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I hereby recommend that the thesis prepared under my supervision by Yee-Tak Yu

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Approved by:

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THE VIRTUAL MASS OF RECTANGULAR PLATES AND PARALLELOPIPEDS IN A LIQUID AS DETERMINED BY DIMENSIONS AND THE DIRECTION OF MOTION

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C O N T E N T S

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Virtual Masses and Moments of Inertia of Disks and Cylinders in Various Liquids

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The effective mass or moment of inertia of an object in a liquid may be considerably greater than the corresponding value in vacuum. Hence many applied dynamical problems require the use of effective rather than actual values. Herein is reported an experimental method, making use of a torsion pendulum, for determining the increase or "virtual" moment of inertia or mass of an object due to a surrounding liquid. Actual data on disks and cylinders in water, gasoline and carbon tetrachloride are given. Experimental and theoretical results are compared where possible.

INTRODUCTION

It is an important, yet not-too-well recognized fact that the effective mass of an object in a liquid is considerably greater than in a vacuum. This, of course, is due to the fact that when the object is set in motion the surrounding liquid is also set in motion. The increase in effective mass will be referred to as its "virtual" mass. Its increase in moment of inertia about any given axis will be similarly designated. *Numerical values of these quantities are, in general, not negligible.* Indeed, for certain objects such as a hollow sphere or cylinder in water, virtual values may far exceed actual values.

There are many dynamical problems in which, if all or part of the system is moving with acceleration in a fluid, it is very necessary to take account of the above-mentioned virtual quantities. As a simple example the under-sea acceleration of a submarine, for a given force applied by the propeller, is considerably less than it would be were it not for its virtual mass due to the water. The same statement is, of course, true regarding the acceleration of a dirigible. In the design of lighter-than-air craft the virtual mass is taken about equal to the mass of the air displaced. In any and all mechanical systems where parts are vibrating in a fluid, virtual moment of inertia and mass may be extremely important. As a simple example, the period of a torsion pendulum may be fifty percent greater in water (depending on its size and shape) than in a vacuum. Hence it is clear that, from the point of view of applied physics and engineering, this problem has considerable importance.

The following is a brief description of an arrangement for the experimental determination of virtual moments of inertia and masses and a summary of results obtained for disks and cylinders in water, gasoline and carbon tetrachloride. These liquids were chosen because they vary widely in density and the viscosity of each is low.

EXPERIMENTAL ARRANGEMENT

The apparatus was essentially a torsion pendulum (see Fig. 1) consisting of a tightly

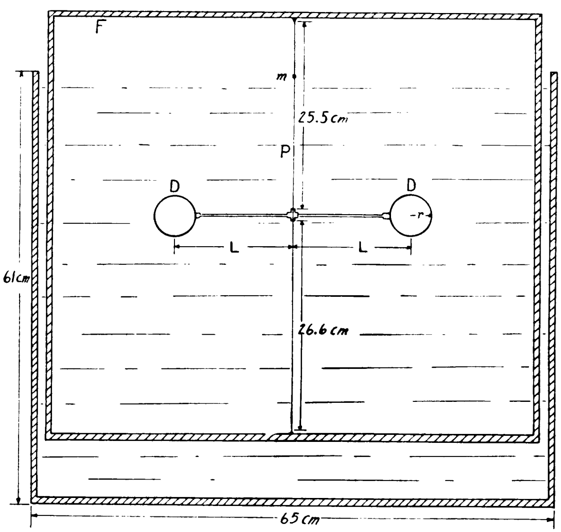

FIG. 1. Apparatus for the determination of virtual moments of inertia and masses in various liquids.

stretched piano wire P, mounted on a rigid frame F, and a thin cross-arm fastened rigidly to the wire near its midpoint. Exactly similar disks or cylinders DD were fastened at either end of the cross-arm. Since the frame was portable, it was easy to submerge the pendulum in any liquid. Dimensions of the apparatus and the cylindrical tank for the liquid are given on the figure. As will be shown in what immediately follows, the only data necessary for the determination of the virtual moment of inertia of the oscillating member and thence the virtual mass of either object D are, except for certain constants, the free periods of oscillation of the pendulum in air (assumed to be the same as in a vacuum) and in the liquid.

The period of a torsion pendulum is given by the well-known expression

$$T=2\pi\Big/\left(\frac{K}{I}-\frac{b^2}{4I^2}\right)^{\frac{1}{2}}, \tag{1}$$

where in this particular case K is the torsional constant of the wire, I the effective moment of inertia of the cross-arm and parts attached and b is the damping factor due to the liquid. The quantity $b/2I$ was determined experimentally, in the usual manner, for several cases and $(b^2/4I^2)$ proved to be negligibly small compared with K/I for the liquids used. As an example, the period of disk A in water was found to be 1.705 sec. and $b/2I$ equal to 0.11. Applying Eq. (1) we find the numerical value of K/I to be about 13.6 which is approximately one thousand times the value of $(b^2/4I^2)$. Neglecting $(b^2/4I^2)$ we get

$$T_a=2\pi\left(\frac{I_a}{K}\right)^{\frac{1}{2}},\quad T_l=2\pi\left(\frac{I_l}{K}\right)^{\frac{1}{2}}, \tag{2}$$

where the subscript a indicates values in air and l values in the liquid. The virtual moment of inertia of the entire oscillating member due to the liquid, is given by

$$I_1=I_l-I_a=\frac{K}{4\pi^2}(T_l^2-T_a^2). \tag{3}$$

Hence, by merely determining T_a, T_l and K the virtual moment of inertia of any pendulum arrangement can be found from (3). In this work K was determined in the usual manner by a separate experiment. In the course of the experimental work two wires were used, one in water and carbon tetrachloride with $K=1.158\times10^5$ and the other in gasoline with $K=1.125\times10^5$ dyne cm.

It is now clear that the virtual moment of inertia of either object D is given by

$$\tfrac{1}{2}(I_1-I_2)=I_v, \tag{4}$$

where I_2 is the virtual moment of inertia of the cross-arm alone without DD attached. Using this

TABLE I. *Data for the determination of the virtual moments of inertia and masses of disks in water (motion normal to face of disk).*

Disk	T_w sec.	T_a sec.	L cm	r cm	I_2 g cm^2	I_v g cm^2	M_v g
A	1.705	1.475	11.63	1.27	$.214\times10^3$	0.965×10^3	7.1
B	2.000	1.690	11.84	1.48	$.214\times10^3$	1.570×10^3	11.2
C	2.310	1.914	12.04	1.68	$.214\times10^3$	2.346×10^3	16.2
D	2.720	2.200	12.29	1.93	$.214\times10^3$	3.645×10^3	24.1
E	3.880	3.002	12.86	2.50	$.214\times10^3$	8.754×10^3	52.9
F	5.030	3.750	13.36	3.00	$.214\times10^3$	16.375×10^3	91.7

value of I_v, the virtual mass M_v of D can be computed with a fair degree of accuracy from the relation $M_vL^2=I_v$ where L is the distance from the wire to the center of D. Making use of (3) and (4) this relation becomes

$$M_v=\frac{1}{L^2}\left[\frac{K}{8\pi^2}(T_l^2-T_a^2)-\frac{I_2}{2}\right]. \tag{5}$$

I_2 is constant for all experiments in a particular liquid.

EXPERIMENTAL RESULTS

Detailed results obtained with lead disks, 3.4 mm thick and of various radii, in water are given in Table I. These data were taken with the plane of each disk parallel to the wire (motion perpendicular to the face). Graphs of I_v and M_v plotted as function of the cube of the radius, r, are shown in Fig. 2. M_v is directly proportional to r^3. This is to be expected from theoretical considerations which will be discussed later. The relation between M_v and r is represented by the simple empirical relation,

$$M_v=3.41r^3\rho, \tag{6}$$

where ρ, the density of the liquid, is unity for water.

An analysis of the results obtained with the plane of each disk perpendicular to the wire shows that M_v varies as the square of the radius. The empirical equation relating these quantities is

$$M_v=0.4r^2\rho. \tag{7}$$

The experiments described above were repeated in gasoline ($\rho=0.734$) and in carbon tetrachloride ($\rho=1.574$). In each case the experimental values of I_v and M_v were found to be the same as those in water multiplied by the density of the corresponding liquid. See Table II. Hence it is clear that I_v and M_v depend directly on the density of the liquid and for these *experiments Eqs.* (*6*) *and* (*7*) *hold for any of the three liquids.*

For the experiments on cylinders, thin-walled brass tubing 1.27 cm in radius were used. A thin metal disk soldered at the center of each tube made it equivalent to two cylindrical cups placed base to base. Table III gives the results in water when the longitudinal axis of each cylinder was placed perpendicular to the wire. Curves 1 and 2 in Fig. 3 show the variation of I_v and M_v with the length of the cylinder, h. For the range of values of h used in these experiments, M_v can be represented by

$$M_v=(7.1+5.1h)\rho. \qquad (8)$$

This has considerable significance since (8) can be written in the form,

$$M_v=(3.41r^3+\pi r^2h)\rho, \qquad (9)$$

where r is the radius of the cylinder. It will be noticed that *the first term of* (8) *is just the virtual mass of a disk having a radius the same as that of*

TABLE II. *Virtual moments of inertia and masses of disks moving normal to their faces in gasoline* ($\rho_g=0.734$) *and in carbon tetrachloride* ($\rho_c=1.574$).

