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I hereby recommend that the thesis prepared under my supervision by A. Wayne McGaughey

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Approved by:

Charles N. Moore

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LACUNARY DOUBLE FOURIER SERIES

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by

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1. Introduction. Lacunary simple trigonometric series, that is, series where the terms different from zero are "very sparse", are series of the form

$$\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x),$$

where the indices n_k satisfy an inequality

$$n_{k+1}/n_k > \lambda > 1 \quad (\lambda \text{ fixed}).$$

These series have been investigated by A. Zygmund (1), (2), S. Szidon (3), (4), (5), (6), A. Kolmogoroff (7), J. Marcinkiewicz (8), and S. Banach (9), (10).

(1) A. Zygmund, On the convergence of lacunary trigonometric series, Fund. Math., 16 (1930), 90-107, corrigenda, Fund. Math., 18 (1932), 312.

(2) A. Zygmund, Quelques théorèmes sur les séries trigonométriques et celles de puissances, Studia Math., 3 (1931), 77-91.

(3) S. Szidon, Einige Sätze und Fragestellungen über Fourierkoeffizunten, Math. Zeit., 34 (1932), 477-480.

(4) S. Szidon, Ein Satz über Fouriersche Reihen mit Lücken, Math. Zeit., 34 (1932), 480-86.

(5) S. Szidon, Ein Satz über trigonometrische Polynome und seine anwendung in der Theorie der Fourierreihen, Math. Ann., 106 (1932), 536-539.

(6) S. Szidon, Verallgemeinerung eines Satzes über die absolute Konvergenz von Fourierreihen mit Lücken, Math. Ann., 97 (1927), 675-676.

(7) A. Kolmogoroff, Une contribution à l'étude de la convergence des séries de Fourièr, Fund. Math., 5 (1924), 96-97.

(8) J. Marcinkiewicz, A new proof of a theorem on Fourier series, L.M.S. Journal, 8 (1933), 279.

(9) S. Banach, Über einige Eigenschaften der lakunären trigonometrischen Reihen, Studia Math., 2 (1930), 207-220.

(10) S. Banach, Sur les séries lacunaires, Bull. International De L'Académie Polonaise Des Sciences et Des Lettres,

April-Oct. 1933.

Some of these results are collected in A. Zygmund, Trigonometrical Series.

This thesis consists of generalizations to lacunary double Fourier series of some of the results that are known for lacunary simple Fourier series. In particular see A. Zygmund, Trigonometrical Series, § 5.4, 6.4, 9.6, 9.601 and 10.31. We shall define a lacunary double trigonometric series to be any series of the form

$$(1.1) \quad \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} (a_{k\ell}\cos m_k x \cos n_\ell y + b_k \cos m_k x \sin n_\ell y + c_{k\ell}\sin m_k x \cos n_\ell y + d_{k\ell}\sin m_k x \sin n_\ell y),$$

where the indices m_k and n_ℓ satisfy the inequalities

$$m_{k+1}/m_k > \lambda > 1, \quad n_{\ell+1}/n_\ell > \lambda > 1, \quad k,\ell = 1,2,3,\ldots$$

We have assumed $a_{oo} = 0$, $a_{ko} = a_{o\ell} = b_{o\ell} = c_{ko} = 0$

for $k, \lambda = 1, 2, \ldots$. We have lost no generality in making this assumption because these coefficients are merely coefficients of simple Fourier series for which the analogous theorems are true. It is clear that the rows and columns of (1.1) in which some of the terms are different from zero are "very sparse". Also, the terms in these rows and columns that are not zero are "very sparse".

2. Generalization of theorem due to Zygmund on lacunary Fourier series. Let $\varphi_{ij}(\alpha)$ be a set of functions defined over a set of points, $G(\alpha)$, in space of any number of dimensions, with coordinates real or complex. Let the $\varphi_{ij}(\alpha)$ satisfy the following conditions*:

$$\lim_{\alpha\to\alpha_0} \sum_{j=0}^{\infty} |\varphi_{ij}(\alpha)| = 0 \qquad \text{(all i)},$$

$$\lim_{\alpha\to\alpha_0} \sum_{i=0}^{\infty} |\varphi_{ij}(\alpha)| = 0 \qquad \text{(all j)},$$

$$\lim_{\alpha\to\alpha_0} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{ij}(\alpha) = 1.$$

(*C. N. Moore, Am. Math. Soc. Coll. Pub., vol. 22, P. 23.)

If $\lim_{\alpha\to\alpha_0} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} S_{ij}\varphi_{ij}(\alpha) = S$, where S_{ij} are the partial sums and S is the sum of the series, then we will say that the series is summable Φ^*.

Given a lacunary series (1.1), let us consider the sum

$$(2.1) \quad \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left(a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2\right).$$

THEOREM I. If a series of the form (1.1) is summable Φ^* in a set of positive measure, then the series (2.1) converges.

If the series (1.1) is summable Φ^* in a set $|E|>0$ ($|E|$ means measure of E), we mean that for every $(x,y) \subset E$

$$(2.2) \quad \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \varphi_{pq}(\alpha)\, S_{pq}(x,y) = \sigma_\alpha(x,y), \quad \alpha \subset G,$$

the left hand side being convergent, and $\lim_{\alpha\to\alpha_0} \sigma_\alpha(x,y)$ exists and is finite. We will first consider the case where each row and each column of $\{\varphi_{ij}(\alpha)\}$ possesses only a finite number of terms different from zero. It will be convenient to consider the series in the complex form

$$\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\left(a_{k\ell} - i b_{k\ell} - i c_{k\ell} - d_{k\ell}\right) e^{i m_k x} e^{i n_\ell y}.$$

We put

$$
\begin{aligned}
4\gamma_{k,l} &= a_{kl} - i b_{kl} - i c_{kl} - d_{kl} = 4\gamma_{kl} \\
4\gamma_{-k,l} &= a_{kl} - i b_{kl} + i c_{kl} + d_{kl} = 4\beta_{kl} \\
4\gamma_{k,-l} &= a_{kl} + i b_{kl} - i c_{kl} + d_{kl} = 4\bar{\beta}_{kl}
\end{aligned}
$$

(2.3)
$$
4\gamma_{-k,-l} = a_{kl} + i b_{kl} + i c_{kl} - d_{kl} = 4\bar{\gamma}_{kl},
$$

$$
m_{-k} = -m_k, \quad n_{-l} = -n_l, \quad k, l = 1, 2, \dots .
$$

Before proceeding we shall establish some equalities and inequalities which we will need.

(I) $$\gamma_{kl}\bar{\gamma}_{kl} + \beta_{kl}\bar{\beta}_{kl} = \frac{a_{kl}^2 + b_{kl}^2 + c_{kl}^2 + d_{kl}^2}{8}$$

We have

$$
\begin{aligned}
&\gamma_{kl}\bar{\gamma}_{kl} + \beta_{kl}\bar{\beta}_{kl} \\
&= \frac{[(a_{kl} - d_{kl}) - i(b_{kl} + c_{kl})][(a_{kl} - d_{kl}) + i(b_{kl} + c_{kl})]}{16} \\
&+ \frac{[(a_{kl} + d_{kl}) - i(b_{kl} - c_{kl})][a_{kl} + d_{kl}) + i(b_{kl} - c_{kl})]}{16}
\end{aligned}
$$

$$= \frac{(a_{kl} - d_{kl})^2 + (b_{kl} + c_{kl})^2 + (a_{kl} + d_{kl})^2 + (b_{kl} - c_{kl})^2}{16}$$

$$= \frac{a_{kl}^2 + b_{kl}^2 + c_{kl}^2 + d_{kl}^2}{8}$$

(11) $$|\gamma_{kl}|^2 + |\beta_{kl}|^2 = \frac{(a_{kl}^2 + b_{kl}^2 + c_{kl}^2 + d_{kl}^2)}{8} .$$

$$|\gamma_{kl}|^2 + |\beta_{kl}|^2 = \frac{(a_{kl} - d_{kl})^2 + (b_{kl} + c_{kl})^2}{16}$$

$$+ \frac{(a_{kl} + d_{kl})^2 + (b_{kl} - c_{kl})^2}{16}$$

$$= \frac{a_{kl}^2 + b_{kl}^2 + c_{kl}^2 + d_{kl}^2}{8} .$$

(III) $$|\gamma_{kl}|^2 \leq \frac{a_{kl}^2 + b_{kl}^2 + c_{kl}^2 + d_{kl}^2}{8}$$

This is a corollary of (II).

(IV) $$|\beta_{kl}|^2 \leq \frac{a_{kl}^2 + b_{kl}^2 + c_{kl}^2 + d_{kl}^2}{8}$$

This is also a corollary of (II).

