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AN ANALYSIS OF

THE RELAY-TYPE SERVOMECHANISM

OF AN ELECTROMAGNETIC UNDERWATER LOG SYSTEM

A dissertation submitted to the

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by

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I. INTRODUCTION

The development of an electromagnetic underwater log system by the Applied Science Research Laboratory, University of Cincinnati, under Navy contracts NObs 45473 and NObs 50435 has required the design and construction of a relay-type servomechanism. The nature of the problem required an experimental solution, and the purpose of this dissertation is to develop a method of analyzing such servomechanisms.

The problem consists of the following six main parts:

1. The representation of the physical system by an equivalent mechanical system and an equivalent electrical circuit, each in terms of measurable servo parameters.
2. The development of a systemmatic method of predicting in terms of measurable servo parameters the operation of a servomechanism represented by these parameters.
3. The measurement and calculation of these servo parameters for the electromagnetic underwater log system mentioned above.
4. The prediction of the operation of this system for a number of typical cases using the method developed herein.
5. The experimental measurement of the actual operation of the servomechanism for the same cases.

6. The comparision of the theoretical and experimental operation of the electromagnetic underwater log system for the typical cases considered.

II. DESCRIPTION OF THE ELECTROMAGNETIC UNDERWATER LOG SYSTEM

An underwater log system is a device to determine the speed of a ship with respect to the water through which it is moving. All systems now in use consist essentially of an apparatus for securing a signal which is a function of the ship's speed, an indicator-transmitter giving direct readings of speed in knots (nautical miles per hour) and distance traveled in nautical miles, and the intermediate apparatus necessary to operate the speed and distance indicators from the input signal.

The electromagnetic underwater log system uses electromagnetic means to obtain a signal voltage which is directly proportional to the ship's speed through the water. Because of the small magnitude of this signal voltage and the fact that the input is slowly varying, a relay-type servomechanism is desirable. The operation of this servomechanism will be explained with the aid of Figure 1, page 7.

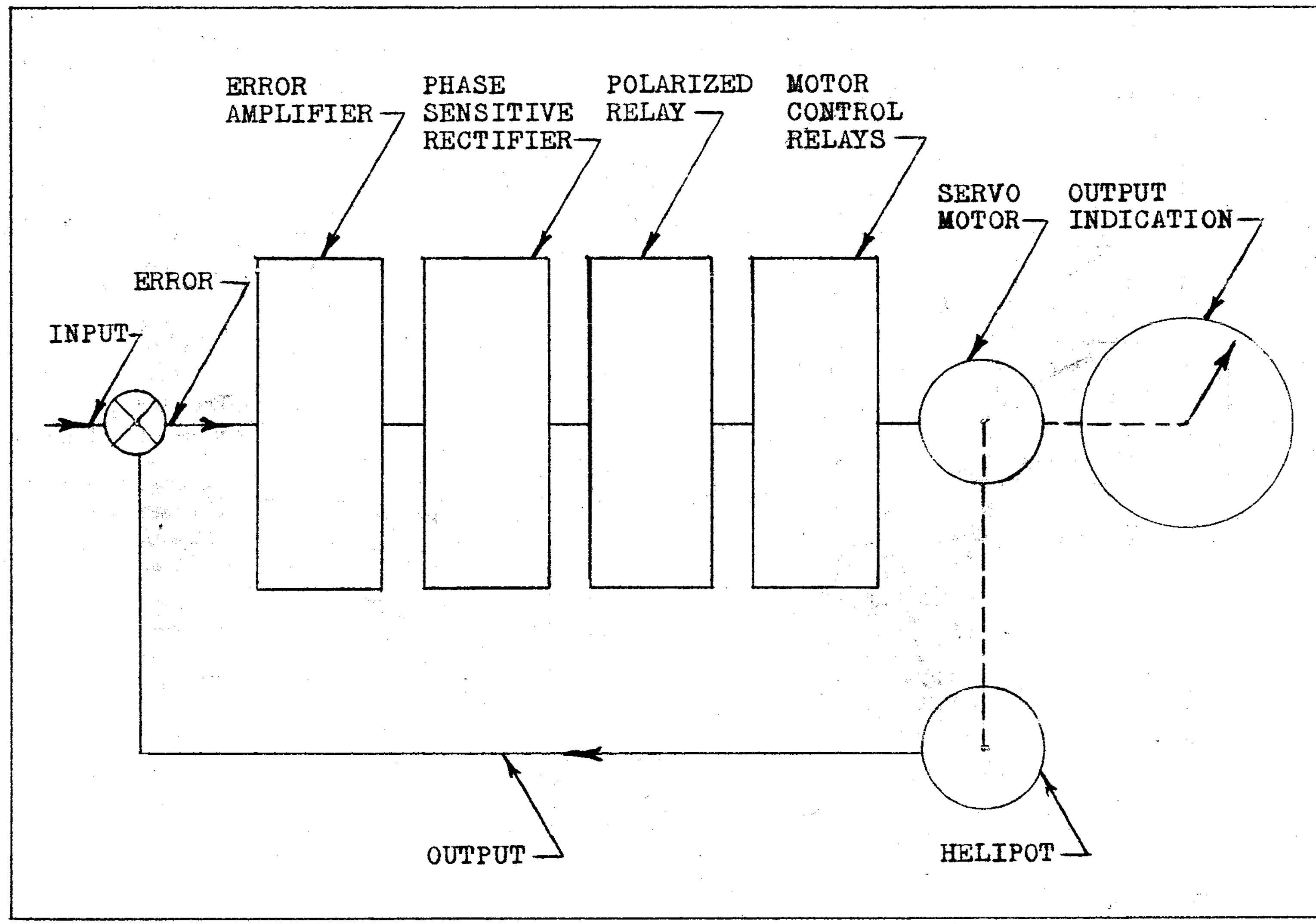
As mentioned before, the signal voltage is very small. In addition, it is detected in the presence of extraneous voltages much larger than itself. Because of these, an error amplifier, a phase sensitive rectifier, and a polarized relay are used to amplify the error and extraneous voltages, to sort out and rectify the error voltage, and to operate the motor control relays.

In any servomechanism the input and output must be compared in the same physical quantity. In the electromagnetic underwater log system the input is a small electrical voltage directly proportional to the ship's speed, and the output is an angular rotation directly proportional to the indicated speed. Consequently, either one must be converted to the same physical quantity as the other, or both to some common third quantity. In this servomechanism the output angle is converted linearly to a voltage for comparision with the input.

The Helipot is excited with a voltage of the same frequency and phase as the signal voltage. The variable tap of this Helipot is connected to the output shaft thus giving a voltage which is directly proportional to the output reading. The system is calibrated so that the voltage per knot of the output is the same as the input voltage per knot. The difference between the input and output voltages as well as the extraneous voltages is admitted to the error amplifier. The output of the error amplifier is placed across the phase sensitive rectifier which is biased so that it rectifies only the component of its input voltage which is of the same frequency as and of the same or opposite phase as the signal voltage. When the indicated speed of the output is the same as the ship's speed, the output of the phase sensitive rectifier is zero. When there is a difference between the two, its rectified output voltage is applied to the polarized relay. The polarity of this output voltage depends on the direction of the error and so closes the sensitive, polarized relay on one side or the other depending on the direction of the error. This polarized relay in turn controls two single throw motor control relays, one of which controls the clockwise circuit and the other the counterclockwise circuit of a capacitor servomotor. This servomotor drives the output shaft which, among other things, positions two concentric pointers, one of which is calibrated to read one knot per revolution and the other twenty-five knots per revolution. The system is phased so that for an error of a given direction, the direct current will have the direction necessary to close the polarized relay on the side which controls the motor control relay which controls the servomotor circuit which will drive the output in the direction that reduces the error.

Because the complications in the analysis of this servomechanism are due to the relay circuit employed, this circuit will be explained in greater detail with the aid of Figure 2, page 8. After the relay circuit was constructed, it was found that the moving contact of relay K-1 often hovered in the vicinity of the stationary contacts without definitely opening or closing. In order to eliminate this excessive chattering a circuit was designed whereby an additional voltage of the same polarity as the error voltage is applied to this circuit as soon as a contact is made and is removed as soon as a break is made. This prevents the moving contact from chattering on the stationary contacts for any appreciable length of time. In order to compensate for this added voltage another voltage of the opposite polarity is applied when relay K-2 or K-3 closes. It has been found desirable to overcompensate for the initial applied voltage in order to reduce overshooting. Consequently, the second applied voltage is larger in magnitude than the first.

The direct voltage E from the phase sensitive rectifier is used to operate the polarized relay K-1. This relay when closed closes either relay K-2 or K-3 depending on which side it is closed. These relays in turn control the servomotor circuits. As explained before, the direction of the error determines the sign of E and hence which servomotor circuit is closed. As can be seen from the circuit diagram, when the relay K-1 closes in either direction, a circuit through one of the six volt power supplies, one of the other relay coils, and R-1 is completed thus impressing a voltage of the same polarity as E on the relay K-1 coil circuit. Similarly, when relay K-2 or K-3 closes, a circuit through one of the six volt power supplies, R-1, R-2, and either R-3 or R-4 is completed thus impressing a voltage of polarity opposite to E on the relay K-1 coil circuit. It is obvious that as the relays open, the corresponding voltages are removed from the relay K-1 coil circuit.



Block Diagram of the Electromagnetic Underwater Log System

Figure 1

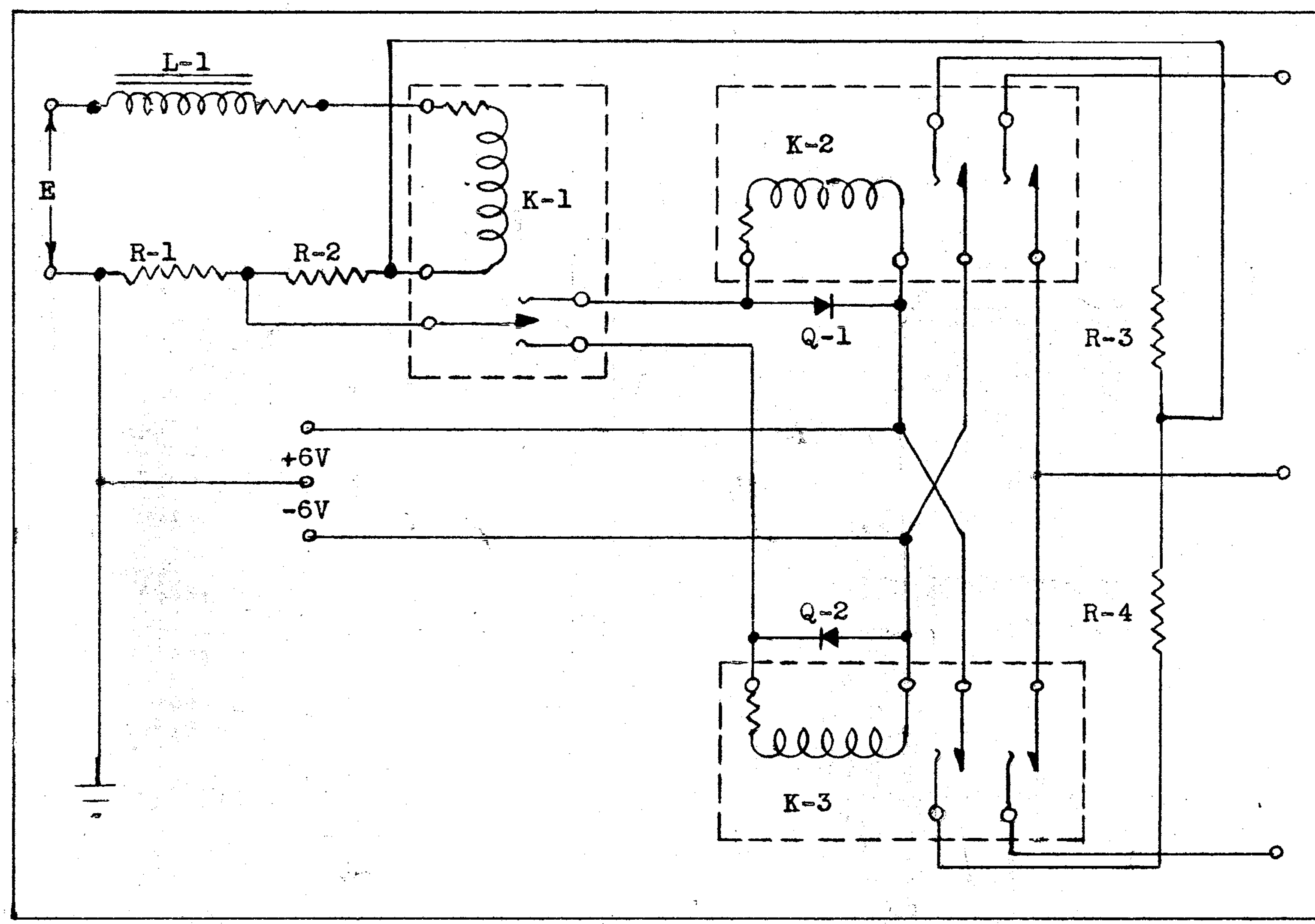


Diagram of the Relay Circuit

Figure 2

III. DEVELOPMENT OF FORMULAE

This section is devoted to representing the physical system by an equivalent mechanical system and an equivalent electrical circuit and to deriving formulae which will be required in analyzing the servomechanism. The mechanical system and the electrical circuit are treated separately in this section, and the solution of the problem consists in systematically combining the mechanical and electrical formulae in accordance with the conditions of the case being considered.

The Mechanical System.

The mechanical system can be represented and treated quite easily in the conventional manner. Let the system be represented by an input shaft and an output shaft driven by a reversible servomotor as shown in Figure 3, page 21. Let all of the system parameters be in a consistent set of units and be referred to the output shaft. Let the input angle be represented by θ_i', the output angle by θ_m', and the error angle by θ', where error is defined as $\theta' = \theta_i' - \theta_m'$. Let the time be denoted by t, and indicate differentation with respect to time by dots over the quantity differentiated. Let the output shaft have a moment of inertia I, a viscous damping coefficient c, and a Coulomb friction F. Let I and c be assumed constant. Let it be assumed that F has the following form:

$F = F_o$, for $\dot{\theta}_m'$ positive.

$F = 0$, for $\dot{\theta}_m'$ zero,

$F = -F_o$, for $\dot{\theta}_m'$ negative,

where F_o is assumed constant. Let it be assumed that the servomotor has a torque T of the following form:

$T = T_o$, for the servomotor turned on upscale,

$T = 0$, for the servomotor turned off,

$T = -T_o$, for the servomotor turned on downscale,

where T_o is assumed constant.

It follows from Newton's second law of motion that the differential equation of motion for the output is

$$(1) \qquad I\ddot{\theta}_m' = T - c\dot{\theta}_m' - F,$$

or

$$(2) \qquad I\ddot{\theta}_m' + c\dot{\theta}_m' = T - F.$$

The right side of this differential equation changes abruptly when either T or F changes, i.e., when the servomotor is turned on or off or $\dot{\theta}_m'$ changes from positive to zero, negative to zero, zero to positive, or zero to negative; otherwise it is constant. Thus this differential equation can be integrated giving an algebraic equation which is valid until either T or F changes in value. When this occurs the differential equation can be reintegrated using the physical point at which T or F changed sign as the initial condition for the new region. Integrating subject to constant T and F and the initial conditions $\dot{\theta}'_m = \dot{\theta}'_{m_o}$, $\theta'_m = \theta'_{m_o}$, at $t = t_o$,

$$(3) \qquad \dot{\theta}'_m = \frac{T - F}{c} + \left(\dot{\theta}'_{m_o} - \frac{T - F}{c}\right) \exp\left[-\frac{c}{I}(t - t_o)\right]$$

and

$$(4) \qquad \theta'_m = \theta'_{m_o} + \frac{T - F}{c}(t - t_o) + \frac{I}{c}\left(\dot{\theta}'_{m_o} - \frac{T - F}{c}\right)\left\{1 - \exp\left[-\frac{c}{I}(t - t_o)\right]\right\} .$$

Consider the case of the servomotor turned on upscale at $t_o = 0$ with $\dot{\theta}'_{m_o} = 0$ and $\theta'_{m_o} = 0$. $T = T_o$ and $F = F_o$ for this case. Equations (3) and (4) give

$$(5) \qquad \dot{\theta}'_m = \frac{T_o - F_o}{c}\left[1 - \exp\left(-\frac{c}{I}t\right)\right]$$

and

$$(6) \qquad \theta'_m = \frac{T_o - F_o}{c}\left\{t - \frac{I}{c}\left[1 - \exp\left(-\frac{c}{I}t\right)\right]\right\} , \; t \geq 0.$$

Consider the case of the servomotor turned off and allowed to coast to rest after it had attained its upscale steady-state velocity. Take the zeros of t and θ'_m at the time when and the position at which the servomotor is turned off. $T = 0$ and $F = F_o$ for this case. Equation (5) gives

$$(7) \qquad \dot{\theta}'_m = \frac{T_o - F_o}{c}$$

at the initial point. Thus the initial conditions are

$t_o = 0$, $\dot{\theta}'_{m_o} = \frac{T_o - F_o}{c}$, $\theta'_{m_o} = 0$. Equations (1) and (2) give

(8) $$\dot{\theta}'_m = -\frac{F_o}{c} + \frac{T_o}{c} \exp\left(-\frac{c}{I}t\right)$$

and

(9) $$\theta'_m = -\frac{F_o}{c}t + \frac{I}{c}\frac{T_o}{c}\left[1 - \exp\left(-\frac{c}{I}t\right)\right], \quad t \geq 0.$$

$\dot{\theta}'_m = 0$ gives

(10) $$t = \frac{I}{c} \ln \frac{T_o}{F_o}$$

as the time at which the output comes to rest. Equation (9) then gives

(11) $$\theta'_m = \frac{I}{c}\frac{T_o - F_o}{c} - \frac{F_o}{c} \ln \frac{T_o}{F_o}$$

as the distance at which the output comes to rest.

