

NOTE TO USERS

This reproduction is the best copy available.

UMI[®]

UNIVERSITY OF CINCINNATI

April 14 1930

I hereby recommend that the thesis prepared under my
supervision by Harold Ward Sibert
entitled Moderately Thick Circular Plates
with Plane Faces

be accepted as fulfilling this part of the requirements for the
degree of Doctor of Philosophy.

Approved by:

Harris Hancock

MODERATELY THICK CIRCULAR PLATES WITH PLANE FACES

**A dissertation submitted in partial
fulfillment of the requirements for the degree of**

DOCTOR OF PHILOSOPHY

**to the Graduate School of the
University of Cincinnati**

1930

by

Harold Ward Sibert

M. E. Cornell University 1914

M. A. University of Cincinnati 1927

UMI Number: DP16766

INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

UMI[®]

UMI Microform DP16766

Copyright 2009 by ProQuest LLC

All rights reserved. This microform edition is protected against unauthorized copying under Title 17, United States Code.

ProQuest LLC
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106-1346

INDEX
PART I--THEORY

<u>Articles</u>	<u>Pages</u>
1. Introduction	1
2. Stress equations of motion	8
3. Surface-traction conditions	11
4. Boundary conditions at an edge	12
5. The U and w systems	17
6. The determination of $\bar{U}_0^*$ and U_0	22
7. The determination of $\bar{w}_0'$ and w_0	34
8. Upper bounds of the constants	47
9. Convergence of the U and w series	50
10. Finite series	52

PART II--APPLICATION

11. Introduction to Part II	64
12. Determination of K_1, K_3, K_4, C_2 for a complete plate with continuous distribution of load	65
13. Determination of K_1, K_3, K_4, C_2 for a complete plate with central load	69
14. Pure stretching of a complete or incomplete plate with continuous distribution of load	74
15. The bending of a complete plate	82
16. The bending of an incomplete plate	92
17. The bending of a plate under bizonal distribution of load	99
18. Conclusion	113

PART I

1. Introduction. Immediately after the propounding of the modern theory of elasticity by Navier in the first quarter of the nineteenth century, a number of mathematicians turned their attention to problems in elasticity. Soon after the work of Navier, Poisson and Cauchy had found the differential equations for the displacements of plates which are infinitesimally thin, but it was not until 1883 that a solution for the stresses in a moderately thick plate was obtained. This solution, which was found by de Saint-Venant, * involves

* Final note of Art. 45 of his translation of Clebsch, 1883.

rational integral functions of the cylindrical coordinates, r and z , where r is measured from the axis of the plate and z from the middle surface. His method consists in finding values for the displacements for several cases of loading of simple type; these solutions are then combined so as to give more-complicated loading situations.

In 1887 C. Chree # found the solution for a rotating

Trans. Cambridge Philosophical Soc., vol. 14.

plate. His method consists in finding sets of complementary solutions for the differential equations which must be satisfied by the displacements. With each complimentary solution

is associated an arbitrary constant; these constants are determined for any particular problem by the loading conditions on the outer surface of the plate.

The problem of a moderately thick circular plate under uniform load was not solved correctly by de Saint-Venant. It was not until after 1900 that the correct solution for this problem was obtained by A. E. H. Love *. Love's solu-

* Love's "The Mathematical Theory of Elasticity", fourth edition, 1927, p. 465.

tion is an extension of the method developed by J. H. Michell in 1900 #, and gives the displacements in terms of rational

Proc. London Math. Soc., vol. 31, 1900, p. 100.

integral powers of r and z .

Subsequent to Love's solution for uniform load, the first important contribution to the literature of moderately thick circular plates was made by A. Nádai ** in 1920. He

** Schweizerische Bauzeitung, vol. 76, Nr. 22, Dec. 4, 1920, pp. 257-260.

obtained, in terms of Bessel functions, the solution for the bending of a moderately thick circular plate under a concentrated load at the center. Several years later, C. A. Garabedian ## found correct solutions for a circular

Trans. Amer. Math. Soc., vol. 25, 1923, pp. 343-398.

plate of constant thickness in terms of polynomials in r and z . The next year, A. Timpe * found the same solutions

* Zeitschrift für angewandte Mathematik und Mechanik, vol. 4, no. 5, 1924, pp. 361-376.

by an entirely different method. In 1926, C. A. Clemmow #

Proc. Royal Soc., London, ser. A. vol. 112, 1926, pp. 559-598.

obtained solutions for the bending of a circular plate by adding solutions involving polynomials in r and z to the solutions in terms of Bessel functions which had been obtained previously by Nádai.

Both Nádai's and Clemmow's solutions are very cumbersome, even for the case of uniform load. The extension of either of these methods to more complicated types of loading would be almost an impossibility on account of the extremely difficult computations which would arise. The methods used by de Saint-Venant, Love, and Timpe could be extended to more complicated types of loading without giving rise to very difficult computations; but all three of these methods have the disadvantage that, for each new type of loading, the problem must be solved from the very beginning by the process of "trial and error". Garabedian's method is much less involved than those of de Saint-Venant, Love, or Timpe; and, in addition, it gives the necessary machinery for solving any

type of loading by means of a single set of equations.

Garabedian's method is based on the assumption that the displacements can be expanded in rational integral powers of a parameter. The device of developing the displacements in ascending powers of a parameter had previously been used by G. D. Birkhoff* in an attempt to solve problems in circu-

* Phil. Mag., London, Edinburgh, and Dublin, ser. 6, vol. 43, 1922.

lar plates by the use of the Calculus of Variations.

Although Garabedian was the first to solve successfully problems in moderately thick circular plates by a method involving the assumption that the displacements can be expanded in convergent series, he was not the first to make use of such an assumption. Before 1827, Cauchy# had made

Cauchy's "Exercices de Mathématiques", vol. 11, pp. 330-348, 1827.

use of this assumption; he had solved problems by assuming that the stresses and displacements could be expressed as convergent series in ascending powers of z . But Cauchy concerned himself only with plates which were infinitesimally thin, and neglected all powers of z higher than the second. In 1887, M. Lévy**, in his study of a thick circular plate

** Liouville's journal, vol. 3, 1877, p. 219.

having no load on either base, made the assumption that the displacements could be expanded in ascending powers of z . He did not attempt to find the solution for any given loading condition, but he was able to prove that the displacements could not contain powers of z greater than the third when the cylinder is weightless and its bases are free of load.

Garabedian, at the close of his paper, conjectured that his series were convergent, and sketched a physical argument in support of the contention that his method could be put on a rigorous basis. Subsequent to the work done on this paper, Garabedian has found the general term of his series, and has been able to establish convergence for a certain class of loading functions. Moreover, he has found connected with his series an infinite set of constants which turn out to be the same set of constants exhibited in this paper.

The present paper was inspired directly by the above-mentioned paper by Garabedian, and hence, indirectly, by the work done by Birkhoff. Although Garabedian's method and the method used in this paper lead to the same results for any given loading conditions, they are quite distinct. On the other hand, it should be said of the two methods that, precisely because of the difference in approach, each method sheds light on the other. Indeed, the two methods, in a sense, complement each other and eventually completely clarify a problem which has waited a full century for solution.

The method of solution employed in this paper is based on the assumption that the components of displacement can be developed in positive integral powers of z . In article 9 we shall justify this assumption in the case of plates of constant thickness by proving that the series defining the displacements are convergent for a certain class of loading functions. It is clear that if solutions can be found which call for only a finite number of terms of the series, no question of rigor can arise. In view of the space which would be required for complete elaboration, we do not propose to study exhaustively the problem of convergence; we shall concern ourselves principally with the case in which the displacements terminate. Moreover, we shall treat only the case of constant thickness, since, in general, problems in variable thickness involve infinite series of considerable complication.

Adopting cylindrical coordinates, we naturally take the axis of the plate to be the z -axis and the middle plane of the plate to be the plane $z = 0$. Let the upper and lower faces be $z = h$ and $z = -h$, respectively; thus the thickness of the plate is $2h$. The plate is taken to be homogeneous, isotropic, and only slightly bent; moreover, the plate must be thin enough so that de Saint-Venant's principle of the elastic equivalence of statically equipollent systems of load can be used at the edge (cf. art. 4). In order to simplify

this first exposition of our method, we shall restrict all stresses and displacements to be independent of θ ; the advantage of this assumption is that the differential equations which determine the coefficients of z will be ordinary instead of partial.

We shall use Love's notation * with some slight

* In this paper, all references to Love are to the fourth edition of his Mathematical Theory of Elasticity, Cambridge, 1927.

modifications. To obtain results in compact form, the star operator introduced by Professor Garabedian # will be used;

Trans. Amer. Math. Soc., Vol. 25, 1923, pp. 343-398.

this operator is defined as follows:

$$A^* = \frac{1}{r} \frac{\partial(rA)}{\partial r} = A' + \frac{A}{r}.$$

U, V, w denote the displacements in the directions of r, θ, z , respectively. The nine components of stress are $\overline{rr}, \overline{\theta\theta}, \overline{zz}, \overline{r\theta}, \overline{\theta r}, \overline{rz}, \overline{zr}, \overline{\theta z}, \overline{z\theta}$, in which the first letter indicates the direction of the stress and the second letter indicates the direction of the normal to the plane across which the stress acts. The six components of strain are defined as follows (Love p. 56):

(1a) $e_{rr} = U',$

$$\begin{aligned}
 (1b) \quad e_{\theta\theta} &= \frac{U}{r}, \\
 (1c) \quad e_{zz} &= \frac{\partial w}{\partial z}, \\
 (1d) \quad e_{zr} &= \frac{\partial v}{\partial z} + w', \\
 (1e) \quad e_{r\theta} &= v' - \frac{v}{r}, \\
 (1f) \quad e_{\theta z} &= \frac{\partial v}{\partial z}.
 \end{aligned}$$

Note that all terms in $\partial/\partial\theta$ have been suppressed.

We shall confine ourselves to the problem in which the surface tractions are known and the displacements are to be found. In this type of problem the displacements must satisfy

- (1) the stress equations of motion throughout the body,
- (2) the surface traction conditions on the upper and lower faces,
- (3) the boundary conditions at the edge.

We proceed to discuss, in the order just indicated, the three requirements which must be met by the displacements.

2. Stress equations of motion. The following relations exist between six of the nine stress components (Love, p. 78):

$$\overline{r\theta} = \overline{\theta r}, \quad \overline{rz} = \overline{zr}, \quad \overline{\theta z} = \overline{z\theta}.$$

The six independent components of stress can be expressed in

terms of the six components of strain as follows (Love, p. 102):

$$(2a) \quad \overline{rr} = \lambda \Delta + 2\mu e_{rr},$$

$$(2b) \quad \overline{\theta\theta} = \lambda \Delta + 2\mu e_{\theta\theta},$$

$$(2c) \quad \overline{zz} = \lambda \Delta + 2\mu e_{zz},$$

$$(2d) \quad \overline{zr} = \mu e_{zr},$$

$$(2e) \quad \overline{r\theta} = \mu e_{r\theta},$$

$$(2f) \quad \overline{\theta z} = \mu e_{\theta z},$$

where $\Delta = e_{rr} + e_{\theta\theta} + e_{zz} = V^* + \frac{\partial w}{\partial z}.$

By means of equations (1), we can express these stress components in terms of the displacements. It is convenient to use the elastic constants E and σ instead of λ and μ , where

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)}.$$

We obtain

$$(3a) \quad \overline{rr} = \frac{E}{1+\sigma} \left\{ \frac{\sigma}{1-2\sigma} (V^* + \frac{\partial w}{\partial z}) + V' \right\},$$

$$(3b) \quad \overline{\theta\theta} = \frac{E}{1+\sigma} \left\{ \frac{\sigma}{1-2\sigma} (V^* + \frac{\partial w}{\partial z}) + \frac{V}{r} \right\},$$

$$(3c) \quad \overline{zz} = \frac{E}{(1+\sigma)(1-2\sigma)} \left\{ \sigma V^* + (1-\sigma) \frac{\partial w}{\partial z} \right\},$$

$$(3d) \quad \widehat{rz} = \frac{E}{2(1+\sigma)} \left(\frac{\partial V}{\partial z} + w' \right),$$

$$(3e) \quad \widehat{r\theta} = \frac{E}{2(1+\sigma)} \left(V' - \frac{V}{r} \right),$$

$$(3f) \quad \widehat{\theta z} = \frac{E}{2(1+\sigma)} \frac{\partial V}{\partial z}.$$

When terms in $\partial/\partial\theta$ have been suppressed, the stress equations of motion are (Love, p. 90)

$$(4a) \quad \frac{\partial \widehat{rr}}{\partial r} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = \rho(f_r - F_r),$$

$$(4b) \quad \frac{\partial \widehat{r\theta}}{\partial r} + \frac{\partial \widehat{\theta z}}{\partial z} + 2 \frac{\widehat{r\theta}}{r} = \rho(f_\theta - F_\theta),$$

$$(4c) \quad \frac{\partial \widehat{rz}}{\partial r} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = \rho(f_z - F_z),$$

where ρ , f , and F denote the density, acceleration, and body force, respectively (Love, p. 75). All important and familiar applications involving accelerations or body forces will be provided for if we let $\rho(f_z - F_z)$ be a constant, and $\rho(f_r - F_r)$ and $\rho(f_\theta - F_\theta)$ be proportional to r . Hence we shall write

$$(4'a) \quad \rho(f_r - F_r) = c_r r,$$

$$(4'b) \quad \rho(f_\theta - F_\theta) = c_\theta r,$$

$$(4'c) \quad \rho(f_z - F_z) = c_z,$$

where c_r , c_θ , c_z are constants. We shall call $c_r r$, $c_\theta r$, and c_z the radial, tangential, and axial mass forces, respectively. Observe that these mass forces may be due to accel-

erations or to body forces.

By means of formulas (3), we can express formulas (4) in terms of U , V , w and the elastic constants E and σ . At the same time, we shall substitute the values which we have just assigned to the components of $\rho(f - F)$. We thus obtain

$$(5a) \quad \frac{\partial^2 U}{\partial z^2} + \frac{1}{1-2\sigma} \frac{\partial w'}{\partial z} + \frac{2(1-\sigma)}{1-2\sigma} U^{*1} = \frac{2(1+\sigma)}{E} c_r r,$$

$$(5b) \quad \frac{\partial^2 V}{\partial z^2} + V^{*1} = \frac{2(1+\sigma)}{E} c_\theta r,$$

$$(5c) \quad \frac{\partial^2 w}{\partial z^2} + \frac{1}{2(1-\sigma)} \frac{\partial U^*}{\partial z} + \frac{1-2\sigma}{2(1-\sigma)} w^{1*} = \frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)E} c_z.$$

3. Surface-traction conditions. By surface tractions are meant the stresses exerted on the bounding surface of a body by some other object having contact with the body in question (Love, p. 75). The surface tractions applied to the upper and lower faces can be resolved into radial, tangential, and normal components. The positive direction of the normal component will be taken for both the upper and the lower faces to be that of the outward drawn normal (Love, p. 75). The positive direction of the tangential component will be taken for both faces to be the clockwise direction when the plate is viewed from above. The positive direction of the radial component will be taken outward on the upper face; it must be taken inward on the lower face since these three axes must form either a right-

handed system, or a left-handed system on both faces. We will designate the radial, tangential, and normal components of the surface tractions by J_1 , K_1 , L_1 on the upper face and by J_2 , K_2 , L_2 on the lower face. Note that L_1 and L_2 are tensions when positive and pressures when negative.

In order to satisfy the surface traction conditions on a face, it is necessary that the components of the internal stress at every point of the face should be equal to the corresponding component of the surface traction at that point. Thus, the surface traction conditions on the two faces are

$$(6a) \quad \overline{zz}|_{z=h} = L_1,$$

$$(6b) \quad \overline{zz}|_{z=-h} = L_2,$$

$$(6c) \quad \overline{rz}|_{z=h} = J_1,$$

$$(6d) \quad \overline{rz}|_{z=-h} = J_2,$$

$$(6e) \quad \overline{\theta z}|_{z=h} = K_1,$$

$$(6f) \quad \overline{\theta z}|_{z=-h} = K_2.$$

4. Boundary conditions at an edge. There are two ways of specifying the boundary conditions at an edge; namely, we may assign at an edge definite values

(i) to the stresses or to functions of the stresses,

(ii) to the displacements or to functions of the displacements.

In applying an edge condition, it is also important to distinguish two cases:

- (a) the stresses or displacements may be assigned values for every z in the interval from $z = -h$ to $z = h$;
- (b) the displacements may have prescribed values at only a limited number of points, or values may be assigned to certain resultant stresses and to certain resultant stress moments taken along a vertical element of an edge.

In the first case, the solution obtained is exact; moreover, this solution applies to a plate of any thickness and may be called a three-dimensional solution. In the second case, the solution is rigorous, but fails to be exact unless for every value of z at the edge the surface tractions are precisely in agreement with the corresponding internal stresses as calculated from the values obtained for the displacements. This type of solution, when not exact, is essentially two-dimensional in character, since this type requires that the thickness of the plate be small as compared with the diameter.

In general, a three-dimensional solution necessitates an elaborate analysis involving Bessel functions -- or, in rectangular plates, Fourier series. If the plate is very thick, a three-dimensional solution is to be desired. On the other hand, when the plate is only moderately thick, a two-dimensional solution may suffice. If a two-dimensional solution is adequate, immediately there are advantages to be

observed; the work is far simpler in both theory and application than that involved in a three-dimensional solution, and the approximate method of satisfying edge conditions permits of the solution of many important problems for which, as yet, no three-dimensional attack had been found. We shall be concerned in the present paper only with the two-dimensional type of solution.

In the two-dimensional attack, we begin by defining the resultant stresses and the resultant stress moments (Love, p. 455) mentioned in case (b) above. Let C be any cylindrical surface with the equation $r = \text{constant}$, and let s be the circle in which this surface meets the middle plane, $z = 0$. In particular, C may be an edge; in this case, s is the "edge-line". We draw the normal ν to s in a chosen sense, and choose the sense of s so that ν , s , z are parallel to the directions of a right-handed system of axes, assuming that r , θ , z form a right-handed system. We consider the action which that part of the plate lying on the side of C towards which ν is drawn exerts upon that part of the plate lying on the other side. Let Δs be a short length of the curve s , and let A be the area marked out upon C by two generating lines of C drawn thru the extremities of Δs . The tractions on A are statically equivalent to a force at the centroid of A and a couple. Resolve this force and couple into components along the ν , s , z directions, and let (T) , (S) , (N) be

the components of the force and (H) , (G) , (K) the components of the couple. Let Δs approach zero, and denote by T_r , S_r , N_r , H_r , G_r , K_r the limits of $(T)/\Delta s$, $(S)/\Delta s$, $(N)/\Delta s$, $(H)/\Delta s$, $(G)/\Delta s$, $(K)/\Delta s$, respectively. $K_r = 0$, but the other five limits may be finite and different from zero. T , S , N are the components of the resultant stress and H , G are the components of the resultant stress moment belonging to an element of the surface C . T is either a tension or a pressure; S and N are shearing forces tangential and normal, respectively, to the middle plane; G is a flexural couple and H is a torsional couple. These resultant stresses and resultant stress moments are given by the following formulas:

$$(7a) \quad T_r = \int_{-h}^h \overline{FF} dz,$$

$$(7b) \quad N_r = \int_{-h}^h \overline{ZF} dz,$$

$$(7c) \quad S_r = \int_{-h}^h \overline{\theta F} dz,$$

$$(7d) \quad G_r = \int_{-h}^h \overline{FF} z dz,$$

$$(7e) \quad H_r = \int_{-h}^h \overline{\theta F} z dz.$$

For moderately thick plates, we know by de Saint-Venant's principle that the actual distribution of the tractions applied to an edge is of no practical importance. Therefore, instead of dealing with the tractions themselves, we consider their force and couple resultants estimated per unit length of the edge-line. Let the components of these

resultants be T, S, N and H, G in the sense previously assigned to T_r, S_r, N_r, H_r, G_r , the normal to the edge-line being drawn away from the center for an outer edge and towards the center for an inner edge. It is necessary that the applied tractions be statically equivalent to the stress resultants and the resultant stress moments at the edge; but this does not require the satisfaction of all the equations

$$T_r = T, S_r = S, G_r = G, N_r = N, H_r = H,$$

since the two last conditions can be replaced by the condition

$$N_r - \frac{\partial H_r}{\partial s} = N - \frac{\partial H}{\partial s}$$

(Love, pp. 459-461). All factors of our problem being independent of θ , both $\partial H / \partial s$ and $\partial H_r / \partial s$ are zero. Hence, so far as the stresses are concerned, the boundary conditions at an edge are given by the equations

$$(8a) \quad T_r = T,$$

$$(8b) \quad N_r = N,$$

$$(8c) \quad G_r = G,$$

$$(8d) \quad S_r = S.$$

It is true that the solutions obtained by means of these equations may not be exact, but, if not exact, they will

sufficiently approximate the exact solutions for all points that are not too close to the edge of the plate (Love, pp. 131, 132).

5. The U and w systems. We are now in position to determine formally the values of U, V, w which satisfy equations (5), (6), (8). In this paper, we shall concern ourselves only with the determination of U and w and shall not attempt to find the value of V. Turning to equations (3), we observe that (5b), (6e), (6f), (7c), (7e) are the only equations involving V and, moreover, they involve V only. Hence we set these equations aside and turn to the remaining equations, which involve U and w only.

We now proceed to the formal development of our method of solution. Let us assume that U and w can be expanded in powers of z; that is,

$$(9a) \quad U = \sum_{n=0}^{\infty} U_n \frac{z^n}{n!},$$

$$(9b) \quad w = \sum_{n=0}^{\infty} w_n \frac{z^n}{n!},$$

in which U_n and w_n are functions of r only. Substitute (9a) and (9b) in formulas (5a) and (5b), and equate to zero the coefficients of like powers of z. We obtain

$$(10a) \quad U_n = -\frac{1}{1-2\sigma} \{w'_{n-1} + 2(1-\sigma)U_{n-2}^*\} + \phi_n(r),$$

$$(10b) \quad w_n = -\frac{1}{2(1-\sigma)} \{U_{n-1}^* + (1-2\sigma)w_{n-2}^*\} + K_n,$$

where $\phi_2(r) = \{2(1+\sigma)/E\} c_r r$ and $\phi_n(r) = 0$ ($n = 3, 4, 5, \dots$);

$K_2 = \{(1-2\sigma)(1+\sigma)/(1-\sigma)E\} c_2$ and $K_n = 0$ ($n = 3, 4, 5, \dots$).

By collapsing the telescopic series (10a) and (10b), it is possible to express U_n and w_n , $n \geq 2$, directly in terms of U_0 , U_1 , w_0 , w_1 . In the formulas for U_n and w_n , it is found necessary to distinguish between odd and even subscripts. Thus, there are four equations; namely,

$$(11a) \quad U_{2i} = \frac{(-1)^i}{1-2\sigma} \{(i+1-2\sigma)U_0^* + i w_1\}^{(*1)^{i-1}} + \phi_i(r),$$

$$(11b) \quad U_{2i+1} = \frac{(-1)^i}{2(1-\sigma)} \{(i+2-2\sigma)U_1 - i w_0'\}^{(*1)^i} \quad (i = 0, 1, 2, \dots),$$

$$(11c) \quad w_{2i} = \frac{(-1)^i}{2(1-\sigma)} \{i U_1 - (i-2+2\sigma)w_0'\}^{*(1*)^{i-1}} + K_i,$$

$$(11d) \quad w_{2i+1} = \frac{-(-1)^i}{1-2\sigma} \{i U_0^* + (i-1+2\sigma)w_1\}^{(1*)^i} + K_i,$$

where $\phi_1(r) = \{2(1+\sigma)/E\} c_r$ and $\phi_i(r) = 0$ ($i = 0, 2, 3, \dots$);

$K_1 = \{(1-2\sigma)(1+\sigma)/(1-\sigma)E\} c_2$ and $K_i = 0$ ($i = 0, 2, 3, \dots$);

$k_1 = \{-2(1+\sigma)/(1-\sigma)E\} c_r$ and $k_i = 0$ ($i = 0, 2, 3, \dots$). These

equations assume a simpler form if we introduce two new functions defined as follows:

$$(12a) \quad \bar{U}_0^* = \frac{U_0^* + w_1}{1-2\sigma},$$

$$(12b) \quad \bar{w}_0' = \frac{U_1 - w_0'}{2(1-\sigma)}.$$

Substituting (12) in (11), we have

$$(13a) \quad U_{2i} = (-1)^i \{i \bar{U}_0^* + U_0\}^{(*1)^{i-1}} + \phi_i(r),$$

$$(13b) \quad U_{2i+1} = (-1)^i \{(i+2-2\sigma)\bar{w}_0' + w_0\}^{(*1)^i} \quad (i = 0, 1, 2, \dots),$$

$$(13c) \quad w_{2i} = (-1)^i \{i \bar{w}_0 + w_0\}^{(*i)} + K_i,$$

$$(13d) \quad w_{2i+1} = -(-1)^i \{(i-1+2\sigma) \bar{U}_0 + U_0\}^{*(i)} + K_i.$$

If (13) is substituted in (9), the result is

$$(14a) \quad U = \sum_{i=0}^{\infty} (-1)^i \{i \bar{U}_0 + U_0\}^{(*i)} \frac{z^{2i}}{(2i)!} + \sum_{i=0}^{\infty} (-1)^i \{(i+2-2\sigma) \bar{w}_0 + w_0\}^{(i)} \frac{z^{2i+1}}{(2i+1)!} \\ + \frac{z(1+\sigma)}{E} C_r r \frac{z^2}{2!},$$

$$(14b) \quad w = \sum_{i=0}^{\infty} (-1)^i \{i \bar{w}_0 + w_0\}^{(i)} \frac{z^{2i}}{(2i)!} - \sum_{i=0}^{\infty} (-1)^i \{(i-1+2\sigma) \bar{U}_0 + U_0\}^{*(i)} \frac{z^{2i+1}}{(2i+1)!} \\ + \frac{(1-2\sigma)(1+\sigma)}{(1-\sigma)E} C_z \frac{z^2}{2!} - \frac{z(1+\sigma)}{(1-\sigma)E} C_r \frac{z^3}{3!}.$$

These expressions for the displacements satisfy formally the stress equations of motion, (5a) and (5c), when U and w are infinite series. It can be shown also that they satisfy (5a) and (5c) when U and w are finite series; in this connection, it is natural to ask what are the necessary and sufficient conditions for U and w to terminate.

