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PART IV. GENERALIZED LINEAR; BERNOULLI AND RICCATI DIFFERENTIAL EQUATIONS

GROUP THEORY APPLIED TO VECTOR DIFFERENTIAL EQUATIONS

INTRODUCTION

The purpose of this paper is to show how the Lie theory* of continuous one - parameter groups of point transformations in four-space may be applied to the solution of vector differential equations of the type

(i) $$\nabla u = \underset{\approx}{F}(\underset{\sim}{r},u),$$

in which $\underset{\approx}{F}$ is a known vector function of the position vector $\underset{\sim}{r}$ and the dependent scalar point function $\underline{u}$. The groups used in solving the various integrable cases of this equation are expressed also in terms of $\underset{\sim}{r}$ and $\underline{u}$.

With the increasing importance of vector analysis in pure and applied mathematics there arises a very definite need for systematic research in the field of vector differential equations. Very little is

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* Lie-Scheffers, Vorlesungen uber Differentialgleichungen and Continuierliche Gruppen, Leipzig 1891,1896; Dickson, Differential Equations from the Group Standpoint, Annals of Mathematics, 1924, Series 2, V25,pp. 287-324; Page, Ordinary Differential Equations; Cohen, The Lie Theory of one - parameter Groups; Ince, Ordinary Differential Equations.

to be found in the literature on this subject. Even in The Collected Works of J. Willard Gibbs only a few short sections* are devoted to

* Vol. 2, Part 2, III, Chap. II, pp. 38-40.

equations of the simplest types. While it is true that a vector differential equation, expressed through its three scalar equivalents, may be solved by the classical theories of simultaneous partial differential equations, the integration is greatly simplified by the use of purely vectorial methods, wherein the form of the equation is left intact.

It is shown in §§ 3,4 that a solvable differential equation of the type (i) possesses either a complete or a singular integral, and that the determination of the latter type requires no integration. A necessary and sufficient condition for complete integrability is established in § 3.

In the following treatment of integrable vector differential equations from the group standpoint, the canonical forms of the groups leaving them invariant, and the canonical variables resulting from them, play a fundamental role. Philip Franklin has said* in reference to ordinary

* Franklin, The Canonical Form of a one-parameter Group, Annals of Mathematics, Vol. 29, Ser. 2, Apr. 1928, p. 113.

differential equations that "the use of these variables to simplify the usual proofs of some standard theorems, and the fact that they furnish the most direct way of applying the Lie theory to known cases of solvable differential equations, seems to have escaped attention." In the writer's first draft of the present paper, no particular stress was laid on the application of canonical variables to the development of the theory. However, upon the suggestion of Professor Louis Brand, the treatment was revised so as to rest almost completely on the use of canonical variables. This effected a material simplification, just as Franklin's work simplified the corresponding theory for ordinary differential equations.

Another fact that seems to have escaped notice is that the theory of ordinary differential equations of the first order, from the group standpoint, may be expounded without the use of Lie's extended transformation. How this might be done may be inferred by anology from the present treatment of vector differential equations, in which the extended transformation is avoided completely.

It will be shown that an integrable equation of the type (1) in general may be solved by the introduction of an integrating factor determined along lines slightly different from those employed by Lie for ordinary differential equations. A similar method is applicable to equations of the latter type.

The writer wishes to express his most sincere thanks to Professor Brand for his careful review of the paper and for his most valuable suggestions.

PART I. ON THE SOLUTION OF THE EQUATION $A\nabla u - B = 0$

1. The Gradient. Change of Variable. If u is a scalar point function of the position vector r, the gradient of a scalar or vector function $\Psi(r,u)$ with respect to r may be written

$$\nabla\Psi(r,u) \equiv \nabla_r\Psi + \nabla u\frac{\partial\Psi}{\partial u}, \tag{1.1}$$

in which $\nabla_r\Psi$ represents the partial gradient of Ψ, obtained by holding u constant.

If Ψ is a vector function, its rotation and partial rotation are defined by the operations

$$\text{rot}\,\Psi \equiv \nabla\times\Psi, \qquad \text{rot}_r\Psi \equiv \nabla_r\times\Psi,$$

whence by (1.1)

$$\text{rot}\,\Psi(r,u) \equiv \text{rot}_r\Psi + \nabla u\times\frac{\partial\Psi}{\partial u}. \tag{1.2}$$

Let two vectors r and v be connected by the inverse relations

$$v = v\,(r), \qquad r = r\,(v).$$

Then if F is a scalar or vector function of v, its gradient with respect to v, denoted by the symbol $\nabla^v F$, is given by the formula*

*J. Spielrein, Vectorrechnung, Zweite Auflage, s. 387.

$$\nabla F\,(v) \equiv \nabla v\cdot\nabla^v F\,(v). \tag{1.3}$$

If r and v are functions also of a scalar point function u, and possess the inverse relationships

$$v = v\,(r,u), \qquad r = r\,(v,u),$$

it follows from (1.1) and (1.3) that

(1.4) $$\nabla \underline{\underline{F}}\,(\underset{\sim}{v}(\underset{\sim}{r},u)) \equiv (\nabla_r \underset{\sim}{v} + \nabla u \frac{\partial \underset{\sim}{v}}{\partial u}) \cdot \nabla^v \underline{\underline{F}}\,(\underset{\sim}{v}).$$

In particular, if $\underline{\underline{F}}\,(\underset{\sim}{v}) \equiv \underset{\sim}{r}\,(\underset{\sim}{v})$, and $\underline{u}$ is held constant, (1.4) shows that

(1.5) $$\nabla_r \underset{\sim}{v}\,(\underset{\sim}{r},u) \cdot \nabla_v \underset{\sim}{r}\,(\underset{\sim}{v},u) \equiv \underline{\underline{I}},$$

where $\underline{\underline{I}}$ is the self-conjugate dyadic idemfactor.

2. Corresponding Ordinary and Partial Vector Differential Equations.

Differential equations of the type

(2.1) $$\nabla u = \underline{\underline{F}}\,(\underset{\sim}{r},u),$$

when simplified algebraically, usually appear in the form

(2.2) $$A\,(\underset{\sim}{r},u)\,\nabla u - \underline{\underline{B}}\,(\underset{\sim}{r},u) = 0 \qquad (A \neq 0)$$

in which $\underline{A}$ and $\underline{\underline{B}}$ are known scalar and vector functions respectively. The latter form therefore will be adopted except where the former expedites the treatment. Obviously any integral is a scalar function explicitly involving $\underline{u}$. The trivial case for which $\underline{\underline{B}} \equiv 0$ will be excluded.

THEOREM 2A. *If $\Phi(\underset{\sim}{r},\underline{u}) = \underline{c}$ is a complete integral* of the ordinary vector differential equation.*

(2.2) $$A(\underset{\sim}{r},u)\,\nabla u - \underline{\underline{B}}\,(\underset{\sim}{r},u) = 0, \qquad (A \neq 0,\ \underline{\underline{B}} \neq 0)$$

$\Phi(\underset{\sim}{r},u)$ is a solution of the corresponding vector partial differential equation

(2.3) $$\underline{\underline{P}}f \equiv A\,(\underset{\sim}{r},u)\,\nabla_r f + \underline{\underline{B}}\,(\underset{\sim}{r},u)\,\frac{\partial f}{\partial u} = 0,$$

*This designation agrees with the accepted classification of integrals of simultaneous partial differential equations of the first order. See Goursat, Vorlesungen uber die Integration der Differentialgleichungen erster Ordnung, Leipzig, 1893.

and conversely.

For by virtue of the assumed integral $\Phi = c$, (1.1) and (2.2) show that

$$\nabla_r \Phi + \nabla u \frac{\partial \Phi}{\partial u} \equiv 0, \quad A\nabla u - \underset{\approx}{B} \equiv 0$$

identically in $\underset{\sim}{r}$ and $\underline{u}$. Therefore

(2.4) $$A\nabla_r \Phi + \underset{\approx}{B} \frac{\partial \Phi}{\partial u} \equiv 0.$$

Conversely, if a function $\Phi(\underset{\sim}{r},u)$, other than a constant satisfies (2.3), the identity (2.4) holds, and by (1.1), $\Phi = c$ implies

(2.5) $$A\nabla_r \Phi + A\nabla u \frac{\partial \Phi}{\partial u} \equiv 0,$$

whence on subtracting (2.4) from (2.5),

$$A\nabla u - \underset{\approx}{B} \equiv 0,$$

since $\partial\Phi/\partial\underline{u} \neq 0$. Therefore $\Phi = \underline{c}$ is a complete integral of (2.2).

THEOREM 2B. *If $\Phi(\underset{\sim}{r},u)$ is a solution of $\underset{\approx}{P}f = 0$, then any scalar function $G(\Phi)$ of Φ is also a solution.*

For by (2.3),

$$\underset{\approx}{P}G(\Phi) \equiv A\nabla_r G + \underset{\approx}{B}\frac{\partial G}{\partial u} \equiv G'(\Phi)\left(A\nabla_r \Phi + \underset{\approx}{B}\frac{\partial \Phi}{\partial u}\right),$$

which vanishes identically by hypothesis.