Consider the case where the servomotor is turned on upscale from rest at $t_o = 0$ and $\theta'_{m_o} = 0$, turned off a time t_d later, and allowed to coast to rest. Equations (5) and (6) give $\dot{\theta}'_m$ and θ'_m for $0 \leq t \leq t_d$. These equations with $t = t_d$ give

(12) $$\dot{\theta}'_{m_o} = \frac{T_o - F_o}{c}\left[1 - \exp\left(-\frac{c}{I}t_d\right)\right]$$

and

(13) $$\theta'_{m_o} = \frac{T_o - F_o}{c}t_d - \frac{I}{c}\frac{T_o - F_o}{c}\left[1 - \exp\left(-\frac{c}{I}t_d\right)\right]$$

as the initial conditions for the coasting period. Equations (3) and (4) with T = 0 and $F = F_o$ and these initial conditions give

$$(14) \qquad \dot{\theta}_m' = -\frac{F_o}{c} + \frac{T_o}{c} \exp\left[-\frac{c}{I}(t - t_d)\right] - \frac{T_o - F_o}{c} \exp\left(-\frac{c}{I} t\right)$$

and

$$(15) \qquad \theta_m' = -\frac{F_o}{c} t - \frac{I}{c}\frac{T_o}{c} \exp\left[-\frac{c}{I}(t - t_d)\right] + \frac{I}{c}\frac{T_o - F_o}{c} \exp\left(-\frac{c}{I} t\right) + \frac{T_o}{c} t_d + \frac{I}{c}\frac{F_o}{c},$$

$$t_d \leq t \leq t_s,$$

for the coasting period. $\dot{\theta}_m' = 0$ gives

$$(16) \qquad t_s = \frac{I}{c} \ln\left\{1 + \frac{T_o}{F_o}\left[\exp\left(\frac{c}{I} t_d\right) - 1\right]\right\}$$

as the time at which the output comes to rest. Equation (15) gives

$$(17) \qquad \theta_m' = \frac{T_o}{c} t_d - \frac{I}{c}\frac{F_o}{c} \ln\left\{1 + \frac{T_o}{F_o}\left[\exp\left(\frac{c}{I} t_d\right) - 1\right]\right\}$$

as the total distance traveled.

These general formulae will be used in the suceeding sections in analyzing the mechanical system of the electromagnetic underwater log system.

The Electrical Circuit.

The coil circuit of relay K-1 can obviously be represented electrically by the circuit shown in Figure 4, page 21. This circuit consists of a resistance R in series with an inductance L. The voltage applied to the circuit is denoted by E and the current through it by i. At the time $t = t_o$ when the current $i = i_o$, a voltage E is applied to the circuit. For any consistent set of units, the differential equation of the circuit is

$$(18) \qquad L \frac{di}{dt} + Ri = E, \; t \geq t_o.$$

Assume that R and L are constants in this circuit. Then this differential equation is easily integrated to give

$$(19) \qquad Ri = \frac{R}{L} \exp\left(-\frac{R}{L} t\right) \int E \exp\left(\frac{R}{L} t\right) dt + c \exp\left(-\frac{R}{L} t\right), \quad t \geq t_o,$$

where c is an arbitrary constant determined by the initial condition.

For $E = E_o$, E_o constant, this gives

$$(20) \qquad Ri = E_o + (Ri_o - E_o) \exp\left[-\frac{R}{L}(t - t_o)\right], \; t \geq t_o.$$

For $E = kt$, k constant, this gives

$$(21) \qquad Ri = k\left(t - \frac{L}{R}\right) + \left[Ri_o - k\left(t_o - \frac{L}{R}\right)\right] \exp\left[-\frac{R}{L}(t - t_o)\right], \quad t \geq t_o.$$

For $E = E_o \exp(-at)$, E_o and a constant, this gives

$$(22) \qquad Ri = E_o \frac{R}{L} \frac{\exp(-at_o)}{a - \frac{R}{L}} \left\{\exp\left[-\frac{R}{L}(t - t_o)\right] - \exp\left[-a(t - t_o)\right]\right\} + Ri_o \exp\left[-\frac{R}{L}(t - t_o)\right], \quad t \geq t_o.$$

Any error input to the error amplifier is amplified, rectified, and applied to the relay K-1 circuit. Assume that the ratio of the direct voltage applied to the relay circuit to the error voltage is a constant, say K. Then error voltages can be multiplied by the factor K and regarded as equivalent direct voltages applied directly to the relay circuit. Alternately, an equivalent relay circuit in which the circuit parameters are unchanged but all of the voltages and currents are divided by this factor K can be used. Thus the current through the relay coil and the voltages impressed when the relays are closed are divided by this factor K while the signal voltages are unchanged in magnitude. The first circuit has the advantage of being the actual condition while in the second circuit the magnitudes of the input and output voltages can be used directly as they appear. In any case, it is obvious that the two circuits are identical except for the constant factor K applied to all voltages and currents. Thus the computations for the two circuits are identical except for the factor K in the voltages and currents, yielding the same values for the quantities desired. In order to work with input and output voltages directly the second circuit has been chosen for the analysis.

Note that in equations (20), (21), and (22) the resistance R, the inductance L, and the current i of the relay circuit appear only in the combinations $\frac{R}{L}$ and Ri. Since these are the quantities actually measured for the servomechanism considered, they will be treated in these forms throughout the analysis. Thus instead of using i in the calculations, Ri will be determined. In the case of a similar analysis of a different servomechanism, Ri for the circuit considered may be determined and used, or i itself may be used. It is obvious that the two methods are the same.

The Determination of the Parameters of the Electromagnetic Underwater Log System.

The values of the parameters given in the Table of Servomechanism Constants, page 81, were obtained as outlined below. Since the output is referred to a shaft which is calibrated at one revolution per knot, 2π radians = 1 knot.

The moment of inertia I of the servomechanism was obtained by measuring the rotor of the servomotor, computing its inertia, and referring it to the output shaft. Because of the large stepdown ratios in the system, the moment of inertia of the servomechanism was taken as that of the servomotor.

The servomotor torque T_o, the Coulomb friction F_o, and the viscous damping coefficient c were obtained by securing distance versus time curves of the servomechanism for its steady state and coasting state, calculating these quantities, and referring them to the output shaft. The distance versus time curves were obtained by means of a double channel brush oscillograph. One pen was used as the time reference by placing it in series with the servomotor and one circuit of a double pole, single throw switch so that the pen motor circuit was energized when and only when the servomotor was excited. The position of the output was determined by means of a circuit through the other switch circuit, a battery, and a wire which made contact with the successive teeth of one of the gears of the output when the servomechanism was running. A number of runs were made and an average steady-state velocity and an average distance versus time curve for the coasting period were obtained. This distance versus time curve was used to obtain a velocity versus time curve for the coasting period. Equation (1) gives for the steady-state

(23) $$c\dot{\theta}_m + F_o = T_o.$$

For the coasting period equation (1) gives

(24) $$I\ddot{\theta}_m + c\dot{\theta}_m + F_o = 0.$$

Using the steady-state velocity of the servomechanism in equation (23), the velocity and acceleration from two points of the velocity versus time curve for the coasting period in equation (24), and the computed moment of inertia, T_o, F_o, and c can be calculated.

The positive and negative voltages applied when the relays close were computed from the circuitry.

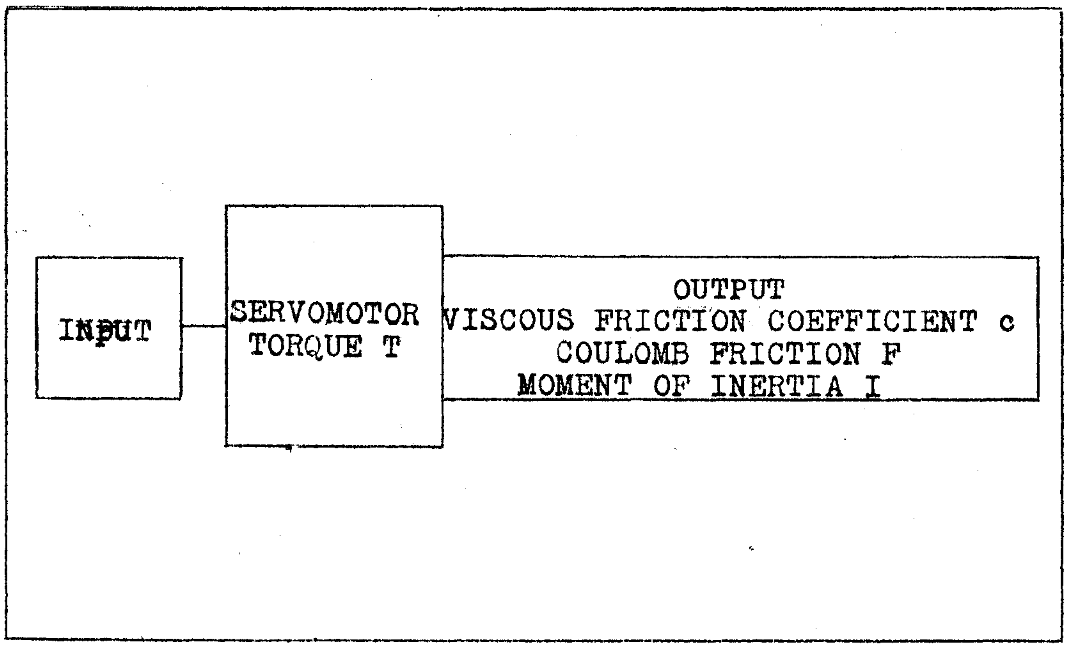
The ratio of the output from the phase sensitive rectifier to the input to the error amplifier which caused it was computed by measuring the input to the error amplifier which was necessary to just compensate for the excess of negative over positive voltage introduced when the relays close. The ratio of this excess, which was already computed, to the voltage measured gave the overall gain from the error input to the relay circuit. This value was then used to convert the relay circuit voltages and currents to the equivalent circuit values.

The various constants of the relay circuits were determined by means of a double channel brush oscillograph. One of the pen motor circuits was connected in series with a single pole single throw switch, a battery, a variable resistance, and a standard resistance. The standard resistance was placed across the input to the relay circuit, and the variable resistance was used to vary the voltage input to the relay circuit. The gain factor of the error amplifier and the phase sensitive rectifier was used to refer the input voltages to their equivalent error voltages. This circuit gave the times when the signal was applied to and removed from the relay circuit. The other pen motor was placed across resistances R-1 and R-2 in Figure 2, thus giving the voltages applied and removed when the relays closed and opened. Oscillograms of these voltages were made for various step input voltages. The ratio of resistance to inductance for the relay K-1 circuit was measured by noting the times necessary for the relay to make first contact from rest at its equilibrium position for these step voltages. The equation of the circuit lumping the mechanical with the electrical parameters and assuming all parameters constant is equation (20). This can be put in the form

$$(25) \qquad \ln\left(1 - \frac{Ri_c}{E}\right) = -\frac{R}{L} t_c,$$

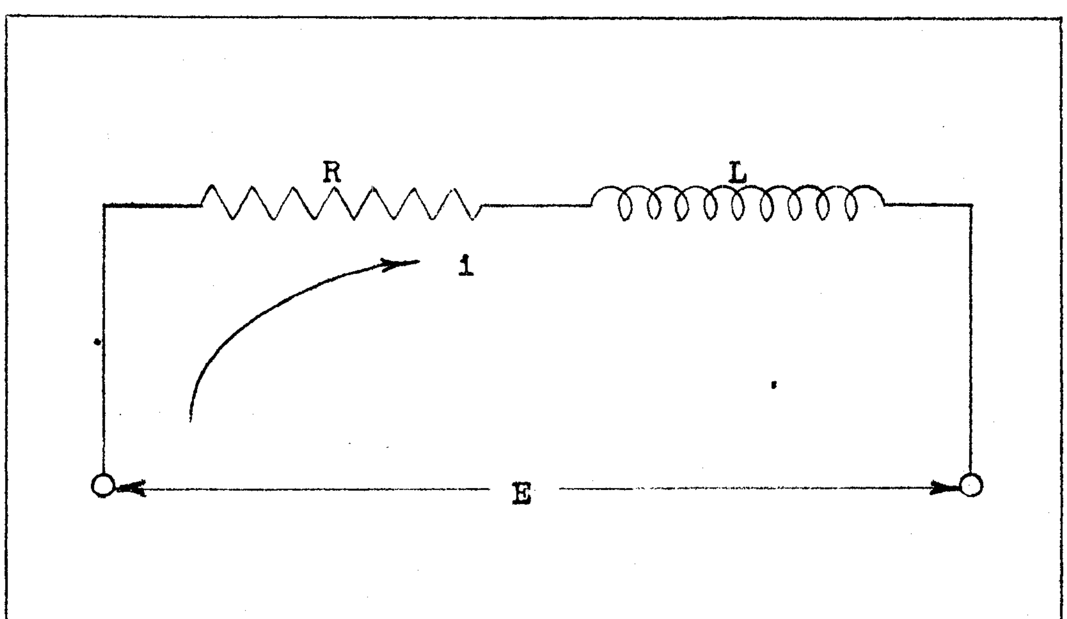
where i_c is the equivalent current at which the relay will close in an infinite time and t_c is the time necessary for the relay K-1 to make first contact after the voltage E has been applied. Ri_c was measured directly as the voltage which just closed the relay. A plot of $\ln\left(1 - \frac{Ri_c}{E}\right)$ versus t_c gave a straight line of slope $-\frac{R}{L}$. These brush recordings also gave t_{d_2}, t_{d_3}, and t_{d_4}. They suggested the assumption that the positive applied voltage be regarded as a step voltage applied when the relay K-2 or K-3 closes. They also verified the assumptions that the negative applied voltage is applied as a step function when relay K-2 or K-3 closes and that the voltages are removed as soon as the relays are opened.

The signal strength V was determined by calibrating the system at the David W. Taylor Model Basin, Carderock, Maryland.



Equivalent Mechanical System

Figure 3



Equivalent Electrical Circuit

Figure 4

IV. METHOD OF SOLUTION

This section is devoted to deriving a method of analyzing this type of servomechanism based on the results of the previous section. Because of the complexity of carrying through the analysis algebraically a semi-graphical method will be developed using the electromagnetic underwater log system as an example. The use of this method in analyzing other similar servomechanisms will be obvious from the example considered.

The specific problem for which a solution is desired is the response of the servomechanism for constant accelerations of the ship from rest. This means a constant input rate to the servomechanism. Consider the problem which is to be solved. A linear input is applied to the servomechanism. Thus a linearly increasing error producing a linearly increasing error voltage is established. This error voltage is amplified, rectified, and applied to the relay K-1 circuit. Since the circuit is inductive, its current lags the applied voltage. However it finally reaches the value necessary to close relay K-1 which closes relay K-2 starting the servomotor upscale. From then on the servomotor may be continually on or alternately on and off depending on the input rate. The problem is to find these on and off periods for various input rates.

Whether the servomotor is running or not, and if running, in which direction, depends on the motor control relays K-2 and K-3 which are controlled by the polarized relay K-1. As stated previously, the position of the moving contact of this relay depends on the current through its coil. Hence any analysis must determine the current through this coil as a function of time as well as determine the output position as a function of time. Since this circuit is highly inductive, the current lags far behind the voltage applied to it. In addition, the voltages applied and removed when the various relays open and close add further complications to the analysis. Since carrying along a simultaneous determination of the output position and the coil current as functions of time would be exceedingly laborious, a method of approximation will be developed. The basis of this method lies in the fact that certain events occur repeatedly, either exactly or approximately. Equations of the output motion and the coil current for each of these events can be developed in advance, plotted, and used whenever these events occur in a particular problem.

Let the following definitions and assumptions regarding the various time delays in the relay circuit be made. Let the relay K-1 make first contact a time t_{d_1} after the input has been applied. Let the relay K-1 close a time t_{d_2} later. Let the relays K-2 and K-3 close a time t_{d_3} after relay K-1 closes. Let relays K-2 and K-3 open a time t_{d_4} after relay K-1 opens. Assume that t_{d_2}, t_{d_3}, and t_{d_4} are constants.

Since the input and output for the electromagnetic underwater log system are actually calibrated in knots, define θ_i as the input in knots, θ_m as the output in knots, and θ as the error in knots, where $\theta = \theta_i - \theta_m$. These units will be used in all of the following calculations. Cgs units will be used for all of the other mechanical units and time. Resistance will be in ohms, inductance in henries, voltage in microvolts, and current in microamperes. Since the output is a shaft which is calibrated at one knot per revolution, 2π radians = 1 knot.

Then the events which occur often enough to be considered in detail are:

1. The application of a linear input.
2. Relay K-1 making first contact and closing firmly a time t_{d2} later, followed by either relay K-2 or K-3 closing an additional time t_{d3} later.
3. Relay K-1 opening followed by relay K-2 or K-3 opening a time t_{d4} later.
4. Relay K-1 closing while the output is stationary, relay K-2 or K-3 closing a time t_{d3} later, relay K-1 opening almost immediately, relay K-2 or K-3 opening a time t_{d4} later, and the output coasting to rest.
5. The output accelerating from rest.
6. The output coasting from its steady-state running condition.

Consider the six events listed on page 24.

