$$= \frac{1}{4\pi^2 P_n} \sum_{k=0}^{n} p_{n-k} F_k(x,y) .$$

The conclusion follows from the hypothesis of the theorem.

We now consider the <u>double</u> Nörlund means of the double sequence of partial sums:

[8.16] $$T_{\mu\nu} = \frac{1}{P_\mu P_\nu} \sum_{m=0}^{\mu} \sum_{n=0}^{\nu} p_{\mu-m}\, p_{\nu-n}\, \bar{S}_{mn} \quad .$$

Cesari (2) has shown that if $f(u,v)$ is periodic of period 2π in u and v and satisfies a Lipschitz condition of order α $(0<\alpha\leq 1)$ in the fundamental square Q, then the (C,1,1) means of the D.C.F.S. converge uniformly, in any rectangle inside Q, to the conjugate function

[8.17] $$\bar{f}(x,y) = \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \psi_{xy} \operatorname{Cot}\frac{u}{2} \operatorname{Cot}\frac{v}{2}\, du\, dv \quad .$$

Cesari has also shown uniform convergence throughout the plane if $f(u,v)$ satisfies a Lipschitz condition of order α throughout the plane. The assumption that the integral form of $\bar{f}(x,y)$, in [8.17], exists is implicit in his work, as it will be here.

We shall improve on Cesari's result in two directions. In order to do so, we prove the following

Theorem 23 If $f(u,v)$ is a periodic function of period 2π with respect to u and v in the fundamental square Q and is such that $\left|\dfrac{\psi_{xy}}{\sin\frac{u}{2}\ \sin\frac{v}{2}}\right|$ is integrable in $(0,0;\pi,\pi)$, then in any inner rectangle with sides parallel to the axes and not containing points on the boundary of Q, the D.C.F.S. converges uniformly according to Nörlund's double means (N, p_μ, p_ν) to the corresponding function $\bar{f}(x,y)$.

We first assume that $f(u,v)$ is continuous in and on the inner rectangle.

<u>Proof</u>: The proof is similar to the proof given by Cesari (2) for the case (C,1,1) with $f(u,v)$ satisfying a Lipschitz condition of order α $(0<\alpha\leq 1)$. At a point (x,y) of continuity, we have, from [8.16], [8.17], and [8.7],

[8.18] $$P_{\mu\nu}(x,y)-\bar f(x,y)=\frac{1}{P_\mu P_\nu}\sum_{m=0}^{\mu}\sum_{n=0}^{\nu}p_{\mu-m}\,p_{\nu-n}\bar S_{mn}-\frac{1}{4\pi^2}\int_0^\pi\!\!\int_0^\pi \psi_{xy}\operatorname{Cot}\frac{u}{2}\operatorname{Cot}\frac{v}{2}\,du\,dv$$

$$=\frac{1}{4\pi^2}\int_0^\pi\!\!\int_0^\pi\psi_{xy}\left\{\left[\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\sum_{K=0}^{m}\sin Ku\right]\left[\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\sum_{K=0}^{n}\sin Kv\right]-\operatorname{Cot}\frac{u}{2}\operatorname{Cot}\frac{v}{2}\right\}du\,dv.$$

Since

[8.19] $$\sum_{K=0}^{m}\sin K\theta=\begin{cases}\frac{1}{2}\operatorname{Cot}\frac{\theta}{2}-\frac{\operatorname{Cos}(n+\frac{1}{2})\theta}{2\operatorname{Sin}\frac{\theta}{2}}, & \theta\neq K\pi\\ 0, & \theta=K\pi\end{cases}$$

we have

[8.20] $$P_{\mu\nu}(x,y)-\bar f(x,y)=$$

$$\frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi\psi_{xy}\left\{\left[\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\left(\frac{1}{2}\operatorname{Cot}\frac{u}{2}-\frac{\operatorname{Cos}(m+\frac{1}{2})u}{2\operatorname{Sin}\frac{u}{2}}\right)\right]\left[\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\left(\frac{1}{2}\operatorname{Cot}\frac{v}{2}-\frac{\operatorname{Cos}(n+\frac{1}{2})v}{2\operatorname{Sin}\frac{v}{2}}\right)-\frac{1}{4}\operatorname{Cot}\frac{u}{2}\operatorname{Cot}\frac{v}{2}\right\}du\,dv.$$

