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ON THE SUMMABILITY OF DOUBLE FOURIER'S SERIES

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CONTENTS

ON THE SUMMABILITY OF DOUBLE FOURIER'S SERIES

The use of double Fourier's series in the solutions of many problems in Mathematical Physics has led a number of writers to consider various aspects of the theory of such series. Both convergence and summability of the developments of certain types of functions have been discussed. The advantages of the process of summability over that of convergence in the case of developments that arise in Mathematical Physics justifies a close study of this process. C. N. Moore* has considered integral orders of the Cesaro method of summability and applied it to the Fourier's series development of a function of two variables at points of discontinuity of the function as well as at points of continuity.

The present paper is concerned with the extension of Moore's results to positive non-integral orders of summability. The advantages of non-integral means lies, not in any aid that it gives in finding the sum of a divergent series, but in the closer information it gives regarding the nature of the divergence of the series. Such information is of value in predicting the behavior of a series when multiplied by a set of convergence factors.

*Trans. Am. Math. Soc. 14 (1913), pp. 73-104. Also Math. Ann. 74 (1913), pp. 555-572.

PART I

Summability of Double Series.

1. Definitions

We take as a basis of the extended theory C. N. Moore's* definitions for summability of the double series

(1) $$\sum a_{mn} = a_{11} + a_{12} + \cdots + a_{21} + a_{22} + \cdots + \cdots$$

Form

(2) $$S_{mn}^{(k)} = \sum_{i=1,j=1}^{m,n} A_{m+1-i,n+1-j}^{(k-1)} s_{ij}$$

where

(3) $$s_{ij} = \sum_{r=1,q=1}^{i,j} a_{pq}$$

and

(4) $$A_{mn}^{(k)} = \frac{\Gamma(m+k)}{\Gamma(k+1)\,\Gamma(m)} \; \frac{\Gamma(n+k)}{\Gamma(k+1)\,\Gamma(n)}$$

where k has any value real or complex except negative integral values. If the quotient

(5) $$S_{mn}^{(k)} / A_{mn}^{(k)}$$

tends to a limit S as m and n become infinite, the series (1) is said to be summable (Ck) to the value S. In order that the above definitions have a meaning for the value k = 0, we assume that the right-hand side of (2) has the value which it approaches as k approaches zero. The discussion in this paper is confined to real values of k greater than -1.

From the definitions of $S_{mn}^{(k)}$ and $A_{mn}^{(k)}$ it follows that

(6) $$\sum_{m=1,n=1}^{\infty,\infty} S_{mn}^{(k)} x^m y^n \sim (1-x)^{-k}(1-y)^{-k} \sum_{m=1,n=1}^{\infty,\infty} s_{mn} x^m y^n$$

and

(7) $$\sum_{m=1,n=1}^{\infty,\infty} A_{mn}^{(k)} x^m y^n \sim xy\,(1-x)^{-(k+1)}(1-y)^{-(k+1)}$$

whether k is integral or not. On equating coefficients of $x^m y^n$ on the two sides of (7), we obtain

*Trans. Am. Math. Soc. 14 (1913), pp. 73-104. Also Math. Ann. 74 (1913), pp. 555-572.

(8) $$A^{(k)}_{mn} = \sum_{i=1,j=1}^{mn} A^{(k-1)}_{ij}$$

from which it is evident that the quotient (5) deals with the weighted mean of the sums of the terms of the double series (1). In the same manner we find from (6) that

(9) $$S^{(k+1)}_{mn} = \sum_{i=1,j=1}^{mn} S^{(k)}_{i,j}$$

Also, from the definition of $A^{(k)}_{mn}$ we have

(10) $$\lim_{m,n\to\infty} \frac{A^{(k)}_{mn}}{m^k n^k} = \frac{1}{[\Gamma(k+1)]^2}$$

so that

(11) $$A^{(k)}_{mn} < M m^k n^k \qquad \left\{ \begin{array}{l} k>-1 \\ m,n=1,2,3,\cdots \end{array} \right\}$$

where M is a positive constant

2. Theorem of Consistency

In order to prove the theorem of consistency we shall first prove the following lemma.

LEMMA 1. If we have two double sequences

(12) $$a_{mn}\ ,\quad b_{mn} \qquad (m,n=1,2,3\cdots)$$

such that

(a) $$\left|\frac{a_{mn}}{m^r n^s}\right| < K_1 \ ,\quad \left|\frac{b_{mn}}{m^p n^q}\right| < K_2 \qquad (r,s,p,q>-1)$$

where K_1 and K_2 are positive constants, and

(b) $$\lim_{m,n\to\infty} \frac{a_{mn}}{m^r n^s} = a, \qquad \lim_{m,n\to\infty} \frac{b_{mn}}{m^p n^q} = b$$

then

(13) $$\lim_{m,n\to\infty} \sum_{i=1,j=1}^{mn} \frac{a_{m+1-i,n+1-j}\, b_{ij}}{m^{r+p+1} n^{s+q+1}} = \frac{\Gamma(r+1)\Gamma(p+1)}{\Gamma(r+p+2)} \cdot \frac{\Gamma(s+1)\Gamma(q+1)}{\Gamma(s+q+2)}\, a b$$

and furthermore this sum remains finite for all values of m and n.

Let

$$\frac{a_{ij}}{i^r j^s} = a + \alpha_{ij}\ , \qquad \frac{b_{ij}}{i^p j^q} = b + \beta_{ij} \qquad (i,j=1,2,3\cdots)$$

Then

$$\alpha_{mn} \to 0 \quad , \quad \beta_{mn} \to 0 \quad \text{as } m \text{ and } n \to \infty$$

and

$$|\alpha_{ij}| < C_1 \quad , \quad |\beta_{ij}| < C_2 \qquad (i,j = 1,2,3,\cdots\cdots)$$

where C_1 and C_2 are positive constants.

Now

$$(14) \quad \sum_{i=1,j=1}^{mn} \frac{a_{m+1-i,n+1-j}\, b_{ij}}{m^{r+p+1}\, n^{s+q+1}} = ab \sum_{i=1,j=1}^{mn} \frac{(m+1-i)^r (n+1-j)^s i^p j^q}{m^{r+p+1}\, n^{s+q+1}}$$

$$+ a \sum_{i=1,j=1}^{mn} \frac{(m+1-i)^r (n+1-j)^s i^p j^q}{m^{r+p+1}\, n^{s+q+1}} \beta_{ij}$$

$$+ b \sum_{i=1,j=1}^{mn} \frac{(m+1-i)^r (n+1-j)^s i^p j^q}{m^{r+p+1}\, n^{s+q+1}} \alpha_{m+1-i,n+1-j}$$

$$+ \sum_{i=1,j=1}^{mn} \frac{(m+1-i)^r (n+1-j)^s i^p j^q}{m^{r+p+1}\, n^{s+q+1}} \alpha_{m+1-i,n+1-j}\, \beta_{ij}$$

For the first term on the right-hand side of (14) we have

$$\sum_{i=1,j=1}^{mn} \frac{(m+1-i)^r (n+1-j)^s i^p j^q}{m^{r+p+1}\, n^{s+q+1}} = \sum_{i=1}^{m} \left(1+\frac{1}{m}-\frac{i}{m}\right)^r \left(\frac{i}{m}\right)^p \frac{1}{m} \sum_{j=1}^{n} \left(1+\frac{1}{n}-\frac{j}{n}\right)^s \left(\frac{j}{n}\right)^q \frac{1}{n}$$

$$= \sum_{i=1}^{m} (1+\Delta x - x_i)^r x_i^p \Delta x \sum_{j=1}^{n} (1+\Delta y - y_j)^s y_j^q \Delta y$$