Let us first consider the expression for U . If U is to have a finite number of terms, both $\{i \bar{U}_0 + U_0\}^{(*i)}$ and $\{(i+2-2\sigma) \bar{w}_0 + w_0\}^{(i)}$ must eventually vanish and, moreover, independently of each other, since the former is associated with even powers of z and the latter with odd powers of z . Hence a necessary condition for U to terminate is that we should be able to find two smallest integers, α and β , for which $\{i \bar{U}_0 + U_0\}^{(*i)} = 0$, $i \geq \alpha$, and $\{(i+2-2\sigma) \bar{w}_0 + w_0\}^{(i)} = 0$, $i \geq \beta$.

Let us define the anti-prime-star operation as that operation which must be performed upon P^{*1} in order to change it to P . By performing this operation upon $\{\alpha \bar{U}_0 + U_0\}^{(*1)\alpha}$ and $\{(\beta + 2 - 2\sigma)\bar{w}_0 + w_0\}^{(*1)\beta}$, we find that both $\{\alpha \bar{U}_0 + U_0\}^{(*1)\alpha-1}$ and $\{(\beta + 2 - 2\sigma)\bar{w}_0 + w_0\}^{(*1)\beta-1}$ must be of the form $\{C_1 r + C_2/r\}$, where C_1 and C_2 are constants. Moreover, since α and β are constants, it follows that the four quantities $\bar{U}_0^{(*1)\alpha-1}$, $U_0^{(*1)\alpha-1}$, $\bar{w}_0^{(*1)\beta-1}$, $w_0^{(*1)\beta-1}$ have the form $\{C_1 r + C_2/r\}$. By performing alternately integrations and inverse-star operations, it may be shown that U_0 and $\bar{w}_0'$ can contain no terms which are not of the form $\{C r^{2m-3} + K r^{2n-1} \log r\}$ and that $\bar{U}_0^*$ and w_0 can contain no terms which are not of the form $\{C' r^{2p-2} + K' r^{2q-2} \log r\}$, m, n, p, q being any positive integers. These limitations on $U_0, \bar{w}_0', \bar{U}_0^*, w_0$ have been found as necessary conditions for the termination of U . On the other hand, we observe that if $U_0, \bar{w}_0', \bar{U}_0^*, w_0$ have the form specified, U will contain only a finite number of terms. Hence these conditions are seen to be both necessary and sufficient for U to terminate.

By a similar argument, it may be shown that the above restrictions on the form of $U_0, \bar{w}_0', \bar{U}_0^*, w_0$ constitute also a necessary and sufficient condition that w have a finite number of terms. Hence it is evident that if U terminate, so also will w ; and vice versa.

We are now in a position to show that U and w satisfy the stress equations of motion when U and w are finite series. We shall not carry through the details since the work is the

same as for the case when U and w are infinite series, except that now we must determine the upper limits for each summation. Although these upper limits may be different for the different summations, all of them are readily found if we observe that the operation by which a term becomes eventually zero is differentiation or the star operation according as the term has the form $C r^m$ or $K r^n \log r$.

We must now impose upon U and w the requirement that they satisfy the surface traction conditions on the upper and lower faces. We begin by substituting formulas (14) in (3c) and (3d), obtaining

$$(15a) \quad \widehat{zz} = -\frac{E}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \{ (i-1+\sigma) \bar{U}_0 + U_0 \}^{(*)i} \frac{Z^{2i}}{(2i)!} \\ - \frac{E}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \{ (i+1-\sigma) \bar{W}_0 + W_0 \}^{(*)i+1} \frac{Z^{2i+1}}{(2i+1)!}$$

$$(15b) \quad \widehat{rz} = \frac{E}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \{ (i+1-\sigma) \bar{W}_0 + W_0 \}^{(*)i} \frac{Z^{2i}}{(2i)!} \\ - \frac{E}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \{ (i+\sigma) \bar{U}_0 + U_0 \}^{(*)i+1} \frac{Z^{2i+1}}{(2i+1)!}$$

These four series of ordinary linear differential equations, determining the displacements over the arbitrary coordinate, Next, we introduce four new quantities defined as follows:-

$$(16a) \quad L = L_1 + L_2,$$

$$(16b) \quad l = L_1 - L_2,$$

$$(16c) \quad J = J_1 + J_2,$$

$$(16d) \quad j = J_1 - J_2.$$

Note that (6a), (6b), (6c), (6d) are equivalent to the following equations:

$$(17a) \quad \overline{z z}|_{z=h} + \overline{z z}|_{z=-h} = L,$$

$$(17b) \quad \overline{z z}|_{z=h} - \overline{z z}|_{z=-h} = \ell,$$

$$(17c) \quad \overline{r z}|_{z=h} + \overline{r z}|_{z=-h} = J,$$

$$(17d) \quad \overline{r z}|_{z=h} - \overline{r z}|_{z=-h} = j.$$

In (17), replace $\overline{z z}$ and $\overline{r z}$ by their values as given in (15). We thus obtain, finally

$$(18) \quad \sum_{i=0}^{\infty} (-1)^i \{ (i+1-\sigma) \overline{U}_0 + U_0 \}^{(*)i} \frac{h^{2i}}{(2i)!} = -\frac{1+\sigma}{2E} L - \frac{1+\sigma}{E} C_r h^2,$$

$$(19) \quad \sum_{i=0}^{\infty} (-1)^i \{ (i+1-\sigma) \overline{W}_0 + W_0 \}^{(*)i+1} \frac{h^{2i}}{(2i+1)!} = -\frac{1+\sigma}{2Eh} \ell + \frac{1+\sigma}{E} C_z,$$

$$(20) \quad \sum_{i=0}^{\infty} (-1)^i \{ (i+1-\sigma) \overline{W}_0 + W_0 \}^{(*)i} \frac{h^{2i}}{(2i)!} = \frac{1+\sigma}{2E} J,$$

$$(21) \quad \sum_{i=0}^{\infty} (-1)^i \{ (i+\sigma) \overline{U}_0 + U_0 \}^{(*)i+1} \frac{h^{2i}}{(2i+1)!} = -\frac{1+\sigma}{2Eh} j + \frac{1+\sigma}{E} C_r r.$$

These four systems of ordinary linear differential equations determine the displacements save for the arbitrary constants of integration; the latter are to be fixed subsequently by the edge conditions (Art. 12).

6. The determination of $\overline{U}_0^*$ and U_0 . From equations

(14), it appears that the displacements are determined as soon as we have found $\overline{U}_0^*$ and U_0 . The function $\overline{U}_0$ does not

occur in any of our formulas. Since the w 's are not involved in (18) and (21), these equations, when considered simultaneously, will suffice for the determination of $\bar{U}_0^*$ and U_0 . Note that the leading terms in the summations of (18) and (21) are, respectively, $\{(-1+\sigma)\bar{U}_0^* + U_0^*\}$ and $\{\sigma\bar{U}_0^* + U_0^*\}$; we therefore solve first for $\bar{U}_0^*$ and U_0^* , and subsequently determine $\bar{U}_0^*$ and U_0 by quadratures.

It turns out, apparently as an immediate consequence of our assumption of developability in powers of z , that it is desirable to break up $\bar{U}_0^*$ and U_0^* into terms ordered according to powers of the ratio h/r . For this purpose, we introduce the following convenient definition. If X and Y are two polynomials in r which contain the same number of terms, Y is defined to be of the n^{th} order of magnitude as compared with X if each term of Y is proportional to $(h/r)^n$ times the corresponding term in X . Let us consider two polynomials, X and Y , which are connected by the relation $Y = X^{*1} h^2$. If $c_m r^m$ is any one term of X , the corresponding term in Y will be $(c_m r^m)^{*1} h^2 = (m^2 - 1) c_m r^{m-2} h^2 = (m^2 - 1) c_m r^m (h/r)^2$. Hence, by definition, Y is of the second order of magnitude as compared with X . Similarly, it may be shown that $X^{(n)} h^{2n}$ is of the $(2n)^{th}$ order of magnitude as compared with X .

Altho we must solve first for $\bar{U}_0^*$ and U_0^* , we shall find it of advantage to begin by making certain assumptions with regard to $\bar{U}_0^*$ and U_0^* ; namely, that these functions are

polynomials in r , and that they involve h in such a manner that their terms may be grouped and arranged in ascending order of magnitude. Since only even powers of h enter in (18) and (21), it is clear that we need provide for only even orders of magnitude in our developments for $\bar{U}_o^*$ and U_o^* . We write, therefore,

$$(22a) \quad \bar{U}_o^* = \sum_{n=0}^{\infty} \bar{U}_{2n,0}^* ,$$

$$(22b) \quad U_o^* = \sum_{n=0}^{\infty} U_{2n,0}^* ,$$

where $\bar{U}_{2n,0}^*$ and $U_{2n,0}^*$ designate terms which, if they exist, are of the $(2n)^{th}$ order of magnitude as compared with either $\bar{U}_{oo}^*$ or U_{oo}^* . Our only assumption with reference to the leading terms $\bar{U}_{oo}^*$ and U_{oo}^* is that they include, in the case that $\bar{U}_o^*$ and U_o^* do not vanish identically, the term of lowest order of magnitude occurring in either development; thus we postulate the existence of at least one of the terms $\bar{U}_{oo}^*$ or U_{oo}^* . Furthermore, we shall assume that $\bar{U}_{oo}^{*'} and $U_{oo}^{*}' cannot both be identically zero unless $\bar{U}_{2n,0}^{*'} \equiv U_{2n,0}^{*'} \equiv 0, n = 0, 1, 2, \dots$. If (22) is substituted in (18) and (21), it follows that$$

$$(23) \quad \sum_{i=0}^{\infty} (-1)^i \left\{ (i-1+\sigma) \sum_{n=0}^{\infty} \bar{U}_{2n,0}^* + \sum_{n=0}^{\infty} U_{2n,0}^* \right\}^{*(1*)^i} \frac{h^{2i}}{(2i)!} \\ = - \frac{1+\sigma}{2E} L - \frac{1+\sigma}{E} C_r h^2 ,$$

$$(24) \quad \sum_{i=0}^{\infty} (-1)^i \left\{ (i+\sigma) \sum_{n=0}^{\infty} \bar{U}_{2n,0}^* + \sum_{n=0}^{\infty} U_{2n,0}^* \right\}^{*(1*)^{i+1}} \frac{h^{2i}}{(2i+1)!} \\ = - \frac{1+\sigma}{2Eh} \mathcal{J} + \frac{1+\sigma}{E} C_r r .$$

Since each of these equations involves in the left-hand member a double series of some complication, it is desirable to write out at length a few of the terms. We have

$$(23') \quad \begin{bmatrix} (-1+\sigma) \bar{U}_{00} + U_{00} \\ + (-1+\sigma) \bar{U}_{20} + U_{20} \\ + (-1+\sigma) \bar{U}_{40} + U_{40} \\ \dots \\ \dots \end{bmatrix} * \begin{bmatrix} \sigma \bar{U}_{00} + U_{00} \\ + \sigma \bar{U}_{20} + U_{20} \\ + \sigma \bar{U}_{40} + U_{40} \\ \dots \\ \dots \end{bmatrix} * 1 * \frac{h^2}{2!}$$

$$+ \begin{bmatrix} (1+\sigma) \bar{U}_{00} + U_{00} \\ + (1+\sigma) \bar{U}_{20} + U_{20} \\ + (1+\sigma) \bar{U}_{40} + U_{40} \\ \dots \\ \dots \end{bmatrix} * (1*)^2 \frac{h^4}{4!} - \dots = -\frac{1+\sigma}{2E} L - \frac{1+\sigma}{E} C_r h^2,$$

$$(24') \quad \begin{bmatrix} \sigma \bar{U}_{00} + U_{00} \\ + \sigma \bar{U}_{20} + U_{20} \\ + \sigma \bar{U}_{40} + U_{40} \\ \dots \\ \dots \end{bmatrix} * 1 \begin{bmatrix} (1+\sigma) \bar{U}_{00} + U_{00} \\ + (1+\sigma) \bar{U}_{20} + U_{20} \\ + (1+\sigma) \bar{U}_{40} + U_{40} \\ \dots \\ \dots \end{bmatrix} * (1*)^2 \frac{h^2}{3!}$$

$$+ \begin{bmatrix} (2+\sigma) \bar{U}_{00} + U_{00} \\ + (2+\sigma) \bar{U}_{20} + U_{20} \\ + (2+\sigma) \bar{U}_{40} + U_{40} \\ \dots \\ \dots \end{bmatrix} * (1*)^3 \frac{h^4}{5!} - \dots = -\frac{1+\sigma}{2Eh} j + \frac{1+\sigma}{E} C_r r.$$

By means of equations (23') and (24'), we shall be able to obtain $\bar{U}_{2n,0}^{*'} and $U_{2n,0}^{*}'$ for $n = 0, 1, 2, \dots$. Our procedure consists in writing out two systems of differential equations by equating all terms in (23') and in (24') which are of the same order of magnitude; $\bar{U}_{2n,0}^{*}'$ and $U_{2n,0}^{*}'$ are obtained by selecting the $(n+1)$ st equation of each system and solving the pair simultaneously.$

For convenience, we may distinguish the following types of loading:

- (i) normal pressure on the face; $c_r = c_z = 0, J \equiv 0, L \neq 0$.
- (ii) shearing traction on the face; $c_r = c_z = 0, L \equiv 0, J \neq 0$.
- (iii) radial mass force; $c_z = 0, J \equiv L \equiv 0, c_r \neq 0$.
- (iv) axial mass force; $c_r = 0, J \equiv L \equiv 0, c_z \neq 0$.

It is desirable to deal with these types of loading one at a time; by the principle of superposition, it will be possible to solve more complicated problems by a synthesis of the separate solutions. We shall begin by considering a distribution of loading of type (i).

Before we can proceed further, we must determine the order of magnitude of $\{-(1+\sigma)/2E\}L$. If we assume that this quantity is of the second or higher order of magnitude as compared with $\bar{U}_{00}^*$ or U_{00}^* , the two equations of lowest order of magnitude in (23') and (24') are, respectively,

$$\{(-1+\sigma)\bar{U}_{00}^* + U_{00}^*\} = 0 \text{ and } \{\sigma\bar{U}_{00}^* + U_{00}^*\} = 0. \text{ If we differentiate}$$

the first of these equations and solve it simultaneously with the second, we obtain $\bar{U}_{00}^{*'} \equiv U_{00}^{*'} \equiv 0$. Since L must appear on the right hand side of some one of the equations derived by equating all terms in (23') which are of the same order of magnitude, it is evident that we cannot have $\bar{U}_{2n,0}^{*'} \equiv U_{2n,0}^{*'} \equiv 0$, $n = 0, 1, 2, \dots$. But by a previous assumption, this implies that $\bar{U}_{00}^{*'}$ and $U_{00}^{*'}$ cannot both be identically zero. From this contradiction, it follows that $\{(-1+\sigma)/2E\}L$ is of the same order of magnitude as $\bar{U}_{00}^{*'}$ or $U_{00}^{*'}$. We are now in a position to obtain the two systems of equations which result from equating all terms of the same order of magnitude in (23') and in (24'). They are, respectively,

$$(25.0) \quad \{(-1+\sigma) \bar{U}_{00} + U_{00}\}^* = -\frac{1+\sigma}{2E} L,$$

$$(25.2) \quad \{(-1+\sigma) \bar{U}_{20} + U_{20}\}^* - \{\sigma \bar{U}_{00} + U_{00}\}^{*1*} \frac{h^2}{2!} = 0,$$

$$(25.2n) \quad \{(-1+\sigma) \bar{U}_{2n,0} + U_{2n,0}\}^* - \{\sigma \bar{U}_{2n-2,0} + U_{2n-2,0}\}^{*1*} \frac{h^2}{2!} + \dots \\ + (-1)^n \{(n-1+\sigma) \bar{U}_{00} + U_{00}\}^{*(1*)^n} \frac{h^{2n}}{(2n)!} = 0,$$

• • • • • Solving these two equations simultaneously, we obtain

$$\bar{U}_{00}^{*1} = - \frac{1+\sigma}{2E} \{a_0 - c_0\} L',$$

$$U_{00}^{*1} = - \frac{1+\sigma}{2E} \{(1-\sigma)a_0 + \sigma c_0\} L'.$$

If we put these values of $\bar{U}_{00}^{*1}$ and U_{00}^{*1} in (25.2)' and (26.2), we have, respectively,

$$(27.2) \quad \{(-1+\sigma)\bar{U}_{20} + U_{20}\}^{*1} = - \frac{1+\sigma}{2E} c_1 h^2 L'^{*1},$$

$$(28.2) \quad \{\sigma \bar{U}_{20} + U_{20}\}^{*1} = - \frac{1+\sigma}{2E} a_1 h^2 L'^{*1},$$

where

$$a_1 = \frac{2a_0 - 1c_0}{3!},$$

$$c_1 = \frac{1a_0 - 0c_0}{2!}.$$

If this pair of equations is solved simultaneously, the result is

$$\bar{U}_{20}^{*1} = - \frac{1+\sigma}{2E} \{a_1 - c_1\} h^2 L'^{*1},$$

$$U_{20}^{*1} = - \frac{1+\sigma}{2E} \{(1-\sigma)a_1 + \sigma c_1\} h^2 L'^{*1}.$$

Substituting in (25.4)' and (26.4) the values just obtained, for $\bar{U}_{00}^{*1}, U_{00}^{*1}, \bar{U}_{20}^{*1}, U_{20}^{*1}$, we find

$$(27.4) \quad \{(-1+\sigma)\bar{U}_{40} + U_{40}\}^{*1} = - \frac{1+\sigma}{2E} c_2 h^4 L'^{(*1)^2},$$

$$(28.4) \quad \{\sigma \bar{U}_{40} + U_{40}\}^{*1} = - \frac{1+\sigma}{2E} a_2 h^4 L'^{(*1)^2},$$

where

$$a_2 = \frac{2a_1 - 1c_1}{3!} - \frac{3a_0 - 2c_0}{5!},$$

$$c_2 = \frac{1a_1 - 0c_1}{2!} - \frac{2a_0 - 1c_0}{4!}.$$

Solving again simultaneously, we obtain

$$\bar{U}_{40}^{*1} = -\frac{1+\sigma}{2E} \{a_2 - c_2\} h^4 L^{1(*)2},$$

$$U_{40}^{*1} = -\frac{1+\sigma}{2E} \{(1-\sigma)a_2 + \sigma c_2\} h^4 L^{1(*)2}.$$

By continuing this procedure, we are led finally to the following general formulas:

$$(29a) \quad \bar{U}_{2n,0}^{*1} = -\frac{1+\sigma}{2E} \{a_n - c_n\} h^{2n} L^{1(*)n}, \quad n=0,1,2,\dots;$$

$$(29b) \quad U_{2n,0}^{*1} = -\frac{1+\sigma}{2E} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{1(*)n}, \quad n=0,1,2,\dots;$$

where a_n and c_n are given by the formulas

$$(29c) \quad a_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+2)a_{n-1-i} - (i+1)c_{n-1-i}}{(2i+3)!}, \quad n=1,2,3,\dots;$$

$$(29d) \quad c_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)a_{n-1-i} - i c_{n-1-i}}{(2i+2)!}, \quad n=1,2,3,\dots.$$

By differentiating (22) and making use of (29), we have

$$(29e) \quad \bar{U}_0^{*1} = -\frac{1+\sigma}{2E} \sum_{n=0}^{\infty} \{a_n - c_n\} h^{2n} L^{1(*)n},$$

$$(29f) \quad U_0^{*1} = -\frac{1+\sigma}{2E} \sum_{n=0}^{\infty} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{1(*)n}.$$

We are not yet in a position to solve for $\bar{U}_0^{*1}$ and U_0^{*1} , since these two quantities are not independent of each other.

To find the relation connecting them, we add up equations (27) and make use of (29a), (29b), and (22), and obtain

$$(29g) \quad -(1-\sigma)\bar{U}_o^* + U_o^* = -\frac{1+\sigma}{2E} \sum_{n=0}^{\infty} c_n h^{2n} L^{(1*)^n}.$$

In solving for $\bar{U}_o^*$ and U_o , one might first find $\bar{U}_o^*$ from (29e) and then determine U_o^* from (29g), U_o being subsequently obtained from U_o^* by anti-starring. An alternative method would be to find U_o from (29f) and then determine $\bar{U}_o^*$ from (29g). We shall use the latter method, since it is the simpler of the two.

Let $\bar{U}_{oc}^*$ and U_{oc} be the complementary solutions of $\bar{U}_o^*$ and U_o , respectively, and let $\bar{U}_{op}^*$ and U_{op} be the corresponding particular solutions. The complementary solutions are found from equations (29e), (29f), (29g) by equating their left-hand members to zero. From (29f), there results

$$(29h) \quad U_{oc} = C_1 r + \frac{C_2}{r},$$

where C_1 and C_2 are arbitrary constants. If the value of U_{oc}^* obtained from (29h) is substituted in (29g), it follows that

$$(29i) \quad \bar{U}_{oc}^* = \frac{2C_1}{1-\sigma}.$$

It may readily be shown that the complementary solutions for all other types of loading will also be given by (29h) and (29i); hence in future we shall record only the particular

solutions.

We shall now find the particular solutions when $\bar{U}_0^{*1}$ and U_0^{*1} are infinite series; the case in which $\bar{U}_0^{*1}$ and U_0^{*1} terminate is more involved and will not be considered until art. 10. It is clear that infinite series occur whenever no integer m can be found such that $L^{1(*)^n} = 0$, $n \geq m$. Let us designate the anti-star and anti-prime-star operations by the symbols $(*)^{-1}$ and $(*1)^{-1}$, respectively; and let us agree that $L^{1(*)^{n-1}}$ is the result obtained by anti-prime-starring $L^{1(*)^n}$ when arbitrary constants are suppressed. Observe that $L^{1(*)^{-1}} = L^{(*)^{-1}}$. From (29f), we have

$$(29j) \quad U_{op} = -\frac{1+\sigma}{2E} \sum_{n=0}^{\infty} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{1(*)^{n-1}}.$$

Substituting in (29g), we find

$$(29k) \quad \bar{U}_{op}^* = -\frac{1+\sigma}{2E} \sum_{n=0}^{\infty} \{a_n - c_n\} h^{2n} L^{(1*)^n}.$$

It is clear that the above value of $\bar{U}_{op}^*$ satisfies (29e).

Let $\bar{a}_0 = 1$, $\bar{c}_0 = 0$. By a process very similar to that employed in the previous case, we obtain the following general formulas for a loading of type (ii):

$$(30a) \quad \bar{U}_{2n,0}^{*1} = -\frac{1+\sigma}{2Eh} \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{(*)^n}, \quad n = 0, 1, 2, \dots;$$

$$(30b) \quad U_{2n,0}^{*1} = -\frac{1+\sigma}{2Eh} \{(1-\sigma)\bar{a}_n + \sigma\bar{c}_n\} h^{2n} j^{(*)^n}, \quad n = 0, 1, 2, \dots;$$

$$(30c) \quad \bar{a}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+2) \bar{a}_{n-1-i} - (i+1) \bar{c}_{n-1-i}}{(2i+3)!}, \quad n = 1, 2, 3, \dots;$$

$$(30d) \quad \bar{c}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1) \bar{a}_{n-1-i} - i \bar{c}_{n-1-i}}{(2i+2)!}, \quad n = 1, 2, 3, \dots;$$

$$(30e) \quad \bar{U}_o^{*I} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{*(I)n};$$

$$(30f) \quad U_o^{*I} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{(1-\sigma)\bar{a}_n + \sigma \bar{c}_n\} h^{2n} j^{*(I)n};$$

$$(30g) \quad -(1-\sigma)\bar{U}_o^{*I} + U_o^{*I} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \bar{c}_n h^{2n} j^{*(I)n}.$$

When $\bar{U}_o^{*I}$ and U_o^{*I} , as defined by (30e) and (30f), are infinite series, we find

$$(30j) \quad U_{op} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{(1-\sigma)\bar{a}_n + \sigma \bar{c}_n\} h^{2n} j^{*(I)n},$$

$$(30k) \quad \bar{U}_{op}^{*I} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{*(I)n}.$$

For the third case the formulas are quite simple; if only a radial mass force is acting, we have

$$(31a) \quad \bar{U}_{oo}^{*I} = \frac{1+\sigma}{E} c_r r \quad ; \quad \bar{U}_{zn,o}^{*I} = 0, \quad n = 1, 2, \dots$$

$$(31b) \quad U_{oo}^{*I} = \frac{1-\sigma^2}{E} c_r r \quad ; \quad U_{zn,o}^{*I} = 0, \quad n = 1, 2, \dots$$

$$(31e) \quad \bar{U}_o^{*I} = \frac{1+\sigma}{E} c_r r$$

$$(31f) \quad U_o^{*I} = \frac{1-\sigma^2}{E} c_r r$$

$$(31g) \quad -(1-\sigma)\bar{U}_o^{*I} + U_o^{*I} = 0.$$

$$(31j) \quad U_{op} = \frac{1-\sigma^2}{8E} c_r r^3$$

$$(31k) \quad \bar{U}_{op}^{*I} = \frac{1+\sigma}{2E} c_r r^2$$

Finally, when the plate is subjected to axial mass force only, the formulas become

$$(32ab) \quad \bar{U}_{2n,0}^{*'} = U_{2n,0}^{*'} = 0, \quad n = 0, 1, \dots;$$

$$(32ef) \quad \bar{U}_0^{*'} = U_0^{*'} = 0;$$

$$(32g) \quad -(1-\sigma)\bar{U}_0^* + U_0^* = 0;$$

$$(32jk) \quad U_{op} = \bar{U}_{op}^* = 0.$$

7. The determination of $\bar{w}_0'$ and w_0 . Since $\bar{w}_0$ does not appear in the formulas for U and w , it is necessary to find only $\bar{w}_0'$ and w_0 . Equations (19) and (20), when considered simultaneously, will suffice for the determination of $\bar{w}_0'$ and w_0 . By analogy with § 6, we might expect to be able to solve directly for $\bar{w}_0'^*$ and $w_0'^*$; but we are prevented from doing so by the fact that, in the leading terms of (19) and (20), the coefficients of $\bar{w}_0'^*$ and $w_0'^*$ are the same. We find that we must solve first for $\bar{w}_0'^{**}$ and $w_0'^{**}$, and subsequently obtain $\bar{w}_0'^*$ and $w_0'^*$ by two quadratures.