Simplifying the expression within the braces and then breaking [8.20] up into three integrals, we get

[8.21] $$P_{\mu\nu}(x,y)-f(x,y)=-\frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi\psi_{xy}\left[\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\frac{\operatorname{Cos}(m+\frac{1}{2})u}{2\operatorname{Sin}\frac{u}{2}}\right]\left(\frac{1}{2}\operatorname{Cot}\frac{v}{2}\right)du\,dv$$

$$+\frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi\psi_{xy}\left[\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\frac{\operatorname{Cos}(m+\frac{1}{2})u}{2\operatorname{Sin}\frac{u}{2}}\right]\left[\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\frac{\operatorname{Cos}(n+\frac{1}{2})v}{2\operatorname{Sin}\frac{v}{2}}\right]du\,dv-\frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi\psi_{xy}\left(\frac{1}{2}\operatorname{Cot}\frac{u}{2}\right)\left[\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\frac{\operatorname{Cos}(n+\frac{1}{2})v}{2\operatorname{Sin}\frac{v}{2}}\right]du\,dv.$$

For brevity, we denote the sum of the three integrals in [8.21] by $I_1 + I_2 + I_3$. Let 2δ be the shortest distance from the point (x,y) to the boundary of Q. Let ρ be such that $0 < \rho < \delta$ be arbitrary momentarily and set

$$[8.22] \qquad I_j = \int_0^{\rho}\int_0^{\rho} + \int_0^{\rho}\int_{\rho}^{\pi} + \int_{\rho}^{\pi}\int_0^{\rho} + \int_{\rho}^{\pi}\int_{\rho}^{\pi} = J_{j1} + J_{j2} + J_{j3} + J_{j4} \quad (j = 1, 2, 3).$$

Let ϵ be arbitrary. Since the functions under the integral sign in the expressions I_j $(j=1,2,3)$ are smaller in absolute value than an integrable function, $\left| \dfrac{\psi_{xy}}{\sin\frac{u}{2}\sin\frac{v}{2}} \right|$, independent of μ and ν, we can select a ρ such that $0 < \rho < \delta$ and sufficiently small so that $|J_{j1}|$, $|J_{j2}|$, $|J_{j3}| < \epsilon$, $j = 1, 2, 3$. Now that ρ has been fixed, the integrals J_{j4} $(j=1,2,3)$ approach zero as $\mu, \nu \to \infty$, by the general convergence theorem (11b). To illustrate this last remark, consider J_{14}, which may be written

$$[8.23] \qquad J_{14} = -\frac{1}{4\pi^2}\,\frac{1}{P_\mu}\sum_{m=0}^{\mu} p_{\mu-m}\int_{\rho}^{\pi}\int_{\rho}^{\pi} \frac{\psi_{xy}\cos\frac{v}{2}}{\sin\frac{u}{2}\sin\frac{v}{2}} \cos(m+\tfrac{1}{2})u \, du\, dv \; .$$

Now $\dfrac{\psi_{xy}\cos\frac{v}{2}}{\sin\frac{u}{2}\sin\frac{v}{2}}$ is integrable in $(\rho, \rho; \pi, \pi)$ since it is $\leq \left| \dfrac{\psi_{xy}}{\sin\frac{u}{2}\sin\frac{v}{2}} \right|$ which is integrable in $(0,0;\pi,\pi)$ by hypothesis. By the general convergence theorem (11b), we have

$$\lim_{m\to\infty} \int_{\rho}^{\pi}\int_{\rho}^{\pi} \frac{\psi_{xy}\cos\frac{v}{2}}{\sin\frac{u}{2}\sin\frac{v}{2}} \cos(m+\tfrac{1}{2})u \, du\, dv = 0 \; .$$

The Nörlund mean of a sequence which tends to zero also tends to zero so that J_{14}, in [8.22], tends to zero as $\mu \to \infty$.