Where $x_i = i/m$, $\Delta x = 1/m$, $y_j = j/n$, $\Delta y = 1/n$

Hence

$$\lim_{m,n\to\infty} \sum_{i=1,j=1}^{mn} \frac{(m+1-i)^r (n+1-j)^s i^p j^q}{m^{r+p+1}\, n^{s+q+1}} = \lim_{m\to\infty} \sum_{i=1}^{m} (1+\Delta x - x_i)^r x_i^p \Delta x \lim_{n\to\infty} \sum_{j=1}^{n} (1+\Delta y - y_j)^s y_j^q \Delta y$$

$$= \int_0^1 (1-x)^r x^p\, dx \int_0^1 (1-y)^s y^q\, dy$$

$$= \frac{\Gamma(r+1)\Gamma(p+1)}{\Gamma(r+p+2)} \cdot \frac{\Gamma(s+1)\Gamma(q+1)}{\Gamma(s+q+2)} \qquad (r,s,p,q > -1)$$

The last three terms on the right-hand side of (14) may be discussed by entirely analogous methods. We shall consider the second term in detail and note that similar arguments apply to the third and fourth terms. For the second term we note that since $\beta_{mn} \to 0$ as m and n $\to \infty$ we can find a μ and ν corresponding to any small positive quantity ϵ such that

$$|\beta_{mn}| < \epsilon/K \qquad (m > \mu,\ n > \nu)$$

where

$$K = \frac{\Gamma(\lambda+1)\,\Gamma(p+1)}{\Gamma(\lambda+p+2)} \cdot \frac{\Gamma(s+1)\,\Gamma(q+1)}{\Gamma(s+q+2)}$$

Then, since $|\beta_{ij}| < C_2$ we have

$$\left|\sum_{i=1,j=1}^{m,n} \frac{(m+1-i)^\lambda (n+1-j)^s i^p j^q}{m^{\lambda+p+1} n^{s+q+1}} \beta_{ij}\right| < C_2 \sum_{i=1,j=1}^{\mu,\nu} \frac{(m+1-i)^\lambda (n+1-j)^s i^p j^q}{m^{\lambda+p+1} n^{s+q+1}}$$

$$+ C_2 \sum_{i=1,j=\nu+1}^{\mu,n} \frac{(m+1-i)^\lambda (n+1-j)^s i^p j^q}{m^{\lambda+p+1} n^{s+q+1}}$$

$$+ C_2 \sum_{i=\mu+1,j=1}^{m,\nu} \frac{(m+1-i)^\lambda (n+1-j)^s i^p j^q}{m^{\lambda+p+1} n^{s+q+1}}$$

$$+ \frac{\epsilon}{K} \sum_{i=\mu+1,j=\nu+1}^{m,n} \frac{(m+1-i)^\lambda (n+1-j)^s i^p j^q}{m^{\lambda+p+1} n^{s+q+1}}$$

It is evident that the first three terms on the right-hand side of the above inequality approach zero as m and n approach infinity while the last sum approaches the value K. It therefore follows that

$$\lim_{m,n\to\infty} \sum_{i=1,j=1}^{m,n} \frac{(m+1-i)^\lambda (n+1-j)^s i^p j^q}{m^{\lambda+p+1} n^{s+q+1}} \beta_{ij} = 0$$

For the third term on the right-hand side of (14) we have

$$\sum_{i=1,j=1}^{m,n} \frac{(m+1-i)^\lambda (n+1-j)^s i^p j^q}{m^{\lambda+p+1} n^{s+q+1}} \alpha_{m+1-i,n+1-j} = \sum_{i=1,j=1}^{m,n} \frac{i^\lambda j^s (m+1-i)^p (n+1-j)^q}{m^{\lambda+p+1} n^{s+q+1}} \alpha_{ij}$$

An argument similar to the preceding one shows that this term approaches zero as m and n become infinite. In a similar manner it can be shown that the fourth term on the right-hand side of (14) approaches zero as m and n become infinite.

From the above results we have

$$\lim_{m,n\to\infty} \sum_{i=1,j=1}^{mn} \frac{a_{m+1-i,n+1-j}\, b_{ij}}{m^{r+p+1} n^{s+q+1}} = \frac{\Gamma(r+1)\Gamma(p+1)}{\Gamma(r+p+2)} \cdot \frac{\Gamma(s+1)\Gamma(q+1)}{\Gamma(s+q+2)}\, ab$$

which proves the first part of our lemma.

For the second part of the lemma, we have

$$\left|\sum_{i=1,j=1}^{mn} \frac{a_{m+1-i,n+1-j}\, b_{ij}}{m^{r+p+1} n^{s+q+1}}\right| < K_1K_2 \sum_{i=1,j=1}^{m,n} \frac{(m+1-i)^r (n+1-j)^s i^p j^q}{m^{r+p+1} n^{s+q+1}} \qquad (m,n=1,2,3,\cdots)$$

$$< K_1K_2K \qquad (r,s,p,q>-1)$$

$$= M$$

where M is a positive constant. This completes the proof of the lemma.

With the aid of the above lemma, we now establish the consistency theorem.

THEOREM 1. *If the series (1) is summable (Cr), where $r>-1$, and if furthermore*

$$\left|\frac{S_{mn}^{(r)}}{A_{mn}^{(r)}}\right| < C \qquad (m,n=1,2,3\cdots)$$

then the series is summable (Cr'), where $r'>r$ and to the same value and furthermore

$$\left|\frac{S_{mn}^{(r')}}{A_{mn}^{(r')}}\right| < C \qquad (m,n=1,2,3\cdots)$$

From (6) we have

$$\sum_{m=1,n=1}^{\infty,\infty} S_{mn}^{(r+s)} x^m y^n \sim (1-x)^{-(r+s)}(1-y)^{-(r+s)} \sum_{m=1,n=1}^{\infty\infty} s_{mn} x^m y^n$$

$$\sim (1-x)^{-s}(1-y)^{-s} \sum_{m=1,n=1}^{\infty\infty} S_{mn}^{(r)} x^m y^n$$

$$\sim \sum_{m=1,n=1}^{\infty,\infty} A^{(s-1)}_{m-1,n-1}\, x^{m-1} y^{n-1} \sum_{m=1,n=1}^{\infty,\infty} S^{(\lambda)}_{mn}\, x^m y^n$$

On equating coefficients of $x^m y^n$ on the two sides of the above expression, we have

(15) $$S^{(\lambda+s)}_{mn} = \sum_{i=1,j=1}^{mn} A^{(s-1)}_{m+1-i,n+1-j}\, S^{(\lambda)}_{ij}$$

Then

$$\frac{S^{(\lambda+s)}_{mn}}{A^{(\lambda+s)}_{mn}} = \frac{\sum_{i=1,j=1}^{mn} A^{(s-1)}_{m+1-i,n+1-j}\, S^{(\lambda)}_{ij}}{A^{(\lambda+s)}_{mn}}$$

$$= \frac{\sum_{i=1,j=1}^{mn} A^{(s-1)}_{m+1-i,n+1-j}\, S^{(\lambda)}_{ij} / m^{\lambda+s} n^{\lambda+s}}{A^{(\lambda+s)}_{mn} / m^{\lambda+s} n^{\lambda+s}}$$