1. Consider the current due to the linearly increasing input starting from $\theta_i = 0$ knot at $t = 0$ second. This is given by equation (21) with $k = V\omega$ and the constants for the servomechanism under consideration.

$$(26) \qquad \frac{Ri}{V\omega} = t - 0.313 \left[1 - \exp\left(- 3.2\ t\right) \right] \text{ seconds,}$$

$$t \geq 0 \text{ second.}$$

This curve is plotted in Figures 5 and 6, pages 32 and 33.

2. Consider relay K-1 making first contact at $t = -0.05$ second and closing firmly 0.02 second later, followed by relay K-2 closing 0.03 second later. Thus $E_P = 305$ microvolts is impressed at $t = -0.03$ second; $E_N = -400$ microvolts is impressed at $t = 0$ second. However, E_P is impressed through an inductive circuit as shown in Figure 2, page 8. The simplifying assumption that E_P is applied as a step function at the time $t = 0$ second is made. This assumption is justified by brush oscillograph recordings. E_N is applied through a resistive circuit so it can be regarded as a step function applied at $t = 0$ second. Thus equation (20) gives

$$(27) \qquad Ri = - 95 \left[1 - \exp\left(- 3.2\ t\right) \right] \text{ microvolts,}$$

$$t \geq 0 \text{ second.}$$

This curve is plotted in Figures 7 and 8, pages 34 and 35.

3. Consider relay K-1 opening at $t = -0.03$ second removing $E_P = 305$ mocrovolts and followed by relay K-2 opening at $t = 0$ second removing $E_N = -400$ microvolts. Equation (20) gives

$$Ri = -305\left\{1 - \exp\left[-3.2\,(t + 0.03)\right]\right\}$$
$$+ 400\left[1 - \exp(-3.2\,t)\right] \text{ microvolts},$$
$$t \geq 0 \text{ second},$$

or

$$(28) \qquad Ri = 95 - 123 \exp(-3.2\,t) \text{ microvolts},$$
$$t \geq 0 \text{ second}.$$

This curve is plotted in Figures 9 and 10, pages 36 and 37.

4. Consider relay K-1 closing at $t = -0.03$ second while the output is stationary. Relay K-2 closes at $t = 0$ second, and -95 microvolts is considered as being impressed at this time. Relay K-1 opens at $t = 0.01$ second impressing $-E_P = -305$ microvolts, and relay K-2 opens at $t = 0.04$ second impressing $-E_N = 400$ microvolts. Refer to this event as a pulse. Equation (20) gives

$$Ri = -95\left[1 - \exp(-3.2\,t)\right]$$
$$-305\left\{1 - \exp\left[-3.2\,(t - 0.01)\right]\right\}$$
$$+400\left\{1 - \exp\left[-3.2\,(t - 0.04)\right]\right\} \text{ microvolts},$$
$$t \geq 0 \text{ second},$$

or

(29) $$Ri = -44.7 \exp(-3.2\,t) \text{ microvolts}, \quad t \geq 0 \text{ second.}$$

Equation (17) converted to knots with $t_d = 0.04$ second gives $\theta_m = 0.012$ knot as the distance advanced by the output during the pulse. Assume that this distance is linearly distributed over the time $t = 0$ second to $t = 0.08$ second when calculating the output position. This advance in the output will add an output voltage of $E = -V\theta_m = -(546)(0.012) = -6.55$ microvolts. Since this is a comparatively small voltage, for convenience assume that it is applied as a step function at the time $t = 0.04$ second. Equation (20) gives

(30) $$Ri = -6.55\left\{1 - \exp\left[-3.2\,(t - 0.04)\right]\right\} \text{ microvolts}, \quad t \geq 0.04 \text{ second.}$$

Thus the current due to a pulse for which the servomotor is turned on at $t = 0$ second is

(31) $$Ri = -\left[6.55 + 37.3 \exp(-3.2\,t)\right] \text{ microvolts}, \quad t \geq 0.04 \text{ second.}$$

This curve is plotted in Figures 11 and 12, pages 38 and 39.

5. Consider the output accelerating from rest starting at $t = 0$ second and $\theta_m = 0$ knot. Equation (6) converted to knots gives

(32) $$\theta_m = 0.445\ t - 0.0852 \left[1 - \exp\ (-\ 5.21\ t)\right] \text{ knots,}$$

$$t \geq 0 \text{ second.}$$

$E = -\ V\theta_m$ gives

(33) $$E = -\ 243\ t + 46.6 \left[1 - \exp\ (-\ 5.21\ t)\right] \text{ microvolts,}$$

$$t \geq 0 \text{ second.}$$

Equations (20), (21), and (22) give

(34) $$Ri = 122.4 - 243\ t - 196.6 \exp\ (-\ 3.2\ t)$$

$$+\ 74.2 \exp\ (-\ 5.21\ t) \text{ microvolts, } t \geq 0 \text{ second.}$$

This curve is plotted in Figure 13, page 40.

6. Consider the servomotor turned off at $t = 0$ second and $\theta_m = 0$ knot after it has reached its steady-state up-scale velocity and allowed to coast to rest. Equations (9), (10), and (11) give

(35) $$\theta_m = -\ 0.0169\ t + 0.0885 \left[1 - \exp\ (-\ 5.21\ t)\right],$$

$$\text{knot, } 0 \leq t \leq 0.635 \text{ second,}$$

and

(36) $\theta_m = 0.075$ knot, $t \geq 0.635$ second.

$E = - V\theta_m$ gives

(37) $$E = 9.24\ t - 48.5 \left[1 - \exp\ (- 5.21\ t)\right] \text{ microvolts,}$$

$$0 \leq t \leq 0.635 \text{ second,}$$

and

(38) $E = - 40.8$ microvolts, $t \geq 0.635$ second.

Equations (20), (21), and (22) give

(39) $$Ri = - 51.4 + 9.24\ t + 128.6 \exp\ (- 3.2\ t)$$

$$- 77.2 \exp\ (- 5.21\ t) \text{ microvolts,}$$

$$0 \leq t \leq 0.635 \text{ second,}$$

and

(40) $$Ri = - 40.8 + 9.3 \exp \left[- 3.2\ (t - 0.635)\right] \text{ micro-}$$

$$\text{volts, } t \geq 0.635 \text{ second.}$$

The Ri curve is given in Figures 14 and 15, pages 41 and 42.

Finally, consider the output after it has been running long enough for the current due to the output to have reached its steady-state condition. When the servomotor is turned off, it is convenient to consider the currents due to the output voltages introduced before and after the servomotor is turned off separately. The current due to the output voltage introduced after the servomotor has been turned off has already been treated on pages 28 and 29. The current due to the output voltage introduced before the servomotor is turned off will continue to rise and will approach a constant value after the servomotor is turned off since the relay circuit is inductive. The current at the time the servomotor is turned off will depend on the length of time the servomotor has been running, but the current added to this initial value will always be the same. Consider this current separately and later superimpose it on the initial value of current. For this steady state, equations (33) and (34) give E = (- 243 t + 46.6) microvolts and Ri = (- 243 t + 122.4) microvolts. Thus E - Ri = - 75.8 microvolts. Take the time at which the servomotor is turned off as t = 0 second. This condition amounts to a step voltage of - 75.8 microvolts applied at t = 0 second. Equation (20) gives

$$(41) \qquad Ri = -75.8 \left[1 - \exp(-3.2\ t)\right] \text{ microvolts,}$$

$$t \geq 0 \text{ second.}$$

This curve is plotted in Figures 16 and 17, pages 43 and 44.

The next four sections will be devoted to the solution of four specific problems illustrating the four main cases which result from this type of input. These are:

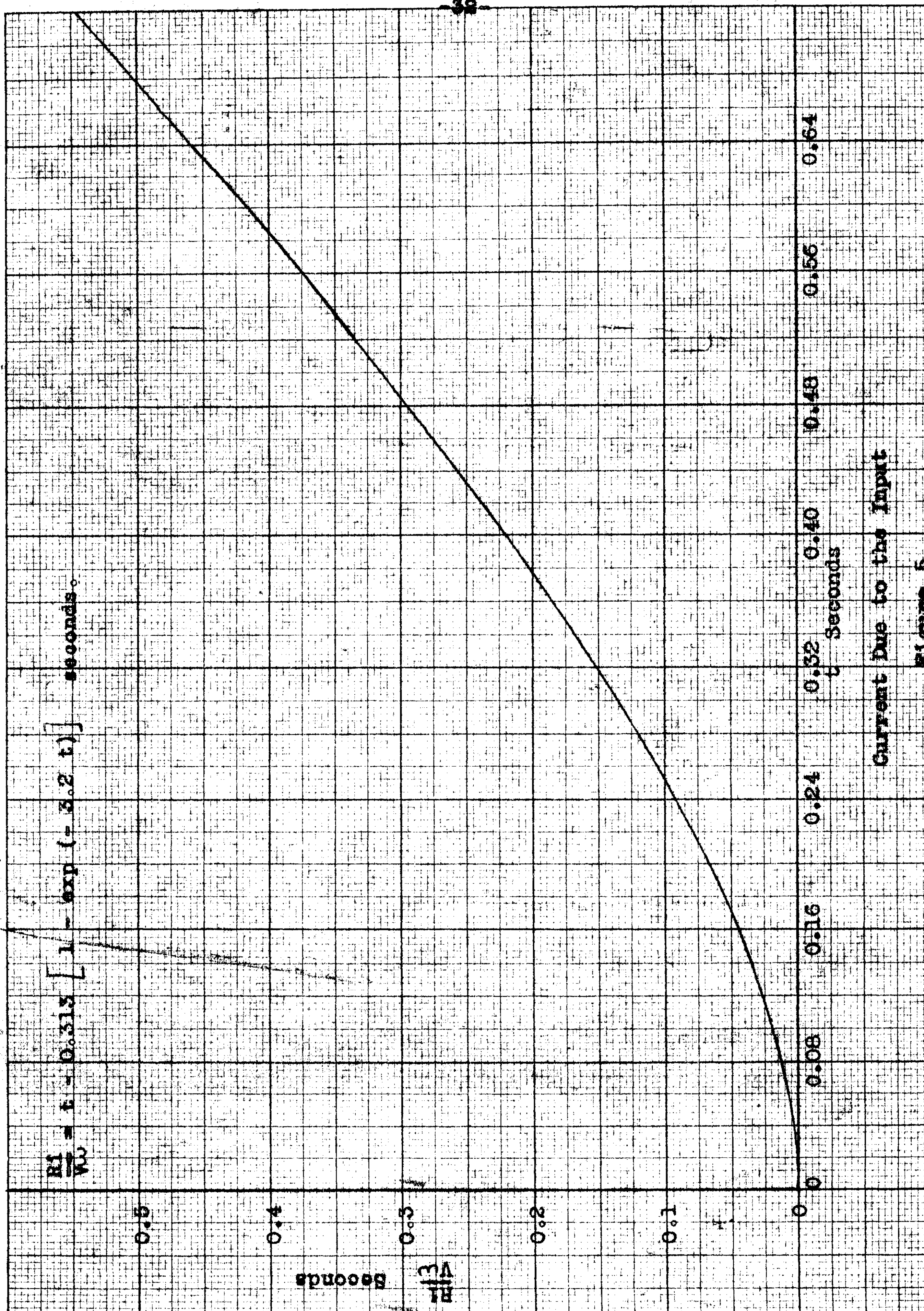
1. Input rate much less than the maximum output rate.

2. Input rate slightly less than the maximum output rate.

3. Input rate equal to the maximum output rate.

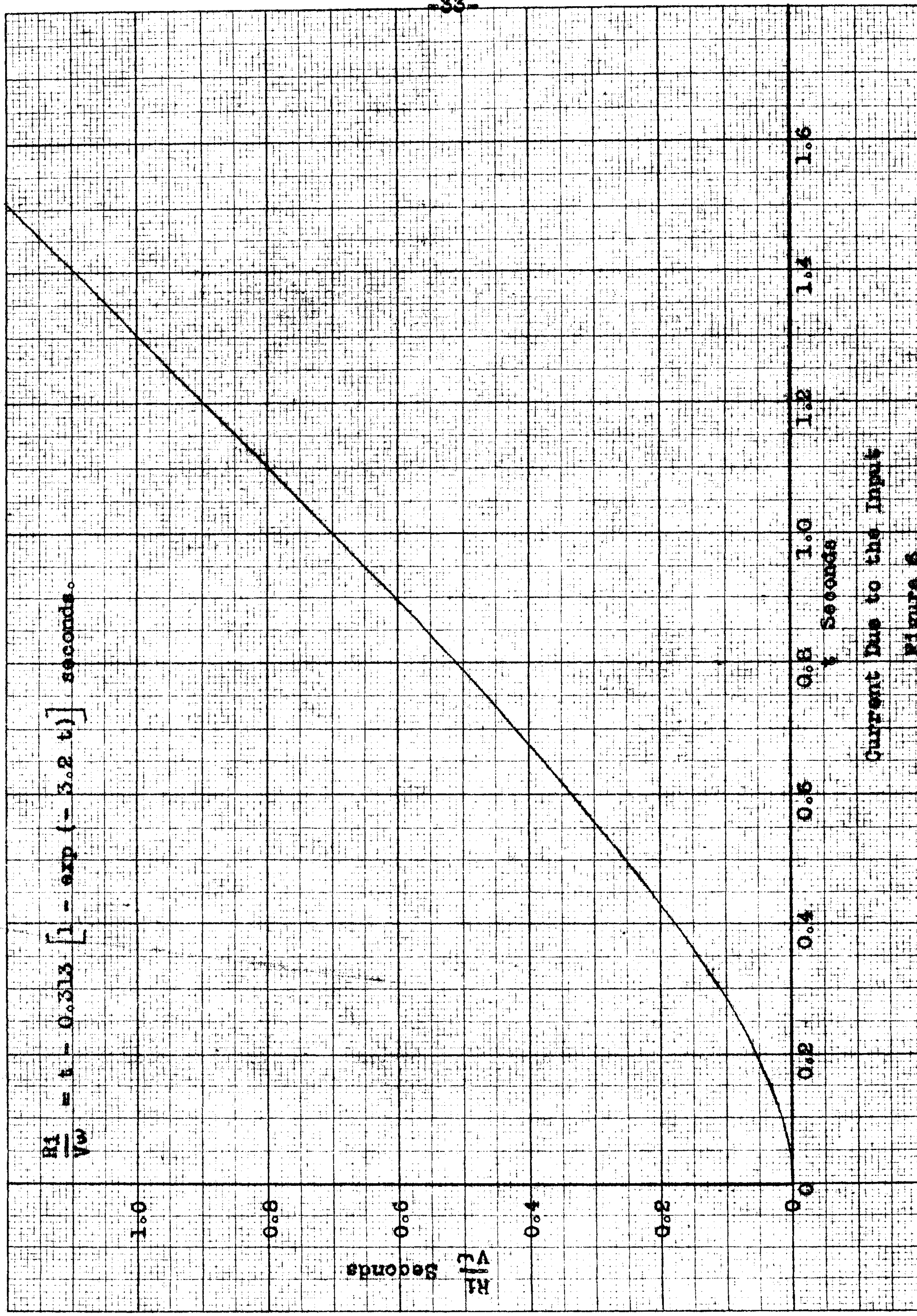
4. Input rate greater than the maximum output rate.

The solution of the problem will consist of determining the currents contributed to the total relay coil current by all of the separate occurrences and linearly superimposing these currents to determine the operation of the relay circuit for the immediate future. Whenever the events developed in this section occur, their contributions to the total relay coil current will be treated graphically thus greatly simplifying the solution.