Disk	I_g G CM²	M_g G	$3.41r^3\rho_g$ G	I_c G CM²	M_c G	$3.41r^3\rho_c$ G
A	0.703×10^3	5.2	5.1	1.542×10^3	11.4	11.0
B	1.135×10^3	8.1	8.1	2.423×10^3	17.3	17.4
C	1.739×10^3	12.0	11.8	3.625×10^3	25.0	25.4
D	2.704×10^3	17.9	18.0	5.758×10^3	38.1	38.6
E	6.533×10^3	39.5	39.1	14.182×10^3	83.8	83.9
F	11.995×10^3	67.2	67.6	25.865×10^3	144.9	144.9

TABLE III. *Data for the determination of the virtual moments of inertia and masses of cylinders moving parallel to longitudinal axis in water* ($I_2=0.214\times10^3$ *g cm*²).

Cyl-inder	T_w SEC.	T_a SEC.	L CM	r CM	h CM	I_v G CM²	M_v G
A	1.520	1.051	11.63	1.27	1.10	1.658×10^3	12.3
B	1.810	1.260	11.63	1.27	2.15	2.369×10^3	17.5
C	2.280	1.580	11.63	1.27	4.11	3.856×10^3	28.5
D	2.660	1.860	11.63	1.27	6.11	5.180×10^3	38.3
E	2.960	2.060	11.63	1.27	8.00	6.519×10^3	48.2
F	3.270	2.260	11.63	1.27	10.00	8.085×10^3	59.8

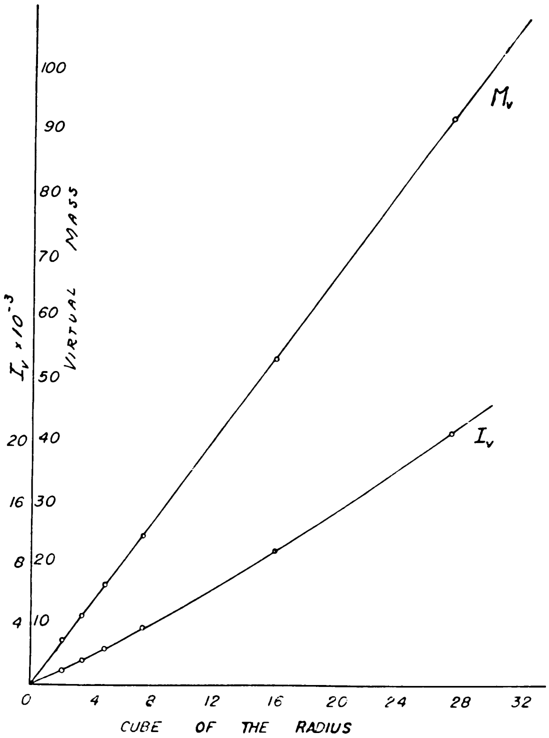

FIG. 2. Variation of I_v and M_v with the cube of the radius. Motion perpendicular to the face of the disk.

the cylinder and the second term is just the mass of the water contained in the cylinder.

Results when the longitudinal axis of each cylinder was placed parallel to the wire are shown graphically by curve 3 in Fig. 3. This straight line is represented rather accurately by

$$M_v=(9.8h-10.1)\rho. \qquad (10)$$

Undoubtedly this curve would not continue on in the direction of the dotted portion, for cylinders of very short lengths (thin disks).

Experimental results with the cylinders in gasoline and carbon tetrachloride were again just the same as those in water except for the proper density factor (see Table IV) so that Eqs. (9) and (10) hold for any of the liquids.

COMPARISON OF RESULTS WITH THEORY

Considering the kinetic energy imparted to a liquid by the motion of a body through the liquid, Lamb[1] has obtained theoretical expres-

[1] For results here given see Horace Lamb, *Hydrodynamics* (Cambridge University Press, 1906), third edition, Chapter V, pp. 102–150.

sions for the virtual mass of variously shaped objects. His theoretical relation for the virtual mass of a disk of radius r, where motion is perpendicular to the face, is

$$M_v = \frac{8}{3} r^3 \rho \tag{11}$$

and for the virtual mass per unit length of a cylinder of radius r, where the motion is perpendicular to its longitudinal axis, he obtains the relation,

$$M_v = \pi r^2 \rho. \tag{12}$$

The theoretical results of Lamb are based on the assumption of a "perfect" non-compressible liquid which extends to infinity in all directions.

Except for a constant factor, experimental values of the virtual mass of a disk moving perpendicular to its face agree well with the results of Lamb. Comparing Eqs. (6) and (11) it is seen that the experimental values are about 1.28 times greater than corresponding theoretical values. This may be due in part to the fact that water is not a "perfect" liquid and that the vessel was by no means infinite in extent.

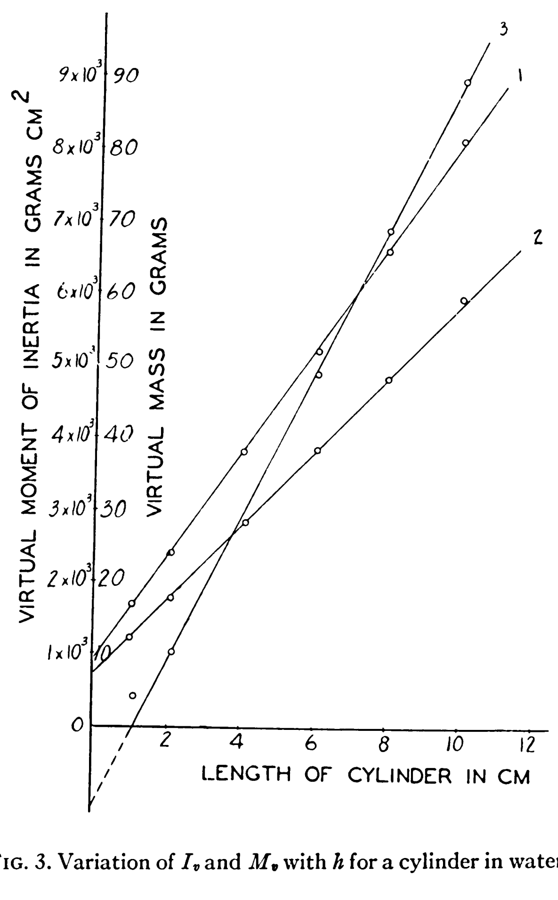

FIG. 3. Variation of I_v and M_v with h for a cylinder in water.

TABLE IV. *Virtual masses of cylinders with motion parallel to their axes in gasoline and carbon tetrachloride.*

CYLINDER	M_g G	$(3.41r^3+\pi r^2 h)\rho_g$ G	M_c G	$(3.41r^3+\pi r^2 h)\rho_c$ G
A	9.2	9.2	19.8	19.8
B	13.0	13.1	28.2	28.2
C	20.9	20.4	43.2	43.8
D	27.9	27.9	59.9	59.9
E	35.1	34.9	75.0	74.8
F	44.2	42.3	92.0	90.8

No known theoretical relations are available with which to check the other results herein reported. However it might be expected that the experimentally determined increase in M_v per unit length of a cylinder moving perpendicular to its longitudinal axis could be compared with Eq. (12). By (10) this increase is 9.8 which is about twice the value of $\pi r^2 \rho$ for the particular size of cylinders used. Considering the fact that the cylinders were hollow and thus filled with the liquid, this result checks very well with the result of Lamb.

CONCLUDING REMARKS

The method herein presented lends itself readily to a rapid and accurate determination of virtual moments of inertia. Indeed the virtual moments of inertia of any object of any shape substituted for the cross-arm of Fig. 1 can be determined at once. For long cross-arms the method employed undoubtedly yields good values of virtual mass for the objects *DD*.

Besides giving actual data with corresponding empirical equations, which it is hoped will be of service in applied problems of physics and engineering, the results here reported verify the general theoretical conclusions that, the virtual mass of an object depends on: (1) its size and shape; (2) the density of the surrounding liquid; (3) The direction in which it is moved relative to certain axes taken in the object. In addition to 1 and 2, the virtual moment of inertia of an object depends, of course, on the axis about which it is taken.

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The Double Torsion Pendulum in a Liquid

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IT is believed that the usual course in analytical dynamics would be greatly improved, from the point of view of student interest and clear understanding, if more specific demonstrations of the topics treated were presented.

The double torsion pendulum in a liquid affords an excellent concrete example of oscillation problems beyond the elementary type, and has the added advantage of showing very clearly the importance of "virtual" moment of inertia and mass in the treatment of any problem in which all or part of the system is moving in a liquid. It is the purpose of this article to describe such a pendulum and to give typical data comparing experimentally determined values of the fundamental periods of oscillation, both in air and in water, with computed values.

Construction details are shown in Fig. 1. A piano wire, 0.673 mm in diameter, is stretched tightly between opposite sides of a rectangular wooden frame. Two crossarms, *AA* and *BB*, each consisting of a slender brass rod with a lead disk attached to either end, are rigidly fastened to the wire as indicated. The lead disks have a thickness of 3.5 mm and the brass rods are each 3.92 mm in diameter. All other dimensions are shown in Fig. 1. Obviously these dimensions may be varied to suit particular needs or materials at hand.

Since this is a problem involving two degrees of freedom, there are two fundamental modes of oscillation. One of these is represented when the crossarms are moving together in phase and the other when their motion is exactly out of phase. With a little practice, it is quite easy to excite either mode of motion alone. Hence the fundamental periods both in air and with the frame submerged in water may be obtained experimentally merely by timing a large number of oscillations with a stop watch.