COROLLARY. *If $\Phi(\underset{\sim}{r},\underline{u}) = \underline{c}$ is a complete integral of $A\nabla\underline{u} - \underset{\approx}{B} = 0$, and $G(\Phi)$ is any function of Φ, then $G(\Phi) = \underline{c}'$ is also a complete integral.*

3. Condition for the Existence of a Complete Integral.

Not every equation of the type (2.1) or (2.2) has an integral. It will develop that if one exists it must be either a complete integral of the form $\Phi(\underset{\sim}{r},\underline{u}) = \underline{c}$, where $\underline{c}$ is an arbitrary constant, or an integral of the form $\Psi(\underset{\sim}{r},\underline{u}) = 0$, involving no arbitrariness. The latter type will be designated as a singular integral, in conformity with the accepted classification* of integrals of simultaneous partial differential equat-

* See Goursat, loc. cit., o. 140.

ions of the first order. What follows is intended to obviate useless attempts to solve an equation of the type (2.1) or (2.2), when no solution exists.

THEOREM 3. *A necessary and sufficient condition that the differential equation*

$$(2.1)\qquad \nabla u = \underline{\underline{F}}\,(\underline{r},u)$$

possess a complete integral is that

$$(3.1)\qquad \underline{\underline{\Omega}}'(\underline{r},u) \equiv \mathrm{rot}_r\,\underline{\underline{F}} + \underline{\underline{F}}\times\frac{\partial \underline{\underline{F}}}{\partial u} = 0,$$

identically in $\underline{r}$ *and* u.

To prove the necessity assume the existence of a complete integral $\Phi = c$ satisfying (2.1). Then $\nabla \underline{u} \equiv \underline{\underline{F}}$, whence rot $\underline{\underline{F}} \equiv 0$*. Therefore by

* Gibb's - Wilson, p.167, (52).

(1.2) and (2.1), $\underline{\underline{\Omega}}'(\underline{r},\underline{u}) \equiv 0$.

The proof of the sufficiency requires the

LEMMA. *The condition* $\underline{\underline{\Omega}}' \equiv 0$ *is satisfied if and only if*

$$\underline{\underline{P}}' \times \underline{\underline{P}}' f \equiv 0$$

in which f *is an arbitrary scalar function of* $\underline{r}$ and u *and* $\underline{\underline{P}}'$ *is defined in accordance with* (2.3) *as the operator*

$$\underline{\underline{P}}' \equiv \frac{1}{A}\,\underline{\underline{P}} \equiv \nabla_r + \underline{\underline{F}}\frac{\partial}{\partial u}\,.$$

First assume $\underline{\underline{\Omega}}' \equiv 0$ and deduce $\underline{\underline{P}}' \times \underline{\underline{P}}' f \equiv 0$. Evidently

$$\mathbf{P}'\times\mathbf{P}'f \equiv \left(\nabla_r+\mathbf{F}\frac{\partial}{\partial u}\right)\times\left(\nabla_r f+\mathbf{F}\frac{\partial f}{\partial u}\right)$$

$$\equiv \nabla_r\times\left(\mathbf{F}\frac{\partial f}{\partial u}\right)+\mathbf{F}\times\frac{\partial}{\partial u}\nabla_r f+\mathbf{F}\times\frac{\partial\mathbf{F}}{\partial u}\,\frac{\partial f}{\partial u}$$

$$\equiv \left(\nabla_r\,\frac{\partial f}{\partial u}\right)\times\mathbf{F}+\frac{\partial f}{\partial u}\,\mathrm{rot}_r\,\mathbf{F}+\mathbf{F}\times\frac{\partial}{\partial u}\nabla_r f+\mathbf{F}\times\frac{\partial\mathbf{F}}{\partial u}\,\frac{\partial f}{\partial u}\ *$$

*See Gibb's - Wilson, p.157,(40) for the expansion of $\nabla_r\times(\mathbf{F}\partial f/\partial u)$.

$$\equiv \left(\mathrm{rot}_r\mathbf{F}+\mathbf{F}\times\frac{\partial\mathbf{F}}{\partial u}\right)\frac{\partial f}{\partial u}\ ,^{*}$$

*The reader will observe that the cancellation of the first and third terms of the preceding line is valid for an extremely broad class of functions, and in fact is quite analogous to Jacobi's cancellation of the second partial derivatives occurring in the deduction of the commutator employed in his classical theory of partial differential equations of the first order (Goursat,loc.cit.,p.140). It may be shown that the cancellation is justified whenever $\frac{\partial}{\partial r}\nabla_r f$ and $\nabla_r\frac{\partial f}{\partial u}$ exist and are continuous.

which vanishes identically by hypothesis.

The converse is implied by the reversibility of these steps.

To prove the sufficiency of the foregoing theorem let the three scalar components of the operator $\mathbf{P}'$ be denoted by

$$P'_x\equiv\frac{\partial}{\partial x}+F_x\frac{\partial}{\partial u}\ ,\quad P'_y\equiv\frac{\partial}{\partial y}+F_y\frac{\partial}{\partial u},\quad P'_z\equiv\frac{\partial}{\partial z}+F_z\frac{\partial}{\partial u}\ ,$$

in which F_x,F_y, F_z are the three components of $\mathbf{F}$. The condition $\mathbf{P}'\times\mathbf{P}'f\equiv 0$ may now be expressed in the form

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ P'_x & P'_y & P'_z\\ P'_xf & P'_yf & P'_zf\end{vmatrix}\equiv 0\ ,$$

in which $\mathbf{i},\mathbf{j},\mathbf{k}$ represent a right-handed set of mutually orthogonal unit vectors in the directions of the axes of x,y,z respectively. The coefficients of $\mathbf{i},\mathbf{j},\mathbf{k}$ must vanish identically. But these coefficients are precisely the three commutators of $P'_x f, P'_y f, P'_z f$. Thus

$$P'_y(P'_z f) - P'_z(P'_y f) \equiv 0,$$

$$P'_z(P'_x f) - P'_x(P'_z f) \equiv 0,$$

$$P'_x(P'_y f) - P'_y(P'_x f) \equiv 0.$$

Hence the linearly independent simultaneous partial differential equations

(3.2) $$P'_x f = 0, \quad P'_y f = 0, \quad P'_z f = 0,$$

form a Jacobian system of three equations in four independent variables (x, y, z, u), whose solutions are therefore all functions of exactly $4 - 3$, or one, existing, common solution $\Phi(\underset{\sim}{r}, u)$ not a constant*. Since the

* Goursat - Hedrick, vol.2, part 2, p.270. Dickson, loc.cit., pp. 329 - 331.

system (3.2) is equivalent to the equation $\underline{\underline{\underline{P}}}'f = 0$, Φ is a solution of the latter. Hence by Theorem 2 *A* there exists a complete integral $\Phi(\underset{\sim}{r}, u) = c$ of (2.1). Furthermore since all solutions of (3.2), and hence also those of $\underline{\underline{\underline{P}}}'f = 0$, are functions of Φ, all integrals of (2.1) are of the form $G(\Phi) = c$.

COROLLARY. *A necessary and sufficient condition that the differential equation*

(2.2) $$A(\underset{\sim}{r}, u)\nabla u - \underline{\underline{\underline{B}}}(\underset{\sim}{r}, u) = 0$$

possess a complete integral is that

(3.3) $$\underline{\underline{\underline{\Omega}}}(\underset{\sim}{r}, u) \equiv \underline{\underline{\underline{B}}} \times \nabla_r A + A\, \mathrm{rot}_r \underline{\underline{\underline{B}}} + \underline{\underline{\underline{B}}} \times \frac{\partial \underline{\underline{\underline{B}}}}{\partial u} \equiv 0,$$

identically in $\underset{\sim}{r}$ *and* u.

EXAMPLE. Consider the differential equation

$$\nabla u - u\phi_1^{-1}\nabla\phi_1 + \phi_1 f(\phi_2)\nabla\phi_2 = 0,$$

in which Φ_1 and Φ_2 are arbitrary scalar point functions and f is known. Evidently $\underline{\underline{F}} \equiv u\phi_1^{-1}\nabla\phi_1 - \phi_1 f(\phi_2)\nabla\phi_2$. It follows that

$$\mathrm{rot}_r \underline{\underline{F}} \equiv -\underline{\underline{F}} \times \frac{\partial \underline{\underline{F}}}{\partial u} \equiv -f(\phi_2)\nabla\phi_1 \times \nabla\phi_2 .$$

Thus $\underline{\underline{\Omega}}' \equiv 0$; hence the differential equation possesses a complete integral. It is readily found to be

$$\frac{u}{\phi_1} + \int f(\phi_2)\,d\phi_2 = c$$

by the introduction of the integrating factor ϕ_1^{-1}.

4. Singular Integral. It is possible that $\underline{\underline{\Omega}}$ (or $\underline{\underline{\Omega}}'$) may not vanish identically, yet may be rendered zero by virtue of a particular relation of the form $\underline{\underline{\Psi}}(\underline{r},\underline{u}) = 0$.* If this relation, obtained without integration and hence involving no arbitrariness, satisfies (2.2)(or(2.1)), it is called a singular integral. No other type of integral can exist. If $\underline{\underline{\Omega}} = 0$ (or $\underline{\underline{\Omega}}' = 0$) has no solution, or if (as may be possible) its solution fails to satisfy (2.2) (or(2.1)), the latter is insolvable.