The preceding article suggests that we should begin by assuming that $\bar{w}_0'$ and w_0' are polynomials in r and that the terms of $\bar{w}_0'$ and w_0' may be arranged in ascending order of magnitude. We write

$$(33a) \quad \bar{w}_0' = \sum_{n=0}^{\infty} \bar{w}_{2n,0}',$$

$$(33b) \quad w_0' = \sum_{n=0}^{\infty} w_{2n,0}',$$

where $\bar{w}'_{2n,0}$ and $w'_{2n,0}$ are of the $(2n)^{\frac{1}{2}}$ order of magnitude as compared with either $\bar{w}'_{00}$ or w'_{00} . We assume that the leading terms, $\bar{w}'_{00}$ and w'_{00} , include, in the case that $\bar{w}'_0$ and w'_0 do not vanish identically, the term of lowest order of magnitude occurring in either development. Furthermore, we shall assume that both $\bar{w}_{00}^{1*1*}$ and w_{00}^{1*1*} cannot be identically zero unless $\bar{w}_{2n,0}^{1*1*} \equiv w_{2n,0}^{1*1*} \equiv 0$, $n = 0, 1, 2, \dots$. Substituting (33) in (19) and (20), we obtain

$$(34) \quad \sum_{i=0}^{\infty} (-1)^i \left\{ (i+1-\sigma) \sum_{n=0}^{\infty} \bar{w}_{2n,0} + \sum_{n=0}^{\infty} w_{2n,0} \right\} (1*)^{i+1} \frac{h^{2i}}{(2i+1)!} = -\frac{1+\sigma}{2Eh} \ell + \frac{1+\sigma}{E} C_z,$$

$$(35) \quad \sum_{i=0}^{\infty} (-1)^i \left\{ (i+1-\sigma) \sum_{n=0}^{\infty} \bar{w}_{2n,0} + \sum_{n=0}^{\infty} w_{2n,0} \right\} (1*)^i \frac{h^{2i}}{(2i)!} = \frac{1+\sigma}{2E} J.$$

Let us write out at length a few terms of each of the above double series. We have

$$(34') \quad \begin{bmatrix} (1-\sigma) \bar{w}_{00} + w_{00} \\ + (1-\sigma) \bar{w}_{20} + w_{20} \\ + (1-\sigma) \bar{w}_{40} + w_{40} \\ \dots \\ \dots \end{bmatrix}^{1*} - \begin{bmatrix} (2-\sigma) \bar{w}_{00} + w_{00} \\ + (2-\sigma) \bar{w}_{20} + w_{20} \\ + (2-\sigma) \bar{w}_{40} + w_{40} \\ \dots \\ \dots \end{bmatrix}^{(1*)^2} \frac{h^2}{3!}$$

$$+ \begin{bmatrix} (3-\sigma) \bar{w}_{00} + w_{00} \\ + (3-\sigma) \bar{w}_{20} + w_{20} \\ + (3-\sigma) \bar{w}_{40} + w_{40} \\ \dots \\ \dots \end{bmatrix}^{(1*)^3} \frac{h^4}{5!} - \dots = -\frac{1+\sigma}{2Eh} \ell + \frac{1+\sigma}{E} C_z,$$

$$\begin{aligned}
 (35') \quad & \left[\begin{array}{c} (1-\sigma) \bar{w}_{00} + w_{00} \\ + (1-\sigma) \bar{w}_{20} + w_{20} \\ + (1-\sigma) \bar{w}_{40} + w_{40} \\ \dots \\ \dots \end{array} \right]' - \left[\begin{array}{c} (2-\sigma) \bar{w}_{00} + w_{00} \\ + (2-\sigma) \bar{w}_{20} + w_{20} \\ + (2-\sigma) \bar{w}_{40} + w_{40} \\ \dots \\ \dots \end{array} \right]^{1*1} \frac{h^2}{2!} \\
 & + \left[\begin{array}{c} (3-\sigma) \bar{w}_{00} + w_{00} \\ + (3-\sigma) \bar{w}_{20} + w_{20} \\ + (3-\sigma) \bar{w}_{40} + w_{40} \\ \dots \\ \dots \end{array} \right]^{1(*)^2} \frac{h^4}{4!} - \dots = \frac{1+\sigma}{2E} J.
 \end{aligned}$$

By means of equations (34') and (35'), we shall be able to solve for $\bar{w}_{2n,0}^{1*1}$ and $w_{2n,0}^{1*1}$, $n = 0, 1, 2, \dots$. The procedure is analogous to that of the preceding article.

Before we can write out the two systems of differential equations obtained from (34') and (35'), we must study the effect of the prime-star operator upon order of magnitude. Let us consider two polynomials, X and Y , connected by the relation $Y = X^{1*} h^2$. If $c_m r^m$ is any term of X , the corresponding term in Y will be $(c_m r^m)^{1*} h^2 = m^2 c_m r^{m-2} h^2 = m^2 c_m r^m (h/r)^2$. Hence, by definition, Y is of the second order of magnitude as compared with X . Similarly, it may be shown that $X^{(1*)^n} h^{2n}$ is of the $(2n)^{th}$ order of magnitude as compared with X .

We distinguish the four types of loading enumerated

in article 6. We deal with these types of loading one at a time; more complicated problems may be solved by a synthesis of separate solutions. We begin by considering a distribution of loading of type (1).

Before we can proceed further, we must determine the order of magnitude of $\{-(1+\sigma)/2Eh\}l$. If we assume that this quantity is of the same order of magnitude as $\bar{w}_{oo}^{1*}$ or w_{oo}^{1*} , the two equations of lowest order of magnitude in (34') and (35') are, respectively, $\{(1-\sigma)\bar{w}_{oo}^{1*} + w_{oo}^{1*}\} = \{-(1+\sigma)/2Eh\}l$ and $\{(1-\sigma)\bar{w}_{oo}' + w_{oo}'\} = 0$. Starring each term of the latter, we obtain an equation which is inconsistent with the former; hence $\{-(1+\sigma)/2Eh\}l$ cannot be of the same order of magnitude as $\bar{w}_{oo}^{1*}$ or w_{oo}^{1*} . For the present, we shall assume that this quantity is of the second order of magnitude; it will be convenient to postpone temporarily the proof that it cannot be of order higher than the second.

We can now write out the two systems of equations which result from equating all terms of the same order of magnitude in (34') and (35'). They are, respectively,

$$(36.0) \quad \{(1-\sigma) \bar{w}_{oo} + w_{oo}\}^{1*} = 0,$$

$$(36.2) \quad \{(1-\sigma) \bar{w}_{2o} + w_{2o}\}^{1*} - \{(2-\sigma) \bar{w}_{oo} + w_{oo}\}^{(1*)^2} \frac{h^2}{3!} = -\frac{1+\sigma}{2Eh} l,$$

$$(36.4) \quad \{(1-\sigma) \bar{w}_{40} + w_{40}\}^{1*} - \{(2-\sigma) \bar{w}_{20} + w_{20}\}^{(1*)^2} \frac{h^2}{3!} \\ + \{(3-\sigma) \bar{w}_{00} + w_{00}\}^{(1*)^3} \frac{h^4}{5!} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(36.2n) \quad \{(1-\sigma) \bar{w}_{2n,0} + w_{2n,0}\}^{1*} - \{(2-\sigma) \bar{w}_{2n-2,0} + w_{2n-2,0}\}^{(1*)^2} \frac{h^2}{3!} + \dots \\ + (-1)^n \{(n+1-\sigma) \bar{w}_{00} + w_{00}\}^{(1*)^{n+1}} \frac{h^{2n}}{(2n+1)!} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(37.0) \quad \{(1-\sigma) \bar{w}_{00} + w_{00}\}' = 0,$$

$$(37.2) \quad \{(1-\sigma) \bar{w}_{20} + w_{20}\}' - \{(2-\sigma) \bar{w}_{00}' + w_{00}'\}^{1*1} \frac{h^2}{2!} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(37.2n) \quad \{(1-\sigma) \bar{w}_{2n,0} + w_{2n,0}\}' - \{(2-\sigma) \bar{w}_{2n-2,0} + w_{2n-2,0}\}^{1*1} \frac{h^2}{2!} + \dots \\ + (-1)^n \{(n+1-\sigma) \bar{w}_{00} + w_{00}\}^{1*(1)^n} \frac{h^{2n}}{(2n)!} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

* On page 358 of Garabedian's "Circular Plates of Constant or Variable Thickness", loc. cit., the groups of equations designated as $Z_{2n+3} = 0$ and $R^{2n+2} = 0$, may be shown, by suitable reductions, to be identical with equations (36) and (37), respectively.

Note that we cannot obtain $\bar{w}_{00}^{1*}$ and w_{00}^{1*} from (36.0) and

(37.0) since the latter equation, when starred, becomes identical with the former. Similarly, it may be shown that $\bar{w}_{2n,0}^{1*}$ and $w_{2n,0}^{1*}$ cannot be obtained from (36.2n) and (37.2n). Although we are unable to solve directly for $\bar{w}_{2n,0}^{1*}$ and $w_{2n,0}^{1*}$, we shall be able to find $\bar{w}_{2n,0}^{1*1*}$ and $w_{2n,0}^{1*1*}$; from the latter, by means of two quadratures, we can find $\bar{w}_{2n,0}^{1*}$ and $w_{2n,0}^{1*}$.

Let us form a third system of equations by starring (37.2n) and subtracting from (36.2n); the equations thus obtained will be in simpler form if we multiply each by $(3/h^2)$. We thus have

$$(38.0) \quad 0 = 0,$$

$$(38.2) \quad \{(2-\sigma) \bar{w}_{00} + w_{00}\}^{(1*)^2} = -\frac{3(1+\sigma)}{2Eh^3} \mathcal{L},$$

$$(38.4) \quad \{(2-\sigma) \bar{w}_{20} + w_{20}\}^{(1*)^2} - \{(3-\sigma) \bar{w}_{00} + w_{00}\}^{(1*)^3} \frac{6 \cdot 2 h^2}{5!} = 0,$$

$$\begin{array}{cccccccccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$(38.2n) \quad \{(2-\sigma) \bar{w}_{2n-2,0} + w_{2n-2,0}\}^{(1*)^2} \\ - \{(3-\sigma) \bar{w}_{2n-4,0} + w_{2n-4,0}\}^{(1*)^3} \frac{6 \cdot 2 h^2}{5!} + \dots \\ - (-1)^n \{(n+1-\sigma) \bar{w}_{00} + w_{00}\}^{(1*)^{n+1}} \frac{6 n h^{2n-2}}{(2n+1)!} = 0,$$

$$\begin{array}{cccccccccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Equations (36) and (38) were obtained on the assumption that $\{-(1+\sigma)/2Eh\} \mathcal{L}$ was of the second order of magnitude

as compared with $\bar{w}_{00}^{1*}$ or w_{00}^{1*} . We have already proved that $\{-(1+\sigma)/2Eh\}l$ is not of the same order of magnitude as $\bar{w}_{00}^{1*}$ or w_{00}^{1*} ; we are now in a position to show that it cannot be of order higher than the second. If it were of the fourth or higher order, equation (38.2) would be $\{(2-\sigma)\bar{w}_{00} + w_{00}\}^{(1*)^2} = 0$. Solving this equation simultaneously either with (36.0) or (37.0), we find $\bar{w}_{00}^{1*1*} \equiv w_{00}^{1*1*} \equiv 0$. Since l must appear on the right hand side of some one of the equations (38), it is evident that we cannot have $\bar{w}_{2n,0}^{1*1*} \equiv w_{2n,0}^{1*1*} \equiv 0$, $n = 0, 1, 2, \dots$. But by a previous assumption, this implies that $\bar{w}_{00}^{1*1*}$ and w_{00}^{1*1*} cannot both be identically zero. From this contradiction, it follows that $\{-(1+\sigma)/2Eh\}l$ is of the second order of magnitude.

By means of (36), (37), and (38), we shall be able to obtain $\bar{w}_{2n,0}^{1*1*}$ and $w_{2n,0}^{1*1*}$ for $n = 0, 1, 2, \dots$. Our procedure consists in solving (38.2n+2) simultaneously with (37.2n). Accordingly, we first perform the star-prime-star operation upon each equation of (37) and designate the resulting set of equations by (37)*^{1*}. In order to secure a notation appropriate to the discussion which follows, we now write (37.0)*^{1*} and (38.2) in the form

$$(39.0) \quad \{(1-\sigma)\bar{w}_{00} + w_{00}\}^{(1*)^2} = -\frac{3(1+\sigma)}{2Eh^3} d_0 l,$$

$$(40.2) \quad \{(2-\sigma)\bar{w}_{00} + w_{00}\}^{(1*)^2} = -\frac{3(1+\sigma)}{2Eh^3} b_0 l,$$

where $b_0 = 1$, $d_0 = 0$. Solving simultaneously, we obtain

$$\bar{w}_{00}^{1*1*} = -\frac{3(1+\sigma)}{2Eh^3} \{b_0 - d_0\} \ell,$$

$$w_{00}^{1*1*} = \frac{3(1+\sigma)}{2Eh^3} \{(1-\sigma)b_0 - (2-\sigma)d_0\} \ell.$$

If we substitute these values of $\bar{w}_{00}^{1*1*}$ and w_{00}^{1*1*} in (37.2)^{*1*} and (38.4), we find

$$(39.2) \quad \{(1-\sigma)\bar{w}_{20} + w_{20}\}^{(1*)^2} = -\frac{3(1+\sigma)}{2Eh^3} d_1 h^2 \ell^{1*},$$

$$(40.4) \quad \{(2-\sigma)\bar{w}_{20} + w_{20}\}^{(1*)^2} = -\frac{3(1+\sigma)}{2Eh^3} b_1 h^2 \ell^{1*},$$

where

$$b_1 = \frac{6 \cdot 2 (2 \cdot b_0 - 1 \cdot d_0)}{5!},$$

$$d_1 = \frac{1 \cdot b_0 - 0 \cdot d_0}{2!}.$$

If this pair of equations is solved simultaneously, the result is

$$\bar{w}_{20}^{1*1*} = -\frac{3(1+\sigma)}{2Eh^3} \{b_1 - d_1\} h^2 \ell^{1*},$$

$$w_{20}^{1*1*} = \frac{3(1+\sigma)}{2Eh^3} \{(1-\sigma)b_1 - (2-\sigma)d_1\} h^2 \ell^{1*}.$$

Substituting in (37.4)^{*1*} and (38.6) the values just found for $\bar{w}_{00}^{1*1*}$, w_{00}^{1*1*} , $\bar{w}_{20}^{1*1*}$, w_{20}^{1*1*} , we have

$$(39.4) \quad \{(1-\sigma)\bar{w}_{40} + w_{40}\}^{(1*)^2} = -\frac{3(1+\sigma)}{2Eh^3} d_2 h^4 \ell^{(1*)^2},$$

$$(40.6) \quad \{(2-\sigma)\bar{w}_{40} + w_{40}\}^{(1*)^2} = -\frac{3(1+\sigma)}{2Eh^3} b_2 h^4 \ell^{(1*)^2},$$

where

$$b_2 = \frac{6 \cdot 2 (2 \cdot b_1 - 1 \cdot d_1)}{5!} - \frac{6 \cdot 3 (3 \cdot b_0 - 2 \cdot d_0)}{7!},$$

$$d_2 = \frac{1 \cdot b_1 - 0 \cdot d_1}{2!} - \frac{2 \cdot b_0 - 1 \cdot d_0}{4!}.$$

Solving again simultaneously, we obtain

$$\bar{w}_{40}^{1*1*} = -\frac{3(1+\sigma)}{2Eh^3} \{b_2 - d_2\} h^4 \ell^{(1*)^2},$$

$$w_{40}^{1*1*} = \frac{3(1+\sigma)}{2Eh^3} \{(1-\sigma)b_2 - (2-\sigma)d_2\} h^4 \ell^{(1*)^2}.$$

By continuing this procedure, we are led finally to the following general formulas:

$$(41a) \quad \bar{w}_{2n,0}^{1*1*} = -\frac{3(1+\sigma)}{2Eh^3} \{b_n - d_n\} h^{2n} \ell^{(1*)^n}, \quad n=0,1,2,\dots;$$

$$(41b) \quad w_{2n,0}^{1*1*} = \frac{3(1+\sigma)}{2Eh^3} \{(1-\sigma)b_n - (2-\sigma)d_n\} h^{2n} \ell^{(1*)^n}, \quad n=0,1,2,\dots;$$

where b_n and d_n are given by the formulas

$$(41c) \quad b_n = 6 \sum_{i=0}^{n-1} (-1)^i \frac{(i+2) \{(i+2)b_{n-1-i} - (i+1)d_{n-1-i}\}}{(2i+5)!}, \quad n=1,2,3,\dots,$$

$$(41d) \quad d_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)b_{n-1-i} - i d_{n-1-i}}{(2i+2)!}, \quad n=1,2,3,\dots.$$

$\bar{w}_{2n,0}^{1*1*}$ and $w_{2n,0}^{1*1*}$ can also be found by solving equations (38.2n+2) simultaneously with equations (36.2n). Again let $b_0 = 1, d_0 = 0$; then, by a process similar to that employed above, we deduce the following values for b_n and d_n :

$$(41'c) \quad b_n = 6 \sum_{i=0}^{n-1} (-1)^i \frac{(i+2) \{(i+2)b_{n-1-i} - (i+1)d_{n-1-i}\}}{(2i+5)!}, \quad n=1,2,3,\dots;$$

$$(41'd) \quad d_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)b_{n-1-i} - i d_{n-1-i}}{(2i+3)!}, \quad n=2,3,4,\dots;$$

$$(41'dd) \quad d_1 = \frac{1 \cdot b_0 - 0 \cdot d_0}{3!} + \frac{1}{3} = \frac{1}{2}.$$

It may be proved without difficulty that the above values of b_n and d_n are precisely those given by formulas (41c) and (41d).

Star-prime-star (33) and make use of (41); the result is

$$(41e) \quad \bar{w}_0'^{1*1*} = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} \{b_n - d_n\} h^{2n} \ell^{(1*)^n},$$

$$(41f) \quad w_0'^{1*1*} = \frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} \{(1-\sigma)b_n - (2-\sigma)d_n\} h^{2n} \ell^{(1*)^n}.$$

To solve for $\bar{w}_0'$ and w_0' , we must first find the relation connecting them. Let us transfer all except the first two terms of equations (37.2n) to the right-hand side; by means of (41a), (41b), and (33), there results

$$(41g) \quad (1-\sigma)\bar{w}_0' + w_0' - \{(2-\sigma)\bar{w}_0' + w_0'\}^{1*1} \frac{h^2}{2} \\ = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} \left\{d_{n+1} - \frac{b_n}{2}\right\} h^{2n+2} \ell^{(1*)^{n+1}}.$$

To obtain equation (41g) in a more convenient form, we star-prime each term. Using (41e) and (41f), we find

$$(41h) \quad \{(1-\sigma)\bar{w}_0' + w_0'\}^{1*1} = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} d_n h^{2n} \ell^{(1*)^{n+1}}.$$

Substitute in (41g) the value of $\bar{w}_0'^{1*1}$ obtained from (41h); we obtain, finally,

$$(41i) \quad (1-\sigma)\bar{w}_0' + w_0' + \frac{h^2}{2(1-\sigma)} w_0'^{1*1} = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} d_n h^{2n} \ell^{(1*)^{n+2}} \\ + \frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} \frac{(1-\sigma)b_n - (2-\sigma)d_n}{2(1-\sigma)} h^{2n+2} \ell^{(1*)^{n+1}}.$$

We are now in position to solve for $\bar{w}_o'$ and w_o . By analogy with art. 6, there are two possible methods of procedure; choosing the simpler of the two, we first find w_o from (41f) and then obtain $\bar{w}_o'$ by means of (41i). Let us designate by $\bar{w}_{oc}'$ and w_{oc} , respectively, the complementary solutions of $\bar{w}_o'$ and w_o , and by $\bar{w}_{op}'$ and w_{op} , respectively, the particular solutions of $\bar{w}_o'$ and w_o . The complementary solutions are obtained by solving the homogeneous equations associated with equations (41e), (41f), (41i). From (41f), we have

$$(41j) \quad w_{oc} = K_1 r^2 \log r + K_2 r^2 + K_3 \log r + K_4,$$

where K_1, K_2, K_3, K_4 are arbitrary constants. The substitution, in (41i) of the values of w_{oc}' and w_{oc}^{*1} obtained from (41j) results in

$$(41k) \quad \bar{w}_{oc}' = -\frac{1}{1-\sigma} \left[K_1 \left\{ 2r \log r + r + \frac{2h^2}{(1-\sigma)r} \right\} + 2K_2 r + \frac{K_3}{r} \right].$$

Since it turns out that the complementary solutions for all types of loading are given by (41j) and (41k), we will henceforth write only the particular solutions.

We shall now find the particular solutions when $\bar{w}_o^{*1}$ and w_o^{*1} are infinite series; the case when they terminate will be considered in art. 10. Let us designate the anti-star-prime-star-prime operation by the symbol $(1*)^{-2}$, and let us agree that $\mathcal{L}^{(1*)^{n-2}}$ is the result obtained when $\mathcal{L}^{(1*)^n}$ is anti-star-prime-star-primed without introduction of arbitrary constants. From (41f), we find

$$(41l) \quad w_{op} = \frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} \{(1-\sigma)b_n - (2-\sigma)d_n\} h^{2n} \ell^{(1*)^{n-2}}.$$

Substituting in (41i), we have

$$(41m) \quad \bar{w}'_{op} = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{\infty} \{b_n - d_n\} h^{2n} \ell^{(1*)^{n-2}}$$

It is clear that this value of $\bar{w}'_{op}$ satisfies (41e).

In the case when there is shearing force only, we solve equations (38) simultaneously with (37). Setting $\bar{b}_0 = 1$, $\bar{d}_0 = 0$, we obtain the following general formulas:

$$(42a) \quad \bar{w}_{2n,0}^{1*1*} = -\frac{3(1+\sigma)}{2Eh^2} \{\bar{b}_n - \bar{d}_n\} h^{2n} J^{*(1*)^n}, \quad n = 0, 1, 2, \dots;$$

$$(42b) \quad w_{2n,0}^{1*1*} = \frac{3(1+\sigma)}{2Eh^2} \{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\} h^{2n} J^{*(1*)^n}, \quad n = 0, 1, 2, \dots;$$

$$(42c) \quad \bar{b}_n = 6 \sum_{i=0}^{n-1} (-1)^i \frac{(i+2)\{(i+2)\bar{b}_{n-1-i} - (i+1)\bar{d}_{n-1-i}\}}{(2i+5)!}, \quad n = 1, 2, 3, \dots;$$

$$(42d) \quad \bar{d}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)\bar{b}_{n-1-i} - i\bar{d}_{n-1-i}}{(2i+2)!}, \quad n = 2, 3, 4, \dots;$$

$$(42dd) \quad \bar{d}_1 = \frac{1 \cdot \bar{b}_0 - 0 \cdot \bar{d}_0}{2!} - \frac{1}{3} = \frac{1}{6}; \quad \dagger$$

$\dagger$ Observe that $\bar{w}_{2n,0}^{1*1*}$ and $w_{2n,0}^{1*1*}$ might also be found by solving equations (38) simultaneously with (36). Recalling that $\bar{b}_0 = 1$, $\bar{d}_0 = 0$, we obtain by this method

$$(42'c) \quad \bar{b}_n = 6 \sum_{i=0}^{n-1} (-1)^i \frac{(i+2)\{(i+2)\bar{b}_{n-1-i} - (i+1)\bar{d}_{n-1-i}\}}{(2i+5)!}, \quad n = 1, 2, 3, \dots;$$

$$(42'd) \quad \bar{d}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)\bar{b}_{n-1-i} - i\bar{d}_{n-1-i}}{(2i+3)!}, \quad n = 1, 2, 3, \dots.$$

It may readily be proved that the above values of $\bar{b}_n$ and $\bar{d}_n$ are the same as those given by (42c), (42d), (42dd),

$$(42e) \quad \bar{w}_o^{1*1*} = -\frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{\infty} \{b_n - d_n\} h^{2n} J^{*(1*)^n};$$

$$(42f) \quad w_o^{1*1*} = \frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{\infty} \{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\} h^{2n} J^{*(1*)^n};$$

$$(42i) \quad (1-\sigma)\bar{w}_o' + w_o' + \frac{h^2}{2(1-\sigma)} w_o^{1*1} = -\frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{\infty} \bar{d}_n h^{2n} J^{*(1*)^{n-1}} \\ + \frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{\infty} \frac{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n}{2(1-\sigma)} h^{2n+2} J^{*(1*)^n}.$$

When $\bar{w}_o^{1*1*}$ and w_o^{1*1*} , as defined by (42e) and (42f), are infinite series, we find that

$$(42l) \quad w_{op} = \frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{\infty} \{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\} h^{2n} J^{*(1*)^{n-2}},$$

$$(42m) \quad \bar{w}_{op}' = -\frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{\infty} \{\bar{b}_n - \bar{d}_n\} h^{2n} J^{*(1*)^{n-1}}.$$

For the third case, the formulas are very simple; if only radial mass force is acting, we have

$$(43ef) \quad \bar{w}_o^{1*1*} = w_o^{1*1*} = 0;$$

$$(43i) \quad (1-\sigma)\bar{w}_o' + w_o' + \frac{h^2}{2(1-\sigma)} w_o^{1*1} = 0;$$

$$(43lm) \quad w_{op} = \bar{w}_{op}' = 0.$$

Finally, when the plate is subjected to axial mass force only, the formulas become

$$(44a) \quad \bar{w}_{oo}^{1*1*} = \frac{3(1+\sigma)}{Eh^2} c_z; \quad \bar{w}_{2n,o}^{1*1*} = 0, \quad n = 1, 2, \dots$$

$$(44b) \quad w_{oo}^{1*1*} = -\frac{3(1-\sigma^2)}{Eh^2} c_z; \quad w_{2n,o}^{1*1*} = 0, \quad n = 1, 2, \dots$$

$$(44e) \quad \bar{w}_o^{1*1*} = \frac{3(1+\sigma)}{Eh^2} c_z.$$

$$(44f) \quad w_o^{1*1*} = - \frac{3(1-\sigma^2)}{Eh^2} c_z.$$

$$(44i) \quad (1-\sigma)\bar{w}_o' + w_o' + \frac{h^2}{2(1-\sigma)} w_o^{1*1} = 0.$$

$$(44l) \quad w_{op} = - \frac{3(1-\sigma^2)}{64Eh^2} c_z r^4.$$

$$(44m) \quad \bar{w}_{op}' = \frac{3(1+\sigma)}{16Eh^2} c_z r^3 + \frac{3(1+\sigma)}{4(1-\sigma)E} c_z r.$$

8. Upper bounds of the constants. Before we can establish convergence of the series for U and w , we must exhibit upper bounds for the eight constants which enter in the formulas for $\bar{U}_o^{*1}$, U_o^{*1} , $\bar{w}_o^{1*1*}$, w_o^{1*1*} .

