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SELF ENERGY AND INTERACTION ENERGY IN PODOLSKY'S GENERALIZED ELECTRODYNAMICS

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by

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ABSTRACT

Using a perturbation method, explicit expressions are derived for the electromagnetic self energy and electromagnetic interaction energy of charged particles on Podolsky's generalized quantum electrodynamics. No infinities arise in the derivation which is justified to the first order of approximation (i.e. to terms quadratic in the charges). Two noteworthy results follow from the calculations. i) The self energy is finite and negative. ii) The interaction energy contains no singularities.

1. INTRODUCTION

In a series of papers [1, 2, 3, 4] a generalized electrodynamics has been developed in which the Lagrangian contains derivatives of the field quantities. This has led to a theory in which an extraordinary field appears in combination with the usual Maxwell-Lorentz field. This extraordinary field is associated with a neutral particle of mass $m=h/ac$, where a is a natural unit of length occurring in the theory. The quantitization rules have been derived and the auxiliary conditions eliminated by methods which are generalizations of those used in the earlier basic work in relativistic quantum electrodynamics.

In GE III the auxiliary conditions are used to eliminate from the wave functional the ordinary scalar potential, the longitudinal

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1 B.Podolsky, Phys. Rev. 62, 68 (1942), previously called GE I.
2 B.Podolsky and C.Kikuchi, Phys. Rev. 65, 228 (1944), previously called GE II.
3 B.Podolsky and C.Kikuchi, Phys. Rev. 67, 184 (1945), previously called GE III.
4 D.J.Montgomery, Phys. Rev. 69, 117 (1946), henceforth will be called GE IV.

component of the ordinary vector potential, the extraordinary scalar potential and part of the longitudinal component of the extra-ordinary vector potential. The consequence of the transformations, which incidently are exact, is the explicit appearance in the wave equation of terms which can be interpreted as the electrostatic self energy and the electrostatic interaction. The present problem is to investigate the effect of the remainder of the ordinary and extra-ordinary vector potentials to obtain what can be interpreted as "transverse" self energy and the spin and retardation inter-action.

2. CALCULATION OF THE PERTURBATION ENERGY

As a starting point we use some equations obtained in GE IV. In this paper a generalization of the functional formalism of Fock [5], leads to the following set of ordinary wave equations in $\underline{k}$ space for a system of charged particles GE IV (2.25), (2.26), and (2.27) respectively.

$$(H-i\hbar\partial/\partial t)\Psi_{oo} = \sum_{j=1}^{3}\int d\underline{k}\left[G_o^*(\underline{k},j)\Psi_{10}(\underline{k},j) + \tilde{G}_o^*(\underline{k},j)\Psi_{01}(\underline{k},j)\right] \quad (2.1)$$

$$(H+\hbar ck-i\hbar\partial/\partial t)\Psi_{10}(\underline{k},j) = G_o(\underline{k},j)\Psi_{oo} \quad (2.2)$$

$$(H+\hbar ck-i\hbar\partial/\partial t)\Psi_{01}(\underline{k},j) = -G_o(\underline{k},j)\Psi_{oo} \quad (2.3)$$

where [6]

$$H = \sum_{s=1}^{n}(\underline{\alpha}_s\cdot c\underline{p}_s + m_s c^2\beta_s) + (1/8\pi a)\sum_s \epsilon_s^2 + \sum\nolimits'_{s,u}(\epsilon_s\epsilon_u/8\pi R)\left[1-\exp(-R/a)\right] \quad (2.4)$$

$$G_o^*(\underline{k},j) = (1/2\pi)^{3/2}(c\hbar/2k)^{1/2}\sum_s \epsilon_s\beta_j(1/k)\underline{\alpha}_s\cdot\underline{k}\times\underline{e}_j\exp(i\underline{k}\cdot\underline{r}_s) \quad (2.5)$$

$$G_o(\underline{k},j) = (1/2\pi)^{3/2}(c\hbar/2\tilde{k})^{1/2}\sum_s\epsilon_s\beta_j(1/a\tilde{k})(\underline{\alpha}_s\cdot a\underline{k}\times\underline{e}_j + \underline{\alpha}_s\cdot\underline{e}_j)\exp(i\underline{k}\cdot\underline{r}_s) \quad (2.6)$$

$$\underline{R} = \underline{r}_s - \underline{r}_u \qquad \tilde{k} = (1+a^2k^2)^{1/2}/a \quad \text{and} \quad \beta_j^2 = \tilde{\beta}_j^2 = 1$$

In the derivation of equations (2.1) (2.2) and (2.3) the series expansion of the wave functional was broken off after three terms which corresponds to limiting the investigation

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5 V.Fock, Physik. Zeits. Sowjetunion 6, 425 (1934).