EXAMPLE 1. Consider the differential equation

$$(\phi_1 + \phi_2)(\nabla u - a\nabla\phi_2) = u\nabla\phi_1$$

in which ϕ_1 and ϕ_2 are arbitrary scalar point functions and a is constant.

$$A \equiv \phi_1 + \phi_2, \qquad \underline{\underline{B}} \equiv u\nabla\phi_1 + a(\phi_1 + \phi_2)\nabla\phi_2,$$

$$\underline{\underline{B}} \times \nabla_r A \equiv [u - a(\phi_1 + \phi_2)]\nabla\phi_1 \times \nabla\phi_2,$$

$$A\, \mathrm{rot}_r \underline{\underline{B}} \equiv a(\phi_1 + \phi_2)\nabla\phi_1 \times \nabla\phi_2 \equiv -\underline{\underline{B}} \times \frac{\partial \underline{\underline{B}}}{\partial u} .$$

Since $\underline{\underline{\Omega}} \equiv [u - a(\phi_1 + \phi_2)]\nabla\phi_1 \times \nabla\phi_2 \not\equiv 0$, (unless $\nabla\phi_1 \times \nabla\phi_2 \equiv 0$), no complete integral exists. However, the solution

$$u = a(\phi_1 + \phi_2),$$

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* Often most easily obtained by solving $\underline{\underline{\Omega}} = 0$ (or $\underline{\underline{\Omega}}' = 0$) algebraically for $\underline{u}$ and thus obtaining the relation in the form $\underline{u} = \underline{f}(\underline{r})$.

of $\Omega = 0$, satisfies the original equation and is therefore a singular integral. By the foregoing theory, no other integral can exist.

If $\nabla\phi_1 \times \nabla\phi_2 \equiv 0$, that is, if ϕ_1 and ϕ_2 are connected by a functional relationship $\phi_1 = \phi$, $\phi_2 = f(\phi)$, the resulting differential equation

$$\nabla u - \frac{\nabla\phi}{\phi + f(\phi)} u = a f'(\phi)\nabla\phi$$

must possess a complete integral, which by § 3 must include the solution $u_1 = a(\phi_1 + \phi_2) = a[\phi + f(\phi)]$ for a particular value of the constant of integration. Evidently

$$\nabla(u - u_1) - \frac{\nabla\phi}{\phi + f(\phi)}(u - u_1) = 0.$$

Hence the complete integral is

$$\ln\left(\frac{u - u_1}{c}\right) = \int\frac{d\phi}{\phi + f(\phi)} \qquad \text{or} \qquad u = a[\phi + f(\phi)] + ce^{\int\frac{d\phi}{\phi + f(\phi)}}.$$

EXAMPLE 2. Consider the equation

$$\nabla u + \phi_1 u \ln u \,\nabla\phi_2 + u\nabla\phi_1 = 0,$$

in which ϕ_1 and ϕ_2 are any scalar point functions satisfying the condition $\nabla\phi_1 \times \nabla\phi_2 \not\equiv 0$. Since

$$\Omega' \equiv u(\phi_1 + \ln u)\nabla\phi_1 \times \nabla\phi_2 \not\equiv 0,$$

no complete integral exists. Nor does the solution $u = e^{\phi_1}$, of $\Omega' = 0$, satisfy the original equation. The latter is therefore insolvable.

PART II. LIE GROUPS OF TRANSFORMATIONS IN A SPACE OF FOUR DIMENSIONS

5. *The Group Concept.* If, in addition to the three ordinary space coordinates associated with a point r, another coordinate, u, is introduced, (r, u) may be regarded as a point in four-space. In what

follows, a systematic study will be made of the effect of transforming such a point by means of transformations belonging to a Lie group.

The set of transformations

$$r_1 = \phi(r,u,a), \quad u_1 = \psi(r,u,a)$$

in which the parameter a varies continuously, and ϕ and ψ are functionally independent, is said to constitute a continuous, one-parameter Lie group provided

(i) the product of any two (distinct or coincident) transformations of the set is itself one of the transformations of the set, and

(ii) the inverse of any transformation of the set is itself one of the transformations of the set.

6. Groups determined by Differential Equations. It will now be shown that each system of differential equations of the type

$$\frac{dr_1}{dt} = \xi(r_1,u_1), \quad \frac{du_1}{dt} = \eta(r_1,u_1) \tag{6.1}$$

in the variables r_1, u_1 and the parameter t, determines a continuous, one-parameter Lie group, and hence that any number of such groups may be produced at will, each corresponding to a chosen pair of values for the functions ξ and η.

Let $v(r_1,u_1) = c$ be an integral of the equation

$$\frac{dr_1}{du_1} = \frac{\xi(r_1,u_1)}{\eta(r_1,u_1)}, \qquad (\eta \neq 0).$$

If $v = c$ is solved for r_1 in terms of u_1 and c and the result $r_1 = r_1(u_1,c)$, is substituted in the second of equations (6.1) and the latter integrated, the result, expressed in terms of $\mathbf{r}_1$ and u_1 takes the form

$$w\,(\mathbf{r}_1,u_1) = t + c'.$$

If $\mathbf{r}_1 = \mathbf{r}$ and $u_1 = u$ when $t = 0$, the solution of the system (6.1) becomes

$$\mathbf{v}(\mathbf{r}_1,u_1) = \mathbf{v}(\mathbf{r},u)\;, \quad w(\mathbf{r}_1,u_1) = w(\mathbf{r},u) + t, \tag{6.2}$$

whose equivalent solved form is

$$\mathbf{r}_1 = \boldsymbol{\phi}(\mathbf{r},u,t), \quad u_1 = \psi(\mathbf{r},u,t). \tag{6.3}$$

It is easy to show that (6.2) and hence (6.3) constitute a set of transformations forming a one-parameter Lie group.

For example if $\boldsymbol{\xi}(\mathbf{r}_1,u_1) \equiv \mathbf{r}_1$, $\eta(\mathbf{r}_1,u_1) \equiv u_1$, the corresponding group is determined by solving the pair of equations

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{r}_1, \quad \frac{du_1}{dt} = u_1$$

subject to the condition that $\mathbf{r}_1 = \mathbf{r}$, $u_1 = u$ when $t = 0$. In this case

$$\mathbf{r}_1 = e^t\mathbf{r},\; u_1 = e^t u, \quad \text{or} \quad \mathbf{r}_1 = a\mathbf{r},\; u_1 = au, \quad (a = e^t).$$

That these transformations form a group may readily be verified independently.

7. The Infinitesimal Transformation. Let a scalar or vector function $\mathbf{f}(\mathbf{r},u)$ be subjected to the transformtion (6.3) and consider the rate at which the transformed function $\mathbf{f}(\mathbf{r}_1,u_1)$ changes with t at a fixed point $(\mathbf{r},u)$ for which $t = 0$. Evidently

$$\frac{d\mathbf{f}(\mathbf{r}_1,u_1)}{dt} \equiv \frac{d\mathbf{r}_1}{dt} \cdot \nabla_{r_1}\mathbf{f}(\mathbf{r}_1,u_1) + \frac{\partial \mathbf{f}(\mathbf{r}_1,u_1)}{\partial u_1}\frac{du_1}{dt}\;;$$

whence by (6.1),

$$\frac{d\mathbf{f}(\mathbf{r}_1,u_1)}{dt} \equiv \boldsymbol{\xi}(\mathbf{r}_1,u_1) \cdot \nabla_{r_1}\mathbf{f}(\mathbf{r}_1,u_1) + \eta(\mathbf{r}_1,u_1)\frac{\partial \mathbf{f}(\mathbf{r}_1,u_1)}{\partial u_1}. \tag{7.1}$$

Place $t = 0$. Then $\mathbf{r}_1,u_1$ become $\mathbf{r},u$. Denote the result by $U\mathbf{f}(\mathbf{r},u)$. Then

$$\left[\frac{d\mathbf{f}(\mathbf{r}_1,u_1)}{dt}\right]_{t=0} \equiv U\mathbf{f}(\mathbf{r},u) \equiv \boldsymbol{\xi}(\mathbf{r},u) \cdot \nabla_r \mathbf{f}(\mathbf{r},u) + \eta(\mathbf{r},u)\frac{\partial \mathbf{f}(\mathbf{r},u)}{\partial u}. \tag{7.2}$$

Uf will be called the (symbol of the) infinitesimal transformation which generates the equations (6.3) of the group; for if Uf is given, ξ and η are known and hence the equations (6.1) yield (6.3). Conversely the equations (6.3) determine Uf; for by (6.1) and (6.3),

$$\left[\frac{dr_1}{dt}\right]_{t=0} \equiv \left[\frac{\partial \phi}{\partial t}\right]_{t=0} \equiv \xi(r,u), \quad \left[\frac{du_1}{dt}\right]_{t=0} \equiv \left[\frac{\partial \psi}{\partial t}\right]_{t=0} \equiv \eta(r,u).$$

8. Series Representation of a Transformed Function. If r and u are regarded as fixed, the transformed function $f(r_1,u_1)$ of any scalar or vector function $f(r,u)$, resulting from the application of (6.3), depends only on t. Hence for sufficiently small values of t it will be assumed expansible into the convergent Maclaurin series

$$f(r_1,u_1) \equiv f_0 + f_0{}' t + \frac{1}{2!} f_0{}'' t^2 + \frac{1}{3!} f_0{}''' t^3 + \cdots + \frac{1}{n!} f_0^{(n)} t^n + \ldots *$$

* Burali - Forti et Marcolongo, Calcul Vectoriel, p.65.

in which

$$f_0 \equiv \left[f(r_1,u_1)\right]_{t=0} \equiv f(r,u), \quad f_0^{(n)} \equiv \left[\frac{d^n f(r_1,u_1)}{dt^n}\right]_{t=0} \equiv U^n f(r,u),$$

where $U^n f$ symbolizes the result of n successive applications of the operator $U \equiv \xi\cdot\nabla_r + \eta\,\partial/\partial u$ to the function $f(r,u)$. Thus

(8.1) $$f(r_1,u_1) \equiv f + t\,Uf + \frac{t^2}{2!}U^2 f + \frac{t^3}{3!}U^3 f + \cdots\cdots + \frac{t^n}{n!}U^n f + \cdots\cdots *$$

* The equations of the finite transformations of the group generated by Uf are found from (8.1) by identifying $f(r_1,u_1)$ with r_1 and u_1 in turn.