(V) $$|\gamma_{k\ell}||\beta_{k\ell}| \leq \frac{a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2}{16}$$

We know that $(a_{k\ell}d_{k\ell} - b_{k\ell}c_{k\ell})^2 \geq 0$, then

$a_{k\ell}^2 d_{k\ell}^2 + b_{k\ell}^2 c_{k\ell}^2 \geq 2a_{k\ell}b_{k\ell}c_{k\ell}d_{k\ell}$. Multiplying both

sides by 4 and transposing we get

$$2a_{k\ell}^2 d_{k\ell}^2 + 2b_{k\ell}^2 c_{k\ell}^2 \geq -2a_{k\ell}^2 d_{k\ell}^2 - 2b_{k\ell}^2 c_{k\ell}^2 + 8a_{k\ell}b_{k\ell}c_{k\ell}d_{k\ell}$$

We add

$$a_{k\ell}^4 + b_{k\ell}^4 + c_{k\ell}^4 + d_{k\ell}^4 + 2a_{k\ell}^2 b_{k\ell}^2 + 2a_{k\ell}^2 c_{k\ell}^2 + 2b_{k\ell}^2 d_{k\ell}^2 + 2c_{k\ell}^2 d_{k\ell}^2$$

to each side getting

$$\frac{(a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2)^2}{256} \geq \left[(a_{k\ell} - d_{k\ell})^2 + (b_{k\ell} + c_{k\ell})^2\right]$$

$$\times \frac{\left[(a_{k\ell} + d_{k\ell})^2 + (b_{k\ell} - c_{k\ell})^2\right]}{256}$$

$$= |\gamma_{k\ell}|^2|\beta_{k\ell}|^2,$$

hence $$\frac{a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2}{16} \geq |\gamma_{k\ell}||\beta_{k\ell}|.$$

Let

$$\left.\begin{array}{l} \varphi_{p,q} + \varphi_{p,q+1} + \varphi_{p,q+2} + \cdots \\ + \varphi_{p+1,q} + \varphi_{p+1,q+1} + \varphi_{p+1,q+2} + \cdots \\ + \varphi_{p+2,q} + \varphi_{p+2,q+1} + \varphi_{p+2,q+2} + \cdots \\ + \quad \cdots \quad \cdots \quad \cdots \quad \cdots \end{array}\right\} = R_\alpha(p,q).$$

We may rewrite (2.2) as

(2.4) $$\sigma_\alpha(x,y) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \gamma_{k\ell}\, e^{im_k x} e^{in_\ell y}\, \pi_\alpha(|m_k|,|n_\ell|),$$

the sum on the right being in reality finite since it includes only a finite number of terms different from zero.

Since $\{\sigma_\alpha(x,y)\}$ converges in E, we can find a subset $\mathcal{E}$ of E such that $|\mathcal{E}|>0$ and a number M such that $|\sigma_\alpha(x,y)| \leqq M$ for $\alpha \subset G$, $(x,y) \subset \mathcal{E}$. In fact, we have $E = E_1 + E_2 + \ldots$ where E_n is the set of values (x,y) such that $|\sigma_\alpha(x,y)| \leqq n$ for $\alpha \subset G$. Since $|E| > 0$, at least one of the sets E_i, say E_M, must have positive measure and may be taken as $\mathcal{E}$. It follows that

$$(2.5) \qquad M^2|\mathcal{E}| \geqq \iint_{\mathcal{E}} \sigma_\alpha^2(x,y)\,dydx.$$

We see from (2.4) that

$$(2.6) \qquad \iint_{\mathcal{E}} \sigma_\alpha^2(x,y)\,dydx$$

$$= \iint_{\mathcal{E}} \left\{ \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \gamma_{kl}\, e^{im_k x} e^{in_l y} R_\alpha(|m_k|, |n_l|) \right\}$$

$$\left\{ \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \gamma_{jk}\, e^{im_j x} e^{in_k y} R_\alpha(|m_j|, |n_k|) \right\} dydx.$$

We may consider each sum of (2.6) as the sum of 4 sums, that is

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} + \sum_{k=1}^{\infty} \sum_{l=-\infty}^{-1} + \sum_{k=-\infty}^{-1} \sum_{l=1}^{\infty} + \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{-1} .$$

By using (2.3) we can use $(1,1;\infty,\infty)$ as our range of summation. Using (2.3) we get

$$(2.7)\quad \iint_{\mathcal{E}} \sigma_\alpha^2(x,y)\,dy\,dx = \iint_{\mathcal{E}} \Big\{ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \gamma_{kl} e^{im_k x} e^{in_l y} R_\alpha(m_k, n_l)$$

$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \bar{\beta}_{kl} e^{im_k x} e^{-in_l y} R_\alpha(m_k, n_l)$$

$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \beta_{kl} e^{-im_k x} e^{in_l y} R_\alpha(m_k, n_l)$$

$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \bar{\gamma}_{kl} e^{-im_k x} e^{-in_l y} R_\alpha(m_k, n_l) \Big\}$$

$$\Big\{ \sum_{j=1}^{\infty}\sum_{h=1}^{\infty} \gamma_{jh} e^{im_j x} e^{in_h y} R_\alpha(m_j, n_h)$$

$$+ \sum_{j=1}^{\infty}\sum_{h=1}^{\infty} \bar{\beta}_{jh} e^{im_j x} e^{-in_h y} R_\alpha(m_j, n_h)$$

$$+ \sum_{j=1}^{\infty}\sum_{h=1}^{\infty} \beta_{jh} e^{-im_j x} e^{in_h y} R_\alpha(m_j, n_h)$$

$$+ \sum_{j=1}^{\infty}\sum_{h=1}^{\infty} \bar{\gamma}_{jh} e^{-im_j x} e^{-in_h y} R_\alpha(m_j, n_h) \Big\}\, dy\,dx.$$

We carry out the indicated multiplication and arrive at

$$(2.8)\quad \iint_{\mathcal{E}} \sigma_\alpha^2(x,y)\,dy\,dx = 2|\mathcal{E}| \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \gamma_{kl}\bar{\gamma}_{kl} + \beta_{kl}\bar{\beta}_{kl} R_\alpha^2(m_k, n_l)$$

$$+ \sum_{1}^{\infty}\sum_{1}^{\infty}\sum_{1}^{\infty}\sum_{1}^{\infty} (\gamma_{kl}\bar{\gamma}_{jh} + \beta_{kl}\bar{\beta}_{jh} + \bar{\beta}_{kl}\beta_{jh} + \bar{\gamma}_{kl}\gamma_{jh}) R_\alpha(m_k, n_l) R_\alpha(m_j, n_h)$$

$k \neq j,\ l \neq h$

$$\iint_{\varepsilon} e^{i(m_k - m_j)x} e^{i(n_\ell - n_h)y}\, dy\, dx$$

$$+ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty \gamma_{k\ell}\, \gamma_{jh}\, R_\alpha(m_k, n_\ell)\, R_\alpha(m_j, n_h)$$

$$\iint_{\varepsilon} e^{i(m_k + m_j)x} e^{i(n_\ell + n_h)y}\, dy\, dx$$

$$+ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty \bar{\gamma}_{k\ell}\, \bar{\gamma}_{jh}\, R_\alpha(m_k, n_\ell)\, R_\alpha(m_j, n_h)$$

$$\iint_{\varepsilon} e^{-i(m_k + m_j)x} e^{-i(n_\ell + n_h)y}\, dy\, dx$$

$$+ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty \bar{\beta}_{k\ell}\, \bar{\beta}_{jh}\, R_\alpha(m_k, n_\ell)\, R_\alpha(m_j, n_h)$$

$$\iint_{\varepsilon} e^{i(m_k + m_j)x} e^{-i(n_\ell + n_h)y}\, dy\, dx$$

$$+ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty \beta_{k\ell}\, \beta_{jh}\, R_\alpha(m_k, n_\ell)\, R_\alpha(m_j, n_\ell)$$

$$\iint_{\varepsilon} e^{-i(m_k + m_j)} e^{i(n_\ell + n_h)y}\, dy\, dx$$

$$+ \sum_1^\infty \sum_1^\infty \sum_1^\infty (\gamma_{k\ell}\, \bar{\beta}_{jh} + \bar{\beta}_{k\ell}\, \gamma_{jh})\, R_\alpha(m_k, n_\ell)\, R_\alpha(m_j, n_h)$$

$$\iint_{\varepsilon} e^{i(m_k + m_j)x} e^{i(n_\ell - n_h)y}\, dy\, dx$$

$$+ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty (\gamma_{k\ell}\, \beta_{jh} + \beta_{k\ell}\, \gamma_{jh})\, R_\alpha(m_k, n_\ell)\, R_\alpha(m_j, n_h)$$

$$\iint_{\varepsilon} e^{i(m_k - m_j)x} e^{i(n_l + n_h)y}\, dy\, dx$$

$$+ \sum \sum \sum \sum (\bar{\beta}_{kl}\bar{\gamma}_{jh} + \bar{\gamma}_{kl}\beta_{jh})\, R_\alpha(m_k, n_l) R_\alpha(m_j, n_h)$$

$$\iint_{\varepsilon} e^{i(m_k - m_j)x} e^{-i(n_l + n_h)y}\, dy\, dx$$

$$+ \sum \sum \sum \sum (\bar{\gamma}_{kl}\beta_{jh} + \beta_{kl}\bar{\gamma}_{jh})\, R_\alpha(m_k, n_l) R_\alpha(m_j, n_h)$$

$$\iint_{\varepsilon} e^{i(m_k - m_j)x} e^{-i(n_l + n_h)y}\, dy\, dx$$

Let us denote the integrals on the right hand side by

$$4\pi^2 A_{jklh}, \qquad 4\pi^2 B_{jklh}, \quad \ldots \quad 4\pi^2 H_{jklh},$$

and $4\pi^2 I_{jklh}$ respectively,

The numbers A_{jklh}, B_{jklh}, etc, are the complex coefficients of a function $\psi(x,y)$ equal to 1 in ε and zero elsewhere. We apply Schwarz's inequality to each of the terms on the right hand side of (2.8), with the exception of the first, and see that they do not exceed.