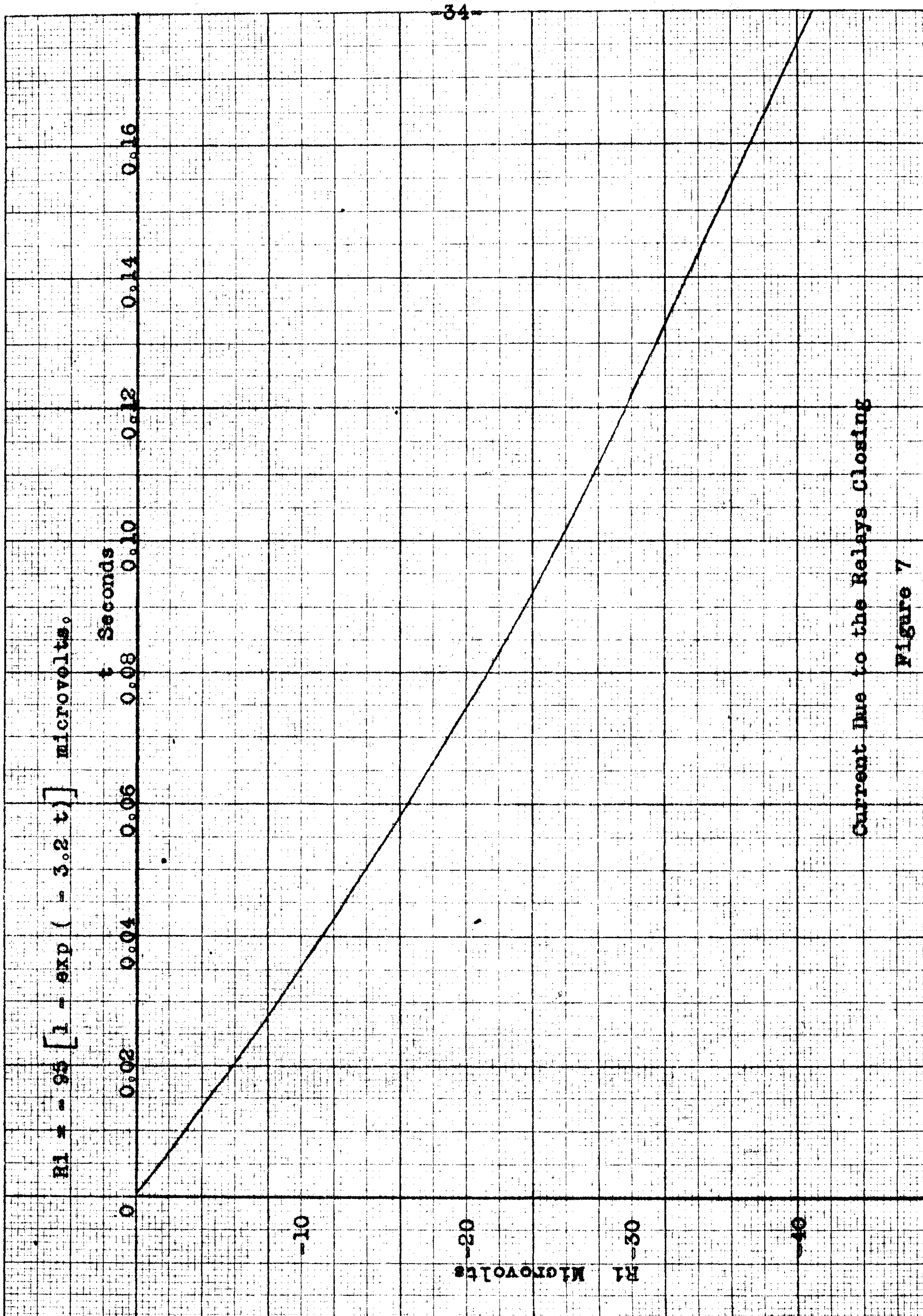
Current Due to the Input

Figure 5



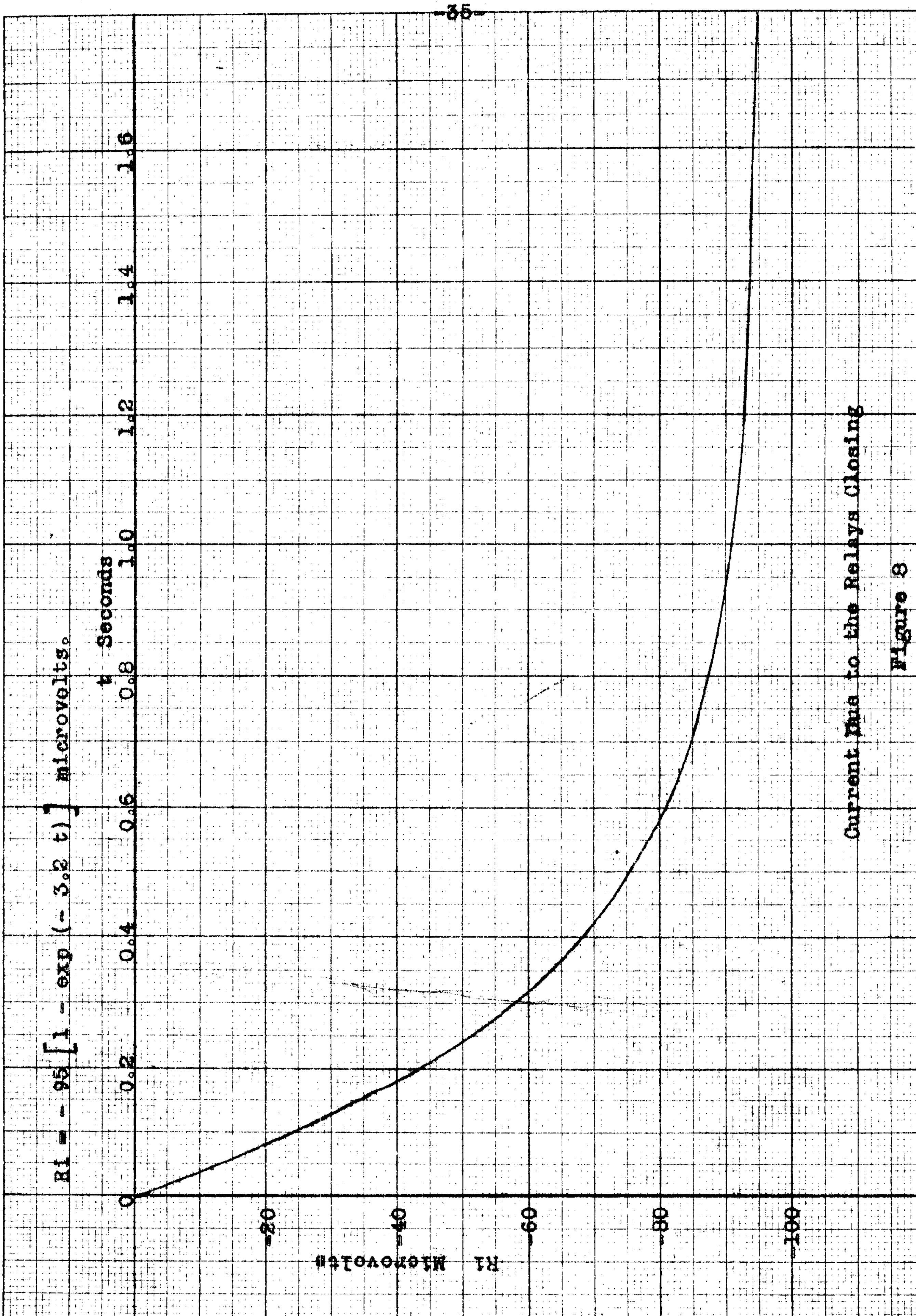
Current Due to the Input

Figure 6



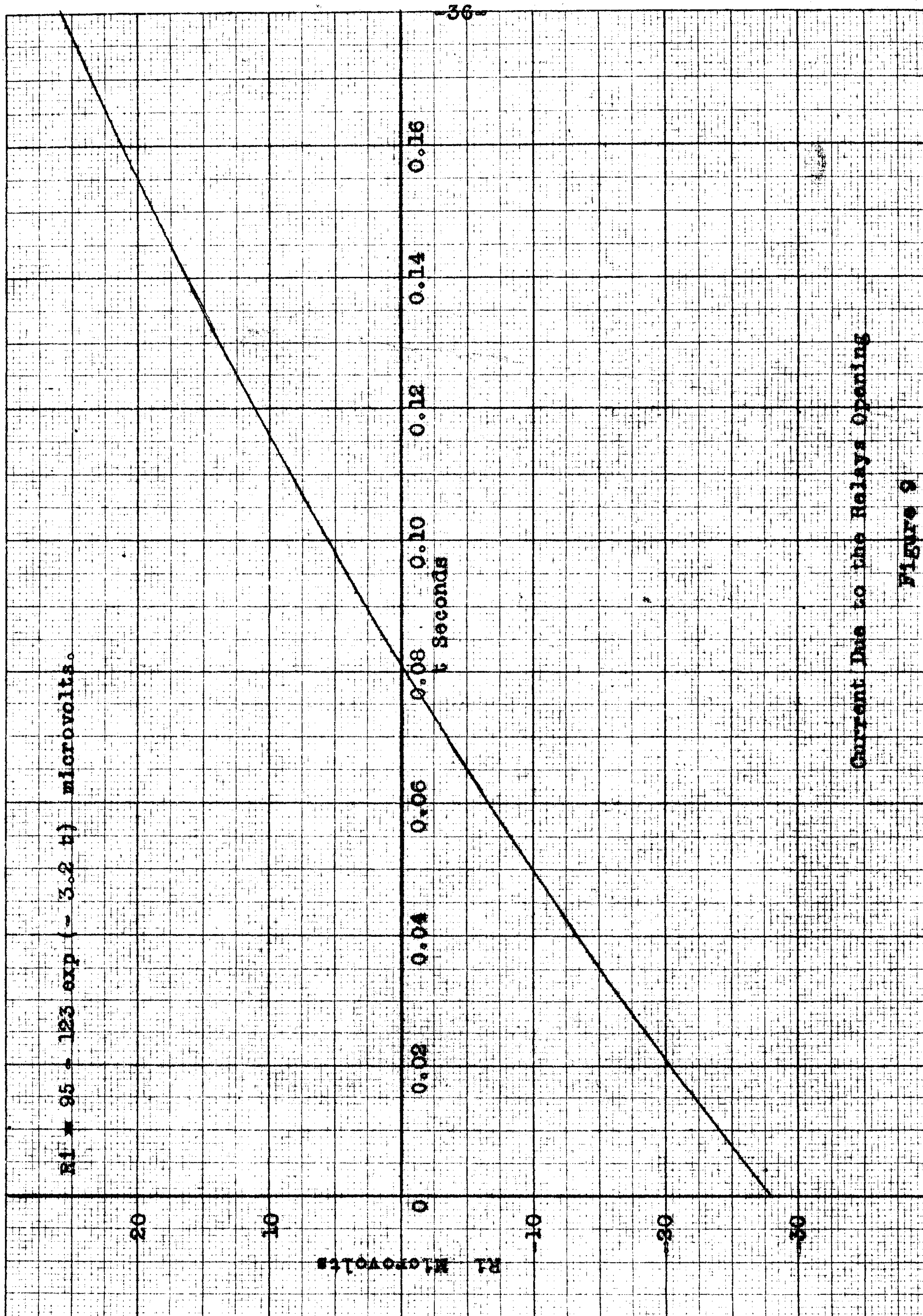
Current Due to the Relays Closing

Figure 7



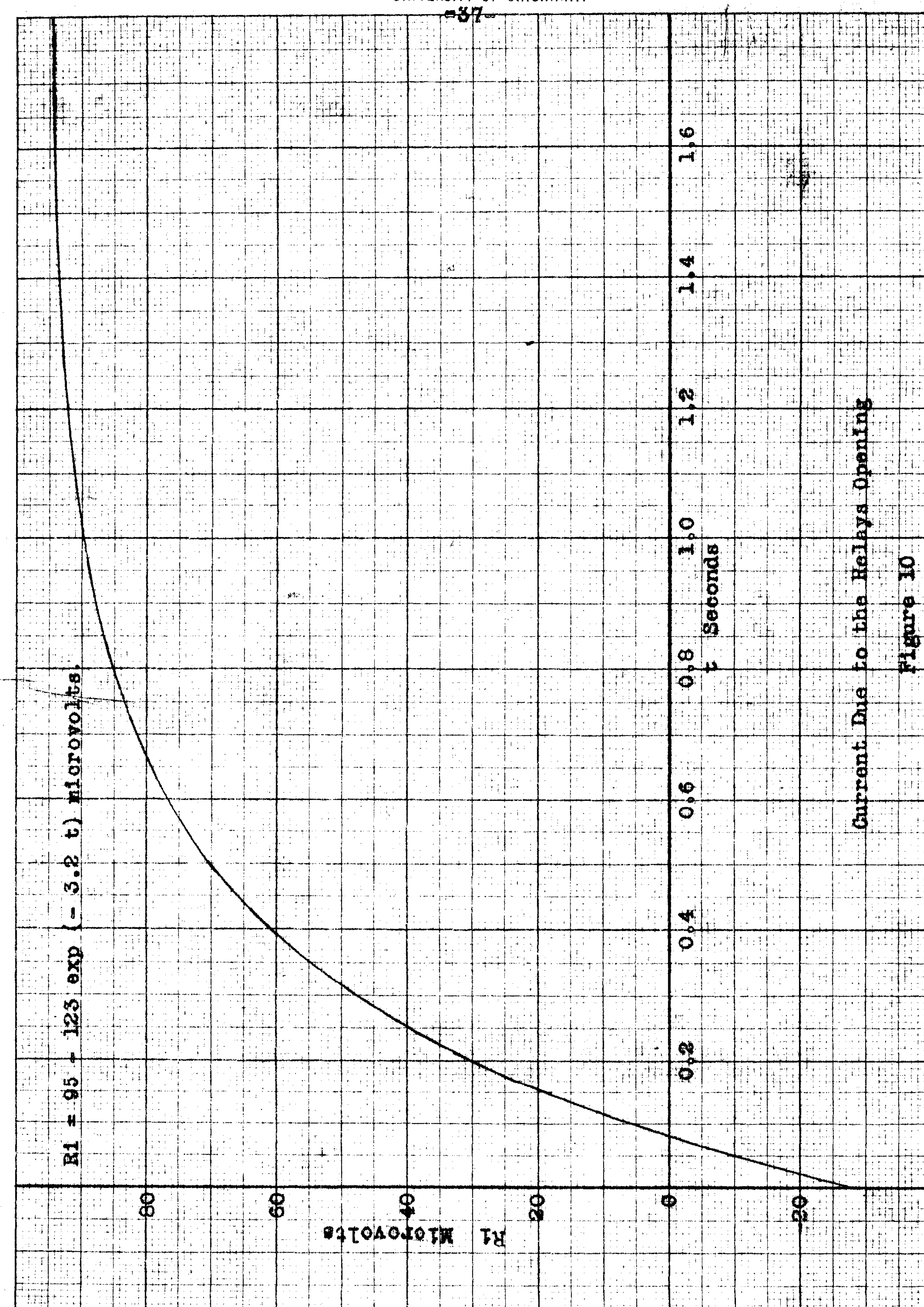
Current Due to the Relays Closing

Figure 8



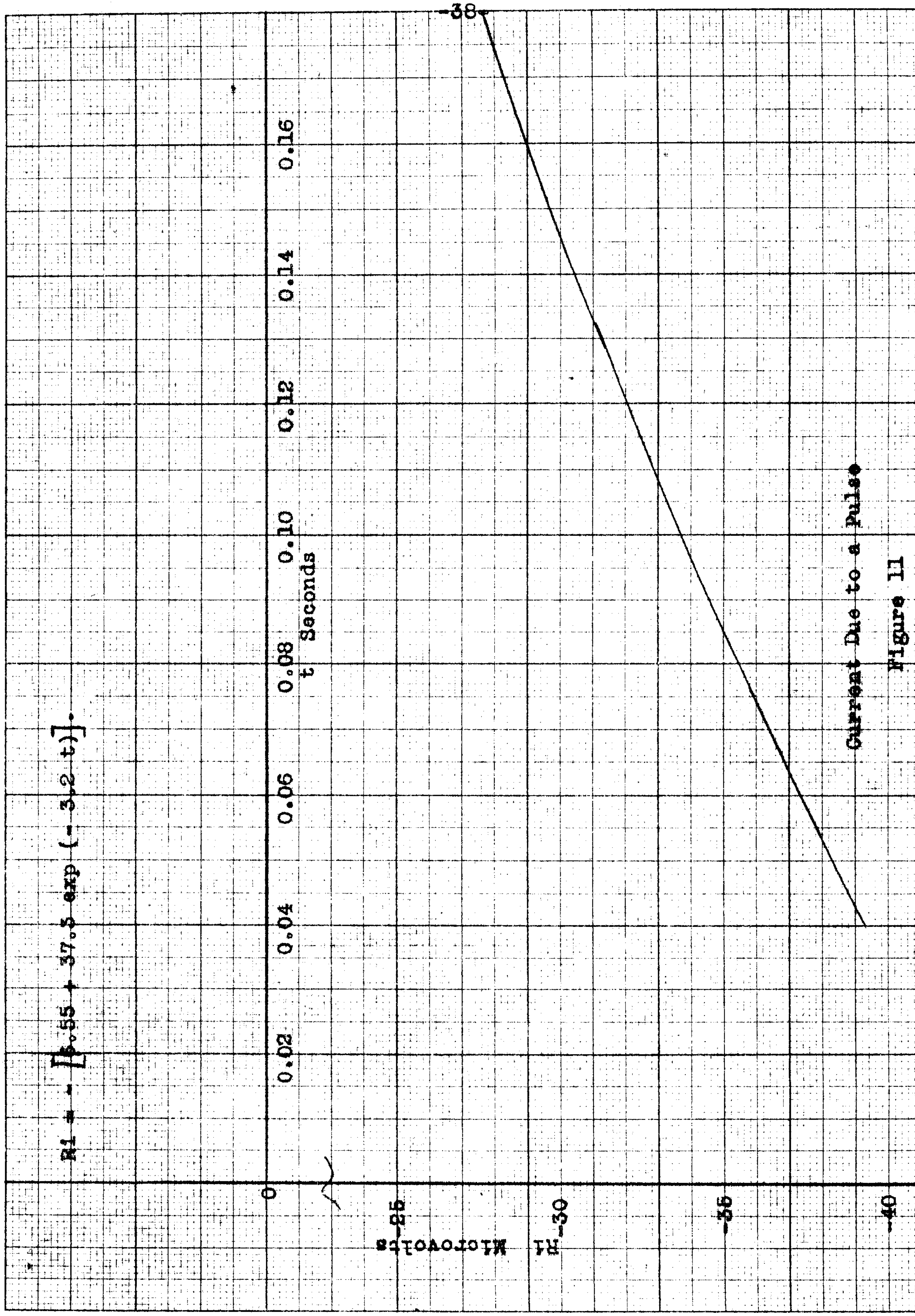
Current Due to the Relays Opening