9. Invariant Surfaces. A family of four-space surfaces $\omega(r,u) = c$ is said to be invariant under a group if each surface is transformed into the same surface, or if the surfaces of the family are permuted among themselves by every transformation of the group.. In the

former case the surfaces are said to be *individually invariant* - in the latter case *collectively invariant*.

THEOREM 9 A. *A family of surfaces $\omega(\mathbf{r},u) = c$ is invariant under the group generated by the infinitesimal transformation Uf if and only if $U\omega$ is a function of ω ; $U\omega \equiv F_1(\omega)$.*

The surfaces are individually invariant if and only if $F_1(\omega) \equiv 0$, and collectively invariant if and only if $F_1(\omega) \not\equiv 0$.

Evidently two families of surfaces $\omega(\mathbf{r},u) = c$, $\omega'(\mathbf{r},u) = c'$ are identical if and only if ω is a function of ω'. Hence the family $\omega = c$ is invariant under Uf if and only if

$$\omega(\mathbf{r}_1,u_1) \equiv F(\omega(\mathbf{r},u),t) \equiv F(c,t) \tag{9.1}$$

for all pairs of solutions $\mathbf{r},u$ of $\omega(\mathbf{r},u) = c$. In (9.1), t may or may not enter, according as the surfaces are collectively or individually invariant. By (7.2),

$$U\omega \equiv \left[\frac{dF(c,t)}{dt}\right]_{t=0}.$$

Hence $U\omega$ is either zero or a function of c, according as the surfaces are individually or collectively invariant. In either case $U\omega$ may be considered a function of ω.

Conversely, if $U\omega \equiv F_1(\omega)$,

$$U^2\omega \equiv U(U\omega) \equiv UF_1(\omega) \equiv U\omega\frac{dF_1}{d\omega} \equiv F_2(\omega).$$

Similarly, $U^n\omega \equiv F_n(\omega)$, $(n = 1,2,3,\cdots\cdots)$. Consequently, by (8.1),

$$\omega(\mathbf{r}_1,u_1) \equiv \omega(\mathbf{r},u) + tF_1(\omega) \quad \frac{t^2}{2!}F_2(\omega) + \cdots\cdots + \frac{t^n}{n!}F_n(\omega) + \cdots\cdots. \tag{9.2}$$

If $F_1(\omega) \equiv 0$, then $U^n\omega \equiv F_n(\omega) \equiv 0$ $(n = 1,2,3,\cdots\cdots)$, and $\omega(\mathbf{r}_1,u_1) \equiv \omega(\mathbf{r},u)$, so that the surfaces $\omega = c$ are individually invariant under

$U\underset{\sim}{f}$. If $F_1(\omega) \not\equiv 0$, (9.2) is of the form (9.1), explicitly involving $\underline{t}$; hence the surfaces $\omega = c$ are collectively invariant under $U\underset{\sim}{f}$.

THEOREM 9 B. *The most general family of surfaces $\omega(\underset{\sim}{r},u) = c$, collectively invariant under the group generated by the infinitesimal transformation $U\underset{\sim}{f}$, are obtained by equating to an arbitrary scalar constant the most general function $\omega(\underset{\sim}{r},u)$ determined by the equation $U\omega = 1$.*

By Theorem 9A, the function $\omega(\underset{\sim}{r},u)$, occuring in the equation $\omega(\underset{\sim}{r},u) = c$ of the most general family of surfaces collectively invariant under $U\underset{\sim}{f}$, satisfies the equation

$$U\omega \equiv F_1(\omega) \not\equiv 0.$$

If ϕ is an arbitrary scalar function, the surfaces $\phi(\omega) = c'$ are evidently identical with the surfaces $\omega = c$. Therefore it is necessary only to establish the existence of a function ϕ, such that

$$1 = U\phi(\omega) \equiv \frac{d\phi}{d\omega} U\omega \equiv \frac{d\phi}{d\omega} F_1(\omega).$$

By hypothesis, $F_1(\omega) \not\equiv 0$. Hence there is a region in which the integral

$$\phi(\omega) = \int \frac{d\omega}{F_1(\omega)}$$

exists.

The most general families of surfaces collectively and individually invariant under the group generated by the infinitesimal transformation $U\underset{\sim}{f} \equiv \partial\underset{\sim}{f}/\partial u$ are readily found from the equations $U\omega = 1$, $U\omega = 0$, to be

$$u + f(\underset{\sim}{r}) = c, \qquad g(\underset{\sim}{r}) = c'$$

respectively. The differential equations of these families have the respective forms

$$\nabla u = \underset{\sim}{\underline{\underline{F}}}(\underset{\sim}{r}), \qquad \nabla g(\underset{\sim}{r}) = 0.$$

PART III. APPLICATION TO THE SOLUTION OF DIFFERENTIAL EQUATIONS

10. Invariant Differential Equations. Canonical Variables. A differential equation $A\nabla u - \underline{B} = 0$ is said to be invariant under a group if it possesses integral surfaces $\Phi = c$ which are individually or collectively invariant under the group.

It follows from §6 (by an interchange of subscripts) that the infinitesimal transformation

$$Uf \equiv \xi(r,u)\cdot\nabla_r f + \eta(r,u)\frac{\partial f}{\partial u} \qquad (\eta \not\equiv 0)$$

furnishes, and is also furnished by, the differential equations

$$\text{(10.1)} \qquad \frac{dr}{dt} = \xi(r,u), \quad \frac{du}{dt} = \eta(r,u),$$

the solutions of which are

$$\text{(10.2)} \qquad v(r,u) = v(r_1,u_1), \quad w(r,u) = w(r_1,u_1) + t.$$

Under the transformation of variables

$$\text{(10.3)} \qquad v = v(r,u), \quad w = w(r,u),$$

the equations (10.2) of the group assume the canonical form

$$\text{(10.4)} \qquad v = v_1, \qquad w = w_1 + t.$$

The variables v and w are called a pair of canonical variables for the group. Evidently

$$0 \equiv \frac{dv}{dt} \equiv \frac{dr}{dt}\cdot\nabla_r v + \frac{du}{dt}\frac{\partial v}{\partial u} \equiv \xi\cdot\nabla_r v + \eta\frac{\partial v}{\partial u},$$
$$1 \equiv \frac{dw}{dt} \equiv \frac{dr}{dt}\cdot\nabla_r w + \frac{du}{dt}\frac{\partial w}{\partial u} \equiv \xi\cdot\nabla_r w + \eta\frac{\partial w}{\partial u}.$$

Therefore

(10.5) $\quad U\underset{\sim}{v} \equiv 0, \quad Uw \equiv 1.$

The infinitesimal transformation $U'\underset{\sim}{f}$, of the canonical form of the group, is found from (10.4) just as $U\underset{\sim}{f}$ is found from (10.1). Thus

$$U'\underset{\sim}{f} \equiv \underset{\sim}{\xi}'\cdot\nabla_v\underset{\sim}{f} + \eta'\frac{\partial \underset{\sim}{f}}{\partial w} \equiv \frac{d\underset{\sim}{v}}{dt}\cdot\nabla_v\underset{\sim}{f} + \frac{dw}{dt}\frac{\partial \underset{\sim}{f}}{\partial w} \equiv \frac{\partial \underset{\sim}{f}}{\partial w}.$$

From the discussion at the end of §9, it follows that the differential equation of the most general family of surfaces collectively invariant under the canonical form (10.4) of the group is of the form

(10.6) $\quad \nabla^v w = \underset{\equiv}{F}(\underset{\sim}{v}),$

while that of the individually invariant surfaces is

(10.7) $\quad \nabla^v g(\underset{\sim}{v}) = 0, \quad \text{or} \quad \nabla g(\underset{\sim}{v}) = 0,$

since $\quad \nabla\underset{\sim}{v}\cdot\nabla^v g(\underset{\sim}{v}) = \nabla g(\underset{\sim}{v})$, by (1.3).