$$(2.9)\quad 8\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|\gamma_{kl}|^2|\gamma_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|A_{jklh}|^2\right\}^{1/2}$$

$$+\,8\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|\beta_{kl}|^2|\beta_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|A_{jklh}|^2\right\}^{1/2}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|\gamma_{kl}|^2|\gamma_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|B_{jklh}|^2\right\}^{1/2}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|\gamma_{kl}|^2|\gamma_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|C_{jklh}|^2\right\}^{\frac{1}{2}}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|\beta_{kl}|^2|\beta_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|D_{jklh}|^2\right\}^{1/2}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|\beta_{kl}|^2|\beta_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|E_{jklh}|^2\right\}^{1/2}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty(|\gamma_{kl}|^2|\beta_{jh}|^2+|\beta_{kl}|^2|\gamma_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|F_{jklh}|^2\right\}^{1/2}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty(|\gamma_{kl}|^2|\beta_{jh}|^2+|\beta_{kl}|^2|\gamma_{jh}|^2 R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|G_{jklh}|^2\right\}^{1/2}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty(|\gamma_{kl}|^2|\beta_{jh}|^2+|\beta_{kl}|^2|\gamma_{jh}|^2)\,R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|H_{jklh}|^2\right\}^{1/2}$$

$$+\,4\pi^2\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty(|\gamma_{kl}|^2|\beta_{jh}|^2+|\beta_{kl}|^2|\gamma_{jh}|^2)R_\alpha^2(m_k,n_l)\,R_\alpha^2(m_j,n_h)\right\}^{1/2}\left\{\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty|F_{jklh}|^2\right\}^{1/2}$$

$$= 8\pi^2 \sum_1^\infty \sum_1^\infty (|\gamma_{kl}|^2 + |\beta_{kl}|^2) R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |A_{jklh}|^2 \Big\}^{1/2}$$

$$j \neq k, \quad l \neq h$$

$$+ 4\pi^2 \sum_1^\infty \sum_1^\infty |\gamma_{kl}|^2 R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |B_{jklh}|^2 \Big\}^{1/2}$$

$$+ 4\pi^2 \sum_1^\infty \sum_1^\infty |\gamma_{kl}|^2 R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |C_{jklh}|^2 \Big\}^{1/2}$$

$$+ 4\pi^2 \sum_1^\infty \sum_1^\infty |\beta_{kl}|^2 R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |D_{jklh}|^2 \Big\}^{1/2}$$

$$+ 4\pi^2 \sum_1^\infty \sum_1^\infty |\beta_{kl}|^2 R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |E_{jklh}|^2 \Big\}^{1/2}$$

$$+ 8\pi^2 \sum_1^\infty \sum_1^\infty |\gamma_{kl}||\beta_{kl}| R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |F_{jklh}|^2 \Big\}^{1/2}$$

$$+ 8\pi^2 \sum_1^\infty \sum_1^\infty |\gamma_{kl}||\beta_{kl}| R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |G_{jklh}|^2 \Big\}^{1/2}$$

$$+ 8\pi^2 \sum_1^\infty \sum_1^\infty |\gamma_{kl}||\beta_{kl}| R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |H_{jklh}|^2 \Big\}^{1/2}$$

$$+ 8\pi^2 |\gamma_{kl}||\beta_{kl}| R_\alpha^2(m_k, n_l) \Big\{ \sum_1^\infty \sum_1^\infty \sum_1^\infty \sum_1^\infty |I_{jklh}|^2 \Big\}^{1/2}$$

in absolute value.

Zygmund has shown* that if $m_{k+1}/m_k > \lambda > 1$,

(*A. Zygmund, Trigonometrical Series, § 5.4)

$n_{\ell+1}/n_\ell > \lambda > 1$, then a number $\Delta = \Delta(\lambda)$ exists such that every integer μ can be represented no more than Δ times in the forms $m_k \pm m_j$ or $n_\ell \pm n_k$, $k > 0$, $j > 0$, $\ell > 0$, $h > 0$.

We observe that $\psi(x,y) \subset L^2$. For fixed j and h the second factor in any term on the right hand side of (2.9) could not exceed $\left(\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} |\nu_{k\ell}|^2\right)^{1/2} < \infty$ where $\nu_{k\ell}$ denotes the complex Fourier coefficient corresponding to that term. When we let j and h range from 1 to ∞ we have the second factor less than $\Delta\left(\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} |\nu_{k\ell}|^2\right)^{1/2} < \infty$.

We choose N sufficiently large such that

$$4\pi^2\left(\sum_{j=N}^{\infty}\sum_{k=N}^{\infty}\sum_{\ell=N}^{\infty}\sum_{h=N}^{\infty} |A_{jk\ell h}|^2\right)^{1/2} < 1/10\,|\varepsilon|$$

$$j \neq k, \quad \ell \neq h$$

$$4\pi^2\left(\sum_{j=N}^{\infty}\sum_{k=N}^{\infty}\sum_{\ell=N}^{\infty}\sum_{h=N}^{\infty} |B_{jk\ell h}|^2\right)^{1/2} < 1/10\,|\varepsilon|,$$

$$\cdots$$

$$4\pi^2\left(\sum_{j=N}^{\infty}\sum_{k=N}^{\infty}\sum_{\ell=N}^{\infty}\sum_{h=N}^{\infty} |B_{jk\ell h}|^2\right) < \tfrac{1}{10}\,|\varepsilon|,$$

There is no loss in generality if we omit the terms of the series (1.1) for $1 \leq k \leq N$, $1 \leq \ell < N$, replacing them by zeros. The series remains summable and this only changes the value of the sum M.

It is made clear by using inequalities III, IV, and V and equality II with the condition $\lim\limits_{\alpha\to\alpha_0} R_\alpha(m_k, n_\ell) = 1$ that the right hand side of (2.9) does not exceed

$$\tfrac{1}{2}|\varepsilon| \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2}{4}$$

We see from (2.5), (2.8) and equality (I) that

$$M^2|\varepsilon| \geqq |\varepsilon| \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{(a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2)}{4} \pm \Omega$$

where

$$|\Omega| \leqq \tfrac{1}{2}|\varepsilon| \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{(a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2)}{4}.$$

Hence

$$8M^2 \geqq \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} (a_{k\ell}^2 + b_{k\ell}^2 + c_{k\ell}^2 + d_{k\ell}^2),$$

which proves the convergence of (2.1).

We will now consider the case where each row and column of $\{\varphi_{ij}(\alpha)\}$ is not restricted to consist of only a finite number of terms different from zero. We will let each row and column have an infinite number of terms different from zero. Let

$$\sigma_\alpha^*(x,y) = \sum_{p=1}^{P}\sum_{q=1}^{Q}\varphi_{pq}(\alpha)s_{pq}(x,y), \qquad \alpha \subset G,$$

where neither P nor Q are infinite but are numbers $P = P(\alpha)$, $Q = Q(\alpha)$. We take P and Q large enough to satisfy the following conditions;

(i) $\lim_{\alpha \to \alpha_0} \sum_{p=1}^{P}\sum_{q=1}^{Q}\varphi_{pq}(\alpha) = 1;$

(ii) $|\sigma_{\alpha_i}(x,y) - \sigma_{\alpha_i}^*(x,y)| \leq \epsilon_{\alpha_i}$, for $(x,y) \subset E - E^{\alpha_i}$, where $\alpha_i \to \alpha_0$ as $i \to \infty$, $\epsilon_{\alpha_i} \to 0$ as $\alpha_i \to \alpha_0$ and the set E^{α_i} is of measure $\leq 2^{-i-2}|E|$. Putting $E^* = E^{\alpha_1} + E^{\alpha_2} + \ldots$, we see that $|E^*| \leq \frac{1}{2}|E|$, hence in the set $E - E^*$ of positive measure $\sigma_{\alpha_i}^*(x,y)$ tends to a finite limit. But (i) insures that the $\sigma_{\alpha_i}^*$ are ϕ^* means, corresponding to a sequence of φ_{ij}'s with only a finite number of terms different from zero in each row and column. In virtue of the special case already dealt with, the theorem is completely established.

Corollary. If the series (1.1) converges in a set of positive measure, the series (2.1) converges.

We will now generalize a theorem of Kolmogoroff which gives us a converse to the corollary. If (2.1) converges, then, by the extension of the Riesz-Fisher theorem* to double Fourier series, there exists a function $f(xy) \subset L^2$.

(*Hobson, The Theory of Functions of a Real Variable, vol. II, § 470)

We will show that the lacunary double Fourier series corresponding to f(x,y) converges almost everywhere.

3. Generalization of Kolmogoroff's theorem. Let f(x,y) be a function of the class L^2 and $s_{mn}(x,y)$ denote the partial sums of the double Fourier series

$$(3.1) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny + c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny)$$

of f(x,y).

THEOREM II. If $m_{k+1}/m_k > \lambda > 1$, $n_{\ell+1}/n_\ell > \lambda > 1$, $k, \ell = 1, 2, \ldots$, then the partial sums $s_{m_k n_\ell}(x,y)$ of (3.1), corresponding to a function $f(x,y) \subset L^2$, converge almost everywhere to f(x,y) .

A series α_{ij} is said to possess a gap (u,v;r,t) if $\alpha_{ij} = 0$ for $u < i \leq r$, $v < j \leq t$. We will need the following lemma.

LEMMA. If a series $\sum\sum \alpha_{ij}$, with partial sums s_{mn}, possesses infinitely many gaps $(M_k, N_\ell; M'_k, N'_\ell)$ such that $M'_k/M_k > \lambda > 1$, $N'_\ell/N_\ell > \lambda > 1$, and is summable $(C, 1)$ to sum S, then $S_{M_k N_\ell}$ converges to S.