For J_{24} and J_{34}, the argument is the same.

Cumulatively for these results, there exists an N_0 such that for $\mu, \nu > N_0$, $|J_{j4}| < \epsilon$, $j = 1,2,3$ whence $|I_{j4}| < 4\epsilon$, $j = 1,2,3$ for $\mu, \nu > N_0$ and finally $|T_{\mu\nu} - \bar{f}(x,y)| < 12\epsilon$.

This theorem includes Cesari's (C,1,1) result and generalizes it in two senses:

(a) from $f(u,v)$ satisfying a Lipschitz condition of order α $(0 < \alpha \leq 1)$ to $f(u,v)$ being continuous within Q and satisfying the condition that

$\left| \dfrac{\varphi_{xy}}{\sin\frac{u}{2} \sin\frac{v}{2}} \right|$ is integrable in $(0,0;\pi,\pi)$,

(b) from (C,1,1) summability to (N, p_n, p_n) summability.

These two more general conditions are of value since there are Nörlund means weaker than (C,α) means for any $\alpha > 0$, and since a much wider class of functions can be considered.

The above proof can also be used to show (N, p_n, q_n) summability with

$$T_{\mu\nu} = \frac{1}{P_\mu Q_\nu} \sum_{m=0}^{\mu} \sum_{n=0}^{\nu} p_{\mu-m} q_{\nu-n} S_{mn} .$$

Herriot (10a) treats double Nörlund means of this type.

Discontinuities Along Straight Lines

We next consider the behavior of $T_{\mu\nu}$ at a point (x,y) of discontinuity of $f(x,y)$ in a manner similar to Professor C.N. Moore's treatment (15) of the (C,1,1) summability of the D.F.S. at a point of discontinuity. We prove the following

Theorem 24 Let $f(u,v)$ be a periodic function of period 2π with respect to u and v in Q and such that $\left|\frac{\psi_{xy}}{\sin\frac{u}{2}\sin\frac{v}{2}}\right|$ is integrable in $(0,0\,;\pi,\pi)$. We assume the point of discontinuity (x_1,y_1) of $f(u,v)$ is interior to Q and all other points of discontinuity in the neighborhood of (x_1,y_1) lie on a straight line passing through the point. The function $f(u,v)$ approaches a definite value as we approach the point (x_1,y_1) from either side of the line. If we enclose the point (x_1,y_1) in a rectangle R_3 with sides parallel to the axes, entirely within the cross neighborhood* such that the line of discontinuities passing through (x_1,y_1) is either a diagonal of R_3 or is parallel to one of its sides, then the Nörlund means $T_{\mu\nu}(x_1,y_1)$ will converge in either case.

Proof: The integral sum to which $T_{\mu\nu}(x_1,y_1)$ converges will be given after we have defined the necessary regions. The proof is similar to the proof of Theorem 23. The integral form of $T_{\mu\nu}(x_1,y_1)$ is, from [8.7], [8.16], and [8.19],

$$[8.24]\quad T_{\mu\nu}(x_1,y_1) = \frac{1}{\pi^2}\int_0^{\pi}\!\!\int_0^{\pi}\psi_{xy}\left\{\left[\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\left(\frac{1}{2}\cot\frac{u}{2}-\frac{\cos(m+\frac{1}{2})u}{2\sin\frac{u}{2}}\right)\right]\left[\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\left(\frac{1}{2}\cot\frac{v}{2}-\frac{\cos(n+\frac{1}{2})v}{2\sin\frac{v}{2}}\right)\right]\right\}du\,dv,$$

---

*The cross neighborhood of a point (x_1,y_1) in a region R is the interior of the region bounded by the lines $x = x_1 \pm \epsilon$, $y = y_1 \pm \delta$, for arbitrary $\epsilon,\delta > 0$. The cross neighborhood is important in D.F.S. in treating difficulties that arise in the generalization from single to D.F.S.

where $\psi_{xy} = f(x+u, y+v) - f(x-u, y+v) - f(x+u, y-v) + f(x-u, y-v)$.