Let us first consider the constants $\bar{a}_n$ and $\bar{c}_n$. Recalling that $\bar{a}_0 = 1$, $\bar{c}_0 = 0$, we compute from (30c) and (30d) the values $\bar{a}_1 = 1/3$, $\bar{c}_1 = 1/2$, $\bar{a}_2 = 1/360$, $\bar{c}_2 = 1/12$. We shall now prove by induction that

$$(45a) \quad |\bar{a}_n| < 1/2^n, \quad |\bar{c}_n| < (5/9)/2^n, \quad n = 2, 3, \dots$$

This relation is seen to be true for $n = 2$. The proof, therefore, will consist of showing that if

$$(45b) \quad |\bar{a}_{k-1-i}| < \frac{1}{2^{k-1-i}}, \quad |\bar{c}_{k-1-i}| < \frac{5}{9} \frac{1}{2^{k-1-i}}, \quad k = 0, 1, \dots, (k-3),$$

then

$$(45c) \quad |\bar{a}_k| < 1/2^k, \quad |\bar{c}_k| < (5/9)/2^k, \quad k = 3, 4, \dots$$

From (30c), we have

$$(45d) \quad |\bar{a}_k| \leq \sum_{i=0}^{k-3} \frac{(i+2)|\bar{a}_{k-1-i}| + (i+1)|\bar{c}_{k-1-i}|}{(2i+3)!} + \frac{|k\bar{a}_1 - (k-1)\bar{c}_1|}{(2k-1)!} + \frac{|(k+1)\bar{a}_0 - k\bar{c}_0|}{(2k+1)!}, \quad k=3,4,\dots.$$

If we substitute in (45d) the values of $|\bar{a}_{k-1-i}|$ and $|\bar{c}_{k-1-i}|$ from (45b) and also the above values of $\bar{a}_0, \bar{c}_0, \bar{a}_1, \bar{c}_1$, the result is

$$(45e) \quad |\bar{a}_k| < \sum_{i=0}^{k-3} \frac{1}{9} \frac{14i+23}{(2i+3)!} \frac{1}{2^{k-1-i}} + \frac{|\frac{1}{2} - \frac{1}{6}k|}{(2k-1)!} + \frac{k+1}{(2k+1)!} \\ < \sum_{i=0}^{k-1} \frac{1}{9} \frac{14i+23}{(2i+3)!} \frac{1}{2^{k-1-i}} < \frac{1}{2^{k-1}} \sum_{i=0}^{\infty} \frac{2^i}{9} \frac{14i+23}{(2i+3)!}, \quad k=3,4,\dots.$$

If we calculate the first three terms of this series and compare the remainder with the series $\sum_{i=0}^{\infty} 1/(1+9)^i$, we find

$$(45f) \quad |\bar{a}_k| < (.4992)/2^{k-1} < 1/2^k, \quad k=3,4,\dots.$$

In a similar manner, we have

$$(45g) \quad |\bar{c}_k| < (.2623)/2^{k-1} < (5/9)/2^k, \quad k=3,4,\dots.$$

In obtaining (45g), the following formula for $\bar{c}_n$ was used instead of (30d):

$$(45h) \quad \bar{c}_n = \sum_{i=0}^{n-2} (-1)^i \frac{(i+1)\{(i+2)\bar{a}_{n-2-i} - (i+1)\bar{c}_{n-2-i}\}}{(2i+4)!}, \quad n=2,3,4,\dots,$$

a formula readily derived from (30c) and (30d).

In a closely analogous manner, we can show that

$$(45i) \quad |a_n| < 1/2^n, \quad |c_n| < (5/9)/2^n, \quad n=1,2,\dots.$$

In obtaining upper bounds for $b_n, d_n, \bar{b}_n, \bar{d}_n$, we use formulas (41c), (41'd), (42c), (42'd), respectively, and find, by a similar procedure,

$$(45j) \quad |b_n| < 1/3^n, \quad |d_n| < (3/4)/3^n, \quad n = 2, 3, \dots;$$

$$(45k) \quad |\bar{b}_n| < 1/3^n, \quad |\bar{d}_n| < (3/4)/3^n, \quad n = 1, 2, \dots.$$

The above upper bounds seem to be the strongest that can be found by the method of absolute values. By means of these inequalities we shall show in the next article that the U and w series are uniformly convergent for a limited class of load functions. We should like to include in this class of load functions all powers of r for which the series do not terminate, but the present inequalities are not strong enough for this purpose. On the other hand, if we calculate a few values of the constants, we find that their absolute values are considerably smaller than the above upper bounds would indicate. This suggests the possibility of securing, by a more refined analysis, the stronger upper bounds desired. In this paper we shall be content to establish convergence on the basis of the inequalities deduced above.

Since there is no particular advantage in using for the c's, b's, and d's stronger upper bounds than for the a's, we shall be content in preparation for the following article, to write

$$(45l) \quad |a_n|, |\bar{a}_n|, |c_n|, |\bar{c}_n|, |b_n|, |\bar{b}_n|, |d_n|, |\bar{d}_n| \leq 1/2^n,$$

$$n = 0, 1, 2, \dots$$

From (45a), (45i), (45j), (45k), we see that the above relations hold for $n = 2, 3, 4, \dots$; recalling the values of the constants for $n = 0$, and calculating their values for $n = 1$, we find that (45l) is also valid for $n = 0, 1$.

9. Convergence of the U and w series. Let U_c and w_c be the complementary solutions of U and w, respectively, and U_p and w_p be the corresponding particular solutions. There is no need to examine U_c and w_c , since we are postponing until the next article the consideration of those cases in which U and w terminate; for the same reason, we are interested only in the values of U_{op} , $\bar{U}_{op}^*$, w_{op} , $\bar{w}_{op}'$ which are defined by infinite series.

Let us first consider U_p when the plate is under normal load only. Substituting (29j), (29k), (41l), (41m) in (14a), we obtain

$$(46a) \quad U_p = -\frac{1+\sigma}{2E} \sum_{i=0}^{\infty} (-1)^i i \left\{ \sum_{n=0}^{\infty} \{a_n - c_n\} h^{2n} L^{(1)(*)^{n-1}} \right\}^{(*)^i} \frac{Z^{2i}}{(2i)!} \\ - \frac{1+\sigma}{2E} \sum_{i=0}^{\infty} (-1)^i \left\{ \sum_{n=0}^{\infty} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{(1)(*)^{n-1}} \right\}^{(*)^i} \frac{Z^{2i}}{(2i)!} \\ - \frac{3(1+\sigma)}{2Eh^3} \sum_{i=0}^{\infty} (-1)^i (i+2-2\sigma) \left\{ \sum_{n=0}^{\infty} \{b_n - d_n\} h^{2n} L^{(1)(*)^{n-2}} \right\}^{(*)^i} \frac{Z^{2i+1}}{(2i+1)!} \\ + \frac{3(1+\sigma)}{2Eh^3} \sum_{i=0}^{\infty} (-1)^i \left\{ \sum_{n=0}^{\infty} \{(1-\sigma)b_n - (2-\sigma)d_n\} h^{2n} L^{(1)(*)^{n-2}} \right\}^{(*)^i} \frac{Z^{2i+1}}{(2i+1)!}.$$

Since we shall be able to show that each of the above series is absolutely convergent for a suitably restricted class of load functions, it will be convenient in the discussion to follow to employ the double series rather than the given iterated series.

Let A be the first iterated series, A_1 the corresponding double series, and A_2 the series formed from the absolute values of the terms of A_1 . Using (45f), we find

$$(45b) \quad A_2 \leq \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{i}{(2i)!} \frac{2}{2^n} h^{2n} |L^{(i)}|^{n-i+i} |z^{2i}|.$$

Making use of the fact that $1/(2i)! \leq 1/2^i$, $i = 0, 1, 2, \dots$, and that $|z| \leq h$, we have

$$(46c) \quad A_2 \leq 2 \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{h^{2(n+i)}}{2^{(n+i)}} |L^{(i)}|^{n-i+i}.$$

Let r_0 be the outer radius of the plate, and let q, q_1, q_2 be constants such that $q_2 = r_0/h$, $0 < q < q_1 < q_2$. Then

$$(46d) \quad A_2 \leq 2 \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{r_0}{\sqrt{2}q}\right)^{2(n+i)} |L^{(i)}|^{n-i+i} \left(\frac{qh}{r_0}\right)^{2(n+i)}.$$

Since $0 < qh/r_0 < q_1 h/r_0 < 1$, the above double series of positive terms will be uniformly convergent if L is such a function of r that a constant M can be found for which

$$(46e) \quad \left(\frac{r_0}{\sqrt{2}q}\right)^{2(n+i)} |L^{(i)}|^{n-i+i} < M, \quad 0 \leq r \leq r_0,$$

for all values of n and i .

It may be shown that if $L = p_1 J_0(\pm \sqrt{2}qr/r_0)$, where p_1 is a constant, then (46e) reduces to $\{p_1 r_0 / (\sqrt{2}q)\} J_1(\pm \sqrt{2}qr/r_0)$

for all values of n and i . Hence we actually exhibit a class of Bessel functions for which the iterated series A is both absolutely and uniformly convergent in the interval $0 \leq r \leq r_0$.

A procedure similar to the foregoing enables us to show that the series defining either U_p or w_p will be absolutely and uniformly convergent for normal load or shearing traction under hypothesis analogous to (46e). In particular, we have, as interesting examples of load functions which give convergent developments, the following:

$$(46f) \quad L = p_1 J_0(\pm \sqrt{2}qr/r_0), \quad \mathcal{L} = p_2 J_0(\pm \sqrt{2}qr/r_0),$$

$$J = p_3 J_1(\pm \sqrt{2}qr/r_0), \quad j = p_4 J_1(\pm \sqrt{2}qr/r_0),$$

where p_1, p_2, p_3, p_4 are constants. *

* Garabedian has found that the U and w series obtained by his method are absolutely and uniformly convergent, and gives, by way of examples, normal loads and shearing tractions proportional to $J_0(r/r_0)$ and $J_1(r/r_0)$, respectively.

In the case of the examples just given, it is readily shown that the series for U_p and w_p may be primed, starred, or differentiated with respect to z , term by term, as many times as may be desired. Thus we have now exhibited examples in which all the operations that have been hitherto applied formally are seen to be justified.

10. Finite series. In article 5 we showed that a

necessary and sufficient condition for the U and w series to terminate is that U_0 and $\bar{w}_0'$ contain no terms which are not of the form

$$C r^{2m-3} + K r^{2n-1} \log r$$

and that $\bar{U}_0^*$ and w_0 contain no terms which are not of the form

$$C' r^{2p-2} + K' r^{2q-2} \log r,$$

m, n, p, q being any positive integers. From formulas (29), (31), (32), (41), (43), (44) it is apparent that the complementary solutions and also the particular solutions for the cases of radial mass force and axial mass force, have the forms which ensure terminating series for the displacements. It remains only to determine the admissible forms of particular solutions in the cases of normal load and shearing traction.

When the plate is subjected to normal load or shearing traction, it is easy to show that a condition sufficient to meet the above requirements on $U_0, \bar{U}_0^*, w_0, \bar{w}_0'$ is that L and ℓ shall contain no terms which are not of the form

$$C' r^{2p-2} + K' r^{2q-2} \log r$$

and that J and j shall contain no terms which are not of the form

$$C r^{2m-3} + K r^{2n-1} \log r,$$

m, n, p, q being any positive integers. Moreover, it may be shown by an argument similar to that employed in article 5 that, assuming $U_0, \bar{U}_0^*, w_0, \bar{w}_0'$ given by series which terminate, the above conditions are also necessary; we shall not attempt in this paper to find necessary conditions in the case in which $U_0, \bar{U}_0^*, w_0, \bar{w}_0'$ are defined by infinite series.

Let us consider first the case in which L, ℓ, J, j contain no logarithmic terms. At this point, we find it desirable to introduce two functions of m and n , namely,

$$(47a) \quad \alpha_{mn} = 4^n \left\{ \frac{m!}{(m-n)!} \right\}^2 ; \quad n \leq m,$$

$$(47b) \quad \alpha'_{mn} = 2(m-n)\alpha_{mn} ; \quad n < m,$$

where $m = 0, 1, 2, \dots$, and n is any integer, or zero. It may readily be shown that

$$(48a) \quad (r^{2m})^{(1*)^n} = \alpha_{mn} r^{2(m-n)}, \quad 0 ; \quad n \leq m, \quad n > m,$$

$$(48b) \quad (r^{2m})^{(1*)^{n'}} = \alpha'_{mn} r^{2(m-n)-1}, \quad 0 ; \quad n < m, \quad n \geq m.$$

We shall not lose in generality if we assume that L, ℓ, J, j contain but one term, since the more complicated types of loading may be considered as the sum of simple loadings which contain one term only. We write, therefore, for the case of normal load

$$(49a) \quad L = e_1 r^{2m}, \quad \ell = e_2 r^{2m}, \quad m = 0, 1, 2, \dots,$$

where e_1 and e_2 are constants. Making use of (49a) and (48),

we obtain from (29e), (29f), (29g), respectively,

$$(49b) \quad \bar{U}_o^{*1} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m-1} \{a_n - c_n\} h^{2n} L^{(1*)^n},$$

$$(49c) \quad U_o^{*1} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m-1} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{(1*)^n},$$

$$(49d) \quad -(1-\sigma) \bar{U}_o^* + U_o^* = -\frac{1+\sigma}{2E} \sum_{n=0}^m c_n h^{2n} L^{(1*)^n}.$$

Performing the inverse prime-star operation upon (49c), we find

$$(49e) \quad U_{op} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m-1} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{(1*)^{n-1}}.$$

Recalling that $U_{oc} = C_1 r + c_2/r$, we observe that U_{op} , as just defined, contains no terms like either of the terms in U_{oc} . But this property is not essential to a particular solution; it is possible to include additional terms which vanish when prime-starred. Instead of (48e) we shall adopt the particular solution,

$$(49f) \quad U_{op} = -\frac{1+\sigma}{2E} \sum_{n=0}^m \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{(1*)^{n-1}};$$

since by this choice our formulas become more-concise and elegant. Note that the difference between (49e) and (49f) is automatically taken care of by the corresponding difference in the complementary solution. Substituting (49f) in (49d), we have

$$(49g) \quad \bar{U}_{op}^* = -\frac{1+\sigma}{2E} \sum_{n=0}^m \{a_n - c_n\} h^{2n} L^{(1*)^n}.$$

Observe that (49f) and (49g) are identical with (29j) and (29k), respectively, except that the upper limits of n in (49f) and (49g) are finite instead of infinite. We shall find it desirable, in the particular solutions of U_o , $\bar{U}_o^*$, w_o , $\bar{w}_o'$, to use the forms already obtained for the corresponding infinite series except that now the upper limit is, in the case of U_o , $\bar{U}_o^*$, $\bar{w}_o'$, the largest integer which yields a non-vanishing term; the upper limit of w_{op} is taken to be the same as that for $\bar{w}_{op}'$. It may be shown that all the particular solutions thus defined will satisfy the appropriate set of differential equations.

Recalling (41l) and (41m), and using (49a) and (48) to determine the upper limits of n , we find

$$(49h) \quad w_{op} = \frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{m+1} \{ (1-\sigma)b_n - (2-\sigma)d_n \} h^{2n} \ell^{(1*)^{n-2}},$$

$$(49i) \quad \bar{w}_{op}' = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{m+1} \{ b_n - d_n \} h^{2n} \ell^{(1*)^{n-2}}.$$

In the case of shear we write

$$(50a) \quad J = s_1 r^{2m+1}, \quad j = s_2 r^{2m+1}, \quad m = 0, 1, 2, \dots,$$

where s_1 and s_2 are constants. The case in which $m = -1$ will be handled separately below. Observe that

$$(50a') \quad J^* = 2(m+1)s_1 r^{2m}, \quad j^* = 2(m+1)s_2 r^{2m}, \quad m = 0, 1, 2, \dots;$$

that is, J^* and j^* are proportional to r^{2m} and, therefore,

we may use formulas (48) to determine the upper limits of n .

We obtain

$$(50f) \quad U_{op} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{m+1} \{(1-\sigma)\bar{a}_n + \sigma\bar{c}_n\} h^{2n} j^{*(*)^{n-1}},$$

$$(50g) \quad \bar{U}_{op}^* = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{m+1} \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{*(*)^{n-1}},$$

$$(50h) \quad w_{op} = \frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{m+1} \{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\} h^{2n} J^{*(*)^{n-2}},$$

$$(50i) \quad \bar{w}'_{op} = -\frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{m+1} \{\bar{b}_n - \bar{d}_n\} h^{2n} J^{*(*)^{n-1}}.$$

For the special case of shear in which

$$(51a) \quad J = s_1/r, \quad j = s_2/r,$$

we have

$$(51f) \quad U_{op} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^1 \{(1-\sigma)\bar{a}_n + \sigma\bar{c}_n\} h^{2n} j^{*(*)^{n-1}},$$

$$(51g) \quad \bar{U}_{op}^* = -\frac{1+\sigma}{2Eh} \sum_{n=0}^0 \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{*(*)^{n-1}},$$

$$(51h) \quad w_{op} = \frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^1 \{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\} h^{2n} J^{*(*)^{n-2}},$$

$$(51i) \quad \bar{w}'_{op} = -\frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^1 \{\bar{b}_n - \bar{d}_n\} h^{2n} J^{*(*)^{n-1}}.$$

In order to treat the case in which L, ℓ, J, j contain logarithmic terms, it is necessary to introduce two additional quantities, namely,

$$(52a) \quad \beta_{mn} = \frac{1}{m} + \frac{1}{m-1} + \dots + \frac{1}{m-n+1}, \quad 0; \quad 0 < n \leq m, \quad n=0,$$

$$(52a') \quad \beta_{mn} = -\frac{1}{m+1} - \frac{1}{m+2} - \dots - \frac{1}{m-n}; \quad n < 0,$$

$$(52b) \quad \beta'_{mn} = \beta_{mn} + \frac{1}{2(m-n)}; \quad n < m,$$

where $m = 0, 1, 2, \dots$, and where n is any integer, or zero.

It may be shown without difficulty that

$$(53a) \quad (r^{2m} \log r)^{(1*)^n} = \alpha_{mn} r^{2(m-n)} \{\log r + \beta_{mn}\}, \quad n \leq m, \\ = 0, \quad n > m;$$

$$(53b) \quad (r^{2m} \log r)^{(1*)^{n-1}} = \alpha'_{mn} r^{2(m-n)-1} \{\log r + \beta'_{mn}\}, \quad n < m, \\ = \alpha_{mm}/r, \quad 0; \quad n = m, \quad n > m.$$

For the case of normal load, we let

$$(54a) \quad L = e_3 r^{2m} \log r, \quad \mathcal{L} = e_4 r^{2m} \log r, \quad m = 0, 1, 2, \dots,$$

where e_3 and e_4 are constants. We find

$$(54f) \quad U_{op} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m+1} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{(1*)^{n-1}},$$

$$(54g) \quad \bar{U}_{op}^* = -\frac{1+\sigma}{2E} \sum_{n=0}^m \{a_n - c_n\} h^{2n} L^{(1*)^n},$$

$$(54h) \quad w_{op} = \frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{m+2} \{(1-\sigma)b_n - (2-\sigma)d_n\} h^{2n} \mathcal{L}^{(1*)^{n-2}},$$

$$(54i) \quad \bar{w}'_{op} = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{m+2} \{b_n - d_n\} h^{2n} \mathcal{L}^{(1*)^{n-2}}.$$

For the case of shearing traction we write

$$(55a) \quad J = s_3 r^{2m+1} \log r, \quad j = s_4 r^{2m+1} \log r, \quad m = 0, 1, 2, \dots,$$

where s_3 and s_4 are constants. Observe that

$$(55a') \quad J^* = 2(m+1)s_3 r^{2m} \log r + s_3 r^{2m}, \quad m = 0, 1, 2, \dots;$$

$$j^* = 2(m+1)s_4 r^{2m} \log r + s_4 r^{2m}, \quad m = 0, 1, 2, \dots.$$

Using (53) and (48) to determine the upper limits of $(r^{2m} \log r)$ and r^{2m} , respectively, we obtain

$$(55f) \quad U_{op} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{m+2} \{(1-\sigma)\bar{a}_n + \sigma\bar{c}_n\} h^{2n} j^{*(1)}^{n-1},$$

$$(55g) \quad \bar{U}_{op}^* = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{m+1} \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{*(1)*}^{n-1},$$

$$(55h) \quad w_{op} = \frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{m+2} \{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\} h^{2n} J^{*(1)*}^{n-2},$$

$$(55i) \quad \bar{w}_{op}' = -\frac{3(1+\sigma)}{2Eh^2} \sum_{n=0}^{m+2} \{\bar{b}_n - \bar{d}_n\} h^{2n} J^{*(1)}^{n-1}.$$

We may now write out the complete formulas for the displacements. It is convenient to give in full the portion involving the mass forces and to use the symbols U_{op} , $\bar{U}_{op}^*$, w_{op} , $\bar{w}_{op}'$ for the particular solutions in the cases of normal load and shearing traction. The displacements take the form

$$(56a) \quad U = C_1 r + \frac{C_2}{r} - \left[K_1 \left\{ 2r \log r + r + \frac{4h^2}{(1-\sigma)r} \right\} + 2K_2 r + \frac{K_3}{r} \right] Z \\ + \frac{2(2-\sigma)}{3(1-\sigma)} \frac{K_1}{r} Z^3 + \frac{1-\sigma^2}{8E} C_1 r^3 + \frac{3(1-\sigma^2)}{16Eh^2} C_2 r^3 Z \\ + \frac{3(1+\sigma)}{2E} C_2 r Z + \frac{\sigma(1+\sigma)}{2E} C_1 r Z^2 \\ - \frac{(2-\sigma)(1+\sigma)}{4Eh^2} C_2 r Z^3 + \sum_{i=0}^{\infty} (-1)^i \{ i \bar{U}_{op} + U_{op} \}^{(1)*} \frac{Z^{2i}}{(2i)!} \\ + \sum_{i=0}^{\infty} (-1)^i \{ (i+2-2\sigma) \bar{w}_{op} + w_{op} \}^{(1)*} \frac{Z^{2i+1}}{(2i+1)!},$$

where C_1 and C_2 are constants, K_1 , K_2 , and K_3 are values which are constants for U_{op} ; we shall find in the next section that the upper limit of m is ∞ for U_{op} . Thus we define

$$\begin{aligned}
(56b) \quad w = & K_1 r^2 \log r + K_2 r^2 + K_3 \log r + K_4 - \frac{2\sigma}{1-\sigma} C_1 Z \\
& + \frac{2\sigma}{1-\sigma} \{K_1(1+\log r) + K_2\} Z^2 - \frac{3(1-\sigma^2)}{64 E h^2} C_2 r^4 \\
& - \frac{\sigma(1+\sigma)}{2E} C_1 r^2 Z - \frac{3\sigma(1+\sigma)}{8 E h^2} C_2 r^2 Z^2 \\
& - \frac{(1+4\sigma)(1+\sigma)}{4(1-\sigma)E} C_2 Z^2 - \frac{\sigma^2(1+\sigma)}{3(1-\sigma)E} C_1 Z^3 \\
& + \frac{(1+\sigma)^2}{8 E h^2} C_2 Z^4 + \sum_{i=0}^{\infty} (-1)^i \{i \bar{W}_{op} + W_{op}\}^{(1*)^i} \frac{Z^{2i}}{(2i)!} \\
& - \sum_{i=0}^{\infty} (-1)^i \{(i-1+2\sigma) \bar{U}_{op} + U_{op}\}^{*(1*)^i} \frac{Z^{2i+1}}{(2i+1)!}.
\end{aligned}$$

The upper limits of i in the above summations are, taken in order, the same as the upper limits of n already found for U_{op} , $\bar{W}'_{op}$, w_{op} , $\bar{U}_{op}^*$, respectively.

