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CESÀRO AND LOGARITHMIC SUMMABILITY OF DOUBLE FOURIER SERIES

AND THEIR DERIVED SERIES

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TABLE OF CONTENTS

INTRODUCTION

In the study of Fourier Series, consideration of the convergence and summability of the series obtained by differentiating term by term naturally finds a place. As early as 1904, Lebesgue [1]* considered Cesàro summability of derived series and his results were improved by Privaloff [1], and Young [2] and are now included in standard works on Fourier Series such as Zygmund's Trigonometric Series and Hardy-Rogosinski's new (1944) Cambridge tract, Fourier Series. Similarly, we can find in Gronwall [1], Young [3] and Zygmund [2] consideration of successively derived series.

It is the concern of part one of this thesis, first, to prove theorems for double Fourier Series similar to those on derived and successively derived series as found

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*Bracketed numbers refer to bibliography.

in Zygmund*; and second, to extend to double Fourier Series a generalization of the theorem on successively derived series, as due to Wang [1] .

In the second part, we consider logarithmic summability and are concerned with the extension to double Fourier Series and their derived series of certain theorems due to Wang [2] and [3] . In his investigations, Wang has used a theorem discovered by W. H. Young in the course of his researches in the theory of Fourier Series, and we are naturally led to the use of an analogous theorem for double series. Hardy [1] has given a simple and direct proof of Young's theorem, and it is his statement and proof which we here follow for double series.

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*Zygmund [1] , page 55 and page 257.

The statement and proof of the extension of the Young-Hardy theorem follows:

THEOREM: Suppose (i) that $f(x,y)$ is summable and periodic with periods 2π; (ii) that $g(x,y)$ is of bounded variation in the cell $(0,0;\infty,\infty)$ and one of the partial derivatives of $g(x,y)$ exists and is bounded on this same cell; and (iii) that the integral

$$\int_0^\infty\int_0^\infty |g(x,y)|\,dx\,dy$$

is convergent. Then the integral

$$\int_0^\infty\int_0^\infty f(x,y)\,g(x,y)\,dx\,dy$$

may be calculated by substituting for $f(x,y)$ its Fourier Series and integrating formally term by term.

If S_{mn} are the partial sums of the double Fourier series of our function $f(x,y)$, then we have

$$\int_0^\infty\int_0^\infty g(x,y)\, S_{mn}\, dx\, dy$$

$$= \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty g(x,y)\, dx\, dy \int_0^{2\pi}\int_0^{2\pi} \frac{\sin(m+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)}\; \frac{\sin(n+\frac{1}{2})(y-v)}{\sin\frac{1}{2}(y-v)}\, f(u,v)\, du\, dv.$$

Taking into account Fubini's theorem, we see that inversion of order of integration is here permissable, and this expression then is equal to

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(u,v)\, du\, dv \int_0^\infty\int_0^\infty \frac{\sin(m+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)}\; \frac{\sin(n+\frac{1}{2})(y-v)}{\sin\frac{1}{2}(y-v)}\, g(x,y)\, dx\, dy.$$

If we now let

$$G(x,y) = \sum_{i=-\infty,\, j=-\infty}^{\infty,\, \infty} g(x+2i\pi,\, y+2j\pi),$$

$$G_{mn}(u,v) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin(m+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)}\;\frac{\sin(n+\frac{1}{2})(y-v)}{\sin\frac{1}{2}(y-v)}\; g(x,y)\,dx\,dy$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\sin(m+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)}\;\frac{\sin(n+\frac{1}{2})(y-v)}{\sin\frac{1}{2}(y-v)}\; G(x,y)\,dx\,dy$$

is the partial sum of the double Fourier series of $G(u,v)$. Thus, if $G_{mn}(u,v)$ remains bounded and tends to $G(u,v)$, by the general theory of the Lebesgue integral

$$\lim_{m,n\to\infty}\int_0^{2\pi}\int_0^{2\pi} f(u,v)\, G_{mn}(u,v)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} f(u,v)\, G(u,v)\,du\,dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\, g(u,v)\,du\,dv,$$

which proves our theorem.

The convergence of the double Fourier series of $G_{\smile}(u,v)$ to $G(u,v)$ follows from a theorem of Hobson* if we show that the function $G(u,v)$ is of bounded variation and has one of its partial derivatives bounded on the cell $(0,0;\ 2\pi,\ 2\pi)$.

Let $u_{ij}(x,y) = g(x + 2i\pi,\ y + 2j\pi)$ and

$$v_{ij}(x,y) = \frac{1}{4\pi^2}\int_{x+2i\pi}^{x+2(i+1)\pi}\int_{y+2j\pi}^{y+2(j+1)\pi} g(u,v)\,du\,dv \quad \text{where} \quad \begin{matrix} 0 \leq x \leq 2\pi \\ 0 \leq y \leq 2\pi. \end{matrix}$$

The series $\sum_{i=0,\,j=0}^{\infty,\,\infty} v_{ij}$ is absolutely and uniformly convergent.

Also

$$u_{ij}(x,y) - v_{ij}(x,y) = \frac{1}{4\pi^2}\int_{x+2i\pi}^{x+2(i+1)\pi}\int_{y+2j\pi}^{y+2(j+1)\pi} \left\{ g(x+2i\pi,\ y+2j\pi) - g(u,v) \right\} du\,dv$$

is not greater in absolute value than V_{ij}, when V_{ij} is the total variation of $g(u,v)$ on the cell

$(x + 2i\pi,\ y + 2j\pi;\ x + 2(i + 1)\pi,\ y + 2(j + 1)\pi)$.

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*Hobson [1] page 710.

As a result, the series

$$\sum_{i=0,\,j=0}^{\infty,\,\infty} \left\{ u_{ij}(x,y) - v_{ij}(x,y) \right\}$$

is absolutely and uniformly convergent as is $\sum_{i=0,\,j=0}^{\infty,\,\infty} u_{ij}$.

Thus $G(x,y)$ is defined for $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2$, and we have

$$|G(x,y)| \leqq \sum_{i=0,\,j=0}^{\infty,\,\infty} |g(x+2i\pi,\, y+2j\pi)| .$$

$$\int_0^{2\pi}\int_0^{2\pi} |G(x,y)|\,dx\,dy \leqq \int_0^{\infty}\int_0^{\infty} |g(x,y)|\,dx\,dy ;$$

and if (x_1, y_1) and (x_2, y_2) are any two points on $(0,0;\ 2\pi,\ 2\pi)$

$$|G(x_1,y_1) - G(x_2,y_2)| \leqq \sum_{i=0,\,j=0}^{\infty,\,\infty} |g(x_1+2i\pi,\, y_1+2j\pi) - g(x_2+2i\pi,\, y_2+2j\pi)|.$$

Therefore, $G(x,y)$ is summable, and if we form one of the sums by means of which the variation of $G(x,y)$ on $(0,0;\ 2\pi,\ 2\pi)$ is defined, it is less than one equal to a corresponding sum formed for $g(x,y)$ on the infinite cell $(0,0;\ \infty,\ \infty)$. The variation of $G(x,y)$ on $(0,0;\ 2\pi,\ 2\pi)$ does not exceed that of $g(x,y)$ on $(0,0;\ \infty,\ \infty)$. Thus the conditions for convergence of G_{mn} are fulfilled.

PART I--CESÀRO SUMMABILITY

1. Definitions. We shall consider the summability of series of the form

$$(1.1)\qquad \sum_{i=0,\, j=0}^{\infty,\,\infty} x^r y^s A_{i,j}^{r,s} \qquad ;$$

where, for r and s integers,

$$(1.2)\qquad A_{ij}^{r,s} = \frac{1}{i^r j^s}\left[\frac{\lambda_{ij}}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\, P_{ij}(x,y,u,v)\, du\, dv\right]_{x^r y^s} \qquad ;$$

The subscripts on the square brackets shall mean the partial taken with respect to x r times and with respect to y s times; λ_{ij} = $\frac{1}{4}$, $\frac{1}{2}$, or 1 accordingly as i and j are both zero, either i or j is zero or neither i nor j is zero; and

$$P_{ij}(x,y,u,v) = \cos i(u-x)\ \cos j(v-y) \quad .$$

We shall let

$$(1.3)\quad \psi_{\alpha\beta}(u,v) = f(x+u,y+v) + (-1)^{\alpha} f(x-u,y+v) + (-1)^{\beta} f(x+u,y-v) + (-1)^{\alpha+\beta} f(x-u,y-v) \ ;$$

$$D_n(u) = \frac{1}{2} + \sum_{i=1}^{n} \cos iu = \frac{\sin(n+1/2)u}{2\sin u/2} ;$$

$$\overline{D}_n(u) = \frac{\sin(n+1/2)u}{u} ;$$

$$A_n^{\alpha} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)\,\Gamma(n)} ;$$

$$K_n^{\alpha}(u) = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha-1} D_k(u)}{A_n^{\alpha}} ;$$

and

$$L_n^{\alpha}(u) = -\sin u \left\{ K_n^{\alpha}(u) \right\}'$$

2. <u>Conditions for Summability</u> $(C, \alpha, 0)$ <u>of x Derived Series.</u> The Cesàro means of orders α, 0, i.e. $(C, \alpha, 0)$ of the series (1.1) for $r = 1$ and $s = 0$ can be written

$$\text{(2.1)} \quad \left[\sigma_{m,n}^{\alpha,0}\right]_x = \frac{1}{\pi^2} \int_0^{\pi}\!\!\int_0^{\pi} \frac{\psi_{10}(u,v)}{u} \cdot 2 L_m^{\alpha}(u) \cdot 2 \overline{D}_n(v)\, du\, dv + o(1),$$

where the prime denotes differentiation with respect to u, and the

o(1) arises from replacing $\frac{1}{2\sin\frac{v}{2}\sin u}$ by $\frac{1}{u\ v}$.

Let $$G(u,v) = \frac{\psi_{1,0}(u,v)}{u}.$$

<u>Theorem 1.</u> If f(x,y) is integrable (L) in the cell $(-\pi, -\pi; \pi, \pi)$ and periodic in both x and y with periods 2π, and if $G(+0,+0)$ exists at a point (x,y) and is finite, say equal to S, then the double series (1.1) with r equal to one and s equal to zero is summable $(C,\alpha, 0)$ for $\alpha > 1$ to the value S at the point providing that G(u,v) is monotonic with respect to v and bounded in a cross-neighborhood of the point.

In particular if f_x exists and is continuous at the point (x,y) and bounded in a cross-neighborhood of the point, then $G(+0,+0)$ is equal to the value of the derivative at the point; G(u,v) is monotonoid; and the series (1.1) with r equal to one and s equal to zero is summable $(C,\alpha, 0)$ for $\alpha > 1$ at the point to the value of the derivative at the point.

Let us assume G(u,v) monotonic decreasing, and consider the double integral in (2.1) broken up into four separate integrals over the following regions:

(2.2)
$$\begin{aligned} A&: (0,0;\ \epsilon,\epsilon) \\ B&: (\epsilon,0;\ \pi,\epsilon) \\ C&: (0,\epsilon;\ \epsilon,\pi) \\ D&: (\epsilon,\epsilon;\ \pi,\pi) \end{aligned}$$

Let us define
$$\Phi_A(u,v) \begin{cases} = G(u,v) - G(\epsilon-0,\epsilon-0) & \text{on } (0,0;\ \epsilon\epsilon) \\ = 0 & \text{elsewhere} \end{cases}$$

Then the integral over the region A is equal to

$$G(\epsilon-0,\epsilon-0)\,\frac{4}{\pi^2}\int_0^\pi\!\!\int_0^\pi L_m^\alpha(u)\,\overline{D}_n(v)\,du\,dv - G(\epsilon-0,\epsilon-0)\,\frac{4}{\pi^2}\int_\epsilon^\pi\!\!\int_\epsilon^\pi L_m^\alpha(u)\,\overline{D}_n(v)\,du\,dv$$

$$+ \frac{4}{\pi^2}\int_0^\epsilon\!\!\int_0^\epsilon \Phi_A(u,v)\,L_m^\alpha(u)\,\overline{D}_n(v)\,du\,dv$$

Since two over π times the integral of $\overline{D}_m(v)$ from zero to π is equal to one and $L_m^\alpha(u)$ is a quasi-positive kernel, the first term is equal to $G(\epsilon - 0, \epsilon - 0)$ which can be made as close to S as we please with proper choice of ϵ, and the second term is o(1). The third term can be written

$$\sum_{i=0}^{s} \frac{4}{\pi^2} \int_{\frac{i\pi}{n+1/2}}^{\frac{(i+1)\pi}{n+1/2}} \int_0^{\epsilon} \phi_A(u,v) \cdot 2 L_m^\alpha(u) \cdot 2 \overline{D}_m(v) \, du \, dv,$$

where s is an integer such that, with m = n + 1/2,

$$s\pi < m\epsilon \leqq (s+1)\pi.$$

If

$$(2.3) \quad u_i = \int_0^{\frac{\pi}{m}} \int_0^{\epsilon} \phi_A\left(u, v + \frac{i\pi}{m}\right) \cdot 2 L_m^\alpha(u) \cdot 2 \overline{D}_m(v) \, du \, dv,$$

the series above can be written $\sum_{i=0}^{s} (-1)^i u_i$.

The u_i are positive and monotonic and decreasing with i since ϕ_A is monotonic decreasing, and therefore the whole sum is less than

$$u_0 \leqq \phi_A(+0, +0) \cdot \frac{2}{\pi} \int_0^{\frac{\pi}{m}} \frac{\sin mv}{v} \, dv \cdot \frac{2}{\pi} \int_0^{\epsilon} L_m^\alpha(u) \, du$$

$$\leqq \phi_A(+0, +0).$$

Now the difference between $G(\epsilon -0, \epsilon -0)$ and $G(+0, +0)$ can be made as small as we please if we choose ϵ small enough; therefore, the integral over A tends to S. We must still show that the three remaining integrals all tend to zero.

If we define
$$\phi_B = \begin{cases} G(u,v) & \text{on } (0,\epsilon;\epsilon,\pi) \\ 0 & \text{elsewhere} \end{cases}$$

we can write the integral over the region B in the form

$$\sum_{i=s}^{p+s} \frac{1}{\pi^2} \int_{\frac{i\pi}{m}}^{\frac{(i+1)\pi}{m}} \int_0^{\epsilon} \phi_B(u,v) \cdot 2\, L_m(u) \cdot 2\, \overline{D}_m(v)\, du\, dv,$$

where s is an integer such that

$$s\pi < \epsilon m \leq (s+1)\pi$$

and p is an integer such that

$$m\pi < p + s + 1$$

where m is, as before, equal to $n + \frac{1}{2}$.

Considering first the region where

$$\epsilon \leqq v \leqq \frac{(s+1)\pi}{m},$$

we have the integral

$$\frac{4}{\pi^2}\int_{\epsilon}^{\frac{s+1}{m}\pi}\int_{0}^{\epsilon}\phi_{\beta}(u,v)\, L_m^{\alpha}(u)\,\frac{\sin mv}{v}\,du\,dv.$$

Since ϕ_{β} (u,v) is bounded and less than some constant K,*
this part of the integral is less than or equal to

$$K\cdot\int_{\epsilon}^{\frac{s+1}{m}\pi}\frac{\sin mv}{v}\,dv\cdot\int_{0}^{\epsilon}L_m^{\alpha}(u)\,du,$$

which, because $L_m^{\alpha}(u)$ is a quasi-positive kernel, is less than
or equal to

$$K\cdot\int_{m\pi}^{(s+1)\pi}\frac{\sin\theta}{\theta}\,d\theta.$$

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* K shall always denote a positive constant, though not necessarily having the same value even when occurring more than once in any continued inequality.

As n becomes infinite so must m and also s so that this integral tends to zero. We have the remainder of the region B giving

$$\sum_{i=s+1}^{p+s} (-1)^i u_i \quad ,$$

where u_i is defined as in (2.3) with ϕ_B in place of ϕ_A and u_i is here again monotonic decreasing with i and the integral over this region is less than $|u_{s+1}|$ which is less than or equal to

$$K \cdot \int_0^{\epsilon} L_m^{\alpha}(u)\, du \int_0^{\frac{\pi}{m}} \frac{\sin mv}{v + \frac{(s+1)\pi}{m}}\, dv$$

$$\leqq \quad K \int_0^{\pi} \frac{d\theta}{\theta + (s+1)\pi}$$

$$\leqq \quad K \log \frac{s+2}{s+1}$$

This last expression converges to zero as s becomes infinite and hence as n becomes infinite.

The integral over the region C in the same manner can be written as a sum:

$$\sum_{i=0}^{\alpha} \frac{4}{\pi^2} \int_{\frac{i\pi}{m}}^{\frac{(i+1)\pi}{m}} \int_{\epsilon}^{\pi} \phi_C(u,v) \, L_m^{\alpha}(u) \frac{\sin nv}{v} \, du \, dv,$$

where $\phi_C(u,v)$ is equal to $G(u,v)$ over the region C and zero elsewhere so that we have the integral over C in absolute value less than

$$U_0 = |U_0| \leq K \int_0^{\frac{\pi}{m}} \frac{\sin mv}{v} \, dv \int_{\epsilon}^{\pi} L_m^{\alpha}(u) \, du$$

$$< K \int_{\epsilon}^{\pi} L_m^{\alpha}(u) \, du,$$

which is $o(1)$ since $L_m^{\alpha}(u)$ is a quasi-positive kernel.

The integral over the region D converges to zero as n and m become infinite by the general limit theorem of Hobson*. Therefore the Cesaro means (2.1) upon choosing ϵ small enough and then letting m and n become infinite can be made to differ from S by less than any positive constant no matter how small.

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* Hobson [1], page 279.