Let us now suppose that a differential equation

(10.8) $\quad A(\underset{\sim}{r},u)\nabla u - \underset{\equiv}{B}(\underset{\sim}{r},u) = 0$

is invariant under the group generated by the infinitesimal transformation

$$U\underset{\sim}{f} \equiv \underset{\sim}{\xi}\cdot\nabla_r\underset{\sim}{f} + \eta\frac{\partial \underset{\sim}{f}}{\partial u}, \qquad (\eta \neq 0)$$

and investigate the conditions under which the canonical variables (10.3) of the group reduce the differential equation to the form of (10.6). Clearly this is a pertinent question; for if such a reduction is possible, a single quadrature yields the integral

(10.9) $\quad \Phi \equiv w - \int \underset{\equiv}{F}(\underset{\sim}{v})\cdot d\underset{\sim}{v} = c.$

Under the transformation to canonical variables, (10,8) will define w as a function of $\underset{\sim}{v}$. Hence by (1.4),

(10.91) $\quad \left(\nabla_r\underset{\sim}{v} + \nabla u\frac{\partial \underset{\sim}{v}}{\partial u}\right)\cdot\nabla^v w = \nabla w.$

It follows from (10.5) and (10.8) that

$$\frac{\partial \underset{\sim}{v}}{\partial u} = -\frac{\underset{\sim}{\xi}}{\eta}\cdot\nabla_r \underset{\sim}{v}, \qquad \nabla u = \frac{\underset{\equiv}{B}}{A}.$$

Hence by (10.91) the differential equation (10.8) becomes

$$(10.92) \qquad (A\eta\nabla_r \underset{\sim}{v} - \underset{\equiv}{B}\,\underset{\sim}{\xi}\cdot\nabla_r \underset{\sim}{v})\cdot\nabla^v w = A\eta\nabla w.$$

A necessary and sufficient condition that (10.92) reduce to the form (10.6) is that the dyadic

$$\underset{\approx}{\Psi} \equiv A\eta\nabla_r \underset{\sim}{v} - \underset{\equiv}{B}\,\underset{\sim}{\xi}\cdot\nabla_r \underset{\sim}{v} \equiv (A\eta\underset{\approx}{I} - \underset{\equiv}{B}\underset{\sim}{\xi})\cdot\nabla_r \underset{\sim}{v}$$

possess a reciprocal, and hence that it be complete.*. Its completeness

* Gibbs - Wilson, p.291. Only complete dyadics have reciprocals.

requires, and is assured by, the completeness of each of the dyadics $A\eta\underset{\approx}{I} - \underset{\equiv}{B}\underset{\sim}{\xi}$ and $\nabla_r \underset{\sim}{v}$.* Now the equation $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{r},u)$ defines $\underset{\sim}{r}$ as a

* Gibbs - Wilson, p.286.

function, $\underset{\sim}{r} = \underset{\sim}{r}(\underset{\sim}{v},u)$. Hence (1.5) holds. Therefore $\nabla_r \underset{\sim}{v}$ is a complete dyadic. Consequently $\underset{\approx}{\Psi}$ will be complete if and only if the dyadic

$$\underset{\approx}{\Psi}' \equiv A\eta\underset{\approx}{I} - \underset{\equiv}{B}\underset{\sim}{\xi}$$

is complete; that is, if and only if its third scalar invariant is not identically zero:

$$(\underset{\approx}{\Psi}')_3 \equiv A^2\eta^2(A\eta - \underset{\sim}{\xi}\cdot\underset{\equiv}{B}) \neq 0.^*$$

* Gibbs - Wilson, p.320; p.315, (69).

Thus <u>a necessary and sufficient condition that an integrable differential equation of the type (10.8) reduce to the form (10.6), and hence that the integral surfaces be collectively invariant under the group generated by $U\underset{\sim}{f}$, is that</u>

$$A\eta - \underset{\sim}{\xi}\cdot\underset{\equiv}{B} \neq 0.$$

It will now be shown that <u>a necessary and sufficient condition that an integrable differential equation of the type (10.8) be expressible in the form (10.7), and hence that the integral surfaces be individually invariant under the group generated by $U\underset{\sim}{f}$, is that</u>

$$A\eta - \underset{\sim}{\xi}\cdot \underset{\equiv}{B} \equiv 0.$$

By Theorems 2A and 9A, the function $\Phi(\underset{\sim}{r},u)$ in the assumed integral $\Phi = c$ of (10.8) satisfies the equations

$$U\Phi \equiv \underset{\sim}{\xi}\cdot\nabla_r \Phi + \eta\frac{\partial\Phi}{\partial u} = 0, \quad \underset{\equiv}{P}\Phi \equiv A\nabla_r\Phi + \underset{\equiv}{B}\frac{\partial\Phi}{\partial u} \equiv 0.$$

Hence

$$(10.93) \qquad AU\Phi - \underset{\sim}{\xi}\cdot\underset{\equiv}{P}\Phi \equiv (A\eta - \underset{\sim}{\xi}\cdot\underset{\equiv}{B})\frac{\partial\Phi}{\partial u} \equiv 0,$$

and since $\partial\Phi/\partial u \not\equiv 0$, $A\eta - \underset{\sim}{\xi}\cdot\underset{\equiv}{B} \equiv 0$.

Conversely, if (10.93) holds, $U\Phi \equiv 0$, since $\underset{\equiv}{P}\Phi \equiv 0$. Hence, by Theorem 9A, the integral surfaces $\Phi = c$ are individually invariant under $U\underset{\sim}{f}$, and have the differential equation (10.7).

These deductions may be summarized in

THEOREM 10. *If the condition for integrability,*

$$(3.3) \qquad \underset{\equiv}{\Omega} \equiv \underset{\equiv}{B}\times\nabla_r A + A\,\mathrm{rot}_r\underset{\equiv}{B} + \underset{\equiv}{B}\times\frac{\partial\underset{\equiv}{B}}{\partial u} \equiv 0,$$

is satisfied identically, a differential equation

$$A(\underset{\sim}{r},u)\nabla u - \underset{\equiv}{B}(\underset{\sim}{r},u) = 0,$$

invariant under the group generated by the infinitesimal transformation

$$U\underset{\sim}{f} \equiv \underset{\sim}{\xi}(\underset{\sim}{r},u)\cdot\nabla_r f + \eta(\underset{\sim}{r},u)\frac{\partial f}{\partial u}, \qquad (\eta \not\equiv 0)$$

may be converted to the form

$$\nabla^v w = \underset{\equiv}{F}(\underset{\sim}{v}),$$

and integrated by a single quadrature, by the introduction of the canonical variables $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{r},u)$, $w = w(\underset{\sim}{r},u)$, *obtained from the differential equations (10.1), if and only if* $A\eta - \xi\cdot\underset{\equiv}{B} \not\equiv 0$.

If $A\eta - \underset{\sim}{\xi}\cdot\underset{\equiv}{B} \equiv 0$, *the differential equation may be written (without*

<u>changing variables) in the form $\nabla g(\underset{\sim}{v}) = 0$, and has the integral $g(\underset{\sim}{v}) = c$.</u>

11. Integrating Factor. Let the conditions of the foregoing theorem be fulfilled. Then (10.8) has the integral $\Phi = c$, where from (10.9),

(11.1) $$\Phi \equiv w - \int \underset{\approx}{F}(\underset{\sim}{v}) \cdot d\underset{\sim}{v}.$$

By Theorem 2A,

(11.2) $$\underset{\approx}{P}\Phi \equiv A\nabla_r \Phi + \underset{\approx}{B}\frac{\partial \Phi}{\partial u} \equiv 0.$$

From (1.4) and the fact that $d\Phi \equiv d\underset{\sim}{v}.\nabla_v \Phi + (\partial\Phi/\partial w)dw$ it follows that

$$U\Phi \equiv \underset{\sim}{\xi}\cdot(\nabla_r \underset{\sim}{v}\cdot\nabla_v \Phi + \nabla_r w\frac{\partial \Phi}{\partial w}) + \eta(\frac{\partial \underset{\sim}{v}}{\partial u}.\nabla_v \Phi + \frac{\partial w}{\partial u}\frac{\partial \Phi}{\partial w}),$$

or, $$U\Phi \equiv (U\underset{\sim}{v})\cdot\nabla_v \Phi + (Uw)\frac{\partial \Phi}{\partial w} \equiv 1,$$

the last step being a consequence of (10.5) and (11.1). Hence

$$AU\Phi - \underset{\sim}{\xi}\cdot\underset{\approx}{P}\Phi \equiv (A\eta - \underset{\sim}{\xi}\cdot\underset{\approx}{B})\frac{\partial \Phi}{\partial u} \equiv A, \qquad \frac{\partial \Phi}{\partial u} = \frac{A}{A\eta - \underset{\sim}{\xi}\cdot\underset{\approx}{B}}.$$

In view of the last result, (11.2) becomes

$$\nabla_r \Phi \equiv \frac{-\underset{\approx}{B}}{A\eta - \underset{\sim}{\xi}\cdot\underset{\approx}{B}}.$$

Therefore, by (1.1),

$$\nabla\Phi \equiv \nabla_r \Phi + \nabla u\frac{\partial \Phi}{\partial u} = \frac{A\nabla u - \underset{\approx}{B}}{A\eta - \underset{\sim}{\xi}\cdot\underset{\approx}{B}}$$

Since the last fraction is a perfect gradient, $(A\eta - \underset{\sim}{\xi}\cdot\underset{\approx}{B})^{-1}$ is an integrating factor of (10.8). This establishes

<u>THEOREM 11.</u> <u>If the condition for integrability,</u>

(.3.3) $$\underset{\approx}{\Omega} \equiv \underset{\approx}{B} \times \nabla_r A + A\,\mathrm{rot}_r\,\underset{\approx}{B} + \underset{\approx}{B}\times\frac{\partial \underset{\approx}{B}}{\partial u} \equiv 0$$

<u>is satisfied identically, the differential equation</u>

$$A(\underset{\sim}{r},u)\nabla u - \underset{\approx}{B}(\underset{\sim}{r},u) = 0,$$

invariant under the group generated by the infinitesimal transformation

$$\mathbf{U}\mathbf{f} \equiv \boldsymbol{\xi}(\mathbf{r},u)\cdot\nabla_r \mathbf{f} + \eta(\mathbf{r},u)\,\frac{\partial \mathbf{f}}{\partial u},$$

has the integrating factor $(A\eta - \boldsymbol{\xi}\cdot\mathbf{B})^{-1}$, *provided* $A\eta - \boldsymbol{\xi}\cdot\mathbf{B} \neq 0.$