We are now in position to express T_r , N_r , G_r in terms of c_r , c_z , U_{op} , $\bar{U}_{op}^*$, w_{op} , $\bar{W}'_{op}$, and the arbitrary constants involved in (56). At the same time, we anticipate the needs of Part II by computing also

$$(57a) \quad T_\theta = \int_{-h}^h \bar{\theta} \theta \, dz,$$

$$(57b) \quad G_\theta = \int_{-h}^h \bar{\theta} \theta \, z \, dz.$$

The formulas for T_r and T_θ can be considerably simplified by making use of the function $\bar{U}_{op}$. This function has hitherto been undefined, since it does not appear in the U or w series. We may assign to $\bar{U}_{op}$ any value which is consistent with the formulas already given for $\bar{U}_{op}^*$; we shall find it convenient, in each case, to take for the upper limit of n in $\bar{U}_{op}$ the value found in U_{op} . Thus we define

$$(49g') \quad \bar{U}_{op} = -\frac{1+\sigma}{2E} \sum_{n=0}^m \{a_n - c_n\} h^{2n} L^{(*)n-1},$$

$$(50g') \quad \bar{U}_{op} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{m+1} \{\bar{a}_n - \bar{c}_n\} h^{2n} J^{(*)n-1},$$

$$(51g') \quad \bar{U}_{op} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^1 \{\bar{a}_n - \bar{c}_n\} h^{2n} J^{(*)n-1},$$

$$(54g') \quad \bar{U}_{op} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m+1} \{a_n - c_n\} h^{2n} L^{(*)n-1},$$

$$(55g') \quad \bar{U}_{op} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{m+2} \{\bar{a}_n - \bar{c}_n\} h^{2n} J^{(*)n-1}.$$

By means of (49), (50), (51), (54), (55), and the formulas which define the a's, b's, c's, and d's, it may be shown that the following relations are valid for all cases of normal load and shearing traction:

$$(58a) \quad \sum_{i=0}^i (-1)^i \{ (i+\sigma) \bar{U}_{op} + U_{op} \} \frac{h^{2i+1}}{(2i+1)!} = -\frac{1+\sigma}{2E} J^{(*)i-1},$$

$$(58b) \quad \sum_{i=0}^i (-1)^i \{ (i+1-\sigma) \bar{W}_{op} + W_{op} \} \frac{h^{2i+1}}{(2i+1)!} = -\frac{1+\sigma}{2E} L^{(*)i-1},$$

$$(58c) \quad \sum_{i=0}^i (-1)^i \{ (i+2-\sigma) \bar{W}_{op} + W_{op} \} \frac{h^{2i+3}}{(2i+3)!} = -\frac{1+\sigma}{4E} h J^{(*)i-1} \\ - \frac{1+\sigma}{4E} L^{(*)i-1},$$

the upper limits of i in the summations being the same as those for n in U_{op} , $\bar{W}'_{op}$, $\bar{W}_{op}$, respectively,

By substituting (56) in (3a), (3b), (3d), we obtain $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{rz}$, respectively; putting these values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{rz}$ in (7a), (7b), (7d), (57a), (57b), and making use of (58), we compute, finally,

$$(59a) \quad T_r = \frac{2Eh}{1-\sigma} C_1 - \frac{2Eh}{1+\sigma} \frac{C_2}{r^2} + \frac{3+\sigma}{4} hr^2 C_r + \frac{\sigma(1+\sigma)}{3(1-\sigma)} h^3 C_r \\ + \frac{2E\sigma}{1+\sigma} \frac{1}{r} \sum_{i=0}^{\infty} (-1)^i \bar{U}_{op}^{(*)i} \frac{h^{2i+1}}{(2i+1)!} - \frac{1}{r} j^{(*)-1},$$

$$(59b) \quad T_\theta = \frac{2Eh}{1-\sigma} C_1 - \frac{2Eh}{1+\sigma} \frac{C_2}{r^2} + \frac{1+3\sigma}{4} hr^2 C_r + \frac{\sigma(1+\sigma)}{3(1-\sigma)} h^3 C_r \\ + \frac{2E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \bar{U}_{op}^{(*)i} \frac{h^{2i+1}}{(2i+1)!} - \frac{1}{r} j^{(*)-1},$$

$$(59c) \quad G_r = -\frac{2Eh^3}{3(1-\sigma)} \left\{ 2 \log r + \frac{3+\sigma}{1+\sigma} - \frac{2(8+\sigma)}{5(1+\sigma)} \frac{h^2}{r^2} \right\} K_1 - \frac{4Eh^3}{3(1-\sigma)} K_2 \\ + \frac{2Eh^3}{3(1+\sigma)} \frac{K_3}{r^2} + \left\{ \frac{3+\sigma}{8} \frac{r^2}{h^2} + \frac{24+23\sigma+3\sigma^2}{30(1-\sigma)} \right\} h^3 C_z \\ + \frac{4E\sigma}{1+\sigma} \frac{1}{r} \sum_{i=0}^{\infty} (-1)^i \bar{W}_{op}^{(*)i} \frac{h^{2i+3}}{(2i+3)!} - h J^{(*)-1} - \ell^{(*)-1},$$

$$(59d) \quad G_\theta = -\frac{2Eh^3}{3(1-\sigma)} \left\{ 2 \log r + \frac{1+3\sigma}{1+\sigma} + \frac{2(8+\sigma)}{5(1+\sigma)} \frac{h^2}{r^2} \right\} K_1 - \frac{4Eh^3}{3(1-\sigma)} K_2 \\ - \frac{2Eh^3}{3(1+\sigma)} \frac{K_3}{r^2} + \left\{ \frac{1+3\sigma}{8} \frac{r^2}{h^2} + \frac{24+23\sigma+3\sigma^2}{30(1-\sigma)} \right\} h^3 C_z \\ + \frac{4E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \bar{W}_{op}^{(*)i} \frac{h^{2i+3}}{(2i+3)!} - \frac{h}{r} J^{(*)-1} - \frac{1}{r} \ell^{(*)-1},$$

$$(59e) \quad N_r = -\frac{8Eh^3}{3(1-\sigma^2)} \frac{K_1}{r} + rh C_z - \ell^{(*)-1},$$

where the upper limits of i in the summations are the same as those of n for $\bar{U}_{op}$, $\bar{U}'_{op}$, $\bar{W}'_{op}$, $\bar{W}''_{op}$, respectively.

There are four other relations which we shall need in Part II, namely,

$$(60a) \quad T_{r\theta} = T_r + T_\theta = \frac{4Eh}{1-\sigma} C_1 + (1+\sigma) hr^2 C_r + \frac{2\sigma(1+\sigma)}{3(1-\sigma)} h^3 C_r \\ + \frac{2E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \bar{U}_{op}^{(*)i} \frac{h^{2i+1}}{(2i+1)!} - \frac{1}{r} j^{(1)-1},$$

$$(60b) \quad t_{r\theta} = T_r - T_\theta = -\frac{4Eh}{1+\sigma} \frac{C_2}{r^2} + \frac{1-\sigma}{2} h r^2 C_r \\ + \frac{2E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \left\{ \frac{1}{r} \bar{U}_{op}^{(i)} - \bar{U}_{op}^{(i)} \right\} \frac{h^{2i+1}}{(2i+1)!} \\ + \frac{1}{r} j^{(i)} - j^{(i)},$$

$$(60c) \quad G_{r\theta} = G_r + G_\theta = -\frac{8Eh^3}{3(1-\sigma)} \{ (1+\log r) K_1 + K_2 \} \\ + \left\{ \frac{1+\sigma}{2} \frac{r^2}{h^2} + \frac{24+23\sigma+3\sigma^2}{15(1-\sigma)} \right\} h^3 C_z \\ + \frac{4E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \frac{\bar{W}_{op}^{(i)} - \bar{W}_{op}^{(i)}}{(2i+3)!} h^{2i+3} - h J^{(i)} - \ell^{(i)},$$

$$(60d) \quad g_{r\theta} = G_r - G_\theta = -\frac{4Eh^3}{3(1-\sigma^2)} \left\{ 1-\sigma - \frac{2(8+\sigma)}{5} \frac{h^2}{r^2} \right\} K_1 + \frac{4Eh^3}{3(1+\sigma)} \frac{K_3}{r^2} \\ + \frac{1-\sigma}{4} h r^2 C_z \\ + \frac{4E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \left\{ \frac{1}{r} \bar{W}_{op}^{(i)} - \bar{W}_{op}^{(i)} \right\} \frac{i+1}{(2i+3)!} h^{2i+3} \\ + \frac{h}{r} J^{(i)} - h J^{(i)} + \frac{1}{r} \ell^{(i)} - \ell^{(i)}.$$

We have now obtained all the formulas necessary for handling a wide class of problems in moderately thick plates. In Part II we shall exhibit the power of our theoretical machinery by applying it to the solution of certain problems of especial interest.

PART II

11. Introduction to Part II. In applying the theory of Part I, we shall consider the following problems:

- (i) a complete plate, the load being
 - (a) a function of r continuous over the whole plate,
 - (b) a pressure concentrated at the center,
 - (c) a distribution continuous in each of two concentric zones but discontinuous at their junction -- a bizonal problem;
- (ii) an incomplete plate, that is, a plate with a concentric hole, the load being
 - (a) a function of r continuous over the whole plate,
 - (b) a uniform shear distributed over the inner edge of the plate.

Further types of problems to which our method is applicable will be mentioned at the close of the paper.

The outer edge of the plate will always be denoted by r_0 ; the radius of an inner edge, or of a junction of two zones, will be called r_1 . In every case, the plate will be fixed in space by demanding that there shall be no axial displacement at the outer edge.

In all problems which we shall consider the surface tractions on the faces will be prescribed; hence the displacements will be completely known as soon as the arbitrary

constants in (56) have been determined. We shall find that K_2 , K_4 , C_1 depend upon conditions at the outer edge; K_1 , K_3 , C_2 are determined by conditions at the center or at the inner edge, according as the plate is complete or incomplete.

In order to center attention on the more important applications, we shall consider, besides radial mass force and axial mass force, only two classes of continuous loads, namely, normal load of type (49a) and shearing traction of type (50a). *

* In point of fact, when cases (51a), (54a), (55a) are worked out, it is found that the procedure is similar to that followed in cases (49a) and (50a).

12. Determination of K_1 , K_3 , K_4 , C_2 for a complete plate with continuous distribution of load. We shall first find K_1 . Let us turn back to the stress-equations of motion as given by formulas (4). The third of these formulas was obtained by requiring that the sum of the z-components of the external and of the internal forces acting on any particle of a body should be equal to $\rho(f_z - F_z)$, where ρ , f , and F denote the density, acceleration, and body force, respectively. Recall that in (4') we assumed that $\rho(f_z - F_z) = c_z = \text{const.}$ Since the sum of the internal forces acting on any portion of a body is zero, we infer from (4c) and (4'c) that the summation of the z-components of all the external forces acting on any portion of a body must equal c_z times its volume.

Recall that we assumed all stresses to be independent of θ , and that we took the outer normal on both the upper and lower faces to be the positive direction for normal load. Hence, for a solid plate of thickness $2h$, we obtain

$$(61a) \quad 2\pi r N_r + \int_0^r 2\pi r (L_1 - L_2) dr = 2\pi r^2 h c_z.$$

Since, by (16b) and (49a),

$$\frac{\int_0^r r (L_1 - L_2) dr}{r} = \frac{e_2 r^{2m+1}}{2(m+1)} = \ell^{(*)^{-1}}, \quad m = 0, 1, 2, \dots,$$

we can conclude, from (61a), that

$$(61b) \quad N_r = rh c_z - \ell^{(*)^{-1}}.$$

Substituting (61b) in (59e), we find

$$(61c) \quad K_r = 0.$$

Observe that we have proved (61c) to be true for radial mass force, axial mass force, and surface loads of types (49a) and (50a); it is worth noting that the proof is valid also for loads of type (51a), (54a), (55a).

Let us now find C_2 and K_2 . It is evident that, as a point approaches the axis of the plate, the direction of $\overline{\theta\theta}$ at that point approaches a radial direction. Since all stresses are independent of θ , we have

$$(62a) \quad \lim_{r \rightarrow 0} \{Fr - \overline{\theta\theta}\} = 0.$$

Observe that (62a) implies

$$(62b) \quad \lim_{r \rightarrow 0} t_{r\theta} = \lim_{r \rightarrow 0} g_{r\theta} = 0.$$

From (62b), (61c), (60b), (60d), we obtain ($r \neq 0$):

$$(62c) \quad C_2 = -\frac{1+\sigma}{4Eh} t_{r\theta} r^2 + \frac{1+\sigma}{4Eh} \left[\frac{1-\sigma}{2} h r^4 C_r + r j^{(*)-1} - r^2 j^{(*)-1} \right. \\ \left. + \frac{2E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \left\{ r \bar{U}_{op}^{(*)i} - r^2 \bar{U}_{op}^{(*)i} \right\} \frac{h^{2i+1}}{(2i+1)!} \right],$$

$$(62d) \quad K_3 = \frac{3(1+\sigma)}{4Eh^3} g_{r\theta} r^2 + \frac{3(1+\sigma)}{4Eh^3} \left[-\frac{1-\sigma}{4} h r^4 C_2 - h r J^{(*)-1} \right. \\ \left. + h r^2 J^{(*)-1} - r \ell^{(*)-1} + r^2 \ell^{(*)-1} \right. \\ \left. - \frac{4E\sigma}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \left\{ r \bar{W}_{op}^{(*)i} - r^2 \bar{W}_{op}^{(*)i} \right\} \frac{h^{2i+3}}{(2i+3)!} \right].$$

It is not difficult to show, in the four loading cases under consideration, that r^4 is the lowest power of r appearing in either of the above brackets. Hence, letting $r \rightarrow 0$ and making use of (62b), we obtain

$$(62e) \quad C_2 = K_3 = 0. \quad *$$

* For the loading cases defined by (51a), (54a), (55a), neither C_2 nor K_3 are zero.

It should be observed that, although (62a) implies (62b), the converse may not be true. If (62a) is not satisfied, our solution will not be valid in the immediate neighborhood of the axis of the plate; on the other hand, by de Saint-Venant's principle, such a solution would be valid for all points whose distance from the axis is at

least equal to the thickness of the plate.

Let us see if the four types of loading we are considering yield solutions which are valid in the neighborhood of the center. From (3a), (3b), (56), we find

$$(62f) \quad \overline{rr} - \overline{\theta\theta} = \frac{E}{1+\sigma} \left\{ U' - \frac{U}{r} \right\}.$$

If the values $K_1 = K_3 = C_2 = 0$ are substituted in (62f), the result is

$$(62g) \quad \overline{rr} - \overline{\theta\theta} = \frac{1-\sigma}{4} C_r r^2 + \frac{3(1-\sigma)}{8h^2} C_z r^2 z \\ + \frac{E}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \left[\{i\overline{V}_{op} + V_{op}\}^{(*)i} - \frac{1}{r} \{i\overline{V}_{op} + V_{op}\}^{(*)i} \right] \frac{z^{2i}}{(2i)!} \\ + \frac{E}{1+\sigma} \sum_{i=0}^{\infty} (-1)^i \left[\{(i+2-2\sigma)\overline{W}_{op} + W_{op}\}^{(*)i} - \frac{1}{r} \{(i+2-2\sigma)\overline{W}_{op} + W_{op}\}^{(*)i} \right] \frac{z^{2i+1}}{(2i+1)!}.$$

It is not difficult to show that the right-hand side of the above equation contains, in the four cases under consideration, no power of r lower than r^2 ; hence (62a) is satisfied and our solution is valid in the neighborhood of the center of the plate. *

* For the cases defined by (51a), (54a), (55a), the condition (62a) is not satisfied, and hence the solution is not valid in the neighborhood of the axis of the plate.

Let us now find K_4 . Recall that we have demanded, in every case, that

$$(63a) \quad w_0|_{r=r_0} = 0.$$

Substituting (63a), (62e), (61c) in (56b), we have

$$(63b) \quad K_4 = -K_2 r_o^2 + \frac{3(1-\sigma^2)}{64Eh^2} C_z r_o^4 - W_{op}|_{r=r_o}.$$

Putting (63b), (62e), (61c) in (56), we obtain

$$(64a) \quad U = C_1 r - 2K_2 r z + \frac{1-\sigma^2}{8E} C_r r^3 + \frac{3(1-\sigma^2)}{16Eh^2} C_z r^3 z \\ + \frac{3(1+\sigma)}{2E} C_z r z + \frac{\sigma(1+\sigma)}{2E} C_r r z^2 \\ - \frac{(2-\sigma)(1+\sigma)}{4Eh^2} C_z r z^3 + \sum_{i=0}^{\infty} (-1)^i \{i \bar{V}_{op} + V_{op}\}^{(*)i} \frac{z^{2i}}{(2i)!} \\ + \sum_{i=0}^{\infty} (-1)^i \{(i+2-2\sigma) \bar{W}_{op} + W_{op}\}^{(*)i} \frac{z^{2i+1}}{(2i+1)!},$$

$$(64b) \quad w = K_2(r^2 - r_o^2 + \frac{2\sigma}{1-\sigma} z^2) - \frac{2\sigma}{1-\sigma} C_1 z - \frac{3(1-\sigma^2)}{64Eh^2} C_z(r^4 - r_o^4) \\ - \frac{\sigma(1+\sigma)}{2E} C_r r^2 z - \frac{3\sigma(1+\sigma)}{8Eh^2} C_z r^2 z^2 \\ - \frac{(1+4\sigma)(1+\sigma)}{4(1-\sigma)E} C_z z^2 - \frac{\sigma^2(1+\sigma)}{3(1-\sigma)E} C_r z^3 \\ + \frac{(1+\sigma)^2}{8Eh^2} C_z z^4 - W_{op}|_{r=r_o} + \sum_{i=0}^{\infty} (-1)^i \{i \bar{W}_{op} + W_{op}\}^{(*)i} \frac{z^{2i}}{(2i)!} \\ - \sum_{i=0}^{\infty} (-1)^i \{(i-1+2\sigma) \bar{V}_{op} + V_{op}\}^{(*)i} \frac{z^{2i+1}}{(2i+1)!}.$$

These formulas are valid for the complete plate, loaded as above described; the arbitrary constants K_2 and C_1 are fixed by further conditions to be imposed at the outer edge.

13. Determination of K_1 , K_3 , K_4 , C_2 for a complete plate with central load. In this article we shall consider a complete plate on which the only load acting is a pressure of $-W$ pounds concentrated at the center of the upper face,

that is,

$$(65a) \quad L_2 = 0, 0 \leq r \leq r_0; \quad L_1 = 0, 0 < r \leq r_0.$$

Let us find the value of the stress L_1 at the center. Recall that the stress at any point is obtained by considering the average stress over an area which includes the point itself and allowing the area to shrink down to the point in question. Consider on the upper face of the plate a circle of radius r and center on the axis. The average stress on this circular area is $-W/(\pi r^2)$. Since W remains constant as $r \rightarrow 0$, it follows that L_1 is infinite at the center of the plate. Thus, from (16a) and (16b), we have

$$(65b) \quad L = \ell = L_1 = 0, 0 < r \leq r_0; \quad L = \ell = L_1 = -\infty, r = 0.$$

We are now in position to find K_1 . Substitute (65b) in (59e) and put $c_z = 0$; there results

$$(66a) \quad N_r = -\frac{8Eh^3}{3(1-\nu^2)} \frac{K_1}{r}, \quad 0 < r \leq r_0,$$

since $\ell^{(*)-1} = 0$ for every point except the center. Let us now consider a section of radius r cut concentrically from the plate. Note that the value of N_r is constant along the circumference of this section since all stresses are independent of θ . Since this section of the plate is in equilibrium under the concentrated load $-W$ on the upper face and the shear N_r along the circumference, we may write

$$(66b) \quad 2\pi r N_r - W = 0.$$

Substituting in (66a) the value of N_r found from (66b), we obtain

$$(66c) \quad K_1 = -\frac{3(1-\sigma^2)W}{16\pi E h^3}.$$

We may now proceed to the determination of K_3 and C_2 . Substituting (65b) in (49f), (49g'), (49h), (49i), we find

$$(67a) \quad U_{op} = \bar{U}_{op} = w_{op} = \bar{w}'_{op} = 0, \quad 0 < r \leq r_0.$$

Substitute (67a) and (66c) in (60b) and in (60d), and recall that $c_r = c_z = j = J = 0$. The resulting equations yield for C_2 and K_3 the values

$$(67b) \quad C_2 = -\frac{1+\sigma}{4Eh} t_{r0} r^2, \quad 0 < r \leq r_0;$$

$$(67c) \quad K_3 = \frac{3(1+\sigma)}{4Eh^3} q_{r0} r^2 + \frac{3(1+\sigma)W}{16\pi E h^3} \left\{ (-1+\sigma)r^2 + \frac{2(\beta+\sigma)}{5} h^2 \right\}, \quad 0 < r \leq r_0.$$

The argument which we used in article 12 to obtain equations (62a) and (62b) is also valid for the case of concentrated load. Hence, if we let $r \rightarrow 0$ in (67b) and (67c) and at the same time make use of (62b), we find

$$(67d) \quad C_2 = 0,$$

$$(67e) \quad K_3 = \frac{3(1+\sigma)(\beta+\sigma)W}{40\pi E h}.$$

* Love (p.475) gives, for the constant corresponding to K_1 , the same value as that found for K_1 in (66c). But he takes the constant corresponding to K_3 to be zero, since that is the only value of K_3 which permits w_0 to remain finite at the center of the plate. Incidentally, this value of K_3 is the only one which makes w'_0 vanish at $r = 0$.

Let us examine the axial displacement in Love's solution for central load. Substituting his value of w_0 in the general formulas given on page 473 and making the proper changes in his arbitrary constants so that they will be identical with our constants, we obtain

$$w = K_2 r^2 + K_4 + \frac{2\sigma}{1-\sigma} K_2 z^2 - \frac{3(1-\sigma^2)W}{16\pi E h^3} r^2 \log r - \frac{3\sigma(1+\sigma)W}{8\pi E h^3} (1 + \log r) z^2.$$

From the above formula for w , we have an impossible situation; in spite of a negatively infinite load at the central point of the upper surface, the displacements are positively infinite at every point of the axis of the plate except at the central point of the middle surface. Moreover, since the deflection of the middle surface is finite throughout the plate, we find that all points on the central axis which, before strain, were below the middle surface assume a position above it after strain. Evidently, Love's solution is incorrect; hence $K_3 \neq 0$ and the displacement at the center of the middle plane is not finite. De Saint-Venant in his "Note du § 45" of the translation of Clebsch, using a wholly different method, had previously made essentially the same mistake. These errors, together with others made by de Saint-Venant, have been pointed out and corrected by Garabedian in a paper published in the Journal de l'Ecole Polytechnique, 26^e cahier, 1927, p. 89.

We obtain K_4 by substituting (63a) and (67a) in (56b), at the same time setting $c_z = 0$; there results

$$(67f) \quad K_4 = -K_1 r_0^2 \log r_0 - K_2 r_0^2 - K_3 \log r_0.$$

Substituting (66c), (67d), (67e), (67f) in (56), we find the following formulas of displacement for the case of central

load:

$$(68a) \quad U = C_1 r - 2K_2 r z + \frac{3(1-\sigma^2)W}{16\pi E h^3} \left\{ 1 + 2 \log r + \frac{2(2-\sigma)h^2}{5(1-\sigma)r^2} \right\} r z \\ - \frac{(1+\sigma)(2-\sigma)W}{8\pi E h^3} \frac{z^3}{r}, \quad 0 < r \leq r_0;$$

$$(68b) \quad w = -K_2(r_0^2 - r^2) - \frac{2\sigma}{1-\sigma} C_1 z + \frac{2\sigma}{1-\sigma} K_2 z^2 \\ + \frac{3(1-\sigma^2)W}{16\pi E h^3} (r_0^2 \log r_0 - r^2 \log r) - \frac{3(1+\sigma)(8+\sigma)W}{40\pi E h} \log \frac{r_0}{r} \\ - \frac{3\sigma(1+\sigma)W}{8\pi E h^3} (1 + \log r) z^2, \quad 0 < r \leq r_0.$$

We shall eventually require T_r and G_r ; substituting (66c), (67d), (67e) in (59a) and in (59c), we have

$$(69a) \quad T_r = \frac{2Eh}{1-\sigma} C_1,$$

$$(69b) \quad G_r = -\frac{4Eh^3}{3(1-\sigma)} K_2 + \frac{W}{8\pi} \{ 3 + \sigma + 2(1+\sigma) \log r \}, \quad 0 < r \leq r_0.$$

Let us see if the solutions for central load are valid in the neighborhood of the axis. Substituting (68) in (62f), we have

$$(70) \quad \overline{rr} - \overline{\theta\theta} = \frac{3(1-\sigma)W}{8\pi h^3} - \frac{(2-\sigma)W}{4\pi h} \left\{ \frac{3}{5} - \frac{z^2}{h^2} \right\} \frac{z}{r^2}, \quad 0 < r \leq r_0.$$

Observe that $\lim_{r \rightarrow 0} \{\overline{rr} - \overline{\theta\theta}\} \neq 0$ for any value of z ; in fact, it is either $+\infty$ or $-\infty$ for all z 's except $z = 0$ and $z = +\sqrt{(3/5)} h$. Hence our solution will not be valid in the neighborhood of the axis of the plate.