3. Conditions for Summability (C, α, β) of x Derived Series. The Cesàro means of orders (α, β) i.e. (C, α, β) for $\alpha > 0$ and $\beta > 0$ with r equal to one and s equal to zero can be written

$$(3.1) \quad \left[\sigma_{m,n}^{\alpha,\beta}\right]_x = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi G(u,v)\cdot 2L_m^\alpha(u)\cdot 2K_n^\beta(v)\,du\,dv + o(1)$$

Theorem 2. If f(x,y) is integrable (L) in the cell $(-\pi, -\pi; \pi, \pi)$ and periodic in both x and y with periods 2π and if $G(+0, +0)$ exists and is finite say equal to S at a point (x,y), then the series (1.1) with r equal to one and s equal to zero is summable (C, α, β), $\alpha > 1$ and $\beta > 0$, to the value S at the point (x,y) providing that G(x,y) is bounded in a cross-neighborhood of the point.

We again break up our integration over the regions (2.2) and the integral over the region A gives us

$$\frac{S}{\pi^2}\int_0^\pi\int_0^\pi 2L_m^\alpha(u)\cdot 2K_n^\beta(v)\,du\,dv - \frac{S}{\pi^2}\int_\epsilon^\pi\int_\epsilon^\pi 2L_m^\alpha(u)\cdot 2K_n^\beta(v)\,du\,dv$$

$$+ \frac{1}{\pi^2}\int_0^\epsilon\int_0^\epsilon\left[G(u,v) - S\right]\cdot 2L_m^\alpha(u)\cdot 2K_n^\beta(v)\,du\,dv.$$

Since $L_m^{\alpha}(u)$ is a quasi-positive kernel for $\alpha > 1$ and $K_n^{\beta}(v)$ is a quasi-positive kernel for $\beta > 1$, the first term on the right is exactly equal to S, the second is o(1), and the third is in absolute value less than or equal to $\text{Max}\,[G(u,v) - S]$, which can be made as small as we please by proper choice of ϵ since G(u,v) tends to S.

If K is the maximum of G(x,y) in the cross-neighborhood of the point (x,y), then the integral over B is in absolute value less than or equal to

$$K \cdot \frac{2}{\pi}\int_0^{\epsilon} |K_n^{\beta}(v)|\,dv \cdot \frac{2}{\pi}\int_{\epsilon}^{\pi} |L_m^{\alpha}(u)|\,du,$$

where the first factor K is independent of m and n, the second factor is bounded for $s > 0$, and for $r > 1$ the third factor tends to zero as m becomes infinite. The integral over the region B, therefore, is o(1).

In exactly the same manner the integral over the region C is less than or equal to

$$K \cdot \frac{2}{\pi}\int_{\epsilon}^{\pi} |K_n^{\beta}(v)|\,dv \cdot \frac{2}{\pi}\int_0^{\epsilon} |L_m^{\alpha}(u)|\,du,$$

where K is independent of m and n, the second factor tends to zero as n becomes infinite, and the third factor is bounded for all m; hence, the integral over C is o(1). Again by the general limit theorem of Hobson, the integral over D converges to zero as m and n become infinite.

Therefore, the Cesàro means (3.1) can be made to differ from S by less than any positive constant by choosing ϵ sufficiently small and then allowing m and n to become infinite.

4. Conditions for Summability (C, α, β) of Successively Derived Series.

Suppose

$$f(x+u, y+v) = \sum_{i+j \leq r+s} \frac{1}{i!\,j!} \gamma_{i,j}\, u^i v^j + R_{r,s} \tag{4.1}$$

for small u and r, where

$$R_{r,s} = \sum_{i+j = r+s+1} u^i v^j h^{(i,j)}(u,v)$$

then the $\gamma_{i,j}$ shall be called the generalized partial of f(x,y) at a point (x,y) of order (i,j) and if the equation (4.1) holds, the generalized partial of order (r,s) of f(x,y) shall be said to exist at the point (x,y). If

$$(4.2)\qquad f(x+u,y) = \sum_{i=0}^{r} \frac{1}{i!}\,\gamma_{i,0}(y)\,u^{i} + u^{r+1} R_r(y)$$

for all y and small u where $R_r(y)$ is bounded for all y, then the rth partial with respect to x shall be said to exist at the point (x,y) and $\gamma_{i,0}$ (y) shall be called the rth generalized partial derivative of f(x,y) at the point (x,y) with respect to x. We also get a similar expression for the sth generalized partial with respect to y. Now $\psi_{r,s}$ (u,v) at a point (x,y) formed for a function satisfying (4.1) will have certain of the terms of (4.1) missing; that is, (4.1) becomes

$$(4.3)\qquad \psi_{r,s}(u,v) = \sum_{i+j \leq \frac{r+s}{2}} \frac{4}{(2i)!\,(2j)!}\,\gamma_{2i,2j}\,u^{2i} v^{2j} + R'_{r,s} \quad \text{for } \begin{matrix} |u| < \epsilon \\ |v| < \epsilon \end{matrix}$$

when both r and s are even. Also

$$\psi_{r,s}(u,v) = \sum_{i+j \leq \frac{r+s-2}{2}} \frac{4}{(2i+1)!\,(2j)!}\,\gamma_{2i+1,2j}\,u^{2i+1} v^{2j} + R''_{r,s} \quad \text{for } \begin{matrix} |u| < \epsilon \\ |v| < \epsilon \end{matrix}$$

when r is odd and s is even and we have similar expressions with r even and s odd, and r and s both odd. The $\gamma_{i,j}$ when formed in this way shall be called the (i,j)th generalized derivative of f(x,y) at the point (x,y) and shall be denoted by $f_{i,j}(x,y)$. For the similar equations with which we replace (4.2) we shall denote these single generalized derivatives with respect to x by $f_{x,i}(x,y)$ and $f_{y,j}(x,y)$ for the single generalized derivative with respect to y. If the regular derivatives exist, they are equal to the generalized derivatives.

The Cesàro means of order (α, β) of the series (1.1) with $r \geqq 1$ and $s \geqq 1$ can be written

$$(4.4)\quad \left[\sigma_{m,n}^{\alpha,\beta}\right]_{x^r y^s} = \frac{(-1)^{r+s}}{\pi^2} \int_0^\pi\int_0^\pi \psi_{r,s}(u,v)\, \frac{d^r K_m^\alpha(u)}{du^r}\, \frac{d^s K_n^\beta(v)}{dv^s}\, du\, dv,$$

Theorem 3. If $f_{r,s}(x,y)$ exists at a point (x,y) the series (1.1) is summable (C, α, β) for $\alpha > r$, $\beta > s$ at the point (x,y) providing the partial derivatives of $f(x+u, y+v)$ with respect to u of order $r+s$ for $|v| \leqq \epsilon$, and of order r for $\epsilon < |v| \leqq \pi$ exist and are of bounded variation for $u = 0$; and the partial derivatives with respect to v of order $r+s$ for $|u| \leqq \epsilon$ and of order s for $\epsilon < |u| \leqq \pi$ exist and are of bounded variation for $v = 0$.

Let us take for convenience the point (x,y) equal to (0,0) and r and s both even. Now since the derived series of a finite trigonometric sum is obviously summable by the Cesaro means, and since we can pick such a sum $T_N(x,y)$, for N large enough, having the values $\gamma_{i,j}$ as partial derivatives at the point (0,0), the summability of the series for f(x,y) is implied by the summability of the function $F(x,y) = f(x,y) - T_N(x,y)$, which has all its generalized partials of order (i,j), where $i + j \leq r + s$, equal to zero, and we can also make $T_N(0,0) = f(0,0)$.

Now the r + sth derivative of F(0,y) and F(x,0) exist and are of bounded variation for points $-\epsilon < x < \epsilon$ and $-\epsilon < y < \epsilon$, and the rth derivative of F(x,0) and the sth derivative of F(0,y) are of bounded variation on the remainder of the intervals $(-\pi, \pi)$.

We then have by a theorem of Jackson* that the Mth partial sums $S_M(x)$ and $S_M(y)$ of the single Fourier series of F(x,0) and F(0,y) considered as functions of x and y respectively are such that

$$\left| F(x,0) - S_M(x) \right| \leq \frac{K}{M^{r+\alpha}} \quad \text{for} \quad 0 \leq x \leq \epsilon$$

$$\left| F(x,0) - S_m(x) \right| \leq \frac{K}{M^{r}} \quad \text{for} \quad \epsilon < x \leq \pi,$$

*Jackson [1] pp. 48 et seq.

and if superscript (h) denotes hth derivative with respect to x,

$$\left| F^{(h)}(x,0) - S_M^{(h)}(x) \right| \leq K \frac{1}{M^{r+\alpha-h}}, \qquad 0 \leq x \leq \epsilon,$$

$$\left| F^{(h)}(x,0) - S_M^{(h)}(x) \right| \leq K \frac{1}{M^{r-h}}, \qquad \epsilon < x \leq \pi$$

with similar expressions for $F(0,y)$ and $S_M(y)$.

Let us now consider a function

$G(x,y) = F(x,y) - S_M(x) - S_M(y)$

Now $F(0,0) = 0$ and therefore

$$\left| G(0,0) \right| \leq \frac{K}{M^{r+\alpha}};$$

$$\left| G(x,0) \right| \leq \left| F(0,0) - S_M(0) \right| + \left| F(x,0) - S_M(x) \right|;$$

(4.5) $$\left| G(x,0) \right| \leq \frac{K}{M^{r+\alpha}}, \qquad 0 \leq x \leq \epsilon;$$

$$(4.5)' \qquad |G(x,0)| \leq \frac{K}{M^{r}}, \qquad \epsilon < x \leq \pi.$$

If G_{h0} represents the (h,0)th generalized derivative of G(u,v) at the point (0,0), we have

$$(4.6) \qquad |G_{h0}| \leq \frac{K}{M^{r+s-h}}.$$

Similarly

$$(4.7) \qquad |G_{xi}(v)| \leq \frac{K}{M^{r+s-i}}, \qquad 0 \leq v \leq \epsilon$$

and

$$(4.7)' \qquad |G_{xi}(v)| \leq \frac{K}{M^{r-i}} \qquad \epsilon < v \leq \pi.$$

We also get similar inequalities for G(0,y), G_{0k}, and G_{yi} (u), and the following expansions for the function G:

$$(4.8) \qquad G(u,v) = \sum_{i+j \leq \frac{r+s}{2}} \frac{1}{(2i)!\,(2j)!} G_{2i,2j}\, u^{2i} v^{2j} + R_{r,s}(G)$$

for u and v small and $R_{r,s}$ (G) similar to $R_{r,s}$ (f) of (4.3);

$$(4.9)\quad G(u,v) = \sum_{i=0}^{\frac{r+s}{2}} \frac{1}{(2i)!}\, G_{x,2i}(v)\; u^{2i} + u^{r+s} e(u,v)$$

for small u, $0 \leqq |v| \leqq \epsilon$, and where $e(u,v) \to 0$ with u; and

$$(4.10)\quad G(u,v) = \sum_{i=0}^{r/2} \frac{1}{(2i)!}\, G_{x,2i}(v)\; u^{2i} + u^{r} e'(u,v)$$

for small u; $\epsilon \leqq |v| \leqq \pi$, and where $e'(u,v) \to 0$ with u.

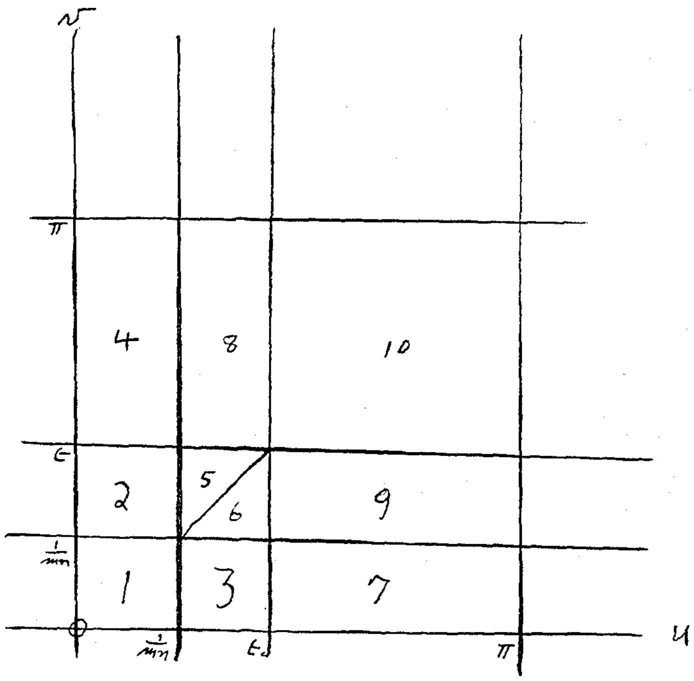
We can also get similar expansions in terms of $G_{y,2j}(u)$.

Now if the double Fourier series of G(x,y) is summable (C, α,β) to zero at the point (0,0), then the double Fourier series of f(x,y) will be summable (C,α ,β) to the value S at the point (0,0), since f(x,y) differs from G(x,y) only by finite trigonometric sums and the (r,s)th partial derivatives of these finite trigonometric sums can be made to differ from $\gamma_{r,s}$ by less than any given positive quantity, no matter how small, and the finite sums are obviously summable (C,α,β).

Therefore. we can consider in place of (4.3) an integral of exactly the same form, where now the $\psi_{r,s}$ (u,v) is formed with with respect to our new function G instead of f.

respect to our new function G instead of f.

Let us break the cell $(0,\ 0,\ +\pi,\ +\pi)$ into the following parts.



The integral over the region 1 we shall call I_1 and similarly for the rest of the parts. Using (4.8) in the integral I_1 we have

$$I_1 = \frac{1}{\pi^2}\int_0^{\frac{1}{mn}}\int_0^{\frac{1}{mn}} \left\{ \sum_{i+j \leq \frac{r+s}{2}} \frac{4}{(2i)!(2j)!}\, G_{2i,2j}\, u^{2i} v^{2j} + R'_{r,s}(G) \right\} \frac{d^r K_m^{(\alpha)}(u)}{du^r}\, \frac{d^s K_n^{(\beta)}(v)}{dv^s}\, du\, dv.$$

Each of the terms in $R'_{r,s}(G)$ contains $u^p v^q$ where $p + q = r + s + 1$. For both i and j not equal to zero $G_{2i,2j} = 0$ because of the structure of the function $G(x,y)$.

Using (4.5), (4.6), and the fact* that

$$\frac{d^r K_m^\alpha(u)}{du^r} \leqq K m^{r+1}$$

we have, when M is greater than m times n, the integral I_1 in absolute value less than or equal to

$$K m^{r+1} n^{s+1} \left[\frac{K}{(m n)^{r+s}} + \frac{K}{(mn)^{r+s+1}} \right] \int_0^{\frac{1}{mn}} \int_0^{\frac{1}{mn}} du\, dv,$$

which is o(1).

In the integral I_2, we use (4.9) so that

$$I_2 = \frac{1}{\pi^2} \int_{\frac{1}{mn}}^{\epsilon} \int_0^{\frac{1}{mn}} \left\{ \sum_{i=0}^{\frac{r+s}{2}} G_{r,2i}\, u^{2i} + u^{r+s}\, e(u,v) \right\} \frac{d^r K_m^\alpha(u)}{du^r}\, \frac{d^s K_n^\beta(v)}{du^s}\, du\, dv,$$

which for $M \geqslant mn$ is less than or equal to, by (4.7),

$$K m^{r+1} n^{s+1} \left\{ \frac{K}{(mn)^{r+s}} + \frac{K}{(mn)^{r+s-2}} \cdot \frac{1}{(mn)^2} + \cdots + K \cdot \frac{1}{(mn)^{r+s}} \right\} \int_{\frac{1}{mn}}^{\epsilon} \int_0^{\frac{1}{mn}} du\, dv,$$

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*Zygmund [1] page 259.

$$\leqq K \cdot \frac{m^{r+1} \quad n^{s+1}}{(mn)^{r+2}} \cdot \epsilon \cdot \frac{1}{mn}$$

which is o(1).

The integral I_4, using now (4.10), gives us

$$\frac{1}{\pi^2} \int_{\epsilon}^{\pi} \int_{0}^{\frac{1}{mn}} \left\{ \sum_{i=0}^{r/2} G_{r,2i}\, u^{2i} + u^{r} e'(u,v) \right\} \frac{d^r K_m^{\alpha}(u)}{du^r} \; \frac{d^s K_n^{\beta}(v)}{dv^s} \, du\, dv,$$

Now on the interval ϵ to π the maximum of $\dfrac{d^s K_n^{\beta}(v)}{dv^s}$

tends uniformly to zero for fixed ϵ and n becoming infinite.
Then, by (4.7),

$$|I_4| \leqq K \cdot \frac{m^{r+1}}{(mn)^{r}} \cdot \epsilon(m) \cdot \frac{1}{mn},$$

which is o(1).

The consideration of I_3 and I_7 is exactly the same as that of I_2 and I_4 where we use in place of (4.9) and (4.10), the corresponding equations on the other axis. For the same reason we need only consider I_5, I_8, and I_{10}.

Using (4.9) again, I_6 becomes in absolute value

$$\leq \int_{\frac{1}{mn}}^{t}\int_{u}^{t} \left| \sum_{i=0}^{r+2} G_{n,2i}(v)\, u^{2i} + u^{r+2}\,\epsilon(u,v) \right| \cdot \left| \frac{d^r K_m^{\alpha}(u)}{du^r} \right| \left| \frac{d^2 K_n^{\beta}(v)}{dv^2} \right| du\, dv.$$

Using (4.7), with $M > (mn)^2$,

$$|I_6| \leq \int_{\frac{1}{mn}}^{t}\int_{u}^{t} O\left\{ \frac{1}{(mn)^{2(r+2)}} + \frac{u^2}{(mn)^{2(r+2)-2}} + \cdots + \frac{u^{(r+2)}}{(mn)^{r+2}} + \epsilon(u,v)\, u^{r+2} \right\}$$

$$\times \left| \frac{d^r K_m^{\alpha}(u)}{du^r} \right| \cdot \left| \frac{d^2 K_n^{\beta}(v)}{dv^2} \right| du\, dv$$

and since $1 < mnu$, $1 < mnv$, and $u < v$,

$$|I_6| \leq \frac{1}{(mn)^{r+2}} \int_{\frac{1}{mn}}^{t} u^r \left| \frac{d^r K_m^{\alpha}(u)}{du^r} \right| du \int_{\frac{1}{mn}}^{t} v^2 \left| \frac{d^2 K_n^{\beta}(v)}{dv^2} \right| dv$$

$$\leq \frac{K}{(mn)^{r+2}},$$

which is $o(1)$.

