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Approved by:

Boris Podolsky

Alex E. S. Green

Harold J. Kersten

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QUANTUM-MECHANICAL POTENTIALS OF BOHM

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by

Walter Richard Steiger

B.S.	Massachusetts Institute of Technology	1948
M.S.	University of Hawaii	1950

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I. Introduction

Although the quantum theory as known and used today has made possible tremendous advances in physical knowledge, it nevertheless suffers from considerable conceptual difficulties, and is found inapplicable when distances involved are of the order of 10^{-13} cm. or less. Most physicists have accepted or adjusted to the conceptual difficulties, but there are still a few, notably among them Einstein[1], who do not accept whole-heartedly the present interpretation of quantum theory. In the years since the development of quantum theory there has been considerable discussion on this question of interpretation. The whole controversy is very well summed up in the book Albert Einstein, Philosopher-Scientist[2].

David Bohm in a recent series of papers[3,4] has suggested a new interpretation of quantum theory in terms of "hidden variables." He defines a "quantum-mechanical" potential such that when added to the ordinary classical potential, this total potential may be used to find the motion of the system using ordinary Newtonian equations of motion. Bohm calls this "quantum-mechanical" potential an objectively real potential and he states that his suggested interpretation, provides a broader conceptual framework than the usual interpretation, because it makes possible a precise and continuous description of all processes. He also suggests that his new interpretation may serve to resolve the difficulties encountered in the domain of distances of the order of 10^{-13} cm. or less.

The purpose of this thesis is to investigate a number of specific examples in terms of Bohm's "quantum-mechanical" potential with a particular view towards providing an increased understanding of the properties of this potential. Having worked through a number of examples we should be in a position to comment upon Bohm's contention that this "quantum-mechanical" potential has an objective reality and that this interpretation provides a broader conceptual framework than the usual interpretation.

II. The Usual Physical Interpretation of Quantum Theory

The usual interpretation of quantum theory is based on two fundamental assumptions. The first assumption is that the wave function determines only probabilities of actual experimental results, but nevertheless provides the most complete possible specification of the quantum state of an individual system. The second assumption is that the process of transfer of a single quantum from observed system to measuring apparatus is inherently unpredictable, uncontrollable, and unanalyzable. From either of these two assumptions the uncertainty principle is readily deduced[5]. The uncertainty principle states that there is an inherent and irreducible limitation on the precision with which one may even conceive of momentum and position as simultaneously defined quantities. Thus, if either the position or momentum is precisely known then all knowledge of the other variable is lost. Not only is this knowledge lost but one may not even conceive of there being a position or momentum, as the case may be, ascribable to the system.

III. Bohm's Interpretation of the Schroedinger Equation

We shall consider here Bohm's physical interpretation of the present mathematical formulation of quantum theory for the case of the single-particle Schroedinger equation. This wave equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{1}$$

where $V = V(\vec{r})$ is the classical potential and $\psi = \psi(\vec{r},t)$ is a complex function and may be written in the form

$$\psi = Re^{iS/\hbar} \tag{2}$$

where R and S are real functions of position and time. Substituting (2) into (1) we obtain a differential equation in terms of R and S instead of ψ :

$$-\frac{\hbar^2}{2m}\nabla^2(Re^{iS/\hbar}) + VRe^{iS/\hbar} = i\hbar\frac{\partial}{\partial t}(Re^{iS/\hbar})$$

which upon expanding and cancelling a factor $e^{iS/\hbar}$ from each term gives

$$-i\frac{\hbar}{2m}\left(R\nabla^2S + 2\nabla R\cdot\nabla S\right) + \frac{1}{2m}R(\nabla S)^2 - \frac{\hbar^2}{2m}\nabla^2R + VR = -R\frac{\partial S}{\partial t} + i\hbar\frac{\partial R}{\partial t}.$$

Upon equating the imaginary terms, we get

$$\frac{\partial R}{\partial t} = -\frac{1}{2m}\left(R\nabla^2S + 2\nabla R\cdot\nabla S\right). \tag{3}$$

And equating the real terms,

$$\frac{\partial S}{\partial t} = -\left[\frac{1}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\frac{\nabla^2R}{R}\right] \tag{4}$$

Let us now write $P(\vec{r},t) = R^2(\vec{r},t)$ or $R = P^{1/2}$, where P therefore is, according to the usual interpretation, the probability density. Then $\nabla R = \nabla P^{1/2} = \frac{1}{2}P^{-\frac{1}{2}}\nabla P$ and $\frac{\partial R}{\partial t} = \frac{\partial P^{1/2}}{\partial t} = \frac{1}{2}P^{-\frac{1}{2}}\frac{\partial P}{\partial t}$.

Equation (3) then becomes

$$\frac{1}{2}P^{-\frac{1}{2}}\frac{\partial P}{\partial t} = -\frac{1}{2m}\left(P^{1/2}\nabla^2 S + P^{-\frac{1}{2}}\nabla P\cdot\nabla S\right).$$

Simplifying:

$$\frac{\partial P}{\partial t} = -\frac{1}{m}\left(P\nabla^2 S + \nabla P\cdot\nabla S\right) = -\frac{1}{m}\nabla\left(P\nabla S\right),$$

which may be written in the form

$$\frac{\partial P}{\partial t} + \nabla\cdot\left(P\frac{\nabla S}{m}\right) = 0. \qquad (5)$$

Now, equation (4) may be written in the form

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0. \qquad (6)$$

Now, in the classical limit, as $\hbar \to 0$, this last equation has the exact form of the Hamilton-Jacobi partial differential equation of classical mechanics. The solution of this equation, S, is the action function defined by $S = 2\int_0^t T\,dt$, where T is the kinetic energy of the particle[6]. A property of S is that $\vec{p} = \nabla S$ is the momentum of the particle.

The classical equation of motion of the particle is then

$$\frac{\partial \vec{p}}{\partial t} = -\nabla V.$$

To show that this leads to the Hamilton-Jacobi equation we write

$$\frac{d\vec{p}}{dt} = \frac{\partial \vec{p}}{\partial t} + \frac{\partial \vec{p}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{p}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{p}}{\partial z}\frac{dz}{dt} = \frac{\partial \vec{p}}{\partial t} + (\vec{v}\cdot\nabla)\vec{p}$$

$$= \frac{\partial}{\partial t}(\nabla S) + \left(\frac{\nabla S}{m}\cdot\nabla\right)\nabla S$$

$$= \nabla\left[\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m}\right] = -\nabla V.$$

Hence

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V = 0 ,$$

which is the Hamilton-Jacobi partial differential equation we sought.

Bohm now makes the assumption that even when $\hbar \neq 0$ Eq. (4) represents a Hamilton-Jacobi Partial differential equation in which the entire quantity $\left[V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right]$ is considered a potential, the first term being the usual classical potential, and the second term a new type of potential which he calls the "quantum-mechanical" potential,

$$U(\vec{r},t) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} . \tag{7}$$

The quantity ∇S can still be regarded as the particle momentum and the motion of the particle may still be treated by an equation of the classical form, but containing the additional force derived from the "quantum-mechanical" potential:

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla\left\{V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right\} . \tag{8}$$

Since $\frac{\nabla S}{m} = \frac{\vec{p}}{m} = \vec{v}$, we may write Eq. (5)

$$\frac{\partial P}{\partial t} + \nabla \cdot (P\vec{v}) = 0 . \qquad (9)$$

This equation says that we may still regard $P(\vec{r},t)$ as the probability density for in that case $P\vec{v}$ is the probability current and this equation expresses the conservation of probability.