Now

$$\lim_{m,n\to\infty} \frac{A^{(s-1)}_{mn}}{m^{s-1} n^{s-1}} = \frac{1}{[\Gamma(s)]^2}$$

and

$$\lim_{m,n\to\infty} \frac{S^{(\lambda)}_{mn}}{m^\lambda n^\lambda} = \lim_{m,n\to\infty} \frac{S^{(\lambda)}_{mn}}{A^{(\lambda)}_{mn}} \frac{A^{(\lambda)}_{mn}}{m^\lambda n^\lambda} = \frac{S}{[\Gamma(\lambda+1)]^2}.$$

where

$$\lim_{m,n\to\infty} \frac{S^{(\lambda)}_{mn}}{A^{(\lambda)}_{mn}} = S$$

Also

$$\lim_{m,n\to\infty} \frac{A^{(\lambda+s)}_{mn}}{m^{\lambda+s} n^{\lambda+s}} = \frac{1}{[\Gamma(\lambda+s+1)]^2}$$

and

$$\left|\frac{S^{(\lambda)}_{mn}}{m^\lambda n^\lambda}\right| = \left|\frac{S^{(\lambda)}_{mn}}{A^{(\lambda)}_{mn}} \frac{A^{(\lambda)}_{mn}}{m^\lambda n^\lambda}\right| = \left|\frac{S^{(\lambda)}_{mn}}{A^{(\lambda)}_{mn}}\right| \frac{A^{(\lambda)}_{mn}}{m^\lambda n^\lambda}$$

$$< CM = K \qquad \text{a positive constant}$$

Hence the conditions of lemma (1) are satisfied and we have

$$\lim_{m,n\to\infty} \frac{S_{mn}^{(\lambda+s)}}{A_{mn}^{(\lambda+s)}} = \left[\Gamma(\lambda+s+1)\right]^2 \left[\frac{\Gamma(\lambda+1)\,\Gamma(s)}{\Gamma(\lambda+s+1)}\right]^2 \frac{1}{[\Gamma(s)]^2}\,\frac{S}{[\Gamma(\lambda+1)]^2}$$

$$= S$$

Putting r+s=r' we get the result as stated. It also follows from lemma (1) that

$$\left|\frac{S_{mn}^{(\lambda+s)}}{A_{mn}^{(\lambda+s)}}\right| < K \quad \text{a positive constant.}$$

PART II

Application to the Double Fourier's Series

We wish to apply this method of summation to the development of a function of two variables, $f(x,y)$, in a double Fourier's series. C. N. Moore* has done this for functions which are finite and integrable or have certain types of infinite discontinuities while remaining absolutely integrable in the region $(-\pi \leqq x \leqq \pi,\ -\pi \leqq y \leqq \pi)$ and he has shown that the double series is summable (C1) at all points for which the function is continuous and also at points where the function has certain types of discontinuities. By the extension of his method to non-integral orders of summability, we find that the Fourier's series development for the same class of functions is summable (Ck) for any value of $k > 0$. The statements of the theorems and lemmas involved in the proofs of these theorems are quite analogous to those of Moore; in fact they become identical with his when k is placed equal to 1. The difference in the form of S requires some modifications of his proofs. In proving the theorems, it is convenient to restrict the value of k so that $0 < k < 1$. This, however, implies no ultimate restriction since a series that is summable (Cr) is also summable (Cr') for any value of $r' > r$.

*Trans. Am. Math. Soc. 14 (1913) pp. 73-104; Math. Ann. 74 (1913) pp. 555-572)

The double Fourier's series corresponding to a function of two variables, f(x,y), may be written in the form

$$(16) \qquad f(x,y) \sim \sum_{i=1,j=1}^{\infty,\infty} \frac{1}{2^{E(\frac{i}{2})+E(\frac{j}{2})}\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y')\left\{\cos[(m-1)(x-x')]\cos[(n-1)(y-y')]\right\}dx'dy'$$

where E(z) is used to designate the largest integer contained in z. For this series we find

$$(17) \qquad \begin{aligned} S_{mn}(x,y) &= \sum_{i=1,j=1}^{m,n} \frac{1}{2^{E(\frac{i}{2})+E(\frac{j}{2})}\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y')\left\{\cos[(m-1)(x-x')]\cos[(n-1)(y-y')]\right\}dx'dy' \\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y')\,\frac{\sin[(2m-1)(x'-x)/2]\,\sin[(2n-1)(y'-y)/2]}{\sin[(x'-x)/2]\,\sin[(y'-y)/2]}\,dx'dy' \end{aligned}$$

and

$$(18) \qquad \frac{S_{mn}^{(k)}}{A_{mn}^{(k)}} = \frac{1}{4A_{mn}^{(k)}\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y')\sum_{i=1,j=1}^{m,n} A_{m+1-i,n+1-j}^{(k-1)}\,\frac{\sin[(2m-1)(x'-x)/2]\,\sin[(2n-1)(y'-y)/2]}{\sin[(x'-x)/2]\,\sin[(y'-y)/2]}\,dx'dy'$$

By the change of variable

$$(19) \qquad x'-x = 2\alpha, \qquad y'-y = 2\beta$$

we obtain

$$(20) \qquad \frac{S_{mn}^{(k)}(x,y)}{A_{mn}^{(k)}} = \frac{1}{A_{mn}^{(k)}\pi^2}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}}\int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}} f(x+2\alpha,y+2\beta)\sum_{i=1,j=1}^{m,n} A_{m+1-i,n+1-j}^{(k-1)}\,\frac{\sin(2i-1)\alpha\,\sin(2j-1)\beta}{\sin\alpha\,\sin\beta}\,d\alpha\,d\beta$$

We shall investigate first the nature of the

$$\lim_{m,n\to\infty} S_{mn}^{(k)}(x,y)/A_{mn}^{(k)}$$

at points of continuity of the function f(x,y). In what follows when we refer to a function as being finite and integrable we shall mean that it has an integral in the sense of Lebesgue.

For convenience in writing we shall introduce the following notations

$$(21) \qquad \sigma_{mn}^{(k)}(\alpha,\beta) = \frac{1}{A_{mn}^{(k)}}\sum_{i=1,j=1}^{m,n} A_{m+1-i,n+1-j}^{(k-1)}\,\frac{\sin(2i-1)\alpha\,\sin(2j-1)\beta}{\sin\alpha\,\sin\beta}$$

$$(22) \qquad A_{m}^{(k)} = \frac{\Gamma(k+m)}{\Gamma(k+1)\,\Gamma(m)}$$

(23) $\sigma_m^{(k)}(\alpha) = \frac{1}{A_m^{(k)}} \sum_{i=1}^{m} A_{m+1-i}^{(k-1)} \frac{\sin(2i-1)\alpha}{\sin\alpha}$

From these definitions it is evident that

$$A_{mn}^{(k)} = A_m^{(k)} A_n^{(k)}$$

and

$$\sigma_{mn}^{(k)}(\alpha,\beta) = \sigma_m^{(k)}(\alpha)\,\sigma_n^{(k)}(\beta)$$

3. Summability at Points of Continuity.

We shall first prove three lemmas which are needed in the proof of the main theorem of this section.

LEMMA 2. Let R be a region in the α,β-plane lying within the square whose sides are $\alpha = \pm(\pi-\rho_1)$, $\beta = \pm(\pi-\rho_1)$ where ρ_1 is a small positive quantity, and such that no point of R lies within the circle whose center is at the origin and whose radius is ρ_2 where ρ_2 is another small positive quantity. Then if $\varphi(\alpha,\beta)$ is a function that is finite and integrable in the region R, the limit

(24) $\lim_{m,n\to\infty} [\iint_R \varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta]$

will exist and be equal to zero.