Using (4.10),

$$I_\gamma = \int_\epsilon^\pi \int_{\frac{1}{mn}}^{\epsilon} \left\{ \sum_{i=0}^{r/2} G_{r,2i}(v)\, u^{2i} + u^r\, e'(u,v) \right\} \frac{d^r K_m^\alpha(u)}{du^r} \frac{d^s K_n^\beta(v)}{dv^s}\, du\, dv$$

For v on the interval ϵ to π, $e''(n)$, equal to the maximum of $\dfrac{d^s K_n^\beta(v)}{dv^s}$ tends uniformly to zero as n becomes infinite, which with (4.7) gives us

$$|I_\gamma| \leqq e''(n) \int_\epsilon^\pi \int_{\frac{1}{mn}}^{\epsilon} O\left\{ \frac{1}{(mn)^r} + \frac{u^2}{(mn)^{r-2}} + \cdots + \frac{u^r}{1} \right\} \cdot \left| \frac{d^r K_m^\alpha(u)}{du^r} \right| du\, dv,$$

but $1 \leqq mnu$, and therefore

$$|I_\gamma| \leqq e''(n) \cdot K \int_\epsilon^\pi dv \cdot \int_{\frac{1}{mn}}^{\epsilon} u^r \left| \frac{d^r K_m^\alpha(u)}{du^r} \right| du.$$

The third factor is bounded as before, the second is constant for fixed ϵ , and the first tends to zero as n becomes infinite, and therefore $I_\gamma = o(1)$.

Since $\frac{d^s K_n^{\beta}(v)}{dv^s}$ and $\frac{d^r K_m^{\alpha}(u)}{du^r}$ tend uniformly to zero on the interval ϵ to π as m and n become infinite the integral I_q tends to zero.

Therefore the integral I can be made to differ from O by less than any positive constant, no matter how small, if only m and n are sufficiently large and M $>$ (mn) . Therefore our theorem is proved.

5. <u>Further Conditions for Summability (C, α, β) of Successively Derived Series.</u>

Let us define a function P(u,v) as follows:

$$P(u,v) = \sum_{i+j \leq r+s}' \gamma_{i,j} \frac{1}{i!\,j!} u^i v^j \qquad \text{for} \quad \begin{matrix} |u| < \epsilon \\ |v| < \epsilon \end{matrix}$$

where the prime denotes the term for which i = r and j = s is missing;

$$P(u,v) = \sum_{i=0}^{r} \gamma_{s,i}(v) \frac{1}{i!} u^i \qquad \text{for} \quad \begin{matrix} |u| < \epsilon \\ |v| > \epsilon \end{matrix} \quad ;$$

$$P(u,v) = \sum_{j=0}^{s} \gamma_{r,j}(u) \frac{1}{j!} v^j \qquad \text{for} \quad \begin{matrix} |v| < \epsilon \\ |u| > \epsilon \end{matrix} \quad ;$$

$$P(u,v) = f(x+u,\ y+v) \qquad \text{for} \quad \begin{array}{l} |u| > \epsilon \\ |v| > \epsilon \end{array}$$

The γ_{rs} are those defined for the function f(x,y) in (4.1) and (4.2).

We will now define a function g(u,v).

$$\frac{4u^r v^s}{r!\,s!}\, g(u,v) = \left[f(x+u,\ y+v) - P(u,v)\right] + (-1)^r \left[f(x-u,\ y+v) - P(-u,v)\right]$$

$$+ (-1)^s \left[f(x+u,\ y-v) - P(u,-v)\right] + (-1)^{r+s}\left[f(x-u,\ y-v) - P(-u,-v)\right],$$

on the cell $(-\pi, -\pi; \pi, \pi)$ and periodic in u and v of periods 2π outside the fundamental rectangle.

We can define the (C, α, β) means of the series (1.1) as follows:

$$(5.1) \qquad \left[\sigma_{m,n}^{\alpha,\beta}\right]_{x^r y^s} = \sum_{\substack{i<m \\ j<n}} \left(1 - \frac{i}{m}\right)^{\alpha} \left(1 - \frac{j}{n}\right)^{\beta} i^r j^s A_{ij}^{r,s}$$

<u>Theorem 4</u>. If $f(x,y)$ and the corresponding $g(u,v)$ is integrable (L) and periodic in the cell $(-\pi, -\pi; \pi, \pi)$ and the double Fourier series of $g(u,v)$ is summable (C, λ, δ) for $\lambda > 0$ and $\delta > 0$ to the sum S at $u = v = 0$, then the series (1.1) is summable $(C, r + \lambda, s + \delta)$ to S at the point (x,y).

For convenience, let us consider r and s both even, then for $\psi_{r,s}(u,v)$, defined as in (1.3), we can easily get the following Fourier expansion:

(5.2) $$\sum_{i=0,\,j=0}^{\infty,\,\infty} A_{i,j}^{r,s} \cos iu \cos jv .$$

If we define

$$Y_{1+\alpha}(u) = \Gamma(1+\alpha)\, t^{-(1+\alpha)}\, C_{1+\alpha}(u)$$

where $C_\gamma(t)$ is Young's function*, then†

(5.3) $$Y_{1+\alpha}^{(r)}(u) = \frac{(-1)^r}{u^r} \sum_{i=0}^{r} C_i^r \frac{\Gamma(\alpha + r - i + 1)}{\Gamma(\alpha - i + 1)} Y_{1+\alpha-i}(u)$$

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*Young [1] pp. 161 et seq.

† Wang, F. T. [1] pp. 400 et seq.

is of bounded variation over the region zero to infinity. So multiplying (5.2) by $Y^{(r)}_{1+\alpha}$ (mu) $Y^{(s)}_{1+\beta}$ (nv) and integrating with respect to u and v from zero to infinity, we have by the extension of a theorem due to Young-Hardy*

$$(5.4)\quad \int_0^\infty\int_0^\infty Y^{(r)}_{1+\alpha}(mu)\, Y^{(s)}_{1+\beta}(nv)\, \psi_{r,s}(u,v)\,du\,dv = \sum_{i=0,\,j=0}^{\infty\ \infty} A^{r,s}_{ij}\int_0^\infty Y^{(r)}_{1+\alpha}(mu)\cos iu\,du \int_0^\infty Y^{(s)}_{1+\beta}(nv)\cos jv\,dv.$$

For r even

$$\int_0^\infty Y^{(r)}_{1+\alpha}(mu)\cos iu\,du = \begin{cases} (-1)^{\frac{r+2}{2}}\, m^{-r-1}\,\frac{2}{\pi}\, i^r\left(1-\frac{i}{m}\right)^\alpha & i<m \\ 0 & i>m \end{cases}$$

and we then have the right hand side of (5.4) equal to

$$\sum_{i<m,\,j<n} A^{r,s}_{ij}\,(-1)^{\frac{r+s}{2}}\, m^{-r-1}\, n^{-s-1}\,\frac{\pi^2}{4}\, i^r j^s\left(1-\frac{i}{m}\right)^\alpha\left(1-\frac{j}{n}\right)^\beta,$$

and (5.4) is equivalent to

$$\frac{4}{\pi^2}(-1)^{\frac{r+s}{2}}\, m^{r+1}\, n^{s+1}\int_0^\infty\int_0^\infty Y^{(r)}_{1+\alpha}(mu)\, Y^{(s)}_{1+\beta}(nv)\,\psi_{r,s}(u,v)\,du\,dv$$

$$= \sum_{\substack{i<m\\ j<n}} A^{r,s}_{ij}\, i^r j^s\left(1-\frac{i}{m}\right)^\alpha\left(1-\frac{j}{n}\right)^\beta.$$

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*Hardy [1] p. 186

This last term is exactly the right side of (5.1) so that

$$(5.5)\quad \left[\sigma_{m,n}^{\alpha,\beta}\right]_{x^r y^s} = \frac{4}{\pi^2}\, m^{r+1} n^{s+1} (-1)^{r+s} \int_0^\infty\!\!\int_0^\infty \gamma_{1+\alpha}^{(r)}(mu)\, \gamma_{1+\beta}^{(s)}(nv)\, \psi_{r,s}(u,v)\, du\, dv.$$

This formula can be derived for all integral r and s $\geqslant$ 0.

Suppose

$$g(u,v) \sim \sum_{i=0,\, j=0}^{\infty,\infty} a_{i,j}^{r,s} \cos iu \cos jv ,$$

where the $\sim$ denotes formal correspondence. However, g(u,v) is summable (C, λ, δ) and we can write the Cesaro means in the form:

$$\sigma_{m,n}^{\alpha-k,\beta-\ell}(G) = \sum_{\substack{i<m\\ j<n}} \left(1-\frac{i}{m}\right)^{\alpha-k} \left(1-\frac{j}{n}\right)^{\beta-\ell} a_{i,j}^{r,s},$$

which from (5.5) is equal to

$$\frac{4}{\pi^2}\, m\, n \int_0^\infty\!\!\int_0^\infty \gamma_{1+\alpha-k}(mu)\, \gamma_{1+\beta-\ell}(nv)\quad g(u,v)\, du\, dv$$

We have g(u,v) in place of the function $\psi_{r,s}$ (u,v) because of the form of g(u,v).

It can be shown* that for $\alpha > r$

$$Y_{1+\alpha}^{(r)}(mu) = O\left[\frac{1}{(mu)^{r+2}}\right] + O\left[\frac{1}{(mu)^{1+\alpha}}\right],$$

and it then follows that

$$m^{r+1}\int_0^\infty u^i\, Y_{1+\alpha}^{(r)}(mu)\, du = o(1).$$

Since $\psi_{r,s}(u,v)$ satisfies equations of the type (4.3), we get in place of (5.5)

$$\left[\sigma_{m,n}^{\alpha,\beta}\right]_{x^r y^s} = \frac{4}{\pi^2}\, m^{r+1} n^{s+1} (-1)^{r+s} \int_0^\infty\int_0^\infty Y_{1+\alpha}^{(r)}(mu)\, Y_{1+\beta}^{(s)}(nv)\, \varphi_{r,s}(u,v)\, du\, dv,$$

where

$$\begin{aligned}
\varphi_{r,s}(u,v) &= \frac{u^r v^s}{r!\, s!}\left[\gamma_{r,s} + \beta'_{r,s}(u,v)\right] && \text{for } |u| < \epsilon,\ |v| < \epsilon \\
&= \frac{u^r}{r!}\, e'(u,v) && \text{for } |u| < \epsilon,\ |v| > \epsilon \\
&= \frac{v^s}{s!}\, e''(u,v) && \text{for } |u| > \epsilon,\ |v| < \epsilon \\
&= 0 && \text{for } |u| > \epsilon,\ |v| > \epsilon
\end{aligned}$$

*Young, loc. cit.

Then, using (5.3),

$$(5.6)\quad \left[\sigma_{m,n}^{\alpha,\beta}\right]_{x^r y^s} = \frac{(-1)^{r+s}}{r!\,s!}\sum_{i=0,\,j=0}^{r,s}(-1)^{i+j}\,C_i^r\,C_j^s\,\frac{\Gamma(\alpha+r-i+1)}{\Gamma(\alpha-i+1)}\,\frac{\Gamma(\beta+s-j+1)}{\Gamma(\beta-j+1)}\,\times$$

$$\times\left\{mn\int_0^{\pi}\int_0^{\pi}\gamma_{1+\alpha-i}(mu)\,\gamma_{1+\beta-j}(nv)\,\frac{r!\,s!}{u^r v^s}\,\varphi_{r,s}(u,v)\,du\,dv\right\}+o(1).$$

The o(1) arises from the integrals over $(0,\pi;\pi,\infty)$, $(\pi,0;\infty,\pi)$, and $(\pi,\pi;\infty,\infty)$ since*

$$\gamma_u^{\ell}(u) = O\left(\frac{1}{u^{\ell+2}}\right).$$

Now $g(u,v) = \dfrac{r!\,s!}{u^r v^s}\,\varphi_{r,s}(u,v)$,

and therefore the quantity in braces in (5.6) is exactly

$\sigma_{m,n}^{\alpha-i,\beta-j}(G)$. Since $\lim\limits_{m,n\to\infty}\sigma_{m,n}^{\alpha-r,\beta-s}\to S$,

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*Young, loc. cit.

by the consistency theorem for Cesàro summability,

$$\lim_{m,n\to\infty} \sigma_{m,n}^{\alpha-i,\beta-j}(G) \longrightarrow S \qquad \text{for } i \leqq r, \quad j \leqq s.$$

We have, therefore,

$$\lim_{m,n\to\infty}\left[\sigma_{m,n}^{\alpha,\beta}\right]_{x^r y^s}$$

$$= \frac{S}{r!\,s!} \cdot \sum_{i=0}^{r} (-1)^i C_i^r \frac{\Gamma(\alpha+r-i+1)}{\Gamma(\alpha-i+1)} \cdot \sum_{j=0}^{s} (-1)^j C_j^s \frac{\Gamma(\beta+s-j+1)}{\Gamma(\beta-i+1)} .$$

But*

$$\sum_{i=0}^{r} (-1)^i C_i^r \frac{\Gamma(\alpha+r-i+1)}{\Gamma(\alpha-i+1)} = r!,$$

$$\text{so} \quad \left[\sigma_{m,n}^{\alpha,\beta}\right]_{x^r y^s} = \left[\sigma_{m,n}^{r+\lambda,\, s+\delta}\right]_{x^r y^s} \longrightarrow S.$$

*Wang, F. T. [1] p. 400

PART II--LOGARITHMIC SUMMABILITY

6. <u>Definitions</u>. We shall say the series $\sum_{i=0,j=0}^{\infty,\infty} A_{i,j}$ is summable by logarithmic means of order k,ℓ, i.e. (R,k,ℓ), to sum S if

$$R_{m,n}^{k,\ell} = A_{0,0} + \frac{1}{(\log m)^k} \sum_{i<m} \left(\log \frac{m}{i}\right)^k A_{i,0} + \frac{1}{(\log n)^\ell} \sum_{j<n} \left(\log \frac{n}{j}\right)^\ell A_{0,j}$$

$$+ \frac{1}{(\log m)^k (\log n)^\ell} \sum_{i<m,\, j<n} \left(\log \frac{m}{i}\right)^k \left(\log \frac{n}{j}\right)^\ell A_{i,j}$$

tends to a limit S.

Let us define

$$\phi(u,v;g) = \phi(u,v;\, g(y,v))$$

$$= \frac{1}{4}\left[f(x+u,y+v) + f(x-u,y+v) + f(x+u,y-v) + f(x-u,y-v) - 4\, \frac{g(y+v) - g(y-v)}{2} \right];$$

$$\chi_{0,0}(u,v;g) = \phi(u,v;g) \quad ;$$

$$\chi_{r,0}(u,v;g) = \int_u^{\pi} \frac{\chi_{r-1,0}(t,v;g)}{t}\, dt ;$$

$$\chi_{0,s}(u,v;g) = \int_v^{\pi} \frac{\chi_{0,s-1}(u,\tau;g)}{\tau}\, d\tau ;$$

where r and s are integers $\geqq 1$;

$$\chi_{r,s}(u,v;g) = \int_u^{\pi} \frac{\chi_{r-1,s}(t,v;g)}{t}\, dt = \int_v^{\pi} \frac{\chi_{r,s-1}(u,\tau;g)}{\tau}\, d\tau ;$$

$$\chi^{*,*}_{r,s}(u,v;g) = \int_0^{v}\int_0^{u} \chi_{r,s}(t,\tau,g)\, dt\, d\tau ;$$

$$\chi^{0,*}_{r,s}(u,v;g) = \int_0^{v} \chi_{r,s}(u,\tau;g)\, d\tau ;$$

$$\chi^{*,0}_{r,s}(u,v;g) = \int_0^{u} \chi_{r,s}(t,v;g)\, dt .$$

7. Conditions for Summability (R,k,ℓ) of Double Fourier Series.

Theorem 5. If for a given point (x,y) there exist functions g(y) and h(x) periodic with periods 2π such that

$$\int_0^u \left| \chi_{k-1,0}(t,v;g(y,v)) \right| dt = o\left[u\left(\log\frac{1}{u}\right)^k\right] \qquad \text{for all } v,$$

$$\int_0^v \left| \chi_{0,\ell-1}(u,\tau;h(x,u)) \right| d\tau = o\left[v\left(\log\frac{1}{v}\right)^\ell\right] \qquad \text{for all } u,$$

$$\chi_{k,0}(u,v;g) = o\left[\left(\log\frac{1}{u}\right)^k\right] \qquad \text{for all } v,$$

and

$$\chi_{0,\ell}(u,v;h) = o\left[\left(\log\frac{1}{v}\right)^\ell\right] \qquad \text{for all } u,$$

where

$$\frac{g(y+v) + g(y-v)}{2} \longrightarrow S \quad \text{as} \quad v \to 0,$$

and

$$\frac{h(x+u) + h(x-u)}{2} \longrightarrow S \quad \text{as} \quad u \to 0,$$

then the series (1.1) with r = 0, s = 0 is summable (R,k,ℓ) at the point (x,y) to sum S.