6 Note the typographical error in GE IV (2.16) which has a factor 1/4 instead of a factor 1/8 before the interaction term.

to processes involving one ordinary quantum and one extraordinary quantum. As in GE IV we treat the term on the right of (2.1) as a perturbation of the unperturbed Hamiltonian. At this point we depart from the procedure of GE IV because we seek an explicit expression for the perturbation energy rather than the corresponding matrix element.

In order to eliminate Ψ_{10} and Ψ_{01} from (2.1) we assume that as a zero order approximation

$$(H-i\hbar\partial/\partial t)\Psi_{10}(\underline{k},j)=0 \text{ and } (H-i\hbar\partial/\partial t)\Psi_{01}(\underline{k},j)=0 \qquad (2.7)$$

From (2.2) and (2.3) we have thus

$$\Psi_{10}(\underline{k},j)=G_o(\underline{k},j)\Psi_{oo}/\hbar ck$$
$$\text{and } \Psi_{01}(\underline{k},j)=-\tilde{G}_o(\underline{k},j)\Psi_{oo}/\hbar c\tilde{k} \qquad (2.8)$$

Introducing these terms into the right side of (2.1) gives as a first order approximation the perturbation energy

$$U=-\Sigma_j\int d\underline{k}\left[G_o^{*}(\underline{k},j)G_o(\underline{k},j)/\hbar ck-\tilde{G}_o^{*}(\underline{k},j)\tilde{G}_o(\underline{k},j)/\hbar c\tilde{k}\right] \qquad (2.9)$$

This expression may be simplified by use of the following identities:

$$\sum_j \beta_j (1/k)\underline{\alpha}_s\cdot\underline{k}\times\underline{e}_j \exp(i\underline{k}\cdot\underline{r}_s)\beta_j(1/k)\underline{\alpha}_u\cdot\underline{k}\times\underline{e}_j\exp(-i\underline{k}\cdot\underline{r}_u)$$

$$= \sum_j (1/k^2)(\underline{\alpha}_s\times\underline{k}\cdot\underline{e}_j)(\underline{\alpha}_u\times\underline{k}\cdot\underline{e}_j)\exp(i\underline{k}\cdot\underline{R})$$

$$= (\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k)\exp(i\underline{k}\cdot\underline{R}) \qquad (2.10)$$

and

$$\sum_j \beta_j(1/a\tilde{k})(\underline{\alpha}_s\cdot a\underline{k}\times\underline{e}_j+\underline{\alpha}_s\cdot\underline{e}_j)\exp(i\underline{k}\cdot\underline{r}_s)\beta_j(1/a\tilde{k})(\underline{\alpha}_u\cdot a\underline{k}\times\underline{e}_j + \underline{\alpha}_u\cdot\underline{e}_j)\exp(-i\underline{k}\cdot\underline{r}_u)$$

$$= \sum_j (1/a^2\tilde{k}^2)(\underline{\alpha}_s\times a\underline{k}+\underline{\alpha}_s)\cdot\underline{e}_j(\underline{\alpha}_u\times a\underline{k}+\underline{\alpha}_u)\cdot\underline{e}_j\exp(i\underline{k}\cdot\underline{R})$$

$$=(1/a^2\tilde{k}^2)\left[a^2(\underline{\alpha}_s\times\underline{k})\cdot(\underline{\alpha}_u\times\underline{k})+\underline{\alpha}_s\cdot\underline{\alpha}_u+a(\underline{\alpha}_s\times\underline{k}\cdot\underline{\alpha}_u+\underline{\alpha}_s\cdot\underline{\alpha}_u\times\underline{k})\right]\exp(i\underline{k}\cdot\underline{R})$$

$$=\left[(\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k)+(1/a^2\tilde{k}^2)(\underline{\alpha}_s\cdot\underline{k}/k)(\underline{\alpha}_u\cdot\underline{k}/k)\right]\exp(i\underline{k}\cdot\underline{R}) \qquad (2.11)$$

In arriving at the last expression (appendix I) we use the known properties of the triple scalar product, and the vector equation

$$(\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k)=\underline{\alpha}_s\cdot\underline{\alpha}_u-(\underline{\alpha}_s\cdot\underline{k}/k)(\underline{\alpha}_u\cdot\underline{k}/k) \qquad (2.12)$$

The perturbation energy thus becomes

$$U = -\sum_{s,u}(\epsilon_s\epsilon_u/16\pi^3)\int d\underline{k}\exp(i\underline{k}\cdot\underline{R})\left[(\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k)(1/k^2) - (\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k)(1/\tilde{k}^2) - (\underline{\alpha}_s\cdot\underline{k}/k)(\underline{\alpha}_u\cdot\underline{k}/k)(1/a^2\tilde{k}^4)\right] \qquad (2.13)$$

It can be shown [7] that the three terms have the following origins. The first term is the contribution of the transverse component of the ordinary field. It is the only term present in the usual electrodynamics, in which case it gives rise to an infinite self energy. The second term is the contribution of the transverse component of the extraordinary field. The third term is the contribution of the longitudinal component of the extraordinary field. Its magnitude has been determined by the treatment of the auxiliary conditions in GE II and GE III.