We can justify the above conclusion in another manner. Recalling that $L_r = -\infty$ at $r = 0$, we can conclude

that $\overline{zz}$ is infinite at the center of the upper face and hence the elastic limit is exceeded at this point. But the stress-equations of motion were derived on the assumption that the stress remains within the elastic limit; hence the stress-equations of motion themselves are not valid at the center of the plate. On the other hand, by de Saint-Venant's principle, the actual stress distribution will be closely approximated by our solution at any point whose distance from the axis exceeds the thickness of the plate.

14. Pure stretching of a complete or incomplete plate with continuous distribution of load. We define pure stretching as that state of strain in which the middle plane is not bent and the deformed plate itself is symmetrical in the middle plane. We find it convenient to have a criterion for pure stretching in terms of surface tractions and mass forces alone, since they generally constitute the data in any given problem. We shall show that, given only surface tractions of types (49a) or (50a) and radial or axial mass forces, a sufficient condition for pure stretching is, for the complete plate,

$$(71a) \quad G_r|_{r=r_0} = c_z = \mathcal{L} \equiv J \equiv 0,$$

and, for the incomplete plate,

$$(71b) \quad G_r|_{r=r_0} = G_r|_{r=r_1} = N_r|_{r=r_1} = c_z = \mathcal{L} \equiv J \equiv 0.$$

Recall that, for a complete plate subjected to these four types of loading, we have already proved $K_1 = K_3 = 0$; hence, by (59c), (71a) implies $K_2 = 0$. In the case of an incomplete plate, it is readily shown, by (59c) and (59e), that (71b) is a sufficient condition for $K_1 = K_2 = K_3 = 0$ whatever the type of loading. Hence, for both the complete and incomplete plate subjected to the above four types of loading, (71) is a sufficient condition that

$$(72) \quad K_1 = K_2 = K_3 = c_z = \mathcal{L} \equiv J \equiv 0.$$

From (56), we find that (72), whether the plate is complete or incomplete, implies that the displacements should have the following form:

$$(73a) \quad U = C_1 r + \frac{C_2}{r} + \frac{1-\sigma^2}{8E} C_r r^3 + \frac{\sigma(1+\sigma)}{2E} C_r r z^2 + \sum_{i=0}^{\infty} (-1)^i \left\{ i \bar{U}_{op} + U_{op} \right\}^{(*)} \frac{z^{2i}}{(2i)!},$$

$$(73b) \quad w = K_4 - \frac{2\sigma}{1-\sigma} C_1 z - \frac{\sigma(1+\sigma)}{2E} C_r r^2 z - \frac{\sigma^2(1+\sigma)}{3(1-\sigma)E} C_r z^3 - \sum_{i=0}^{\infty} (-1)^i \left\{ (i-1+2\sigma) \bar{U}_{op} + U_{op} \right\}^{*(1*)} \frac{z^{2i+1}}{(2i+1)!}.$$

From (73b), we see that $w_o = K_4$, $U(r, z) = U(r, -z)$, $w(r, z) = -w(r, -z)$. Obviously, these conditions are sufficient for pure stretching as defined above.

We are now in position to solve some problems in pure stretching. In all such problems, we shall assume

that the middle plane is not displaced axially; that is, $w(r,0) = 0$. Applying this condition to (73b), we find, for both the complete and incomplete plate, that

$$(74) \quad K_4 = 0.$$

Furthermore, in dealing with a complete plate under the above four types of loading, it should be remembered that we have already proved $C_2 = 0$. In each of the following problems, it is assumed that the distribution of load is such that either (71a) or (71b) is satisfied.

Problem I :- Complete plate whose outer edge is free to expand, that is, $T_r|_{r=r_0} = 0$. From (59a), we find

$$(75a) \quad C_1 = -\frac{(3+\sigma)(1-\sigma)}{8E} r_0^2 C_r - \frac{\sigma(1+\sigma)h^2}{6E} C_r \\ + \left[\frac{1-\sigma}{2Eh} j^{(*)-1} - \frac{\sigma(1-\sigma)}{1+\sigma} \frac{1}{r} \sum_{i=0}^{\infty} (-1)^i \bar{U}_{op}^{(*)i} \frac{h^{2i}}{(2i+1)!} \right]_{r=r_0}.$$

Substituting (75a) in (73), we have for the displacements

$$(75b) \quad U = -\frac{1-\sigma}{8E} C_r r \{ (3+\sigma)r_0^2 - (1+\sigma)r^2 \} - \frac{\sigma(1+\sigma)}{6E} C_r r (h^2 - 3z^2) \\ + r \left[\frac{1-\sigma}{2Eh} j^{(*)-1} - \frac{\sigma(1-\sigma)}{1+\sigma} \frac{1}{r} \sum_{i=0}^{\infty} (-1)^i \bar{U}_{op}^{(*)i} \frac{h^{2i}}{(2i+1)!} \right]_{r=r_0} \\ + \sum_{i=0}^{\infty} (-1)^i \{ i \bar{U}_{op} + U_{op} \}^{(*)i} \frac{z^{2i}}{(2i)!},$$

$$(75b') \quad w = \frac{\sigma}{4E} C_r z \{ (3+\sigma)r_0^2 - 2(1+\sigma)r^2 \} + \frac{\sigma^2(1+\sigma)}{3(1-\sigma)E} C_r z (h^2 - z^2) \\ + z \left[\frac{2\sigma^2}{1+\sigma} \frac{1}{r} \sum_{i=0}^{\infty} (-1)^i \bar{U}_{op}^{(*)i} \frac{h^{2i}}{(2i+1)!} - \frac{\sigma}{Eh} j^{(*)-1} \right]_{r=r_0} \\ - \sum_{i=0}^{\infty} (-1)^i \{ (i-1+2\sigma) \bar{U}_{op} + U_{op} \}^{(*)i} \frac{z^{2i+1}}{(2i+1)!}.$$

Case Ia :- If in (75b) and (75b') we set $\bar{U}_{op} \equiv j \equiv 0$, we have the solution for radial mass force only. If we assume that the radial mass force is due to the normal acceleration caused by the rotation of the plate about the z axis with an angular velocity of ω rad./sec., we find from (4'a) that $c_r r = \rho f_r = -\rho \omega^2 r$; hence $c_r = -\rho \omega^2$. If we use this value of c_r , we obtain the well-known formulas for a rotating plate (Love, p. 148). It is readily shown that $\widehat{rr}$ does not vanish at the edge for all z; hence our solution is not valid for points too close to the edge.

Case Ib :- Consider a plate with no loads other than equal constant tensions on both faces; that is, $L_1 = L_2 = p$, $L = 2p$, $\ell = 0$. From (49f) and (49g'),

$$U_{op} = -\frac{\sigma(1+\sigma)}{2E} pr, \quad \bar{U}_{op} = \frac{1+\sigma}{2E} pr;$$

and from (75b) and (75b'),

$$U = -\frac{\sigma}{E} pr, \quad w = \frac{pz}{E}.$$

This is the well-known formula for a cylinder under axial tension. We find that $\widehat{rr}$ vanishes at the edge for all z, and hence our solution is valid throughout the plate. In this particular case, the ratio of h to r_0 need not be small.

Case Ic :- To show the power of our method, we shall now consider a plate on which the only loads are equal tensions which are proportional to r^2 ; that is, $L_1 = L_2 = pr^2$, $L = 2pr^2$, $\ell = 0$. We find

$$U_{op} = -\frac{\sigma(1+\sigma)}{4E} pr^3 + \frac{1-\sigma^2}{3E} h^2 pr,$$

$$\bar{U}_{op} = \frac{1+\sigma}{4E} pr^3 + \frac{1+\sigma}{3E} h^2 pr,$$

$$U = -\frac{\sigma}{4E} pr \{ (1-\sigma)r_o^2 + (1+\sigma)r^2 \} + \frac{1-\sigma^2}{3E} pr (h^2 - 3z^2),$$

$$w = \frac{p}{2E} z \{ \sigma^2 r_o^2 + 2(1-\sigma^2)r^2 \} - \frac{2\sigma(1+\sigma)}{3E} pz (h^2 - z^2).$$

In this problem, $\widehat{rr}$ does not vanish at the edge for all z ; consequently, the solution is not good in the neighborhood of the edge.

Case Id :- We now consider a plate on which the only loads are equal outward shearing tractions proportional to r ; that is, $J_1 = -J_2 = pr$, $J = 0$, $j = 2pr$. The results are

$$U_{op} = -\frac{1-\sigma^2}{8Eh} pr^3 - \frac{(2+\sigma)(1+\sigma)}{6E} hpr,$$

$$\bar{U}_{op} = -\frac{1+\sigma}{8Eh} pr^3 + \frac{1+\sigma}{6E} hpr,$$

$$U = \frac{1-\sigma}{8Eh} pr \{ (3+\sigma)r_o^2 - (1+\sigma)r^2 \} - \frac{2+5\sigma-\sigma^2}{6E} hpr \\ + \frac{(2-\sigma)(1+\sigma)}{2Eh} pr z^2,$$

$$w = -\frac{\sigma p}{4Eh} z \{ (3+\sigma)r_o^2 - 2(1+\sigma)r^2 \} + \frac{3+2\sigma+\sigma^2}{3E} hpz \\ - \frac{(1+\sigma)^2}{3Eh} pz^3.$$

Again we find that our solution is not good near the edge, since $\widehat{rr}$ does not vanish there for all z .

Problem II :- Incomplete plate whose outer and inner edges are free to expand, that is, $T_r|_{r=r_o} = T_r|_{r=r_i} = 0$.

Here we could work out four cases corresponding to the four given under Problem I; it will suffice to consider the first case only. From (59a), we have

$$\frac{2E}{1-\sigma} C_1 - \frac{2E}{1+\sigma} \frac{C_2}{r_0^2} + \frac{3+\sigma}{4} r_0^2 C_r + \frac{\sigma(1+\sigma)}{3(1-\sigma)} h^2 C_r = 0,$$

$$\frac{2E}{1-\sigma} C_1 - \frac{2E}{1+\sigma} \frac{C_2}{r_1^2} + \frac{3+\sigma}{4} r_1^2 C_r + \frac{\sigma(1+\sigma)}{3(1-\sigma)} h^2 C_r = 0.$$

Solving these two equations simultaneously, we obtain

$$C_1 = - \frac{(3+\sigma)(1-\sigma)}{8E} (r_0^2 + r_1^2) C_r - \frac{\sigma(1+\sigma)}{6E} h^2 C_r,$$

$$C_2 = - \frac{(3+\sigma)(1+\sigma)}{8E} r_0^2 r_1^2 C_r.$$

Substituting these values of C_1 and C_2 in (73), there results

$$\begin{aligned} U &= - \frac{1-\sigma}{8E} C_r r \{ (3+\sigma)(r_0^2 + r_1^2) - (1+\sigma)r^2 \} \\ &\quad - \frac{(3+\sigma)(1+\sigma)}{8E} \frac{r_0^2 r_1^2}{r} C_r - \frac{\sigma(1+\sigma)}{6E} C_r r (h^2 - 3z^2), \\ W &= \frac{\sigma}{4E} C_r z \{ (3+\sigma)(r_0^2 + r_1^2) - 2(1+\sigma)r^2 \} \\ &\quad + \frac{\sigma^2(1+\sigma)}{3(1-\sigma)E} C_r z (h^2 - z^2). \end{aligned}$$

If we put $c_r = -\rho\omega^2$, we obtain the formulas given by Love (p. 148).

Problem III :- Consider a complete plate with no load other than equal constant pressures on the two faces; that is, $L_1 = L_2 = -p$, $L = -2p$, $\ell = 0$. We have

$$U_{op} = \frac{\sigma(1+\sigma)}{2E} pr,$$

$$\bar{U}_{op} = -\frac{1+\sigma}{2E} pr.$$

Hence, from (73) and (3a), we find

$$(76a) \quad U = C_1 r + \frac{\sigma(1+\sigma)}{2E} pr,$$

$$(76b) \quad w = -\frac{2\sigma}{1-\sigma} C_1 z - \frac{1-\sigma^2}{E} pz,$$

$$(76c) \quad \bar{r}r = \frac{E}{1-\sigma} C_1 - \frac{\sigma}{2} p.$$

Case IIIa :- The radius of the outer edge is to remain unchanged for all z ; that is, $U|_{r=r_0} = 0$. We find

$$C_1 = -\frac{\sigma(1+\sigma)}{2E} p,$$

$$U \equiv 0, \quad w = -\frac{(1-2\sigma)(1+\sigma)}{(1-\sigma)E} pz,$$

$$\bar{r}r = -\frac{\sigma}{1-\sigma} p.$$

Thus a pressure of $-\sigma p/(1-\sigma)$ must be applied at the outer edge if this edge is to remain unchanged.

Case IIIb :- The thickness of the plate is to remain unchanged; that is, $w|_{z=\pm h} = 0$. We find

$$C_1 = -\frac{(1-\sigma)^2(1+\sigma)}{2E\sigma} p,$$

$$U = - \frac{(1-2\sigma)(1+\sigma)}{2E\sigma} p r \quad , \quad w \equiv 0,$$

$$\overline{rr} = - \frac{p}{2\sigma}.$$

Hence a pressure of $-p/(2\sigma)$ is required at the outer edge if the thickness of the plate is not to be altered. The formulas in IIIa and IIIb are valid throughout the plate; hence there is no limitation on the ratio of thickness to diameter. A similar remark applies to the problem which follows.

Problem IV :- Consider an incomplete plate with no loads other than constant pressures, $-p_o$ and $-p_i$, at the outer and inner edges, respectively. We have $L_1 \equiv L_2 \equiv L_r \equiv \ell \equiv 0$. Hence, from (73) and (3a), we find

$$U = C_1 r + \frac{C_2}{r} \quad , \quad w = - \frac{2\sigma}{1-\sigma} C_1 z \quad ,$$

$$\overline{rr} = \frac{E}{1-\sigma} C_1 - \frac{E}{1+\sigma} \frac{C_2}{r^2}.$$

Since $\overline{rr}|_{r=r_o} = -p_o$ and $\overline{rr}|_{r=r_i} = -p_i$, we obtain

$$C_1 = - \frac{1-\sigma}{E} \frac{p_o r_o^2 - p_i r_i^2}{r_o^2 - r_i^2} \quad ,$$

$$C_2 = - \frac{1+\sigma}{E} \frac{(p_o - p_i) r_o^2 r_i^2}{r_o^2 - r_i^2}.$$

Most of the solutions just obtained are well known; only cases Ic and Id seem to be given for the first time. Our object in obtaining anew the known solutions has been to exhibit the

power and elegance of a method which brings all these solutions, and many others, together under one uniform method of treatment.

15. The bending of a complete plate. We find in the literature only a limited number of problems dealing with the bending of moderately thick circular plates. The plate bent by its own weight has been solved; the problem of the plate loaded uniformly over one face has been solved for several types of edge conditions; a plate whose faces are subjected to shearing tractions seems not to have been considered. Also, certain problems involving central load have been solved, but, as we shall see, some of these have been in error. Since it will be interesting to check our results, so far as possible, with the work of others, we propose to consider in this article those problems involving axial mass force, uniform load, and central load, which have already been solved by methods which yield solutions of the same form as ours.

We find it convenient to write out separate formulas for each of these three cases. For the case of central load, the formulas defining U , w , T_r , G_r are given by (68) and (69). If we put $c_r = U_{op} = \bar{U}_{op} = w_{op} = \bar{w}'_{op} = 0$ in (64), we obtain the displacements for the case of axial mass force; the corresponding formulas for T_r and G_r are found from (59a) and (59c). The results for axial mass force are

$$(77a) \quad U = C_1 r - 2K_2 rz + \frac{3(1-\sigma^2)C_2}{16Eh^2} \left\{ r^2 + \frac{8h^2}{1-\sigma} \right\} rz - \frac{(1+\sigma)(2-\sigma)}{4Eh^2} C_2 r z^3,$$

$$(77b) \quad w = -K_2(r_0^2 - r^2) - \frac{2\sigma}{1-\sigma} C_1 z + \frac{2\sigma}{1-\sigma} K_2 z^2 + \frac{3(1-\sigma^2)}{64Eh^2} (r_0^4 - r^4) C_2 - \frac{3\sigma(1+\sigma)C_2}{8Eh^2} \left\{ r^2 + \frac{2(1+4\sigma)h^2}{3\sigma(1-\sigma)} \right\} z^2 + \frac{(1+\sigma)^2}{8Eh^2} C_2 z^4,$$

$$(77c) \quad T_r = \frac{2Eh}{1-\sigma} C_1,$$

$$(77d) \quad G_r = -\frac{4Eh^3}{3(1-\sigma)} K_2 + \left\{ \frac{3+\sigma}{8} \frac{r^2}{h^2} + \frac{24+23\sigma+3\sigma^2}{30(1-\sigma)} h^3 C_2 \right\}.$$

At this point, it will be interesting to check formulas (77a) and (77b) against the displacements given by Prof. G. H. Bryan for a plate bent by its own weight (Love, p. 486). If, in (77), we set $c_z = \rho g$, we have the solution for a plate bent by its own weight. If, in addition, we set $C_1 = \rho g(1-\sigma)h/(2E)$ and $K_2 = \rho g(3+7\sigma)/(8E)$, we obtain the results given by G. H. Bryan except for a constant term in w , accounted for by an axial translation of the plate as a whole.

For the case of uniform pressure on the upper face, we have

$$(78a) \quad L_2 = 0, L_1 = L = \ell = -p;$$

$$(78b) \quad U_{op} = \frac{\sigma(1+\sigma)}{4E} p r,$$

$$(78c) \quad \bar{U}_{op} = -\frac{(1+\sigma)}{4E} p r,$$

$$(78d) \quad w_{op} = -\frac{3(1+\sigma)}{16Eh^3} p \left\{ \frac{(1-\sigma)r^4}{8} - \frac{(8-3\sigma)h^2 r^2}{5} \right\},$$

$$(78e) \quad \bar{w}'_{op} = \frac{3(1+\sigma)}{8Eh^3} p \left\{ \frac{r^3}{4} - \frac{3}{5} h^2 r \right\}.$$

Substituting (78) in (64), (59a), (59c), we obtain

$$(79a) \quad U = C_1 r - 2K_2 r z + \frac{\sigma(1+\sigma)}{4E} p r \\ + \frac{3(1-\sigma^2)}{32Eh^3} p \left\{ r^2 + \frac{4(2+3\sigma)h^2}{5(1-\sigma)} \right\} r z - \frac{(1+\sigma)(2-\sigma)}{8Eh^3} p r z^3,$$

$$(79b) \quad w = -K_2(r_0^2 - r^2) - \frac{2\sigma}{1-\sigma} C_1 z + \frac{2\sigma}{1-\sigma} K_2 z^2 \\ + \frac{3(1-\sigma^2)}{128Eh^3} p \left\{ r_0^2 + r^2 - \frac{8(2-3\sigma)h^2}{5(1-\sigma)} \right\} (r_0^2 - r^2) - \frac{1-\sigma^2}{2E} p z \\ - \frac{3\sigma(1+\sigma)}{16Eh^3} p \left\{ r^2 + \frac{2(5-3\sigma)h^2}{5\sigma} \right\} z^2 + \frac{(1+\sigma)^2}{16Eh^3} p z^4,$$

$$(79c) \quad T_r = \frac{2Eh}{1-\sigma} C_1 - \frac{\sigma p h}{2},$$

$$(79d) \quad G_r = -\frac{4Eh^3}{3(1-\sigma)} K_2 + \frac{3+\sigma}{16} p r^2 - \frac{\sigma p h^2}{5}.$$

When the constants C_1 and K_2 have been fixed by the conditions at the outer edge, the displacements will be completely determined. We shall consider nine types of conditions at the outer edge; these we shall order according to the magnitude of the deflection at the center of the middle plane of the plate. Since, for the types of loads we shall consider, the deflection at the center is in the direction of the negative z -axis, the formulas for w show us that the central deflection of the middle plane increases as K_2 increases. Since the thickness of our plate is small compared to the outer radius, the term in K_2 having the smallest order of magnitude in h/r_0 will be used as the basis for comparing the relative magnitude of K_2 .

In each case, we shall give, in order, the values of

the constants for axial mass force, central load, and normal load, respectively. We shall find, in each of the nine cases, that $C_1 = 0$ for both axial mass force and central load; when the load is uniform, it turns out that there is a value of C_1 associated with each of three groups to be described below. The nine cases follow.

Case S. This is the case of classical support. The outer edge is free to bend and free to move radially; that is,

$$G_r|_{r=r_0} = T_r|_{r=r_0} = 0.$$

We find that

$$(80a) \quad K_2 = \frac{3(1-\sigma^2)}{32Eh^2} C_2 \left\{ \frac{3+\sigma}{1+\sigma} r_0^2 + \frac{4(24+23\sigma+3\sigma^2)}{15(1-\sigma^2)} h^2 \right\}, \quad C_1 = 0;$$

$$(80b) \quad K_2 = \frac{3(1-\sigma^2)W}{32\pi Eh^3} \left\{ \frac{3+\sigma}{1+\sigma} + 2 \log r_0 \right\}, \quad C_1 = 0;$$

$$(80c) \quad K_2 = \frac{3(1-\sigma^2)p}{64Eh^3} \left\{ \frac{3+\sigma}{1+\sigma} r_0^2 - \frac{16\sigma h^2}{5(1+\sigma)} \right\}, \quad C_1 = \frac{\sigma(1-\sigma)p}{4E}.$$

For the case of axial mass force, the displacements are

$$U = - \frac{3(1-\sigma)}{16Eh^2} \left\{ (3+\sigma)r_0^2 - (1+\sigma)r^2 - \frac{4(6+7\sigma-3\sigma^2)}{15(1-\sigma)} h^2 \right\} C_2 r Z \\ - \frac{(1+\sigma)(2-\sigma)}{4Eh^2} C_2 r Z^3,$$

$$W = - \frac{3(1-\sigma)}{64Eh^2} \left\{ (5+\sigma)r_0^2 - (1+\sigma)r^2 + \frac{8(24+23\sigma+3\sigma^2)}{15(1-\sigma)} h^2 \right\} C_2 (r_0^2 - r^2) \\ + \frac{3\sigma}{16Eh^2} \left\{ (3+\sigma)r_0^2 - 2(1+\sigma)r^2 - \frac{4(5+6\sigma+3\sigma^2)}{15\sigma} h^2 \right\} C_2 Z^2 \\ + \frac{(1+\sigma)^2}{8Eh^2} C_2 Z^4.$$

If we set $c_z = \rho g$, we obtain the solution for a plate bent by its own weight. This result is apparently new; but the same result could be obtained from G. H. Bryan's formulas (loc. cit.) by compounding other solutions so chosen as to make $T_r|_{r=r_0} = G_r|_{r=r_0} = 0$.

For central load, the displacements are

$$\begin{aligned}
 U &= -\frac{3(1-\sigma^2)W}{8\pi E h^3} \left\{ \frac{1}{1+\sigma} + \log \frac{r_0}{r} - \frac{2-\sigma}{5(1-\sigma)} \frac{h^2}{r^2} \right\} r z \\
 &\quad - \frac{(1+\sigma)(2-\sigma)W}{8\pi E h^3} \frac{z^3}{r}, \\
 w &= -\frac{3(1-\sigma^2)W}{16\pi E h^3} \left[\frac{3+\sigma}{2(1+\sigma)} (r_0^2 - r^2) - \left\{ r^2 - \frac{2(8+\sigma)}{5(1-\sigma)} h^2 \right\} \log \frac{r_0}{r} \right] \\
 &\quad + \frac{3\sigma(1+\sigma)W}{8\pi E h^3} \left\{ \frac{1-\sigma}{2(1+\sigma)} + \log \frac{r_0}{r} \right\} z^2.
 \end{aligned}$$

This solution was first given correctly by Garabedian. *

* Comptes rendues, vol. 186, 1928, p. 1518.

For the case of uniform load, we have #

Cf Love, p. 481.

$$\begin{aligned}
 U &= \frac{\sigma p r}{2E} - \frac{3(1-\sigma)}{32E h^3} \left\{ (3+\sigma)r_0^2 - (1+\sigma)r^2 - \frac{4(2+9\sigma-\sigma^2)}{5(1-\sigma)} h^2 \right\} p r z \\
 &\quad - \frac{(1+\sigma)(2-\sigma)}{8E h^3} p r z^3, \\
 w &= -\frac{3(1-\sigma)}{128E h^3} \left\{ (5+\sigma)r_0^2 - (1+\sigma)r^2 + \frac{8(8+\sigma+\sigma^2)}{5(1-\sigma)} h^2 \right\} p (r_0^2 - r^2) \\
 &\quad - \frac{p z}{2E} + \frac{3\sigma}{32E h^3} \left\{ (3+\sigma)r_0^2 - 2(1+\sigma)r^2 - \frac{4(5+2\sigma+\sigma^2)}{5\sigma} h^2 \right\} p z^2 \\
 &\quad + \frac{(1+\sigma)^2}{16E h^3} p z^4.
 \end{aligned}$$

In the eight remaining cases, we shall not attempt to write the complete formulas for U and w ; we shall be content to give the values of K_2 and C_1 . These eight cases fall into two groups. Group C comprises, besides the case of classical clamping, three closely allied cases. The four remaining cases, referred to as Group M, are characterized by the fact that they yield central deflections intermediate between those of Case S and Group C. We shall consider first the cases under Group M.