Let us designate by M the upper limit of the absolute value of $\varphi(\alpha,\beta)$ in the region R, and by ρ the smaller of the two quantities ρ_1 and ρ_2/r_2. The region R may be divided into two portions R_1 and R_2 such that the portion R_1 contains all points for which

$$|\alpha| < \rho_2/r_2$$

and the portion R_2 contains all points for which

$$|\alpha| \geq \rho_2/r_2$$

Since the region R contains no points which lie within a circle with center at the origin and of radius ρ_2 we have

$$\alpha^2+\beta^2 \geq \rho_2^2 \qquad in\ R.$$

For all points of R, we have

$$\alpha^2 < \rho^2/2$$

So that in this region

$$\beta^2 > \rho^2/2 \geqslant \rho^2$$

It is evident that

$$(25)\qquad \left|\iint_R \varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right| \leqq \left|\iint_{R_1} \varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right| + \left|\iint_{R_2} \varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right|$$

The two terms on the right-hand side of (25) may be discussed by entirely analogous methods. We shall consider the first term in detail and note that a similar argument applies to the second. For the first term we have

$$(26)\qquad \begin{aligned} \left|\iint_{R_1} \varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right| &\leqq \iint_{R_1} |\varphi(\alpha,\beta)|\,|\sigma_{mn}^{(k)}(\alpha,\beta)|\,d\alpha\,d\beta \\ &\leqq M\iint_{R_1} |\sigma_m^{(k)}(\alpha)|\,|\sigma_n^{(k)}(\beta)|\,d\alpha\,d\beta \end{aligned}$$

Since, in the region R_1, $\beta^2 \geqslant \rho^2$, we have

$$\begin{aligned} |\sigma_n^{(k)}(\beta)| &= \frac{1}{A_n^{(k)}}\left|\sum_{j=1}^{n} A_{n+1-j}^{(k-1)} \frac{\sin(2j-1)\beta}{\sin\beta}\right| \\ &= \frac{1}{A_n^{(k)}}\left|\sum_{j=1}^{n} A_{n+1-j}^{(k-1)} \frac{\sin(2j-1)\beta\,\sin\beta}{\sin^2\beta}\right| \\ &\leqq \frac{1}{A_n^{(k)}\sin^2\rho}\left|\sum_{j=1}^{n} A_{n+1-j}^{(k-1)}[\sin^2 j\beta - \sin^2(j-1)\beta]\right| \\ &= \frac{1}{A_n^{(k)}\sin^2\rho}\left|\sum_{j=1}^{n} A_{n+1-j}^{(k-2)}\sin^2 j\beta\right| \end{aligned}$$

Now

$$\begin{aligned} 0 \geqslant \sum_{j=1}^{n-1} A_{n+1-j}^{(k-2)}\sin^2 j\beta \geqslant \sum_{j=1}^{n-1} A_{n+1-j}^{(k-2)} &= \sum_{j=1}^{n} A_{n+1-j}^{(k-2)} - A_1^{(k-2)} \\ &= A_n^{(k-1)} - 1 > -1 \qquad (0<k<1) \end{aligned}$$

and

$$0 \leqq A_1^{(k-1)}\sin^2 n\beta \leqq 1 \qquad (0<k<1)$$

for all values of β.

Hence

(27) $$|\sigma_n^{(k)}(\beta)| < \frac{1}{A_m^{(k)} \sin^2 \rho} \qquad \rho \leq |\beta| < \pi - \rho$$

Then the right-hand side of (26) is less than

(28) $$M \iint_{R_1} |\sigma_m^{(k)}(\alpha)| \frac{1}{A_m^{(k)} \sin^2 \rho} d\alpha\, d\beta$$
$$< \frac{M}{A_m^{(k)} \sin^2 \rho} \int_{-\pi}^{\pi} |\sigma_m^{(k)}(\alpha)| d\alpha \int_{\pi}^{\pi} d\beta$$
$$= \frac{2\pi M}{A_n^{(k)} \sin^2 \rho} \int_{\pi}^{\pi} |\sigma_m^{(k)}(\alpha)| d\alpha$$

Now

(29) $$\int_{\pi}^{\pi} |\sigma_m^{(k)}(\alpha)| d\alpha = 4\int_0^{\pi/2} |\sigma_m^{(k)}(\alpha)| d\alpha$$
$$= 4\int_0^{\pi/4} |\sigma_m^{(k)}(\alpha)| d\alpha + 4\int_{\pi/4}^{\pi/2} |\sigma_m^{(k)}(\alpha)| d\alpha$$

T. H. Gronwall* has shown that

(30) $$\int_0^{\pi/4} |\sigma_m^{(k)}(\alpha)| d\alpha < K$$

where K is a positive constant independent of m. The second integral on the right-hand side of (29) is also finite for all values of m since its integrand is less than $[1/\sin^2 \frac{1}{4}\pi]$ Collecting our results, we see that the left-hand side of (26) is less than $C/A_n^{(k)}$ where C is a positive constant.

From the definition of $A_n^{(k)}$ it follows that

$$\lim_{n \to \infty} \frac{A_n^{(k)}}{n^k} = \frac{1}{\Gamma(k+1)}$$

Consequently

$$C/A_n^{(k)} < C'/n^k$$

Hence, for any arbitrarily small positive quantity ϵ, we can find an integer q such that

$$C/A_n^{(k)} < \tfrac{1}{2}\epsilon \qquad n \geqq q$$

*Amer. Math. Soc. Bul. 20(1913-14), pp.141-142.

We therefore have

(31) $$\left|\iint_{R_1} \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right| < \tfrac{1}{2}\epsilon \qquad n \geq q$$

Since for all points in R_2

$$\alpha^2 > \beta_2^2/2 \geq \rho^2$$

we can show by a similar argument that

(32) $$\left|\iint_{R_2} \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right| < \tfrac{1}{2}\epsilon \qquad m \geq q$$

Combining the inequalities (31) and (32) we have

$$\left|\iint_{R} \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right| < \epsilon \qquad m, n \geq q$$

Hence

(33) $$\lim_{m,n\to\infty} \left[\iint_R \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right] = 0.$$

and our lemma is established.

LEMMA 3. <u>If h, h_1, k and k_1 are positive numbers less than π, the limit</u>

$$\lim_{m,n\to\infty} \left[\frac{1}{\pi^2}\int_{-h_1}^{h}\int_{-k_1}^{k} \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right]$$

<u>will exist and be equal to unity.</u>

Let R be the region which must be added to or subtracted from the square whose sides are $\alpha = \pm\pi/2$, $\beta = \pm\pi/2$ to produce the rectangle whose sides are $\alpha = h$, $\alpha = -h_1$, $\beta = k$, $\beta = -k_1$. Then

(34) $$\frac{1}{\pi^2}\int_{-h_1}^{h}\int_{-k_1}^{k} \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta = \frac{1}{\pi^2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta$$
$$+ \frac{1}{\pi^2}\iint_R \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta$$

It follows at once from lemma (2) that

$$\lim_{m,n\to\infty}\frac{1}{\pi^2}\iint_R \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta = 0$$

For the first term on the right-hand side of (34) we have

$$\frac{1}{\pi^2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$

$$= \frac{1}{A_{mn}^{(k)}\,\pi^2}\sum_{i=1,j=1}^{m,n}\left\{A_{m+1-i,n+1-j}^{(k-1)}\int_{-\pi/2}^{\pi/2}\frac{\sin(2i-1)\alpha}{\sin\alpha}\,d\alpha\int_{-\pi/2}^{\pi/2}\frac{\sin(2j-1)\beta}{\sin\beta}\,d\beta\right.$$

$$= 1$$

Since

$$\int_{-\pi/2}^{\pi/2}\frac{\sin n\alpha}{\sin\alpha}\,d\alpha = \pi \qquad \text{when } n \text{ is odd}$$

and

$$\sum_{i=1,j=1}^{mn} A_{m+1-i,n+1-j}^{(k-1)} = A_{mn}^{(k)}$$

Combining these results we have

$$(35)\qquad \lim_{m,n\to\infty}\left[\frac{1}{\pi^2}\int_{-k_1}^{k}\int_{-k_1}^{k}\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right] = 1$$

which proves our lemma.