The foregoing interpretation leads Bohm to regard the wave function $\psi(\vec{r},t)$ of an individual electron as a mathematical representation of an objectively real field. From its absolute value, R, can be found the "quantum-mechanical" force acting on the particle. But since the initial conditions of an individual electron cannot, for purely practical reasons, be determined the equation of motion (8) cannot be completely solved and the motion of an individual electron is not precisely determined. However, the probability density of a statistical ensemble of electrons is given by $P(\vec{r},t)$.

Hence all the results of the usual interpretation are obtained from Bohm's interpretation if the following three special assumptions are made which are mutually consistent:

(1) That the ψ-field satisfies Schroedinger's equation.

(2) That the particle momentum is restricted to $\vec{p} = \nabla S(\vec{r},t)$

(3) That we do not predict or control the precise location of the particle, but have, in practice, a statistical ensemble with probability density $P(\vec{r},t) = |\psi(\vec{r},t)|^2$. The use of statistics is, however, not inherent in the conceptual structure, but merely a consequence of our ignorance of the precise initial conditions of the particle.

IV. Stationary States of the Hydrogen Atom

In order to learn more about the properties of this "quantum-mechanical" potential it is desirable to study a number of specific examples.

The wave functions for the five lowest states of the hydrogen-like atom are as follows[7]:

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0} - iE_1t/\hbar} \tag{10}$$

$$\psi_{200} = \frac{1}{2\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(1 - \frac{Zr}{2a_0}\right) e^{-\frac{Zr}{a_0} - iE_2t/\hbar} \tag{11}$$

$$\psi_{210} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} \cos\theta \; e^{-\frac{Zr}{a_0} - iE_2t/\hbar} \tag{12}$$

$$\psi_{21\pm1} = \frac{1}{8\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} \sin\theta \; e^{-\frac{Zr}{a_0} - iE_2t/\hbar \pm i\varphi} \tag{13}$$

where Z is the atomic number of the atom, $a_0 = \frac{\hbar^2}{me^2}$ the Bohr radius, and E is the energy.

Let the absolute value of ψ_{100} be R_1. Then

$$R_1 = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0}}.$$

We wish to find now the "quantum-mechanical" potential defined by $U(\vec{r}) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$.

$$\nabla R_1 = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(-\frac{Z}{a_0}\right) e^{-\frac{Zr}{a_0}} \nabla r$$

And

$$\nabla\cdot\nabla R_1 = \nabla^2 R_1 = -\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{5/2}\left[\nabla^2 r - \frac{Z}{a_0}(\nabla r)^2\right] e^{-\frac{Zr}{a_0}} .$$

But

$$\nabla r = \frac{\vec{r}}{r} \quad , \quad (\nabla r)^2 = 1$$

$$\nabla\cdot\nabla r = \frac{1}{r}\nabla\cdot\vec{r} + \nabla\frac{1}{r}\cdot\vec{r} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r} .$$

Thus

$$\frac{\nabla^2 R_1}{R_1} = \frac{\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{5/2} e^{-\frac{Zr}{a_0}}\left(-\frac{2}{r} + \frac{Z}{a_0}\right)}{\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0}}} = \frac{Z}{a_0}\left(\frac{Z}{a_0} - \frac{2}{r}\right) .$$

Finally

$$U_1(\vec{r}) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R_1}{R_1} = -\frac{\hbar^2}{2m}\frac{Z}{a_0}\left(\frac{Z}{a_0} - \frac{2}{r}\right).$$

Letting $a_0 = \frac{\hbar^2}{me^2}$ this becomes

$$U_1(\vec{r}) = \frac{Ze^2}{r} - \frac{Z^2 me^4}{2\hbar^2} . \tag{14}$$

The equation of motion for this electron is therefore

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla(V + U_1) = -\nabla\left(-\frac{Ze^2}{r} + \frac{Ze^2}{r} - \frac{Z^2 me^4}{2\hbar^2}\right) = 0 .$$

Hence $\frac{dr}{dt}$ = constant = 0, since if it is not zero the electron will leave the atom, which it cannot do. Therefore the electron in this state of zero angular momentum also has zero radial momentum. The electron is simply at rest at some distance r from the nucleus. The probability density for r is given by $|\psi(\vec{r})|^2$. The coulomb attraction force between the nucleus and electron is just cancelled by the "quantum-mechanical" force.

For the ψ_{200} state the "quantum-mechanical" potential turns out to be

$$U_2(\vec{r}) = -\frac{\hbar^2}{2m}\left(\frac{Z}{2a_0}\right)\frac{\left(\frac{5}{2}\frac{Z}{a_0} - \frac{4}{r} - \frac{Z^2 r}{4a_0}\right)}{1 - \frac{Zr}{2a_0}} . \qquad (15)$$

This has quite a different form than $U_1(r)$, but the force turns out to be the same.

For

$$-\nabla U_2(\vec{r}) = \frac{\hbar^2}{2m}\left(\frac{Z}{a_0}\right)\frac{\vec{r}}{r^3} = \frac{Ze^2}{r^3}\vec{r} .$$

Thus the equation of motion again gives

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla(V + U_2) = -\nabla\left(-\frac{Ze^2}{r} + U_2\right)$$

$$= \left(-\frac{Ze^2}{r^3} + \frac{Ze^2}{r^3}\right)\vec{r} = 0$$

and the same interpretation as before applies.

The ψ_{210} state again has zero angular momentum, and the forces on the electron again cancel. But the probability density for the position of the electron now involves θ as well as r.

The $\psi_{21\pm1}$ state has angular momentum of magnitude $\pm\hbar$. The "quantum-mechanical" potential turns out to be

$$U_4(\vec{r},\theta) = -\frac{\hbar^2}{2m}\frac{Z}{a_0}\left(\frac{Z}{4a_0} - \frac{2}{r}\right) - \frac{\hbar^2}{2mr^2\sin\theta} . \qquad (16)$$

If $\vec{u},\vec{v},\vec{w}$ are the unit vectors in the direction of increasing r,θ,φ respectively, then

$$\nabla U = \frac{\partial U}{\partial r}\vec{u} + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{v} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}\vec{w} .$$

The "quantum-mechanical" force thus becomes

$$-\nabla U_q = \left(\frac{Ze^2}{r^2} - \frac{\hbar^2}{m\sin^2\theta\, r^3}\right)\vec{u} - \frac{\hbar^2\cos\theta}{mr^3\sin\theta}\vec{v} .$$

The first term in the $\vec{u}$ direction is again just equal and opposite to the coulomb force. The last two terms can be written

$$-\frac{\hbar^2}{mr^3\sin^3\theta}\left(\sin\theta\,\vec{u} + \cos\theta\,\vec{v}\right) .$$

The absolute magnitude of this is

$$\frac{\hbar^2}{mr^3\sin^3\theta} .$$

If we note that the angular momentum $p_\varphi = m\dot{\varphi}\rho^2$, and that the centrifugal force $F_c = m\dot{\varphi}^2\rho$, then we may eliminate $\dot{\varphi}$ and write

$$F_c = \frac{p_\varphi^2}{m\rho^3} ,$$

where ρ is the radius of rotation. In this particular case $p_\varphi = \pm\hbar$ and $F_c = \frac{\hbar^2}{m\rho^3}$. Thus we see that the last two terms of the "quantum-mechanical" force can be interpreted as composing a force equal and opposite to the centrifugal force, provided $r\sin\theta = \rho$.