Using (2.12) and rearranging terms, it follows that

$$\begin{aligned} U = -\textstyle\sum_{s,u}(\epsilon_s\epsilon_u/16\pi^3)\Big[&\int d\underline{k}\exp(i\underline{k}\cdot\underline{R})\underline{\alpha}_s\cdot\underline{\alpha}_u(1/k^2-1/\tilde{k}^2) \\ &-\int d\underline{k}\exp(i\underline{k}\cdot\underline{R})(\underline{\alpha}_s\cdot\underline{k}/k)(\underline{\alpha}_u\cdot\underline{k}/k)(1/k^2-1/\tilde{k}^2) \\ &-\int d\underline{k}\exp(i\underline{k}\cdot\underline{R})(\underline{\alpha}_s\cdot\underline{k}/k)(\underline{\alpha}_u\cdot\underline{k}/k)(1/a^2\tilde{k}^4)\Big] \end{aligned} \qquad (2.14)$$

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7 To do so requires a formalism (appendix II) in which all vector quantities are resolved into transverse and longitudinal components rather than the formalism of GE IV. Such a development has been carried our. It also leads to equation (2.13).

To evaluate the integrals we use polar coordinates in $\underline{k}$ space taking $\underline{R}$ as the direction of the polar axis. The volume element is $k^2 \sin\theta\, dk\, d\theta\, d\varphi$ where θ and φ are the polar and azimuth angles respectively. As $\underline{\alpha}_s \cdot \underline{\alpha}_u$ is independent of $\underline{k}$ it is unaffected by the integrations. The factor $(\underline{\alpha}_s \cdot \underline{k}/k)(\underline{\alpha}_u \cdot \underline{k}/k)$ however, requires more careful attention. Introducing α_x, α_y and α_z, the well known Dirac matrices, and

$$k_x/k = \sin\theta\cos\varphi \qquad k_y/k = \sin\theta\sin\varphi \qquad k_z/k = \cos\theta$$

we have

$$\begin{aligned}(\underline{\alpha}_s \cdot \underline{k}/k)(\underline{\alpha}_u \cdot \underline{k}/k) \doteq\ & \alpha_{sx}\alpha_{ux}\sin^2\theta\cos^2\varphi + \alpha_{sy}\alpha_{uy}\sin^2\theta\sin^2\varphi \\ &+ \alpha_{sz}\alpha_{uz}\cos^2\theta + (\alpha_{sx}\alpha_{uy} + \alpha_{sy}\alpha_{ux})\sin^2\theta\sin\varphi\cos\varphi \\ &+ (\alpha_{sy}\alpha_{uz} + \alpha_{sz}\alpha_{uy})\sin\theta\cos\theta\sin\varphi \\ &+ (\alpha_{sz}\alpha_{ux} + \alpha_{sx}\alpha_{uz})\sin\theta\cos\theta\cos\varphi \qquad (2.15)\end{aligned}$$

Upon integration with respect to φ over the range $0 - 2\pi$ the cross products all vanish. Using

$$\alpha_{sx}\alpha_{ux} + \alpha_{sy}\alpha_{uy} + \alpha_{sz}\alpha_{uz} = \underline{\alpha}_s \cdot \underline{\alpha}_u \text{ and } \alpha_{sz}\alpha_{uz} = (\underline{\alpha}_s \cdot \underline{R}/R)(\underline{\alpha}_u \cdot \underline{R}/R)$$

the remaining terms give

$$U = - \sum_{s,u}(\epsilon_s \epsilon_u/16\pi^2)\Big[\underline{\alpha}_s \cdot \underline{\alpha}_u(2\ I_1\ -\ I_1 + I_2\ -\ I_3 + I_4)$$
$$(\underline{\alpha}_s \cdot \underline{R}/R)(\underline{\alpha}_u \cdot \underline{R}/R)(I_1\ -\ 3\ I_2 + I_3\ -\ 3\ I_4)\Big] \tag{2.16}$$

where

$$I_1 = \int_0^\infty \int_0^\pi \frac{\exp(ikR\cos\theta)\sin\theta\, dk\, d\theta}{1 + a^2k^2} \tag{2.17}$$

$$I_2 = \int_0^\infty \int_0^\pi \frac{\cos^2\theta \exp(ikR\cos\theta)\sin\theta\, dk\, d\theta}{1 + a^2k^2} \tag{2.18}$$

$$I_3 = \int_0^\infty \int_0^\pi \frac{a^2k^2 \exp(ikR\cos\theta)\sin\theta\, dk\, d\theta}{(1 + a^2k^2)^2} \tag{2.19}$$

$$I_4 = \int_0^\infty \int_0^\pi \frac{a^2k^2 \cos^2\theta \exp(ikR\cos\theta)\sin\theta\, dk\, d\theta}{(1 + a^2k^2)^2} \tag{2.20}$$