In order to compute the periods it is necessary to know (*a*) the torsional constant of each section of the wire, (*b*) the moments of inertia of the crossarms *AA* and *BB* in air and (*c*) their moments of inertia in water. Each of these quantities is determined experimentally by making use of a simple torsion pendulum similar to the one shown in Fig. 1 but having only one crossarm. When the free period T of oscillation of this pendulum in air is known, the torsional constant k of the wire can be obtained from the usual relation,

$$T=2\pi(I/k)^{\frac{1}{2}}, \tag{1}$$

where I is the computed moment of inertia of a long slender rod used as a crossarm. To determine α, the torsional constant per unit length of the wire, use is made of the relation,

$$k=\alpha(1/l_1+1/l_2), \tag{2}$$

where l_1 and l_2 represent the lengths of the wire above and below the crossarm, respectively. The numerical value of α for our apparatus was found to be 1.51×10^6 dyne cm^2/rad. Hence the torsional constant for each section of the double pendulum is given by α/L_1, α/L_2 and α/L_3, respectively, where L_1, L_2 and L_3 represent the lengths of the wire indicated in Fig. 1.

TABLE I. Comparison of experimental and computed values of the fundamental periods.

Period (sec)	In air Experimental value	Computed value	In water Experimental value	Computed value
T_1	1.18	1.17	1.45	1.43
T_2	1.98	1.93	2.36	2.33

The slender rod of the simple torsion pendulum is now replaced by the arm *AA* from the double pendulum and the periods of oscillation both in air and in water are determined. Applying Eq. (1) again, this time with k known, the moment of inertia of the arm and its effective moment of inertia—real plus virtual—in water are computed. A repetition of this procedure gives the corresponding values for the arm *BB*. One can, of course, compute the moments of inertia of these arms from the known masses and dimensions. Likewise, the effective values in water can be computed from the known masses, dimensions and virtual masses.[1] However, the accuracy of directly determined experimental values is probably greater.

A brief outline of the way in which the fundamental periods of the double pendulum are computed is given below. The Lagrangian function in generalized coordinates,

$$L=E_k-E_p=\tfrac{1}{2}\sum_{ij}^{n} a_{ij}\dot{q}_i\dot{q}_j-\tfrac{1}{2}\sum_{ij}^{n} c_{ij}q_iq_j,$$

becomes, for the double pendulum in air,

$$L=\tfrac{1}{2}(I_1\dot{\theta}_1^2+I_2\dot{\theta}_2^2)-\frac{1}{2}\left[\frac{\alpha}{L_1}\theta_1^2+\frac{\alpha}{L_2}(\theta_1-\theta_2)^2+\frac{\alpha}{L_3}\theta_2^2\right], \quad (3)$$

where I_1 and θ_1 represent, respectively, the moment of inertia and angular displacement from equilibrium of arm AA, and I_2 and θ_2 have similar meanings for BB. Applying the Lagrangian equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)-\frac{\partial L}{\partial q_r}=0,$$

to Eq. (3), we obtain the two equations of motion,

$$a_{11}\ddot{\theta}_1+c_{11}\theta_1+c_{12}\theta_2=0, \quad (4)$$

$$a_{22}\ddot{\theta}_2+c_{21}\theta_1+c_{22}\theta_2=0, \quad (5)$$

where

$a_{11}=I_1=0.701\times10^4$ g cm^2,
$a_{22}=I_2=1.085\times10^4$ g cm^2,
$c_{11}=\alpha(1/L_1+1/L_2)=1.760\times10^5$ dyne cm/rad,
$c_{12}=c_{21}=-\alpha/L_2=0.641\times10^5$ dyne cm/rad,
$c_{22}=\alpha(1/L_2+1/L_3)=1.557\times10^5$ dyne cm/rad.

From Eqs. (4) and (5) the fundamental determinant of the system can be written down at once as

$$\begin{vmatrix} a_{11}\omega^2-c_{11} & -c_{12} \\ -c_{21} & a_{22}\omega^2-c_{22} \end{vmatrix}=0. \quad (6)$$

The two natural periods of motion are obtained by inserting the two values of ω that satisfy Eq. (6) in the relations $T_1=2\pi/\omega_1$ and $T_2=2\pi/\omega_2$.

[1] Yee-Tak Yu, "Virtual masses and moments of inertia of disks and cylinders in various liquids," J. App. Phys. **13**, 66–69 (1942).

To obtain the natural periods in water an exactly similar procedure is followed except that I_1 and I_2 are replaced by the "effective" values, which were $I'_1=1.056\times10^4$ g cm^2 and $I'_2=1.576\times10^4$ g cm^2.

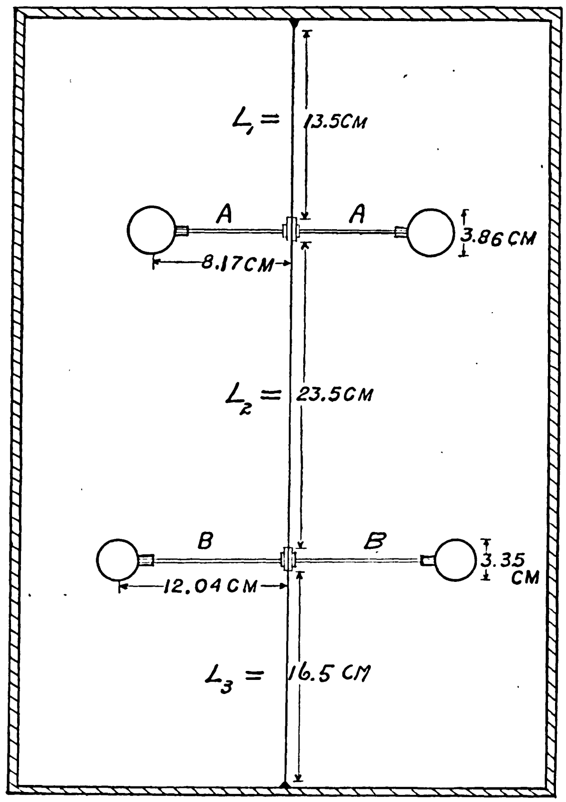

FIG. 1. A double torsion pendulum for experiments in air or in a liquid.

In treating the double pendulum, no account of damping is taken since previous experience with a single torsion pendulum has shown that its effect on the fundamental periods of this particular apparatus is very small.

A comparison of experimental and computed values of the fundamental periods in air and in water is afforded by Table I, where T_1 and T_2 refer, respectively, to the periods when the motion of the two arms is out of phase and in phase. The agreement between experimental and computed values is quite close. It seems certain that a class demonstration of such a pendulum and of the results that can be obtained with it will not fail to promote interest in the subject at hand and confidence in the methods employed.

THE VIRTUAL MASS OF RECTANGULAR PLATES AND PARALLELOPIPEDS IN A LIQUID AS DETERMINED BY DIMENSIONS AND THE DIRECTION OF MOTION

Introduction and Scope of the Work:

Whenever an object is set in motion in a liquid the surrounding liquid is also set in motion. This means that part of the work done on the object in changing its velocity is transferred to the surrounding liquid and (for non-viscous fluids) appears entirely as kinetic energy of the fluid particles. This would imply that the effective mass of an object (so far as Newton's Laws of Motion are concerned) should be greater in a liquid than in empty space. Furthermore, elementary considerations will show that this increase in effective mass should depend on the size and shape of the object, the extent of the liquid, the nearness of the moving object to containing walls or to other objects in the liquid, the density of the liquid and finally on the direction of motion relative to axes fixed in the object. Indeed it can be shown by straightforward analytical methods that such is the case and actual expressions for the increase in effective mass or "virtual mass" of certain objects having simple geometrical shapes such as a sphere or a long thin strip, etc. have been determined.[1]

Excepting expressions for the virtual mass of a few simply shaped objects, very little theoretical work has been done in this

1. "Hydrodynamics" by Horace Lamb, Third Edition, 1906. Chaps. IV and V, pp. 60 - 149.

field and a search through the literature reveals that even less has been accomplished experimentally. Yet the importance of both theretical and experimental results becomes evident when it is recalled that <u>the motion of any mechanical system moving in any fluid must involve a consideration of the virtual masses of the component parts of the system</u>. This is true even for motion in air although in many (but not all) cases the virtual mass of an object in air can be neglected.

Previous experimental work by the author of this paper on the virtual mass of disks, spheres and cylinders in various liquids has been reported.[2] This is an account of experimental results obtained for the virtual mass of rectangular plates and rectangular parallelopipeds as determined by the dimensions of these objects and the direction of motion relative to various faces.

Apparatus Employed:

The apparatus consisted essentially of a simple torsion pendulum constructed as follows (see Fig. 1). A piano wire W having a diameter of about 1 mm. was stretched tightly between opposite legs of a strong metal frame F. A cross-arm 'A' consisting of a small brass rod was fastened rigidly to the wire and served as a support of the pair of rectangular plates or parallelopipeds BB. The large screws SS served to regulate the tension on the wire. A beam of light reflected from the small mirror 'm' to a scale some

2. "Virtual Masses and Moments of Inertia of Disks and Cylinders in Various Liquids," Journal of Applied Physics, 13, 66-69 (1942)
"The Double Torsion Pendulum in a Liquid," A. J. Phys. Vol. 10, No. 3, 152-153 (1942)