Let 2δ be the shortest distance from (x_1, y_1) to the boundary of Q. Denote the cross neighborhood of (x_1, y_1) by R_1 and let R_2 be the remainder of the region of integration. Break off from R_2 the rectangles $R_2'(\tau, 0; \pi, \tau)$, $R_2''(0, \tau; \tau, \pi)$, $R_2'''(0, 0; \tau, \tau)$, where τ is such that $0 < \tau < \delta$ is arbitrary momentarily.

The cross neighborhood of (x_1, y_1) is R_1. The remainder of the region of integration is

$$R_2 = \left\{ \begin{array}{ll} R_2' : & (\tau, 0; \pi, \tau) \\ R_2'' : & (0, \tau; \tau, \pi) \\ R_2''' : & (0, 0; \tau, \tau) \end{array} \right\} + A + B + C + D .$$

The deleted cross neighborhood, R_1^*, is cross hatched. Within R_1^* is R_3 which encloses the point (x_1, y_1). The sum of the regions A, B, C and D is

$$R_2 - R_2' - R_2'' - R_2''' = A + B + C + D .$$

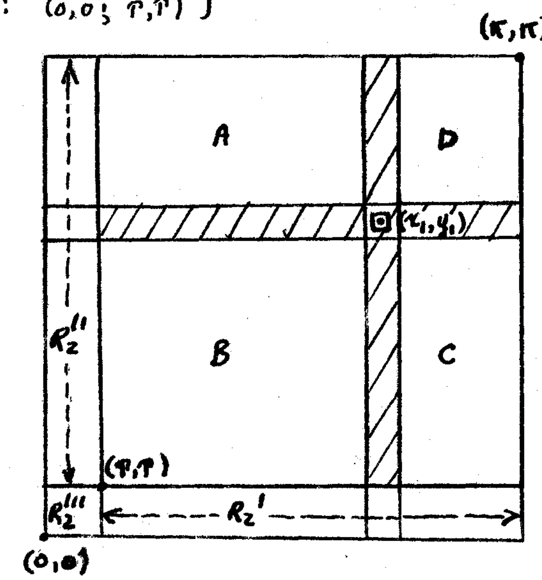

Fig.1

Let ϵ be arbitrary and write [8.24] as

[8.25] $$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi = \frac{1}{\pi^2}\left\{ \int_0^\tau\int_0^\tau + \int_0^\tau\int_\tau^\pi + \int_\tau^\pi\int_0^\tau + \int_\tau^\pi\int_\tau^\pi \right\} ,$$

where the integrand is understood to be that in [8.24]. Let us consider the second integral on the right side of [8.25]. By an argument similar to that used with J_{j2} in the previous theorem (see [8.22]), this integral, i.e.,

$$[8.26] \qquad \frac{1}{\pi^2}\int_0^{\tau}\int_{\tau}^{\pi}\psi_{xy}\left\{\left[\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\left(\frac{\cos\frac{u}{2}-\cos(m+\frac{1}{2})u}{2\sin\frac{u}{2}}\right)\right]\left[\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\left(\frac{\cos\frac{v}{2}-\cos(n+\frac{1}{2})v}{2\sin\frac{v}{2}}\right)\right]\right\}du\,dv,$$

can be shown to tend to

$$[8.27] \qquad \frac{1}{4\pi^2}\int_0^{\tau}\int_{\tau}^{\pi}\psi_{xy}\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv \ ,$$

by considering the difference between these two integrals, [8.26] and [8.27], and choosing τ sufficiently small so that this difference in absolute value, can be made less than ϵ. This difference between the two integrals [8.26] and [8.27] has exactly the same form as is found in [8.21] except that integration is now over the range $(0,\tau;\tau,\pi)$.