These integrals may be evaluated by standard methods. (Appendix III) No special limiting processes are necessary. The results are

$$I_1 = (\pi/R)\left[1\ -\ \exp(-R/a)\right] \tag{2.21}$$

$$I_2 = -\pi\exp(-R/a)(1/R + 2a/R^2 + 2a^2/R^3) + 2\pi a^2/R^3 \tag{2.22}$$

$$I_3 = (\pi/2a)\exp(-R/a) \tag{2.23}$$

$$I_4 = (\pi/2a)\exp(-R/a) + \pi\exp(-R/a)(1/R + 2a/R^2 + 2a^2/R^3)$$
$$-2\pi a^2/R^3 \tag{2.24}$$

Introducing these expressions into (2.16) we observe that the effect of the terms which come from the residual longitudinal component of the extraordinary field is to cancel all the singularities which arise from the terms representing the effect of the transverse components of the ordinary and extraordinary fields. We obtain finally an amazingly simple expression for the perturbation of the Hamiltonian.

$$U = -\sum_{s,u} \frac{\epsilon_s \epsilon_u}{16\pi} \left\{ \underline{\alpha}_s \cdot \underline{\alpha}_u \left[\frac{1-\exp(-R/a)}{R} \right] + (\underline{\alpha}_s \cdot \underline{R}/R)(\underline{\alpha}_u \cdot \underline{R}/R) \left[\frac{1-\exp(-R/a)}{R} - \frac{\exp(-R/a)}{a} \right] \right\} \tag{2.25}$$

3. SELF ENERGY OF CHARGED PARTICLES

To obtain the self energy we consider only those terms in the summation corresponding to $s = u$ and let $\underline{R}$ go to zero. The second term in (2.25), which incidentally can be interpreted as the effect of retardation, vanishes. For a single particle we have left only [8]

$$U = -\frac{\epsilon^2}{16\pi a}\underline{\alpha}\cdot\underline{\alpha} = -\frac{3}{4}\frac{\epsilon^2}{4\pi a} \tag{3.1}$$

Combining this with the electrostatic self energy we obtain for the total, electromagnetic self energy of a charged particle

$$U_s = -\frac{1}{4}\frac{\epsilon^2}{4\pi a} \quad \text{(in Heaviside units)} \tag{3.2}$$

or $$U_s = -\frac{1}{4}\frac{\epsilon^2}{a} \quad \text{(in electrostatic units)} \tag{3.3}$$

Thus on this theory the electromagnetic self energy of a charged particle is necessarily distinct from the mass energy.

The wave equation for an isolated charged particle is

$$(c\underline{\alpha}\cdot\underline{p} + mc^2\beta - \epsilon^2/4a - i\hbar\partial/\partial t)\Psi = 0 \tag{3.4}$$

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8 This result may be obtained more directly by letting $s = u$ and $R = 0$ in equation (2.14). The integrations can then be carried out very simply. (Appendix IV)

4. THE ELECTROMAGNETIC INTERACTION ENERGY

Omitting the self energy terms in (2.25) and considering the electrostatic interaction we have for the complete electromagnetic interaction energy the following equation:

$$U_i = \sum_{su}{}' \frac{\epsilon_s \epsilon_u}{8\pi} \left\{ \left(1 - \frac{\alpha_s \cdot \alpha_u}{2}\right) \left[\frac{1-\exp(-R/a)}{R}\right] - \frac{1}{2} (\underline{\alpha}_s \cdot \underline{R}/R)(\underline{\alpha}_u \cdot \underline{R}/R) \left[\frac{1-\exp(-R/a)}{R} - \frac{\exp(-R/a)}{a}\right] \right\} \tag{4.1}$$

Note the appearance of the term $(1/a)\exp(-R/a)$ which is only effective at short range. If we let $a \to 0$, equation (4.1) reduces to Breit's Formula. (Appendix IV)

The remarkable simplicity of the above results as well as its freedom from singularities has led the writer to the opinion that the theory upon which it is based has a deep seated validity.