Fig. 1 shows ρ and the orbit of the electron. This orbit is a circle lying in a plane parallel to the XY plane and a distance

$r\cos\theta$ above or below it. This is a rather surprising result for it says that the electron does not necessarily revolve about the center of coulomb attraction but rather about a statistical ensemble of points at a distance $r\cos\theta$ from the center.

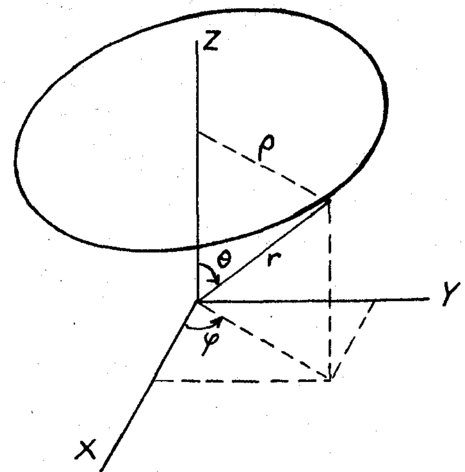

Fig. 1

We have seen that for the states we have considered the "quantum-mechanical" potential is of such a nature that the force derivable from it is in each case equal and opposite to the classical forces acting on the electron. Thus the net force on the electron is zero and the term "stationary state" is very appropriate.

Let us investigate the general form of the "quantum-mechanical" potential for stationary states. For stationary states the wave function has the form

$$\psi(\vec{r},t) = u(\vec{r})\, e^{-iEt/\hbar} . \tag{17}$$

The wave equation then becomes

$$-\frac{\hbar^2}{2m}\nabla^2 u + Vu = Eu . \tag{18}$$

In general $u(\vec{r})$ is complex and may be written in the form

$$u(\vec{r}) = R(\vec{r})\, e^{iS(\vec{r})/\hbar} . \tag{19}$$

Substituting this into the wave equation and equating the real and imaginary parts, we get

$$R\nabla^2 S + 2\nabla S \cdot \nabla R = 0 \tag{20}$$

and

$$\frac{1}{2m}\left[(\nabla S)^2 - \hbar^2 \frac{\nabla^2 R}{R}\right] = E - V. \tag{21}$$

The "quantum-mechanical" potential can be found from the last equation and is

$$U = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = E - V - \frac{1}{2m}(\nabla S)^2. \tag{22}$$

The "quantum-mechanical" force is

$$\begin{aligned} -\nabla U &= \nabla V + \frac{1}{2m}\nabla(\nabla S)^2 \\ &= -F + \frac{1}{2m}\nabla(\nabla S)^2 \end{aligned}$$

where F is the classical force on the electron. If the total force on an electron in a stationary state is the sum of the classical and "quantum-mechanical" forces, then the total force F_T is

$$F_T = -\nabla V - \nabla U = F - F + \frac{1}{2m}\nabla(\nabla S)^2. \tag{23}$$

Since $\nabla S = \vec{p}$, the total force may be written

$$F_T = \frac{1}{2m}\nabla\vec{p}^{\,2} = \frac{1}{2m}\nabla p^2 = \nabla\frac{p^2}{2m} = \nabla T \tag{24}$$

where T is the kinetic energy.

We arrive at the interesting result that in a stationary state the net force on a particle is always equal to the gradient of the kinetic energy. In the case of uniform rotational motion, as was the case with the $\psi_{21\pm1}$ state of the hydrogen-like system, the kinetic energy may be written $T = \frac{m\omega^2 r^2}{2} = \frac{p_\varphi^2}{2mr^2}$, where p_φ is a constant of the motion. Then

$$\nabla T = \frac{p_\varphi^2}{2m} \nabla\left(\frac{1}{r^2}\right) = -m\omega^2 \vec{r}$$

which is the negative of the centrifugal force.

V. Diffraction Problems

When a collimated beam of particles such as electrons is incident upon a narrow slit, the particles passing through the slit would, by classical considerations, produce an image of the slit on a screen such as a photographic film placed behind the slit. Experiment shows, however, that this is not the case but instead a pattern of light and dark fringes is produced on the film very similar to that of an optical diffraction pattern. If a second slit is opened parallel to and near the first slit, the pattern on the screen will be altered. It is possible that some region on the screen that is "illuminated" with only one slit open may become "dark" with both slits open.

The treatment of this problem by quantum mechanics leads to results that correspond to experiment, but one must give up the particle picture in favor of a wave picture. Yet in the final analysis it is the particle that strikes the film and produces an image. So although the quantum mechanical treatment predicts the proper density distribution on the screen, it is no help in understanding how it is possible for the opening of a second slit to influence the behaviour of the particles going through the first slit.

The treatment of the problem by Bohm's interpretation may be more satisfying from a conceptual point of view. To employ this interpretation we must first find the exact wave function from which we can find the "quantum-mechanical" potential. Using this potential in a classical equation of motion we can determine the motion of particles coming through the slit. The difficulty in carrying out this procedure is, however, that the exact wave function cannot be found, but only approximations which are good at some distance from the slit. But if we wish to determine the trajectory of a particle we must know the exact force on it at every instant of time, and hence we must know the exact wave function for all points from the slit to screen.

However, let us assume that we have an exact wave function ψ_A for a beam of particles passing through a slit. We can then find a "quantum-mechanical" potential, U_A, which is exact at all points. Now if we open a second slit at a distance d from the first, the wave function for the system becomes $\psi = \psi_A + \psi_B$ where ψ_B is the wave function when only the second slit is open. Although the wave functions are additive, the total "quantum-mechanical" potential is not the sum $U_A + U_B$ since the total "quantum-mechanical" potential depends on the absolute value of ψ. But

$$|\psi| = |\psi_A + \psi_B| \neq |\psi_A| + |\psi_B| .$$

Instead

$$|\psi_A + \psi_B| = \left\{ |\psi_A|^2 + |\psi_B|^2 + 2|\psi_A|\cdot|\psi_B| \cos(\psi_A, \psi_B) \right\}^{1/2}.$$

Hence the "quantum-mechanical" potential for the system of two slits will contain interaction terms between the two slits which will account for the influence of the second slit on the diffraction of the first slit.

VI. Potential Barrier Problems

The classical treatment of the problem of a free particle of energy E approaching a region of potential V gives as its solution the result that if V is greater than E the particle will be totally reflected, and if V is less than E the particle will be totally transmitted, but with energy E-V. Experiment shows that electrons and other elementary particles do not behave in this manner. It is found that when the potential barrier has a height V greater than E a certain fraction of particles will penetrate to any finite depth. If the depth of the barrier is finite, those penetrating to that depth will be transmitted, while all others will be reflected. If the barrier is infinite in depth all particles will eventually be reflected. And it is found that when the potential barrier has a height V less than E that a certain fraction of the particles approaching the barrier will be reflected and the remainder will be transmitted with energy E-V. Obviously classical considerations do not agree with experiment. The treatment of the problem by quantum mechanics, however, leads to results that agree very well with experiment. But in making use of the quantum-mechanical treatment we are forced to give up completely the particle-picture and treat the particle as if it were a wave. The only justification for this is that it gives results that agree with experiment. But it is certainly not a satisfactory treatment from a conceptual point of view.