Analogously, we can treat the first and third integrals on the right side of [8.25] and we can show that these tend to

$$[8.28] \qquad \frac{1}{4\pi^2}\int_0^{\tau}\int_0^{\tau}\psi_{xy}\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv \quad \text{and} \quad \frac{1}{4\pi^2}\int_{\tau}^{\pi}\int_0^{\tau}\psi_{xy}\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv,$$

respectively. The last integral on the right side of [8.25] remains to be considered. Within its region of integration, $(\tau,\tau;\pi,\pi)$ lies the cross neighborhood of (x_1,y_1), the region R_1, minus the two small rectangular portions of this cross neighborhood already accounted for by the second and third integrals on the right side of [8.25]. Now the region $R_2-R_2'-R_2''-R_2'''$ accounts for all the region $(\tau,\tau;\pi,\pi)$ except this deleted cross neighborhood of (x_1,y_1), which we call R_1^*. We can write the last integral on the right side of [8.25] as

[8.29] $$\frac{1}{\pi^2}\int_{\tau}^{\pi}\int_{\tau}^{\pi} = \frac{1}{\pi^2}\iint_{R_2-R_2'-R_2''-R_2'''} + \frac{1}{\pi^2}\iint_{R_1^*} \quad ,$$

where the integrand is understood to be that found in [8.24] . Let us consider the first integral of [8.29] . Now that τ has been fixed, if we expand the integrand of

[8.30] $$\frac{1}{\pi^2}\iint_{R_2-R_2'-R_2''-R_2'''}\psi_{xy}\left\{\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\left(\frac{\cos\frac{u}{2}-\cos(m+\frac{1}{2})u}{2\sin\frac{u}{2}}\right)\right\}\left\{\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\left(\frac{\cos\frac{v}{2}-\cos(n+\frac{1}{2})v}{2\sin\frac{v}{2}}\right)\right\}du\,dv,$$

we see that one of the integrals is

[8.31] $$\frac{1}{4\pi^2}\iint_{R_2-R_2'-R_2''-R_2'''}\psi_{xy}\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv \quad ,$$

and the other three integrals tend to zero as $\mu,\nu\to\infty$, by applying the general convergence theorem (11b), and arguing as in the previous theorem with J_{34} (see [8.23]). Thus far we have shown, from [8.27], [8.28], and [8.31] that

[8.32] $$\frac{1}{\pi^2}\iint_{R_2}\psi_{xy}\left[\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\left(\frac{\cos\frac{u}{2}-\cos(m+\frac{1}{2})u}{2\sin\frac{u}{2}}\right)\right]\left[\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\left(\frac{\cos\frac{v}{2}-\cos(n+\frac{1}{2})v}{2\sin\frac{v}{2}}\right)\right]du\,dv$$

tends to

[8.33] $$\frac{1}{4\pi^2}\iint_{R_2}\psi_{xy}\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv \quad ,$$

as $\mu,\nu\to\infty$.

Let us now consider the deleted cross neighborhood of (x_1,y_1), i.e., R_1^*, since there remains only to consider the second integral on the right side of [8.29] . We enclose (x_1,y_1) in a rectangle R_3 , with sides parallel to the axes, entirely within the cross neighborhood such that the line of discontinuities passing through (x_1, y_1) is either a diagonal of R_3 or is parallel to either side. The point (x_1, y_1) is at the center of R_3.