APPENDIX I

THE DERIVATION OF EQ (2.11)

Starting with the expression

$$(1/a^2\tilde{k}^2)\left[a^2(\underline{\alpha}_s\times\underline{k})\cdot(\underline{\alpha}_u\times\underline{k}) + \underline{\alpha}_s\cdot\underline{\alpha}_u + a(\underline{\alpha}_s\times\underline{k}\cdot\underline{\alpha}_u + \underline{\alpha}_s\cdot\underline{\alpha}_u\times\underline{k})\right]$$

we expand the first triple scalar product by the formula

$$\underline{\alpha}_s\times\underline{k}\cdot\underline{\alpha}_u = (\alpha_{sy}k_z - \alpha_{sz}k_y)\alpha_{ux} + (\alpha_{sz}k_x - \alpha_{sx}k_z)\alpha_{uy} + (\alpha_{sx}k_y - \alpha_{sy}k_x)\alpha_{uz}$$

Although the components of $\underline{\alpha}_u$ do not always commute with the components of $\underline{\alpha}_s$ they do commute with the components of $\underline{k}$. We may therefore rearrange the terms on the right to obtain the relations

$$\underline{\alpha}_s\times\underline{k}\cdot\underline{\alpha}_u = -\alpha_{sx}(\alpha_{uy}k_z - \alpha_{uz}k_y) - \alpha_{sy}(\alpha_{uz}k_x - \alpha_{ux}k_z) - \alpha_{sz}(\alpha_{ux}k_y - \alpha_{uy}k_x) = -\underline{\alpha}_s\cdot\underline{\alpha}_u\times\underline{k}$$

The two triple scalar products therefore cancel.

To the remainder we add and subtract

$$(1/a^2\tilde{k}^2)(\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k)$$

Using $\quad 1 + a^2k^2 = a^2\tilde{k}^2$

and $\quad (\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k) = \underline{\alpha}_s\cdot\underline{\alpha}_u - (\underline{\alpha}_s\cdot\underline{k}/k)(\underline{\alpha}_u\cdot\underline{k}/k)$

We may now regroup the terms to obtain in place of the original expression

$$(\underline{\alpha}_s \times \underline{k}/k) \cdot (\underline{\alpha}_u \times \underline{k}/k) + (1/a^2 k^2)(\underline{\alpha}_s \cdot \underline{k}/k)(\underline{\alpha}_u \cdot \underline{k}/k)$$

APPENDIX II

THE IDENTIFICATION OF THE TERMS IN EQ (2.13)

From GE IV Eq (2.9) the wave equation for a system of charged particles may be written as

$$[H - i\hbar\partial/\partial t]\Omega = [\Sigma_s \epsilon_s \underline{\alpha}_s \cdot \underline{D}(\underline{r}_s, t)]\Omega$$

where H is given by Eq (2.4) and by GE III Eq (3.6)

$$\underline{D}(\underline{k}) = \underline{A}(\underline{k}) - \underline{k}\underline{k}\cdot\underline{A}(\underline{k})/k^2 = \underline{A}_t(\underline{k}) \quad \text{and}$$

$$\tilde{\underline{D}}(\underline{k}) = \tilde{\underline{A}}(\underline{k}) - \underline{k}\underline{k}\cdot\tilde{\underline{A}}(\underline{k})\tilde{k}^2 + \underline{k}\tilde{B}_0(\underline{k})/a\tilde{k}^2 = \tilde{\underline{A}}_t(\underline{k}) + \tilde{\underline{D}}_1(\underline{k})$$

Thus $\underline{D}(\underline{r}_s, t)$ is equal to $\underline{A}_t(\underline{r}_s, t)$ the transverse component of the vector potential at the position of the s^{th} particle plus $\underline{D}_1(\underline{r}_s, t)$ a vector having the direction of propagation of the field and a magnitude which involves only the residual longitudinal component of the vector potential which remains after elimination of the auxiliary conditions, and the function $B(\underline{r}_s, t)$.

We may now represent

$$\underline{A}_t(\underline{k}) = (c\hbar/2k)^{1/2}\sum_{j=1}^{2}\underline{e}_j b(\underline{k},j)$$

$$\underline{D}_1(\underline{k}) = \underline{e}_3 b(\underline{k},3)\times 0$$

$$\hat{\underline{A}}_t(\underline{k}) = (c\hbar/2\tilde{k})^{1/2}\sum_{j=1}^{2}\underline{e}_j\tilde{b}(\underline{k},j)$$

$$\tilde{\underline{D}}_1(\underline{k}) = (c\hbar/2a^2\tilde{k}^3)^{1/2}\underline{e}_3\tilde{b}(\underline{k},3)$$

where $\left[b^*(\underline{k},j),b(\underline{k}',j')\right] = -\delta_{jj'}\delta(\underline{k}-\underline{k}')$