Since

$$u\,\chi_{k,0}(u,v;g) = \int_0^u \chi_{k,0}(t,v;g)\,dt - \int_0^u \chi_{k-1,0}(t,v;g)\,dt,$$

and

$$v\,\chi_{0,\ell}(u,v;h) = \int_0^v \chi_{0,\ell}(u,\tau;h)\,d\tau - \int_0^v \chi_{0,\ell-1}(u,\tau;h)\,d\tau,$$

theorem 5 is an immediate corollary of section 8.

8. <u>Further Conditions for Summability (R,k,ℓ) of Double Fourier Series.</u>

<u>Theorem 6.</u> If there exist positive integers k and ℓ and periodic functions g(y) and h(x) such that

$$(8.1)\qquad \int_0^u |\chi_{k-1,0}(t,v;g)|\,dt = O\left[u\left(\log\tfrac{1}{u}\right)^k\right] \quad \text{for all } v;$$

$$\int_0^v |\chi_{0,\ell-1}(u,\tau;h)|\,d\tau = O\left[v\left(\log\tfrac{1}{v}\right)^\ell\right] \quad \text{for all } u;$$

(8.2) $$\chi_{k,0}(u,v;g) = O\left[\left(\log\frac{1}{u}\right)^{k}\right] \quad \text{for all } v,$$

$$\chi_{0,\ell}(u,v;h) = O\left[\left(\log\frac{1}{v}\right)^{\ell}\right] \quad \text{for all } u;$$

and

$$\chi^{*,0}_{k,0}(u,v;g) = o\left[u\left(\log\frac{1}{u}\right)^{k}\right] \quad \text{for all } v,$$

(8.3) $$\chi^{0,*}_{0,\ell}(u,v;h) = o\left[v\left(\log\frac{1}{v}\right)^{\ell}\right] \quad \text{for all } u;$$

$$\lim_{v\to 0} \frac{g(y+v)+g(y-v)}{2} \to S \; ; \quad \lim_{u\to 0} \frac{h(x+u)+h(x-u)}{2} \to S \; ,$$

then the series (1.1) for $r = 0$ and $s = 0$ is summable (R,k,ℓ) at the point (x,y) to sum S.

Let us suppose without loss of generality that the point $(x,y) = (0,0)$; $a_{0,0} = 0$; $S = 0$ and $f(x,y)$ is even-even. Then $\phi(u,v;S) = f(u,v)$. If we integrate (8.1), (8.2), and (8.3) with respect to v from 0 to π and use Fubini's theorem to interchange the order of integration, we have these relations holding for a function of one variable $\int_{-\pi}^{\pi} f(x,y)\,dy$, since they held for all v. The simple Fourier series of this function is summable (R,k).* In other words, the series $\sum_{i=1}^{\infty} A_{i,0}^{0,0}$ is summable (R,k) and similarly $\sum_{j=1}^{\infty} A_{0,j}^{0,0}$ is summable (R,ℓ). We can therefore assume without loss of generality that the $A_{i,0} = A_{0,j} = 0$ for all i and all j. We then have for the function under consideration

$$\phi(u,v) = f(u,v) \sim \sum_{i=1, j=1}^{\infty,\infty} a_{i,j} \cos iu \cos jv.$$

The $\sim$ sign denotes formal correspondence and $a_{i,j} =$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) \cos x \cos y \, dx \, dy.$$

Now $a_{i,0} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi(u,v) \cos iu \, du \, dv = 0$ for all i

and $a_{0,j} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi(u,v) \cos jv \, du \, dv = 0$ for all j.

Since the trigonometric system is complete, we must have

$\int_{-\pi}^{\pi} \phi(u,v) \, du = 0$ and $\int_{-\pi}^{\pi} \phi(u,v) \, dv = 0$ which makes the

*Wang, F. T. [3] pp. 142-143.

function $\phi^{*,*}(t,\tau) = \int_0^{\tau}\int_0^{t} \phi(u,v)\,du\,dv$ doubly periodic

with periods 2π for both t and τ.

The integrated Fourier series converges and we have

$$(8.4) \qquad \phi^{*,*}(t,\tau) = \sum_{i=1,\,j=1}^{\infty,\,\infty} \frac{a_{ij}}{ij} \sin it \sin j\tau .$$

We will consider a function similar to Young's function used in part one. The function

$$(8.5) \qquad L_k(t) = \Gamma(1+k) \sum_{p=0}^{\infty} (-1)^p \frac{t^{2p}}{(2p)!}$$

has the following properties.*

$$(8.6) \qquad L_0(t) = \frac{\sin t}{t} ;$$

$$(8.7) \qquad L_k(t) = O(1) \qquad \text{for } t \geqslant 0 , \ k > -1 ;$$

*Wang, F. T. [3] p. 145

(8.8) $$L_k(t) = O\left[\frac{(\log t)^{k-1}}{t}\right] \quad \text{for } t \geqq 2, \quad k \geqq 1;$$

(8.9) $$L_k'(t) = O\left[\frac{(\log t)^{k-1}}{t^2}\right] \quad \text{for } t \geqq 2, \quad k \geqq 1;$$

(8.10) $$t\,L_k'(t) + L_k(t) = k\,L_{k-1}(t) \quad ;$$

(8.11) $$L_k'(t) = O(1) \quad \text{for } 0 \leqq t \leqq 2, \quad k > -1;$$

(8.12) $$\int_0^\infty L_k'(mt)\sin it\,dt = -\frac{i}{m^2}\,\frac{\pi}{2}\left(\log\frac{m}{i}\right)^k, \quad i < m;$$

$$= 0 \qquad\qquad i \geqq m.$$

F. T. Wang has also proved that $L_k'(t)$ and $L_k''(t)$ are absolutely integrable and of bounded variation on the interval $(0, \infty)$. By an extension of the theorem due to Young-Hardy, we may multiply by such a function and integrate over the region of bounded variation.

Integrating (8.4) term by term, after multiplication by $L_k'(mt)\, L_\ell'(n\tau)$,

$$\int_0^\infty\!\!\int_0^\infty L_k'(mt)\, L_\ell'(n\tau)\, \phi^{*,*}(t,\tau)\, dt\, d\tau = \sum_{i=1,\, j=1}^{\infty,\infty} \frac{a_{ij}}{ij} \int_0^\infty\!\!\int_0^\infty L_k'(mt)\, L_\ell'(n\tau) \sin it \sin j\tau \, dt\, d\tau,$$

and by (8.12), we get

$$\sum_{i < m,\, j < n} \left(\log \frac{m}{i}\right)^k \left(\log \frac{n}{j}\right)^\ell a_{ij} = \frac{4}{\pi^2} m^2 n^2 \int_0^\infty\!\!\int_0^\infty L_k'(mt)\, L_\ell'(n\tau)\, \phi^{*,*}(t,\tau)\, dt\, d\tau.$$

But for our function the left hand side is equal to

$$(\log m)^k (\log n)^\ell\, R_{m,n}^{k,\ell}$$

and therefore

$$R_{m,n}^{k,\ell} = \frac{4}{\pi^2} \frac{m^2 n^2}{(\log m)^k (\log n)^\ell} \int_0^\infty\!\!\int_0^\infty \phi^{*,*}(t,\tau)\, L_k'(mt)\, L_\ell'(n\tau)\, dt\, d\tau.$$

To prove our theorem, we must show that the right side is o(1).

Let us break the integral into integrals over the regions

$$\text{(8.13)} \qquad \begin{aligned} A&: (0,0\,;\,\eta,\eta)\,, & B&: (\eta,0\,;\,\infty,\eta)\,,\\ C&: (0,\eta\,;\,\eta,\infty)\,, & D&: (\eta,\eta\,;\,\infty,\infty) \end{aligned}$$

and denote by the capital letters, the integrals over the corresponding regions. Then, using (8.9)

$$|D| \leqq \frac{K\, m^2 n^2}{(\log m)^k (\log n)^l} \int_\eta^\infty\!\!\int_\eta^\infty \frac{(\log mt)^{k-1}}{(mt)^2}\,\frac{(\log n\tau)^{l-1}}{(n\tau)^2}\,\phi^{**}(t,\tau)\,dt\,d\tau,$$

for $n\eta \geq 2$. This in turn is

$$\leq \frac{2^{k+l-2} K}{(\log m)^k (\log n)^l} \int_0^\infty\!\!\int_0^\infty \frac{[(\log m)^{k-1} + |\log t|^{k-1}][(\log n)^{l-1} + |\log \tau|^{l-1}]}{t^2\,\tau^2}\,\phi^{**}(t,\tau)\,dt\,d\tau$$

$$= \frac{K(\log m)^{k-1}(\log n)^{l-1}}{(\log m)^k (\log n)^l} \int_\eta^\infty\!\!\int_\eta^\infty \frac{\phi^{**}(t,\tau)}{t^2\,\tau^2}\,dt\,d\tau \;+$$

$$+ \frac{K\,(\log m)^{k-1}}{(\log m)^{k}(\log n)^{\ell}} \int_{\eta}^{\infty}\!\!\int_{\eta}^{\infty} \frac{|\log \tau|^{\ell-1}}{t^{2}\tau^{2}}\,\left|\phi^{*,*}(t,\tau)\right| dt\,d\tau$$

$$+ \frac{K\,(\log m)^{\ell-1}}{(\log m)^{k}(\log n)^{\ell}} \int_{\eta}^{\infty}\!\!\int_{\eta}^{\infty} \frac{|\log t|^{k-1}}{t^{2}\tau^{2}}\,\left|\phi^{*,*}(t,\tau)\right| dt\,d\tau$$

$$+ \frac{K}{(\log m)^{k}(\log n)^{\ell}} \int_{\eta}^{\infty}\!\!\int_{\eta}^{\infty} \frac{|\log t|^{k-1}\,|\log \tau|^{\ell-1}}{t^{2}\tau^{2}}\,\left|\phi^{*,*}(t,\tau)\right| dt\,d\tau .$$

K depends only on k and ℓ, and therefore, all four terms above are o(1).

Now

$$C = \frac{4}{\pi^{2}}\,\frac{m^{2}n^{2}}{(\log m)^{k}(\log n)^{\ell}} \int_{\eta}^{\infty} L_{\ell}'(n\tau) \int_{0}^{\eta} L_{k}'(mt)\,\phi^{*,*}(t,\tau)\,dt\,d\tau,$$

and integrating the inner integral by parts with

$$\phi^{0*}(t,\tau) = \int_0^\tau \phi(t,u)\,du,$$

we get

$$\int_0^\eta L_k'(mt)\,\phi^{**}(t,\tau)\,dt = \frac{L_k(m\eta)}{m}\,\phi^{**}(\eta,\tau) - \int_0^\eta \frac{L_k(mt)}{m}\,\phi^{0*}(t,\tau)\,dt,$$

since $\phi^{**}(0,t) = 0$ and $L_k(t) = O(1)$. Therefore, C is equal to

$$\frac{4}{\pi^2}\,\frac{m^2 n^2}{(\log m)^k (\log n)^\ell}\left\{\int_\eta^\infty \frac{L_k(m\eta)}{m}\,L_\ell'(n\tau)\,\phi^{**}(\eta,\tau)\,d\tau - \int_\eta^\infty\int_0^\eta \frac{L_\ell'(n\tau)\,L_k(mt)}{m}\,\phi^{0*}(t,\tau)\,dt\,d\tau.\right.$$

Now the first term, again using (8.9), is in absolute value less than or equal to

$$\frac{m^2 n^2 K}{(\log m)^k (\log n)^\ell}\;\frac{(\log m\eta)^{k-1}}{m^2\eta^2}\int_\eta^\infty \frac{(\log n\tau)^{\ell-1}}{n^2\tau^2}\,\left|\phi^{**}(\eta,\tau)\right|\,d\tau,$$

where K depends only on k and ℓ. As before, this is less than or equal to

$$\frac{K(\log m\eta)^{k-1}}{(\log m)^k}\left\{\frac{1}{\log m}\int_\eta^\infty \frac{|\phi^{*,*}(\eta,\tau)|}{\tau^2}\,d\tau + \frac{1}{(\log m)^\ell}\int_\eta^\infty \frac{|\log\tau|^{\ell}}{\tau^2}|\phi^{*,*}(\eta,\tau)|\,d\tau\right\},$$

which is o(1).

The second term is, by the same procedure, in absolute value less than or equal to

$$(8.14)\quad K\left\{\frac{1}{\log n}\int_\eta^\infty \frac{d\tau}{\tau^2} + \frac{1}{(\log n)^\ell}\int_\eta^\infty \frac{|\log\tau|^{\ell-1}}{\tau^2}\,d\tau\right\}\frac{m}{(\log m)^k}\int_c^\eta L_k(mt)\,\phi^{c*}(t,\tau)\,dt.$$

Let us call the last integral E and integrate E by parts. Then,

$$E = \int_0^\eta t\,L_k(mt)\left\{\frac{\int_0^\tau \chi_{0,0}(t,u)\,du}{t}\right\}dt,$$

$$E = \left[\chi_{1,0}^{0,*}(t,\tau)\;t\,L_k(mt)\right]_c^\eta - \int_c^\eta k\,L_{k-1}(mt)\,\chi_{1,0}^{c,*}(t,\tau)\,dt.$$

Now* $t \cdot \chi_{1,0}^{0,*}(t,\tau) \longrightarrow 0$ as $t \to 0$ and therefore

$$E = \chi_{1,0}^{0,*}(\eta,\tau)\,\eta\, L_k(m\eta) - k\int_0^{\eta} t\, L_{k-1}(mt)\, \frac{\chi_{1,0}^{0,*}(t,\tau)}{t}\, dt.$$

Calling the first term E_1, and integrating by parts, we have

$$E = E_1 - k\Big[\chi_{2,0}^{0,*}(t,\tau)\; t\, L_{k-1}(mt)\Big]_0^{\eta} + k(k-1)\int_0^{\eta} L_{k-2}(mt)\; \chi_{2,0}^{0,*}(t,\tau)\, dt,$$

where again $t \cdot \chi_{2,0}^{0,*}(t,\tau) \longrightarrow 0$ as $t \to 0$. We can continue defining the E_i, step by step, k times until we arrive at

$$|E| \leq \sum_{i=1}^{k} |E_i| + \Big| k! \int_0^{\eta} L_0(mt)\; \chi_{k,0}^{0,*}(t,\tau)\, dt\Big|.$$

If $n\eta > 2$, the first k terms when substituted in (8.14) give us terms which, using (8.8), are o(1). The remainder of E is all that is left of C to be shown o(1).

Now $L_0(mt) = \dfrac{\sin mt}{mt}$, and

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*Hardy [2] pp. 50-52.

$$\left| \frac{1}{(\log m)^k} \int_0^{\eta} \frac{\sin mt}{t} \chi_{k,0}^{0,k}(t,\tau)\, dt \right|$$

$$\leq \left| \frac{1}{(\log m)^k} \int_0^{\eta} \frac{\sin m_1 t}{t} \chi_{k,0}^{0,k}(t,\tau;\xi)\, dt \right| + \left| \frac{\int_0^{\tau} G(u)\, du}{(\log m)^k} \int_{\tau}^{h} \frac{\sin mt}{t} \sum_{i=1}^{k} K\left(\log \frac{1}{t}\right)^i dt \right| ,$$

where now $G(u) = \dfrac{g(u) + g(-u)}{2}$

The conditions (8.1), (8.2), and (8.3) give us the first term on the right o(1).* The argument used by F. T. Wang, applied to the second term gives us something $O\left[\int_0^{\tau} G(u)\, du\right]$ and therefore

$$|C| \leq o(1) + \left\{ \frac{1}{\log n} \int_{\eta}^{\infty} \frac{\left| \int_0^{\tau} G(u) du \right|}{\tau^2} d\tau + \frac{1}{(\log n)^{\ell}} \int_{\eta}^{\infty} \frac{|\log \tau|^{\ell-1}}{\tau^2} \left| \int_0^{\tau} G(u) du \right| d\tau \right\} ,$$

which is o(1). The same procedure will also give B = o(1).

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*Wang, F. T. [3] p. 150

We have now only to show $A = o(1)$. If we integrate A by parts, we get

$$\frac{4}{\pi^2}\frac{m^2 n^2}{(\log m)^k(\log n)^\ell}\left\{\left[\frac{\phi^{*,*}(t,\tau)L_k(mt)L_\ell(n\tau)}{mn}\right]_{0,0}^{\eta,\eta} - \int_0^\eta \left[\frac{L_k(mt)L_\ell(n\tau)}{mn}\,\phi^{0,*}(t,\tau)\right]_{\tau=0}^{\tau=\eta} dt\right.$$

$$\left. - \int_0^\eta \left[\frac{L_k(mt)L_\ell(n\tau)}{mn}\,\phi^{*,0}(t,\tau)\right]_{t=0}^{t=\eta} d\tau + \int_0^\eta\!\!\int_0^\eta \frac{L_k(mt)L_\ell(n\tau)}{mn}\,\phi(t,\tau)\,dt\,d\tau\right\}$$

but, as before, $\phi^{*,*}(0,0) = \phi^{*,0}(0,\tau) = \phi^{0,*}(t,0) = 0$, and so A is equal to

$$\frac{4}{\pi^2}\frac{mn}{(\log m)^k(\log n)^\ell}\left\{\phi^{*,*}(\eta,\eta)L_k(m\eta)L_\ell(n\eta) - \int_0^\eta L_k(mt)L_\ell(n\eta)\,\phi^{0,*}(t,\eta)\,dt\right.$$

$$\left. - \int_0^\eta L_k(m\eta)L_\ell(n\tau)\,\phi^{*,0}(\eta,\tau)\,d\tau + \int_0^\eta\!\!\int_0^\eta L_k(mt)L_\ell(n\tau)\,\phi(t,\tau)\,dt\,d\tau\right\}.$$

Let us call these terms, with the factor multiplied through, A_1, A_2, A_3, and A_4, respectively. Using (8.8), we find that for $m\eta > 2$, $n\eta > 2$

$$|A_1| \leq \frac{K\; mn\; (\log m\eta)^{k-1} (\log n\eta)^{\ell-1}}{(\log m)^k (\log n)^\ell\; m\eta\; n\eta},$$

which is $o(1)$.