There is no loss in generality if we let $S = 0$. Then, if we let Δ be the domain

$$M_k < M_i \leq M_k', \qquad N_\ell < N_j \leq N_\ell' ,$$

we have

$$(M_k' - M_k)(N_\ell' - N_\ell) S_{M_k N_\ell} = \sum_\Delta S_{M_i N_j}$$

$$= (M_k'+1)(N_\ell'+1)\sigma_{M_k' N_\ell'} - (M_k+1)(N_\ell+1)\sigma_{M_k N_\ell}$$

$$= o(M_k' N_\ell') + o(M_k N_\ell) = o(M_k' N_\ell').$$

Hence $S_{M_k N} = o(1)$ and our lemma is established. In particular we have

THEOREM III. If the double Fourier series of a function $f(x,y) \subset L^2$ possesses infinitely many gaps $(M_k, N_\ell ; M_k', N_\ell')$ such that $M_k'/M_k > \lambda > 1$, $N_\ell'/N_\ell > \lambda > 1$, then the partial sums $S_{M_k N_\ell}(x,y)$ converge almost everywhere to $f(x,y)$.

In order to prove theorem II we split (3.1) into rectangular blocks consisting of terms $m_k \leq m < m_{k+1}$, $n_\ell \leq n < n_{\ell+1}$. We then break (3.1) into two series, one consisting of the terms of rectangular blocks where the sum $k+\ell$ is even, the other consisting of terms of rectangular blocks where the sum $k + \ell$ is odd. Applying the extended Riesz-Fisher theorem we see that these series are Fourier series of functions $f'(x,y) \subset L^2$ and $f''(x,y) \subset L^2$. Theorem II tells us that the partial sums $S'_{m_k n_\ell}(x,y)$ and $S''_{m_k n_\ell}(x,y)$ of the two series converge almost everywhere to $f'(x,y)$ and $f''(x,y)$ respectively. Hence $S_{m_k n_\ell} = S'_{m_k n_\ell} + S''_{m_k n_\ell}$ converges almost everywhere to $f'(x,y) + f''(x,y) = f(x,y)$.

4. Generalization of theorem in A. Zygmund, Trigonometrical Series. § 9.601. We let

$$(4.1) \qquad M_p[f(x,y);\ a,\ b,\ c,\ d] = \left(\int_a^b\int_c^d |f(x,y)|^p \, dydx\right)^{1/p},$$

$$(4.2) \qquad A_p[f(x,y);\ a,\ b,\ c,\ d] = \left(\frac{1}{b-a} \cdot \frac{1}{d-c} \int_a^b\int_c^d |f(x,y)|^p \, dydx\right)^{1/p}.$$

when the domain $(a, c;\ b, d)$ is fixed, we shall write $M_p[f]$, $A_p[f]$. We shall need to make use of Hölder's inequality

$$(4.3) \qquad M[fg] \leq M_p[f]\, M_{p'}[g], \qquad p > 1,$$

where

$M[fg] = M_1[fg]$ and where $f, g \geqq 0$,

(4.4) $$\frac{1}{P} + \frac{1}{P'} = 1 .$$

We shall also need the following lemma.

LEMMA. Given a function $f(x,y)$, the expression $A_\alpha[f]$ is a non-decreasing function of α.

In (4.3) let $f = |f|^\alpha$, $\alpha > 0$, and $g = 1$.

Then

$$M[|f|] \leqq M_P[|f|]\, M_{P'}[1] ,$$

from which

$$\frac{1}{(b-a)(d-c)}\left(\int_a^b\int_c^d |f|^\alpha dydx\right) \leqq \left(\frac{1}{(b-a)(d-c)}\right)^{1-1/P'}\left(\int_a^b\int_c^d |f|^{P\alpha} dydx\right)^{1/P} .$$

From (4.4) we see that $1 - 1/P' = 1/P$. We take the α root of each side getting

$$\left\{\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |f|^\alpha dydx\right\}^{1/\alpha} \leqq \left\{\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |f|^{P\alpha} dydx\right\}^{1/P\alpha},$$

hence

$$A_\alpha[f] \leqq A_{\alpha P}[f].$$

Let γ_{kl} be as defined in (2.3).

THEOREM IV. If $m_{k+1}/m_k > \lambda > 1$, $n_{\ell+1}/n_\ell > \lambda > 1$, $k, \ell = 1, 2, \ldots$, and if the series $\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} |\gamma_{k\ell}|^2$ converges, then (1.1) is the Fourier series of a function f(x,y) belonging to every class L^P, and

(4.5) $$\left\{ \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} |f(x,y)|^P \, dy \, dx \right\} \leq C_{P,\lambda} \left\{ \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} |\gamma_{k\ell}|^2 \right\}^{1/2},$$

where $C_{P,\lambda}$ depends only on P and λ.

Applying the lemma we see that
$$\left\{ \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |f(x,y)|^P \, dy \, dx \right\}^{1/P}$$
is a non-decreasing function of P. Let
$$G(P) = \left\{ \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} |f(x,y)|^P \, dy \, dx \right\}^{1/P}$$
then $(1/4)^{1/P}$ G(P) is a non-decreasing function of P but $(1/4)^{1/P}$ is a decreasing function of P, hence G(P) is an increasing function of P.

Since the left hand side of (4.5) is an increasing function of P we may choose, for odd values of P, $A_{P,\lambda}$ the same as $A_{P+1,\lambda}$. Hence it is sufficient to consider the values $P = 2h$, $h = 1, 2, \ldots$. We first suppose that (1.1) converges absolutely and we let

$$F(z, Z) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \gamma_{k\ell} \, z^{m_k} Z^{n_\ell}$$

Be the power series of which the real part, for $z = e^{ix}$ and $Z = e^{iy}$, is (1.1). Then

$$F^h(z,Z) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} D_{\mu\nu} z^\mu Z^\nu,$$

where the series on the right converges, and where $D_{\mu\nu} = 0$, if μ is not of the form

(4.6) $$\alpha_1 m_{k_1} + \alpha_2 m_{k_2} + \cdots,$$

$$\text{with } m_{k_1} > m_{k_2} > \cdots;$$

$$\alpha_i > 0, \ \alpha_1 + \alpha_2 + \cdots = h,$$

and if ν is not of the form

(4.7) $$\beta_1 n_{\ell_1} + \beta_2 n_{\ell_2} + \cdots,$$

$$\text{with } n_{\ell_1} > n_{\ell_2} > \cdots,$$
$$\beta_i > 0, \qquad \beta_1 + \beta_2 + \cdots = h.$$

We observe that if λ is sufficiently large, $\lambda > \lambda_0$, every positive integer can be represented at most once in the form (4.6) and at most once in the form (4.7). If this were not so it would be possible to represent μ as $\delta_0 m_{k_0} + \delta_1 m_{k_1} + \cdots$ where at least one $\delta_i \neq a_i$. Let j be the smallest value of i for which $\delta_i \neq a_i$. We subtract the representation with δ_i as coefficients from (4.6) getting

$$(\alpha_j - \delta_j) m_{k_j} + (\alpha_{j+1} - \delta_{j+1}) m_{k_{j+1}} + \cdots = 0,$$

where

$$m_{k_j} > m_{k_{j+1}} > \cdots, \quad 0 \leq |\alpha_i - \delta_i| \leq h \qquad i = j, j+1, \cdots.$$

Then

$$(\alpha_j - \delta_j) m_{k_j} = - \left\{ (\alpha_{j+1} - \delta_{j+1}) m_{k_{j+1}} + (\alpha_{j+2} - \delta_{j+2}) m_{k_{j+2}} + \cdots \right\}$$

$$|\alpha_j - \delta_j| m_{k_j} \leq h (m_{k_{j+1}} + m_{k_{j+2}} + \cdots),$$

but $|\alpha_j - \delta_j|$ is an integer ≥ 1, hence

$$m_{k_j} \leq h(m_{k_{j+1}} + m_{k_{j+2}} + \cdots),$$

$$1 \leq h(\lambda^{-1} + \lambda^{-2} + \quad \cdots)$$

which is impossible if $\lambda \geq \lambda_0 = h + 1$.

Applying the extended Riesz-Fisher theorem we obtain

$$\frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} |F^h(e^{ix}, e^{iy})|^2 \, dydx = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} |D_{\mu\nu}|^2,$$

hence $f(x,y)$ is of class L^{2h} or L^{P}. If μ is of the form (4.6) and ν is of the form (4.7) we may write

$$\mu = \omega_1 m_{k_i} + \omega_2 m_{k_s} + \omega_3 m_{k_t} + \cdots,$$

$$\nu = \omega_1 m_{k_q} + \omega_2 m_{k_u} + \omega_3 m_{k_v} + \cdots,$$

where i, s, and t are fixed indices equal to some number 1, 2, 3, ..., possibly equal to each other. The indices q, ω, and ν have the same properties as i, s, and t. The ω' s are the powers to which each successive γ_{kl} is raised in the product $D_{\mu\nu}$, hence

$$\omega_1 + \omega_2 + \cdots = h$$

We have

$$D_{\mu\nu} = \frac{h!}{\omega_1!\,\omega_2!\,\omega_3!\cdots}\,\gamma_{k_1 l_1}^{\omega_1}\,\gamma_{k_1 l_2}^{\omega_2}\,\gamma_{k_2 l_1}^{\omega_3}\cdots,$$

$$|D_{\mu\nu}| \leq h!\,\frac{h!}{\omega_1!\,\omega_2!\,\omega_3!\cdots}\,|\gamma_{k_1 l_1}|^{2\omega_1}\,|\gamma_{k_1 l_2}|^{2\omega_2}\,|\gamma_{k_2 l_1}|^{2\omega_3}\cdots,$$

$$\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}|D_{\mu\nu}|^2 = h!\,\Big(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|\gamma_{kl}|^2\Big)^h.$$

Hence, if

$$\lambda = \lambda_0$$

(4.7)

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|F(e^{ix}, e^{iy})|^{2h}\,dy\,dx \leq h!\,\Big(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|\gamma_{kl}|^2\Big)^h.$$

Since the real part of $F(e^{ix}, e^{iy})$ is $f(x,y)$ we have $\left|f(x,y)\right| \leq \left|F(e^{ix},e^{iy})\right|$ and the inequality (4.5) follows with

$$C^{2h}_{P,\lambda} = 4h!$$

In order to remove the condition concerning the absolute convergence of (1.1), we apply (4.5) to the function

$$f(r,\rho, x,y) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} (a_{k\ell}\cos m_k x \cos n_\ell y + b_{k\ell}\cos m_k x \sin n_\ell y$$
$$+ c_{k\ell}\sin m_k x \cos n_\ell y + d_{k\ell}\sin m_k x \sin n_\ell y) r^{m_k}\rho^{n_\ell}$$

and then let $r, \rho \to 1$. We see that (4.5) holds if both ρ and r are less than 1. However, the right hand side of (4.5) is convergent, hence (4.5) is true in the limit as r and ρ approach 1.