12. Determination of all the Differential Equations Invariant under a Group. An example will show how to obtain all the differential equations whose integral surfaces are collectively invariant under a griven group. Consider the infinitesimal transformation

$$\mathbf{U}\mathbf{f} \equiv r^{1-m}\,\mathbf{R}\cdot\nabla_r \mathbf{f} \mp u^{1-n}\,\frac{\partial \mathbf{f}}{\partial u}\,, \qquad (m \neq 0,\ n \neq 0)$$

in which $r \equiv |\mathbf{r}|$, $\mathbf{R} \equiv \mathbf{r}/r$. One finds readily the canonical variables

$$\mathbf{v} = \left(\frac{r^m}{m} \pm \frac{u^n}{n}\right)\mathbf{R}, \qquad w = \frac{r^m}{m}\,.$$

In §§ 9,10 it was shown that the most general differential equation possessing integral surfaces collectively invariant under the canonical form of the group, is of the form

$$\nabla^v w = \mathbf{F}(\mathbf{v}). \tag{12.1}$$

As a change of variable can have no effect on the property of invariance, the required differential equation, invariant under the given form of $\mathbf{U}\mathbf{f}$, is found by expressing (12.1) in terms of $\mathbf{r}$ and u. By (1.4),

$$\left(\nabla_r \mathbf{v} + \nabla u\,\frac{\partial \mathbf{v}}{\partial u}\right)\cdot\nabla^v w \equiv \nabla w.$$

Hence (12.1) becomes

$$\left[r^m\mathbf{R}\,\mathbf{R} + v(\mathbf{I} - \mathbf{R}\,\mathbf{R}) \pm u^{n-1}(\nabla u)\mathbf{r}\right]\cdot\mathbf{F}(\mathbf{v}) = r^m\mathbf{R}. \qquad (v = \mathbf{v}\cdot\mathbf{R}) \tag{12.2}$$

If $\mathbf{F}(\mathbf{v})$ is of the form $\phi(v)\mathbf{R}$, (12.2) easily reduces to

$$r^{1-m}\,u^{n-1}\,\nabla u = f(v)\mathbf{R}, \tag{12.3}$$

in which $f(v) \equiv \pm(\phi^{-1} - 1)$.

Thus any differential equation of the form (12.2) or (12.3) which satisfies the condition for integrability (§ 3), is invariant under the group generated by Uf.

By this method the accompanying table was constructed, showing a number of differential equations of quite general form, together with the groups leaving them invariant. The group are quite analogous to those given by Professor Dickson* for ordinary scalar differential equations.

* l.c.p.324

13. Solution of Differential Equations. It must be understood that not all differential equations of the forms given in the table will be integrable. The first step in the solution of any differential equation should be therefore to test it for integrability by the method of § 3. If the condition $\underline{\underline{\Omega}} \equiv 0$ (3.3) is satisfied identically, an attempt should be made to identify the equation with one of the forms of the table. If such an identification is possible, the integration may be effected in two ways - either by the introduction of an integrating factor, determined as in Theorem 11, from the corresponding group listed in the table, or by a transformation to a pair of canonical variables for the group. If $\underline{\underline{\Omega}} \neq 0$, the possibility of a singular integral should be investigated by the method of § 4.

The following formulas, deduced from (1.4), either directly or by symmetry, will be found useful in making the transformation required in the second method.

$$(13.1)\qquad \nabla w \equiv \left(\nabla_r \underset{\sim}{v} + \nabla u \frac{\partial \underset{\sim}{v}}{\partial u}\right)\cdot \overset{v}{\nabla} w,$$

$$(13.2)\qquad \overset{v}{\nabla} w \equiv \left(\nabla_v \underset{\sim}{r} + \overset{v}{\nabla} w \frac{\partial \underset{\sim}{r}}{\partial w}\right)\cdot \nabla w,$$

$$(13.3)\qquad \nabla u \equiv \left(\nabla_r \underset{\sim}{v} + \nabla u \frac{\partial \underset{\sim}{v}}{\partial u}\right)\cdot \overset{v}{\nabla} u,$$

$$(13.4)\qquad \overset{v}{\nabla} u \equiv \left(\nabla_v \underset{\sim}{r} + \overset{v}{\nabla} w \frac{\partial \underset{\sim}{r}}{\partial w}\right)\cdot \nabla u.$$

In these formulas, $\nabla_r \underset{\sim}{v}$ is computed by holding $\underline{u}$ constant, and $\nabla_v \underset{\sim}{r}$ by holding $\underline{w}$ constant, while superscripts designate total gradients.

EXAMPLE 1. Solution by Integrating Factor. If $\underset{\sim}{\alpha}$ is a constant vector, the differential equation

$$(u - \underset{\sim}{r}\cdot\underset{\sim}{\alpha}\, e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u})\nabla u + u\underset{\sim}{\alpha}\, e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u} = 0$$

satisfies the condition (3.3) for integrability. It is identified with type II of the table for $a = 1$, $b = -1$ and is therefore invariant under the group for which $\underset{\sim}{\xi} \equiv \underset{\sim}{r}$, $\eta \equiv u$. One finds readily the integrating factor $(A\eta - \underset{\sim}{\xi}\cdot\underset{\approx}{B})^{-1} \equiv u^{-2}$. When this is introduced the differential equation takes the form

$$\nabla\Phi \equiv \frac{\underset{\sim}{\alpha}}{u} e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u} + \nabla u\left(\frac{1}{u} - \frac{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}}{u^2} e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u}\right) = 0,$$

whence, by (1.1),

$$\nabla_r \Phi \equiv \frac{\underset{\sim}{\alpha}}{u} e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u}, \qquad \frac{\partial\Phi}{\partial u} \equiv \frac{1}{u} - \frac{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}}{u^2} e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u}.$$

The first of these shows that

$$\Phi \equiv e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u} + f(u), \quad \frac{\partial\Phi}{\partial u} \equiv -\frac{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}}{u^2} e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u} + f'(u).$$

Hence $$f'(u) \equiv \frac{1}{u}, \quad f(u) \equiv \ln u,$$

$$\Phi \equiv e^{\underset{\sim}{r}\cdot\underset{\sim}{\alpha}/u} + \ln u.$$

Thus the complete integral $\Phi = c$ becomes

$$e^{\mathbf{r}\cdot\boldsymbol{\alpha}/u} + \ln u = c.$$

<u>Solution by Canonical Variables.</u> A pair of canonical variables for the group are $\mathbf{v} = \mathbf{r}/u$, $w = \ln u$. Hence, by (13.2),

$$\nabla^v w \equiv \left[\mathbf{I} + (\nabla^v w)\mathbf{v}\right]\cdot\nabla u.$$

From the differential equation one finds

$$\nabla u = \frac{u\boldsymbol{\alpha} e^{\mathbf{r}\cdot\boldsymbol{\alpha}/u}}{\mathbf{r}\cdot\boldsymbol{\alpha} e^{\mathbf{r}\cdot\boldsymbol{\alpha}/u} - u} = \frac{\boldsymbol{\alpha} e^{\mathbf{v}\cdot\boldsymbol{\alpha}}}{\mathbf{v}\cdot\boldsymbol{\alpha} e^{\mathbf{v}\cdot\boldsymbol{\alpha}} - 1}.$$

Therefore
$$\nabla^v w + \boldsymbol{\alpha} e^{\mathbf{v}\cdot\boldsymbol{\alpha}} = 0,$$

$$w + e^{\mathbf{v}\cdot\boldsymbol{\alpha}} = c, \qquad \ln u + e^{\mathbf{r}\cdot\boldsymbol{\alpha}/u} = c.$$

<u>EXAMPLE 2.</u> The differential equation

$$(r + u)\boldsymbol{\alpha} + (\mathbf{R}\cdot\boldsymbol{\alpha})(r\nabla u - u\,\mathbf{R}) = \mathbf{r} \qquad (\boldsymbol{\alpha} \text{ constant})$$

may be identified with type VI e, by taking $\mathbf{F}(\mathbf{v}) \equiv \boldsymbol{\alpha}$, and has the complete integral

$$(\mathbf{R}\cdot\boldsymbol{\alpha})(u + r) = r + c.$$

<u>EXAMPLE 3.</u> The differential equation

$$\left[1 - \mathbf{a}\cdot(b\mathbf{r} + \mathbf{a}u)\right]\nabla u = b(b\mathbf{r} + \mathbf{a}u) \qquad (\mathbf{a}, b \text{ constants})$$

is seen at once to be invariant under the group I of the table and has the complete integral

$$u = \frac{1}{2}(b\mathbf{r} + \mathbf{a}u)^2 + c.$$

<u>EXAMPLE 4.</u> The differential equation

$$\left(r - \frac{r}{u}\right)\nabla u - 2u\mathbf{R} + \frac{2u^2}{r^2}\mathbf{R} - \frac{u^2}{r^2}\frac{\boldsymbol{\alpha}}{\mathbf{R}\cdot\boldsymbol{\alpha}} = 0 \qquad (\boldsymbol{\alpha} \text{ constant})$$

may be identified with VII a of the table, by taking $m = 1$, $\gamma = -2$, $n = 1$, $\mathbf{F} \equiv v\boldsymbol{\alpha}/\mathbf{v}\cdot\boldsymbol{\alpha}$, and has the complete integral

$$\frac{r^2}{u} + \ln\left(\frac{u}{r}\mathbf{R}\cdot\boldsymbol{\alpha}\right) = c.$$