LEMMA 4. Let R be a region in the α,β-plane lying within the square whose sides are $\alpha = \pm(\pi-\rho_1)$, $\beta = \pm(\pi-\rho_1)$ where ρ_1 is a small positive quantity and such that the point $\alpha=0, \beta=0$ lies within or on the boundary of R. Then, if $\varphi(\alpha,\beta)$ is a function that is finite and integrable in R, the limit

$$\lim_{m,n\to\infty}\left[\iint_R \varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right]$$

will exist and be equal to zero, provided

$$(36)\qquad \lim_{\alpha,\beta\to 0}\left[\varphi(\alpha,\beta)\right] = 0$$

Since $\lim_{\alpha,\beta\to 0}[\varphi(\alpha,\beta)]=0$

we may surround the point $\alpha=0, \beta=0$ with a circle of radius ρ_2 so small that corresponding to any arbitrarily small positive quantity ϵ

$$|\varphi(\alpha,\beta)| < \frac{\epsilon}{2K^2} \qquad (\alpha^2+\beta^2 \leq \rho_2^2)$$

where K is a positive constant such that

$$(37) \qquad \int_0^{\pi/2} |\sigma_m^{(k)}(\alpha)|\, d\alpha < \tfrac{1}{2}K$$

Let us designate by R_1 that portion of the circle of radius ρ_2 which lies within R, and by R_2 the remainder of the region R. Evidently

$$(38) \qquad \iint_R \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta = \iint_{R_1} \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta + \iint_{R_2} \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta$$

For the first term on the right hand side of (38) we have

$$(39) \qquad \begin{aligned} \left|\iint_{R_1} \varphi(\alpha,\beta)\, \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right| &\leq \iint_{R_1} |\varphi(\alpha,\beta)|\,|\sigma_{mn}^{(k)}(\alpha,\beta)|\, d\alpha\, d\beta \\ &< \frac{\epsilon}{2K^2} \iint_{R_1} |\sigma_{mn}^{(k)}(\alpha,\beta)|\, d\alpha\, d\beta \\ &< \frac{\epsilon}{2K^2} \int_{-\pi/2}^{\pi/2} |\sigma_m^{(k)}(\alpha)|\, d\alpha \int_{-\pi/2}^{\pi/2} |\sigma_n^{(k)}(\beta)|\, d\beta \\ &< \frac{\epsilon}{2} \end{aligned}$$

since $\int_{-\pi/2}^{\pi/2} |\sigma_m^{(k)}(\alpha)|\, d\alpha = 2\int_0^{\pi/2} |\sigma_m^{(k)}(\alpha)|\, d\alpha < K$

Since, by lemma 2

$$\lim_{m,n\to\infty}\left[\iint_{R_2}\varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right]=0$$

it follows that

(40) $$\left|\iint_{R_2}\varphi(\alpha,\beta)\,\sigma_{m,n}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right|<\frac{\epsilon}{2}\qquad(m,n\geqq q)$$

and hence

(41) $$\left|\iint_{R}\varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right|<\epsilon\qquad(m,n\geqq q)$$

Therefore

(42) $$\lim_{m,n\to\infty}\left[\iint_{R}\varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right]=0$$

which proves our lemma.

We are now ready to prove the following theorem regarding the summability of a double Fourier's series development of a function at points where the function is continuous.

<u>THEOREM 2</u>. <u>If the function f(x,y) is finite and integrable in the region</u> $(-\pi\leqq x\leqq\pi,-\pi\leqq y\leqq\pi)$ <u>the development of the function in a double Fourier's series will be summable (Ck), for any value of k > 0, to the value of the function at every interior point of the region at which the function is continuous.</u>

In equation (20) introduce the new function

(43) $$\varphi(\alpha,\beta)=f(x+2\alpha,y+2\beta)-f(x,y)$$

Then, for the double Fourier's series corresponding to the function f(x,y), we will have

(44) $$\frac{S_{mn}^{(k)}(x,y)}{A_{mn}^{(k)}}=f(x,y)\frac{1}{\pi^2}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}}\int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}}\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta+\frac{1}{\pi^2}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}}\int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}}\varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$

From its definition, it is evident that $\varphi(\alpha,\beta)$ satisfies the conditions of lemma 4 in the region of integration of the integrals in (44), provided (x,y) is a point interior to the region $(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ and f(x,y) is continuous at this point. It therefore follows from this lemma that the second term on the right hand side of (44) approaches zero as m and n become infinite. From lemma 3, it follows that the first term on the right hand side of (44) approaches f(x,y) as a limit as m and n approach infinity. We have then

$$\lim_{m,n\to\infty} \frac{S_{mn}^{(k)}(x,y)}{A_{mn}^{(k)}} = f(x,y) \qquad (k>0) \tag{45}$$

at all points interior to the region $(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ at which f(x,y) is continuous, which proves our theorem.

4. Summability at Points of Discontinuity that lie along Straight Lines.

Before proceeding with the discussion of the nature of the

$$\lim_{m,n\to\infty} \frac{S_{mn}^{(k)}(x,y)}{A_{mn}^{(k)}}$$

at points of discontinuity of the function, we shall first prove two lemmas which are needed in the investigation of this limit.

LEMMA 5. If h and k are positive numbers less than π the four limits

$$\begin{aligned}
&\lim_{m,n\to\infty} \left[\int_0^h \int_0^k \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right] \\
(46)\qquad &\lim_{m,n\to\infty} \left[\int_h^0 \int_0^k \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right] \\
&\lim_{m,n\to\infty} \left[\int_h^0 \int_{-k}^0 \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right]
\end{aligned}$$

$$\lim_{m,n\to\infty}\left[\int_0^h\int_{-k}^0 \sigma_{mn}^{(h)}(\alpha,\beta)\,d\alpha\,d\beta\right]$$

will each exist and be equal to $\frac{1}{4}$.

The proof that each of the limits exist and is equal to $\frac{1}{4}$ may be made in a manner quite analogous to the method used in the proof of lemma 3. The existence and value of the last three limits may be obtained from the first by a suitable change of variables in the integrals involved.

LEMMA 6. If h and k are positive numbers less than π and we divide the rectangle whose sides are $\alpha=\pm h$, $\beta=\pm k$ into two parts R_1 and R_2 by drawing either diagonal, the limits

$$\lim_{m,n\to\infty}\left[\frac{1}{\pi^2}\iint_{R_1} \sigma_{mn}^{(h)}(\alpha,\beta)\,d\alpha\,d\beta\right]$$

(47)

$$\lim_{m,n\to\infty}\left[\frac{1}{\pi^2}\iint_{R_2} \sigma_{mn}^{(h)}(\alpha,\beta)\,d\alpha\,d\beta\right]$$

will each exist and be equal to $\frac{1}{2}$.

For the case in which the diagonal joins the points (-h,-k) and (h,k) let us designate by R the triangle which has its right angle at the point (h,-k). Then

$$\frac{1}{\pi^2}\iint_{R_1} \sigma_{mn}^{(h)}(\alpha,\beta)\,d\alpha\,d\beta = \frac{1}{\pi^2}\int_0^h \sigma_m^{(h)}(\alpha)\left(\int_0^{k\alpha/h}\sigma_n^{(h)}(\beta)\,d\beta\right)d\alpha$$

(48)

$$+\frac{1}{\pi^2}\int_0^h\int_{-k}^0 \sigma_{mn}^{(h)}(\alpha,\beta)\,d\alpha\,d\beta$$

$$+\frac{1}{\pi^2}\int_{-k}^0 \sigma_n^{(h)}(\beta)\left(\int_{h\beta/k}^0 \sigma_m^{(h)}(\alpha)\,d\alpha\right)d\beta$$

If, in the third term on the right hand side of (48), we make the change of variable

$$\alpha=-\alpha',\quad \beta=-\beta'$$

we obtain the expression

$$\frac{1}{\pi^2}\int_0^{h} \sigma_n^{(k)}(\beta')\left(\int_0^{h\beta'/k}\sigma_m^{(k)}(\alpha')\,d\alpha'\right)d\beta'$$

Dropping the primes and combining this term with the first term on the right side of (48), we get

$$\frac{1}{\pi^2}\iint_{R_1}\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta = \frac{1}{\pi^2}\int_0^{h}\int_0^{k}\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$
$$+\frac{1}{\pi^2}\int_0^{h}\int_{-k}^{0}\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$

It follows at once from lemma 5 that the limit of the first expression in (47) exists and its value is $\frac{1}{2}$.