In order to prove (4.5) for general $\lambda > 1$, we break (1.1) up into a finite number, say N, of series for each of which the number λ is $\geq h+1$. Correspondingly

$$f = f_1 + \dots + f_N$$

Applying the above result to each f_i we have

$$M_{2h}[f_i] \leq (4h!)^{1/2h}\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}|\gamma_{k\ell}|^2\right\}^{1/2}$$

However

$M_{2h}[f] \leq \sum_{i=1}^{N} M_{2h}[f_i]$. We have obtained (4.5)

with

$$C_{2h,\lambda} = C_{P,\lambda} = N(4h!)^{1/2h}.$$

5. Generalization of Szidon's theorem on lacunary series. This theorem on absolute convergence tells us far more than we can expect for ordinary double Fourier series.

THEOREM V. If a lacunary double trigonometrical series (1.1), where

$$m_{k+1}/m_k > \lambda > 1, \quad n_{\ell+1}/n_\ell > \lambda > 1,$$

is the double Fourier series of a bounded function

$$f(x,y), \quad |f| \leq B,$$

then the series converges absolutely.

Taking, instead of $f(x,y)$, combinations of the functions $f(x,y) \pm f(x,-y) \pm f(-x,y) \pm f(-x,-y)$ where the number of terms that are preceeded by a positive sign is always even, we may restrict ourselves to purely cosine cosine series, sine sine series, cosine sine series or sine cosine series, e.g., the former. We shall need to consider the non-negative polynomials.

(5.1) $$R_{pq}(x,y) = \prod_{k=1,\,\ell=1}^{p\,q} (1+\epsilon_k\epsilon_\ell \cos M_k x \cos N_\ell y)$$

where
$$\epsilon_k,\ \epsilon_\ell = \pm 1$$

and the positive integers M_k and N_ℓ satisfy the inequalities
$$M_{k+1}/M_k > \mu > 3 \ , \quad N_{\ell+1}/N_\ell > \mu > 3 \ .$$

Performing the indicated multiplication in (5.1) and using elementary reduction formulas we see that the product consists of the constant term 1 and of the terms
$$A_{\nu\omega} \cos \nu x \cos \omega y$$

where

$$\nu = \pm M_{k_1} \pm M_{k_2} \pm \cdots \pm M_{k_i} > 0,$$

$$M_{k_1} < M_{k_2} < \cdots < M_{k_i} \leqq M_{k_p} \ ,$$

$$\omega = \pm N_{\ell_1} \pm N_{\ell_2} \pm \cdots \pm N_{\ell_j} > 0,$$

$$N_{\ell_1} < N_{\ell_2} < \cdots < N_{\ell_j} \leqq N_{\ell_q} .$$

We see that

$$(M_{k_i} + M_{k_{i-1}} + \dots + M_{k_2} + M_{k_1}) \geq \nu \geq (M_{k_i} - M_{k_{i-1}} - \dots - M_{k_2} - M_{k_1})$$

and

$$(N_{\ell_j} + N_{\ell_{j-1}} + \dots + N_{\ell_2} + N_{\ell_1}) \geq \omega \geq (N_{\ell_j} - N_{\ell_{j-1}} - \dots - N_{\ell_2} - N_{\ell_1}).$$

We readily obtain

$$M_{k_i}(1+\mu^{-1}+\mu^{-2}+\dots) > \nu > M_{k_i}(1-\mu^{-1}-\mu^{-2}-\dots),$$

$$N_{\ell_j}(1+\mu^{-1}+\mu^{-2}+\dots) > \omega > N_{\ell_j}(1-\mu^{-1}-\mu^{-2}-\dots),$$

then

$$M_{k_i}\left(\frac{\mu}{\mu-1}\right) > \nu > M_{k_i}\left(\frac{\mu-2}{\mu-1}\right),$$

$$N_{\ell_j}\left(\frac{\mu}{\mu-1}\right) > \omega > N_{\ell_j}\left(\frac{\mu-2}{\mu-1}\right)$$

Since $\mu \geq 3$ we have $M_{k+1} > 2\sum_{i=1}^{k} M_i$, $N_{\ell+1} > 2\sum_{j=1}^{\ell} N_j$

Hence, the numbers

$$\pm M_{k_1} \pm M_{k_2} \pm \dots \pm M_{k_i}$$

corresponding to various sequences $\{k_s\}$ are all different. Similarly, the numbers $\pm N_{\ell_1} \pm N_{\ell_2} \pm \dots \pm N_{\ell_j}$ corresponding to various sequences $\{\ell_s\}$ are all different.

If μ is large enough, $\mu \geq \mu_0(\epsilon)$, the indices ν and ω corresponding to $A_{\nu\omega}$ concentrate in the neighborhoods

$$[M_k(1-\epsilon),\ M_k(1+\epsilon)]$$

and

$$[N_\ell(1-\epsilon),\ N_\ell(1+\epsilon)]$$

of the numbers M_k and N_ℓ. We choose $\epsilon > 0$ and arbitrarily small.

Returning to the series (1.1), take take $\epsilon > 0$ so small that the intervals

$$[m_k(1-\epsilon),\ m_k(1+\epsilon)],$$

$$[n_\ell(1-\epsilon),\ n_\ell(1+\epsilon)\,)\ ,\quad k,\ell = 1,2,\ldots,$$

do not overlap and choose an integer r such that

$$\lambda^r = \mu_0(\epsilon)$$

Put

$$M_k^{(s)} = m_{kr+s},\quad N_\ell^{(t)} = n_{\ell r+t},\quad k,\ell = 1,2,\ldots,$$

$$0 \leq s \leq r-1, \qquad 0 \leq t \leq r-1.$$

Let

$$R_{pq}^{(st)}(x,y) = \prod_{k=1,\ell=1}^{p,q} (1+ \epsilon_k \epsilon_\ell \cos M_k^{(s)} x \cos N_\ell^{(t)}(y)),$$

where

$$\epsilon_k \epsilon_\ell = \text{sign } a_{kr+s,\, \ell r+t} \, .$$

We assumed $a_{00} = 0$, hence

$$\int_0^{2\pi}\int_0^{2\pi} f(x,y)\,dydx \quad 0.$$

Since

$$M_{k+1}^{(s)}/M_k^{(s)} > \lambda > \mu(\epsilon) \quad ,$$

$$N_{\ell+1}^{(t)} / N^{(t)} > \lambda > \mu_0(\epsilon)$$

the only terms of

$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y) R_{pq}^{(st)}(x,y)\,dydx$$

different from zero are those that define the coefficients $a_{k\ell}$ in (1.1). This is true because each term defines a Fourier coefficient but we have assumed the coefficients a_k in (1.1) to be the only coefficients different from zero.

Hence

$$(5.2) \qquad \sum_{k=1}^{p} \sum_{l=1}^{q} \left| a_{kr+s,\, lr+t} \right|$$

$$= \frac{1}{\pi^2} \int_0^{3\pi}\int_0^{2\pi} f(x,y) R_{pq}^{(st)}(x,y)\,dydx \;\leq \frac{B}{4\pi^2}\int_0^{3\pi}\int_0^{3\pi} R_{pq}^{(st)}(x,y)\,dydx = 4B.$$

Letting p, q $\to \infty$ we see that the resulting infinite series defined by the left hand side of (5.2) is convergent for each choice of s and t. Since s and t are bounded integers we have

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty .$$

In the case of an odd-odd function we consider, instead of (5.1), analogous polynomials with cosines replaced by sines. However, in this case ν and ω will not necessarily be positive. We can show, if M_{k_i} is the largest of the M_k's, that

$$\left|\epsilon_{i+1} M_{k_{i+1}}\right| > \nu > \sum_{u=1}^{i-1} \left|\epsilon_u M_{k_u}\right| > \left|\epsilon_{i-1} M_{k_{i-1}}\right|$$

where

$$\epsilon_u = \pm 1 .$$

This is seen by making use of the inequality

$$M_{k_{i\,1}} > 2\sum_{u=1}^{i} M_{k_u} .$$

We also have

$$|\nu| = \left|\epsilon_i M_{k_i}\right| \pm \left|\sum_{u=1}^{i-1} \epsilon M_{k_u}\right| ,$$

but

$$\sum_{u=1}^{i-1} \epsilon_u M_{k_u} \neq 0$$

because

$$\left|\epsilon_{i-1} M_{k_{i-1}}\right| > 2\sum_{u=1}^{i-1} \left|\epsilon_u M_{k_u}\right|.$$

Hence

$$|\nu| \neq |\epsilon_i M_{k_i}|.$$

We have shown that $|\nu|$ is between $M_{k_{i-1}}$ and $M_{k_{i+1}}$ but is not equal to M_{k_i}. Similarly we can show that $|\omega|$ is between $N_{\ell_{j-1}}$ and $N_{\ell_{j+1}}$ but is not equal to N_{ℓ_j}. These added results enable us to see that all of the terms of our sine sine polynomial corresponding to

$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y) R_{pq}^{(st)}(x,y)\,dydx$$

are zero except those that define the coefficients $d_{k\ell}$ in (1.1) Of course the product of an even number of sine terms can be represented as a cosine term but since we are assuming an odd-odd function, the integral of f(x,y) multiplied by a cosine cosine term or by a cosine sine term is zero for all m_k and n_ℓ.