$\left[b^*(\underline{k},j),b(\underline{k}',j')\right] = \delta_{jj'}\,\delta(\underline{k}-\underline{k}')$

and $\underline{e}_1$ and $\underline{e}_2$ are two orthogonal unit vectors perpendicular to $\underline{e}_3$ a unit vector in the direction of the field. Defining for

$j = 1$ and 2

$$G^*(\underline{k},j) = (c\hbar/16\pi^3 k)^{1/2}\sum_s \epsilon_s\underline{\alpha}_s\cdot\underline{e}_j \exp i(\underline{k}\cdot\underline{r}_s - ckt)$$

$G^*(\underline{k},3) = 0$ for $j = 1$ and 2 again

$$\tilde{G}^*(\underline{k},j) = (c\hbar/16\pi^3\tilde{k})^{1/2}\sum_s \epsilon_s\underline{\alpha}_s\cdot\underline{e}_j \exp i(\underline{k}\cdot\underline{r}_s - ckt)$$

and $\tilde{G}^*(\underline{k},3) = (c\hbar/16\pi^3 a^2\tilde{k}^3)^{1/2}\sum_s \epsilon_s\underline{\alpha}_s\cdot\underline{e}_3 \exp i(\underline{k}\cdot\underline{r}_s - ckt)$

we then obtain GE III (2.17). The elimination of the time factors in the exponentials and the transformation to $\underline{k}$ space given on pages 119 and 120 GE IV then results in equations (2.1), (2.2) and (2.3), the starting point of the present paper. The altered proceedure presented here however enables us to identify the subscripts $j = 1,2$ and 3 with the three directions given above. Thus from Eq (2.9) we have for $j = 1$ and 2

$$G_o^*(\underline{k},j)G_o(\underline{k},j)/\hbar ck = \sum_{su}(\underline{\alpha}_s\cdot\underline{e}_j)(\underline{\alpha}_u\cdot\underline{e}_j)(\epsilon_s\epsilon_u/16\pi^3)\exp(i\underline{k}\cdot\underline{R})/k^2$$

$$G_o^*(\underline{k},3)G_o(\underline{k},3)/\hbar ck = 0$$

$$\tilde{G}_o^*(\underline{k},j)\tilde{G}_o(\underline{k},j)/\hbar c\tilde{k} = \sum_{su}(\underline{\alpha}_s\cdot\underline{e}_j)(\underline{\alpha}_u\cdot\underline{e}_j)(\epsilon_s\epsilon_u/16\pi^3)\exp(i\underline{k}\cdot\underline{R})/\tilde{k}^2$$

$$\tilde{G}_o^*(\underline{k},3)\tilde{G}_o(\underline{k},3)/\hbar c\tilde{k} = \sum_{su}(\underline{\alpha}_s\cdot\underline{e}_3)(\underline{\alpha}_u\cdot\underline{e}_3)(\epsilon_s\epsilon_u/16\pi^3)\exp(i\underline{k}\cdot\underline{R})/a^2\tilde{k}^4$$

To carry out the integrations we let $\underline{e}_3 = \underline{k}/k$ but remove the dependence of the terms in (2.9) on our particular coordinate system by using the relations

$$\begin{aligned}(\underline{\alpha}_s\cdot\underline{e}_1)(\underline{\alpha}_u\cdot\underline{e}_1) + (\underline{\alpha}_s\cdot\underline{e}_2)(\underline{\alpha}_u\cdot\underline{e}_2) &= \underline{\alpha}_s\cdot\underline{\alpha}_u - (\underline{\alpha}_s\cdot\underline{e}_3)(\underline{\alpha}_u\cdot\underline{e}_3)\\ &= \underline{\alpha}_s\cdot\underline{\alpha}_u - (\underline{\alpha}_s\cdot\underline{k}/k)(\underline{\alpha}_u\cdot\underline{k}/k)\\ &= (\underline{\alpha}_s\times\underline{k}/k)\cdot(\underline{\alpha}_u\times\underline{k}/k)\end{aligned}$$

We thus arrive at (2.13) in which the three terms may be identified as the contributions of the transverse component of the ordinary field, the transverse components of the extraordinary field and the residual longitudinal component of the extraordinary field respectively.