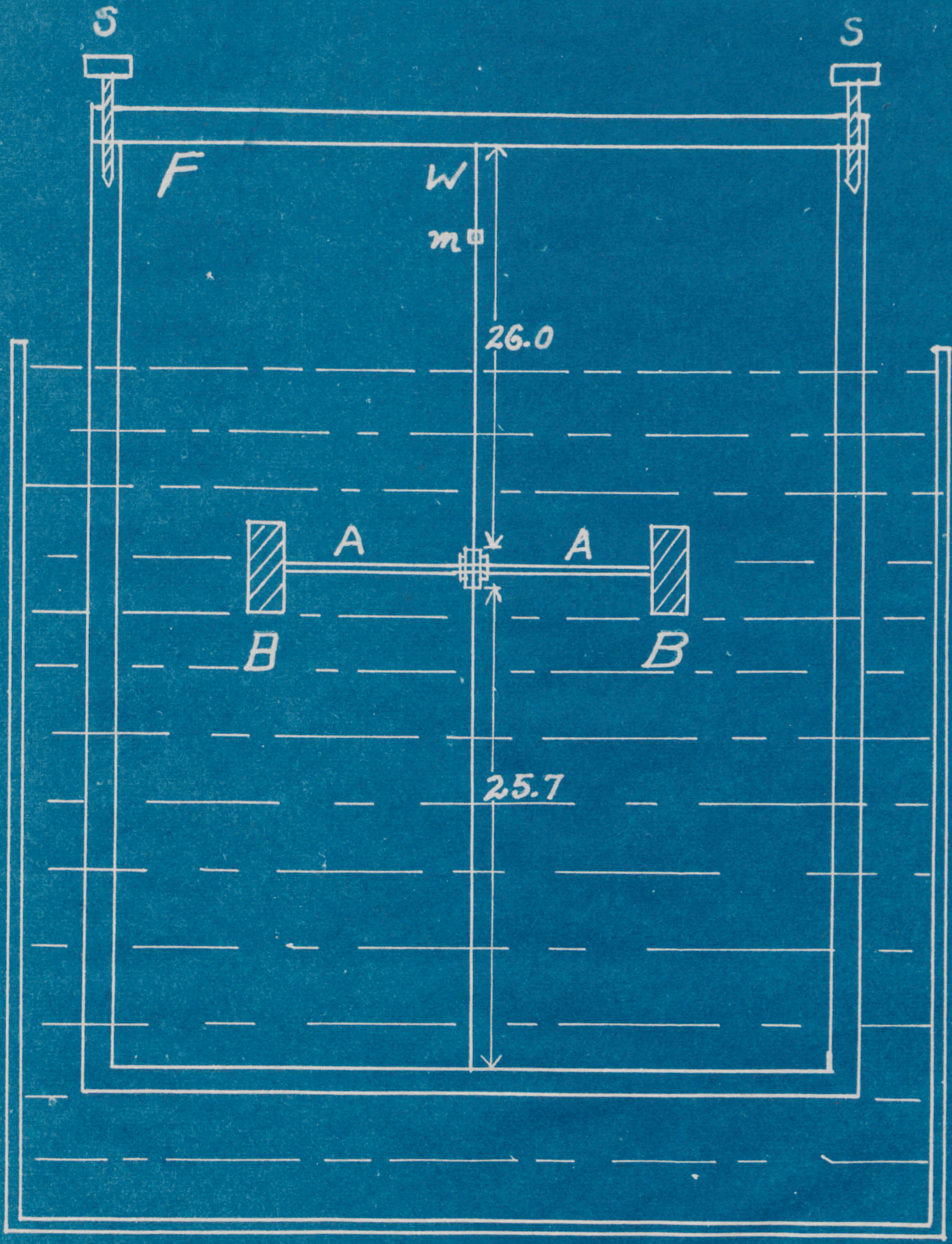

Fig. 1 DIAGRAM OF APPARATUS. The apparatus consists of a torsion pendulum the periods of which can be determined in air and in water and from which the virtual masses of objects BB can be computed.

fifteen or twenty feet away made it quite easy to obtain the period of oscillation of the arm for very small amplitude of motion. Since the entire frame was portable it was an easy matter to obtain periods of oscillation in air and in water. (Water was the only liquid used for this work since theoretical considerations as well as previous experimental work had shown that all virtual masses are directly proportional to the density of the liquid.) For the latter purpose the apparatus was submerged in a large circular tank 4 ft. in diameter and 5 ft. high, to a point where the mirror 'm' was just above the surface of the water.

Experimental Procedure:

The general procedure was simple. Exactly similar rectangular plates or rectangular parallelopipeds BB were rigidly fastened to either end of the cross-arm of the torsion pendulum. Periods of oscillation, first in air and then in water, were carefully determined by timing a large number of oscillations with an accurate stop watch. These two periods together with certain other data, to be mentioned presently, make it possible to compute the virtual mass of the objects BB.

The period of a simple torsion pendulum of the above type in air is given by:

$$T_a = 2\pi\sqrt{I_a/K} \qquad (1)$$

and neglecting the damping due to viscosity (which previous experience has shown to be insufficient appreciably to change the period) the period of this pendulum in water is given by:

$$T_w = 2\pi\sqrt{I_w/K} \qquad (2)$$

where I_a and I_w are the moments of inertia of the cross-arm with objects BB attached in air and in water respectively and K is the torsional constant of the wire W. Now let I'_a and I'_w represent the moments of inertia of the cross arm only in air and water respectively. Then the increase in moment of inertia of the entire cross-arm in water over and above its value in air, due to either of the objects BB, is given by:

$$I = 1/2 \left[I_w - I_a - (I'_w - I'_a) \right] \qquad (3)$$

This difference we shall refer to as the virtual moment of inertia of the object in the water. Letting L represent the distance from the center of the wire W to the center of the object B we can write (assuming that L is relatively long),

$$I = ML^2 \qquad (4)$$

where M is now the virtual mass associated with the object B. Hence assuming that the torsional constant K is known, equations (1), (2), (3), and (4) give the final result for which we are seeking. It will be noticed that the same procedure can be used for finding virtual mass regardless of how the objects are oriented on the ends of the cross-arms. Thus this same apparatus and procedure was used as a means of determining the virtual masses of rectangular plates and rectangular parallelopipeds as a function of direction of motion relative to a normal to one face. To do this it is only necessary to take periods of oscillation with the objects set at various known angles between a face and the plane containing the wire W and the cross-arm.

The constant K was determined experimentally by replacing the cross-arm shown in the figure with a long uniform brass rod of known moment of inertia I_k. Then I_k is related to the period of oscillation T_k in air by,

$$K = 4\pi^2 I_k/T_k \, .$$

With this and corresponding relations it is easy to derive the following convenient working equation for M in which only I_k, L and five periods of motion occur.

$$M = I_k/2T_k^2L^2 \left[T_w^2 - T_a^2 - (T_w'^2 - T_a'^2) \right] \qquad (5)$$

T_w' and T_a' represent the period of the pendulum in water and in air respectively without objects BB attached. For a fixed tension on the wire and given rods supporting BB, the values of T_k, T_w', and T_a' should remain constant. However, it was found that relatively small changes in the tension would change these values. Hence they were frequently checked.

The models for this work were made as follows. Thin rectangular plates were cut from sheet lead. Thicker models were cast in a simple mold from plaster of paris. After hardening and drying, the plaster of paris casts were boiled in paraffin wax for some time to render them impervious to water. This method proved quite successful since models so made were not too heavy, it was relatively easy to obtain accurate dimensions, and a brass nut could be cast in the face of each so that it could be screwed directly to the end of the cross-arm.

EXPERIMENTAL RESULTS

The Virtual Mass of Rectangular Parallelopipeds With Motion Perpendicular to One Face:

Following the procedure just outlined the virtual mass of a number of parallelopipeds each presenting the same area in the direction of motion but having various thicknesses were determined. These results are shown in table 1. Plotting values of virtual mass against thickness the graph of Fig. 2 was obtained. As is evident from the graph we can write,

$$M = M_1 + M_2 \tag{6}$$

where M_1 is constant and M_2 is a function of thickness only since for this particular data the area presented to the direction of motion is always the same. The general shape of the curve in Fig. 2 indicates that M_2 can be represented by an equation of the form,

$$M_2 = B\,S^n$$

or

$$\log M_2 = \log B + n \log S.$$

A plot of $\log M_2$ against $\log S$, Fig. 3, gives a straight line from which B is found to be 21.7 and n is very approximately 1/2. But this value of B is a particular one for parallelopipeds presenting a face area of 8x4 square centimeters to the direction of motion. In general B will depend on the area of the face and of course on the density of the surrounding liquid. For the moment (to be justified later) we shall assume that B is directly proportional to this area and to the density ρ. Then the above equation for M_2 with proper modification of the constant factor can be written as,

$$M_2 = 3.84\,\rho\,a\,b\,c^{\frac{1}{2}} \tag{7}$$

Table 1

S in cm. Thickness of parallelopipeds	L in cm.	T in sec. in air	in water	M by eqn. (5)
.5	22.9	2.37	4.60	105.7
1	23	3.21	5.18	111.7
2	23.1	3.90	5.77	120.8
3	22.9	5.04	6.64	128.0
5	22.9	6.32	7.78	140.5
7	23.1	8.34	6.90	147.7

Data for the parallelopipeds with the area 8 x 4 $cm.^2$ moving into water perpendicularly to their faces. (Thickness = .3 cm. in each case.)