Figure 9



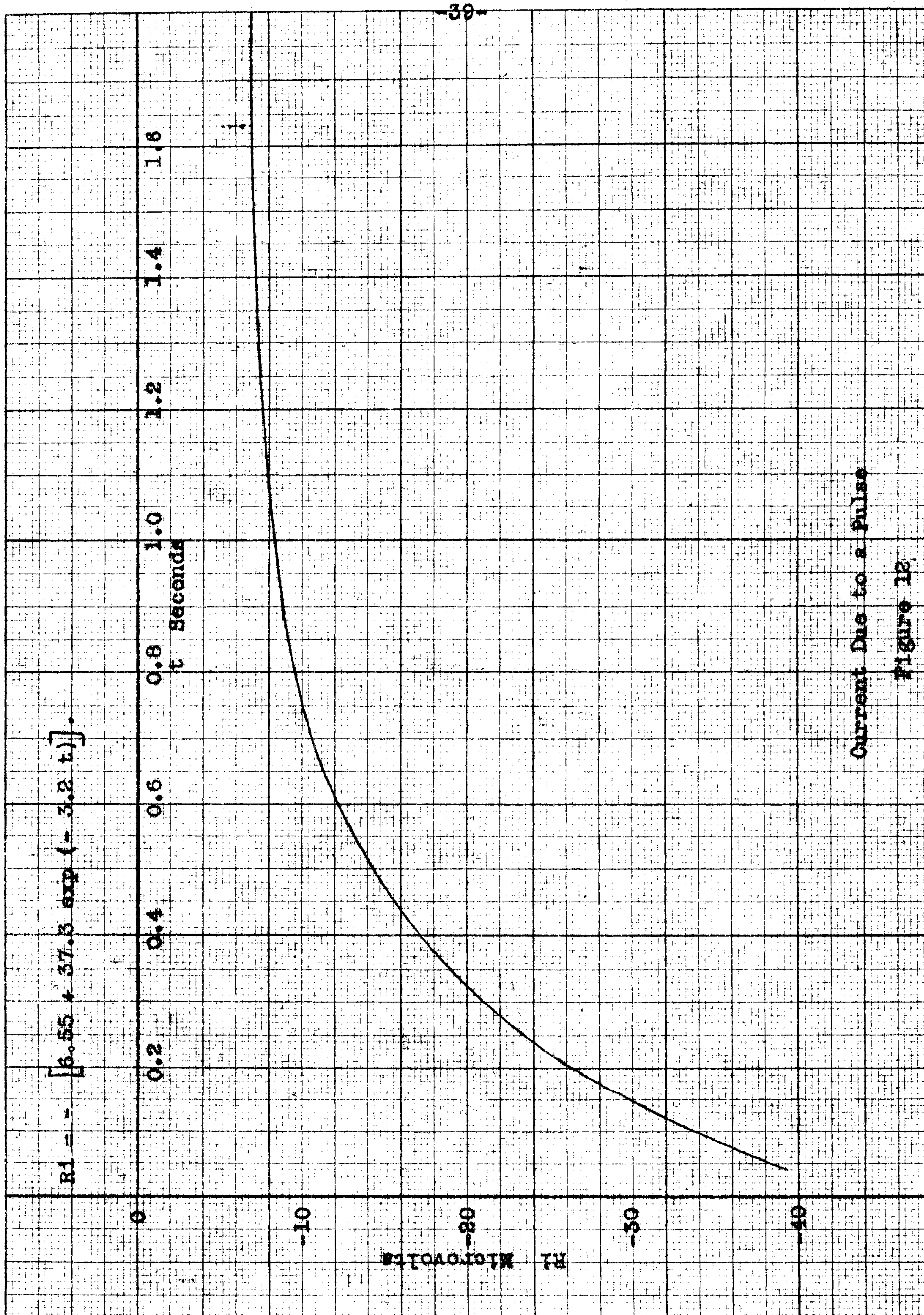
Current Due to the Relays Opening

Figure 10



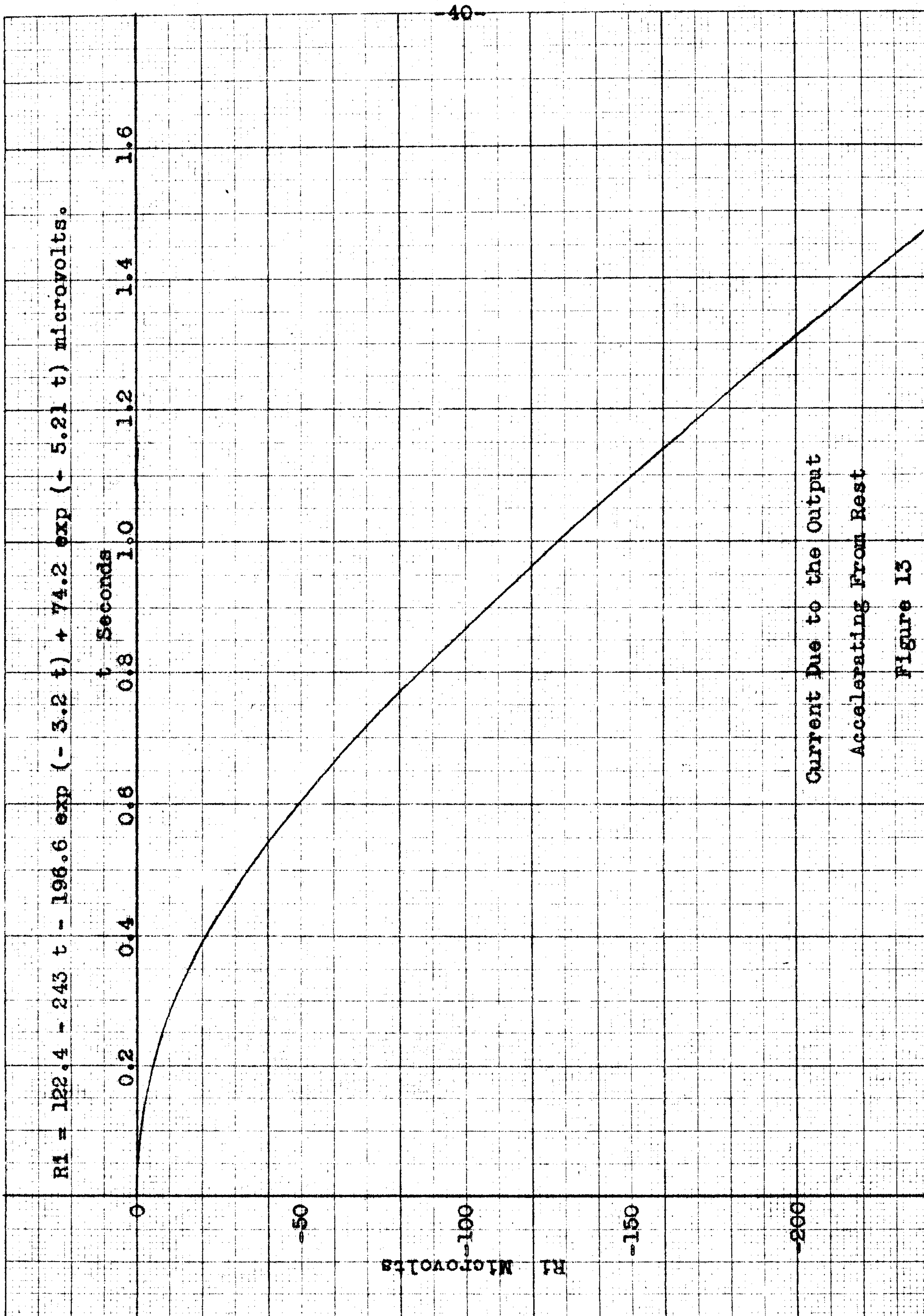
Current Due to a Pulse

Figure 11

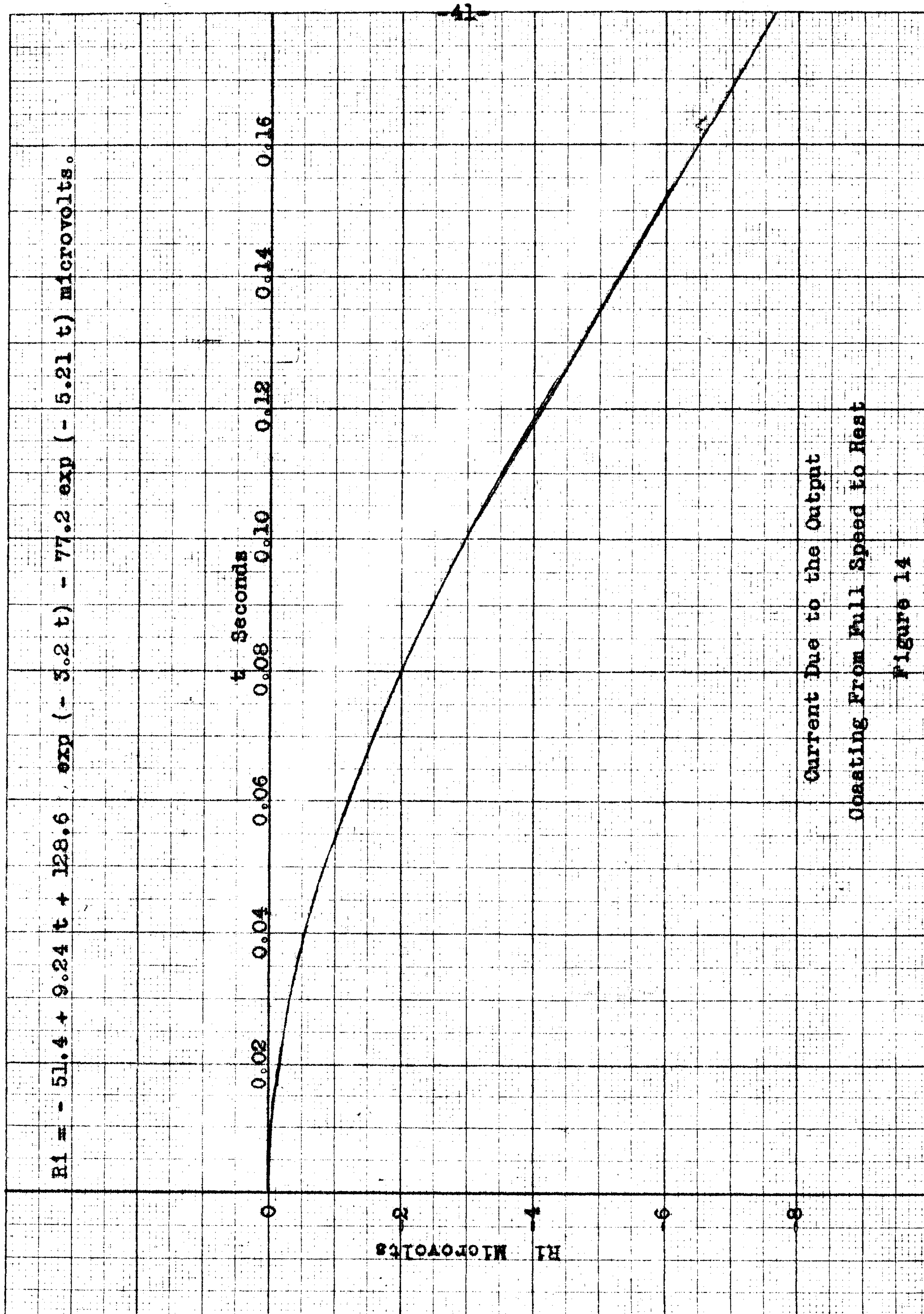


Current Due to a Pulse

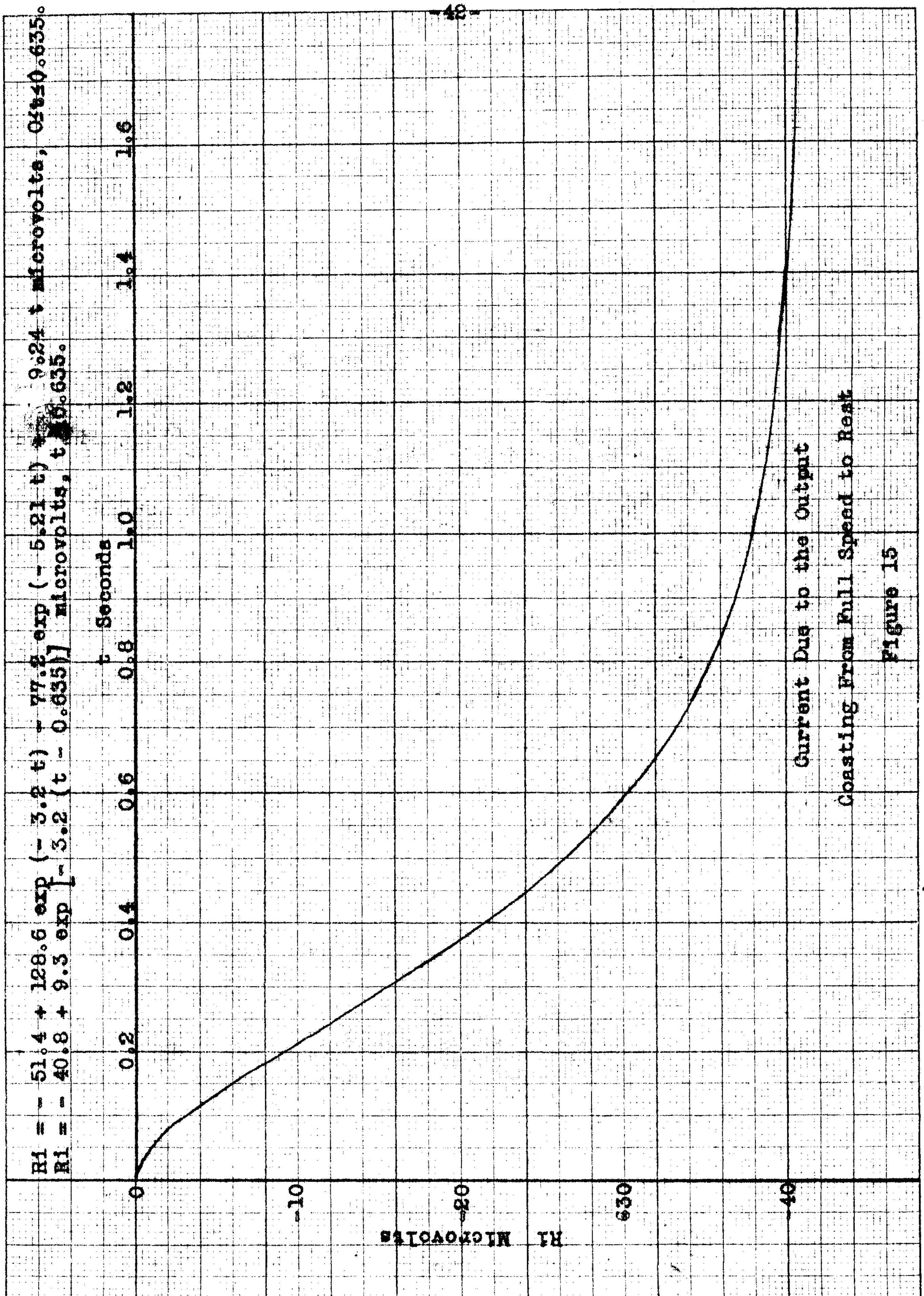
Figure 12



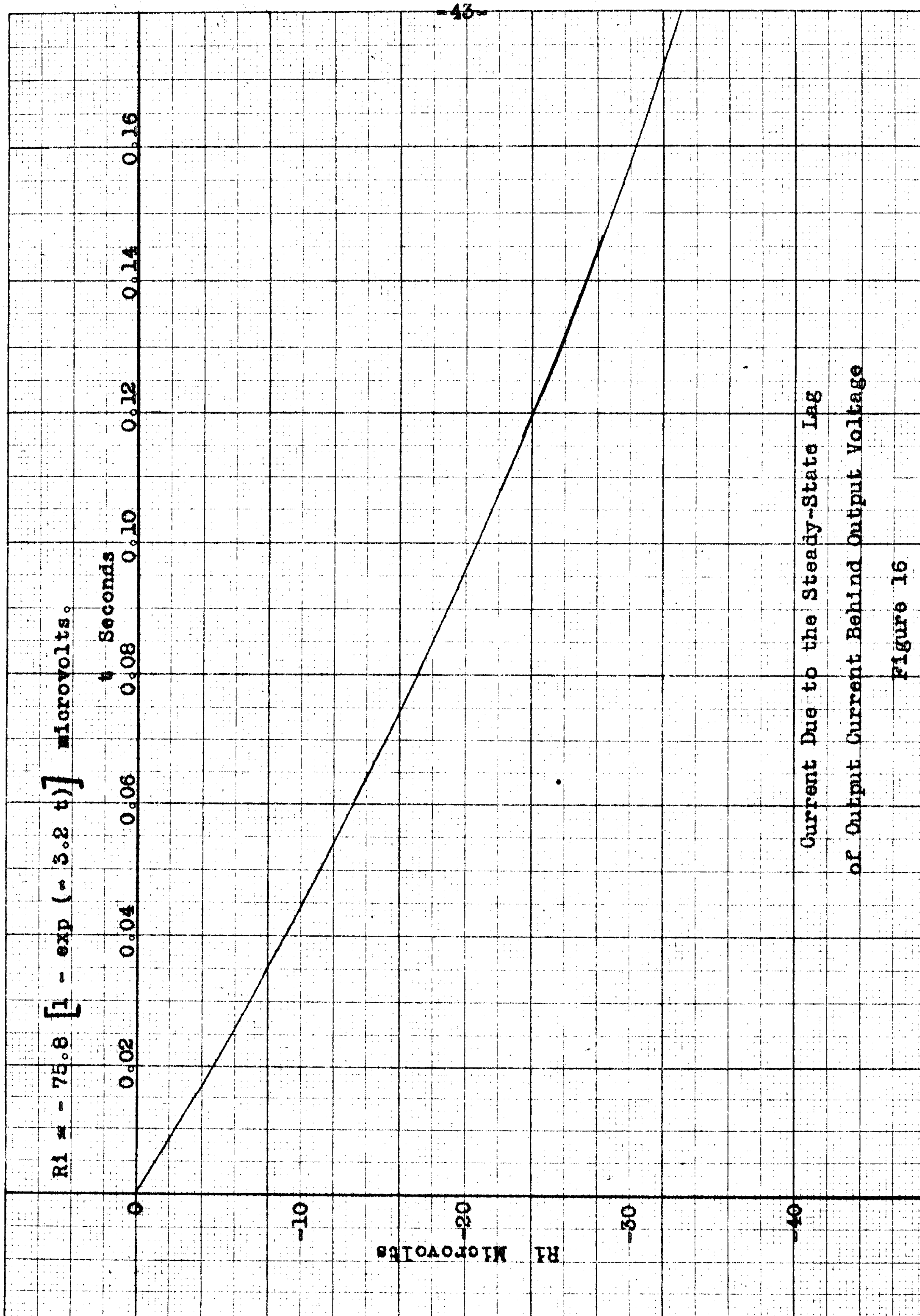
Current Due to the Output
Accelerating From Rest
Figure 13