If Bohm's interpretation is employed for this problem we are assured of results that agree with experiment. But we do not have to give up the particle picture, for the "wave-function", ψ, no longer needs to be interpreted as dealing with a wave. It is now simply a field from which is derived the "quantum-mechanical" force on the particle.

We shall apply this interpretation to a number of potential barrier problems.

<u>Case 1</u>.

Let us consider first the case of a stream of particles of mass m and energy E directed toward a potential barrier of width <u>a</u> and of height $V > E$, as shown in Fig. 2.

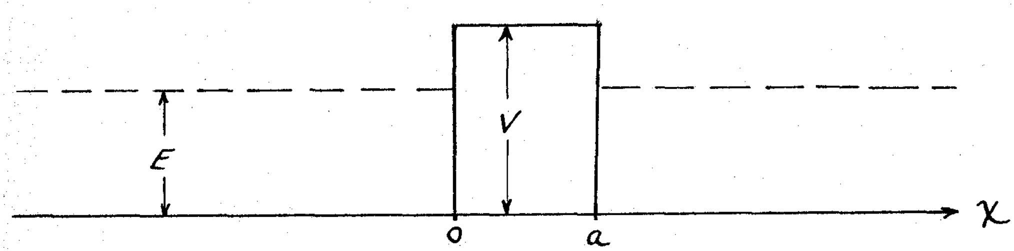

Fig. 2

Following the usual procedure we set down the ψ-functions in terms of constants which must be evaluated by the boundary conditions:

$$\left.\begin{array}{lll} \text{For } x<0 & \psi_1(x) = De^{ik_1x} + Ee^{-ik_1x} \\ \text{For } 0<x<a & \psi_2(x) = Be^{-ik_2x} + Ce^{ik_2x} \\ \text{For } x>0 & \psi_3(x) = Ae^{ik_1x} \end{array}\right\} \quad (25)$$

where $k_1 = \sqrt{2mE}$ and $k_2 = \sqrt{2m(E-V)}$.

Since one of the five constants is arbitrary, let us set $D = 1$. Then

A, B, C, E are complex constants to be determined by the boundary conditions. Let us write

$$\left.\begin{aligned} A &= a_1 + i a_2 \\ B &= b_1 + i b_2 \\ C &= c_1 + i c_2 \\ E &= e_1 + i e_2 \end{aligned}\right\} \quad (26)$$

Then

$$\left.\begin{aligned} \psi_1(x) &= e^{ik_1 x} + (e_1 + i e_2) e^{-ik_1 x} \\ \psi_2(x) &= (b_1 + i b_2) e^{-ik_2 x} + (c_1 + i c_2) e^{ik_2 x} \\ \psi_3(x) &= (a_1 + i a_2) e^{ik_1 x} \end{aligned}\right\} \quad (27)$$

We wish to find the "quantum-mechanical" potential in region 1, where $x<0$, given by

$$U_1(x) = -\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_1|}{|\psi_1|} .$$

The absolute value of $\psi_1(x)$ is

$$|\psi_1| = |(1+e_1)\cos k_1 x + e_2 \sin k_1 x + i(1-e_1)\sin k_1 x + i e_2 \cos k_1 x|$$

$$= \left\{[(1+e_1)\cos k_1 x + e_2 \sin k_1 x]^2 + [(1-e_1)\sin k_1 x + e_2 \cos k_1 x]^2\right\}^{1/2}$$

$$= \left\{(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1 x + 2e_2 \sin 2k_1 x\right\}^{1/2} .$$

The gradient of $|\psi_1|$ is

$$\nabla|\psi_1| = \frac{d|\psi_1|}{dx} = -\frac{2e_1k_1 \sin 2k_1x - 2e_2k_1 \cos 2k_1x}{\{(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1x + 2e_2 \sin 2k_1x\}^{1/2}} .$$

The Laplacian of $|\psi_1|$ is

$$\nabla\cdot\nabla|\psi_1| = \frac{d^2|\psi_1|}{dx^2} = -\frac{4k_1^2[-e_1 \sin 2k_1x + e_2 \cos 2k_1x]^2}{\{(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1x + 2e_2 \sin 2k_1x\}^{3/2}}$$

$$-\frac{4k_1^2[e_1 \cos 2k_1x + e_2 \sin 2k_1x]}{\{(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1x + 2e_2 \sin 2k_1x\}^{1/2}} .$$

Upon simplifying and multiplying by $-\frac{\hbar^2}{2m}$ we obtain for the "quantum-mechanical" potential:

$$U_1(x) = \frac{\hbar^2k_1^2}{2m}\left\{\frac{4e_1^2+4e_2^2-(1+e_1^2+e_2^2)^2}{[(1+e_1^2+e_2^2)+2e_1 \cos 2k_1x + 2e_2 \sin 2k_1x]^2} + 1\right\} \qquad (28)$$

Let us now find the "quantum-mechanical" potential in region 2, $o<x<a$, given by

$$U_2(x) = -\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_2|}{|\psi_2|} .$$

In order to find the absolute value of ψ_2 we must note that since $E-V$ is negative k_2 will be an imaginary quantity and hence $e^{\pm ik_2x}$ is a real quantity. Thus

$$|\psi_2| = \left\{[b_1e^{-ik_2x} + c_1e^{ik_2x}]^2 + [b_2e^{-ik_2x} + c_2e^{ik_2x}]^2\right\}^{1/2}$$

$$= \left\{(b_1^2+b_2^2)e^{-2ik_2x} + (c_1^2+c_2^2)e^{2ik_2x} + 2(c_1b_1+c_2b_2)\right\}^{1/2} ..$$

The gradient of $|\psi_2|$ is

$$\nabla|\psi_2| = \frac{d|\psi_2|}{dx} = \frac{-ik_2(b_1^2+b_2^2)e^{-2ik_2x}+ik_2(c_1^2+c_2^2)e^{2ik_2x}}{\left\{(b_1^2+b_2^2)e^{-2ik_2x}+(c_1^2+c_2^2)e^{2ik_2x}+2(c_1b_1+c_2b_2)\right\}^{1/2}} .$$

The Laplacian of $|\psi_2|$ is

$$\nabla\cdot\nabla|\psi_2| = \frac{d^2|\psi_2|}{dx^2} = -\frac{\left[-ik_2(b_1^2+b_2^2)e^{-2ik_2x}+ik_2(c_1^2+c_2^2)e^{2ik_2x}\right]^2}{\left\{(b_1^2+b_2^2)e^{-2ik_2x}+(c_1^2+c_2^2)e^{2ik_2x}+2(c_1b_1+c_2b_2)\right\}^{3/2}}$$
$$-\frac{2k_2^2(b_1^2+b_2^2)e^{-2ik_2x}+2k_2^2(c_1^2+c_2^2)e^{2ik_2x}}{\left\{(b_1^2+b_2^2)e^{-2ik_2x}+(c_1^2+c_2^2)e^{2ik_2x}+2(c_1b_1+c_2b_2)\right\}^{1/2}} .$$

Upon simplifying and multiplying by $-\frac{h^2}{2m}$ we obtain for the "quantum-mechanical" potential

$$U_2(x) = \frac{\hbar^2k_2^2}{2m}\left\{\frac{4(c_1b_2-c_2b_1)^2}{\left[(b_1^2+b_2^2)e^{-2ik_2x}+(c_1^2+c_2^2)e^{2ik_2x}+2(c_1b_1+c_2b_2)\right]^2}+1\right\} \quad (29)$$

In region 3, $x > a$, it is immediately seen that $|\psi_3| = A =$ constant and hence $U_3(x) = 0$.