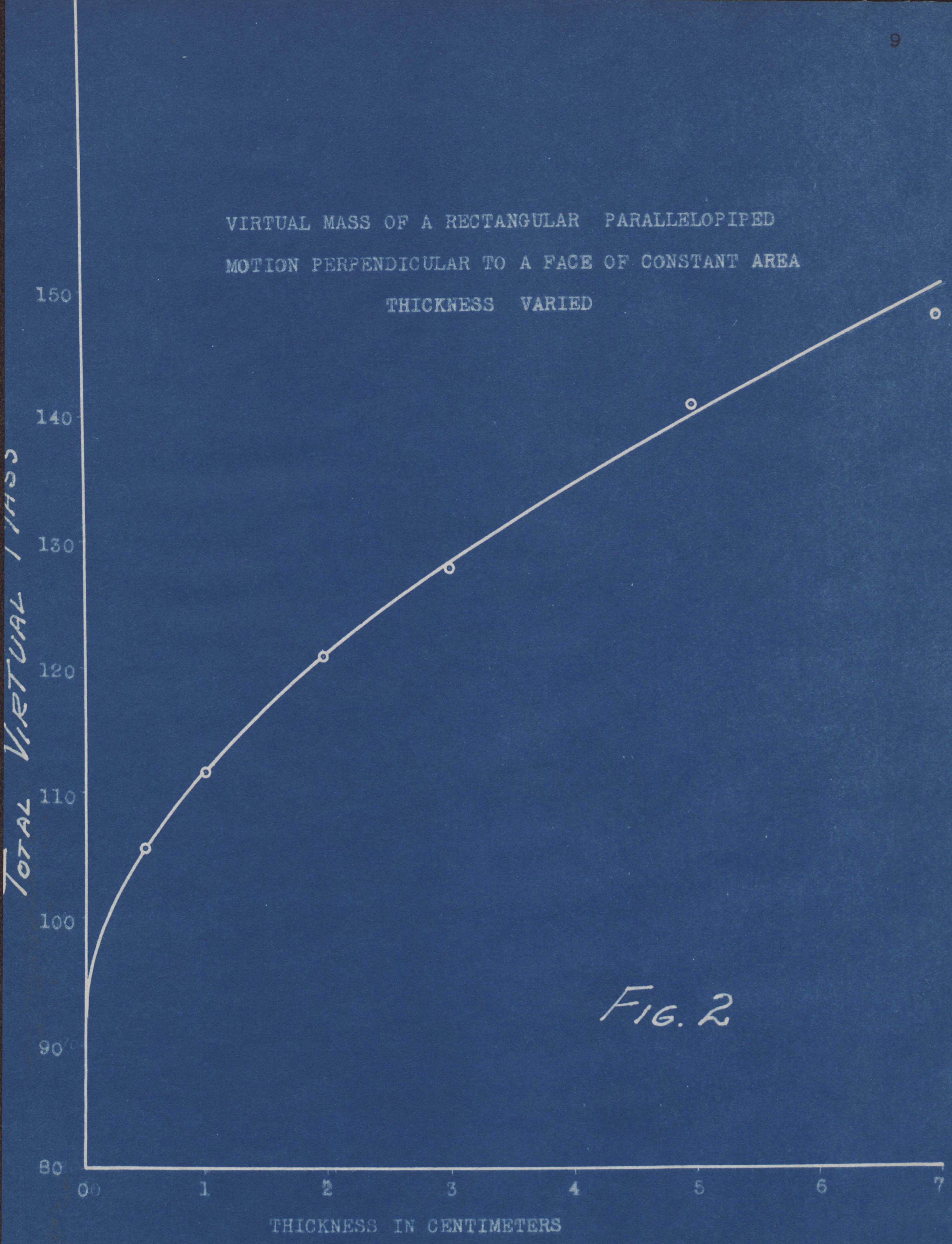
VIRTUAL MASS OF A RECTANGULAR PARALLELOPIPED
MOTION PERPENDICULAR TO A FACE OF CONSTANT AREA
THICKNESS VARIED
TOTAL VIRTUAL MASS
150
140
130
120
110
100
90
80
0
1
2
3
4
5
6
7
THICKNESS IN CENTIMETERS

FIG. 2

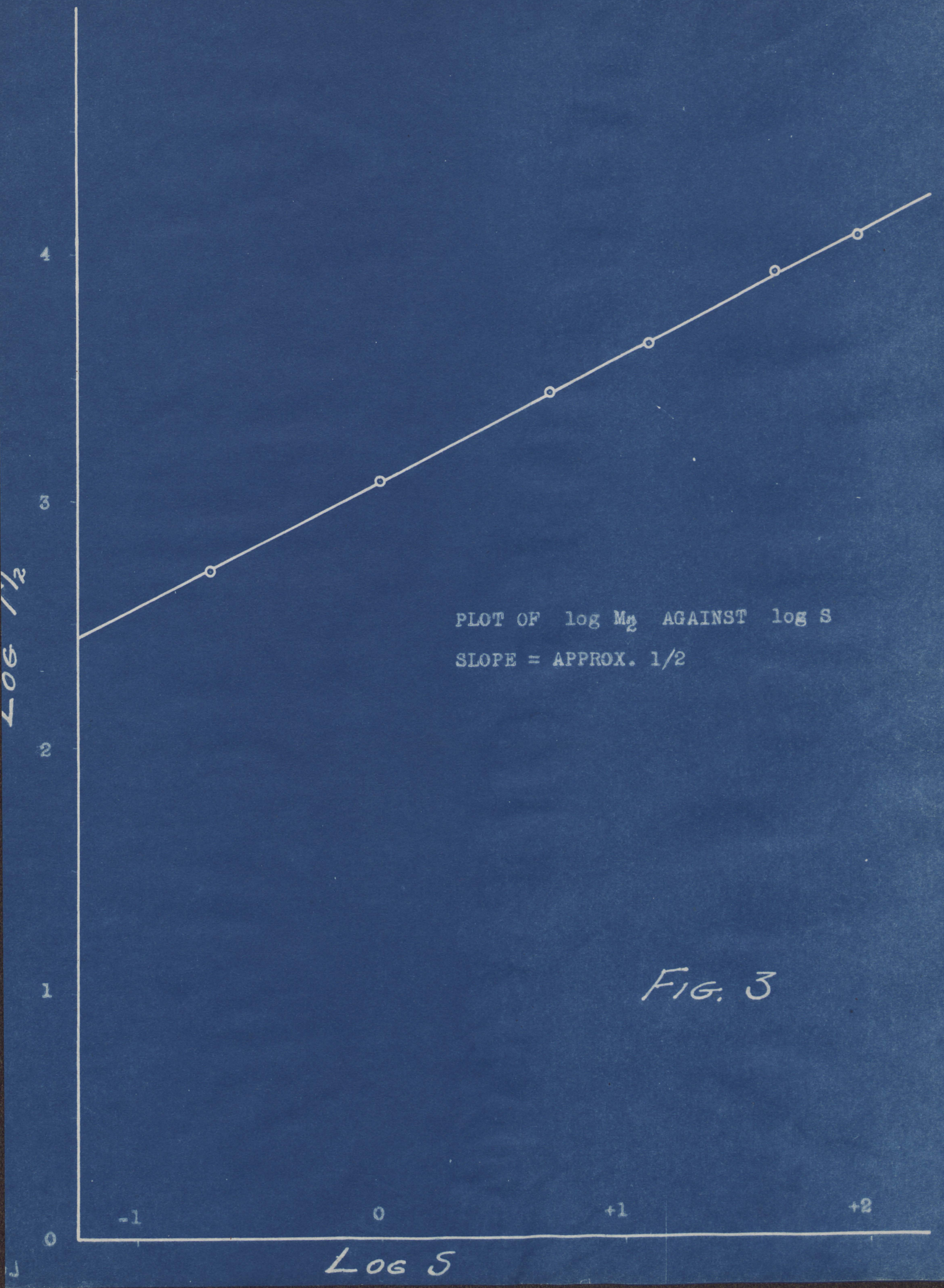
LOG $T_{1/2}$
4
3
2
1
0
-1
0
+1
+2
LOG S
PLOT OF log $M_{1/2}$ AGAINST log S
SLOPE = APPROX. 1/2

Fig. 3

Table 2

Angle between normal to the plate and direction of motion. (n,v)		0°		15°		30°	
Size of plate.	4 x 12	5.46	7.40	5.47	7.30	5.48	7.00
	M exp.	161.0		150.9		122.0	
	M comp.	161.0		150.2		121.3	
Size of plate.	4 x 10	5.02	6.77	5.02	6.66	5.03	6.40
	M exp.	132.2		123.2		100.2	
	M comp.	131.0		122.2		99.3	
Size of plate.	4 x 8	4.56	6.07	4.56	5.99	4.56	5.73
	M exp.	102.8		95.4		76.4	
	M comp.	102.0		95.0		77.0	
Size of plate.	4 x 7	4.28	5.68	4.28	5.6	4.29	5.4
	M exp.	89.0		83.1		68.0	
	M comp.	88.0		82.9		66.6	
Size of plate.	4 x 6	3.96	5.18	3.96	5.12	3.97	4.94
	M exp.	70.6		65.7		54.2	
	M comp.	71.7		67.1		54.6	
Size of plate.	4 x 4	3.35	4.22	3.35	4.18	3.35	4.06
	M exp.	40.1		38.4		31.8	
	M comp.	41.0		38.0		31.2	

The Variation of Virtual Mass with Direction.

For these data all plates have the same thickness, .3cm and the effective arm in each case was 23.55cm. (n,v) designates the angle between the nornal to the face and the direction of motion.

Table 2
(Cont'd)

(n,v) 45°		60°		75°		90°	
5.49	6.56	5.50	6.10	5.51	5.72	5.52	5.61
82.1		43.1		M = 12.9		3.9	
82.1		42.5		13.7		3.1	
5.03	6.00	5.04	3.57	5.04	5.24	5.04	5.13
67.7		34.8		11.4		3.4	
66.9		34.8		11.4		2.8	
4.56	5.40	4.57	5.05	4.57	5.05	4.57	4.66
52.3		27.7		9.6		3.1	
52.3		27.4		9.3		2.6	
4.29	5.08	4.29	4.77	4.30	4.50	4.30	4.40
46.0		25.4		8.3		3.1	
45.2		24.0		8.3		2.4	
3.98	4.70	3.98	4.40	4.0	4.2	4.0	4.10
38.4		20.5		8.0		2.7	
37.1		20.0		7.0		2.3	
3.35	3.86	3.36	3.68	3.36	3.53	3.36	3.47
21.5		12.1		5.1		2.3	
21.5		12.0		4.6		2.0	

The Variation of Virtual Mass of a Rectangular Plate with the Direction of Motion.