Case M-I. In this case, we attempt to make w vanish at the edge for all z . We can accomplish this if the load is concentrated at the center; but if the load is uniform, or if the plate is subjected to radial mass force, we can make all powers of z vanish except the last, a term in z^4 . The constant C_1 is readily found from the condition that the coefficient of the term in z shall vanish. To ask that the term in z^2 vanish is coextensive with demanding that

$$\left. \frac{\partial^2 w_0}{\partial z^2} \right|_{r=r_0} = 0.$$

We find

$$(81a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{16Eh^2} \left\{ r_0^2 + \frac{2(1+4\sigma)}{3\sigma(1-\sigma)} h^2 \right\}, \quad C_1 = 0;$$

$$(81b) \quad K_2 = \frac{3(1-\sigma^2)W}{16\pi Eh^3} (1 + \log r_0), \quad C_1 = 0;$$

$$(81c) \quad K_2 = \frac{3(1-\sigma^2)P}{32Eh^3} \left\{ r_0^2 + \frac{2(5-3\sigma)}{5\sigma} h^2 \right\}, \quad C_1 = -\frac{(1-\sigma)(1-\sigma^2)P}{4E\sigma}.$$

Case M-II. We demand that the upper and lower faces shall not be displaced axially at the outer edge; that is,

$$w \Big|_{\substack{r=r_0 \\ z=\pm h}} = 0.$$

We obtain

$$(82a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{16Eh^2} \left\{ r_0^2 + \frac{1+8\sigma+\sigma^2}{3\sigma(1-\sigma)} h^2 \right\}, \quad C_1 = 0;$$

$$(82b) \quad K_2 = \frac{3(1-\sigma^2)W}{16\pi E h^3} (1 + \log r_0), \quad C_1 = 0;$$

$$(82c) \quad K_2 = \frac{3(1-\sigma^2)p}{32Eh^3} \left\{ r_0^2 + \frac{25-23\sigma}{15\sigma} h^2 \right\}; \quad C_1 = -\frac{(1-\sigma)(1-\sigma^2)}{4E\sigma} p.$$

Case M-III. In this case, the axial displacement at the edge is to be a maximum at $z = \pm h$; that is,

$$\frac{\partial w}{\partial z} \Big|_{\substack{r=r_0 \\ z=\pm h}} = 0.$$

We find

$$(83a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{16Eh^2} \left\{ r_0^2 + \frac{2(4+\sigma)}{3(1-\sigma)} h^2 \right\}, \quad C_1 = 0;$$

$$(83b) \quad K_2 = \frac{3(1-\sigma^2)W}{16\pi E h^3} (1 + \log r_0), \quad C_1 = 0;$$

$$(83c) \quad K_2 = \frac{3(1-\sigma^2)p}{32Eh^3} \left\{ r_0^2 + \frac{4(5-7\sigma)}{15\sigma} h^2 \right\}, \quad C_1 = -\frac{(1-\sigma)(1-\sigma^2)}{4E\sigma} p.$$

Case M-IV. There is no simple geometrical interpretation for this case; the conditions at the edge are closely related to those of Case M-I. We take the same value for C_1 and require that

$$\left. \frac{\partial^2 W}{\partial z^2} \right|_{\substack{r=r_0 \\ z=\pm h}} = 0.$$

The constants are found to be

$$(84a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{16Eh^2} \left\{ r_0^2 - \frac{2(2-4\sigma-3\sigma^2)}{3\sigma(1-\sigma)} h^2 \right\}, \quad C_1 = 0;$$

$$(84b) \quad K_2 = \frac{3(1-\sigma^2)W}{16\pi Eh^3} (1 + \log r_0), \quad C_1 = 0;$$

$$(84c) \quad K_2 = \frac{3(1-\sigma^2)p}{32Eh^3} \left(r_0^2 - \frac{16}{5} h^2 \right), \quad C_1 = -\frac{(1-\sigma)(1-\sigma^2)}{4E\sigma} p.$$

Case C-I. In this case, we attempt to make U vanish at the edge for all z ; we find that we can make all powers of z vanish except the term in z^3 . Requiring that the terms of the zeroth and first powers of z should vanish is the same as demanding that

$$U_0|_{r=r_0} = \frac{\partial U_0}{\partial z} \Big|_{r=r_0} = 0.$$

The above conditions have obvious geometric interpretations. The constants are

$$(85a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{32Eh^2} \left(r_0^2 + \frac{8h^2}{1-\sigma} \right), \quad C_1 = 0;$$

$$(85b) \quad K_2 = \frac{3(1-\sigma^2)W}{32\pi Eh^3} \left\{ 1 + 2 \log r_0 + \frac{2(2-\sigma)}{5(1-\sigma)} \frac{h^2}{r_0^2} \right\}, \quad C_1 = 0;$$

$$(85c) \quad K_2 = \frac{3(1-\sigma^2)p}{64Eh^3} \left\{ r_0^2 + \frac{4(2+3\sigma)}{5(1-\sigma)} h^2 \right\}, \quad C_1 = -\frac{\sigma(1+\sigma)}{4E} p.$$

Case C-II. The requirement in this case is that the upper and lower surfaces shall not be stretched; that is,

$$U \Big|_{\substack{r=r_0 \\ z=\pm h}} = 0.$$

The results are

$$(86a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{32Eh^2} \left\{ r_0^2 + \frac{4(4+\sigma)}{3(1-\sigma)} h^2 \right\}, \quad C_1 = 0;$$

$$(86b) \quad K_2 = \frac{3(1-\sigma^2)W}{32\pi Eh^3} \left\{ 1 + 2 \log r_0 - \frac{4(2-\sigma)}{15(1-\sigma)} \frac{h^2}{r_0^2} \right\}, \quad C_1 = 0;$$

$$(86c) \quad K_2 = \frac{3(1-\sigma^2)p}{64Eh^3} \left\{ r_0^2 - \frac{8(2-7\sigma)}{15(1-\sigma)} h^2 \right\}, \quad C_1 = -\frac{\sigma(1+\sigma)}{4E} p.$$

Case C-III. We demand that the middle surface shall not be stretched, and that both faces shall have, after strain, a horizontal tangent at the edge; that is,

$$U_0|_{r=r_0} = w'|_{r=r_0, z=\pm h} = 0. *$$

* For all problems in which there is no shearing traction on either face, the condition $w'|_{r=r_0, z=\pm h} = 0$ is coextensive with $\frac{\partial v}{\partial z}|_{r=r_0, z=\pm h} = 0$, since $\overline{rr} = (\frac{\partial v}{\partial z} + w')E/\{2(1+\sigma)\}$.

We obtain

$$(87a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{32Eh^2} \left\{ r_0^2 + \frac{4\sigma}{1-\sigma} h^2 \right\}, \quad C_1 = 0;$$

$$(87b) \quad K_2 = \frac{3(1-\sigma^2)W}{32\pi Eh^3} \left\{ 1 + 2 \log r_0 - \frac{8(2-\sigma)}{5(1-\sigma)} \frac{h^2}{r_0^2} \right\}, \quad C_1 = 0;$$

$$(87c) \quad K_2 = \frac{3(1-\sigma^2)p}{64Eh^3} \left(r_0^2 - \frac{32}{5} h^2 \right), \quad C_1 = -\frac{\sigma(1+\sigma)p}{4E}.$$

Case C-IV. This is the case of classical clamping; the middle surface is unstretched and remains horizontal at the outer edge; that is,

$$U_0|_{r=r_0} = w'|_{r=r_0} = 0.$$

$$(88a) \quad K_2 = \frac{3(1-\sigma^2)C_2}{32Eh^2} r_0^2, \quad C_1 = 0;$$

$$(88b) \quad K_2 = \frac{3(1-\sigma^2)W}{32\pi Eh^3} \left\{ 1 + 2 \log r_0 - \frac{2(8+\sigma)}{5(1-\sigma)} \frac{h^2}{r_0^2} \right\}, \quad C_1 = 0;$$

$$(88c) \quad K_2 = \frac{3(1-\sigma^2)p}{64Eh^3} \left\{ r_0^2 - \frac{4(8-3\sigma)}{5(1-\sigma)} h^2 \right\}, \quad C_1 = -\frac{\sigma(1+\sigma)}{4E} p.$$

Cases S and C-IV are classical. The thin plate solutions for Cases M-I, C-I, C-II have been given in a paper by C. A. Clemmow; *

* Proc. Royal Soc., London, series A, vol. 112, 1929, pp. 559-598.

the four remaining cases appear to be new. Cases M-I and C-I are of especial interest. The former gives the closest approximation which it is possible to obtain with a solution involving polynomials only to the edge condition in which w vanishes for all z , the latter gives the closest approximation to the edge condition in which U vanishes for all z . These two edge conditions are important since they are the only boundary conditions possible when Bessel functions alone are used. #

Nádaí, Schweizerische Bauzeitung, vol. 76, Nr. 22, Dec. 4, 1920, pp. 257-260.

It is interesting to note that, in the investigation made by Clemmow upon clamped plates, the best agreement with experiment was given by Case C-I. The circular plate tested was an integral part of a cylinder, the cylinder and

head having been cut out from a solid piece of metal. It was expected that, with such a plate, the clamping obtained would be as rigid as that of classical clamping. This expectation was not realized, since the experimental results were not very close to those calculated from Case C-IV even when the ratio of thickness to diameter was small. It thus appears that it is virtually impossible to construct a physical type of clamping which will agree with a set of analytical conditions previously advanced. Hence it becomes necessary to devise new analytical conditions which will approximate the physical situations arising in practice. Herein lies a justification for considering Cases C-I, C-II, C-III. It should be borne in mind, however, that only in exceptional cases is the head of a cylinder an integral part of the cylinder itself. Ordinarily, the head is fastened to the cylinder by means of bolts; and this type of fastening is certainly less rigid than that used by Clemmow. It would appear desirable, therefore, to study also the additional cases, M-I, M-II, M-III, M-IV, intermediate between S and C-I.

16. The bending of an incomplete plate. We shall consider only two problems in this article. In each problem, we shall assume that (i) there is no axial displacement of the middle surface at the outer edge, (ii) the inner edge is free, (iii) the mass force is nil and there is no shearing traction on the faces; that is,

$$(89a) \quad w_o|_{r=r_o} = 0,$$

$$(89b) \quad T_r|_{r=r_i} = G_r|_{r=r_i} = 0,$$

$$(89c) \quad c_r = c_z = J_1 = J_2 = 0.$$

Substitute (89a) and (89c) in (56b) and solve for K_4 . We obtain, for both problems,

$$(90) \quad K_4 = -K_1 r_o^2 \log r_o - K_2 r_o^2 - K_3 \log r_o - w_{op}|_{r=r_o}.$$

The remaining constants depend upon the problem under consideration.

Problem I. In this problem, the only load is a uniform pressure, $-p$, on the upper face; that is,

$$(91) \quad N_r|_{r=r_i} = 0, L_2 = 0, L_1 = L = \ell = -p.$$

Let us first find K_1 . Substituting $\ell = -p$ in (59e), we obtain

$$(92a) \quad N_r = -\frac{BEh^3}{3(1-\sigma^2)} \frac{K_1}{r} + \frac{pr}{2}.$$

Of the annular rings cut out by a concentric cylindrical surface of radius r , consider the inner section. Since this ring is held in equilibrium by the downward pressure $-p$ on the upper face and the upward shear N_r on the outer edge, we have

$$(92b) \quad 2\pi r N_r - \pi(r^2 - r_i^2)p = 0.$$

Substituting in (92a) the value of N_r just found, we obtain

$$(92c) \quad K_1 = \frac{3(1-\sigma^2)}{16Eh^3} p r_1^2.$$

We shall find it convenient to express C_2 and K_3 in terms of C_1 and K_2 , respectively. To accomplish this, we first compute T_r and G_r . Since L and ℓ in (91) are the same as in (78a), the values of U_{op} , $\bar{U}_{op}$, w_{op} , $\bar{w}'_{op}$ may be found from (78b), (78c), (78d), (78e), respectively. Substituting (78), (89c), (92c) in (59a) and (59c), we have

$$(93a) \quad T_r = \frac{2Eh}{1-\sigma} C_1 - \frac{2Eh}{1+\sigma} \frac{C_2}{r^2} - \frac{\sigma p h}{2},$$

$$(93b) \quad G_r = -\frac{4Eh^3}{3(1-\sigma)} K_2 + \frac{2Eh^3}{3(1+\sigma)} \frac{K_3}{r^2} + \frac{(3+\sigma)p}{16} (r^2 - 2r_1^2) \\ - \frac{(1+\sigma)p r_1^2}{4} \log r + \frac{(8+\sigma)h^2 p r_1^2}{20r^2} - \frac{\sigma p h^2}{5}.$$

From (89b) and (93), there results

$$(94a) \quad C_2 = \frac{1+\sigma}{1-\sigma} r_1^2 C_1 - \frac{\sigma(1+\sigma)p r_1^2}{4E},$$

$$(94b) \quad K_3 = \frac{2(1+\sigma)r_1^2}{1-\sigma} K_2 \\ + \frac{3(1+\sigma)p r_1^4}{8Eh^3} \left\{ \frac{3+\sigma}{4} + (1+\sigma) \log r_1 - \frac{(8-3\sigma)h^2}{5r_1^2} \right\}.$$

We can now express U , w , T_r , G_r in terms of C_1 and K_2 . Substituting (78), (90), (92c), (94) in (56), (59a), and in (59c), we obtain

$$\begin{aligned}
 (95a) \quad U = & \left\{ r + \frac{(1+\sigma)r_1^2}{(1-\sigma)r} \right\} (C_1 - 2K_2 z) + \frac{\sigma(1+\sigma)p}{4Eh} (r^2 - r_1^2) \\
 & + \frac{3(1-\sigma^2)p}{32Eh^3} \left\{ r^2 - 2r_1^2(1+2\log r) + \frac{4(2+3\sigma)h^2}{5(1-\sigma)} - \frac{8h^2r_1^2}{(1-\sigma)r^2} \right\} r z \\
 & - \frac{3(1+\sigma)p r_1^4}{8Eh^3 r} \left\{ \frac{3+\sigma}{4} + (1+\sigma)\log r_1 - \frac{(8-3\sigma)h^2}{5r_1^2} \right\} z \\
 & - \frac{(1+\sigma)(2-\sigma)p}{8Eh^3} (r^2 - r_1^2) \frac{z^3}{r},
 \end{aligned}$$

$$\begin{aligned}
 (95b) \quad W = & -K_2 \left\{ r_0^2 - r^2 + \frac{2(1+\sigma)}{1-\sigma} r_1^2 \log \frac{r_0}{r} \right\} - \frac{2\sigma}{1-\sigma} C_1 z \\
 & + \frac{2\sigma}{1-\sigma} K_2 z^2 - \frac{3(1-\sigma^2)p r_1^2}{16Eh^3} (r_0^2 \log r_0 - r^2 \log r) \\
 & - \frac{3(1+\sigma)p r_1^4 \log \frac{r_0}{r}}{8Eh^3} \left\{ \frac{3+\sigma}{4} + (1+\sigma)\log r_1 - \frac{(8-3\sigma)h^2}{5r_1^2} \right\} \\
 & + \frac{3(1-\sigma^2)p}{128Eh^3} \left\{ r_0^2 + r^2 - \frac{8(8-3\sigma)h^2}{5(1-\sigma)} \right\} (r_0^2 - r^2) - \frac{1-\sigma^2}{2E} p z \\
 & - \frac{3\sigma(1+\sigma)p}{16Eh^3} \left\{ r^2 - 2r_1^2(1+\log r) + \frac{2(5-3\sigma)h^2}{5\sigma} \right\} z^2 \\
 & + \frac{(1+\sigma)^2}{16Eh^3} p z^4,
 \end{aligned}$$

$$(95c) \quad T_r = \left(1 - \frac{r_1^2}{r^2}\right) \left(\frac{2Eh}{1-\sigma} C_1 - \frac{\sigma p h}{2}\right),$$

$$\begin{aligned}
 (95d) \quad G_r = & \left(1 - \frac{r_1^2}{r^2}\right) \left\{ -\frac{4Eh^3}{3(1-\sigma)} K_2 + \frac{(3+\sigma)p}{16} (r^2 - r_1^2) - \frac{\sigma p h^2}{5} \right\} \\
 & - \frac{(1+\sigma)p r_1^2}{4r^2} (r^2 \log r - r_1^2 \log r_1).
 \end{aligned}$$

We are now in position to find the values of K_2 and C_1 for the nine cases of edge conditions given in the last article. We here record the value of these constants only for the case of classical support. Setting

$T_r|_{r=r_0} = G_r|_{r=r_0} = 0$, we find

$$(96a) \quad C_1 = \frac{\sigma(1-\sigma)p}{4E},$$

$$(96b) \quad K_2 = \frac{3(1-\sigma)P}{4Eh^3} \left\{ \frac{3+\sigma}{16} (r_0^2 - r_1^2) - \frac{\sigma h^2}{5} \right\} \\ - \frac{3(1-\sigma^2)Pr_1^2}{16Eh^3(r_0^2 - r_1^2)} (r_0^2 \log r_0 - r_1^2 \log r_1),$$

Observe that (95) becomes identical with (79) when $r_1 = 0$. Hence, for uniform load, the complete plate is a limiting case of the incomplete plate. We shall comment on this situation further after we have solved Problem II.

Problem II. The only load is a downward shearing force of $-W$ pounds distributed uniformly along the inner edge; that is,

$$(97) \quad L_1 = L_2 = L = \mathcal{L} = 0.$$

Substituting (97) in (59e), we have

$$(98a) \quad N_r = - \frac{8Eh^3}{3(1-\sigma^2)} \frac{K_1}{r}.$$

Consider an annular ring bounded by the inner edge and a concentric cylinder of radius r . Since this portion of the plate is in equilibrium, we may write

$$(98b) \quad 2\pi r N_r - W = 0.$$

The elimination of N_r between (98a) and (98b) results in

$$(98c) \quad K_1 = - \frac{3(1-\sigma^2)W}{16\pi E h^3}.$$

Let us now find T_r and G_r . Substituting (89c), (97), (98c) in (59a) and (59c), we find

$$(99a) \quad T_r = \frac{2Eh}{1-\sigma} C_1 - \frac{2Eh}{1+\sigma} \frac{C_2}{r^2},$$

$$(99b) \quad G_r = -\frac{4Eh^3}{3(1-\sigma)} K_2 + \frac{2Eh^3}{3(1+\sigma)} \frac{K_3}{r^2} \\ + \frac{W}{8\pi} \left\{ 3+\sigma + 2(1+\sigma) \log r - \frac{2(8+\sigma)h^2}{5r^2} \right\}.$$

From (89b) and (99), we compute

$$(100a) \quad C_2 = \frac{1+\sigma}{1-\sigma} r_1^2 C_1,$$

$$(100b) \quad K_3 = \frac{2(1+\sigma)}{1-\sigma} r_1^2 K_2 \\ - \frac{3(1+\sigma)W r_1^2}{16\pi E h^3} \left\{ (3+\sigma) + 2(1+\sigma) \log r_1 - \frac{2(8+\sigma)h^2}{5r_1^2} \right\}.$$

Substituting (90), (97), (98c), (100) in (56), (59a), and in (59c), we obtain, finally,

$$(101a) \quad U = \left(r + \frac{1+\sigma}{1-\sigma} \frac{r_1^2}{r} \right) (C_1 - 2K_2 Z) + \frac{3(1-\sigma^2)W}{16\pi E h^3} (1+2 \log r) r Z \\ + \frac{3(1+\sigma)W}{16\pi E h^3} r_1^2 \left\{ 3+\sigma + 2(1+\sigma) \log r_1 + \frac{2(2-\sigma)h^2}{5r_1^2} \right\} \frac{Z}{r} \\ - \frac{(1+\sigma)(2-\sigma)W}{8\pi E h^3} \frac{Z^3}{r},$$

$$(101b) \quad W = -K_2 \{ r_0^2 - r^2 + \frac{2(1+\sigma)}{1-\sigma} r_1^2 \log \frac{r_0}{r} \} - \frac{2\sigma}{1-\sigma} C_1 Z \\ + \frac{2\sigma}{1-\sigma} K_2 Z^2 + \frac{3(1-\sigma^2)W}{16\pi E h^3} (r_0^2 \log r_0 - r^2 \log r) \\ + \frac{3(1+\sigma)W r_1^2}{16\pi E h^3} \log \frac{r_0}{r} \left\{ 3+\sigma + 2(1+\sigma) \log r_1 - \frac{2(8+\sigma)h^2}{5r_1^2} \right\} \\ - \frac{3\sigma(1+\sigma)W}{8\pi E h^3} (1+\log r) Z^2,$$

$$(101c) \quad T_r = \left(1 - \frac{r_1^2}{r^2} \right) \frac{2Eh}{1-\sigma} C_1,$$

$$(101d) \quad G_r = \left(1 - \frac{r_1^2}{r^2}\right) \left\{ -\frac{4Eh^3}{3(1-\sigma)} K_2 + \frac{(3+\sigma)W}{8\pi} \right\} \\ + \frac{(1+\sigma)W}{4\pi r^2} (r^2 \log r - r_1^2 \log r_1).$$

We are now ready to find the values of K_2 and C_1 for the nine cases of edge conditions given in article 16. As an example, consider the case of classical support.

Letting $T_r|_{r=r_0} = G_r|_{r=r_0} = 0$, we obtain

$$(102) \quad K_2 = \frac{3(1-\sigma)W}{16\pi E h^3} \left\{ \frac{3+\sigma}{2} + (1+\sigma) \frac{r_0^2 \log r_0 - r_1^2 \log r_1}{r_0^2 - r_1^2} \right\}, \quad C_1 = 0.$$

Observe that if W is kept constant and r_1 is set equal to zero, (101) becomes precisely the solution for central load given by formulas (68) and (69). Hence the centrally loaded plate is a limiting case of the incomplete plate with vertical shear distributed around the inner edge.

It should be observed that we have no right to assume in advance that the complete plate will be a limiting case of the incomplete plate, since, in the case of a complete plate, the constants C_2 and K_3 were determined from the conditions $\lim_{r \rightarrow 0} t_{r0} = \lim_{r \rightarrow 0} g_{r0} = 0$, while, for the incomplete plate, the constants were fixed by the conditions $T_r|_{r=r_1} = G_r|_{r=r_1} = 0$.

It appears that A. Timpe * was the first to find

* Zeitschrift für angewandte Mathematik und Mechanik, vol. 4, no. 5, 1924, pp. 361-376.

the solution for an incomplete plate with vertical shear

distributed uniformly around the inner edge. Timpe did not attempt to obtain the solution for central load by letting $r \rightarrow 0$, since he thought that such a procedure would not lead to the correct solution. (cf. Garabedian, loc. cit.). In his opinion, the true solution for central load could be found only as a limiting case of a bizonal plate having constant load on the inner zone and no load on the outer zone.

17. The bending of a plate under bizonal distribution of load. In this article, we shall use the subscripts i and e to indicate the interior and exterior zones, respectively. We shall consider only two problems; in the first, the only load will be a total pressure of $-W$ pounds distributed uniformly over the upper surface of the inner zone; in the second, the only load will be a total downward pressure of $-W$ pounds distributed uniformly over the junction of the two zones. For each problem, therefore, we have

$$(103a) \quad c_{ri} = c_{zi} = J_{1i} = J_{2i} = 0, \quad r_i \leq r;$$

$$(103b) \quad c_{re} = c_{ze} = J_{1e} = J_{2e} = L_{1e} = L_{2e} = 0, \quad r_i < r \leq r_o.$$

In both problems, the inner zone is a complete plate on which the loading conditions are the same as in article 12. Hence, for both problems, we have

$$(104) \quad K_{1i} = K_{3i} = C_{2i} = 0.$$

is zero. Since that K_{1e} has the same value in each problem,

Substituting (1011) in (100), we obtain

$$(1014) \quad K_{1e} = - \frac{2\pi A_1}{(1-\nu^2)h} \frac{A_2}{r_0}$$

Consider a section of the plate meeting at the outer edge a cylinder of radius r , $r_0 < r < r_1$. Since this portion of the plate is in equilibrium, we have

$$(1015) \quad 2\pi r K_{1r} = N = 0.$$

The elimination of K_{1r} between (1014) and (1015) yields, for both problems,

$$(1016) \quad K_{1e} = - \frac{1(1-\nu^2)N}{16\pi E h^2}.$$

It should be observed that if the normal load is distributed uniformly over the inner zone, as in the first problem, the principle of the equality of action and reaction requires that

$$\lim_{r \rightarrow r_1} K_{1e} = \lim_{r \rightarrow r_1} K_{1r}.$$

On the other hand, if the normal load is distributed uniformly along the junction of the two zones, as in the second problem, $K_{1e}/r_0 r_1$ must differ from $K_{1r}/r = r_1$ by an amount equal to the normal load per unit length of the junction. It is not difficult to show that the above values of K_{1e} and K_{1r} ensure the satisfaction, in each problem, of the

appropriate condition,

to the contrary, has been proposed, that (1) there is no axial displacement of the elastic surface of the outer edge, (2) the elastic surface is continuous at the junction of the two tubes, and (3),

$$(104a) \quad \sigma_{\theta\theta}|_{r=r_0} = 0.$$

$$(104b) \quad \sigma_{\theta\theta}|_{r=r_0} = \sigma_{\theta\theta}|_{r=r_0}.$$

Let us now determine the other conditions which must hold at the junction of the two tubes. The principle of the equality of action and reaction requires that

$$(107) \quad \frac{\partial}{\partial r} \tau_r = \frac{\partial}{\partial r} \tau_r,$$

and hence that

$$(108a) \quad \frac{\partial}{\partial r} \tau_r = \frac{\partial}{\partial r} \tau_r.$$

$$(108b) \quad \frac{\partial}{\partial r} \sigma_{rr} = \frac{\partial}{\partial r} \sigma_{rr}.$$

It should be noted that although (107) implies (108), the converse may not be true. If (107) is not satisfied, our solution will not be valid in the immediate neighborhood of the junction; however, if (108) is satisfied, our solution, according to de Saint-Venant's principle, will be valid for all points not too near the junction.