Using the same inequality for $L_k(n\tau)$, we find that

$$|A_2| \leq \frac{4}{\pi^2} \frac{mn\; K}{(\log m)^k (\log n)^\ell} \frac{(\log n\eta)^{\ell-1}}{n\eta} \left| \int_0^\eta L_k(mt)\, \phi^{\ast}_{\eta}(t,\eta)\, dt \right|.$$

But the quantity in absolute values can be treated exactly as the similar quantity in E and can be shown to be

$$o\left[\frac{(\log m)^k}{m}\right] + O\left[\frac{(\log m)^k}{m}\right]$$

since the factor, $\frac{1}{2}\int_0^\eta [g(u) + g(-u)]\,du$ of E is in this case a constant. As a result, $A_2 = o(1)$, and a similar argument gives us at once $A_3 = o(1)$.

Integrating A_4 by parts, we get $\dfrac{4}{\pi^2} \dfrac{mn\,F}{(\log m)^k (\log n)^\ell}$,

where

$$F = \int_0^\eta \int_0^\eta t\, L_k(mt)\, \tau L_\ell(n\tau)\, \frac{\phi(t,\tau)}{t\tau}\, dt\, d\tau,$$

$$= \Big[L_k(mt)\, L_\ell(n\tau)\, t\tau\, \chi_{1,1}(t,\tau) \Big]_{0,0}^{\eta,\eta}$$

$$- k \int_0^\eta \tau L_{k-1}(mt)\, L_\ell(n\tau)\, \chi_{1,1}(t,\tau) \Big]_{\tau=0}^{\tau=\eta} dt$$

$$- \ell \int_0^\eta t\, L_k(mt)\, L_{\ell-1}(n\tau)\, \chi_{1,1}(t,\tau) \Big]_{t=0}^{t=\eta} d\tau$$

$$+ k\ell \int_0^\eta \int_0^\eta L_{k-1}(mt)\, L_{\ell-1}(n\tau)\, \chi_{1,1}(t,\tau)\, dt\, d\tau .$$

Let us call these terms F_1, F_2, F_3, and G_1 so that $F = F_1 + F_2 + F_3 + G_1$. Now, G_1 is of the same form as F, and upon integrating G_1 by parts, we get four terms similar to those of F. We shall call them F_4, F_5, F_6, and G_2 so that

$G_1 = F_4 + F_5 + F_6 + G_2$.

Let us assume $k < \ell$. If $k = \ell$, we would already have (8.15). We then have by (8.6),

$$G_k = \frac{k!\,\ell!}{(\ell-k)!}\int_0^b\int_0^b \frac{\sin mt}{mt} L_{\ell-k}(m\tau)\,\chi_{k,k}(t,\tau)\,dt\,d\tau,$$

and our F contains 3k F_i all similar in form to one of F_1, F_2, or F_3. If we integrate G_k, by parts $\ell - k$ times with respect to τ, we get

$$(8.15)\qquad G_\ell = k!\,\ell!\,(-1)^{\ell-k}\int_0^b\int_0^b \frac{\sin mt}{mt}\;\frac{\sin m\tau}{m\tau}\;\chi_{k,\ell}(t,\tau)\,dt\,d\tau,$$

with $\ell - k$ additional F_i's.

We now show that

$$\sum_{i=1}^{2k+\ell} F_i\ \frac{4}{\pi^2}\ \frac{mn}{(\log m)^k(\log n)^\ell} = o(1).$$

The terms involving the F_{3i+1} for i = 0,1,2 • • •, k - 1 are all of the type

$$K \frac{mn}{(\log m)^k (\log n)^\ell} \Big[L_{k-3i}(mt) L_{\ell-3i}(n\tau)\, t\tau\, X_{3i+1,3i+1}(t,\tau) \Big]_{0,0}^{\eta,\eta}$$

but $t\tau X_{3i+1,3i+1}(t,\tau) \longrightarrow 0$ as $t \to 0$ and $\tau \to 0$ and $L_{k-3i}(mt)$ and $L_{\ell-3i}(n\tau)$ are $O(1)$, so we have only to consider the upper limits. Now for $m\eta > 2$ and $n\eta > 2$, using (8.8),

$$\left| \frac{mn\,K}{(\log m)^k (\log n)^\ell} L_{k-3i}(m\eta) L_{\ell-3i}(n\eta)\, \eta\eta\, X_{3i+1,3i+1}(\eta,\eta) \right|$$

$$\leqq \frac{K\,mn}{(\log m)^k (\log n)^\ell} \frac{(\log m\eta)^{k-1-3i}}{m\eta} \frac{(\log n\eta)^{\ell-1-3i}}{n\eta},$$

which is o(1).

The terms containing F_{2+3i}, i = 0,1,2,• • •, k - 1 are all of the type

$$\frac{mK}{(\log m)^k (\log n)^\ell} \int_0^\eta \Big[L_{k-2-3i}(mt) L_{\ell-1-3i}(n\tau)\, \tau\, X_{2+3i,2+3i}(t,\tau) \Big]_{\tau=0}^{\tau=\eta} dt.$$

Again using (8.8), and since $\tau \chi_{i,i}(t,\tau) \to 0$ as $\tau \to 0$, this is in absolute value less than or equal to

$$\frac{m K (\log m \mathfrak{z})^{\ell-1-3i}}{(\log m)^{k} (\log m)^{\ell}} \left| \int_0^{\mathfrak{z}} t L_{k-2-3i}(mt) \, \chi_{2+3i,2+3i}(t,\mathfrak{z}) \, dt \right|.$$

The quantity inside absolute value signs, integrated by parts $k - 2 - 3i$ times, gives us something which is less than or equal to

$$\frac{m K (\log m \mathfrak{z})^{\ell-1-3i}}{(\log m)^{k} \cdot (\log m)^{\ell}} \Bigg| \Big[t L_{k-2-3i}(mt) \, \chi_{3+3i,2+3i}(t,\mathfrak{z}) -$$

$$(k-2-3i) \, t L_{k-1-3i}(mt) \, \chi_{4+3i,2+3i}(t,\mathfrak{z}) + \ldots \pm (k-2-3i)! \, t L_{k-1}(mt) \, \chi_{k,2+3i}(t,\mathfrak{z}) \Big]_{t=0}^{t=\mathfrak{z}}$$

$$\mp (k-2-3i)! \int_0^{\mathfrak{z}} L_0(mt) \, \chi_{k,2+3i}(t,\mathfrak{z}) \, dt \Bigg|.$$

Each of the terms containing a factor from the square brackets is o(1) as in the consideration of F_{3i+1} and the last term, using (8.6), is

$$(8.16)\quad K\,\frac{(\log n\eta)^{\ell-1-3i}}{(\log n)^{\ell}}\cdot\frac{1}{(\log n)^{k}}\left|\int_0^{\eta}\frac{\sin nt}{t}\,\chi_{k,2+3i}(t,\eta)\,dt\right|.$$

The first factor is o(1), and we can show* that the second factor is $O(1)$ if the following relations are satisfied:

$$\int_0^u \left|\chi_{k-1,\,2+3i}(t,\eta)\right| dt = O\left[u\left(\log\tfrac{1}{u}\right)^k\right],$$

$$\int_0^u \chi_{k,\,2+3i}(t,\eta)\, dt = O\left[u\left(\log\tfrac{1}{u}\right)^k\right],$$

$$\chi_{k,\,2+3i}(t,\eta)\, dt = O\left[\left(\log\tfrac{1}{u}\right)^k\right].$$

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*Wang, F. T. [3] page 150.

These are satisfied since

$$\chi_{k-1,\,2+3i}(t,b) = \chi_{k-1,\,2+3i}(t,b;g) - K_{2+3i}(b)\sum_{i=1}^{k-1} K\left(\log\frac{1}{t}\right)^{i},$$

and the first term on the right satisfies the inequalities upon integrating (8.1), (8.2), and (8.3), and the second obviously satisfies the three conditions. Therefore, the term (8.16) is o(1).

In exactly the same manner, we can, in general, show that

(8.17) $$\chi^{*,*}_{k,\ell}(u,v) = o\left[v\left(\log\frac{1}{v}\right)^{\ell} u\left(\log\frac{1}{u}\right)^{k}\right],$$

(8.18) $$\chi_{k,\ell}(u,v) = O\left[\left(\log\frac{1}{u}\right)^{k}\left(\log\frac{1}{v}\right)^{\ell}\right],$$

(8.19) $$\int_0^u\int_0^v \chi_{k-1,\,\ell-1}(t,\tau)\,dt\,d\tau = O\left[v\left(\log\frac{1}{v}\right)^{\ell} u\left(\log\frac{1}{u}\right)^{k}\right],$$

since

$$(8.20) \qquad \chi^{**}_{k,\ell}(u,v) = \chi^{**}_{k,\ell}(u,v;g) - K^{*}_{\ell}(v) \sum_{i=1}^{?} K\, u \left(\log \frac{1}{u}\right)^{i},$$

where $$K^{*}_{\ell}(v) = \int_0^{v}\int_{v_\ell}^{\pi} \frac{1}{v_{\ell-1}} \int_{v_{\ell-1}}^{\pi} \frac{1}{v_{\ell-2}} \int_{v_{\ell-2}}^{\pi} \cdots \int_{v_1}^{\pi} \frac{G(v_0)}{v_0}\, dv_0\, dv_1 \ldots dv_\ell .$$

The first term on the right of (8.20), upon integrating (8.1), (8.2) and (8.3), obviously satisfies (8.17), (8.18), and (8.19). The second term with $g(v) + g(-v) \to 0$ gives

$$K^{*}_{\ell}(v) = \int_0^{v}\int_{v_\ell}^{b} \cdots \frac{1}{v_1}\int_{v_1}^{b} \frac{G(b)}{v_0}\, dv_0 \ldots dv_\ell + \sum_{i=1}^{\ell-1} O\left[v \left(\log \frac{1}{v}\right)^{i}\right]$$

$$(G(b) \to 0)$$

$$= G(b)\, v \left(\log \frac{1}{v}\right)^{\ell} + \sum_{i=1}^{\ell-1} O\left[v \left(\log \frac{1}{v}\right)^{i}\right],$$

multiplied by

$$\sum_{i=1}^{k} K_u \left(\log \frac{1}{u}\right)^i$$

Therefore, (8.17) holds and, in a similar manner, (8.18) and (8.19) also.

Now the terms of F containing F_{3+3i} $(i = 0,1,\cdots, k-1)$; and the terms F_i $i > 3k$, which are of the type

$$\frac{n\, K\, L_{\ell-k-j}(n\delta)}{(\log m)^k (\log n)^\ell} \int_0^\delta \frac{\sin mt}{t}\, X_{k,\,k+j}(t,\delta)\, dt,$$

are o(1) as above.

We are then left with the term

$$K\, \frac{m\, n}{(\log m)^k (\log n)^\ell} \int_0^\delta \int_0^\delta \frac{\sin mt}{mt}\, \frac{\sin n\tau}{n\tau}\, X_{k\ell}(t,\tau)\, dt\, d\tau,$$

which must be shown o(1).

Let

$$H = \int_0^{b}\int_0^{b} \frac{\sin mt}{t} \, \frac{\sin n\tau}{\tau} \, X_{k,l}(t,\tau) \, dt\, d\tau,$$

and break up the integral over the following regions, calling the four integrals H_1 , H_2 , H_3 , and H_4 :

$$H_1 \; : \; \left(0,0 \, ; \, \tfrac{p}{m}, \tfrac{q}{m} \right) \qquad\qquad H_2 \; : \; \left(0, \tfrac{q}{m} \, ; \, \tfrac{p}{m}, b \right)$$

$$H_3 \; : \; \left(\tfrac{p}{m}, 0 \, ; \, b, \tfrac{q}{m} \right) \qquad\qquad H_4 \; : \; \left(\tfrac{p}{m}, \tfrac{q}{m} \, ; \, b, b \right)$$

Integrating H_1 by parts, we have

$$H_1 = X_{k,l}^{*,*}(t,\tau) \, \frac{\sin mt}{t} \, \frac{\sin n\tau}{\tau} \Bigg]_{0,0}^{\frac{p}{m},\frac{q}{m}} + \int_\delta^{\frac{p}{m}} X_{k,l}^{*,*}(t,\tau) \, \frac{\sin mt}{t^2} \, \frac{\sin n\tau}{\tau} \Bigg]_{\tau=0}^{\tau=\frac{q}{m}} dt \; -$$

$$-m\int_0^{\frac{p}{m}} \chi^{*,*}_{k,\ell}(t,\tau)\,\frac{\sin n\tau}{\tau}\,\frac{\cos mt}{t}\Big]_{\tau=0}^{\tau=\frac{q}{n}} dt + \int_0^{\frac{q}{n}} \chi^{*,*}_{k,\ell}(t,\tau)\,\frac{\sin mt}{t}\,\frac{\sin n\tau}{\tau^2}\Big]_{t=0}^{t=\frac{p}{m}} d\tau$$

$$-n\int_0^{\frac{q}{n}} \chi^{*,*}_{k,\ell}(t,\tau)\,\frac{\sin mt}{t}\,\frac{\cos n\tau}{\tau}\Big]_{t=0}^{t=\frac{p}{m}} d\tau + \int_0^{\frac{q}{n}}\int_0^{\frac{p}{m}} \chi^{*,*}_{k,\ell}(t,\tau)\,\frac{\sin mt}{t^2}\,\frac{\sin n\tau}{\tau^2}\,dt\,d\tau$$

$$-m\int_0^{\frac{q}{n}}\int_0^{\frac{p}{m}} \chi^{*,*}_{k,\ell}(t,\tau)\,\frac{\sin n\tau}{\tau^2}\,\frac{\cos mt}{t}\,dt\,d\tau - n\int_0^{\frac{q}{n}}\int_0^{\frac{p}{m}} \chi^{*,*}_{k,\ell}(t,\tau)\,\frac{\sin mt}{t^2}\,\frac{\cos n\tau}{\tau}\,dt\,d\tau$$

$$+\,mn\int_0^{\frac{q}{n}}\int_0^{\frac{p}{m}} \chi^{*,*}_{k,\ell}(t,\tau)\,\frac{\cos mt}{t}\,\frac{\cos n\tau}{\tau}\,dt\,d\tau,$$

let us say $I_1 + I_2 + \cdots + I_9$.

Obviously $\chi^{*,*}_{k,\ell}(0,0) = 0$, and by (8.17),

$$|I_1| \leq c \left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^\ell \frac{m}{p} \frac{n}{q} \cdot \frac{p}{m} \frac{q}{n} \cdot \sin p \sin q .$$

$$I_1 = o\left[(\log m)^k (\log n)^\ell \right].$$

$$|I_2| \leq \frac{|\sin q|}{q} m n \int_0^{p/m} \frac{\chi^{**}_{k\ell}(t, \frac{q}{n})}{t} \, dt$$

$$\leq c \left(\log \frac{n}{q}\right)^\ell m \int_0^{p/m} \left(\log \frac{1}{t}\right)^k dt ,$$

$$\leq m c \left(\log \frac{n}{q}\right)^\ell \left[t \left\{ \left(\log \frac{1}{t}\right)^k + \sum_{i=1}^{k} k(k-1)\cdots(k-i+1) \left(\log \frac{1}{t}\right)^{k-i} \right\} \right]_0^{p/m}$$

$$= c\, p \left(\log \frac{n}{q}\right)^\ell \left\{ \left(\log \frac{m}{p}\right)^k + \sum_{i=1}^{k} K \left(\log \frac{m}{p}\right)^{k-i} \right.$$

$I_2 \quad = \quad o\left[(\log m)^{k} \quad (\log n)^{\ell}\right].$

$$|I_3| \leq \frac{mn}{\delta}\int_0^{\frac{\delta}{m}} \frac{|\chi^{**}_{k,\ell}(t,\frac{\delta}{n})|}{t}\,dt,$$

and as in I_2

$I_3 \quad = \quad o\left[(\log m)^{k} \quad (\log n)^{\ell}\right].$

$I_4 \quad = \quad o\left[(\log m)^{k} \quad (\log n)^{\ell}\right]$, in exactly the same manner as I_2.

$I_5 \quad = \quad o\left[(\log m)^{k} \quad (\log n)^{\ell}\right]$, in exactly the same manner as I_3.

$$|I_6| \leq \epsilon\, mn \int_0^{\frac{\delta}{n}} \left(\log \frac{1}{\tau}\right)^{\ell} d\tau \int_0^{\frac{\delta}{m}} \left(\log \frac{1}{t}\right)^{k} dt,$$

and, then, integrating as in I_2, we get

$I_6 \quad = \quad o\left[(\log m)^{k} \quad (\log n)^{\ell}\right].$

$$|I_7| \leq \epsilon\, mn \int_0^{\frac{\delta}{n}} \left(\log \frac{1}{\tau}\right)^{\ell} d\tau \int_0^{\frac{\delta}{m}} \left(\log \frac{1}{t}\right)^{k} dt.$$

$I_7 \quad = \quad o\left[(\log m)^{k} \quad (\log n)^{\ell}\right]$

$I_8 \quad = \quad o\left[(\log m)^{k} \quad (\log n)^{\ell}\right]$, as I_7.

$$|I_9| \leq \epsilon\, m n \int_0^{\frac{\delta}{n}} \left(\log \frac{1}{\tau}\right)^{\rho} d\tau \int_0^{\frac{\delta}{m}} \left(\log \frac{1}{t}\right)^{k} dt,$$

$I_9 = o\left[(\log m)^{k} (\log n)^{\rho} \right]$

Therefore, $E_1 = o\left[(\log m)^{k} (\log n)^{\rho}\right]$, and we turn to H_2.