Let us now consider the boundary conditions. Since the ψ-function and its first derivative must everywhere be continuous, this requires that at $x = 0$

$$1+E = B+C$$
$$ik_1(1-E) = -ik_2(B-C) ,$$

and at $x = a$

$$Be^{-ik_2a}+Ce^{ik_2a} = Ae^{ik_1a}$$
$$-ik_2(Be^{-ik_2a}-Ce^{ik_2a}) = ik_1Ae^{ik_1a} .$$

Eliminating A from the last two, we get

$$i(k_1+k_2)\,Be^{-ik_2 x} + i(k_1-k_2)\,Ce^{ik_2 a} = 0.$$

Eliminating E from the first two, we get

$$2ik_1 = i(k_1-k_2)B + i(k_1+k_2)C\,.$$

Solving these last two for B, we get

$$B = \frac{k_1(k_1-k_2)\,e^{ik_2 a}}{(k_1^2+k_2^2)\sinh ik_2 a - 2k_1k_2\cosh ik_2 a} \tag{30}$$

Then

$$C = \frac{-k_1(k_1+k_2)\,e^{-ik_2 a}}{(k_1^2+k_2^2)\sinh ik_2 a - 2k_1k_2\cosh ik_2 a} \tag{31}$$

And

$$E = \frac{(k_1^2-k_2^2)\sinh ik_2 a}{(k_1^2+k_2^2)\sinh ik_2 a - 2k_1k_2\cosh ik_2 a} \tag{32}$$

In order to simplify the analysis of the problem we shall consider a specific example. Let us choose E = 9 electron volts, V = 13 ev., and a = 10^{-8} cm. Then

$$k_1 = \sqrt{\frac{2m}{\hbar^2}E} = 1.5 \times 10^{8}\ cm^{-1}$$

$$ik_2 = i\sqrt{\frac{2m}{\hbar^2}(E-V)} = -1.0 \times 10^{8}\ cm^{-1}$$

Substituting these values of k_1, k_2, and a into the expressions for B, C, and E, we get

$$B = 0.0567 + i\,0.197$$

$$C = 1.178 - i\,0.945$$

$$E = 0.235 - i\,0.748$$

For this example the "quantum-mechanical" potentials are

$$U_1(x) = 9\left\{\frac{-0.15}{[1.614 + 0.470\cos(3\times10^8 x) - 1.496\sin(3\times10^8 x)]^2} + 1\right\}\ ev.$$

and

$$U_2(x) = -4\left\{\frac{0.326}{[0.042\,e^{2\times10^8 x} + 2.278\,e^{-2\times10^8 x} - 0.238]^2} + 1\right\}\ ev.$$

In Table 1 we have tabulated U versus x, and in the graph of Fig. 3 is plotted the net potential, V + U, versus x. This plot shows that at no point is the net potential greater than E. A particle moving from left to right, obeying a classical equation of motion would alternately slow down and speed up as it passed the high and low regions of the net potential. As it passed through the "barrier" its velocity would increase from a small value at x = 0 to the value $v = \sqrt{2mE}$ at x = a and remain constant thereafter. There really is no longer a barrier since the "quantum-mechanical" potential cancelled just enough of the classical potential to bring the net potential below the energy of the particles. Furthermore this is not just an accident due to the way we chose the numbers in this particular example. It is inherent in the way U depends on V through the ψ-function.

TABLE I

$$U_1(x) = \frac{h^2 k_1^2}{2m}\left[\frac{-.15}{(1.614 + 0.470\cos 3\times10^8 x - 1.496\sin 3\times10^8 x)^2} + 1\right]$$

$$U_2(x) = -\frac{h^2 k_2^2}{2m}\left[\frac{0.326}{(0.042e^{2\,10^8 x} + 2.278e^{-2\,10^8 x} - 0.238)^2} + 1\right]$$

x - cm.	$U_1(x)$-(ev.)	x - cm.	$U_2(x)$-ev.
$-\frac{2\pi}{3}\times10^{-8}$	+8.70	0	-4.29
$-\frac{7\pi}{12}\times10^{-8}$	+7.29	$\frac{a}{4} = \frac{1}{4}\times10^{-8}$	-4.88
$-\frac{\pi}{2}\times10^{-8}$	-88.1	$\frac{a}{2} = \frac{1}{2}\times10^{-8}$	-6.55
$-\frac{5\pi}{12}\times10^{-8}$	-17.9	$\frac{3a}{4} = \frac{3}{4}\times10^{-8}$	-10.2
$-\frac{\pi}{3}\times10^{-8}$	+ 7.97	$a = 1\times10^{-8}$	-13.0
$-\frac{\pi}{4}\times10^{-8}$	+ 8.75		
$-\frac{\pi}{6}\times10^{-8}$	+ 8.86		
$-\frac{\pi}{12}\times10^{-8}$	+ 8.85		
0	+ 8.70		