APPENDIX III a

THE EVALUATION OF I_1

$$I_1 = \int_0^\infty \int_0^\pi \frac{\exp(ikR\cos\theta)\sin\theta\, dk\, d\theta}{1 + a^2k^2}$$

Let $x = \cos\theta$, $y = ak$ and $b = R/a$. The integral becomes

$$I_1 = \frac{1}{a}\int_0^\infty \int_{-1}^1 \frac{\exp(ixyb)\,dy\,dx}{1 + y^2}$$

$$= \frac{1}{a}\int_0^\infty \frac{dy}{1+y^2}\left[\int_{-1}^1 \cos(xyb)\,dx + i\int_{-1}^1 \sin(xyb)\,dx\right]$$

The second integrand is odd in x, therefore this term vanishes leaving

$$I_1 = \frac{2}{a}\int_0^\infty \frac{dy}{1+y^2}\int_0^1 \cos(xyb)\,dx$$

$$= \frac{2}{a}\int_0^\infty \frac{\sin(yb)\,dy}{y(1+y^2)}$$

Consider now the integral

$$J = \int_\Gamma \frac{\exp(izb)\,dz}{z(1+z^2)}$$

evaluates around an infinite semicircular contour in the upper half of the complex plane, indented at the origin. Dividing the integral into parts we have

$$J = \int_{-\infty}^{-\epsilon} \frac{\exp(ixb)\,dx}{x(1+x^2)} + \int_{-\epsilon}^{\epsilon} \frac{\exp(izb)\,dz}{z(1+z^2)} + \int_{\epsilon}^{\infty} \frac{\exp(ixb)\,dx}{x(1+x^2)}$$

The integral along the infinite semicircular arc vanishes.

To evaluate the integral around the indent let $z = \epsilon e^{i\theta}$. Thus

$$J_1 = \lim_{\epsilon \to 0} \int_{-\pi}^{0} \frac{i d\theta \epsilon e^{i\theta} \exp(ib\epsilon e^{i\theta})}{\epsilon e^{i\theta}(1 + \epsilon^2 e^{2i\theta})} = -\pi i$$

The residue at $z = i$ is $-e^{-b}/2$. We therefore have

$$J = 2\pi i R = -\pi i e^{-b} = -\pi i + \int_{-\infty}^{\infty} \frac{\exp(ixb)}{x(1 + x^2)} dx$$

Equating imaginary parts gives the formula

$$\int_0^{\infty} \frac{\sin(xb)}{x(1+x^2)} dx = \frac{(1-e^{-b})}{2}$$

Using this result in the work above we get

$$I_1 = (\pi/R)\left[1-\exp(-R/a)\right]$$

APPENDIX III b

THE EVALUATION OF I_2

$$I_2 = \int_0^\infty \int_0^\pi \frac{\cos^2\theta \exp(ikR\cos\theta)\sin\theta\, dk\, d\theta}{1 + a^2k^2}$$

$$= \frac{1}{a}\int_0^\infty \int_{-1}^1 \frac{x^2 \exp(ixyb)\, dx\, dy}{1 + y^2}$$

$$= \frac{2}{a}\int_0^1 x^2 dx \int_0^\infty \frac{\cos(xyb)}{1+y^2}\, dy$$

Where we have used the fact that sin(xyb) is odd in x. Consider now the integral

$$J = \int_\Gamma \frac{e^{imz}dz}{1+z^2}$$

evaluated around an infinite semicircular contour in the upper half of the complex plane. The residue at $z = i$ is $e^{-m}/2i$. The integral along the infinite arc vanishes. Thus

$$J = 2\pi i R = \pi e^{-m} = \int_{-\infty}^\infty \frac{e^{imx}}{1+x^2}\, dx$$

Equating real parts gives the formula

$$\int_0^\infty \frac{\cos(xyb)\, dy}{1 + y^2} = \frac{\pi}{2} e^{-xb}$$

Letting $u = -xb$ we have therefore in the work above

$$I_2 = -\frac{\pi}{ab^3}\int_0^{-b} u^2 e^u du$$

$$= -\frac{\pi}{ab^3}\left[e^u(u^2-2u+2)\right]_0^{-b}$$

Inserting the limits and the original quantities we arrive at the result

$$I_2 = -\pi \exp(-R/a)(1/R + 2a/R + 2a^3/R^3) + 2\pi a^2/R^3$$

APPENDIX III c

THE EVALUATION OF I_3

As in the previous cases we simplify our problem by several straightforward steps. Thus

$$I_3 = \int_0^\infty \int_0^\pi \frac{a^2k^2 \exp(ikR\cos\theta)\sin\theta\, dk\, d\theta}{(1+a^2k^2)^2}$$

$$= \frac{1}{a}\int_0^\infty \int_{-1}^{1} \frac{y^2 \exp(ixyb)\, dy\, dx}{(1+y^2)^2}$$

$$= \frac{2}{a}\int_0^\infty \frac{y^2 dy}{(1+y^2)^2}\int_0^1 \cos(xyb)\, dx$$

$$\frac{2}{ab}\int_0^\infty \frac{y\sin(yb)\, dy}{(1+y^2)^2}$$

Consider now the integral

$$J = \int_\Gamma \frac{z e^{izb} dz}{(1+z^2)^2}$$

around an infinite semicircular contour in the upper half plane. The residue at the pole $z = i$ is $\pi b e^{-b}/4$. The integral along the infinite semicircular arc vanishes. Thus

$$J = 2\pi i R = \frac{\pi i b e^{-b}}{2} = \int_{-\infty}^{\infty} \frac{x e^{ixb} dx}{(1+x^2)^2}$$