(In these data all plates have the same thickness, .3 cm. The effective arm L was taken as 23.55 cm. for all of them. (n,v) = angle between the normal and direction of motion.)

where, for reasons which will appear later, a,b, and c are taken to represent half values of the dimensions. Dimension c is in the direction of motion and the face $2a \times 2b$ is perpendicular to the direction of motion.

In order to find M_1 as a function of the dimensions a and b, rectangular plates having different areas but all of the same thickness (lead plates .3 cm. thick) were used. Results of the total mass M thus obtained are shown in the first column of table 2. Assuming the correctness of (7), a value of M_1 is computed for each value of M by the relation,

$$M_1 = M - 3.84(\rho\ ab)c^{1/2} \qquad (8)$$

It is to be noted that a value of M_1 represents the virtual mass of an infinitely thin rectangular plate of area 2ax2b since M_1 is the value of total virtual mass M where the curve of Fig. 2 is extrapolated to zero thickness.

From general theoretical considerations to be discussed later, it appeared likely that the virtual mass of an infinitely thin rectangular plate should be given by:

$$M_1 = \frac{2\pi\ a^2b^2}{\sqrt{a^2 + b^2}} \qquad (9)$$

Thus values of M_1 as determined from (8) were plotted against

$$\frac{a^2b^2}{\sqrt{a^2 + b^2}}\ .$$

This gives the straight line shown in Fig. 4 the slope of which is very nearly 2π (actual value = 6.28). Hence we write as the total virtual mass of a rectangular parallelopiped having dimensions 2a, 2b, and 2c and moving perpendicular to the face 2a x 2c:

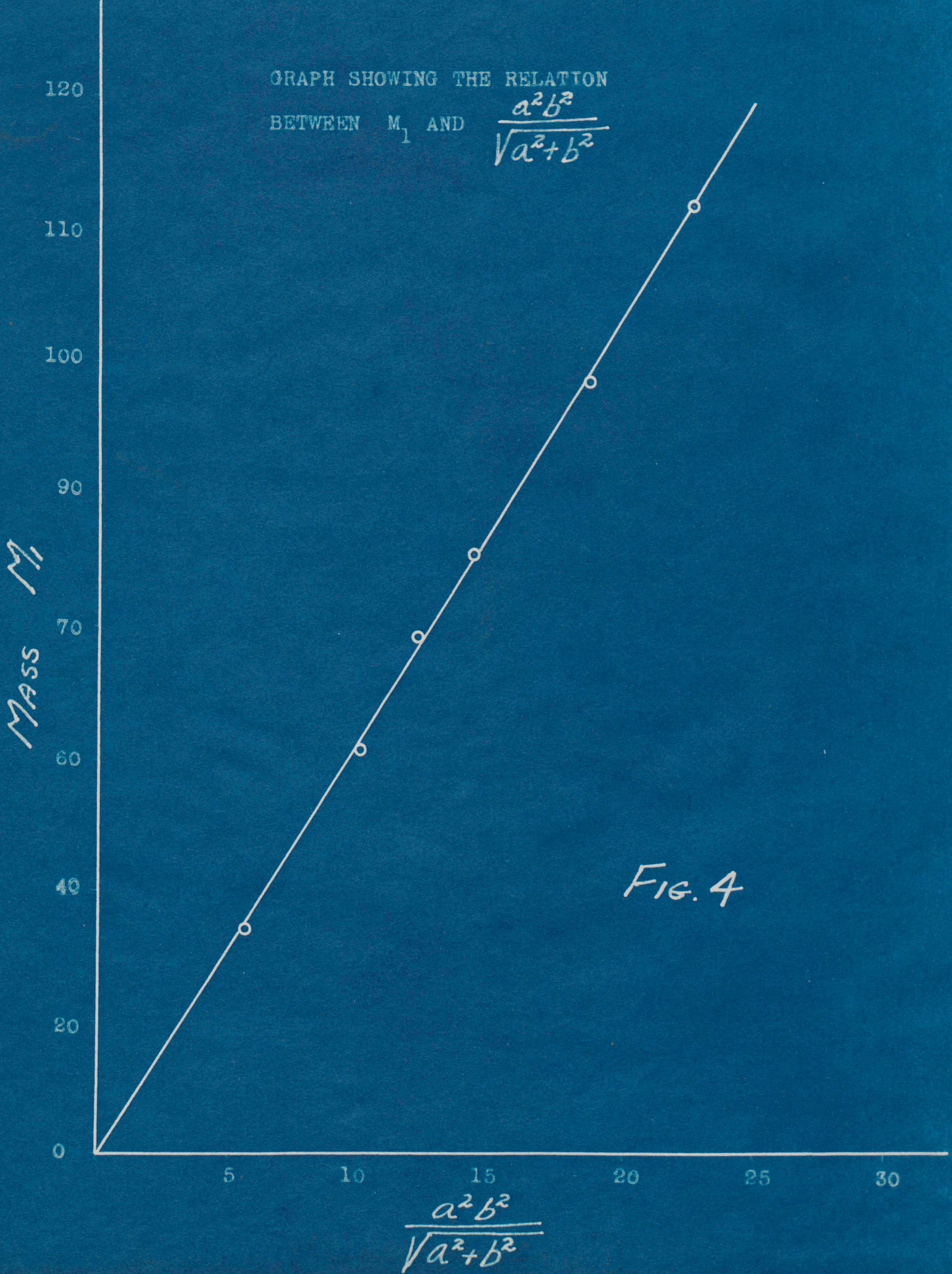
GRAPH SHOWING THE RELATION
BETWEEN M_1 AND $\frac{a^2b^2}{\sqrt{a^2+b^2}}$
MASS M_1
120
110
100
90
70
60
40
20
0
5
10
15
20
25
30
$\frac{a^2b^2}{\sqrt{a^2+b^2}}$

FIG. 4

$$M = \rho \left[\frac{2\pi^2 a^2 b^2}{a^2 + b^2} + 3.84 a\, b\, c^{1/2} \right] \qquad (10)$$

As a test of the correctness of this equation values of M for various dimensions have been computed and tabulated with corresponding experimental values in table 1. It is seen that the agreement between experiment and equation (10) is excellent. Since the agreement is so close it appears certain that the tentative assumptions made in writing equations (8) and (9) cannot be far from true.

<u>Variation of Virtual Mass of a Thin Rectangular Plate with Direction of Motion.</u>

It is evident that the virtual mass of a rectangular plate moving edgewise is less than when moving perpendicular to its face. Hence it seems evident that the virtual mass of parallelopipeds in general must depend on the direction of motion relative to axes fixed in them.

In order to determine virtual mass as a function of direction of motion the following procedure was adopted. Exactly similar rectangular plates (the thin lead plates previously used) were rigidly fastened to either end of the cross-arm A of the original apparatus (Fig. 1) in such positions that the normal to the face of each was at a definitely known angle with respect to the direction of motion of the arm. This angle could be set to within an accuracy of about $\pm$ .5 degrees by means of a protractor arrangement which it seems need not be described here. Thence by taking the same set of periods of oscillation as for the previous work the virtual mass for any particular angle could be computed by equation (5). A set of values of virtual masses for rectangles of various dimensions a and b, but all of the same thickness (.3 cm.), for angles 0, 15, 30, 45, 60, 75, and 90 degrees between the normal to the face and the direction of motion are given in table 3. These rectangles are so thin that they can almost (but not quite) be regarded as having only two dimensions. Plotting experimentally determined values of virtual mass against corresponding angles for each rectangle the curves of Fig. 5 were obtained. In plotting these graphs values of mass on one side of the zero angle are repeated for corresponding angles on the other side since it is obvious that the virtual mass must be the same for motion at a given angle on either side of the normal to the face.

Any one of these curves is represented quite accurately, except for angles near $\pm$ $\pi/2$, by the equation:

$$M_\theta = M_o \operatorname{Cos}^2\theta \qquad (11)$$

That (11) does not give results in close agreement with experiment for angles near $\pm$ $\pi/2$ is due largely to the fact that the rectangles have a finite thickness (.3 cm. in each case). This fact will be taken account of in the work that follows. From the results just discussed it seems certain that, neglecting thickness, the virtual mass of a rectangle varies as the square of the cosine of the angle between the direction of motion and a normal to the face and is represented by:

$$M = \rho \left[\frac{2\pi a^2 b^2}{\sqrt{a^2 + b^2}} \right] \cos^2\theta \, . \qquad (12)$$

<u>The Virtual Mass of a Rectangular Parallelopiped as a Function of the Direction of Motion:</u>

Consider a rectangular plate having dimensions 2a, 2b, and 2c moving parallel to the y axis as shown in Fig. 6 with the faces