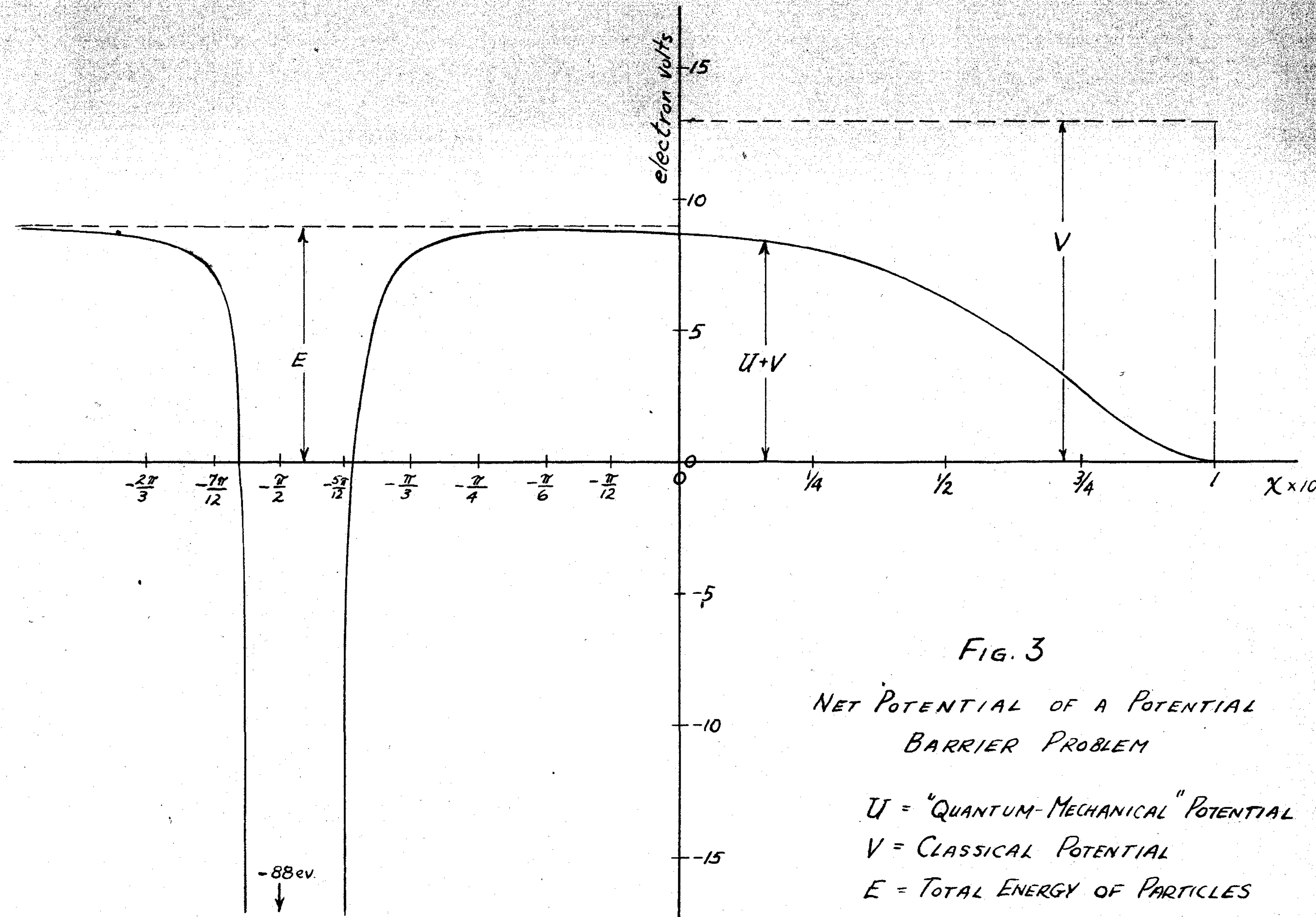

FIG. 3

NET POTENTIAL OF A POTENTIAL BARRIER PROBLEM

U = "QUANTUM-MECHANICAL" POTENTIAL
V = CLASSICAL POTENTIAL
E = TOTAL ENERGY OF PARTICLES

This interpretation has enabled us to keep the particle-picture and still allow a particle to penetrate a potential barrier. But it tells us nothing about any reflection at $x = 0$. Does this interpretation then give us less information than the usual interpretation? If we consider that the only physically measurable quantity in such an experiment is the particle density, then we find that both interpretations give identical results, as indeed we have seen they must. The coefficients of reflection and transmission generally discussed in the usual treatment of potential barrier problems are not physically measurable quantities.

Suppose the electron beam has an intensity of N electrons per second. Then

$$N = \rho v \tag{33}$$

where $\rho = \rho(x)$ is the electron density or number of electrons per unit length of beam at x, and $v = v(x)$ is the velocity of the electrons at x. ρdx is proportional to the probability of finding an electron in dx at x. If ρ_1 and ρ_2 are the densities at two different points along the beam, then

$$\frac{\rho_1}{\rho_2} = \frac{P_1}{P_2} \tag{34}$$

where P_1 and P_2 are the probabilities of finding an electron in dx at the same two points. Also, from conservation of electrons, we have

$$\rho_1 v_1 = \rho_2 v_2 \quad .$$

Hence

$$\frac{\rho_1}{\rho_2} = \frac{v_2}{v_1} = \frac{P_1}{P_2} = \frac{|\psi_1|^2}{|\psi_2|^2} \quad .$$

where $|\psi_1|^2$ and $|\psi_2|^2$ are the probability densities as obtained from the usual interpretation of quantum mechanics.

We have here a relationship between the velocities and probability densities. Let us see if this holds for the problem just considered in the region $x < 0$. We must show then that

$$\frac{v(x_2)}{v(x_1)} = \frac{|\psi_1(x_1)|^2}{|\psi_1(x_2)|^2} \quad . \tag{35}$$

First let us find $v(x)$. The equation of motion of a particle in the region $x < 0$ is

$$m\frac{d^2x}{dt^2} = F(x)$$

where $F(x)$ is the force on the particle as derived from the sum of the classical and "quantum-mechanical" potentials which in this case, since $V(x) = 0$ in $x < 0$, is simply the "quantum-mechanical" potential, $U(x)$. In order to integrate the equation of motion let $v = \frac{dx}{dt}$. Then $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$, and

$$v\frac{dv}{dx} = \frac{1}{m}F(x).$$

Integrating, we have

$$\int_{x_0}^{x} v\,dv = \frac{1}{m}\int_{x_0}^{x} F(x')dx'$$

$$\frac{1}{2}\left[v(x)^2 - v(x_0)^2\right] = \frac{1}{m}\int_{x_0}^{x} -\nabla U_1(x')dx' = -\frac{1}{m}U_1(x')\Big|_{x_0}^{x}$$

$$= -\frac{1}{m}U_1(x) + \frac{1}{m}U_1(x_0)$$

Or

$$v(x)^2 - v(x_0)^2 = \frac{2}{m}\left[U_1(x_0) - U_1(x)\right] .$$

If x_o is chosen so that $U_1(x_o) = 0$, then the velocity at x_o will be $v(x_o) = \sqrt{2mE} = \hbar k_1$. Solving for $v(x)$ we have then

$$v(x) = \frac{1}{m}\sqrt{\hbar^2 k_1^2 - 2m U_1(x)} \, . \tag{36}$$

Recalling Eq. (28) for $U_1(x)$, Eq. (36) becomes

$$v(x) = \frac{\hbar k_1}{m} \frac{\left[-4e_1^2 - 4e_2^2 + (1+e_1^2+e_2^2)^2\right]^{1/2}}{\left[(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1 x + 2e_2 \sin 2k_1 x\right]} \, .$$

Then

$$\frac{v(x_2)}{v(x_1)} = \frac{\left[(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1 x_1 + 2e_2 \sin 2k_1 x_1\right]}{\left[(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1 x_2 + 2e_2 \sin 2k_1 x_2\right]} \, .$$

We recall also that

$$|\psi_1| = \left[(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1 x + 2e_2 \sin 2k_1 x\right]^{1/2}$$

and therefore

$$\frac{|\psi_1(x_1)|^2}{|\psi_1(x_2)|^2} = \frac{\left[(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1 x_1 + 2e_2 \sin 2k_1 x_1\right]}{\left[(1+e_1^2+e_2^2) + 2e_1 \cos 2k_1 x_2 + 2e_2 \sin 2k_1 x_2\right]}$$

which is identical to $v(x_2)/v(x_1)$ above. Hence we have shown that relationship Eq. (35) holds in this problem, indicating that the two interpretations give the same physical results.

<u>Case 2.</u>

Let us consider now the same problem but with the barrier of infinite extent. That is, let $a \to \infty$, as shown in Fig. 4.