We know from lemma 3 that the limit of the sum of the two expressions in (47) exists and is equal to 1. Hence the limit of the second expression in (47) must exist and be equal to $\frac{1}{2}$.

In a similar manner we can show that the lemma is true in the case dealing with the other diagonal.

We are now ready to prove the following theorem regarding the summability of a double Fourier's series at points of discontinuity.

<u>THEOREM 3.</u> <u>If $f(x,y)$ is finite and integrable in the region $(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ and (x_1, y_1) is a point of discontinuity of $f(x,y)$ such that every other point of discontinuity of the function in the neighborhood of (x_1, y_1) lies on a straight line thru that point and the function approaches a definite value as we approach the point from either side of this line, then the series (16) will be summable (Ck), for</u>

<u>any value of $k > 0$, at the point (x_1, y_1) and its value will be half way between the limiting values of the function.</u>

Since all the points of discontinuity of f(x,y) lie on a straight line passing thru the point (x_1, y_1), let us choose a rectangle with center at the point (x_1, y_1) and with sides of length 2h and 2k respectively parallel to the coordinate axes and such that all points of discontinuity of f(x,y) in the neighborhood of (x_1, y_1) lie either on one of the diagonals of the rectangle or on a line parallel to one of the coordinate axes. The corresponding rectangle in the α,β-plane obtained by the transformation

$$x' - x_1 = 2\alpha, \quad y' - y_1 = 2\beta$$

will be divided by the transformed line of discontinuities into two portions which we will designate by R_1' and R_2'. Let us designate by R'' the rest of the region of integration in the α,β-plane. We have then for the double Fourier's series at the point (x_1, y_1)

$$\begin{aligned}(49)\qquad \frac{S_{mn}^{(k)}(x_1,y_1)}{A_{mn}^{(k)}} &= \frac{1}{\pi^2}\iint_{R_1'} f(x_1+2\alpha, y_1+2\beta)\ \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2}\iint_{R_2'} f(x_1+2\alpha, y_1+2\beta)\ \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2}\iint_{R''} f(x_1+2\alpha, y_1+2\beta)\ \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\end{aligned}$$

It follows at once from lemma 2 that the third term on the right hand side of (49) approaches zero as m and n become infinite. Let us designate by $f_1(x_1, y_1)$ the value which f(x,y) approaches as we approach the point (x_1, y_1) thru the region R_1' and by $f_2(x_1, y_1)$ the value which it approaches as we approach the point thru the region R_2'. Introduce into the first and second terms on the right hand side of (49) the new functions

$$\varphi_1(\alpha,\beta) = f(x_1+2\alpha, y_1+2\beta) - f_1(x_1,y_1)$$

(50) and

$$\varphi_2(\alpha,\beta) = f(x_1+2\alpha, y_1+2\beta) - f_2(x_1,y_1)$$

Equation (49) may then be written

(51)

$$\begin{aligned}\frac{S_{mn}^{(k)}(x_1,y_1)}{A_{mn}^{(k)}} &= \frac{1}{\pi^2} f_1(x_1,y_1)\iint_{R_1'} \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2} f_2(x_1,y_1)\iint_{R_2'} \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2}\iint_{R_1'} \varphi_1(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2}\iint_{R_2'} \varphi_2(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\end{aligned}$$

It is evident that $\varphi_1(\alpha,\beta)$ and $\varphi_2(\alpha,\beta)$ satisfy the conditions of lemma 4 in the regions of integration. Hence each of the last two terms on the right hand side of (51) approach zero as m and n become infinite. That the first two terms approach the limits $\frac{1}{2} f_1(x_1,y_1)$ and $\frac{1}{2} f_2(x_1,y_1)$ respectively as m and n become infinite is easily shown by the application of lemma 6 when the line of discontinuities is a diagonal of the rectangle or by lemma 5 in the case when the line of discontinuities is parallel to one of the coordinate axes. Hence

(52)
$$\lim_{m,n\to\infty} \frac{S_{mn}^{(k)}(x_1,y_1)}{A_{mn}^{(k)}} = \frac{1}{2}\left[f_1(x_1,y_1) + f_2(x_1,y_1)\right]$$

which proves our theorem.

It may be noted that in the proofs of lemmas 2 to 6 inclusive no restrictions were placed upon the manner in which m and n become infinite. The conclusions still hold if m is held fixed while n becomes infinite and then m is allowed to become infinite or vice versa.

Moore has defined summability (Ck) of a double series by rows and by columns as follows:

"Definition. If each row of a double series is summable (Ck) and the series formed from the values of the rows is summable (Ck), we say that the series is summable (Ck) by rows, and has the value of the series formed from the values of the rows." A series summable (Ck) by columns is defined in a similar manner.

This definition enables us to state the following corollary:

Corollary 1. *If $f(x,y)$ satisfies the conditions of theorem 3 and (x_1, y_1) is a point of discontinuity of $f(x,y)$ of the type described in theorem 3, the series (16) will be summable (Ck), for any value of $k > 0$, by rows or by columns at the point (x_1, y_1) and its value will be one-half the sum of the limiting values of the function.*

Since the lemmas involved in the proof of theorem 3 hold equally well when m and n become infinite in any manner, this corollary follows immediately from theorem 3.

The investigation of the behavior of the series at points on the boundary of the region $(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ with the exception of the points at the four corners leads to the following corollary.

COROLLARY 2. If x_1 is a value of x lying in the interval $(-\pi < x < \pi)$ and (x,y) satisfies the conditions of theorem 3 and furthermore is such that the two limits

(53) $$\lim_{x \to x_1, y \to \pi} f(x,y), \qquad \lim_{x \to x_1, y \to -\pi} f(x,y)$$

exist, the Fourier's development of f(x,y) will be summable (Ck) for any value of k > 0, at the points (x_1, π) and $(x_1, -\pi)$ and its value will be one-half the sum of the two limits (53). Similarly, if y_1 lies in the interval $(-\pi < y < \pi)$ and the two limits

(54) $$\lim_{x \to \pi, y \to y_1} f(x,y), \qquad \lim_{x \to -\pi, y \to y_1} f(x,y)$$

exist, the series will be summable (Ck), for any value of $k > 0$ at the points (π, y_1) and $(-\pi, y_1)$ and to a value half way between the limits (54). Moreover, in all these cases the series will be summable (Ck), for any value of k > 0, by rows or by columns, and to the same value.

We have for the series (16) at the point (x_1, π)

$$\frac{S_{mn}^{(k)}(x_1,\pi)}{A_{mn}^{(k)}} = \frac{1}{\pi^2}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_{-\pi}^{0} f(x_1+2\alpha,\ \pi+2\beta)\ \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$

(55) $$= \frac{1}{\pi^2}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_{-\pi}^{-\pi/2} f(x_1+2\alpha,\ \pi+2\beta)\ \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$

$$+ \frac{1}{\pi^2}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_{-\pi/2}^{0} f(x_1+2\alpha,\ \pi+2\beta)\ \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$

In the first term, on the right hand side of (55) make the change of variable

$$\beta' = \beta + \pi$$

Then we have, after dropping the primes

$$(56)\quad \frac{S_{mn}^{(k)}(x_1,\pi)}{A_{mn}^{(k)}} = \frac{1}{\pi^2}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_0^{\pi/2} f(x_1+2\alpha, -\pi+2\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta + \frac{1}{\pi^2}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_{-\pi/2}^{0} f(x_1+2\alpha, \pi+2\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta$$