The above, or equivalent, conditions at the junction have been used by each person who has attempted to find the

solution for a moderately thick plate under bizonal distribution of load. These conditions, however, are not sufficient for the determination of the unknown arbitrary constants. De Saint-Venant, in obtaining the solution for a bizonal plate supported at the outer edge * ,

* Final note to par. 45 of the translation of Clebsch.

demanding, in addition, that w'_0 (the slope of the middle surface) should be continuous at the junction. G. Fish, in solving the same problem by an approximate method # ,

Trans.Amer.Soc. of Mech.Eng., vol. 43, 1921, p. 615.

assumed that the value of $\overline{\theta\theta}|_{z=0}$ should be continuous at the boundary between the two zones. Garabedian, in correcting de Saint-Venant's solution † , demanded that, at the

† Journal de l'École Polytechnique, vol.26, 1927, p.89.

junction, both T_0 and G_0 should be continuous.

If the approximate nature of Fish's solution is taken into account, his assumption may be considered the equivalent of Garabedian's. However, the assumptions made by Garabedian and by de Saint-Venant are not equivalent; and naturally there arises the question: which of the two assumptions is correct?

It should be observed that we have no more right,

a priori, to demand the continuity of T_θ and G_θ at the junction than we have to demand the continuity of w'_θ , because the law of the equality of action and reaction does not apply in the case of T_θ and G_θ when the direction of $\theta\theta$ is tangent to the junction. * We must find some other test for deter-

* However, this law does apply to T_θ and G_θ in the case of a radial junction.

mining which of these assumptions is correct. We have such a test in the case of the two problems we are considering in this article, since each of them reduces to the problem of the complete plate loaded centrally when r_1 is allowed to approach zero while the total load, W , is kept constant.

Let us first consider de Saint-Venant's assumption of the continuity of w'_θ at the junction. It may readily be shown, for both problems, that so long as $r_1 \neq 0$, $w'_{\theta 1}$ vanishes at $r = 0$. Hence de Saint-Venant's assumption results in $w'_{\theta 1}|_{r=0} = 0$ for the case of a centrally loaded plate. But this condition is the same as that obtained by Love for a complete plate centrally loaded, and we have already shown (art. 13) that Love's solution is incorrect. Consequently, the assumption of de Saint-Venant is incorrect.

Let us now turn to Garabedian's assumption of the continuity of T_θ and G_θ at the junction. Since, for both problems, both $t_{r\theta 1}$ and $g_{r\theta 1}$ vanish at $r = 0$ when $r_1 \neq 0$, Garabedian's assumption results in the condition

$t_{roe}|_{r=0} = g_{roe}|_{r=0} = 0$ for the case of central load.

Observe that this condition is the same as that obtained by us for the case of a complete plate loaded centrally (art.13). Since, furthermore, the value of K_{1e} given by (105c) is identical with the value found for K_1 for the centrally loaded plate, Garabedian's assumption is a sufficient condition for obtaining the correct solution for central load if r_1 is allowed to approach zero.

We shall use Garabedian's assumption even though it has not been proved to be a necessary condition for the centrally loaded plate to be a limiting case of the above bizonal problems, since such a proof would seem to involve a difficult problem in the calculus of variations. Thus, we shall assume that

$$(108c) \quad \lim_{r \rightarrow r_1} T_{oe} = \lim_{r \rightarrow r_1} T_{oi},$$

$$(108d) \quad \lim_{r \rightarrow r_1} G_{oe} = \lim_{r \rightarrow r_1} G_{oi}.$$

Observe that conditions (108) are equivalent to

$$(109a) \quad \lim_{r \rightarrow r_1} (T_{oe} - T_{oi}) = \lim_{r \rightarrow r_1} (t_{roe} - t_{roi}) = 0,$$

$$(109b) \quad \lim_{r \rightarrow r_1} (G_{oe} - G_{oi}) = \lim_{r \rightarrow r_1} (g_{roe} - g_{roi}) = 0.$$

Let us turn now to the first problem.

Problem I. The only load is a total pressure of - W pounds distributed uniformly over the upper surface of the inner zone; that is,

$$(110a) \quad L_{zi} = 0, L_{1i} = L_i = \ell_i = -\frac{W}{\pi r_i^2}, \quad r \leq r_1.$$

Observe that (78a) and (110a) would be identical if $p = W/(\pi r_i^2)$. Hence, by the aid of (78), we have the following values of U_{opi} , $\bar{U}_{opi}$, w_{opi} , $\bar{w}'_{opi}$:

$$(110b) \quad U_{opi} = \frac{\sigma(1+\sigma)W}{4E\pi r_i^2} r;$$

$$(110c) \quad \bar{U}_{opi} = -\frac{(1+\sigma)W}{4E\pi r_i^2} r,$$

$$(110d) \quad w_{opi} = -\frac{3(1+\sigma)W}{16E\pi r_i^2 h^3} \left\{ \frac{1-\sigma}{8} r^4 - \frac{8-3\sigma}{5} h^2 r^2 \right\},$$

$$(110e) \quad \bar{w}'_{opi} = \frac{3(1+\sigma)W}{8E\pi r_i^2 h^3} \left\{ \frac{r^3}{4} - \frac{3}{5} h^2 r \right\}.$$

Let us now find $T_{r\theta}$, $t_{r\theta}$, $G_{r\theta}$, $g_{r\theta}$ for each zone. Substituting (104) and (110) in (60), we have

$$(111ab) \quad T_{r\theta i} = \frac{4Eh}{1-\sigma} C_{1i} - \frac{\sigma Wh}{\pi r_i^2}, \quad t_{r\theta i} = 0,$$

$$(111c) \quad G_{r\theta i} = -\frac{8Eh^3}{3(1-\sigma)} K_{2i} + \frac{W}{4\pi r_i^2} \left\{ (1+\sigma)r^2 - \frac{8}{5}\sigma h^2 \right\},$$

$$(111d) \quad g_{r\theta i} = \frac{(1-\sigma)W}{8\pi r_i^2} r^2.$$

Substituting (103b) and (105c) in (60), we find

$$(112ab) \quad T_{r\theta e} = \frac{4Eh}{1-\sigma} C_{1e}, \quad t_{r\theta e} = -\frac{4Eh}{1+\sigma} \frac{C_{2e}}{r^2},$$

$$(112c) \quad G_{r\theta e} = -\frac{8Eh^3}{3(1-\sigma)} K_{2e} + \frac{(1+\sigma)W}{2\pi} (1 + \log r),$$

$$(112d) \quad g_{r\theta e} = \frac{4Eh^3}{3(1+\sigma)} \frac{K_{3e}}{r^2} + \frac{W}{4\pi} \left\{ 1-\sigma - \frac{2(8+\sigma)}{5} \frac{h^2}{r^2} \right\}.$$

We are now ready to determine C_{2e} and K_{3e} and also

to find the relations between C_{1i} , C_{1e} , K_{2i} , K_{2e} . Substituting (111) and (112) in (109), there results

$$(113a) \quad C_{2e} = 0, \quad K_{3e} = -\frac{3(1-\sigma^2)W}{32\pi E h^3} \left\{ r_1^2 - \frac{4(8+\sigma)}{5(1-\sigma)} h^2 \right\},$$

$$(113b) \quad C_{1i} = C_{1e} + \frac{\sigma(1-\sigma)W}{4E\pi r_1^2},$$

$$(113c) \quad K_{2i} = K_{2e} - \frac{3(1-\sigma^2)W}{32\pi E h^3} \left\{ 1 + 2 \log r_1 + \frac{8\sigma h^2}{5(1+\sigma)r_1^2} \right\}.$$

Let us now find K_{4i} and K_{4e} . Substituting (104), (110d), (113c) in (56b), we obtain

$$(114a) \quad w_i|_{z=0} = K_{2e} r^2 + K_{4i} - \frac{3(1-\sigma^2)W}{32\pi E h^3} r^2 \left\{ \frac{r^2 + 4r_1^2}{4r_1^2} + 2 \log r_1 - \frac{2(8+\sigma+\sigma^2)h^2}{5(1-\sigma^2)r_1^2} \right\},$$

Substituting (103b), (105c), (113a) in (56b), we have

$$(114b) \quad w_e|_{z=0} = K_{2e} r^2 + K_{4e} - \frac{3(1-\sigma^2)W}{32\pi E h^3} \log r \left\{ 2r^2 + r_1^2 - \frac{4(8+\sigma)}{5(1-\sigma)} h^2 \right\}.$$

From (114b) and (106a), there results

$$(115a) \quad K_{4e} = -K_{2e} r_0^2 + \frac{3(1-\sigma^2)W}{32\pi E h^3} \log r_0 \left\{ 2r_0^2 + r_1^2 - \frac{4(8+\sigma)}{5(1-\sigma)} h^2 \right\}.$$

Substituting (114) and (115a) in (106b), we find

$$(115b) \quad K_{4i} = -K_{2e} r_0^2 - \frac{3(1-\sigma^2)W}{128\pi E h^3} \frac{r_0^2 - r_1^2}{r_1^2} \left\{ r_0^2 + r_1^2 - \frac{8(8-3\sigma)h^2}{5(1-\sigma)} \right\} + \frac{3(1-\sigma^2)W}{32\pi E h^3} \log \frac{r_0}{r_1} \left\{ r_1^2 - \frac{4(8+\sigma)}{5(1-\sigma)} h^2 \right\} + \frac{3(1-\sigma^2)W}{32\pi E h^3} \left\{ 2r_0^2 \log r_0 + r_1^2 + \frac{8\sigma h^2}{5(1+\sigma)} \right\}.$$

We may now express the displacements in terms of W , C_{1e} , and K_{2e} . Substituting (104), (110), (113), (115b) in (56), we have

$$(116a) \quad U_i = C_{1e} r - 2K_{2e} r z + \frac{\sigma W r}{2E\pi r_1^2} \\ + \frac{3(1-\sigma^2)W}{32\pi E h^3} \left\{ \frac{r^2 + 2r_1^2}{r_1^2} + 4 \log r_1 + \frac{4(2+9\sigma-\sigma^2)h^2}{5(1-\sigma^2)r_1^2} \right\} r z \\ - \frac{(1+\sigma)(2-\sigma)W}{8E\pi r_1^2 h^3} r z^3,$$

$$(116b) \quad w_i = -K_{2e}(r_0^2 - r^2) - \frac{2\sigma}{1-\sigma} C_{1e} z + \frac{2\sigma}{1-\sigma} K_{2e} z^2 \\ - \frac{3(1-\sigma^2)W}{128\pi E h^3} \frac{r^2 - r_1^2}{r_1^2} \left\{ r^2 + 5r_1^2 - \frac{8(8+\sigma+\sigma^2)h^2}{5(1-\sigma^2)} \right\} \\ + \frac{3(1-\sigma^2)W}{16\pi E h^3} (r_0^2 \log r_0 - r^2 \log r_1) \\ + \frac{3(1-\sigma^2)W}{32\pi E h^3} \log \frac{r_0}{r_1} \left\{ r_1^2 - \frac{4(8+\sigma)h^2}{5(1-\sigma)} \right\} - \frac{Wz}{2E\pi r_1^2} \\ - \frac{3\sigma(1+\sigma)W}{16\pi E h^3} \left\{ \frac{r^2 + r_1^2}{r_1^2} + 2 \log r_1 + \frac{2(5+2\sigma+\sigma^2)h^2}{5\sigma(1+\sigma)r_1^2} \right\} z^2 \\ + \frac{(1+\sigma)^2 W}{16\pi E h^3} \frac{z^4}{r_1^2}.$$

Substituting (103b), (105c), (113a), (115a) in (56), we obtain

$$(117a) \quad U_e = C_{1e} r - 2K_{2e} r z \\ + \frac{3(1-\sigma^2)W}{32\pi E h^3} \left\{ r_1^2 + 2r^2(1+2 \log r) + \frac{4(2-\sigma)h^2}{5(1-\sigma)} \right\} \frac{z}{r} \\ - \frac{(1+\sigma)(2-\sigma)W}{8\pi E h^3} \frac{z^3}{r},$$

$$\begin{aligned}
 (117b) \quad w_e = & -K_{2e}(r_0^2 - r^2) - \frac{2\sigma}{1-\sigma} C_{1e} z + \frac{2\sigma}{1-\sigma} K_{2e} z^2 \\
 & + \frac{3(1-\sigma^2)W}{16\pi E h^3} (r_0^2 \log r_0 - r^2 \log r) \\
 & + \frac{3(1-\sigma^2)W}{32\pi E h^3} \log \frac{r_0}{r} \left\{ r_1^2 - \frac{4(8+\sigma)}{5(1-\sigma)} h^2 \right\} \\
 & - \frac{3\sigma(1+\sigma)W}{8\pi E h^3} (1 + \log r) z^2.
 \end{aligned}$$

Let us now find T_{re} and G_{re} . The substitution of (103b), (105c), (113a) in (59a) and in (59c) yields

$$(118a) \quad T_{re} = \frac{2Eh}{1-\sigma} C_{1e},$$

$$(118b) \quad G_{re} = -\frac{4Eh^3}{3(1-\sigma)} K_{2e} + \frac{W}{8\pi} \left\{ 3 + \sigma + 2(1+\sigma) \log r - \frac{1-\sigma}{2} \frac{r_1^2}{r^2} \right\}.$$

We are now in position to find the values of C_{1e} and K_{2e} for the nine cases of edge conditions given in article 15. We shall consider only the case of classical support. Substituting the conditions $T_{re}|_{r=r_0} = G_{re}|_{r=r_0} = 0$ in (118), we find

$$(119) \quad K_{2e} = \frac{3(1-\sigma)W}{32\pi E h^3} \left\{ 3 + \sigma + 2(1+\sigma) \log r_0 - \frac{1-\sigma}{2} \frac{r_1^2}{r_0^2} \right\}, \quad C_{1e} = 0.$$

It is important to know whether or not our solution is valid in the vicinity of the junction of the two zones. It is not difficult to show, by means of (116), (117), (3a), and (3b), that conditions (107) do not hold. Hence our solution is valid only for points which are not too near the junction.

In this connection, it is interesting to ask whether

the solution for the complete plate with uniform load on the upper face can be obtained from our bizonal problem by expanding the inner zone to include the entire plate. In (116), replace $W/(\pi r_i^2)$ by p and, at the same time, set $r_i = r_o$. We find that the resulting values of U_i and w_i are not identical with the displacements given in (79) for a complete plate. But there is no reason why we should expect an agreement. The formulas for U_i and w_i do not hold at $r = r_i$; hence, when we set $r_i = r_o$, the formulas for U_i and w_i may not be valid at the outer edge. However, if we substitute (119) in (116) and then let $r_i = r_o$, we do obtain the solution for the complete plate loaded uniformly on its upper face and supported at the outer edge. However, for no other type of edge condition is the complete plate a limiting case of the bizonal plate. Agreement in the case of the supported plate is explained by the fact that, although we do have a discontinuity of U and w at the junction, the imposing of conditions (108) ensures the continuity of T_r and G_r .

Let us see what happens when we shrink r_i to zero but keep W constant. We find that formulas (117) and (118) become identical with (68) and (69), respectively. Hence the centrally loaded plate is a limiting case of the bizonal plate with uniform load on the upper face of the inner zone. But, since U_e and w_e are not valid in the vicinity of the junction, the limiting values of the displacements will not

hold in the neighborhood of the origin. Recall that we had reached a similar conclusion in article 13 in our discussion of a complete plate loaded at its center.

Problem II. The only load is a total downward pressure of $-W$ pounds distributed uniformly over the junction of the two zones; that is,

$$(120a) \quad L_{zi} = L_{1i} = L_i = \mathcal{L}_i = 0, \quad r < r_1.$$

It is evident that L_{1i} is $-\infty$ at the junction; thus

$$(120b) \quad L_{zi} = 0, \quad L_{1i} = L_i = \mathcal{L}_i = -\infty, \quad r = r_1.$$

Our procedure is similar to that followed in Problem I. We first find T_{re} , t_{re} , G_{re} , g_{re} for each zone. Substituting (104) and (120a) in (60), we have

$$(121a) \quad T_{rei} = \frac{4Eh}{1-\sigma} C_{1i}, \quad t_{rei} = 0, \quad r_1 < r;$$

$$(121b) \quad G_{rei} = -\frac{8Eh^3}{3(1-\sigma)} K_{2i}, \quad g_{rei} = 0, \quad r_1 < r.$$

The values of T_{ree} , t_{ree} , G_{ree} , g_{ree} are given by formulas (112). Substituting (112) and (121) in (109), we obtain

$$(122a) \quad C_{2e} = 0, \quad K_{3e} = -\frac{3(1-\sigma^2)W}{16\pi Eh^3} \left\{ r_1^2 - \frac{2(8+\sigma)}{5(1-\sigma)} h^2 \right\};$$

$$(122b) \quad C_{1i} = C_{1e}, \quad K_{2i} = K_{2e} - \frac{3(1-\sigma^2)W}{16\pi Eh^3} (1 + \log r_1).$$

Let us now find K_{4i} and K_{4e} . Substituting (104), (120a), (122b) in (56b), we find

$$(123a) \quad w_i|_{z=0} = K_{2e} r^2 + K_{4i} - \frac{3(1-\sigma^2)W}{16\pi E h^3} r^2 (1 + \log r_i).$$

Substituting (103b), (105c), (122a) in (56b), we have

$$(123b) \quad w_e|_{z=0} = K_{2e} r^2 + K_{4e} - \frac{3(1-\sigma^2)W}{16\pi E h^3} \log r \left\{ r^2 + r_i^2 - \frac{2(8+\sigma)}{5(1-\sigma)} h^2 \right\}.$$

From (123b) and (106a), it follows that

$$(124a) \quad K_{4e} = -K_{2e} r_0^2 + \frac{3(1-\sigma^2)W}{16\pi E h^3} \log r_0 \left\{ r_0^2 + r_i^2 - \frac{2(8+\sigma)}{5(1-\sigma)} h^2 \right\}.$$

If (123) and (124a) are substituted in (106b), the result is

$$(124b) \quad K_{4i} = -K_{2e} r_0^2 + \frac{3(1-\sigma^2)W}{16\pi E h^3} (r_0^2 \log r_0 + r_i^2) + \frac{3(1-\sigma^2)W}{16\pi E h^3} \log \frac{r_0}{r_i} \left\{ r_i^2 - \frac{2(8+\sigma)}{5(1-\sigma)} h^2 \right\}.$$

We are now prepared to find the displacements.

Substituting (104), (120a), (122b), (124b) in (56), we have

$$(125a) \quad U_i = C_{1e} r - 2K_{2e} r z + \frac{3(1-\sigma^2)W}{8\pi E h^3} (1 + \log r_i) r z,$$

$$(125b) \quad w_i = -K_{2e} (r_0^2 - r^2) - \frac{2\sigma}{1-\sigma} C_{1e} z + \frac{2\sigma}{1-\sigma} K_{2e} z^2 + \frac{3(1-\sigma^2)W}{16\pi E h^3} (r_0^2 \log r_0 - r^2 \log r_i + r_i^2 - r^2) + \frac{3(1-\sigma^2)W}{16\pi E h^3} \log \frac{r_0}{r_i} \left\{ r_i^2 - \frac{2(8+\sigma)}{5(1-\sigma)} h^2 \right\} - \frac{3\sigma(1+\sigma)W}{8\pi E h^3} (1 + \log r_i) z^2.$$

Substituting (103b), (105c), (122a), (124a) in (56), we find

$$(126a) \quad U_e = C_{1e} r - 2K_{2e} r z$$

$$+ \frac{3(1-\sigma^2)W}{16\pi E h^3} \left\{ r^2(1+2\log r) + r_1^2 + \frac{2(2-\sigma)h^2}{5(1-\sigma)} \right\} \frac{z}{r} \\ - \frac{(1+\sigma)(2-\sigma)W}{8\pi E h^3} \frac{z^3}{r},$$

$$(126b) \quad w_e = -K_{2e}(r_0^2 - r^2) - \frac{2\sigma}{1-\sigma} C_{1e} z + \frac{2\sigma}{1-\sigma} K_{2e} z^2$$

$$+ \frac{3(1-\sigma^2)W}{16\pi E h^3} (r_0^2 \log r_0 - r^2 \log r) \\ + \frac{3(1-\sigma^2)W}{16\pi E h^3} \log \frac{r_0}{r} \left\{ r_1^2 - \frac{2(8+\sigma)}{5(1-\sigma)} h^2 \right\} \\ - \frac{3\sigma(1+\sigma)W}{8\pi E h^3} (1+\log r) z^2.$$

Let us now find T_{re} and G_{re} . The substitution of (103b), (105c), (122a) in (59a) and in (59c) results in

$$(127a) \quad T_{re} = \frac{2Eh}{1-\sigma} C_{1e},$$

$$(127b) \quad G_{re} = -\frac{4Eh^3}{3(1-\sigma)} K_{2e} + \frac{W}{8\pi} \left\{ 3+\sigma+2(1+\sigma)\log r - (1-\sigma)\frac{r_1^2}{r^2} \right\}.$$

We might now proceed to find the values of C_{1e} and K_{2e} for the nine cases of edge conditions given in article 15. We propose here to determine these constants only for the case of classical support. Substituting the condition

$T_{re}|_{r=r_0} = G_{re}|_{r=r_0} = 0$ in (127), we find

$$(128) \quad K_{2e} = \frac{3(1-\sigma)W}{32\pi E h^3} \left\{ 3+\sigma+2(1+\sigma)\log r_0 - (1-\sigma)\frac{r_1^2}{r_0^2} \right\}, \quad C_{1e} = 0.$$

Observe that z^2 is the highest power of z in U_e , w_e , or w_e , whereas U_e contains a term in z^3 ; it is evident, there-

fore, that at the junction of the two zones, $\overline{rr}_e \neq \overline{rr}_i$ and $\overline{\theta\theta}_e \neq \overline{\theta\theta}_i$. Consequently, our solution is not valid in the immediate neighborhood of the zonal boundary.

It is of interest to note that if we keep W constant and let $r_i = 0$, equations (126) and (127) become identical with (68) and (69), respectively. Hence the complete plate loaded centrally is a limiting case of Problem II as well as of Problem I.

18. Conclusion. The generality of our method has been clearly exhibited in Part II, since all the solutions there obtained were determined from a single formula for the displacements, namely, formula (56). Moreover, the power of our method has been demonstrated by the diversity of the problems solved. These problems have been concerned with both complete and incomplete plates and with conditions of load which give not only pure stretching but also combined stretching and bending. The class of loads treated has been extensive; we have considered not only radial and axial mass force but also uniform normal load, normal load proportional to r^2 , concentrated normal load, and shearing traction proportional to r . Furthermore, with each type of load a large number of edge conditions have been studied.

Further evidence of both the power and generality of this method is seen in the ease and rigor with which it has

been possible to find the values of the arbitrary constants K_1 , K_3 , and C_2 in the case of a complete plate. Certain writers, in attempting to determine these constants, have allowed themselves to be influenced by speculation concerning the physical nature of the problem. For instance, C_2 and K_3 have been put equal to zero in order to keep U and w from being infinite at the center of the plate. That intuition is a poor substitute for rigorous mathematical analysis is evidenced by the fact that C_2 and K_3 are not zero for the loading cases defined by formulas (51a), (54a), (55a), and by the fact that K_3 is not zero for the case of central load. Indeed, it was by assuming $K_3 = 0$ for the case of load concentrated at the center of a plate that both de Saint-Venant and Love were led to give erroneous solutions in the problem of central load.

Since our solutions are of two-dimensional type, a comparison with solutions of three-dimensional character is natural, and there arises at once the question of relative accuracy. We have already pointed out that Clemmow's experiments show that the best agreement with the actual deflection of a clamped plate is given by Case C-I, which is a two-dimensional solution, and not by either Clemmow's three-dimensional solution or Nádaí's. Clemmow has attributed this to a mere coincidence, since he-- and Nádaí also -- has taken the stand that the correct solution

for a plate can only be found from a three-dimensional solution, a two-dimensional solution being, of necessity, a less accurate approximation to the true physical situation. The position taken by Clemmow and Nádai would be a justifiable one if it were possible to give a correct mathematical definition of the boundary conditions at an edge and subsequently to find a three-dimensional solution in agreement with this mathematical definition. But it is not possible to give an accurate mathematical description of the boundary conditions at an edge, even for such a simple case as that of a plate clamped at the edge.

Furthermore, there are only two types of boundary conditions which can be handled by Nádai's method; Clemmow is able to go beyond these two types, but only at the expense of extremely complicated computations. On the other hand, there seems to be no limit to the number of boundary conditions which can be studied by means of our two-dimensional method. Moreover, the computations are much simplified when we seek a two-dimensional instead of a three-dimensional solution. Finally, in practice, a two-dimensional solution is actually to be preferred to a three-dimensional solution; for not only is the former simpler in structure, but, what is more, the results obtained from it may be truer physically -- when a proper mathematical definition of the boundary conditions at the edge is used --

than results obtained from a three-dimensional solution whose boundary conditions do not so closely approximate those actually existing.

The application of our method is not limited to circular plates of constant thickness. The method may also be applied to circular plates of variable thickness, to rectangular plates of either constant or variable thickness, and, under the heading of one-dimensional problems, to moderately thin rods of either constant or variable thickness; in fact, this method may be used in any problem where Garabedian's method is applicable. So far as the author is aware, no method other than Garabedian's has been developed which can be applied to such broad classes of problems.

A careful examination of the literature, and a detailed study of the various methods thus far devised, has convinced the author that his method, or the closely parallel method of Garabedian, affords the most natural and satisfactory machinery for handling the two-dimensional problems of Elasticity. Not only may these two methods be said to yield the solution most to be desired in practice, but also they seem to be in every way superior -- by virtue of their power, generality, and simplicity -- to any two-dimensional method hitherto advanced.