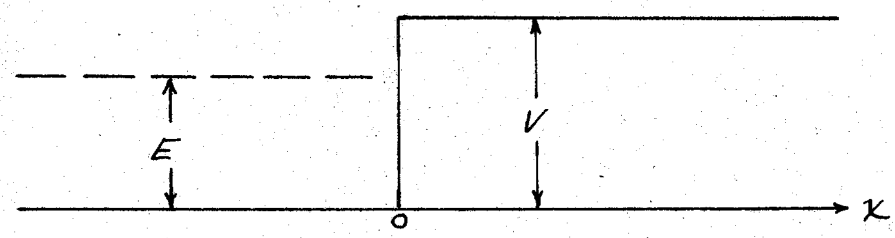

Fig. 4

The ψ-functions can be written

$$\text{for } x<0 \qquad \psi_1(x) = e^{ik_1 x} + E e^{-ik_1 x}$$
$$\text{for } x>0 \qquad \psi_2(x) = C e^{ik_2 x} \tag{37}$$

where k_1 and k_2 are defined as before. Imposing the boundary conditions, we find that

$$E = \frac{k_1-k_2}{k_1+k_2}, \quad C = \frac{2k_1}{k_1+k_2}. \tag{38}$$

Since we have used the same notation as in the previous problem, we may use Eq. (28) to find $U_1(x)$ and Eq. (29) for $U_2(x)$, where

$$e_1 = \frac{k_1^2+k_2^2}{k_1^2-k_2^2} \qquad c_1 = \frac{2k_1^2}{k_1^2-k_2^2} \qquad b_1 = 0$$

$$e_2 = -i\frac{2k_1k_2}{k_1^2-k_2^2} \qquad c_2 = -i\frac{2k_1k_2}{k_1^2-k_2^2} \qquad b_2 = 0 .$$

Since $e_1^2 + e_2^2 = 1$

$$U_1(x) = \frac{\hbar^2 k_1^2}{2m} = E \tag{39}$$

and since $b_1 = b_2 = 0$

$$U_2(x) = \frac{\hbar^2 k_2^2}{2m} = E - V . \tag{40}$$

Consequently the net potential is everywhere equal to E, as shown in Fig. 5. This means that a particle with total energy $E < V$, no matter

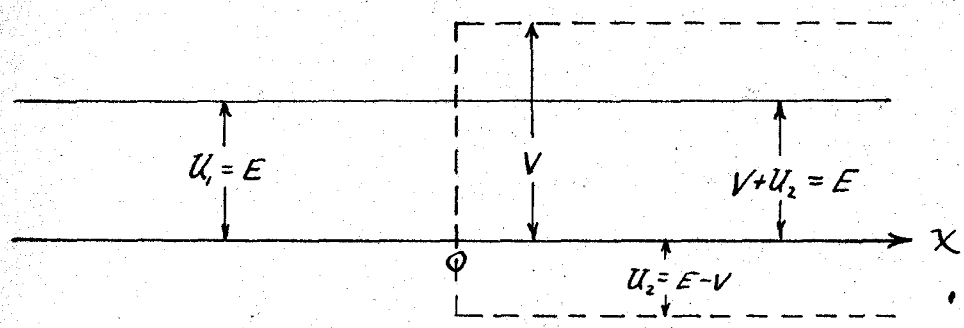

Fig. 5

how far from the barrier to the left, will have zero kinetic energy and hence will be at rest. This would seem to indicate that the influence of the potential barrier is felt all the way to $-\infty$. When considering only a single particle such a phenomenon is contrary to our experience. But when a beam of particles is directed at the barrier, information about the barrier is obtained by the approaching particles from the reflected particles. This result really corresponds to the fact already mentioned that for an infinite barrier of this type all the particles are reflected and the net flow of particles, or net velocity, is everywhere zero.

<u>Case 3</u>.

We have found that the Bohm treatment gives us no information about reflection in the problems already treated. If we consider a wave-packet instead of a beam of particles approaching the barrier, we are certain to observe the splitting of the wave packet into reflected and transmitted components. It would be interesting to see if and how the Bohm interpretation accounts for this reflection.

For simplicity we shall consider an infinitely high but infinitely thin barrier. This type of barrier depends on one parameter only (the area under the curve) and thus leads to a more straightforward treatment[8]. Thus the potential energy function can be written

$$V(x) = \gamma\,\delta(x) \tag{41}$$

where

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

is the Dirac delta function, and has the property that

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

γ is the parameter that determines the area under the potential energy function.

To further simplify the problem we shall employ atomic units[9]. In these units the Schroedinger equation is written

$$-\frac{1}{2}\nabla^2\psi + V\psi = i\frac{\partial\psi}{\partial t}\ .$$

In our case this becomes

$$-\frac{1}{2}\psi'' = -\gamma\psi\,\delta(x) + i\dot{\psi}\ . \tag{42}$$

Since the ψ-function must be everywhere continuous it is necessary that

$$\psi(0-) = \psi(0+) = \psi(0)\ . \tag{43}$$

This is one boundary condition on ψ. Another is found by integrating the Schroedinger equation (42) within a small interval about the origin.

$$-\frac{1}{2}\int_{0-}^{0+}\psi''\,dx = -\gamma\int_{0-}^{0+}\psi\,\delta(x)\,dx + i\int_{0-}^{0+}\dot{\psi}\,dx$$

$$-\frac{1}{2}\left[\psi'(0+) - \psi'(0-)\right] = -\gamma\psi(0)\ ,$$

The last integral being zero because ψ is continuous. The second boundary condition is thus

$$\psi'(0+) - \psi'(0-) = 2\gamma\,\psi(0). \tag{44}$$

This tells us that there is a discontinuity in the derivative of ψ at the origin.

Since the Schroedinger equation (42) is a free particle equation everywhere except at x = 0, the problem consists of finding a solution of the free particle problem with a break in the derivative at x = 0 as given by (44). Such a solution may be written as

$$\psi(x,t) = \psi_1(x,t) + \Delta(x)\,\psi_2(x,t) \tag{45}$$

where

$$\Delta(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

and both $\psi_1(x,t)$ and $\psi_2(x,t)$ are solutions of the free particle problem. That is, they satisfy the Schroedinger equation

$$-\tfrac{1}{2}\,\psi_k''(x,t) = i\,\dot{\psi}_k(x,t)\quad,\quad k = 1,2 \tag{46}$$

In order that $\psi(x,t)$ satisfy the boundary conditions given in (43) and (44) it is necessary that

$$\psi_2(0,t) \equiv 0 \tag{47a}$$

$$\psi_2'(0,t) \equiv -2\gamma\,\psi_1(0,t) \tag{47b}$$

If a function $\Psi(x,t)$ is a continuous solution of (46), then $\Psi(-x,t)$ is also. The odd part of $\Psi(x,t) = \frac{1}{2}\left[\Psi(x,t) - \Psi(-x,t)\right]$ and

$$\psi_2(x,t) = k\left[\Psi(x,t) - \Psi(-x,t)\right] \tag{48a}$$

will satisfy (47a). Then

$$\psi_2'(x,t) = k\left[\Psi'(x,t) + \Psi'(-x,t)\right]$$

$$\psi_2'(0,t) = 2k\,\Psi'(0,t) = -2\gamma\,\psi_1(0,t)$$

by (47b). We may take

$$\psi_1(x,t) = \Psi'(x,t) \tag{48b}$$

provided we set $k = -\gamma$, since $\Psi'(x,t)$ is also a continuous solution of (46). Thus we have

$$\left.\begin{aligned} \psi_1(x,t) &= \Psi'(x,t) \\ \psi_2(x,t) &= -\gamma\left[\Psi(x,t) - \Psi(-x,t)\right] \end{aligned}\right\} \tag{48}$$

and hence

$$\psi(x,t) = \Psi'(x,t) - \Delta(x)\,\gamma\left[\Psi(x,t) - \Psi(-x,t)\right] \tag{49}$$

is a solution of our problem, where $\Psi(x,t)$ is an arbitrary continuous solution of the free particle Schroedinger equation.