In the case of an odd-even function or an even-odd function we consider, instead (5.1), analogous polynomials with cos $M_k x$ replaced by sin $M_k x$ or cos $N_\ell y$ replaced by sin $N_\ell y$. However, as in the previous case

$$M_{k_{i-1}} < |\nu| < M_{k_{i+1}}, \quad N_{\ell_{j-1}} < |\omega| < N_{\ell_{j+1}},$$

and

$$\nu \neq M_{k_i}, \ \omega \neq N_{\ell_j}.$$

Using these facts we conclude

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |b_{ij}| < \infty, \ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |c_{ij}| < \infty,$$

hence (1.1) converges absolutely.

6. We know that a necessary condition for a double sequence $\{a_{mn}, b_{mn}, c_{mn}, d_{mn}\}$ to be that of the Fourier coefficients of an integrable function f(x,y) is $|a_{mn}|+|b_{mn}|+|c_{mn}|+|d_{mn}| \to 0$. However

$$|a_{mn}|+|b_{mn}|+|c_{mn}|+|d_{mn}| \to 0$$

is not a sufficient condition that there exists an integrable function f(x,y) with

$$\{a_{mn}, b_{mn}, c_{mn}, d_{mn}\}$$

as the coefficients of its double Fourier series, but we shall prove that if a_{mn}, b_{mn}, c_{mn}, d_{mn} tend to zero rapidly enough, then we may choose the coefficients of the double Fourier series, for some value of m and n, as the above sequence. The result that we shall obtain is a weaker result than the result for simple Fourier series which was stated by Banach. We shall prove the following theorem.

THEOREM VI. Let $\{m_i\}$, $\{n_j\}$ be sequences of positive integers such that $m_{i+1}/m_i > \lambda > 1$, $n_{j+1}/n_j > \lambda > 1$, i, j = 1,2,..., and let $\{u_{ij}\}$, $\{v_{ij}\}$, $\{w_{ij}\}$, and $\{z_{ij}\}$ be sequences of real numbers such that $u_{ij} \log i \log j \to 0$,

$$v_{ij} \log i \log j \to 0, \quad w_{ij} \log j \cancel{\log} i \to 0,$$

and

$$z_{ij} \log i \log j \to 0,$$

then there exists an integrable function f(x,y) such that the Fourier coefficients $a_{m_i n_j}$, $b_{m_i n_j}$, $c_{m_i n_j}$, and $d_{m_i n_j}$ are u_{ij}, v_{ij}, w_{ij}, and z_{ij} respectively.

In proving this theorem we shall need to make use of several lemmas which are interesting in themselves and have wider application. We shall use the following lemma.

LEMMA I. Le $\{m_i\}$, $\{n_j\}$ be sequences satisfying the same conditions as above. If

$$\{u_{ij}\}, \quad \{v_{ij}\}, \quad \{w_{ij}\},$$

and $\{z_{ij}\}$ are any bounded double sequences, then there exists a double Fourier-Stieltjes series of a non-decreasing function having u_{ij} , v_{ij} , w_{ij} , and z_{ij} as the coefficients with indices $m_i n_j$.

However, to prove this lemma we shall use some other lemmas., First we shall define bounded variation for a function of two variables. We are using Hardy's definition.*

(*E. W. Hobson, The Theory of Functions of a Real V ariable, vol. I, P.345.)

We have the following lemma due to R. J. Dunholter.*

(*Doctor's dessertation, University of Cincinnati, 1939).

LEMMA II. Given a double sequence of functions $\{F_{mn}(x,y)\}$ defined in $(0,0;\ 2\pi, 2\pi)$ and of uniformly, bounded variation, either there exists a uniformly bounded subsequence $\{F_{m_k n}(x,y)\}$ which converges everywhere to a function $F(x,y)$ of bounded variation, or $\{|F_{mn}(x,y)|\}$ diverges uniformly to ∞ as m, n tend toward ∞.

LEMMA III. If $\sigma_{m_k n_\ell} \geq 0$, $\{\sigma_{m_k n_\ell}\}$ being a subsequence of $\{\sigma_{mn}\}$, m, n = 0, 1, 2, ..., where σ_{mn} is the first arithmetic mean of the first m rows and n columns of the double trigonometrical series

$$(6.1)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (a_{ij}\cos ix\cos jy + b_{ij}\cos ix\sin jx$$
$$+ c_{ij}\sin ix\cos jy + d_{ij}\sin ix\sin jy),$$

then (6.1) is a Fourier-Stieltjes series of a non-decreasing function.

Let

$$F_{m_k n_\ell}(x,y) = \int_0^x\int_0^y \sigma_{m_k n_\ell}(u,v)\,du\,dv.$$

We have

$$F_{m_k n_\ell}(2\pi,2\pi) = 4\pi^2 a_{00},$$

hence the functions $F_{m_k n_\ell}(x,y)$ are of uniform bounded variation over $(0,0;\ 2\pi,2\pi)$. Applying lemma II we find that there exists a uniformly bounded subsequence $\{F_{M_k N}(x,y)\}$ converging everywhere to a function

F(x,y) of bounded variation.

By carrying out the integration on the right and using the orthogonal properties we see that

$$\left(1-\frac{|\alpha|}{M_k+1}\right)\left(1-\frac{|\beta|}{N_l+1}\right)\gamma_{\alpha\beta}$$
$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sigma_{M_kN_l}e^{-i\alpha x}e^{-i\beta y}\,dy\,dx$$

where

$$M_k > |\alpha|, \quad N_l > |\beta|$$

$\gamma_{\alpha\beta}$ was defined in (2.3). Considering the double integral as an iterated integral and integrating by parts we find that it is equal to

$$\frac{1}{4\pi^2}F_{M_kN_l}(2\pi,2\pi)+\frac{i\beta}{4\pi^2}\int_0^{2\pi}F_{M_kN_l}(2\pi,y)e^{-i\beta y}\,dy$$
$$+\frac{i\alpha}{4\pi^2}\int_0^{2\pi}F_{M_kN_l}(x,2\pi)e^{-i\alpha x}\,dx-\frac{\alpha\beta}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}F_{M_kN_l}(x,y)e^{-i\alpha x}e^{-i\beta y}\,dy\,dx.$$

Letting k, $l\to\infty$ we find that

$$\gamma_{\alpha\beta}=\frac{1}{4\pi^2}F(2\pi,2\pi)+\frac{i\beta}{4\pi^2}\int_0^{2\pi}F(2\pi,y)e^{-i\beta y}\,dy$$
$$+\frac{i\alpha}{4\pi^2}\int_0^{2\pi}F(x,2\pi)e^{-i\alpha x}\,dx-\frac{\alpha\beta}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}F(x,y)e^{-i\alpha x}e^{-i\beta y}\,dy\,dx$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}e^{-i\alpha x}e^{-i\beta y}\,d_{xy}F(x,y),$$

for α, $\beta = 0, \pm 1, \pm 2, \ldots$, so that (6.1) is a Fourier-Stieltjes series.

LEMMA IV. If $\int_{0}^{2\pi}\int_{0}^{2\pi} \sigma_{mn}(x,y)\,dy\,dx < V$, where V is a finite constant independent of m and n, then (6.1) is a Fourier-Stieltjes series.

Since $F_{mn}(0,0) = 0$, $m, n = 0, 1, 2, \ldots$, $\{|F_{mn}(x,y)|\}$ cannot diverge to $+\infty$ and so there exists a sequence $\{F_{M_k N}(x,y)\}$ uniformly bounded and converging everywhere to a function $F(x,y)$ of bounded variation. The proof follows the same asthe proof of lemma III. We are now ready to prove lemma I.

It will be convenient to write $u_{m_i n_j}$, $v_{m_i n_j}$, $w_{m_i n_j}$, and $z_{m_i n_j}$ instead of u_{ij}, v_{ij}, w_{ij}, and z_{ij}. There will be no loss in generality if we assume

$$\rho^2_{m_i n_j} = u^2_{m_i n_j} + v^2_{m_i n_j} + w^2_{m_i n_j} + z^2_{m_i n_j} \leq 1.$$

We first assume that $\lambda > 3$ and put

$$u_{m_i n_j} \cos m_i x \cos n_j y + v_{m_i n_j} \cos m_i x \sin n_j y$$
$$+ w_{m_i n_j} \sin m_i x \cos n_j y + z_{m_i n_j} \sin m_i x \sin n_j y$$
$$= \rho_{m_i n_j}[\cos \psi_{m_i n_j} \cos m_i x \cos (n_j y + \theta_{m_i n_j})$$
$$+ \sin \psi_{m_i n_j} \sin m_i x \cos (n_j y + \theta_{m_i n_j})]$$

and consider the partial products p_{kl} of the product

$$(6.2) \qquad P = \prod_{i=1,j=1}^{\infty,\infty} \Big\{ 1 + \rho_{m_i n_j} [\cos \psi_{m_i n_j} \cos m_i x \cos(n_j y + \theta_{m_i n_j}) + \sin \psi_{m_i n_j} \sin m_i x \sin(n_j y + \theta_{m_i n_j})] \Big\}.$$

Multiplying out these factors and making use of elementary trigonometrical reduction formulas we see that all the terms can be arranged in groups, where each group is a sum of two terms of the form as those in the brackets in (6.2). We notice also that the polynomial $p_{k\ell}$ is a partial sum of any polynomial where wither k or ℓ has been increased. Making k, $\ell \to \infty$, we obtain, quite formally, a trigonometrical series. Since some of the partial sums are non-negative, e.g. $p_{k\ell}$, we may apply lemma III, hence this series is a Fourier-Stieltjes series of a non-decreasing function. Moreover the coefficients with suffixes $m_i n_j$ are $u_{m_i n_j}$, $v_{m_i n_j}$, $w_{m_i n_j}$, and $z_{m_i n_j}$. In section 5 we found that if λ is large enough, $\lambda > \lambda_0(\epsilon)$, the indices of terms different from zero concentrate in the neighborhoods

$$[m_k(1-\epsilon),\ m_k(1+\epsilon)]$$

and

$$[n_\ell(1-\epsilon),\ n_\ell(1+\epsilon)]$$

of the numbers m_k and n_ℓ, where $\epsilon > 0$ is arbitrary.