Introduce the new functions

$$(57)\quad \varphi_1(\alpha,\beta) = f(x_1+2\alpha, -\pi+2\beta) - f(x_1,-\pi) \quad \text{and} \quad \varphi_2(\alpha,\beta) = f(x_1+2\alpha, \pi+2\beta) - f(x_1,\pi)$$

into the first and second terms respectively on the right hand side of (56). Then (56) becomes

$$(58)\quad \begin{aligned} \frac{S_{mn}^{(k)}(x_1,\pi)}{A_{mn}^{(k)}} &= \frac{1}{\pi^2} f(x_1,-\pi)\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_0^{\pi/2} \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2} f(x_1,\pi)\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_{-\pi/2}^{0} \sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_0^{\pi/2} \varphi_1(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \\ &+ \frac{1}{\pi^2}\int_{-\frac{\pi+x_1}{2}}^{\frac{\pi-x_1}{2}}\int_{-\pi/2}^{0} \varphi_2(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta \end{aligned}$$

It is evident that $\varphi_1(\alpha,\beta)$ and $\varphi_2(\alpha,\beta)$ satisfy the conditions of lemma 4 in the region of integration involved. Hence each of the two terms above which involve these factors approaches zero as m and n become infinite. It follows from lemma 5 that as m and n become infinite the first and second terms on the right hand side of (58) approach the limits $\frac{1}{2}f(x_1,-\pi)$ and $\frac{1}{2}f(x_1,\pi)$ respectively. Hence

$$(59)\quad \lim_{m,n\to\infty} \frac{S_{mn}^{(k)}(x_1,\pi)}{A_{mn}^{(k)}} = \frac{1}{2}\left[f(x_1,\pi)+f(x_1,-\pi)\right]$$

By a similar treatment we can obtain the results stated above at the three other points $(x_1,-\pi)$, (π,y_1) and $(-\pi,y_1)$.

Since, in lemmas 4 and 5, no restrictions were placed upon the manner in which m and n become infinite, it follows that in each case the series is also summable (Ck), for any value of $k > 0$, by rows or by columns and to the same value.

The behavior of the double Fourier's series development of a function at a point of discontinuity (x_1, y_1) such that all other points of discontinuity of the function lie on two straight lines intersecting at right angles at the point (x_1, y_1) and parallel to the coordinate axes is given in the following theorem.

THEOREM 4. If (f(x,y) satisfies the conditions of theorem 3, and (x_1, y_1) is a point of discontinuity of f(x,y) such that every other point of discontinuity in its neighborhood lies on one of two straight lines parallel to the coordinate axes and intersecting each other at the point (x_1, y_1), and if furthermore f(x,y) approaches a definite value as we approach the point (x_1, y_1) thru each of the four regions into which the lines of discontinuity divide the neighborhood of (x_1, y_1), then the development of f(x,y) in a double Fourier's series will be summable (Ck), for any value of $k > 0$, at the point (x_1, y_1) and to a value which is one-fourth the sum of the four limiting values of f(x,y).

The proof of this theorem can be carried out by methods similar to those used in the proof of theorem 3 when the line of discontinuity was parallel to one of the coordinate axes.

The behavior of the series at the four corners of the region $(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ is expressed in the following corollary to theorem 4 which can be established by similar methods.

COROLLARY. *If $f(x,y)$ is a function that satisfies the conditions of theorem 3, and if furthermore the four limits*

$$\lim_{x\to\pi,\, y\to\pi} f(x,y), \quad \lim_{x\to\pi,\, y\to-\pi} f(x,y), \quad \lim_{x\to-\pi,\, y\to-\pi} f(x,y), \quad \lim_{x\to-\pi,\, y\to\pi} f(x,y) \tag{60}$$

exist, the Fourier's development of $f(x,y)$ will be summable (Ck), for any value of $k > 0$, at each of the points (π,π), $(\pi,-\pi)$, $(-\pi,-\pi)$ and $(-\pi,\pi)$ and to a value which is one-fourth the sum of the limits (60).

In the cases considered in theorem 4 and its corollary it is also true that the series is summable (Ck) for any value of $k > 0$ by rows or by columns and to the same value to which it is summable when summed as a double series.

5. Summability at Points of Discontinuity that lie on Curves.

We will consider here the summability of a double Fourier's series at a point of discontinuity of the developed function such that all other points of discontinuity of the function in the neighborhood of that point lie on a curve which passes thru that point. We will assume that the curve has a tangent at the point of discontinuity of the function and that any line passing thru that point will intersect the curve only in a finite number of points in the neighborhood of the point. We will also assume that the function approaches a definite limit as we approach the point from either side of the curve.

In discussing the summability of the series we will make use of the following three lemmas.

For the sake of brevity we have introduced the following notations

(61) $$T_{mn}^{(k)}(\alpha,\beta) = \frac{1}{A_{mn}^{(k)}} \sum_{i=1,j=1}^{m,n} A_{m+1-i,n+1-j}^{(k-1)} \frac{\sin(2i-1)\alpha \sin\alpha \sin(2j-1)\beta \sin\beta}{\alpha^2\beta^2}$$

(62) $$T_m^{(k)}(\alpha,\beta) = \frac{1}{A_m^{(k)}} \sum_{i=1}^{m} A_{m+1-i}^{(k-1)} \frac{\sin(2i-1)\alpha \sin\alpha}{\alpha^2}$$

From these definitions it is obvious that

$$T_{mn}^{(k)}(\alpha,\beta) = T_m^{(k)}(\alpha) T_n^{(k)}(\beta)$$

LEMMA 7. <u>If R is a region of the same nature as in lemma 4, the limits</u>

(63) $$\lim_{m,n\to\infty} \left[\iint_R \sigma_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta - \iint_R T_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right]$$

(64) $$\lim_{m,n\to\infty} \left[\iint_R \sigma_{mn}^{(k)}(\alpha,\beta) \frac{\sin\alpha \sin\beta}{\alpha\,\beta}\, d\alpha\, d\beta - \iint_R T_{mn}^{(k)}(\alpha,\beta)\, d\alpha\, d\beta\right]$$

<u>will each exist and be equal to zero.</u>

This lemma follows immediately from lemma 4 on writing $\varphi(\alpha,\beta)$ equal to

$$1 - \frac{\sin^2\alpha \sin^2\beta}{\alpha^2\beta^2}$$

for the first expression and

$$\frac{\sin\alpha \sin\beta}{\alpha\,\beta} - \frac{\sin^2\alpha \sin^2\beta}{\alpha^2\beta^2}$$

for the second expression.

LEMMA 8. <u>If $g(\alpha)$ is a function of α that has a derivative equal to λ at the point $\alpha=0$, and is such that the curve $\beta=g(\alpha)$ is intersected only a finite number of times in the neighbor-</u>

hood of the point ($\alpha=0, \beta=0$) by any line passing thru that point, the limit

(65) $$\lim_{m,n\to\infty}\left[\int_0^h T_m^{(k)}(\alpha)\left(\int_0^{g(\alpha)} T_n^{(k)}(\beta)\,d\beta\right)d\alpha - \int_0^h T_m^{(k)}(\alpha)\left(\int_0^{\lambda\alpha} T_n^{(k)}(\beta)\,d\beta\right)d\alpha\right]$$

where h is a constant between 0 and π, will exist and be equal to zero provided λ is different from zero.

Suppose, for definiteness, that $\lambda>0$ and the curve $\beta=g(\alpha)$ lies above the tangent line to the right of the origin until perhaps it intersects the tangent line. It is evident that λ' can be chosen greater than λ so that the line $\beta=\lambda'\alpha$ cuts the curve $\beta=g(\alpha)$ in a point to the right of the origin which is nearer the origin than any point in which the tangent line may cut the curve to the right of the origin. Let us designate by ℓ the α-coordinate of the first point to the right of the origin in which the line $\beta=\lambda'\alpha$ cuts the curve.