Integrating H_2 by parts, we get the terms

$$\chi^{**}_{k,\rho}(t,\tau)\, \frac{\sin mt}{t}\, \frac{\sin n\tau}{\tau}\Bigg]_{0,\,\frac{\delta}{m}}^{\frac{\delta}{m},\,\delta} - \int_0^{\frac{\delta}{m}} \chi^{**}_{k,\rho}(t,\tau)\, \frac{\sin n\tau}{\tau}\left(\frac{m \cos mt}{t} - \frac{\sin mt}{t^2}\right)\Bigg]_{\tau=\frac{\delta}{n}}^{\delta} dt$$

$$-\int_{\frac{\delta}{n}}^{\delta} \chi^{**}_{k,\rho}(t,\tau)\, \frac{\sin mt}{t}\left(\frac{n\cos n\tau}{\tau} - \frac{\sin n\tau}{\tau^2}\right)\Bigg]_{t=0}^{\frac{\delta}{m}} d\tau + \int_{\frac{\delta}{n}}^{\delta}\int_0^{\frac{\delta}{m}} \chi^{**}_{k,\rho}(t,\tau)\left(\frac{m\cos mt}{t} - \frac{\sin mt}{t^2}\right)\left(\frac{n\cos n\tau}{\tau} - \frac{\sin n\tau}{\tau^2}\right) dt\, d\tau,$$

which we shall call J_1, J_2, J_3, and J_4 respectively.

$$|J_1| \leq \epsilon\, t\, \tau \left(\log \frac{1}{t}\right)^{k}\left(\log \frac{1}{\tau}\right)^{\rho} \frac{|\sin mt|}{t}\, \frac{\sin n\tau}{\tau}\Bigg|^{\frac{\delta}{m},\,\delta}$$

$$\leq \epsilon \left(\log \frac{m}{\delta}\right)^{k}\left(\log \frac{1}{\delta}\right)^{\rho}.$$

$J_1 = o\left[(\log m)^{k} (\log n)^{\rho} \right].$

$$|J_2| \leqq \left| \int_0^{\pi/m} \chi_{k,l}^{*,*}(t,\eta) \frac{\sin n\eta}{\eta} \left[\frac{m \cos mt}{t} - \frac{\sin mt}{t^2} \right] dt \right|$$

$$+ n \left| \int_0^{\pi/m} \chi_{k,l}^{*,*}\left(t, \frac{q}{n}\right) \frac{\sin q}{q} \left[\frac{m \cos mt}{t} - \frac{\sin mt}{t^2} \right] \right|$$

$$\leqq \left(\log \frac{1}{\eta}\right)^l \epsilon\, m \int_0^{\pi/m} \left(\log \frac{1}{t}\right)^k \left| \cos mt + \frac{\sin mt}{mt} \right| dt$$

$$+ q \left(\log \frac{n}{q}\right)^l \epsilon\, m \int_0^{\pi/m} \left(\log \frac{1}{t}\right)^k \left| \cos mt + \frac{\sin mt}{mt} \right| dt$$

$$\leqq \left[\left(\log \frac{1}{\eta}\right)^l + q \left(\log \frac{n}{q}\right)^l \right] m\, \epsilon \int_0^{\pi/m} \left(\log \frac{1}{t}\right)^k dt.$$

$$J_2 = o\left[(\log m)^k (\log n)^l \right].$$

$$J_3 = -m \int_{\frac{q}{n}}^{b} X^{*,*}_{k,\rho}\left(\frac{p}{m}, \tau\right) \frac{\sin p}{p} \left(\frac{n \cos n\tau}{\tau} - \frac{\sin n\tau}{\tau^2} \right) d\tau .$$

Integrating by parts,

$$J_3 = -m \frac{\sin p}{p} \left\{ X^{*,*}_{k,\rho}\left(\frac{p}{m}, \tau\right) \frac{\sin n\tau}{\tau} \Big]_{\frac{q}{n}}^{b} - \int_{\frac{q}{n}}^{b} X^{*,0}_{k,\rho}\left(\frac{p}{m}, \tau\right) \frac{\sin n\tau}{\tau} d\tau \right\}.$$

$$|J_3| \leq \epsilon \frac{m}{p} \left[\frac{p}{m} \left(\log \frac{m}{p}\right)^k b \left(\log \frac{1}{b}\right)^{\ell} \frac{1}{b} \right] + \epsilon \frac{m}{p} \left[\frac{p}{m} \left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^{\ell} \right]$$

$$+ \frac{m}{p} \left| \int_{\frac{q}{n}}^{b} X^{*,0}_{k,\rho}\left(\frac{p}{m}, \tau\right) \frac{\sin n\tau}{\tau} d\tau \right| .$$

The first two terms are $o\left[(\log m)^k (\log n)^{\ell}\right]$; and integrating the third term by parts, we have

$$\frac{m}{p}\left|-\frac{1}{n}\,\chi^{*,0}_{k,\ell}\left(\frac{p}{m},\tau\right)\frac{\cos n\tau}{\tau}\right]^{b}_{\frac{q}{n}}-\frac{1}{n}\int_{\frac{q}{n}}^{b}\chi^{*,0}_{k,\ell}\left(\frac{p}{m},\tau\right)\frac{\cos n\tau}{\tau^2}\,d\tau$$

$$-\frac{1}{n}\int_{\frac{q}{n}}^{b}\chi^{*,0}_{k,\ell-1}\left(\frac{p}{m},\tau\right)\frac{\cos n\tau}{\tau}\,d\tau\,\Big|.$$

Let us call these J_3' , J_3'' , and J_3''' respectively.

Upon integrating (8.18), we have

$$\left|\chi^{*,0}_{k,\ell}(u,v)\right| \leq K\,u\left(\log\frac{1}{u}\right)^{k}\left(\log\frac{1}{v}\right)^{\ell};$$

$$|J_3'| \leq K\left(\log\frac{m}{p}\right)^{k}\left(\log\frac{1}{q}\right)^{\ell}+K\left(\log\frac{m}{p}\right)^{k}\frac{1}{q}\left(\log\frac{n}{q}\right)^{\ell}.$$

Therefore, $|J_3'|/(\log m)^{k}\cdot(\log n)^{\ell}$ can be made less than any arbitrarily small positive constant by choosing q sufficiently large.

Using again the bound for $\left|\chi_{k,\ell}^{*,0}(u,v)\right|$,

$$\left|J_3''\right| \leq \frac{K}{n}\left(\log \frac{m}{p}\right)^k \int_{\frac{g}{n}}^{b} \frac{(\log 1/\tau)^\ell}{\tau^2}\, d\tau$$

$$= \frac{K}{n}\left(\log \frac{m}{p}\right)^k \left[\frac{1}{\tau}\left\{\left(\log \frac{1}{\tau}\right)^\ell + \sum_{i=1}^{\ell} K\left(\log \frac{1}{\tau}\right)^{\ell-i+1}\right\}\right]_{\frac{g}{n}}^{b}.$$

It therefore follows, for g sufficiently large, that*

$$\frac{\left|J_3''\right|}{(\log m)^k (\log n)^\ell} < \epsilon.$$

If

$$\Phi(\tau) = \int_0^\tau \left|\chi_{k,\ell-1}^{*,0}\left(\frac{\xi}{m}, u\right)\right| du,$$

*Here, and in what follows, ϵ shall designate any arbitrarily small positive constant.

$$|J_3'''| \leq \frac{m}{np} \int_{\frac{q}{m}}^{b} \frac{X_{k,\ell-1}^{t,\tau}\left(\frac{p}{m}, \tau\right)}{\tau^2}\, d\tau$$

$$= \frac{m}{np}\left[\bar{\Phi}(\tau)\cdot\frac{1}{\tau^2}\right]_{\frac{q}{m}}^{b} + \frac{2m}{np}\int_{\frac{q}{m}}^{b} \frac{\bar{\Phi}(\tau)}{\tau^3}\, d\tau .$$

$\bar{\Phi}$ **(t, τ) can easily be shown to be, in absolute value, less than or equal to**

$$K\, t\, \tau \left(\log \frac{1}{t}\right)^{k}\left(\log \frac{1}{\tau}\right)^{\ell},$$

and therefore $|J_3'''|$ **is less than or equal to**

$$K \cdot \frac{1}{q}\left(\log \frac{m}{p}\right)^{k}\left(\log \frac{m}{q}\right)^{\ell} + K\left(\log \frac{m}{p}\right)^{k}\int_{\frac{q}{m}}^{b} \frac{\tau\left(\log \frac{1}{\tau}\right)^{\ell}}{\tau^3}\, d\tau .$$

As a result, we have, for q sufficiently large,

$$\frac{|J_3'''|}{(\log n)^k (\log n)^\ell} < \epsilon .$$

$$J_4 = \int_0^{\frac{\#}{2m}} \left(\frac{m\cos mt}{t} - \frac{\sin mt}{t^2}\right) dt \int_{\frac{4}{t}/n}^{b} X_{k\ell}^{*,*}(t,\tau) \left(\frac{n\cos n\tau}{\tau} - \frac{\sin n\tau}{\tau^2}\right) d\tau .$$

Integrating the second integral by parts twice, we have

$$J_4 = \int_0^{\frac{\#}{m}} \left(\frac{m\cos mt}{t} - \frac{\sin mt}{t^2}\right) dt \left\{ X_{k\ell}^{**}(t,\tau) \frac{\sin n\tau}{\tau}\Big]_{\tau=\frac{q}{n}}^{b} + \frac{1}{n} X_{k\ell}^{*,0}(t\tau) \frac{\cos n\tau}{\tau}\Big]_{\tau=\frac{q}{n}}^{b} \right.$$

$$\left. + \frac{1}{n}\int_{\frac{q}{n}}^{b} X_{k\ell}^{*,0}(t,\tau) \frac{\cos n\tau}{\tau^2} d\tau + \frac{1}{n}\int_{\frac{q}{n}}^{b} X_{k,\ell-1}^{*,0}(t,\tau) \frac{\cos n\tau}{\tau^2} d\tau \right\}.$$

$$J_4 = J_4' + J_4'' + J_4''' + J_4'''' .$$

$$|J_+'| \leq \int_0^{\frac{p}{m}} \frac{m}{t\,p} \left|\chi_{k,\rho}^{**}(t,p)\right| dt + \int_0^{\frac{p}{m}} \frac{n}{t\,q} \left|\chi_{k,\rho}^{**}(t,\tfrac{n}{q})\right| dt$$

$$+ \int_0^{\frac{p}{m}} \frac{m}{t\,p} \left|\chi_{k,\rho}^{**}(t,p)\right| dt - \int_0^{\frac{p}{m}} \frac{n}{t\,q} \left|\chi_{k,\rho}^{**}(t,\tfrac{n}{q})\right| dt$$

$$\leq \epsilon \int_0^{\frac{p}{m}} \left(\log\frac{1}{t}\right)^k dt \left\{ \left(\log\frac{1}{p}\right)^{\rho} \frac{m\,p}{p} - \frac{m\,n}{q}\,\frac{q}{n} \left(\log\frac{n}{q}\right)^{\rho} \right\}.$$

$$J_1' = o\left[(\log m)^k (\log n)^{\rho}\right].$$

$$|J_+''| \leq \int_0^{\frac{p}{m}} \left\{ \frac{2m}{n\,t} \left|\chi_{k,\rho}^{*,c}(t,p)\right| \frac{1}{p} + \frac{2m}{q\,t} \left|\chi_{k,\rho}^{*,c}(t,\tfrac{n}{q})\right| \right\} dt$$

As can easily be shown,

$$\chi_{k,\rho}^{*,c}(u,v) = o\left[u \left(\log\frac{1}{u}\right)^k \left(\log\frac{1}{v}\right)^{\rho}\right],$$

from which it follows that

$$|J_+'| \leq \epsilon \frac{m}{n}\int_0^{\vartheta_m} \left(\log\frac{1}{t}\right)^k dt + \epsilon \frac{m}{T}\left(\log\frac{n}{T}\right)^{\ell}\int_0^{\vartheta_m}\left(\log\frac{1}{t}\right)^k dt.$$

$$J_+'' = o\left[(\log m)^k\ (\log n)^{\ell}\right].$$

$$|J'''| \leq \int_0^{\vartheta_m} \frac{2m\,dt}{t}\int_{\frac{T}{n}}^{b} \left|\chi^{*,0}_{k,\ell}(t,\tau)\right| \cdot \frac{1}{\tau^2}\,d\tau$$

$$\leq \epsilon\, m\int_0^{\vartheta_m}\left(\log\frac{1}{t}\right)^k dt \cdot \frac{1}{n}\int_{\frac{T}{n}}^{b}\frac{\left(\log\frac{1}{\tau}\right)^{\ell}}{\tau^2}\,d\tau.$$

$$J_+''' = o\left[(\log m)^k\ (\log n)^{\ell}\right].$$

$$|J_+''''| \leq \frac{1}{n}\int_0^{\vartheta_m}\frac{2m}{t}\,dt\int_{\frac{T}{n}}^{b}\left|\chi^{*,0}_{k,\ell-1}(t,\tau)\right| \frac{1}{\tau^2}\,d\tau.$$

If again $\Phi(t,\tau) = \int_0^{\tau}\left|\chi^{*,0}_{k,\ell-1}(t,\sigma)\right| d\sigma$ then,

$$|J_4''''| \leq \frac{2m}{n}\int_0^{\vartheta_m}\frac{dt}{t}\left\{\left[\frac{\Phi(t,\tau)}{\tau^2}\right]_{\frac{q}{n}}^{b} + 2\int_{\frac{q}{n}}^{b}\frac{\Phi(t,\tau)}{\tau^3}\,d\tau\right\}.$$

But $\Phi(u,v) = O\left[u(\log \frac{1}{u})^k\, v\,(\log \frac{1}{v})^\ell\right]$, which makes

$$|J_4''''| \leq K\,\frac{m}{n}\int_0^{\vartheta_m}(\log \tfrac{1}{t})^k\,dt + K\,\frac{m}{q}\,(\log \tfrac{n}{q})^\ell\int_0^{\vartheta_m}(\log \tfrac{1}{t})^k\,dt$$

$$+ K\,m\int_0^{\vartheta_m}(\log \tfrac{1}{t})^k\cdot\frac{1}{n}\int_{\frac{q}{n}}^{b}\frac{(\log \tau)^\ell}{\tau^2}\,d\tau$$

$$\leq o(1) + \frac{K\,(\log \frac{n}{q})^\ell\,(\log \frac{m}{p})^k}{q},$$

from which it follows that, for sufficiently large q,

$$\frac{|J_4''''|}{(\log m)^k\,(\log n)^\ell} < \epsilon\,.$$

It then follows that H_2 can be made less than $\overset{(\log m)^k(\log n)^l}{\epsilon_1}$ by properly choosing p and q and allowing m and n to increase. H_3 can be handled in exactly the same manner as H_2. We have still to consider H_4.

$$H_4 = \int_{\frac{q}{n}}^{p}\int_{\frac{q}{m}}^{p} \frac{\sin mt}{t}\,\frac{\sin n\tau}{\tau}\,\chi_{k,l}(t,\tau)\,dt\,d\tau .$$

It can be shown that

$$\int_{\frac{q}{m}}^{t} \frac{\sin mu}{u}\,du = \left[-\frac{\cos mt}{mt} - \frac{\cos p}{p} - \int_{\frac{q}{m}}^{t} \frac{\cos mu}{mu^2}\,du\right].$$

If we let

$$R(t,\tau) = \int_{\frac{q}{m}}^{t} \frac{\sin mu}{u}\,du \int_{\frac{q}{n}}^{\tau} \frac{\sin nv}{v}\,dv ,$$

integrating H_4 by parts, gives

$$\chi_{k,l}(t,\tau)\,R(t,\tau)\Big]_{\frac{q}{m},\ \frac{q}{n}}^{p,\ p} - \int_{\frac{q}{m}}^{p} \frac{\chi_{k-1,l}(t,\tau)}{t}\,R(t,\tau)\Big]_{\tau=\frac{q}{n}}^{p} dt$$

$$- \int_{\frac{q}{n}}^{p} \frac{\chi_{k,l-1}(t,\tau)}{\tau}\,R(t,\tau)\Big]_{t=\frac{q}{m}}^{p} d\tau + \int_{\frac{q}{n}}^{p}\int_{\frac{q}{m}}^{p} \frac{\chi_{k-1,l-1}(t,\tau)}{t\,\tau}\,R(t,\tau)\,dt\,d\tau .$$

If we substitute the expressions for $R(t,\tau)$ into these four terms, we get, upon expanding, the following eighteen terms:

$$X_{kl}(p,p)\,\frac{\cos mp \cos np}{mn\,p\,p} - \frac{\cos np}{np}\int_{p/m}^{p}\frac{\cos mt}{mt^2}\,X_{k-1,l}(t,p)\,dt$$

$$+\frac{\cos p}{p}\,\frac{\cos q}{q}\,X_{k,l}\left(p,\frac{q}{m}\right) - \frac{\cos q}{q}\,\frac{\cos mp}{mp}\,X_{k,l}\left(p,\frac{q}{n}\right)$$

$$+\frac{\cos p}{p}\,\frac{\cos q}{q}\,X_{k,l}\left(\frac{p}{m},p\right) - \frac{\cos p}{p}\,\frac{\cos np}{np}\,X_{k,l}\left(\frac{p}{m},p\right)$$

$$+\frac{\cos np}{np}\int_{p/m}^{p}X_{k,l}(t,p)\,\frac{\cos mt}{mt^2}\,dt + \frac{\cos mp}{mp}\int_{q/n}^{p}X_{k,l}(p,\tau)\,\frac{\cos n\tau}{\tau^2}\,d\tau$$

$$-\frac{\cos mp}{mp}\int_{q/n}^{p}\frac{\cos n\tau}{n\tau^2}\,X_{k,l-1}(p,\tau)\,d\tau + \int_{q/n}^{p}\int_{p/m}^{p}X_{k-1,l-1}(t,\tau)\,\frac{\cos mt}{mt^2}\,\frac{\cos n\tau}{n\tau^2}\,dt\,d\tau +$$

$$+\int_{q/n}^{p}\int_{q/m}^{p} X_{k,l}(t,\tau)\,\frac{\cos mt}{mt^2}\,\frac{\cos n\tau}{n\tau^2}\,dt\,d\tau-\int_{q/n}^{p}\int_{q/m}^{p} X_{k,l-1}(t,\tau)\,\frac{\cos mt}{mt^2}\,\frac{\cos n\tau}{n\tau^2}\,dt\,d\tau$$

$$-\int_{q/n}^{p}\int_{q/m}^{p} \frac{\cos mt}{mt^2}\,\frac{\cos n\tau}{n\tau^2}\,X_{k-1,l}(t,\tau)\,dt\,d\tau-\frac{\cos q}{q}\int_{q/m}^{p}\frac{\cos mt}{mt^2}\,X_{k-1,l}\left(t,\frac{q}{n}\right)dt$$

$$+\frac{\cos p}{p}\int_{q/n}^{p} X_{k,l-1}\left(\frac{p}{m},\tau\right)\frac{\cos n\tau}{n\tau^2}\,d\tau-\frac{\cos p}{p}\,\frac{\cos q}{q}\,X_{k,l}\left(\frac{p}{m},\frac{q}{n}\right)$$

$$-\frac{\cos p}{p}\int_{q/n}^{p} X_{k,l}\left(\frac{p}{m},\tau\right)\frac{\cos n\tau}{n\tau^2}\,d\tau-\frac{\cos q}{q}\int_{q/m}^{p} X_{k,l}\left(t,\frac{q}{n}\right)\frac{\cos mt}{mt^2}\,dt.$$

We shall designate the above terms by M_1, M_2, M_3, . . . , M_{15}.