In the general case $\lambda > 1$, we choose r as the smallest integer such that the inequality $\lambda^r \geq \mu$ is satisfied, then break up $\{m_i\}$ into r sequences

$$m_1^1,\ m_2^1,\ \ldots,\quad m_1^2,\ m_2^2,\ \ldots,\quad m_1^r,\ m_2^r,\ \ldots,$$

and $\{n_j\}$ into r sequences

$$n_1^1,\ n_2^1,\ \ldots,\quad n_1^2,\ n_2^2,\ \ldots,\quad n_1^r,\ n_2^r,\ \ldots$$ in

such a way that

$$m_{i+1}^s/m_i^s > \mu, \qquad n_{j+1}^t/n_j^t > \mu,$$

$i,\ j = 1,2,\ \ldots,$

$$1 \leq s \leq r, \quad 1 \leq t \leq r, \qquad \mu > 3$$

being a large number which we shall define later. Let P_{st} denote the product, analogous to (6.2) consisting of the factors

$$1 + \rho_{MN}[\cos\psi_{MN} \cos Mx \cos(Ny + \varphi_{MN})$$
$$+ \sin\psi_{MN} \sin Mx \cos(Ny + \theta_{MN})$$

where M runs through the sequence

$$m_1^s, m_2^s,\ \ldots,$$

N runs through the sequence

$$n_1^t,\ n_2^t,\ \ldots\ .$$

We shall prove that $\sum_{s=1}^{r}\sum_{t=1}^{r} P_{st}$ gives the required Fourier-Stieltjes series. In fact, if μ is large enough, the indices occurring in the series obtained from P_{st} all belong to the intervals

$$(m_i^s/\sqrt{\lambda}\ ,\ m_i^s\sqrt{\lambda}\ ),\quad (n_j^t/\sqrt{\lambda}\ ,\ n_j^t\sqrt{\lambda}\ ),\quad i,\ j = 1,2,\ \ldots,$$

so that the series P_{st}, $s,\ t = 1,2,\ldots r$, do not overlap. Each product P_{st} is a Fourier-Stieltjes series of a non-decreasing function in which the terms with indices $m_i^s n_j^t$ have the coefficients

$$u_{m_i^s n_j^t}\ ,\quad v_{m_i^s n_j^t},\quad w_{m_t^s n_j^t}\ ,$$

and

$$z_{m_i^s n_j^t}\ .$$

Considering $\sum_{s=1}^{r}\sum_{t=1}^{r} P_{st}$, we see that the lemma follows.

LEMMA V. If $S_{mn}(x,y)$ are the partial sums of (6.1) and if $M\,[f(x,y) - S_{mn}(x,y)] \rightarrow 0$, then

(6.1) is the double Fourier series of $f(x,y)$.

$M[f - S]$ is used in the same sense as $M[f]$ in section 4. Put

$$\varphi_{kl}^{(1)}(x,y) = \cos kx \cos ly$$

$$\varphi_{kl}^{(2)}(x,y) = \cos kx \sin ly$$

$$\varphi_{kl}^{(3)}(x,y) = \sin kx \cos ly$$

$$\varphi_{kl}^{(4)}(x,y) = \sin kx \sin ly$$

Then

$$\lim_{m,n\to\infty} \int_0^{2\pi}\int_0^{2\pi} \left\{f(x,y) - S_{mn}(x,y)\right\} \varphi_{kl}^{(i)}(x,y)\,dy\,dx \to 0,$$

$$i = 1,2,3,4.$$

Hence

$$\lim_{m,n\to\infty} \int_0^{2\pi}\int_0^{2\pi} S_{mn}(x,y)\ \varphi_{kl}^{(i)}(x,y)\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^{2\pi} f(x,y)\ \varphi_{kl}^{(i)}(x,y)\,dy\,dx,$$

$$i = 1,2,3,4.$$

$$\therefore a_{kl} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)\cos kx \cos ly \, dy\, dx,$$

$$b_{kl} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)\cos kx \sin ly \, dy\, dx.$$

$$c_{kl} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)\sin kx \cos ly \, dy\, dx,$$

$$d_{kl} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)\sin kx \sin ly \, dy\, dx.$$

LEMMA VI. A necessary and sufficient condition that (6.1) should be a Fourier series is that

$$M\left[\sigma_{mn} - \sigma_{rs}\right] \to 0$$

as

$$m,\ n,\ r,\ s \to \infty$$

Let us suppose that (6.1) is the Fourier series of f(x,y). Integrating the inequality

$$\left|\sigma_{mn}(x,y) - f(x,y)\right| \leqq \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|f(x+u, y+v) - f(x,y)\right| K_{mn}(u,v)\, du\, dv.$$

where

$$(m+1)(n+1)K_{mn}(u,v)$$

$$= \frac{1}{4}\left(\frac{\sin(m+1)\frac{1}{2}u}{\sin\frac{1}{2}u}\right)^2 \left(\frac{\sin\,(n+1)\frac{1}{2}v}{\sin\frac{1}{2}v}\right)^2,$$

over

$$(0,0,;2\pi\,,2\pi)$$

we find that

$$M[\sigma_{mn} - f] \leq \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \eta(u,v)K_{mn}(u,v)\,du\,dv$$

where

$$\eta(u,v) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+u,y+v) - f(x,y)|\,dy\,d\psi$$

We notice that $\eta(u,v)$ is continuous and vanishes for $u = 0$, $v = 0$. Also we notice that the right hand side of the last inequality is mn-th Fejér sum of the double Fourier series of $\eta(u,v)$. Since the Fejér sum tends to the value of the function at all points of continuity where the point has a cross neighborhood in which the function is bounded, we see that

$$M[\sigma_{mn} - f] \to 0$$

as m , n $\to \infty$. Using the inequality

$$M[\sigma_{mn} - \sigma_{rs}] \leq M[\sigma_{mn} - f] + M[\sigma_{rs} - f]$$

we see that

$$M[\sigma_{mn} - \sigma_{rs}] \to 0$$

as m, n, r, s $\to \infty$.

Conversely, the condition $M[\sigma_{mn} - \sigma_{rs}] \to 0$ implies

$$M[\sigma_{mn}] = O(1).$$

Hence, by lemma IV, (6.1) is a Fourier-Stieltjes series of F(x,y). If we can show that F(x,y) is absolutely continuous, then $\frac{\partial^2 F(x,y)}{\partial x \, \partial y}$ will exist almost everywhere and will be equal to f(x,y), hence (6.1) will be the Fourier series of f(x,y).

In order to show that F(x,y) is absolutely continuous it is sufficient to show that the functions

$$F_{mn}(xy) = \int_0^x \int_0^y \sigma_{mn}(uv)\,du\,dv$$

are uniformly absolutely continuous, i.e. that, given an $\epsilon > 0$, there exists a $\delta > 0$ such that, for any finite system A of non-overlapping rectangular domains (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), ...,

$$(c_1 - a_1)(d_1 - b_1) + (c_2 - a_2)(d_2 - b_2) + \dots < \delta$$

we have

$$\sum_i \Big| F_{mn}(c_i, d_i) - F_{mn}(c_i, b_i)$$

$$- F_{mn}(a_i, d_i) + F_{mn}(a_i, b_i)\Big| < \epsilon$$

$m > m_0$, $n > n_0$.

In fact if, for fixed A, the inequality is satisfied by the functions F_{mn}, it is satisfied by $F = \lim F_{mn}$. Now

$$M[\sigma_{mn};A] \leq M[\sigma_{mn} - \sigma_{rs};A] + M[\sigma_{rs};A]$$

$$\leq M[\sigma_{mn} - \sigma_{rs}] + M[\sigma_{rs};A].^*$$

(* $M[\sigma_{m,n};A]$ denotes the double integral of $|\sigma_{mn}|$ over A).

Let r, s be so large that $M[\sigma_{mn} - \sigma_{rs}] < \frac{1}{2}\epsilon$

for $m > r$, $n > s$. For fixed r, s we have

$$M[\sigma_{rs};A] < \tfrac{1}{2}\epsilon$$

if only

$$|A| < \delta = \delta(\epsilon)$$

Therefore

$$M[\sigma_{mn};A] < \epsilon$$

for $m > r$, $n > s$, $|A| < \delta$, i.e.

$$\sum_i |F_{mn}(c_i, d_i) - F_{mn}(c_i, b_i)$$

$$- F_{mn}(a_i, d_i) + F_{mn}(a_i, b_i)| < \epsilon$$

for $m > r$, $n > s$, $|A| < \delta$.

LEMMA VII. If the series

$$(6.3) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny + c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny)$$

is a Fourier-Stieltjes series and if the series

$$(6.4) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn} \cos mx \cos ny$$

is a Fourier series, then

$$(6.5) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn} (a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny + c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny)$$

is a Fourier series.