Current Due to the Output
Coasting From Full Speed to Rest
Figure 14

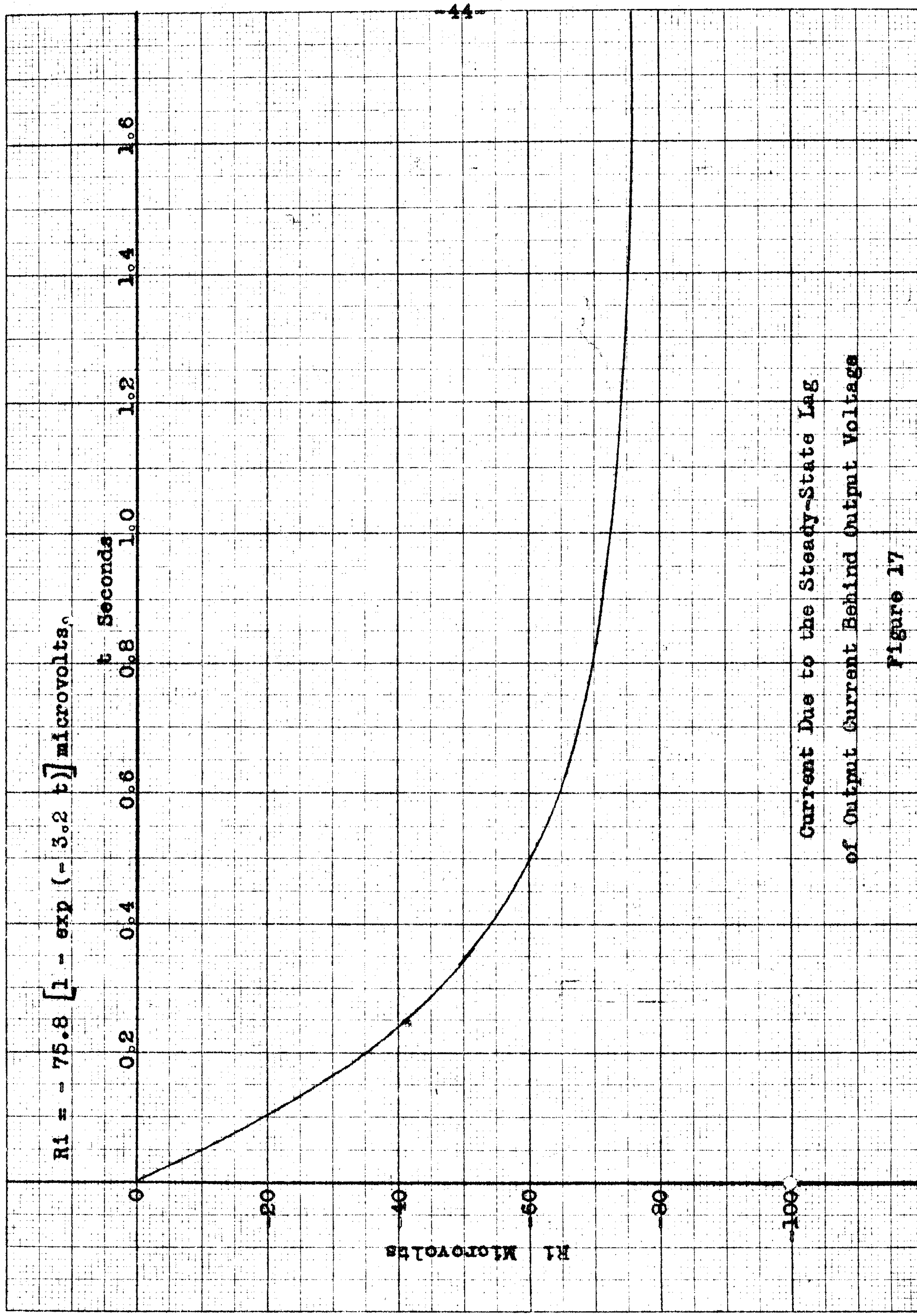


Current Due to the Output
Coasting From Full Speed to Rest
Figure 15



Current Due to the Steady-State Lag of Output Current Behind Output Voltage

Figure 16



Current Due to the Steady-State Lag of Output Current Behind Output Voltage

Figure 17

V. CASE OF THE INPUT RATE MUCH LESS THAN THE MAXIMUM OUTPUT RATE

The maximum output rate of the servomechanism considered is 0.445 knot per second. This section will deal with the response of the servomechanism to an input rate of 0.1 knot per second. It will be assumed that whenever the output receives a pulse that it is at rest. In addition, whenever the servomotor is running for less than 0.08 second, this event will be treated as a pulse.

$V\omega = (0.1)\ (546) = 54.6$ microvolts per second.

Relay K-1 makes first contact when the Ri of the equivalent relay K-1 coil circuit is 10 microvolts.

$$\frac{Ri}{V\omega} = \frac{10}{54.6} = 0.183 \text{ second.}$$

Figure 5 gives $t_{d_1} = 0.41$ second.

$t_{d_1} + t_{d_2} + t_{d_3} = 0.46$ second. Hence relay K-2 closes at $t = 0.46$ second.

The Ri due to the various events will be handled as much as possible by means of Figures 5 through 17 and superimposed linearly. Since $t = 0$ was taken arbitrarily for each of the events considered in the preceeding section, each time one of these events is used in a calculation , its $t = 0$ must be referred to the corresponding time of that particular calculation. This will be done by placing the actual time which is equivalent to the $t = 0$ for which the event has been considered in parenthesis after the definition of the Ri. The figure numbers after an Ri give the graphs from which the Ri is obtained.

Proceeding with the calculations.

Ri_1 due to input ($t = 0$). Figures 5 and 6.

Ri'_2 due to relays closing ($t = 0.46$). Figures 7 and 8.

Ri'_3 due to output accelerating ($t = 0.46$). Figure 13.

$Ri = Ri_1 + Ri'_2 + Ri'_3$.

At $t = 0.46$ second, $Ri = 12.0$ microvolts.

At $t = 0.47$ second, $Ri = 9.5$ microvolts.

Thus a pulse occurs at $t = 0.46$ second.

Ri_2 due to a pulse ($t = 0.46$). Figures 11 and 12.

$Ri = Ri_1 + Ri_2$.

At $t = 0.81$ second, $Ri = 9.6$ microvolts.

At $t = 0.82$ second, $Ri = 10.5$ microvolts.

Relay K-1 makes first contact at $t = 0.81$ second, and relay K-2 closes at $t = 0.86$ second.

Ri''_3 due to relays closing ($t = 0.86$). Figures 7 and 8.

Ri'_4 due to output accelerating ($t = 0.86$). Figure 13.

$Ri = Ri_1 + Ri_2 + Ri''_3 + Ri'_4$.

At $t = 0.87$ second, $Ri = 11.9$ microvolts.

At $t = 0.88$ second, $Ri = 9.9$ microvolts.

A pulse occurs at $t = 0.86$ second.

Ri_3 due to a pulse ($t = 0.86$). Figures 11 and 12.

$Ri = Ri_1 + Ri_2 + Ri_3$.

At $t = 1.11$ seconds, $Ri = 9.5$ microvolts.

At t = 1.12 seconds, Ri = 10.7 microvolts.

Relay K-1 makes first contact at t = 1.11 seconds, and relay K-2 closes at t = 1.16 seconds.

Ri''_4 due to relays closing (t = 1.16). Figures 7 and 8.

Ri'_5 due to output accelerating (t = 1.16). Figure 13.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri''_4 + Ri'_5.$

At t = 1.20 seconds, Ri = 8.1 microvolts.

A pulse occurs at t = 1.16 seconds.

Ri_4 due to a pulse (t = 1.16). Figures 11 and 12.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4.$

At t = 1.37 seconds, Ri = 9.8 microvolts.

At t = 1.38 seconds, Ri = 11.3 microvolts.

Relay K-1 makes first contact at t = 1.37 seconds, and relay K-2 closes at t = 1.42 seconds.

Ri''_5 due to relays closing (t = 1.42). Figures 7 and 8.

Ri'_6 due to output accelerating (t = 1.42). Figure 13.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri''_5 + Ri'_6.$

At t = 1.46 seconds, Ri = 10.1 microvolts.

At t = 1.47 seconds, Ri = 8.7 microvolts.

A pulse at t = 1.42 seconds.

Ri_5 due to a pulse (t = 1.42). Figures 11 and 12.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri_5.$

At t = 1.60 seconds, Ri = 9.4 microvolts.

At t = 1.61 seconds, Ri = 11.0 microvolts.

Relay K-1 makes first contact at t = 1.60 seconds, and relay K-2 closes at t = 1.65 seconds.

Ri_6 due to relays closing (t = 1.65). Figures 7 and 8.

Ri_7' due to output accelerating (t = 1.65). Figure 13.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri_5 + Ri_6 + Ri_7'$.

At t = 1.69 seconds, Ri = 11.3 microvolts.

At t = 1.70 seconds, Ri = 9.9 microvolts.

Relay K-1 opens at t = 1.70 seconds, and relay K-2 opens at t = 1.73 seconds.

Equation (6) gives

$$\theta_m = 0.445\ t - 0.0852 \left[1 - \exp\ (-\ 5.21\ t)\right] \text{ knot}$$

$$(t = 1.65),\ 1.65 \leq t \leq 1.73 \text{ seconds.}$$

Equation (16) gives $t_s = 0.51$ second.

Equation (15) gives

$$\theta_m = 0.0402 + 0.0169\ t - 0.0492 \exp\ (-\ 5.21\ t)$$

knot (t = 1.65), $1.73 \leq t \leq 2.24$ seconds or until the servomotor is turned on, whichever is sooner.

$E = -\ V\theta_m$ gives

$$E = -\ 22.0 - 8.69\ t - 39.4 \exp\ (-\ 5.21\ t) \text{ micro-}$$

volts (t = 1.65), $1.73 \leq t \leq 2.24$ seconds or until the servomotor is turned on, whichever is sooner.

This gives

$$Ri_7'' = -19.3 - 8.69\,t - 42.79 \exp(-3.2\,t)$$

$$+ 62.6 \exp(-5.21\,t) \text{ microvolts } (t = 1.65),$$

$1.73 \leq t \leq 2.24$ seconds or until the servomotor is turned on, whichever is sooner.

Ri_8 due to relays opening ($t = 1.73$). Figures 9 and 10.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri_5 + Ri_6 + Ri_7'' + Ri_8$.

At $t = 2.00$ seconds, $Ri = 8.4$ microvolts.

At $t = 2.01$ seconds, $Ri = 10.6$ microvolts.

Relay K-1 makes first contact at $t = 2.01$ seconds, and relay K-2 closes at $t = 2.06$ second.

At $t = 2.06$ seconds, E due to the output coasting $= -30.2$ microvolts, and $Ri_7'' = -26.2$ microvolts. This gives

$$Ri_7 = -\left\{26.2 + 4\left[1 - \exp(-3.2\,t)\right]\right\} \text{ microvolts, } (t = 2.06).$$

Ri_9' due to relays closing ($t = 2.06$). Figures 7 and 8.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri_5 + Ri_6 + Ri_7 + Ri_8$
Ri_9'.

At $t = 2.06$ seconds, $Ri = 2.7$ microvolts. Assume that the servomotor has come to rest by this time.

A pulse at $t = 2.06$ seconds.

Ri_9 due to a pulse $(t = 2.06)$. Figures 11 and 12.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri_5 + Ri_6 + Ri_7 + Ri_8 + Ri_9$.

In a similar manner, the following results are obtained.

A pulse at $t = 2.30$ seconds.

The servomotor turned on at $t = 2.52$ seconds.

The servomotor turned off at $t = 2.62$ seconds.

$$\theta_m = 0.445\ t - 0.0852 \left[1 - \exp\ (-\ 5.21\ t)\right] \text{ knot}$$

$$(t = 2.52),\ 2.52 \leq t \leq 2.62 \text{ seconds.}$$

These calculations can be carried along in a similar manner as far as desired.

$\theta = \theta_i - \theta_m$ gives the following theoretical results.

$$\theta = 0.1\ t \text{ knot},\ 0 \leq t \leq 0.46 \text{ second.}$$

$$\theta = -\ 0.05\ t + 0.069 \text{ knot},\ 0.46 \leq t \leq 0.54 \text{ second.}$$

$$\theta = 0.1\ t - 0.012 \text{ knot},\ 0.54 \leq t \leq 0.86 \text{ second.}$$

$$\theta = -\ 0.05\ t + 0.117 \text{ knot},\ 0.86 \leq t \leq 0.94 \text{ second.}$$

$$\theta = 0.1\ t - 0.024 \text{ knot},\ 0.94 \leq t \leq 1.16 \text{ seconds.}$$

$$\theta = -0.05\ t + 0.160 \text{ knot}, \ 1.16 \leq t \leq 1.24 \text{ seconds.}$$

$$\theta = 0.1\ t - 0.036 \text{ knot}, \ 1.24 \leq t \leq 1.42 \text{ seconds.}$$

$$\theta = -0.05\ t + 0.177 \text{ knot}, \ 1.42 \leq t \leq 1.50 \text{ seconds.}$$

$$\theta = 0.1\ t - 0.048 \text{ knot}, \ 1.50 \leq t \leq 1.65 \text{ seconds.}$$

$$\theta = -0.345\ t + 0.771$$

$$- 0.0852 \exp\left[-5.21\ (t - 1.65)\right] \text{ knot},$$

$$1.65 \leq t \leq 1.73 \text{ seconds.}$$

$$\theta = 0.0831\ t + 0.0492 \exp\left[-5.21\ (t - 1.65)\right]$$

$$- 0.061 \text{ knot}, \ 1.73 \leq t \leq 2.06 \text{ seconds.}$$

$$\theta = -0.05\ t + 0.219 \text{ knot}, \ 2.06 \leq t \leq 2.14 \text{ seconds.}$$

$$\theta = 0.1\ t - 0.102 \text{ knot}, \ 2.14 \leq t \leq 2.30 \text{ seconds.}$$

$$\theta = -0.05\ t + 0.233 \text{ knot}, \ 2.30 \leq t \leq 2.38 \text{ secomds.}$$

$$\theta = 0.1\ t - 0.114 \text{ knot}, \ 2.38 \leq t \leq 2.52 \text{ seconds.}$$

$$\theta = -0.345\ t - 0.0852 \exp\left[-5.21\ (t - 2.52)\right]$$

$$+ 1.093 \text{ knot}, \ 2.52 \leq t \leq 2.62 \text{ seconds.}$$

Experimental values of error for an input rate of 0.1 knot per second are given in Tables 1 and 2, pages 53 and 54. The experimental values of error were determined as outlined below.

A dummy signal voltage of the same frequency and phase as the input signal voltage of the electromagnetic underwater log system was impressed on a Helipot driven by a constant speed motor. The linearly increasing output of this Helipot was used as the input signal to the servomechanism. A double channel brush oscillograph was used to record the time of application of this input as well as the output position. One pen motor was placed across the input to the servomechanism thus giving the time of application of the input voltage. The other pen motor was placed in series with a battery and a wire which made contact with the successive teeth of one of the gears of the output when the servomechanism was running. This gave an oscillogram from which the output was determined as a function of time. The input rate in microvolts per second and hence in knots per second was known thus giving the input as a function of time. The input minus the output at any time gave the experimental value of the error at that time.

Plots of the theoretical error and the two experimental runs are given in Figure 18, page 55.