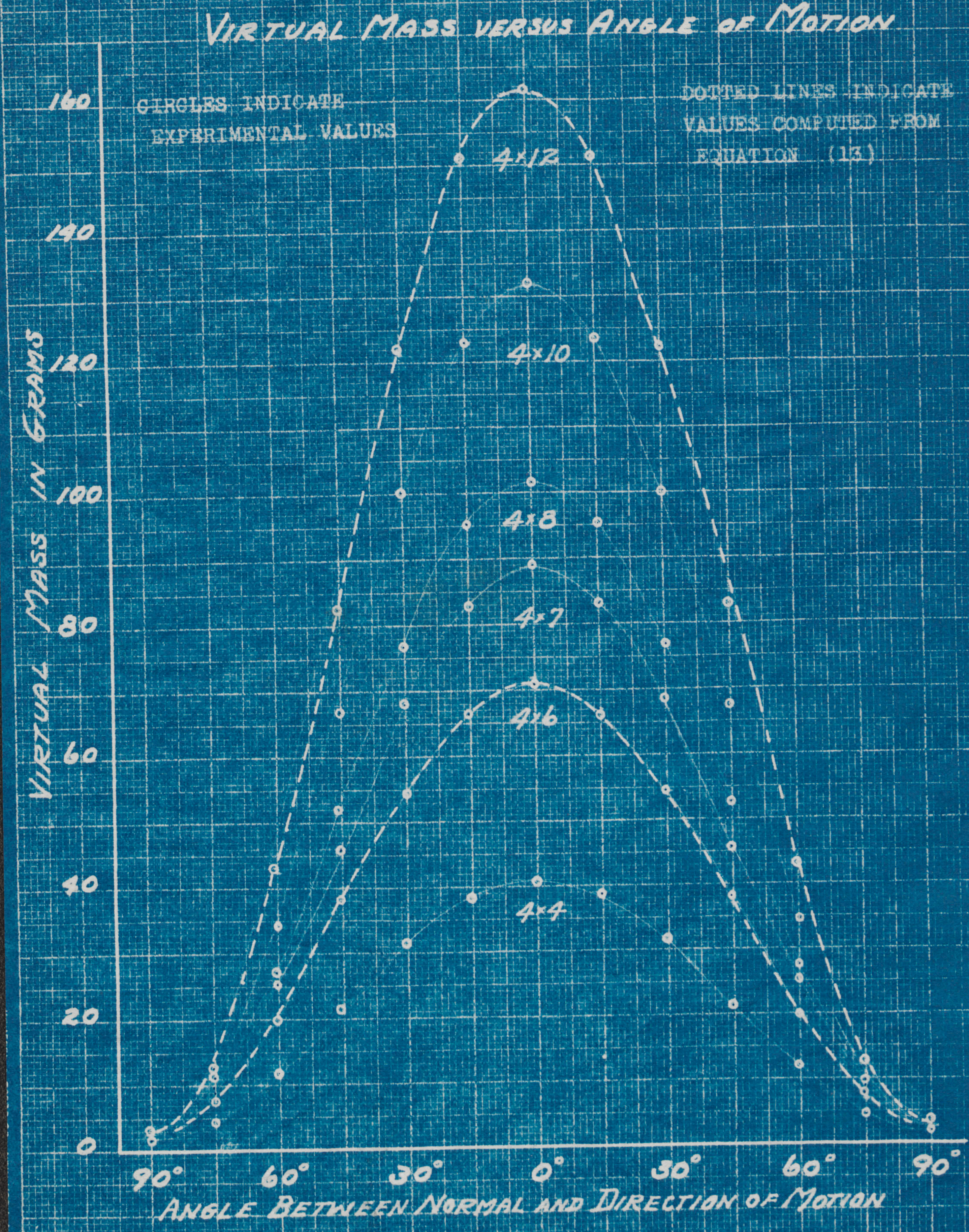
VIRTUAL MASS VERSUS ANGLE OF MOTION
CIRCLES INDICATE
EXPERIMENTAL VALUES
DOTTED LINES INDICATE
VALUES COMPUTED FROM
EQUATION (13)
VIRTUAL MASS IN GRAMS
160
140
120
100
80
60
40
20
0
4×12
4×10
4×8
4×7
4×6
4×4
90°
60°
30°
0°
30°
60°
90°
ANGLE BETWEEN NORMAL AND DIRECTION OF MOTION

FIG. 5

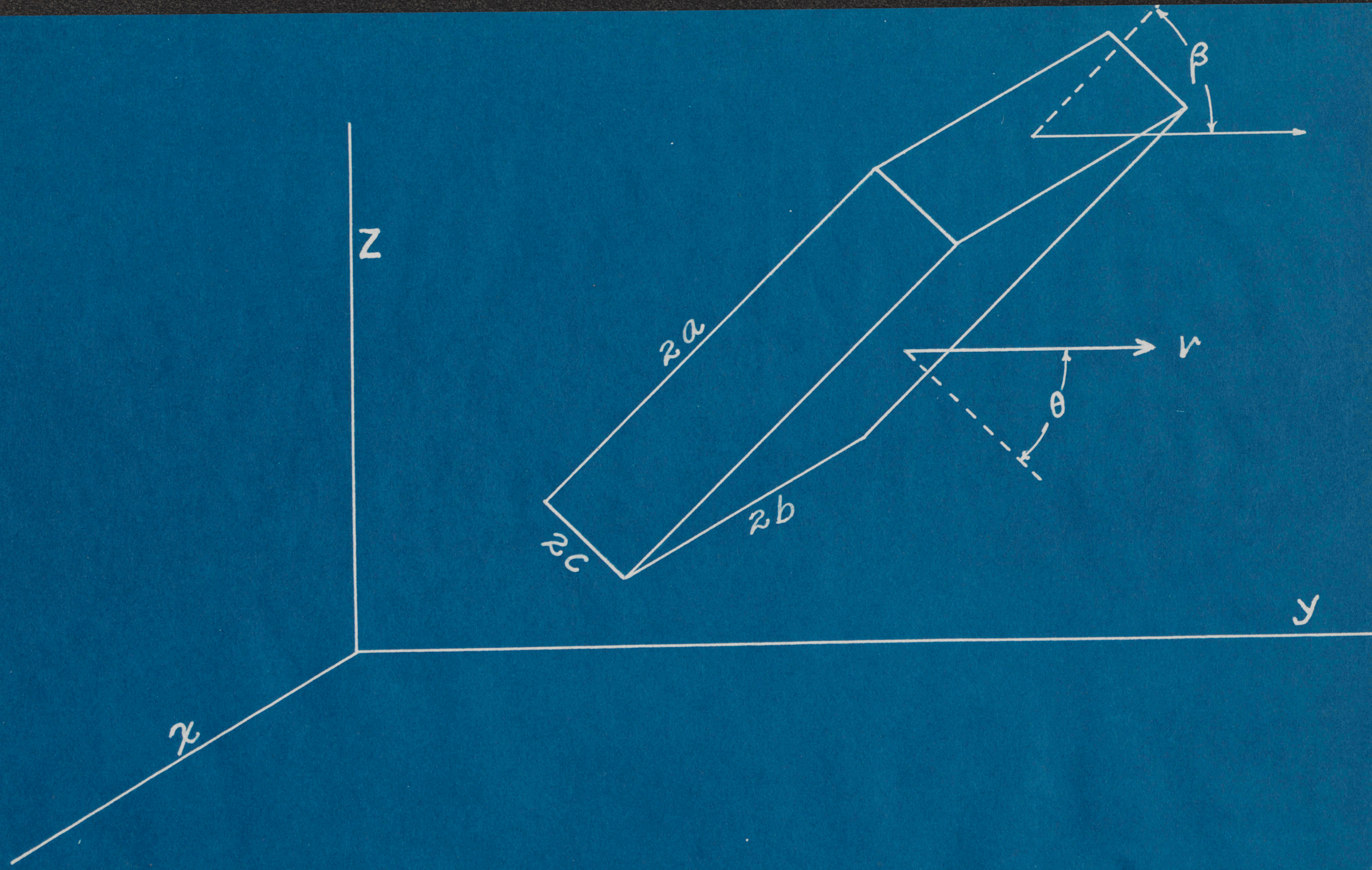

Fig. 6

Angles in degree	$(n_{4x5},v) = 60^{o}$ $(n_{4x8},v) = 30^{o}$		$(n_{4x5},v) = 30$ $(n_{4x8},v) = 60$		$(n_{4x5},v) = 0$ $(n_{4x8},v) = 90$	
Periods in sec.	in water	in air	in water	in air	in water	in air
	7.66	6.33	7.44	6.34	7.28	6.34
Experimental values M in gm.	125.7		100.8		86.9	
Values computed by eqn. (13) M in gm.	126		100.3		87	

<u>Table 3</u>

2a x 2c parallel to the xy plane and the normals to the faces 2ax2b and 2cx2b making angles θ and β respectively with the direction of motion. For motion of this type it seems likely that the object can be regarded as two plates, one with an area 2ax2b and thickness 2c with an angle θ between the normal to 2ax2c and another of thickness 2a with an angle β between the normal to 2bx2c and the direction of motion. Thus since equation (10) gives good agreement with experiment for motion perpendicular to one face and since the experimental results just discussed indicate that virtual mass varies as the square of the cosine of the angle between the normal to a face and the direction of motion we shall, in an attempt to arrive at a suitable empirical equation, compare experimental results with values obtained from the following relation:

$$M = \rho \left[\left(\frac{2\pi a^2 b^2}{\sqrt{a^2 + b^2}} + 3.84abc^{1/2} \right) \cos^2\theta + \left(\frac{2\pi b^2 c^2}{\sqrt{b^2 + c^2}} + 3.84bca^{1/2} \right) \right] \cos^2\beta \qquad (13)$$

That the agreement between (13) and experiment is close is shown by the dotted curves of Fig. 5.

As further evidence of the correctness of this equation consider a rectangular parallelopiped with dimensions 4x8x5 cm^3. Let us set the normal to the 8x5 face at 90 degrees with respect to the direction of motion and determine the virtual mass for various known angles between the 4x5 and 4x8 faces and the direction of motion. Experimental results and results obtained by computation from equation (13) are tabulated in table 2. It is seen that for each angular setting, experimental and computed results agree very closely.