Let $l_{mn}(x,y)$ and $\sigma_{mn}(x,y)$ denote the (c, 1) means of the series (6.4) and (6.5) respectively. We have

$$\sigma_{mn} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} l_{mn}(u,v)\, d_{uv}F(x+u, y+v).$$

Then

$$\pi^2 M[\sigma_{mn} - \sigma_{rs}]$$

does not exceed

$$M[l_{mn} - l_{rs}]$$

multiplied by the total variation of F over $(0, 0; 2\pi, 2\pi)$. Applying lemma VI we see that

$$M[l_{mn} - l_{rs}] \to 0$$

as m, n, r, s $\to \infty$.

Hence

$$M[\sigma_{mn} - \sigma_{rs}] \to 0$$

as m, n, r, s $\to \infty$.

Applying lemma VI again we see that (6.5) is a Fourier series.

We shall need to make use of the generalization to double series of the Abel transformation*. Consider

(*C. N. Moore, Am. Math. Soc. Coll. Pub. vol. 22, P. 16*) the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{ij}\, f_{ij}\,(\alpha).$$

We set

$$\Delta_{10} f_{ij} = f_{ij} - f_{i+1,\,j},$$

$$\Delta_{01} f_{ij} = f_{ij} - f_{i,\,j+1}$$

$$\Delta_{11} f_{ij} = f_{ij} - f_{i,\,j+1} - f_{i+1,\,j} + f_{i+1,\,j+1}.$$

Then

$$\sum_{i=0}^{p}\sum_{j=0}^{q} u_{ij} f_{ij}(\alpha) = \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} S_{ij}\,\Delta_{11} f_{ij}\,(\alpha)$$

$$+ \sum_{i=0}^{p-1} S_{iq}\,\Delta_{10}\, f_{iq}\,(\alpha)$$

$$+ \sum_{j=0}^{q-1} S_{pj}\,\Delta_{01} f_{pj}\,(\alpha) + S_{pq}\, f_{pq}\,(\alpha).$$

We shall also introduce a convex double sequence. A double sequence $\{\lambda_{mn}\}$ is said to be convex if

$$\Delta_{01}^{2}\lambda_{mn} \geqq 0, \qquad \Delta_{10}^{2}\lambda_{mn} \geqq 0, \qquad \Delta_{11}^{2}\lambda_{mn} \geqq 0,$$

m, n = 0, 1, 2, ...,

where

$$\Delta_{01}\lambda_{mn} = \lambda_{mn} - \lambda_{m,n+1}$$

$$\Delta_{10}\lambda_{mn} = \lambda_{mn} - \lambda_{m+1,n}$$

$$\Delta_{11}\lambda_{mn} = \Delta_{10}(\Delta_{01}\lambda_{mn}) = \Delta_{01}(\Delta_{10}\lambda_{mn})$$

$$\Delta^2_{01}\lambda_{mn} = \Delta_{01}(\Delta_{11}\lambda_{mn}) = \Delta_{11}(\Delta_{01}\lambda_{mn})$$

$$\Delta^2_{10}\lambda_{mn} = \Delta_{10}(\Delta_{11}\lambda_{mn}) = \Delta_{11}(\Delta_{10}\lambda_{mn}).$$

$$\Delta^2_{11}\lambda_{mn} = \Delta_{11}(\Delta_{11}\lambda_{mn})$$

LEMMA VIII. If $\lambda_{mn} \to 0$ as m, n $\to \infty$ and if λ_{mn} log m log n $\to 0$ where $\{\lambda_{mn}\}$ is convex, the series

(6.6) $$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn} \cos mx \cos ny$$

converges, save for x = 0, y = 0 to an integrable, non-negative function f(x,y), and is the double Fourier series of f(x,y).

Applying the generalized Abel transformation to (6.6) we obtain

$$S_{mn}(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} D_{ij}(x,y)\Delta_{11}\lambda_{ij}$$

$$+ \sum_{i=0}^{m-1} D_{in}(x,y)\Delta_{10}\lambda_{in} + \sum_{j=0}^{n-1} D_{mj}(x,y)\lambda_{mj}$$

$$+ D_{mn}(x,y)\lambda_{mn},$$

where $D_{ij}(x,y)$ denotes Dirichlet's kernel for double series. Applying Abel's transformation the second time we obtain

$$S_{mn}(x,y) = \sum_{i=0}^{m-2}\sum_{j=0}^{n-2}(i+1)(j+1)K_{ij}(x,y)\Delta_{11}^{2}\lambda_{ij}$$

$$+\sum_{i=0}^{m-2} n(i+1)K_{i,n-1}(x,y)\Delta_{10}^{2}\lambda_{i,n-1}$$

$$+\sum_{j=0}^{n-2} m(j+1)K_{m-1,j}(x,y)\Delta_{01}^{2}\lambda_{m-1,j}$$

$$+mnK_{m-1,n-1}(x,y)\lambda_{m-1,n-1}$$

$$+\sum_{i=0}^{m-2}(n+1)(i+1)K_{in}(x,y)\Delta_{10}^{2}\lambda_{in}$$

$$+m(n+1)K_{m-1,n}(x,y)\lambda_{m-1,n}$$

$$+\sum_{j=0}^{n-2}(m+1)(j+1)K_{mj}(x,y)\Delta_{01}^{2}\lambda_{mj}$$

$$+(m+1)nK_{m,n-1}(x,y)\lambda_{m,n-1}$$

$$+D_{mn}(x,y)\lambda_{mn}.$$

If $x \neq o$, $y \neq o$, we have that Dirichlet's and Fejér's kernels are bounded, hence each term on the right, except the first, tends to zero with $1/mn$.
Therefore

$$S_{mn} \to f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (i+1)(j+1)K_{ij}(x,y)\Delta_{11}^2 \lambda_{ij},$$

which is non-negative. Since the last series integrated over ($-\pi$, $-\pi$; π, π) gives the finite value

$$\pi^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta_{11}^2 \lambda_{ij} = \pi^2 a_{00},$$ f(x,y) is integrable.

From the expressions for f(x,y) and $S_{mn}(x,y)$ we see that

$$\begin{aligned}
|f(x,y) - S_{mn}(x,y)| \leq & \sum_{i=m-1}^{\infty}\sum_{j=n-1}^{\infty} (i+1)(j+1)K_{ij}(x,y)\Delta_{11}^2 \lambda_{ij} \\
& + \sum_{i=0}^{m-2} n(i+1)K_{i,n-1}(x,y)\Delta_{10}^2 \lambda_{i,n-1} \\
& + \sum_{j=0}^{m-2} m(j+1)K_{m-1,j}(x,y)\Delta_{01}^2 \lambda_{n-1,j} \\
& + mnK_{m-1,n-1}(x,y)\lambda_{m-1,n-1} \\
& + \sum_{i=0}^{m-2} (n+1)(i+1)K_{in}(x,y)\Delta_{10}^2 \lambda_{in} \\
& + m(n+1)K_{m-1,n}(x,y) \\
& + \sum_{j=0}^{n-2} (m+1)(j+1)K_{mj}(x,y)\Delta_{01}^2 A_{mj} \\
& + (m+1)nK_{m,n-1}(x,y)\lambda_{m,n-1} \\
& + |D_{mn}(x,y)|\lambda_{mn}.
\end{aligned}$$

Integrating over $(-\pi, -\pi; \pi, \pi)$ we find that

$$M[f(x,y) - S_{mn}(x,y)] = o(1) + 4\lambda_{mn} L_{mn},$$

where L_{mn} is analogous to Lebesgue's constant. Since

$$L_{mn} = \frac{16}{\pi^2} \log m \log n + O(1)$$

and

$$\lambda_{mn} \log m \log n \to 0$$

we have

$$M[f(x,y) - s_{mn}(x,y)] \to 0.$$

Using lemma V we see that (6.6) is the Fourier series of f(x,y)In order to prove the theorem, Let $\{\lambda_{kl}\}$

In order to prove the theorem, Let $\{\lambda_{kl}\}$
be a convex double sequence such that

$$\lambda_{kl} \log k \log l \to 0$$

and such that

$$\left\{\frac{u_{m_i n_j}}{\lambda_{m_i n_j}}\right\}, \qquad \left\{\frac{v_{m_i n_j}}{\lambda_{m_i n_j}}\right\}, \qquad \left\{\frac{w_{m_i n_j}}{\lambda_{m_i n_j}}\right\},$$

and

$$\left\{ \frac{z_{m_i n_j}}{\lambda_{m_i n_j}} \right\}$$

are bounded. Let us consider a double trigonometrical series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (A_{mn} \cos mx \cos ny + B_{mn} \cos mx \sin ny$$

$$+ C_{mn} \sin mx \cos ny + D_{mn} \sin mx \sin ny)$$

which by lemma I is a Fourier-Stieltjes series if we choose

$$A_{m_i n_j} = \frac{u_{m_i n_j}}{\lambda_{m_i n_j}}, \qquad B_{m_i n_j} = \frac{v_{m_i n_j}}{\lambda_{m_i n_j}},$$

$$C_{m_i n_j} = \frac{\omega_{m_i n_j}}{\lambda_{m_i n_j}}$$

and

$$D_{m_i n_j} = \frac{z_{m_i n_j}}{\lambda_{m_i n_j}}$$

Applying lemma VIII we see that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn} \cos mx \cos ny$$

is a double Fourier series. Hence, by lemma VII, we see that if we set

$$a_{mn} = \lambda_{mn} A_{mn}, \qquad b_{mn} = \lambda_{mn} B_{mn},$$

$$c_{mn} = \lambda_{mn} C_{mn}, \qquad d_{mn} = \lambda_{mn} D_{mn}$$

then

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny$$

$$+ c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny)$$

is the required double Fourier series.