We consider first the diagonal line. Let this line divide R_3 into two regions R_3' and R_3'' and let $f_1(x_1, y_1)$ and $f_2(x_1, y_1)$ be the two values which $f(x,y)$ approach when (x,y) approaches (x_1, y_1) through the regions R_3' and R_3''. Since only the second integral on the right side of [8.29] remains to be considered, we write this as

[8.34] $$\frac{1}{\pi^2}\iint_{R_1^*} \psi_{xy}\left\{\frac{1}{P_\mu}\sum_{m=0}^{\mu} p_{\mu-m}\left(\frac{\cos\frac{u}{2} - \cos(m+\frac{1}{2})u}{2\sin\frac{u}{2}}\right)\right\}\left\{\frac{1}{P_\nu}\sum_{n=0}^{\nu} p_{\nu-n}\left(\frac{\cos\frac{v}{2} - \cos(n+\frac{1}{2})v}{2\sin\frac{v}{2}}\right)\right\} du\, dv$$

$$= \frac{1}{\pi^2}\iint_{R_1^*-R_3} + \frac{1}{\pi^2}\iint_{R_3} = \frac{1}{\pi^2}\iint_{R_1^*-R_3} + \frac{1}{\pi^2}\iint_{R_3'} + \frac{1}{\pi^2}\iint_{R_3''} .$$

We then use the general convergence theorem to show that the last three integrals in [8.34] tend respectively to

[8.35] $$\frac{1}{4\pi^2}\iint_{R_1^*-R_3} \psi_{xy} \cot\frac{u}{2}\cot\frac{v}{2}\, du\, dv, \quad \frac{1}{4\pi^2}\iint_{R_3'} \psi_{xy} \cot\frac{u}{2}\cot\frac{v}{2}\, du\, dv, \quad \frac{1}{4\pi^2}\iint_{R_3''} \psi_{xy} \cot\frac{u}{2}\cot\frac{v}{2}\, du\, dv,$$

as $\mu, \nu \to \infty$, by the familiar argument of considering the respective differences between these integrals and proceeding as before.

We have thus shown that the Nörlund means of the D.C.F.S. at a point of discontinuity (x_1, y_1) converge to the sum of the four integrals

[8.36] $$\frac{1}{4\pi^2}\left(\iint_{R_2} + \iint_{R_1-R_3} + \iint_{R_3'} + \iint_{R_3''}\right)\psi_{xy} \cot\frac{u}{2}\cot\frac{v}{2}\, du\, dv ,$$

provided each of these integrals exists.

For the case in which the line through (x_1, y_1) is parallel to one of the sides of R_3 and divides R_3 into R_3''' and $R_3^{(iv)}$, we require the existence of the integrals

$$\frac{1}{4\pi^2}\left(\iint_{R_3'''} + \iint_{R_3^{(iv)}}\right) \varphi_{xy} \cot\frac{u}{2} \cot\frac{v}{2}\, du\, dv ,$$

where $f(x,y)$ tends to $f_1(x_1,y_1)$ and $f_2(x_1,y_1)$ as (x,y) tends to (x_1,y_1) through R_3''' and $R_3^{(iv)}$, respectively. The proof for this case is similar to the proof for the diagonal line.

Discontinuities That Lie Along Curves

We consider the behavior of the Nörlund means of the D.C.F.S. at a point of discontinuity such that all other points of discontinuity in the neighborhood of that point lie on a curve which passes through the point, and the function $f(x,y)$ approaches a definite value as we approach the point from either side of the curve.

Assume the curve is of such a nature that it has a tangent at the point of discontinuity of $f(x,y)$ and that any line passing through that point will intersect the curve in only a finite number of points in the neighborhood of the point. From [8.24] we again consider

$$[8.37] \quad T_{\mu\nu}(x_1,y_1) = \frac{1}{4\pi^2}\int_0^{\pi}\!\!\int_0^{\pi} \varphi_{xy} \left\{\frac{1}{P_\mu}\sum_{m=0}^{\mu} p_{\mu-m}\left(\frac{\cos\frac{u}{2} - \cos(m+\frac{1}{2})u}{\sin\frac{u}{2}}\right)\right\}\left\{\frac{1}{P_\nu}\sum_{n=0}^{\nu} p_{\nu-n}\left(\frac{\cos\frac{v}{2} - \cos(n+\frac{1}{2})v}{\sin\frac{v}{2}}\right)\right\} du\, dv$$