Table 1

0.1 Knot per Second

Run 1

t Seconds	θ Knots	t Seconds	θ Knots
0.000	0.000	2.232	0.108
0.400	0.040	2.272	0.102
0.464	0.036	2.292	0.084
0.896	0.069	2.328	0.087
0.960	0.065	2.368	0.081
1.296	0.088	2.428	0.086
1.344	0.082	2.568	0.080
1.424	0.078	2.856	0.098
1.856	0.113	2.928	0.095
1.928	0.109	3.176	0.093
2.148	0.121	3.456	0.127
2.196	0.115	3.500	0.121

Table 2

0.1 Knot per Second

Run 2

t Seconds	θ Knots	t Seconds	θ Knots
0.000	0.000	2.032	0.088
0.544	0.054	2.088	0.083
0.848	0.074	2.192	0.084
0.952	0.074	2.272	0.081
1.488	0.118	2.544	0.098
1.560	0.114	2.800	0.113
1.816	0.130	2.856	0.109
1.856	0.123	2.904	0.103
1.896	0.117	2.976	0.100
1.924	0.109	3.188	0.111
1.952	0.101	3.456	0.127
1.992	0.095	3.512	0.123

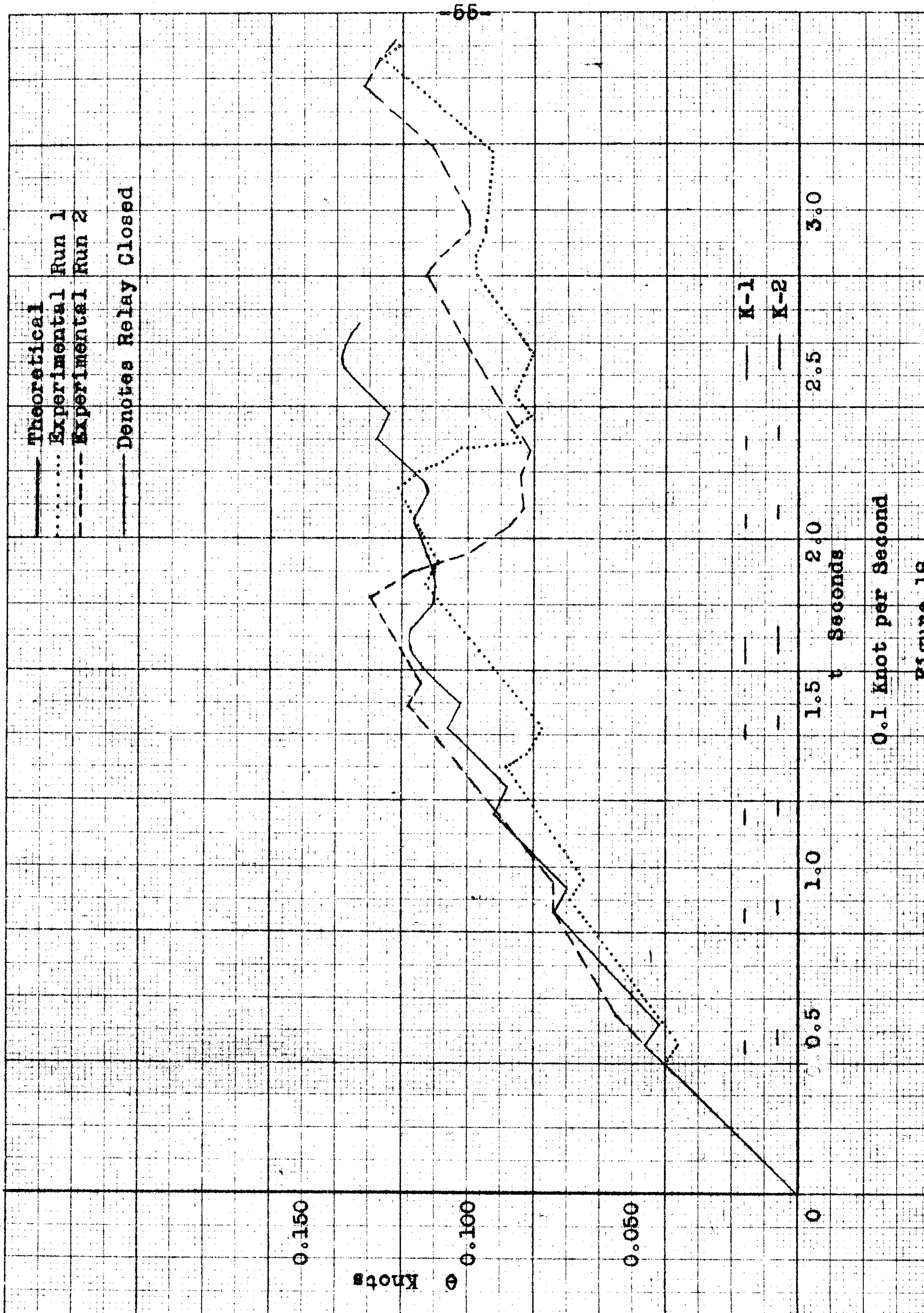

Theoretical
Experimental Run 1
Experimental Run 2
Denotes Relay Closed
K-1
K-2
0.150
0.100
0.050
0
θ Knots
0.5
1.0
1.5
2.0
2.5
3.0
t Seconds
0.1 Knot per Second

Figure 18

VI. CASE OF THE INPUT RATE SLIGHTLY LESS THAN THE MAXIMUM OUTPUT RATE

This section will deal with the response of the servomechanism to an input rate of 0.4 knot per second.

$V\omega = (0.4)\ (546) = 218$ microvolts per second.

$\dfrac{Ri}{V\omega} = \dfrac{10}{218} = 0.0459$ second.

Figure 5 gives $t_{d_1} = 0.19$ second.

$t_{d_1} + t_{d_2} + t_{d_3} = 0.24$ second. Hence relay K-2 closes at $t = 0.24$ second.

Ri_1 due to input ($t = 0$). Figures 5 and 6.

Ri_2' due to relays closing ($t = 0.24$). Figures 7 and 8.

Ri_3' due to output accelerating ($t = 0.24$). Figure 13.

$Ri = Ri_1 + Ri_2' + Ri_3'$.

At $t = 0.27$ second, $Ri = 10.9$ microvolts.

At $t = 0.28$ second, $Ri = 9.0$ microvolts.

A pulse occurs at $t = 0.24$ second.

Ri_2 due to a pulse ($t = 0.24$). Figures 11 and 12.

$Ri = Ri_1 + Ri_2$.

At $t = 0.40$ second, $Ri = 8.0$ microvolts.

At $t = 0.41$ second, $Ri = 11.7$ microvolts.

Relay K-1 makes first contact at $t = 0.41$ second, and relay K-2 closes at $t = 0.46$ second.

Ri_3 due to relays closing ($t = 0.46$). Figures 7 and 8.

Ri_4 due to output accelerating ($t = 0.46$). Figure 13.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4$.

Computation of Ri indicates that the relays remain closed until the transients die out. This gives

$$Ri = 218\ (t - 0.313) - 6.55 - 95$$
$$+ 122.4 - 243\ (t - 0.46) \text{ microvolts,}$$

or

$$Ri = -25\ t + 64.3 \text{ microvolts.}$$

Ri = 10 gives t = 2.17 seconds.

Relay K- 1 opens at t = 2.17 seconds, and relay K-2 opens at t = 2.20 seconds.

$Ri_2 = -6.6$ microvolts.

$Ri_3 = -95$ microvolts.

$Ri_4 = -300.6$ microvolts.

Ri_5 due to steady-state lag of output current behind output voltage (t = 2.20). Figures 16 and 17.

Ri_6 due to relays opening (t = 2.20). Figures 9 and 10.

Ri_7' due to output coasting (t = 2.20). Figures 14 and 15.

$$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri_5 + Ri_6 + Ri_7'.$$

At t = 2.28 seconds, Ri = 7.8 microvolts.

At t = 2.29 seconds, Ri = 10.7 microvolts.

Relay K-1 makes first contact at t = 2.29 seconds, and relay K-2 closes at t = 2.34 seconds.

At the end of the coasting run, equations (14) and (15) give $\dot{\theta}_m = 0.206$ knot per second and $\theta_m = 0.043$ knot. $E = -V\dot{\theta}_m$ gives E = - 23.5 microvolts, and Figure 14 gives $Ri_7' = -5.3$ microvolts at the end of the coasting run. This gives

$Ri_7 = -23.5 + 18.2 \exp(-3.2\ t)$ microvolts $(t = 2.34)$.

For the acceleration period equation (4) gives

$$\theta_m = 0.445\ t - 0.0458\left[1 - \exp(-5.21\ t)\right]$$

knots $(t = 2.34)$.

$E = -V\theta_m$ gives

$$E = -243\ t + 20.85\left[1 - \exp(-5.21\ t)\right]$$ micro-

volts $(t = 2.34)$.

$Ri_8 = 97.0 + 33.2 \exp(-5.21\ t) - 130.2 \exp(-3.2\ t)$

$- 243\ t$ microvolts $(t = 2.34)$.

Ri_9 due to relays closing $(t = 2.34)$. Figures 7 and 8.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4 + Ri_5 + Ri_6 + Ri_7 + Ri_8$

$+ Ri_9$.

At $t = 3.37$ seconds, $Ri = 10.0$ microvolts.

Relay K-1 opens at $t = 3.37$ seconds, and relay K-2 opens at 3.40 seconds.

These calculations can be carried along in a similar manner as far as desired.

$\theta = \theta_i - \theta_m$ gives the following theoretical results.

$\theta = 0.4\ t$ knot, $0 \leq t \leq 0.24$ second.

$\theta = 0.25\ t + 0.036$ knot, $0.24 \leq t \leq 0.32$ second.

$\theta = 0.4\ t - 0.012$ knot, $0.32 \leq t \leq 0.46$ second.

$$\theta = -0.045\,t - 0.0852 \exp\left[-5.21\,(t - 0.46)\right]$$

$$+ 0.278 \text{ knot}, \quad 0.46 \leq t \leq 2.20 \text{ seconds.}$$

$$\theta = 0.383\,t + 0.0885 \exp\left[-5.21\,(t - 2.20)\right]$$

$$- 0.752 \text{ knot}, \quad 2.20 \leq t \leq 2.34 \text{ seconds.}$$

$$\theta = -0.045\,t - 0.0458 \exp\left[-5.21\,(t - 2.34)\right]$$

$$+ 0.338 \text{ knot}, \quad 2.34 \leq t \leq 3.40 \text{ seconds}$$

Experimental values of error for an input rate of 0.4 knot per second are given in Tables 3 and 4, pages 60 and 61.

Plots of the theoretical error and the two experimental runs are given in Figure 19, page 62.

Table 3

0.4 Knot per Second

Run 1

t Seconds	θ Knots	t Seconds	θ Knots
0.000	0.000	1.788	0.194
0.496	0.198	1.952	0.249
0.548	0.209	2.000	0.258
0.580	0.211	2.036	0.262
0.608	0.212	2.064	0.263
0.636	0.213	2.092	0.264
0.656	0.210	3.240	0.202
0.688	0.213	3.264	0.201
0.712	0.212	3.292	0.202
0.736	0.211	3.320	0.203
0.760	0.210	3.356	0.207
0.784	0.209	3.396	0.213
1.588	0.166	3.444	0.221
1.612	0.166	3.480	0.225
1.640	0.166	3.520	0.231
1.680	0.172	3.548	0.232
1.720	0.178	4.528	0.176

Table 4

0.4 Knot per Second

Run 2

t Seconds	θ Knots	t Seconds	θ Knots
0.000	0.000	2.504	0.158
0.344	0.138	2.540	0.162
0.420	0.158	2.576	0.166
0.624	0.229	2.624	0.175
0.672	0.238	2.672	0.183
0.704	0.240	2.704	0.186
0.736	0.242	2.732	0.187
0.760	0.242	2.760	0.187
0.788	0.242	3.408	0.144
0.816	0.243	3.436	0.145
0.840	0.242	3.464	0.146
0.864	0.241	3.492	0.147
2.304	0.161	3.528	0.151
2.328	0.160	3.564	0.155
2.352	0.160	3.608	0.162
2.376	0.159	3.664	0.174
2.396	0.156	3.708	0.181
2.420	0.156	3.740	0.184
2.444	0.155	3.764	0.183
2.472	0.156	4.400	0.145

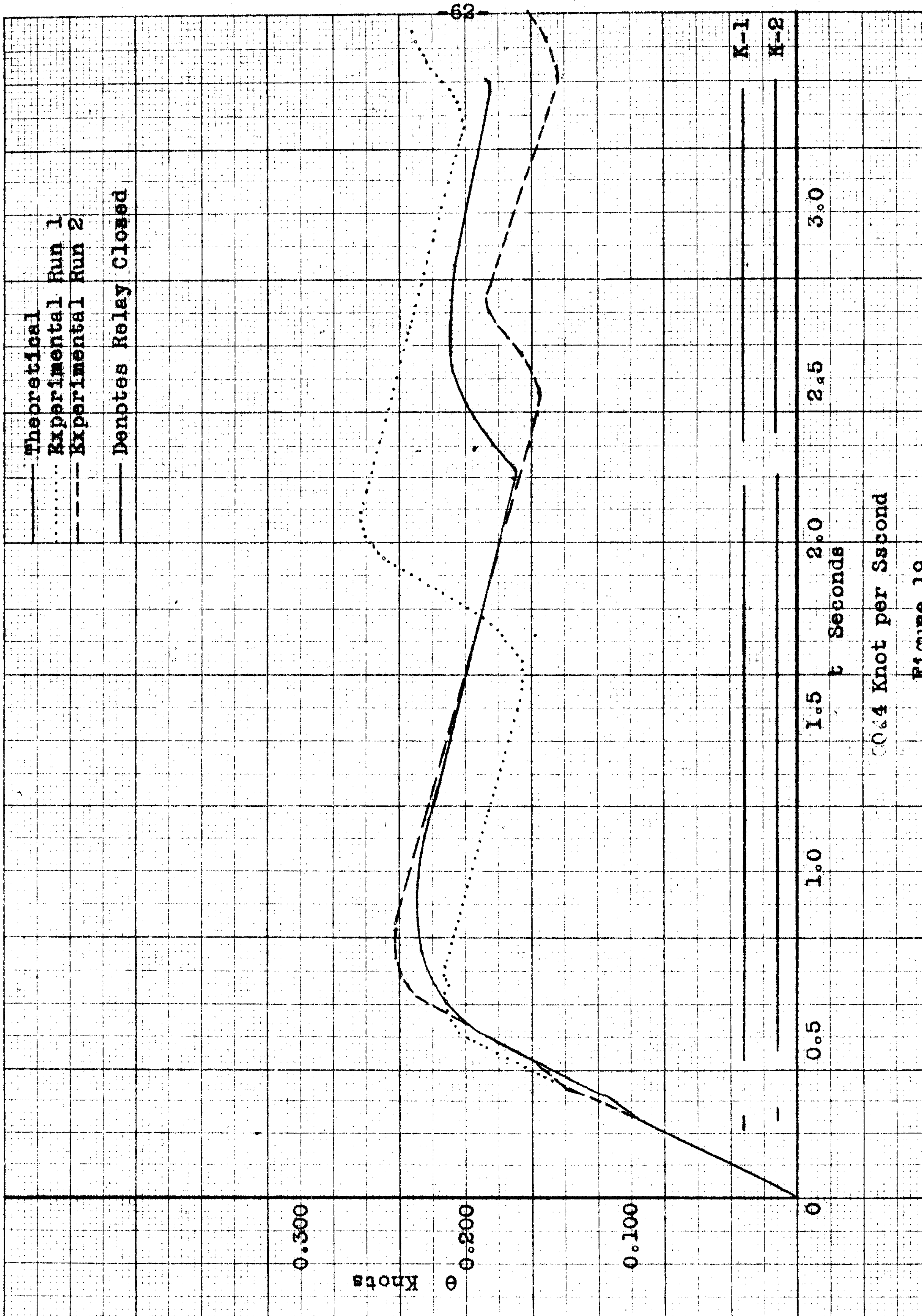

Theoretical
Experimental Run 1
Experimental Run 2
Denotes Relay Closed
K-1
K-2
0
0.5
1.0
1.5
2.0
2.5
3.0
t Seconds
0.100
0.200
0.300
θ Knots
0.4 Knot per Second

Figure 19

VII. CASE OF THE INPUT RATE EQUAL TO THE MAXIMUM OUTPUT RATE

This section will deal with the response of the servo-mechanism to an input rate of 0.445 knot per second.

$V\omega = (0.445)(546) = 243$ microvolts per second.

$\frac{Ri}{V\omega} = \frac{10}{243} = 0.0412$ second.

Figure 5 gives $t_{d_1} = 0.18$ second.

$t_{d_1} + t_{d_2} + t_{d_3} = 0.23$ second. Hence relay K-2 closes at $t = 0.23$ second.

Ri_1 due to input ($t = 0$). Figures 5 and 6.

Ri_2' due to relays closing ($t = 0.23$). Figures 7 and 8.

Ri_3' due to output accelerating ($t = 0.23$). Figure 13.

$Ri = Ri_1 + Ri_2' + Ri_3'$.

At $t = 0.27$ second, $Ri = 10.4$ microvolts.

At $t = 0.28$ second, $Ri = 9.3$ microvolts.

A pulse occurs at $t = 0.23$ second.

Ri_2 due to a pulse ($t = 0.23$). Figures 11 and 12.

$Ri = Ri_1 + Ri_2$.

At $t = 0.38$ second, $Ri = 9.6$ microvolts.

At $t = 0.39$ second, $Ri = 11.9$ microvolts.

Relay K-1 makes first contact at $t = 0.38$ second, and relay K-2 closes at $t = 0.43$ second.

Ri_3 due to relays closing ($t = 0.43$). Figures 7 and 8.

Ri_4 due to output accelerating ($t = 0.43$). Figure 13.

$Ri = Ri_1 + Ri_2 + Ri_3 + Ri_4$.

A check of the Ri reveals that relay K-1 remains closed since Ri approaches 49.3 microvolts never dropping to 10 microvolts. For the acceleration period equation (4) gives

$$\theta_m = 0.445\ t - 0.0852\ \left[1 - \exp\ (-\ 5.21\ t)\right]\ \text{knot}$$

$$(t = 0.43).$$

$\theta = \theta_i - \theta_m$ gives the following theoretical results.

$$\theta = 0.445\ t\ \text{knot},\ 0 \leq t \leq 0.23\ \text{second}.$$

$$\theta = 0.295\ t + 0.0345\ \text{knot},\ 0.23 \leq t \leq 0.31\ \text{second}.$$

$$\theta = 0.445\ t - 0.012\ \text{knot},\ 0.31 \leq t \leq 0.43\ \text{second}.$$

$$\theta = 0.264 - 0.0852\ \exp\left[-\ 5.21\ (t - 0.43)\right]\ \text{knot},$$

$$t \geq 0.43\ \text{second}.$$

Experimental values of error for an input rate of 0.445 knot per second are given in Tables 5 and 6, pages 65 and 66.