These terms can be handled in the following manner:

$M_1 = o(1) = o\left[(\log m)^k (\log n)^l\right].$

$$|M_2| \leqq \frac{1}{mn\eta} \int_{\frac{\pi}{m}}^{b} |\chi_{k-1,\ell}(t,\eta)| \frac{dt}{t^2} .$$

If we let

$$\Phi(u,v) \equiv \int_0^u |\chi_{k-1,\ell}(t,v)|\, dt = O\left[u \left(\log \frac{1}{u}\right)^k \left(\log \frac{1}{v}\right)^\ell \right],$$

$$|M_2| \leqq \frac{K}{mn\eta} \left[\frac{1}{t} \left(\log \frac{1}{t}\right)^k \left(\log \frac{1}{\eta}\right)^\ell \right]_{\frac{\pi}{m}}^{b} + \frac{K}{mn\eta} \left(\log \frac{1}{\eta}\right)^\ell \int_{\frac{\pi}{m}}^{b} \frac{t \left(\log \frac{1}{t}\right)^k}{t^3} dt;$$

$M_2 = o(1) + o\left[(\log m)^k \right] + o\left[(\log m)^k \right]$.

$M_2 = o\left[(\log m)^k (\log n)^\ell \right]$.

$$|M_3| \leqq \frac{K}{pq} \left(\log \frac{1}{\eta}\right)^k \left(\log \frac{n}{\pi}\right)^\ell$$

$M_3 = o\left[(\log m)^k (\log n)^\ell \right]$.

$$|M_4| \leqq \frac{K}{mhq}\left(\log\frac{1}{h}\right)^k\left(\log\frac{n}{q}\right)^\ell.$$

$$M_4 = o\left[(\log m)^k (\log n)^\ell\right].$$

$$|M_5| \leqq \frac{K}{pq}\left(\log\frac{m}{p}\right)^k\left(\log\frac{1}{h}\right)^\ell.$$

$$M_5 = o\left[(\log m)^k (\log n)^\ell\right].$$

$$|M_6| \leqq \frac{1}{mhp}\left(\log\frac{m}{p}\right)^k\left(\log\frac{1}{h}\right)^\ell.$$

$$M_6 = o\left[(\log m)^k (\log n)^\ell\right].$$

$$|M_7| \leqq \frac{1}{mnh}\int_{\frac{2}{3m}}^{h}\frac{|X_{k,\ell}(t,h)|}{t^2}\,dt$$

$$= \frac{1}{mnh}\left[\frac{1}{t^2}\int_0^t |X_{k,\ell}(u,h)|\,du\right]_{\frac{2}{3m}}^{h} + \frac{1}{mnh}\int_{\frac{2}{3m}}^{h}\frac{\int_0^t |X_{k,\ell}(u,h)|\,du}{t^3}\,dt,$$

and $\int_0^t |\chi_{8,\rho}(u,\eta)|\,du \leq K\,t\,(\log \frac{1}{t})^{k}(\log \frac{1}{\eta})^{\rho}$.

$M_7 = o(1) + o\left[(\log m)^{k}\right] + o\left[(\log m)^{k}\right]$.

$M_7 = o\left[(\log m)^{k}\,(\log n)^{\rho}\right]$.

$M_8 = o\left[(\log m)^{k}\,(\log n)^{\rho}\right]$ in the same manner as M .

$M_9 = o\left[(\log m)^{k}\,(\log n)^{\rho}\right]$ in the same manner as M .

$$|M_{10}| \leq \frac{1}{mn}\int_{\frac{\delta}{n}}^{\eta}\int_{\frac{\rho}{m}}^{\eta}\frac{|\chi_{k-1,\rho-1}(t,\tau)|}{t^2\tau^2}\,dt\,d\tau .$$

Integration by parts gives us $|M_{10}|$

$$\leq \frac{1}{mn}\left\{\int_0^\tau\int_0^t|\chi_{k-1,\rho-1}(u,v)|\,du\,dv\right\}\frac{1}{t^2}\,\frac{1}{\tau^2}\Bigg]_{\frac{\rho}{m},\frac{\delta}{n}}^{\eta,\eta}$$

$$+\frac{1}{mn}\int_{\frac{\delta}{m}}^{\eta}\frac{1}{\tau^2t^3}\left\{\int_0^\tau\int_0^t|\chi_{k-1,\rho-1}(u,v)|\,du\,dv\right\}\Bigg]_{\tau=\frac{\delta}{n}}^{\eta}dt$$

$$+\frac{1}{mn}\int_{\frac{\delta}{n}}^{\eta}\frac{1}{\tau^3t^2}\left\{\int_0^\tau\int_0^t|\chi_{k-1,\rho-1}(u,v)|\,du\,dv\right\}\Bigg]_{t=\frac{\rho}{m}}^{\eta}d\tau$$

$$+\frac{1}{mn}\int_{\frac{\delta}{n}}^{\eta}\int_{\frac{\rho}{m}}^{\eta}\frac{1}{t^3\tau^3}\left\{\int_0^\tau\int_0^t|\chi_{k-1,\rho-1}(u,v)|\,du\,dv\right\}dt\,d\tau$$

which, using (8.19), is equal to

$$o(1) + \frac{K}{pq}\left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^l + o\left[(\log m)^k\right]$$

$$+ \frac{K}{q}\left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^l + o\left[(\log n)^l\right]$$

$$+ \frac{K}{p}\left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^l + \frac{K}{pq}\left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^l .$$

These terms can be made less than $\epsilon\,(\log m)^k\,(\log n)^l$, if we choose p and q large enough. Both the terms M_{12} and M_{13} are similar to and can be handled in exactly the same way as M_{10} .

$$|M_{11}| \leqq \frac{K}{mn}\int_{\frac{q}{n}}^{b}\int_{\frac{p}{m}}^{b} \frac{\left(\log\frac{1}{t}\right)^k\left(\log\frac{1}{\tau}\right)^l}{t^2\,\tau^2}\,dt\,d\tau$$

$$\leqq \frac{K}{mn}\left[\frac{1}{t}\left(\log\frac{1}{t}\right)^k + \frac{1}{t}\sum_{i=1}^{k} K\left(\log\frac{1}{t}\right)^{k-i+1}\right]_{\frac{p}{m}}^{b}\left[\frac{1}{\tau}\left(\log\frac{1}{\tau}\right)^l + \frac{1}{\tau}\sum_{i=1}^{l} K\left(\log\frac{1}{\tau}\right)^{l-i+1}\right]_{\frac{q}{n}}^{b}$$

$$= o(1) + \frac{K}{pq}\left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^\ell + o\left[(\log m)^k (\log n)^\ell\right].$$

By choosing p and q sufficiently large and then letting m and n become infinite, we can make

$$\frac{|M_{11}|}{(\log m)^k (\log n)^\ell} < \epsilon.$$

$$|M_{14}| \leqq \frac{K}{qm}\left[\frac{1}{t^2}\int_0^t |\chi_{k-1,\ell}(u, \tfrac{q}{n})|\, du\right]_{\frac{1}{m}}^{p}$$

$$+ \frac{K}{qm}\int_{\frac{1}{m}}^{p} \frac{1}{t^3}\left\{\int_0^t |\chi_{k-1,\ell}(u, \tfrac{q}{n})|\, du\right\} dt$$

$$\leqq o(1) + \frac{K}{pq}\left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^\ell.$$

Therefore, for p, q, m, and n large enough

$$\frac{|M_{14}|}{(\log m)^k (\log n)^l} < \epsilon .$$

M_{15} can be handled in the same manner as M_{14} .

$$|M_{16}| \leqq \frac{K}{pq} \left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^l$$

which behaves as the above terms.

$$|M_{17}| \leqq \frac{K}{pn} \left(\log \frac{m}{p}\right)^k \int_{\frac{q}{n}}^{1} \frac{1}{\tau^2} \left(\log \frac{1}{\tau}\right) d\tau .$$

The integral, when handled as in M_{11} , gives us

$$|M_{17}| \leqq \frac{K}{pq} \left(\log \frac{m}{p}\right)^k \left(\log \frac{n}{q}\right)^l$$

and $|M_{17}|$ can be made less than $\epsilon (\log m)^k (\log n)^l$ by choice of large p and q. M_{18} can be handled in the same manner as M_{17} .

It now follows that H_4 divided by $(\log m)^k (\log n)^\ell$ can be made less than any positive quantity by choosing p and q sufficiently large and then letting m and n become infinite.

If we now choose η sufficiently small, and p and q sufficiently large, and then let m and n become infinite, $R_{m,n}^{k,\ell}$ can be made to differ from zero by as little as we please, which proves our theorem.

9. Conditions for Summability $(R, 2, \ell)$ of x Derived Series.

We now turn to logarithmic summability of derived series. Let the (m,n)th logarithmic means of order (k, ℓ) of the series (1.1) with $r = 1$, $s = 0$ be

$$(9.1)\quad \left[R_{m,n}^{k,\ell}\right]_x = \frac{1}{(\log m)^k} \sum_{i<m} \left(\log \frac{m}{i}\right)^k i A_{i0}^{10} + \frac{1}{(\log m)^k (\log n)^\ell} \sum_{\substack{i<m \\ j<n}} \left(\log \frac{m}{i}\right)^k \left(\log \frac{n}{j}\right)^\ell i A_{ij}^{10}.$$

If this tends to a limit S, we say the partially derived Fourier series of a function f(x,y) with respect to x is summable (R, k, ℓ) to sum S.

Theorem 7. Suppose $\dfrac{\psi_{10}^{1,x}(u,v)}{u}$ is integrable for all v and all u on $(0, \eta)$ where $\eta > 0$; if there exists an integrable function g(y) periodic in y such that $G(v) = \frac{1}{2}[G(y + v) + G(y-v)] \to S$ as $v \to 0$ for some y where the function G(v) satisfies the inequality

(9.2) $$\int_0^u \left\{ \frac{\psi_{1,0}'(t,v)}{4t} - G(v) \right\}^2 dT = o(u) \qquad \text{for all } v$$

and if there exists a periodic integrable function h(x) which satisfies the conditions,

(9.3)
$$\int_0^v |\chi_{0,\ell-1}(u,T;h)| dT = O\left[v(\log \tfrac{1}{v})^\ell\right] \qquad \text{for all } u$$

$$\chi_{0,\ell}(u,v;h) = O\left[(\log \tfrac{1}{v})^\ell\right] \qquad \text{for all } u$$

$$\int_0^v \chi_{0,\ell}(u,T;h)\, dT = o\left[v(\log \tfrac{1}{v})^\ell\right] \qquad \text{for all } u,$$

then the series (1.1) with r = 1, s = 0 is summable (R,2,ℓ).

If we integrate (9.2) with respect to y from zero to π, we have the condition

$$\int_0^u \left\{ \frac{\psi_{1,0}^{0,*}(t)}{4t} - A \right\} dt = o(1)$$

holding for a function of x, i.e. $\int_{-\pi}^{\pi} f(x,y)\,dy$, where $\psi_{1,0}(\tau)$ is formed with this function of x and A is a constant. As a result,* the series $\sum_{i=1}^{\infty} i\,A_{i,0}^{1,0}$ is summable (R,2), and we may therefore consider, without loss of generality, that these terms are zero. Let us now take

$$\chi(t,\tau) = \frac{1}{4}\psi_{1,0}(t,\tau) - S \cos\tau \sin t.$$

Using the symbol * as a superscript in the same manner as before,

$$\chi^{0,*}(t,\tau) + S \sin t \sin\tau = \frac{1}{4}\psi_{1,0}^{0,*}(t,\tau) \sim \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij}^{1,0} \frac{\sin it \sin j\tau}{j}$$

We now multiply, as before, by the functions $L_2'(mt)$ and $L_2'(n\tau)$

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*Wang, F. T. [2] page 238.

and integrate from 0 to ∞ . Then

$$\int_0^\infty\!\!\int_0^\infty L_2'(mt)\,L_\ell'(n\tau)\,\chi^{0*}(t,\tau)\,dt\,d\tau + S\int_0^\infty L_2'(n\tau)\sin\tau\,d\tau\int_0^\infty L_\ell'(mt)\sin t\,dt$$

$$= \sum_{i=1,\,j=1}^{\infty,\,\infty} A_{ij}^{1,0}\int_0^\infty \sin it\,L_2'(mt)\,dt\int_0^\infty \frac{\sin j\tau}{j}\,L_\ell'(n\tau)\,d\tau$$

and, using (8.12),

$$\int_0^\infty\!\!\int_0^\infty L_2'(mt)\,L_\ell'(n\tau)\,\chi^{0*}(t,\tau)\,dt\,d\tau + S\,\frac{(\log m)^2(\log n)^\ell}{m^2\,n^2}\,\frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4}\,\frac{1}{m^2n^2}\sum_{\substack{i<m\\ j<n}}\left(\log\frac{m}{i}\right)^2\left(\log\frac{n}{j}\right)^\ell i\,A_{ij}^{1,0}.$$

Therefore,

$$\left[R_{m,n}^{2,\ell}\right] - S = -\frac{4}{\pi^2}\,\frac{m^2n^2}{(\log m)^2(\log n)^\ell}\int_0^\infty\!\!\int_0^\infty L_2'(mt)\,L_\ell'(n\tau)\,\chi^{0*}(t,\tau)\,dt\,d\tau.$$

We break the integral up into integrals over the regions (8.13) and call the integrals A, B, C, and D. D is o(1) as in section 8.

$$B = \frac{m^2 n^2}{(\log m)^2 (\log n)^\ell} \int_0^b \int_b^\infty L_1'(m\tau)\, L_2'(mt)\, X^{0,1}(t,\tau)\, dt\, d\tau$$

$$= \frac{m^2 n^2}{(\log m)^2 (\log n)^\ell} \int_0^b \int_b^\infty L_1'(m\tau)\, L_2'(mt)\, X^{0,1}(t,\tau,h)\, dt\, d\tau + \int_0^b \int_b^\infty L_1'(m\tau) L_2'(mt)\, \Phi(t,\tau)\, dt\, d\tau,$$

where

$$\Phi(t,\tau) = \tau H(t) + \int \sin t \sin \tau.$$

The first term is o(1) by conditions (9.3) and the methods of section 8. The second term is

$$\int_0^b L_1'(m\tau)\, \tau\, d\tau \int_b^\infty L_2'(mt)\, H(t)\, dt + \int \int_0^b L_1'(m\tau) \sin\tau\, d\tau \int_b^\infty L_2'(mt) \sin t\, dt.$$

But

$$\left|\int_0^\eta L_\ell'(m\tau)\sin\tau\,d\tau\right| \leq \frac{2}{m}\int_0^{\frac{2}{m}}|L_\ell'(m\tau)|\,d\tau + \eta\int_{\frac{2}{m}}^{\eta}|L_\ell'(m\tau)|\,d\tau$$

and also

$$\left|\int_0^\eta \tau L_\ell'(m\tau)\,d\tau\right| \leq \frac{2}{m}\int_0^{\frac{2}{m}}|L_\ell'(m\tau)|\,d\tau + \eta\int_{\frac{2}{m}}^{\eta}|L_\ell'(m\tau)|\,d\tau .$$

The first term in each of the above is $O(\frac{1}{m^2})$ by (8.11). In each case, the second term, using (8.9) is

$$\leq \eta\int_{\frac{2}{m}}^{\eta}\frac{(\log m\tau)^{\ell-1}}{m^2\tau^2}\,d\tau = O\left[\frac{(\log m)^{\ell-1}}{m^2}\right].$$

Again using (8.9),

$$\int_\eta^\infty L_2'(mt)\left[H(t)+\sin t\right]dt = O\left[\frac{(\log m)^2}{m^2}\right],$$

from which it follows that $B = o(1)$.

$$C = \frac{-m^2 n^2}{(\log m)^2 (\log n)^\rho} \frac{4}{\pi^2} \int_\eta^\infty L_1'(m\tau)\,d\tau \int_0^\eta t\,L_2'(mt)\,\chi^{0*}(t,\tau)\,dt.$$

Let

$$\chi_{c0}^{0*}(t,\tau) = \int_0^t \frac{\chi^{0*}(u,\tau)}{u}\,du$$

and integrate by parts. Then

$$C = \frac{K m^2 n^2}{(\log m)^2 (\log n)^\rho} \int_\eta^\infty \Big[L_1'(m\tau)\; t\,L_2'(mt)\,\chi_{c0}^{0*}(t\tau) \Big]_{t=0}^{\eta} d\tau$$

$$- \frac{K m^2 n^2}{(\log m)^2 (\log n)^\rho} \int_\eta^\infty \int_0^{\frac{2}{m}} L_1'(m\tau)\left[mt\,L_2''(mt) + L_2'(mt) \right] \chi_{c0}^{0*}(t,\tau)\,dt\,d\tau$$

$$- \frac{K m^2 n^2}{(\log m)^2 (\log n)^\rho} \int_\eta^\infty \int_{\frac{2}{m}}^{\eta} L_1'(m\tau) \left[mt\,L_2''(mt) + L_2'(mt) \right] \chi_{c0}^{0*}(t,\tau)\,dt\,d\tau.$$

$$C = C_1 + C_2 + C_3.$$

Since, by (8.9),

$$L_2'(m y) = o\left[\frac{(\log m)^2}{m^2}\right],$$

and, by (9.2), $\Phi_{c,0}^{0*}(u) = o(u)$, it follows at once that $C_1 = o(1)$. Now,* for $mt \geqslant 0$,

$$mt\left[mt\, L_2''(mt) + L_2'(mt)\right] = O(1) \tag{9.4}$$

and for $mt > 2$

$$m^2 t^2\left[mt\, L_2''(mt) + L_2'(mt)\right] = O(\log mt). \tag{9.5}$$

*Wang, F. T. [2] page 239.