DIFFERENTIAL EQUATIONS INVARIANT UNDER ACCOMPANYING GROUPS

$\underline{\underline{\underline{I}}}$ = Dyadic Idemfactor $\qquad$ $a, \underset{\sim}{a}, b, l, m, n,$ are constants $\qquad$ $r \equiv |\underset{\sim}{r}|,\ \underline{\underline{\underline{R}}} \equiv \underset{\sim}{r}/r$

Number		Differential Equation	Group
I		$\nabla u = \underline{\underline{\underline{F}}}(b\underset{\sim}{r} + \underset{\sim}{a}u)$	$\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{r}} f - b\dfrac{\partial f}{\partial u}$
II		$r\nabla u = u\,\underline{\underline{\underline{F}}}(r^b u^a \underline{\underline{\underline{R}}})$	$a\underset{\sim}{r}\cdot\nabla_{\underset{\sim}{r}} f - bu\dfrac{\partial f}{\partial u}$
III		$\nabla u + \underset{\sim}{p}(\underset{\sim}{r})u + \underset{\sim}{q}(\underset{\sim}{r})u^n = 0$	$e^{\int(n-1)\underset{\sim}{p}\cdot d\underset{\sim}{r}}\, u^n \dfrac{\partial f}{\partial u}$
IV	a	$\underline{\underline{\underline{F}}}(\underset{\sim}{v})\cdot[u\underline{\underline{\underline{I}}} - u(n+1)\underline{\underline{\underline{R}}}\,\underline{\underline{\underline{R}}} + \underset{\sim}{r}\nabla u] = g(r)\underline{\underline{\underline{R}}}$	$\dfrac{r^n}{g(r)}(\underset{\sim}{r}\cdot\nabla_{\underset{\sim}{r}} f + nu\dfrac{\partial f}{\partial u})$
	b	$r\nabla u - nu\underline{\underline{\underline{R}}} = g(r)f(v)\underline{\underline{\underline{R}}}$	$\underset{\sim}{v} \equiv \dfrac{u}{r^n}\underline{\underline{\underline{R}}}, \qquad v \equiv \underset{\sim}{v}\cdot\underline{\underline{\underline{R}}}$
V	a	$[v(\underline{\underline{\underline{I}}} - \underline{\underline{\underline{R}}}\,\underline{\underline{\underline{R}}}) + 2\underset{\sim}{r}\underset{\sim}{r} \pm 2u\nabla u\,\underset{\sim}{r}]\cdot\underline{\underline{\underline{F}}}(\underset{\sim}{v}) = \underset{\sim}{r}g(r)$	$\dfrac{\underline{\underline{\underline{R}}}}{g(r)}\cdot(\nabla_{\underset{\sim}{r}} f \mp \dfrac{r}{u}\dfrac{\partial f}{\partial u})$
	b	$\underset{\sim}{r} \pm u\nabla u = g(r)f(v)\underline{\underline{\underline{R}}}$	$\underset{\sim}{v} \equiv (r^2 \pm u^2)\underline{\underline{\underline{R}}},\ v \equiv \underset{\sim}{v}\cdot\underline{\underline{\underline{R}}}$
VI	a	$[v(\underline{\underline{\underline{I}}} - \underline{\underline{\underline{R}}}\,\underline{\underline{\underline{R}}}) + r^m\underline{\underline{\underline{R}}}\,\underline{\underline{\underline{R}}} \pm u^{n-1}\nabla u\,\underset{\sim}{r}]\cdot\underline{\underline{\underline{F}}}(\underset{\sim}{v}) = r^m\underline{\underline{\underline{R}}}$	$r^{1-m}\underline{\underline{\underline{R}}}\cdot\nabla_{\underset{\sim}{r}} f \mp u^{1-n}\dfrac{\partial f}{\partial u}$ $m \neq 0,\ n \neq 0$
	b	$r^{1-m}u^{n-1}\nabla u = f(v)\underline{\underline{\underline{R}}}$	$\underset{\sim}{v} \equiv (\dfrac{r^m}{m} \pm \dfrac{u^n}{n})\underline{\underline{\underline{R}}},\ v \equiv \underset{\sim}{v}\cdot\underline{\underline{\underline{R}}}$

Differential Equations Invariant under Accompanying Groups

$\underset{\equiv}{I}$ = Dyadic Idemfactor $a, \underset{\sim}{a}, b, l, m, n$, are constants $r \equiv |\underset{\sim}{r}|$, $\underset{\equiv}{R} = \underset{\sim}{r}/r$

Number		Differential Equation	Group
VI	c	$[v(\underset{\equiv}{I} - \underset{\equiv}{R}\underset{\equiv}{R}) + nr^n \underset{\equiv}{R}\underset{\equiv}{R} \pm nu^{n-1} \nabla u \, \underset{\sim}{r}] \cdot \underset{\equiv}{G}(v) = r^n \underset{\equiv}{R}$	$r^{1-n} \underset{\equiv}{R} \cdot \nabla_r \underset{\sim}{f} \mp u^{1-n} \frac{\partial \underset{\sim}{f}}{\partial u}$ $\underset{\sim}{v} \equiv (r^n \pm u^n) \underset{\equiv}{R}$, $v \equiv \underset{\sim}{v} \cdot \underset{\equiv}{R}$
	d	$\left(\frac{u}{r}\right)^{n-1} \nabla u = g(v) \underset{\equiv}{R}$	
	e	$(v \underset{\equiv}{I} \mp u \underset{\equiv}{R}\underset{\equiv}{R} \pm \nabla u \, \underset{\sim}{r}) \cdot \underset{\equiv}{F}(\underset{\sim}{v}) = \underset{\sim}{r}$	$\underset{\equiv}{R} \cdot \nabla_r \underset{\sim}{f} \mp \frac{\partial \underset{\sim}{f}}{\partial u}$ $\underset{\sim}{v} \equiv (r \pm u) \underset{\equiv}{R}$, $v \equiv \underset{\sim}{v} \cdot \underset{\equiv}{R}$
	f	$\nabla u = g(v) \underset{\equiv}{R}$	
VII	a	$[mr - u^m r^{l+1} \underset{\equiv}{R} \cdot \underset{\equiv}{F}(\underset{\sim}{v})] \nabla u + [lu \underset{\equiv}{R} + r^l u^{m+1} \{(n+1) \underset{\equiv}{R}\underset{\equiv}{R} - \underset{\equiv}{I}\} \cdot \underset{\equiv}{F}(\underset{\sim}{v})] = 0$	$r^l u^m (\underset{\sim}{r} \cdot \nabla_r \underset{\sim}{f} + nu \frac{\partial \underset{\sim}{f}}{\partial u})$ $\underset{\sim}{v} \equiv ur^{-n} \underset{\equiv}{R}$, $v \equiv \underset{\sim}{v} \cdot \underset{\equiv}{R}$
	b	$[mr - u^m r^{l+1} f(v)] \nabla u + [lu + nr^l u^{m+1} f(v)] \underset{\equiv}{R} = 0$	
VIII	a	$(r \pm \sqrt{n}\, u)(\underset{\equiv}{R} \mp \sqrt{n} \nabla u) = (\underset{\sim}{r} - nu \nabla u) f(v)$	$nu \underset{\equiv}{R} \cdot \nabla_r \underset{\sim}{f} + r \frac{\partial \underset{\sim}{f}}{\partial u}$ $v \equiv r^2 - nu^2$
	b	$r \nabla u - u \underset{\equiv}{R} = (nu \nabla u - \underset{\sim}{r}) g(v)$	

PART IV. GENERALIZED LINEAR, BERNOULLI AND RICCATI DIFFERENTIAL EQUATIONS

14. The Generalized Linear Differential Equation. The differential equation

$$(14.1) \qquad \nabla u + \underset{\sim}{p}u + \underset{\sim}{q} = 0,$$

in which $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are functions of $\underset{\sim}{r}$ only, may be considered a generalization of the ordinary linear differential equation of the first order. By Theorem 3, a complete integral will exist if and only if

$$(14.2) \qquad u \operatorname{rot} \underset{\sim}{p} \equiv \underset{\sim}{q} \times \underset{\sim}{p} - \operatorname{rot} \underset{\sim}{q},$$

identically in $\underset{\sim}{r}$ and $\underline{u}$; that is, if and only if

$$(14.3) \qquad \operatorname{rot} \underset{\sim}{p} \equiv 0, \qquad \operatorname{rot} \underset{\sim}{q} \equiv \underset{\sim}{q} \times \underset{\sim}{p}.$$

A comparison with type III of the table shows that (14.1) is invariant (when integrable) under the group for which

$$U\underset{\sim}{f} \equiv e^{-\int \underset{\sim}{p}\cdot d\underset{\sim}{r}} \frac{\partial \underset{\sim}{f}}{\partial u},$$

and therefore by Theorem 11 has the integrating factor $e^{\int \underset{\sim}{p}\cdot d\underset{\sim}{r}}$. If this is introduced in (14.1) the latter becomes

$$\nabla(ue^{\int \underset{\sim}{p}\cdot d\underset{\sim}{r}}) \equiv -\underset{\sim}{q}e^{\int \underset{\sim}{p}\cdot d\underset{\sim}{r}}.$$

Hence the complete integral is

$$ue^{\int \underset{\sim}{p}\cdot d\underset{\sim}{r}} = c - \int e^{\int \underset{\sim}{p}\cdot d\underset{\sim}{r}}\, \underset{\sim}{q}\cdot d\underset{\sim}{r},$$

in which the two line integrals will be independent of the path by virtue of (14.3).

Two linear differential equations were solved in §§ 3,4.