Plots of the theoretical error and the two experimental runs are given in Figure 20, page 67.

Table 5

0.445 Knot per Second

Run 1

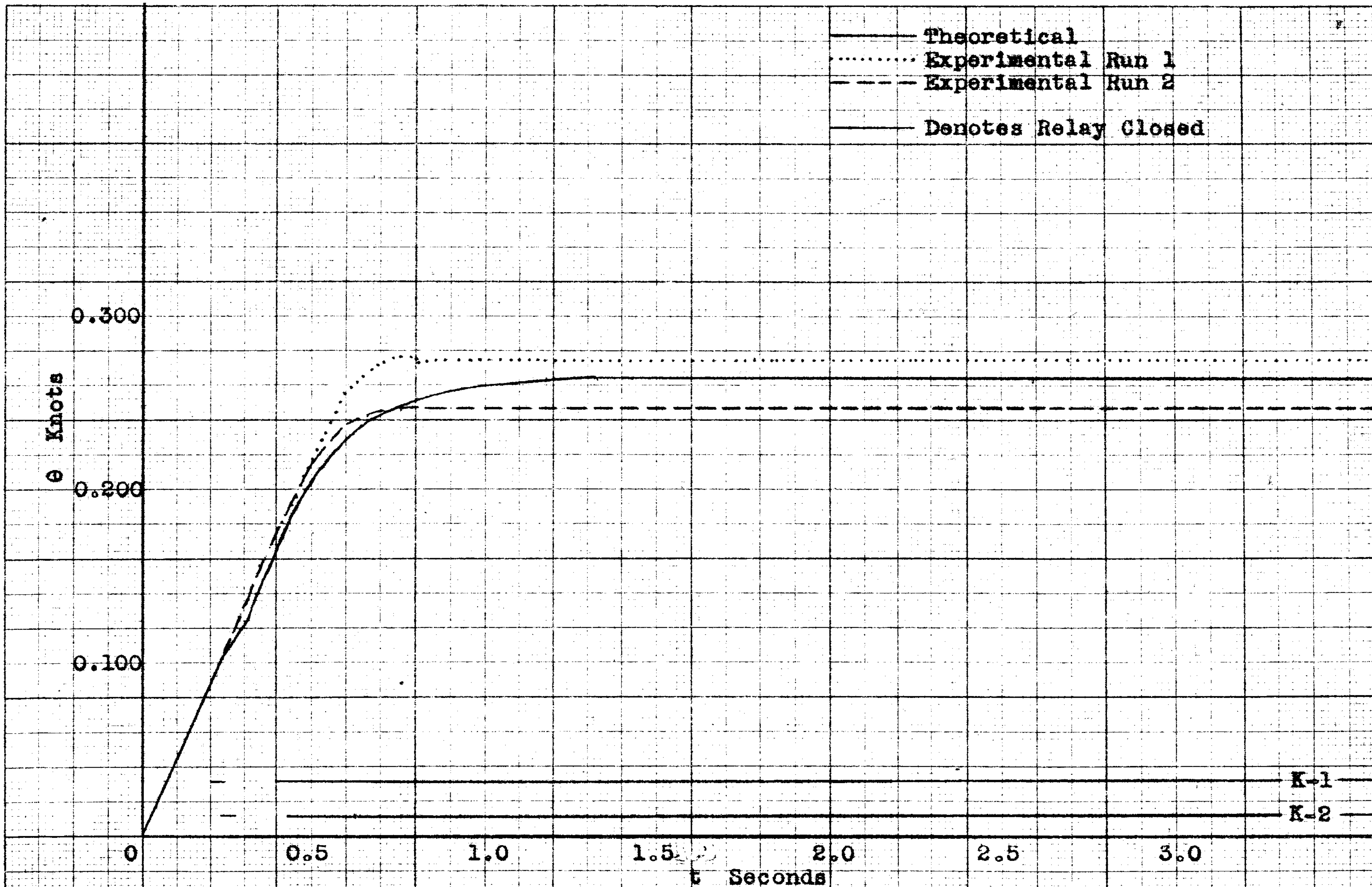
t Seconds	θ Knots	t Seconds	θ Knots
0.000	0.000	0.737	0.276
0.352	0.157	0.761	0.276
0.596	0.255	0.788	0.277
0.644	0.265	0.804	0.274
0.676	0.269	0.828	0.274
0.708	0.273	0.852	0.275

Table 6

0.445 Knot per Second

Run 2

t Seconds	0 Knots	t Seconds	0 Knots
0.000	0.000	0.660	0.242
0.284	0.127	0.688	0.245
0.504	0.215	0.712	0.244
0.568	0.232	0.738	0.246
0.604	0.238	0.762	0.245
0.632	0.240	0.788	0.247



0.445 Knot per Second

Figure 20

VIII. CASE OF THE INPUT RATE GREATER THAN THE MAXIMUM OUTPUT RATE

This section will deal with the response of the servomechanism to an input rate of 0.8 knot per second.

$V\omega = (0.8)\ (546) = 437$ microvolts per second.

$\frac{Ri}{V\omega} = \frac{10}{437} = 0.0229$ second.

Figure 5 gives $t_{d_1} = 0.13$ second.

$t_{d_1} + t_{d_2} + t_{d_3} = 0.18$ second. Hence relay K-2 closes at $t = 0.18$ second.

Ri_1 due to input ($t = 0$). Figures 5 and 6.

Ri_2 due to relays closing ($t = 0.18$). Figures 7 and 8.

Ri_3 due to output accelerating ($t = 0.18$). Figure 13.

$Ri = Ri_1 + Ri_2 + Ri_3$.

A check of the Ri reveals that relay K-1 remains closed since Ri approaches (437 t - 231.8) microvolts never dropping to 10 microvolts. For the acceleration period equation (4) gives

$$\theta_m = 0.445\ t - 0.0852\ [1 - \exp\ (-\ 5.21\ t)]\ \text{knots}\quad (t = 0.18).$$

$\theta = \theta_i - \theta_m$ gives the following theoretical results.

$$\theta = 0.8\ t\ \text{knot},\ 0 \leq t \leq 0.18\ \text{second}.$$

$$\theta = 0.335\,t + 0.165 - 0.0852 \;\exp\left[- 5.21\,(t - 0.18)\right]$$

knots, $t \geq 0.18$ second.

Experimental values of error for an input rate of 0.8 knot per second are given in Tables 7 and 8, pages 70 and 71.

Plots of the theoretical error and the two experimental runs are given in Figure 21, page 72.

Table 7

0.8 Knot per Second

Run 1

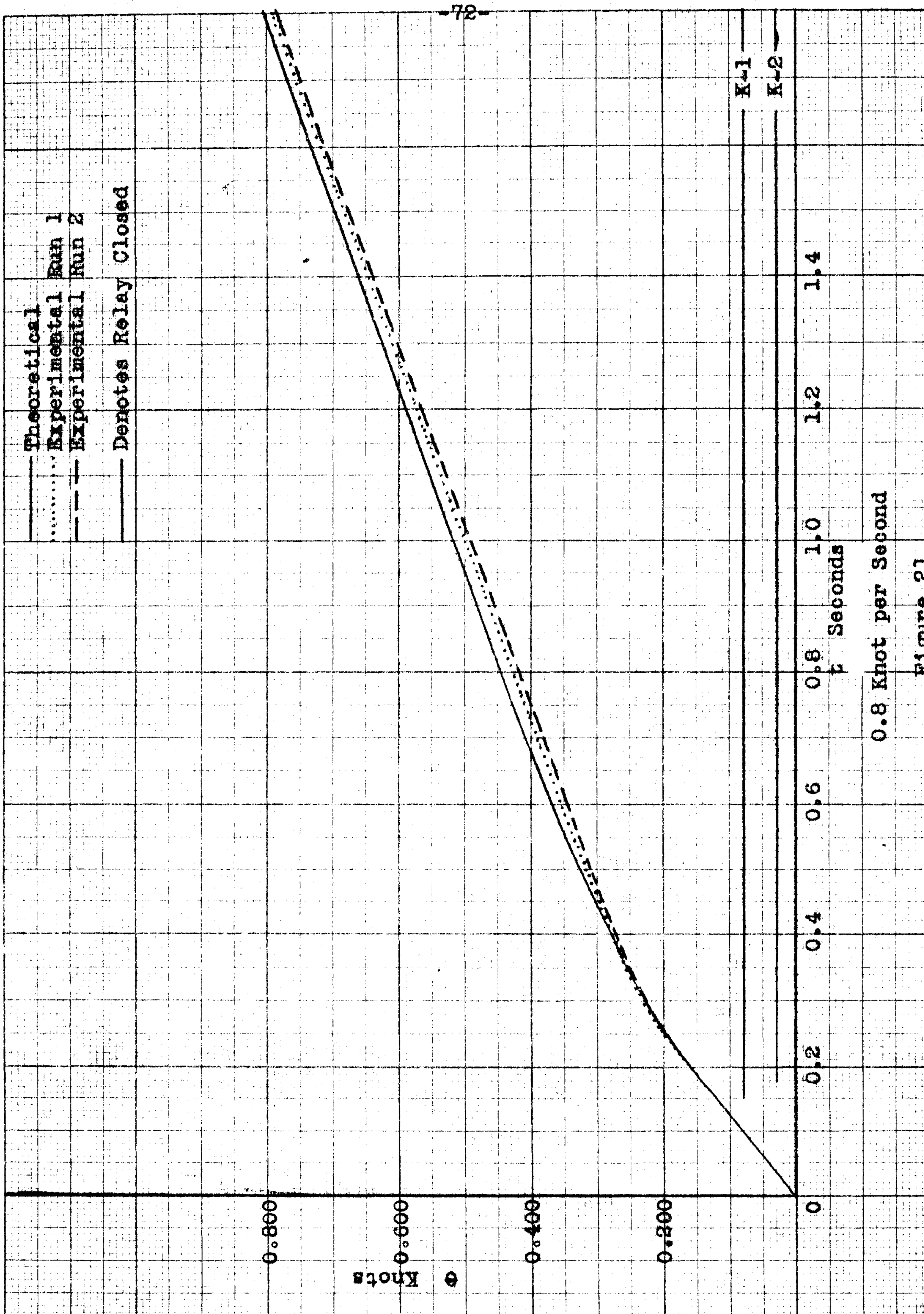
t Seconds	θ Knots	t Seconds	θ Knots
0.000	0.000	0.424	0.287
0.260	0.208	0.450	0.297
0.308	0.236	0.476	0.308
0.344	0.254	0.498	0.315
0.372	0.267	0.520	0.322
0.398	0.276	0.544	0.331

Table 8

0.8 Knot per Second

Run 2

t	θ	t	θ
Seconds	Knots	Seconds	Knots
0.000	0.000	0.408	0.274
0.232	0.186	0.436	0.287
0.288	0.220	0.464	0.298
0.320	0.235	0.488	0.307
0.352	0.251	0.512	0.316
0.380	0.262	0.535	0.324



0.8 Knot per Second

Figure 21

IX. DISCUSSION

The method of analysis developed herein and applied to several cases in connection with the electromagnetic underwater log system can be applied to other similar servomechanisms. Since the important factor in the analysis is the relay circuit employed, this method will be applicable to most servomechanisms employing a similar relay circuit.

The solution of the four specific cases considered have far greater significance than at first seems apparent. The solutions have only been carried far enough to outline the method of solution and to obtain sufficient results to verify the validity of the method of solution by comparision with experimental results. Although the solutions were obtained for the problem of the ship starting from rest, they will apply equally well for the ship starting from any constant velocity provided the ship has been at that velocity long enough for the relay K-1 coil current to go to zero. This follows immediately because the input and output voltages will both be increased by the same voltage, that corresponding to the initial constant velocity of the ship. Thus these voltages will compensate for each other at all times giving the same Ri as if the ship had started from rest. Similarly, since $\theta = \theta_i - \theta_m$, the error versus time curve will be the same irrespective of the initial starting velocity.

In a similar manner, the actual results obtained can be readily extended to the problem of negative linear accelerations applied to the ship when it is traveling at any constant velocity. For this problem it is obvious that all voltages applied to the relay circuit for a given negative acceleration will be the same in magnitude but opposite in sign to the corresponding voltages applied during a positive acceleration of the same magnitude. Hence the total current through the realy circuit will be the same in magnitude but opposite in direction. Since the electromagnetic underwater log system is assumed to have identical upscale and downscale characteristics, the magnitudes of the increment to the input, the increment to the output, the error, and the various relay circuit voltages and currents will all be the same in magnitude but opposite in sign. Thus the error curve for any downscale acceleration will be the negative of the error curve for the corresponding upscale acceleration.

The method developed has been applied to the entire range of input rates with satisfactory results over the entire range. No assumptions were made which obviously depended on the input rate, and the close correlation of experimental and theoretical errors over the entire range of input rates verifies that the approximations made were essentially independent of the input rate.

The method of solution is not limited to constant accelerations of the ship but can be applied to any known motions of the ship. The problems treated were considered because these are the simplest problems of interest for the electromagnetic underwater log system.

X. CONCLUSIONS

A comparision of the theoretical and experimental values of error in Figures 18, 19, 20, and 21 shows for each of the cases a correlation between the theoretical and experimental results which is comparible to the deviations between the different experimental runs for that case. The complexity of carrying through an algebraic solution indicated the desirability of developing a semi-graphical method of analysis; this correlation between theoretical and experimental results indicates the correctness of the method of solution developed and justifies the many assumptions made in developing it. In addition, these figures indicate that nothing would be gained by a more exact analysis since the theoretical results in some cases agree with the different experimental runs better than these experimental runs agree with each other and in no case does the theoretical result deviate seriously from the experimental runs.

LIST OF ASSUMPTIONS

The following assumptions have been made in analyzing the electromagnetic underwater log system:

The output can be represented by a moment of inertia I, a viscous damping coefficient c, and a Coulomb friction F where I and c are constants and F has the following form:

$F = F_o$, for $\dot{\theta}_m$ positive,

$F = 0$, for $\dot{\theta}_m$ zero,

$F = - F_o$, for $\dot{\theta}_m$ negative,

where F_o is constant. The moment of inertia of the servomotor can be taken as the moment of inertia of the output.

The output is driven by a servomotor that has a torque of the following form:

$T = T_o$, for the servomotor turned on upscale,

$T = 0$, for the servomotor turned off,

$T = - T_o$, for the servomotor turned on downscale,

where T_o is constant.

The various resistances and inductances in the relay circuits are constant.

The ratio of the output of the phase sensitive rectifier at any time to the error voltage at that time is constant.

Relay K-1 always makes first contact for the same value of current through its coil.

The time delays of the relay circuit t_{d2}, t_{d3}, and t_{d4} are constants.

The signal strength V is a constant.

The mechanical parameters of the relay K-1 can be lumped with the mechanical parameters.

When the servomotor is turned on for less than twice the time of a pulse, the event is regarded as a pulse.

The output always comes to rest after a pulse.

TABLE OF SYMBOLS

The symbols used herein are tabulated below. For convenience the units used in the numerical analysis are given.

t = time in seconds

θ_i' = input position in radians

θ_m' = output position in radians

θ' = error in radians

ω' = constant input rate in radians per second

θ_i = input position in knots

θ_m = output position in knots

θ = error in knots

ω = constant input rate in knots per second

I = moment of inertia of the servomechanism in gram-centimeters^2

T_o = magnitude of the servomotor torque in dyne-centimeters

F_o = magnitude of the Coulomb friction torque in dyne-centimeters

c = viscous friction coefficient in dyne-centimeters per radian per second

E_i = input voltage in microvolts

E_m = output voltage in microvolts

E = error voltage in microvolts

E_p = equivalent positive feedback voltage in microvolts

E_N = equivalent negative feedback voltage in microvolts

Ri = resistance of relay K-1 equivalent circuit multiplied by the current through the circuit in microvolts

$\frac{R}{L}$ = ratio of resistance to inductance of relay K-1 circuit in ohms per henry or seconds^{-1}

V = signal strength in microvolts per knot

t_{d_1} = delay time of relay K-1 before making first contact after the input has been applied in seconds

t_{d_2} = delay time of relay K-1 before closing firmly after making first contact in seconds

t_{d_3} = delay time of relay K-2 in closing after relay K-1 has closed in seconds

t_{d_4} = delay time of relay K-2 in opening after relay K-1 has opened in seconds

TABLE OF SERVOMECHANISM CONSTANTS

A table of the experimental constants necessary in the analysis of the electromagnetic underwater log system are given below. All mechanical quantities are referred to the one knot output shaft, and all electrical quantities are referred to the equivalent relay K-1 circuit diagram.

I = 341,000 gram-centimeters^2

T_o = 5,180,000 dyne-centimeters

F_o = 188,800 dyne-centimeters

c = 1,780,000 dyne-centimeters per radian per second

E_P = 305 microvolts

E_N = -400 microvolts

$\frac{R}{L}$ = 3.2 ohms per henry (seconds^{-1})

V = 546 microvolts per knot

t_{d2} = 0.02 second

t_{d3} = 0.03 second

t_{d4} = 0.03 second

2π radians = 1 knot

Ri = 10 microvolts to close relay K-1

K = 1165

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