We then have

$$C_2 = \frac{K n^2}{(\log m)^2 (\log n)^\ell} \int_\eta^\infty L_\ell'(n\tau)\, O\left\{ m \int_0^{\frac{2}{m}} \frac{1}{t} \left| X_{\zeta_0}^{o*}(t,\tau;\varsigma) \right| dt \right\} d\tau$$

$$+ \frac{K n^2}{(\log m)^2 (\log n)^\ell} \int_\eta^\infty L_\ell'(n\tau) \left\{ \int_0^\tau \cos v\, dv - A \sin\tau \right\} O\left\{ m \int_0^{\frac{2}{m}} \frac{\int_0^t \frac{\sin u}{u}\, du}{t} \right\} dt\, d\tau.$$

The first term, since $X_{\zeta_0}^{o*}(t\tau;\varsigma) = o(t)$ for all τ, is equal to

$$\frac{K n^2}{(\log m)^2 (\log n)^\ell} \int_\eta^\infty L_\ell'(n\tau) \quad o\left\{ m \int_0^{\frac{2}{m}} dt \right\} d\tau$$

which is $o(1)$, since

$$\frac{n^2}{(\log n)^\ell} \int_\eta^\infty L_\ell'(n\tau)\, d\tau = o(1)$$

The second is less than or equal to

$$\frac{K n^2}{(\log m)^2 (\log n)^\ell} \int_\eta^\infty \frac{(\log n\tau)^{\ell-1}}{n^2 \tau^2}\, \Phi(\tau)\, d\tau$$

where $\Phi(\tau)$ is integrable on 0 to ∞, and this term is therefore o(1). In a similar manner, using (9.5),

$$C_3 = \frac{K n^2}{(\log m)^2(\log n)^\epsilon}\int_\eta^\infty L_\epsilon'(m\tau)\, O\left\{\int_{\frac{2}{m}}^{\eta} \frac{\log mt}{t^2}\,|\alpha_{\epsilon\epsilon}^{\cdots}(t,\tau,g)|\,dt\right\}d\tau$$

$$+ \frac{K n^2}{(\log m)^2(\log n)^\epsilon}\int_\eta^\infty L_\epsilon'(m\tau)\,\Phi(\tau)\, O\left\{\int_{\frac{2}{m}}^{\eta} \frac{\log mt}{t^2}\left|\int_0^t \frac{\sin u}{u}\,du\right|dt\right\}d\tau.$$

The first term then is equal to

$$\frac{K n^2}{(\log m)^2(\log n)^\epsilon}\int_\eta^\infty L_\epsilon'(m\tau)\; o\left\{\int_{\frac{2}{m}}^{\eta} \frac{\log mt}{t}\,dt\right\}d\tau$$

$$= o\left\{\Big[(\log m\tau)^2\Big]_{\frac{2}{m}}^{\eta}\cdot\frac{K n^2}{(\log m)^2(\log n)^\epsilon}\int_\eta^\infty L_\epsilon'(m\tau)\,d\tau\right\}$$

but this is o(1), and the second term is equal to

$$\frac{K n^2}{(\log m)^2(\log n)^\epsilon}\int_\eta^\infty \Phi(\tau)\,L_\epsilon'(m\tau)\; o\left\{\int_{\frac{2}{m}}^{\eta} \frac{\log mt}{t}\,dt\right\}d\tau$$

which is o(1), and we finally have C itself o(1), and only A left to consider.

$$A = -\frac{m^2 n^2}{(\log m)^2 (\log n)^4} \frac{4}{\pi^2} \int_0^b \int_0^b L_2'(mt)\, L_2'(n\tau)\, \chi^{0*}(t,\tau)\, dt\, d\tau.$$

It is easily shown that

$$\int_0^b L_2'(mt)\, \chi^{0*}(t,\tau)\, dt = o\left[\frac{(\log m)^2}{m^2}\right] - \int_0^b \frac{d}{dt}\left\{t\, L_2'(mt)\right\} \chi_{c,0}^{0*}(t,\tau)\, dt$$

and

$$\int_0^b L_2'(n\tau)\, \chi_{c,0}^{0*}(t,\tau)\, d\tau = o\left[\frac{(\log n)^2}{n^2}\right] + \frac{K}{n} \int_0^b \frac{\sin n\tau}{n\tau}\, \chi_{c,0}(t,\tau)\, d\tau.$$

Therefore,

$$A = \frac{K m^2}{(\log m)^2 (\log n)^4} \int_0^b \int_0^b \frac{\sin n\tau}{\tau} \frac{d}{dt}\left\{t\, L_2'(mt)\right\} \chi_{c,0}(t,\tau)\, dt\, d\tau$$

$$= \frac{K m}{(\log m)^2 (\log m)^{\ell}} \int_0^{\frac{2}{m}} m t \frac{d}{dt}\left\{ t L_2'(mt) \right\} \int_0^b \frac{\chi_{c\ell}(t,\tau)}{t} \frac{\sin n\tau}{\tau} \, d\tau \, dt$$

$$+ \frac{K}{(\log m)^2 (\log m)^{\ell}} \int_{\frac{2}{m}}^{b} \frac{m^2 t^2}{t} \frac{d}{dt}\left\{ t L_2'(mt) \right\} \int_0^b \frac{\chi_{c\ell}(t,\tau)}{t} \frac{\sin n\tau}{\tau} \, d\tau \, dt.$$

Since $G(v) \longrightarrow S$ as $v \longrightarrow 0$, we can use (9.2) with $G(v)$ replaced by S and we now have $o(u)$ as v tends to zero as well as u. Then $\chi_{c,\ell}(t,\tau;S)$ can easily be shown to be $o\left[t(\log \tau)^{\ell}\right]$ as t and tend to zero. Then

$$\int_0^b \frac{\chi_{c\ell}(t,\tau)}{t} \frac{\sin n\tau}{\tau} \, d\tau$$

$$= \int_0^b \frac{\chi_{c\ell}(t,\tau;S)}{t} \frac{\sin n\tau}{\tau} \, d\tau + \int_0^b \left\{ \frac{1}{t} \left[t - \sin t \cos \tau \right]_{c,\ell} \right\} \frac{\sin n\tau}{\tau} \, d\tau.$$

The subscripts c,ℓ on the square bracket demand the same process of integration as was performed on $\chi_{c,\ell}$, and the function $\chi_{c,\ell}(t,\tau,S)$. The function in braces satisfies inequalities of the type (9.3) which makes* each of the integrals, and hence, the whole term $o\left[(\log n)^{\ell}\right]$.

Using this inequality in A, we find the following expression for A with the help of (9.4) and (9.5)

$$o\left\{\frac{1}{(\log m)^2}\, m\int_0^{\frac{1}{m}} dt + \frac{1}{(\log m)^2}\int_{\frac{1}{m}}^{\eta} \frac{\log mt}{t}\, dt.\right.$$

This is equal to

$$o\left\{\frac{K}{(\log m)^2} + \frac{(\log m\eta)^2}{(\log m)^2}\right\}$$

which is o(1). We therefore finally have

$$\left[R_{m,n}^{\alpha,\beta}(t,\eta)\right]_A - \int = o(1).$$

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* Wang, F. T. [3] page 150.

10. Conditions for Summability (R,2,2) of (x,y) Derived Series. We now take the series (1.1) for r = 1, s = 1 and prove:

Theorem 8. If $\exists$ $g(u)$ and $h(v)$ $\ni$

(10.1) $$\int_0^u\int_0^v \left\{ \frac{\psi_{11}(t\tau)}{t\tau} - S \right\} dt\, d\tau = o(uv) \; ;$$

(10.2) $$\int_0^u \left\{ \frac{\psi_{11}(tv)}{4t} - G(v) \right\} dt = o(u) \quad \text{for all } v;$$

and

(10.3) $$\int_0^v \left\{ \frac{\psi_{11}(u\tau)}{4\tau} - H(u) \right\} d\tau = o(v) \quad \text{for all } u$$

where G(v) and H(u) are defined as above, then the series (1.1) with r = 1, s = 1 is summable (R,2,2) to sum S. Let

$$\chi(t,\tau,S) = \frac{\psi_{11}(t\tau)}{4} - S \sin t \sin \tau .$$

Then

$$X(t,\tau;S) \sim -S \sin t \sin \tau + \sum_{i=1,\,j=1}^{\infty\ \infty} ij\, A_{ij}'' \sin it \sin j\tau$$

multiplying by $L_2'(mt)$ and $L_2'(n\tau)$, and integrating as before, we have

$$\int_0^\infty\!\!\int_0^\infty L_2'(mt)\,L_2'(n\tau)\,X(t,\tau;S)\,dt\,d\tau - S\int_0^\infty L_2'(mt)\sin t\,dt\int_0^\infty L_2'(n\tau)\sin\tau\,d\tau$$

$$+ \sum_{i=1,\,j=1}^{\infty\ \infty} ij\,A_{ij}''\int_0^\infty L_2'(mt)\sin it\,dt\int_0^\infty L_2'(n\tau)\sin j\tau\,d\tau.$$

The use of condition (8.12) gives us

$$\int_0^\infty\!\!\int_0^\infty L_2'(mt)\,L_2'(n\tau)\,X(t,\tau;S)\,dt\,d\tau = -\frac{S}{m^2 n^2}(\log m)^2(\log n)^2\,\frac{\pi^2}{4}$$

$$+ \frac{\pi^2}{4}\,\frac{1}{m^2 n^2}\sum_{i<m,\,j<n} ij\,A_{ij}''\left(\log\frac{m}{i}\right)^2\left(\log\frac{n}{j}\right)^2.$$

But

$$[R_{m,n}^{2,2}]_{x,y} = \frac{1}{(\log m)^2(\log n)^2} \sum_{i<m,\, j<n} (\log \tfrac{m}{i})^2 (\log \tfrac{n}{j})^2\, ij\, A_{ij}^{1,1}$$

so that

$$[R_{m,n}^{2,2}]_{x,y} - S = \frac{4}{\pi^2} \frac{m^2 n^2}{(\log m)^2(\log n)^2} \int_0^\infty\!\!\int_0^\infty L_2'(mt) L_2'(n\tau)\, \chi(t,\tau;S)\, dt\, d\tau.$$

Let us break the integration up over the regions (8.13) and name the integrals A, B, C, and D. We find that D is o(1), as in section 8.

$$\frac{\pi^2}{4} \frac{(\log m)^2(\log n)^2}{m^2 n^2} C = \int_\eta^\infty L_2'(n\tau)\, d\tau \int_0^\delta L_1'(mt)\, \chi(t,\tau;S)\, dt.$$

If we define

$$\chi(t\tau;G) = \frac{\Psi_{1,1}(t,\tau)}{t} - G(\tau)$$

and

$$\chi(t,\tau;H) = \frac{\psi_{11}'(t,\tau)}{4} - H(t)$$

we find that

$$\chi(t,\tau;\zeta) = \chi(t,\tau;G) + \left[G(\tau) - \zeta \sin t \sin \tau\right]$$

and a similar expression for $\chi(t,\tau;\zeta)$ in terms of $\chi(t,\tau,H)$. Substituting this expression for $\chi(t,\tau;\zeta)$ in the integral for C, we get, after an integration by parts,

$$\int_{-\eta}^{\infty} L_2'(m\tau)\,d\tau \left\{ -\int_0^{\eta} \frac{d}{dt}\left[t L_1'(mt)\right] \chi_{c0}(t,\tau;G)\,dt + \left[t L_1'(mt)\,\chi_{c0}(t,\tau)\right]_0^{\eta} \right\}$$

where

$$\chi_{c0}(t,\tau;G) = \int_0^t \frac{1}{u}\,\chi'(u,\tau,G)\,du$$

By our hypotheses $\chi_{\varphi}(t,\tau,G) = o(t)$. This with the inequalities (9.4) and (9.5) makes the first integral

$$o\left[\frac{(\log m)^2 (\log n)^2}{m^2 n^2}\right].$$

Since

$$\int_0^u \frac{\sin t}{t} - \int_0^u dt$$

is obviously $O(u)$;

$$\int_\eta^\infty L_2'(m\tau)\sin\tau\, d\tau = o\left[\frac{(\log m)^2}{m^2}\right];$$

and

$$\int_\eta^\infty L_2'(m\tau)\, G(\tau)\, d\tau = o\left[\frac{(\log n)^2}{n^2}\right],$$

the second term satisfies the same inequality as the first and $C = o(1)$. B, in the same manner, is also $o(1)$.

If we integrate A by parts, we obtain

$$A = \frac{K m^2 n^2}{(\log m)^2 (\log n)^2} \Big[t\tau\, L_2'(mt)\, L_2'(n\tau)\, \chi_{c,c}(t,\tau;\delta) \Big]_{0,0}^{\eta,\eta}$$

$$- \frac{K m^2 n^2}{(\log m)^2 (\log n)^2} \int_0^{\eta} \chi_{c,c}(t,\tau;\delta)\, \tau L_2'(n\tau) \frac{d}{dt}\Big[t L_2'(mt) \Big] \Big]_{\tau=0}^{\eta} dt$$

$$- \frac{K m^2 n^2}{(\log m)^2 (\log n)^2} \int_0^{\eta} \chi_{c,c}(t,\tau;\delta)\, t L_2'(mt) \frac{d}{d\tau}\Big[\tau L_2'(n\tau) \Big] \Big]_{t=0}^{\eta} d\tau$$

$$+ \frac{K m^2 n^2}{(\log m)^2 (\log n)^2} \int_0^{\eta}\!\!\int_0^{\eta} \chi_{c,c}(t,\tau;\delta) \frac{d}{dt}\Big[t L_2'(mt) \Big] \frac{d}{d\tau}\Big[\tau L_2'(n\tau) \Big]\, dt\, d\tau .$$

We shall designate the terms on the right by A_1, A_2, A_3, and A_4. Since $L_2'(mt) \cdot L_2'(n\tau) = O(1)$; $t\tau\,\chi_{c,c}(t,\tau) \to 0$ as t and $\tau \to 0$; and $\frac{m^2 n^2}{(\log m)^2 (\log n)^2} \cdot L_2'(m\eta)\ L_2'(n\eta) = o(1)$ for $m\eta > 2$ and $n\eta > 2$; $A_1 = o(1)$.

$$A_2 = o(1) + \frac{K\, m^2 n^2}{(\log m)^2 (\log n)^2} \int_0^b \chi_{c,c}(t\,b;\delta) L_2'(nb) \frac{d}{dt}\{t\, L_2'(mt)\}\, dt$$

$$= o(1) + o\left\{ \frac{K\, m^2}{(\log m)^2} \int_0^b \chi_{c,c}(t\,b;\delta) \frac{d}{dt}\{t\, L_2'(mt)\}\, dt \right\}.$$

Using now (10.1),

$$A_2 = o(1) + o\left\{ \frac{K\, m^2}{(\log m)^2} \int_0^b \left| t \frac{d}{dt}\{t\, L_2'(mt)\} \right| dt \right\}$$

and by (9.4) and (9.5),

$$A_2 = o(1) + o\left\{ O\left[\frac{m}{(\log m)^2} \int_0^{\frac{2}{m}} dt\right] + O\left[\frac{1}{(\log m)^2} \int_{\frac{2}{m}}^b \frac{\log mt}{t}\, dt\right] \right\}$$

$A_2 = o(1)$. In the same manner $A_3 = o(1)$.

$$A_4 = o\left\{ \frac{m^2 n^2}{(\log m)^2 (\log n)^2} \int_0^b t \frac{d}{dt}\{ t L_2'(mt)\}\, dt \int_0^b \tau \frac{d}{d\tau}\{\tau L_2'(n\tau)\}\, d\tau. \right.$$

Since $\lambda_{cc}(t,\tau,S) = o(t\ \tau)$, A_4 now is o(1) in a manner similar to A_2.

We have thus shown that

$$\left[R_{m,n}^{2,2} \right]_{x,y} - S = o(1),$$

which is the requirement for summability (R,2,2).

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