Equating imaginary parts gives the formula

$$\int_0^\infty \frac{y\sin(yb)\, dy}{(1+y^2)^2} = \frac{\pi}{4} b e^{-b}$$

and therefore

$$I_3 = \frac{\pi}{2a}\exp(-R/a)$$

APPENDIX III d

THE EVALUATION OF I_4

Again by straightforward steps we have

$$I_4 = \int_0^\infty \int_0^\pi \frac{\cos^2\theta \exp(ikR\cos\theta) a^2k^2 \sin\theta dk d\theta}{(1+a^2k^2)^2}$$

$$\frac{2}{a}\int_0^1 x^2 dx \int_0^\infty \frac{y^2\cos(xby)}{(1+y^2)^2}dy$$

Consider the integral

$$J = \int_\Gamma \frac{z^2 e^{izm}}{(1+z^2)} dz$$

evaluated around an infinite semicircular contour in the upper half of the complete plane. The residue at $z = i$ is $ie^{-m}(m-1)/4$. Therefore

$$J = (\pi/2)e^{-m}(1-m)$$

Thus

$$\int_0^\infty \frac{y^2\cos(xby)dy}{(1+y^2)} = \frac{\pi}{4}e^{-bx}(1-bx)$$

Letting $bx = -u$ the original integral becomes

$$I_4 = -\frac{\pi}{2ab^3}\left[\int_0^{-b} u^3 e^u du + \int_0^{-b} u^2 e^u du\right]$$

$$= -\frac{\pi}{2ab^3}\left[e^u(u^3-3u^2+6u-6) + e^u(u^2-2u+2)\right]_0^{-b}$$

Substituting the limits and the original quantities gives finally

$$I_4 = (\pi/2a)\exp(-R/a) + \pi\exp(-R/a)(1/R + 2a/R^2 + 2a^2/R^3) - 2\pi a^2/R^3$$

APPENDIX IV

THE DYNAMIC SELF ENERGY

Letting $s=u$ and $R=0$ in Eq (2.14) we have

$$U_s = -(\epsilon_s^2/16\pi^3)\int\Big[\,(\underline{\alpha}_s\cdot\underline{\alpha}_s)/k^2(1+a^2k^2)-(\underline{\alpha}_s\cdot\underline{k}/k)^2/k^2(1+a^2k^2)$$
$$-(\underline{\alpha}_s\cdot\underline{k}/k)^2a^2/(1+a^2k^2)^2\Big]\,d\underline{k}$$

Using the properties of the Dirac matrices we note that in an arbitrary cartesian coordinate system

$$\underline{\alpha}\cdot\underline{\alpha}=\alpha_x^2+\alpha_y^2+\alpha_z^2=3$$

and

$$(\underline{\alpha}\cdot\underline{k}/k)^2=(1/k^2)(\alpha_xk_x+\alpha_yk_y+\alpha_zk_z)(\alpha_xk_x+\alpha_yk_y+\alpha_zk_z)$$
$$=(1/k^2)(\alpha_x^2k_x^2+\alpha_y^2k_y^2+\alpha_z^2k_z^2)+k_xk_y(\alpha_x\alpha_y+\alpha_y\alpha_x)$$
$$+k_yk_z(\alpha_y\alpha_z+\alpha_z\alpha_y)+k_zk_x(\alpha_z\alpha_x+\alpha_x\alpha_z)$$
$$=(1/k^2)(k_x^2+k_y^2+k_z^2)=1$$

where we use the fact that the matrices anti-commute. As the integrals no longer have any angular dependence they may be readily evaluated in polar coordinates in $\underline{k}$ space using $4\pi k^2dk$ as a volume element.

We have thus

$$U_s = -(\epsilon_s{}^2/16\pi^3)\left[3\cdot 4\int\frac{dk}{1+a^2k^2} -4\pi\int\frac{dk}{1+a^2k^2} -4\pi\int\frac{a^2k^2dk}{(1+a^2k^2)}\right]$$

A trigonometric substitution simplifies these integrals. We obtain finally

$$U_s = -(\epsilon_s{}^2/4\pi a)(3/2-1/2-1/4) = -3\epsilon_s{}^2/16\pi a$$