Extending this approach to the problem of finding an equation for the virtual mass of a rectangular parallelopiped moving (without rotation) in any direction relative to axes fixed in the object we will write:

$$M = \rho \Big[\left(\frac{2\pi a^2 b^2}{\sqrt{a^2 + b^2}} + 3.84\ abc^{1/2} \right) \cos^2 (n_{ab}, v) + \left(\frac{2\pi a^2 c^2}{\sqrt{a^2 + c^2}} + 3.84\ acb^{1/2} \right) \cos^2 (n_{ac}, v) + \left(\frac{2\pi b^2 c^2}{\sqrt{b^2 + c^2}} + 3.84\ bca^{1/2} \right) \cos^2 (n_{bc}, v) \Big] \qquad (14)$$

where n_{ab}, n_{ac}, and n_{bc} are the normals to faces 2ax2b, 2ax2c, and 2bx2c respectively and v represents the direction of motion. For motion parallel to two sets of faces (14) reduces to (10) and for motion parallel to only one set of faces it reduces to (13). Applying equation (14) to a cube, since $a = b = c$, it reduces to,

$$M = \rho \left(\frac{2\pi a^3}{2} + 3.84\ a^{5/2} \right) \Big[\cos^2 (n_{ab}, v) + \cos^2 (n_{ac}, v) + \cos^2 (n_{bc}, v) \Big]$$

But it is clear that $\cos^2 (n_{ab}, v) + \cos^2 (n_{ac}, v) + \cos^2 (n_{bc}, v) = 1.$

Hence the virtual mass of a cube should be independent of its direction of motion and given by,

$$M = \rho \left(\frac{2\pi a^3}{2} + 3.84 a^{5/2} \right) \qquad (15)$$

This equation was tested experimentally and the results are given in tables 4 and 5. The results of table 4 were obtained by fastening a

θ	0		$\frac{\pi}{6}$		$\frac{\pi}{3}$	
Periods in sec.	in air 5.53	in water 6.84	in air 5.53	in water 6.84	in air 5.53	in water 6.84
M exp.	106		107		106	
M comp.	107		107		107	

Table 4

θ	$\frac{\pi}{2}$		$\frac{\pi}{4}$		$\frac{\pi}{6}$	
Periods in sec.	in air 5.86	in water 7.26	in air 5.86	in water 7.26	in air 5.86	in water 7.26
M exp.	107		107.2		107	
M comp.	107		107		107	

Table 5

pair of exactly similar cubes to the ends of the cross-arms (Fig. 1) so that one pair of faces in each case was parallel to the direction of motion. The angle θ was measured between the normal to a second pair of faces and the direction of motion. Data for table 5 were obtained with the cubes mounted with their diagonals in line with the cross-arms. Imagine a diagonal passing through a cube perpendicular to the diagonal just mentioned. The angles 90, 45, and 30 degrees given in table 5 were measured between this diagonal line and the direction of motion. It is seen that all values of virtual mass listed in these two tables are essentially the same. Indeed it was observed experimentally that the period of oscillation of the pendulum was, within experimental limits, the same for any arbitrary orientation of the cubes, showing that the virtual mass is independent of the direction of motion. It will also be noted that values computed from (15) show close agreement with the experiment. This is important since it gives strong support to the correctness of equation (14).

Theoretical Considerations:

The general theory of the virtual mass of an object moving in a perfect fluid has been completely developed[3] and is relatively easy to apply in certain very special cases. Unfortunately, however, it is quite difficult to apply to most objects even though their geometrical shapes may be quite simple. This is true even of so simple an object as a rectangular plate or rectangular parallelopiped.

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3. **Hydrodynamics; Lamb, Horace, Third edition, Chapt. 111, pp 28-59**

But for the sake of completeness and for purposes of reference a brief outline of the general theory is here given.

(a) The potential function.

The motion of a perfect fluid is characterized by the existence of a velocity potential function Φ which must satisfy Laplace's equation,

$$\nabla^2 \phi = 0 \qquad (16)$$

and the existing boundary conditions for any particular problem.

(b) If ϕ is determined so as to satisfy Laplace's equation and the boundary conditions, the velocity components of the fluid particles in the direction of increasing ϕ are given by:

$$v_x = -\frac{\partial\phi}{\partial x}, \quad v_y = -\frac{\partial\phi}{\partial y} \qquad v_x = -\frac{\partial\phi}{\partial z}$$

Now since the kinetic energy per unit volume of fluid is 1/2 it follows that the total kinetic energy of the fluid is given by:

$$T = \frac{\rho}{2}\iiint_V \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] dx\,dy\,dz$$

and this, as shown by Lamb, Hydrodynamics, p. 44, can be written as:

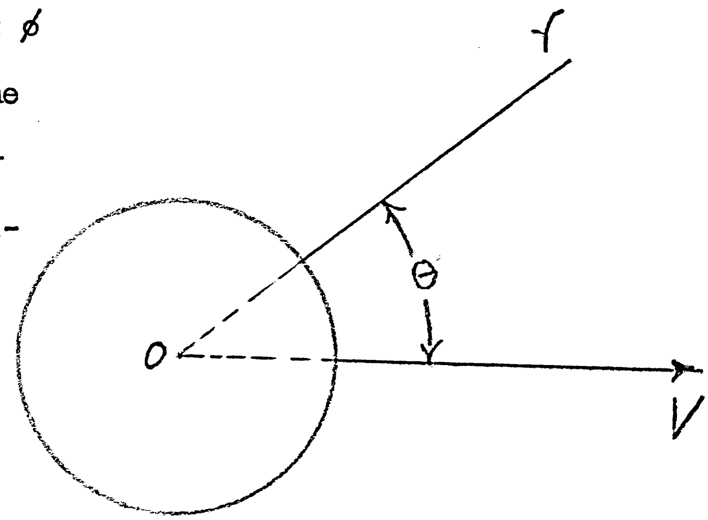
$$T = -\frac{\rho}{2}\iint_S \phi \frac{\partial\phi}{\partial n}\, dS$$

where $-\frac{\partial\phi}{\partial n}$ is the normal velocity of the fluid particles to the element of surface dS. For an object moving in a liquid with velocity V the kinetic energy of the surrounding liquid is given by:

$$T = -\frac{\rho}{2}\iint_S \phi\frac{\partial\phi}{\partial n}\, dS = \frac{1}{2} M V^2 \qquad (17)$$

M represents the increase in effective mass of an object due to the motion of the surrounding fluid and is what we have referred to as "virtual mass." As a simple example of the use of (17) M let us compute the virtual mass of a sphere moving in a perfect liquid of infinite extent. Assume that the sphere of radius 'a' is moving with velocity V as shown in the sketch. Boundary conditions for this problem are: (a) on the surface of the sphere (for $r = a$) the normal component of the fluid particles in contact with the sphere is at any point given by $V_n = V \cos \phi$ and (b) at $r = \infty$ the velocity of the liquid must be zero. Now it is easily seen that a solution of Laplace's equation which ~~also~~ satisfies these boundary conditions is:

$$\phi = \frac{1}{2} V \frac{a^3}{r^2} \rho \cos \theta$$



Hence applying equation (17) we have:

and thus the virtual mass of the sphere is given by:

$$M = 2/3 \pi \rho a^3$$

Expressions for the virtual mass of other objects have been obtained.[4] For example: the virtual mass of a circular disk of radius 'a' with motion perpendicular to one face is equal to $8/3 \rho a^3$ and the virtual mass per unit length of an infinitely long strip of breadth

4. Strock, Lamb .

2b is given by

$$M = \pi \rho b^2 \qquad (18)$$

So far as is known no theoretical work on rectangular plates or rectangular parallelopipeds has been published. Such work has been started by the author of this paper and two remarks in this general connection should be made. First the form of equation (9) was suggested by a series expansion of an assumed form for ϕ and term by term integration of the general equation (17) as applied to a thin rectangle. The factor 2π occurring in (9) was obtained approximately as the slope of the straight line Fig. 4 and suggested specifically by the following. Let $a >> b$. Then neglecting b in the denominator of equation (9) and dividing through by 2a we have $M = \rho \pi b^2$ which should represent the virtual mass per unit length of a very long strip. But this is exactly the relation given by Lamb (see above) for the mass per unit length of an infinite strip. Secondly the $\cos^2 (n,v)$ occurring in (11) and subsequent equations is to be expected from theoretical considerations. One of the boundary conditions over any surface is that the normal component of velocity of the fluid particles in contact with the surface must be equal to $V \cos (n,v)$. Over a plane surface where n and v remain constant (assuming no rotation) $V \cos (n,v)$ remains constant. Hence it can be seen that the general expression for T will contain $\cos^2 (n,v)$ as a factor and thus the expression for virtual mass will also contain this factor.

Conclusions:

(1) The technique of employing a torsion pendulum, developed for this work, lends itself well to an accurate determination

of the virtual mass of any object moving linearly in any direction relative to axes fixed in the object. The casting of models in plaster of paris also proved quite successful.

(2) Experimental values of virtual mass of rectangular parallelopipeds of various sizes were determined for various directions of motion and from this data empirical equations which give results in close agreement with the experiment were obtained for: (a) a very thin rectangular plate moving in any direction relative to a normal to its face, equation (12), (b) a rectangular plate of finite thickness moving parallel to one set of faces and in any direction relative to the other two, equation (13), and finally (c) the general case of a rectangular parallelopiped of any dimensions moving in any direction relative to axes fixed in itself, equation (14).

(3) In so far as comparison of the above equations with general theoretical considerations can be made, there is qualitative agreement. Specific theoretical results for a rectangular parallelopiped are not available. In the one case where a direct comparison can be made, namely the mass per unit length of an infinite strip, the agreement is exact in form and very close as to the one constant involved.