$$= \frac{1}{4\pi^2}\iint_{R_1} + \frac{1}{4\pi^2}\iint_{R_2} ,$$

where R_1 is the rectangle with (x_1,y_1) as center and the tangent to the curve at (x_1,y_1) as one of its diagonals and such that all points of discontinuity of $f(x,y)$ in the rectangle lie on the curve. The region R_2 is the remainder of the region of integration. We choose R_1 so small that the curve of discontinuities does not intersect the diagonal which is tangent to it within the rectangle, except perhaps at the point of tangency. In case the tangent is parallel to one of the axes, it will divide R_1 into two parts, as does the diagonal, and the treatment is the same.

To treat the second integral of [8.37] we need only refer to the previous theorem, [8.32], [8.33] and part of [8.34], [8.35], since the discussion is the same to show that

$$[8.38] \qquad \frac{1}{4\pi^2}\iint_{R_2}\psi_{xy}\left\{\frac{1}{P_\mu}\sum_{m=0}^{\mu}p_{\mu-m}\left(\frac{\cos\frac{u}{2}-\cos(m+\frac{1}{2})u}{\sin\frac{u}{2}}\right)\right\}\left\{\frac{1}{P_\nu}\sum_{n=0}^{\nu}p_{\nu-n}\left(\frac{\cos\frac{v}{2}-\cos(n+\frac{1}{2})v}{\sin\frac{v}{2}}\right)\right\}du\,dv$$

tends to

$$[8.39] \qquad \frac{1}{4\pi^2}\iint_{R_2}\psi_{xy}\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv \quad ,$$

as $\mu, \nu \to \infty$.

Let R_1' and R_1'' be the two portions into which the tangent to the curve at (x_1,y_1) divides R_1 and let f_1 and f_2 be the limiting values of $f(x,y)$ as we approach (x_1,y_1) through R_1' and R_1'' respectively. Then

$$[8.40] \qquad \frac{1}{4\pi^2}\iint_{R_1} = \frac{1}{4\pi^2}\iint_{R_1'} + \frac{1}{4\pi^2}\iint_{R_1''} \quad ,$$

where the integrand is understood to be that of [8.37]. By the familiar argument of applying the general convergence theorem (11b) to both integrals on the right side of [8.40] we can show that these integrals tend, as $\mu, \nu \to \infty$, to

[8.41] $$\frac{1}{4\pi^2}\iint_{R_1'} \psi_{xy} \cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv\ , \quad \frac{1}{4\pi^2}\iint_{R_1''} \psi_{xy} \cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv\ ,$$

respectively. We then have the following

<u>Theorem 25</u> Let $f(u,v)$ be a periodic function of period 2π with respect to u and v in Q and such that $\left|\frac{\psi_{xy}}{\sin\frac{u}{2}\sin\frac{v}{2}}\right|$ is integrable in $(0,0;\pi,\pi)$. We assume the point of discontinuity (x_1,y_1) is interior to Q and such that every other point of discontinuity in the neighborhood of (x_1,y_1) lies on a curve which satisfies the following conditions: (a) the curve has a tangent at the point (x_1,y_1), (b) no line through the point intersects the curve in an infinite number of points in the neighborhood of that point, (c) the function $f(x,y)$ approaches a definite value as we approach the point (x_1,y_1) from either side of the curve. Then the D.C.F.S. is (N, p_n, p_n) summable at the point (x_1,y_1) to the sum

$$\frac{1}{4\pi^2}\left(\iint_{R_1'} + \iint_{R_1''} + \iint_{R_2}\right)\psi_{xy} \cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv\ ,$$

provided each of these integrals exists.

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