15. Singular Integral. If (14.3) is not satisfied identically, (14.1) can have no complete integral. However, if rot $\underset{\sim}{p} \neq 0$, so that $\underline{u}$ actually occurs in (14.2), and if rot $\underset{\sim}{p}$ is paralled to, but not a constant multiple of * $\underset{\sim}{q} \times \underset{\sim}{p} - \text{rot } \underset{\sim}{q}$,

* Since by 14.2, $\underline{u}$ would be a constant.

the algebraic solution

$$u = \frac{(\underset{\sim}{q} \times \underset{\sim}{p} - \text{rot } \underset{\sim}{q})\cdot \text{rot } \underset{\sim}{p}}{(\text{rot } \underset{\sim}{p})^2},$$

of (14.2), may be a singular integral of (14.1). If by direct substitution this fails to satisfy (14.1), no solution exists.

In Example 1, of §4, the singular integral of a generalized linear differential equation was obtained by a slightly different method.

16. The Generalized Bernoulli Differential Equation. Bernoulli's ordinary differential equation of the first order may be generalized in the form

(16.1) $$\nabla u + \underset{\sim}{p}u + \underset{\sim}{q}u^n = 0,$$

in which $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are functions of $\underset{\sim}{r}$ and $\underline{n}$ is a constant. By Theorem 3, a complete integral will exist if and only if

(16.2) $$u^{1-n} \text{rot } \underset{\sim}{p} \equiv (n-1)\, \underset{\sim}{p} \times \underset{\sim}{q} - \text{rot } \underset{\sim}{q},$$

identically in r and $\underline{u}$; that is, if and only if

(16.3) $$\text{rot } \underset{\sim}{p} \equiv 0, \qquad \text{rot } \underset{\sim}{q} \equiv (n-1)\, \underset{\sim}{p} \times \underset{\sim}{q}.$$

If (16.1) is written in the form

$$u^{-n}\nabla u + \underset{\sim}{p}u^{1-n} + \underset{\sim}{q} = 0,$$

it is clear that the substitution $v = u^{1-n}$ reduces the equation to the generalized linear form

(16.4) $$\frac{1}{n-1}\nabla v + pv + q = 0,$$

porvided $n \neq 1$.* The integration may be effected therefore by the method of § 14.

The equation (16.1) is invariant under group III of the table, and hence may be integrated also by the method of § 13. Provided $n \neq 1$, the complete integral is

$$u^{1-n} = (1-n)e^{\phi}\left(c - \int e^{\phi} q \cdot dr\right), \quad \phi \equiv \int (n-1)\, p \cdot dr,$$

in which the two line integrals will be independent of the path if and only if the conditions (16.3) are fulfilled.

If no complete integral exists it is possible that (16.2), regarded as an equation in the unknown u, may furnish a singular integral of (16.1).

17. The Generalized Riccati Differential Equation.

The differential equation

(17.1) $$\nabla u = p + qu + tu^2,$$

in which p, q, t are functions of r only, may be regarded as a generalization of Riccati's ordinary differential equation of the first order.

By Theorem 3, a complete integral will exist if and only if

(17.2) $$u^2\rho + u\sigma + \tau \equiv 0, \text{ where}$$
$$\rho \equiv \operatorname{rot} t + q \times t, \quad \sigma \equiv \operatorname{rot} q + 2p \times t, \quad \tau \equiv \operatorname{rot} p + p \times q;$$

that is, if and only if

$$\rho \equiv \sigma \equiv \tau \equiv 0.$$

* If $n = 1$, (16.1) becomes $\nabla \ln u = p + q$, while (16.2) reduces to the form $\operatorname{rot}(p + q) \equiv 0$. This is precisely a necessary and sufficient condition for the independence of the path of the line integral occurring in the complete integral $\ln u/c = \int (p+q)\cdot dr$.

Let this condition be fulfilled and suppose that a particular solution u_1 of (17.1) is known. Then by subtracting from (17.1) the result of substituting u_1 in it, one obtains

$$\nabla(u - u_1) = \underset{\sim}{q}(u - u_1) + \underset{\sim}{t}(u^2 - u_1^2)$$
$$\equiv (\underset{\sim}{q} + 2u_1\underset{\sim}{t})(u - u_1) + \underset{\sim}{t}(u - u_1)^2.$$

This is a generalized Bernoulli equation in the dependent variable $u - u_1$ and hence by the substitution $v = (u - u_1)^{-1}$ (§16) reduces to the generalized linear form of (16.4),

$$(17.3) \qquad \nabla v + (\underset{\sim}{q} + 2u_1\underset{\sim}{t})v + \underset{\sim}{t} = 0,$$

which may be integrated by two quadratures by the method of § 14.

If $\underset{\sim}{\rho} \equiv \underset{\sim}{\sigma} \equiv \underset{\sim}{\tau} \equiv 0$ and two particular solutions u_1 and u_2 of (17.1) are known, the complete integral may be found by a single quadrature; for the relation $v = (u - u_1)^{-1}$ implies that $v_1 = (u_2 - u_1)^{-1}$ is a particular solution of (17.3), and the result of subtracting from (17.3) the expression obtained by substituting v_1 in it is obviously

$$\nabla(v - v_1) + (\underset{\sim}{q} + 2u_1\underset{\sim}{t})(v - v_1) = 0$$

whence, by a single quadrature,

$$\ln \frac{v - v_1}{C} = -\int (\underset{\sim}{q} + 2u_1\underset{\sim}{t})\cdot d\underset{\sim}{r}.$$

It may be shown that the complete integral of (17.1) may be obtained algebraically when three particular solutions are known.

EXAMPLE. The differential equation

$$\nabla u = \frac{u}{r}\underset{\equiv}{R} + (\frac{u^2}{r} - a^2 r)\nabla\phi,$$

in which ϕ is an arbitrary known scalar point function, satisfies the condition $\underset{\sim}{\rho} \equiv \underset{\sim}{\sigma} \equiv \underset{\sim}{\tau} \equiv 0$ for complete integrability, and may be integrated by one or two quadratures according as use is made of both or only one of the particular solutions $u = \pm ar$. Its complete integral is

$$u = \frac{C\,ar - are^{2a\phi}}{C + e^{2a\phi}}.$$

18. Singular Integral. If $\underset{\sim}{\rho}$ and $\underset{\sim}{\sigma}$ are not both identically zero, and $\underset{\sim}{\rho}, \underset{\sim}{\sigma}, \underset{\sim}{\tau}$, are coplanar vectors,* (17.2) becomes an equation in u, with vector coefficients, whose solution may prove to be a singular integral of (17.1).

If these conditions are fulfilled and $\underset{\sim}{m}$ and $\underset{\sim}{n}$ are two non-parallel vectors in the plane of $\underset{\sim}{\rho}, \underset{\sim}{\sigma}, \underset{\sim}{\tau}$, then any algebraic solution of the equation $u^2\underset{\sim}{\rho} + u\underset{\sim}{\sigma} + \underset{\sim}{\tau} = 0$ satisfies also the equations

(18.1) $$\underset{\sim}{m} \cdot (u^2\underset{\sim}{\rho} + u\underset{\sim}{\sigma} + \underset{\sim}{\tau}) = 0, \quad \underset{\sim}{n} \cdot (u^2\underset{\sim}{\rho} + u\underset{\sim}{\sigma} + \underset{\sim}{\tau}) = 0,$$

and conversely.

Therefore to solve (17.1) when $\underset{\sim}{\rho}$ and $\underset{\sim}{\sigma}$ are not both identically zero, one obtains the common algegraic solutions of (18.1), if such exist, and rejects any not satisfying (17.1).

EXAMPLE. Consider the differential equation

(18.2) $$\nabla u = \frac{\underset{\sim}{j}}{z} + \frac{u\underset{\sim}{i} - u^2\underset{\sim}{k}}{x + y},$$

in which $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ form the usual right-handed set of mutually orthogonal unit vectors. It follows from (17.2) that

$$\underset{\sim}{\rho} \equiv \frac{\underset{\sim}{i}}{(x+y)^2}, \quad \underset{\sim}{\sigma} \equiv \frac{\underset{\sim}{k}}{(x+y)^2} - \frac{2\underset{\sim}{i}}{z(x+y)}, \quad \underset{\sim}{\tau} \equiv \frac{\underset{\sim}{i}}{z^2} - \frac{\underset{\sim}{k}}{z(x+y)}.$$

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* If one of the vectors $\underset{\sim}{\rho}, \underset{\sim}{\sigma}, \underset{\sim}{\tau}$ is zero, the remaining two must be collinear, and the following discussion is easily modified

Since $\underset{\sim}{\rho}, \underset{\sim}{\sigma}, \underset{\sim}{\tau}$ lie in the plane of $\underset{\sim}{i}$ and $\underset{\sim}{k}$, (18.1) may be written

$$\underset{\sim}{i} \cdot (u^2\underset{\sim}{\rho} + u\underset{\sim}{\sigma} + \underset{\sim}{\tau}) \equiv \frac{u^2}{(x+y)^2} - \frac{2u}{z(x+y)} + \frac{1}{z^2} = 0,$$

$$\underset{\sim}{k} \cdot (u^2\underset{\sim}{\rho} + u\underset{\sim}{\sigma} + \underset{\sim}{\tau}) \equiv \frac{u}{(x+y)^2} - \frac{1}{z(x+y)} = 0.$$

The common solution $u = (x+y)/z$ of these equations is found by substitution in (18.2) to be a singular integral. No other integral can exist.