The expression in brackets in (65) can be written in the form

(66) $$\int_0^{\ell} T_m^{(k)}(\alpha)\left(\int_{\lambda\alpha}^{g(\alpha)} T_n^{(k)}(\beta)\,d\beta\right)d\alpha + \int_{\ell}^{h} T_m^{(k)}(\alpha)\left(\int_{\lambda\alpha}^{g(\alpha)} T_n^{(k)}(\beta)\,d\beta\right)d\alpha$$

It is easily seen that for any fixed value of ℓ the second term of (66) approaches zero as m and n become infinite, since

$\left|\sum_{j=1}^{n} A_{n+1-j}^{(k-1)}\sin(2j-1)\gamma\sin\gamma\right|<1$ for all values of γ.

The absolute value of the first term is less than

$$\int_0^l |T_m^{(k)}(\alpha)| \left(\int_{\lambda\alpha}^{\lambda'\alpha} |T_n^{(k)}(\rho)|\, d\rho\right) d\alpha < \int_0^l |T_m^{(k)}(\alpha)| \left(\int_{\lambda\alpha}^{\lambda'\alpha} \frac{d\rho}{\rho}\right) d\alpha$$

$$= \int_0^l |T_m^{(k)}(\alpha)| \ln\frac{\lambda'}{\lambda}\, d\alpha \qquad (\lambda > 0)$$

$$< K \ln\frac{\lambda'}{\lambda}$$

since

$$\int_0^l |T_m^{(k)}(\alpha)|\, d\alpha < \int_0^l |T_m^{(k)}(\alpha)|\, d\alpha < K \qquad (0 < l < \pi)$$

Hence, by choosing λ' near enough to λ we can make the first term in (66) as small as we please. It therefore follows that the limit as m and n become infinite of the expression in brackets in (65) exists and is equal to zero.

LEMMA 9. If the tangent to the curve $\rho = g(\alpha)$ is parallel to the α-axis, then the limit

(67) $$\lim_{m,n\to\infty} \left[\int_0^h T_m^{(k)}(\alpha) \left(\int_0^{g(\alpha)} T_n^{(k)}(\rho)\, d\rho\right) d\alpha\right]$$

will exist and be equal to zero provided m and n become infinite in such a manner that

(68) $$n/m < K \quad \text{a positive constant.}$$

The expression in brackets can be written in the form

(69) $$\int_0^l T_m^{(k)}(\alpha) \left(\int_0^{g(\alpha)} T_n^{(k)}(\rho)\, d\rho\right) d\alpha + \int_l^h T_m^{(k)}(\alpha) \left(\int_0^{g(\alpha)} T_n^{(k)}(\rho)\, d\rho\right) d\alpha$$

where l is determined as in lemma 8. It is easily seen that for any fixed value of l, the second term approaches zero as m and n become infinite in any manner. It follows from lemma 7 that the first term will approach the same limit as

$$I = \int_0^{\ell} \sigma_m^{(k)}(\alpha)\frac{\sin\alpha}{\alpha}\left(\int_0^{g(\alpha)}\sigma_n^{(k)}(\beta)\frac{\sin\beta}{\beta}\,d\beta\right)d\alpha \tag{70}$$

if this limit exists. If we place k=0 and replace 2m - 1 and 2n - 1 by m and n respectively, the integral I becomes

$$I_1 = \int_0^{\ell}\frac{\sin m\alpha}{\alpha}\left(\int_0^{g(\alpha)}\frac{\sin n\beta}{\beta}\,d\beta\right)d\alpha$$

It follows from the work of Titchmarsh* that the integral I_1 will approach the same limit as

$$I_2 = \int_0^{m\ell}\frac{\sin\alpha}{\alpha}\left(\int_0^{\frac{n\lambda'\alpha}{m}}\frac{\sin\beta}{\beta}\,d\beta\right)d\alpha$$

where $\lambda'\ell = g(\ell)$ provided ℓ is chosen sufficiently small and m and n become infinite subject to the restriction (66). Titchmarsh* has also shown that

$$\lim_{m,n\to\infty} I_2 < \int_0^{\lambda' k} \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t}$$

Since $\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$ remains finite for $0 \leqq t < 1$ it follows that

$$I_2 < \lambda' A$$

where A is a positive constant, provided m and n satisfy the restriction (68). By choosing ℓ sufficiently small we can make λ' and hence I_2 as small as we please. Hence, the lemma is true for the special case k = 0. It follows then, from theorem I, that the lemma is true for any value of k greater than zero.

THEOREM 5. If f(x,y) satisfies the conditions of theorem 3 and (x_1, y_1) is a point of discontinuity of f(x,y) such that every other point of discontinuity in the neighborhood of (x_1, y_1) lies on a curve satisfying the following conditions: a) the curve has a tangent at the point (x_1, y_1) that is not parallel to either of the coordinate axes, b) no line thru the point (x_1, y_1) intersects the curve in an infinite number

*Proc. Royal Soc. Lon. Series A, 106 (1924), p. (307)

of points in the neighborhood of that point, c) the function $f(x,y)$ approaches a definite value as we approach the point (x_1, y_1) from either side of the curve, then the development of $f(x,y)$ into a double Fourier's series will be summable (Ck), for any value of $k > 0$, at the point (x_1, y_1) and its value will be half way between the limiting values of the function at that point. If all the other conditions are fulfilled, but the tangent line to the curve at (x_1, y_1) is parallel to one of the axes, the series will be restrictedly summable (Ck), for any value of $k > 0$, to the same value at the point (x_1, y_1).

The proof of this theorem follows precisely on the lines of Moore's* proof for the case when $k = 1$ if we substitute the expression

$$T_{mn}^{(k)}(\alpha,\beta)$$

for the expression

$$\frac{1}{mn}\,\frac{\sin^2\alpha\,\sin^2\beta}{\alpha^2\beta^2}$$

wherever it occurs in his proof.

We have considered so far the summability of the Fourier's series development corresponding to a function which remains finite and integrable in the region $(-\pi < x < \pi, -\pi < y < \pi)$. By means of the following lemma the results may be easily extended to certain cases in which the function to be developed becomes infinite in this region.

* Math. Ann. 74(1913), pp. 567-570.

LEMMA 10. *Let R be a region satisfying the conditions of lemma 2 and $\varphi(\alpha,\beta)$ be a function that is finite and integrable in R except along certain lines of discontinuity where it becomes infinite while remaining absolutely integrable. Then if these lines are finite in number and such that they are either parallel to the coordinate axes or cut such parallels in a finite number of points, the limit*

$$\lim_{m,n\to\infty}\left[\iint_R \varphi(\alpha,\beta)\,\sigma_{mn}^{(k)}(\alpha,\beta)\,d\alpha\,d\beta\right] \tag{71}$$

will exist and be equal to zero provided no infinite discontinuities lie on or in the neighborhood of the coordinate axes.

The region R can be divided into a finite number of regions by adjoining to each line of discontinuity the portion of R which is in its immediate neighborhood. For each of the regions in which $\varphi(\alpha,\beta)$ remains finite it follows from lemma 2 that the limit of (71) exists and is equal to zero. Since there are no points of discontinuity in the neighborhood of the coordinate axes, we will have

$$\left|\sigma_{mn}^{(k)}(\alpha,\beta)\right| < \frac{1}{A_{mn}^{(k)}}\,\frac{1}{\sin^2\alpha\,\sin^2\beta}$$

$$< \frac{1}{A_{mn}^{(k)}}M \text{ a positive constant}$$

in each of the regions in which $\varphi(\alpha,\beta)$ becomes infinite. Hence, for each of these regions, the expression in (71) is less than

$$\frac{M}{A_{mn}^{(k)}}\iint_R |\varphi(\alpha,\beta)|\,d\alpha\,d\beta < \frac{M'}{m^k n^k}$$

since $\varphi(\alpha,\beta)$ is absolutely integrable in R. It is easily seen that the limit of this expression as m and n become infinite exists and is zero. Since there are only a finite number of regions, it follows that the limit in (71) exists and is zero.