The particular choice of $\Psi(x,t)$ that we make depends upon the initial conditions that we desire. A convenient choice for our purposes is a Gaussian function:

$$\Psi(x,t) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{1+it}} \exp\left\{-\frac{(x-x_0-vt)^2}{2(1+it)} + i(x-x_0)v - i\frac{v^2}{2}t\right\} \qquad (50)$$

where x_0 is the initial position of the peak of the function, and v is the velocity of propagation of the peak.

Let us denote for $x<0$

$$\psi(x,t) = \psi_L(x,t) = \Psi'(x,t) - \gamma\left[\Psi(x,t) - \Psi(-x,t)\right]$$

and for $x>0$

$$\psi(x,t) = \psi_R(x,t) = \Psi'(x,t) .$$

Then

$$\psi_L(x,t) = \frac{1}{\pi^{1/4}\sqrt{1+it}}\left[\frac{x_0-x+iv}{1+it} - \gamma\right] \exp\left\{-\frac{(x-x_0-vt)^2}{2(1+it)} + i(x-x_0)v - i\frac{v^2}{2}t\right\}$$

$$+ \gamma \frac{1}{\pi^{1/4}\sqrt{1+it}} \exp\left\{-\frac{(x+x_0-vt)^2}{2(1+it)} - i(x+x_0)v - i\frac{v^2}{2}t\right\} \qquad (51)$$

and

$$\psi_R(x,t) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{1+it}}\left[\frac{iv-x+x_0}{1+it}\right] \exp\left\{-\frac{(x-x_0-vt)^2}{2(1+it)} + i(x-x_0)v - i\frac{v^2}{2}t\right\} \qquad (52)$$

We are now ready to find the "quantum-mechanical" potentials. The ψ -functions are so complex here that a direct approach is very tedious and the results difficult to interpret. To simplify the work we shall carry out the calculations for specific values of x_0, v, and γ and at only certain values of x and t.

The "quantum-mechanical" potential in atomic units is

$$U(x,t) = -\frac{\nabla^2|\psi|}{2|\psi|} . \tag{53}$$

Since $|\psi|^2$ is a simpler function to work with than $|\psi|$, we should like to get (53) in terms of $|\psi|^2$. We note that

$$\nabla^2|\psi|^2 = \nabla\cdot\nabla|\psi|^2 = \nabla\cdot 2|\psi|\,\nabla|\psi|$$

$$= 2|\psi|\,\nabla^2|\psi| + 2\,\nabla|\psi|\cdot\nabla|\psi| = 2\left[|\psi|\nabla^2|\psi| + (\nabla|\psi|)^2\right]$$

Therefore

$$\frac{\nabla^2|\psi|}{|\psi|} = \frac{1}{2}\,\frac{\nabla^2|\psi|^2}{|\psi|^2} - \left(\frac{\nabla|\psi|}{|\psi|}\right)^2 .$$

Also

$$\nabla|\psi|^2 = 2|\psi|\,\nabla|\psi|$$

or

$$\frac{\nabla|\psi|}{|\psi|} = \frac{\nabla|\psi|^2}{2|\psi|^2}$$

So we finally get

$$U(x,t) = \frac{1}{2}\left(\frac{\nabla|\psi|^2}{2|\psi|^2}\right) - \frac{\nabla^2|\psi|^2}{4|\psi|^2} . \tag{54}$$

The procedure to be followed now is to calculate the functions $|\psi|^2$, $\nabla|\psi|^2$, and $\nabla^2|\psi|^2$ and evaluate them individually at the desired points. From the tabulated results U(x,t) can be constructed and plotted as a function of x for various times.

Considering first the functions on the left of the origin, we have for the square of the absolute value of ψ_L :

$$|\psi_L|^2 = \psi_L^* \psi_L = \frac{1}{\sqrt{\pi}\sqrt{1+t^2}}\left[\frac{(x_0-x)^2+v^2}{1+t^2} + \gamma^2 - 2\gamma\frac{(x_0-x+vt)}{1+t^2}\right]\exp\left\{-\frac{(x-x_0-vt)^2}{1+t^2}\right\}$$

$$+\frac{\gamma^2}{\sqrt{\pi}\sqrt{1+t^2}}\exp\left\{-\frac{(x+x_0+vt)^2}{1+t^2}\right\}$$

$$+\frac{2\gamma}{\sqrt{\pi}\sqrt{1+t^2}}\left[\left(\frac{x_0-x+vt}{1+t^2}-\gamma\right)\cos\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)\right.$$

$$\left.+\left(\frac{(x_0-x)t-v}{1+t^2}\right)\sin\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)\right]\exp\left\{-\frac{x^2+(x_0+vt)^2}{1+t^2}\right\} \qquad (55)$$

The gradient of $|\psi_L|^2$ is:

$$\nabla^2|\psi_L|^2 = \frac{1}{\sqrt{\pi}\sqrt{1+t^2}}\left[\frac{-2(x_0-x)+2\gamma}{1+t^2} - \frac{2(x-x_0-vt)}{1+t^2}\left(\frac{(x_0-x)^2+v^2-2\gamma(x_0-x+vt)}{1+t^2}+\gamma^2\right)\right]$$

$$\times\exp\left\{-\frac{(x-x_0-vt)^2}{1+t^2}\right\} + \frac{\gamma^2}{\sqrt{\pi}\sqrt{1+t^2}}\left[-\frac{2(x+x_0+vt)}{1+t^2}\right]\exp\left\{-\frac{(x+x_0+vt)^2}{1+t^2}\right\}$$

$$+\frac{2\gamma}{\sqrt{\pi}\sqrt{1+t^2}}\left\{\left[-\frac{1}{1+t^2}\cos\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)-\left(\frac{x_0-x+vt}{1+t^2}-\gamma\right)\left(\frac{-2t(x_0+vt)}{1+t^2}\right)\sin\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)\right.\right.$$

$$\left.-\frac{t}{1+t^2}\sin\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)-\left(\frac{(x_0-x)t-v}{1+t^2}\right)\left(\frac{2t(x_0+vt)}{1+t^2}\right)\cos\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)\right]$$

$$\left.+\left[-\frac{2x}{1+t^2}\right]\left[\left(\frac{(x_0-x-vt}{1+t^2}-\gamma\right)\cos\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)+\left(\frac{(x_0-x)t-v}{1+t^2}\right)\sin\left(-\frac{2tx(x_0+vt)}{1+t^2}\right)\right]\right\}$$

$$\times\exp\left\{-\frac{x^2+(x_0+vt)^2}{1+t^2}\right\}. \qquad (56)$$

The Laplacian of $|\psi_L|^2$ is:

$$\nabla^2|\psi_L|^2 = \frac{1}{\sqrt{\pi}\sqrt{1+t^2}}\left\{\frac{2}{1+t^2} - \frac{4(x-x_0-vt)}{1+t^2}\left(\frac{-2(x_0-x)+2\gamma}{1+t^2}\right)\right.$$

$$+\left[4\left(\frac{x-x_0-vt}{1+t^2}\right)^2 - \frac{2}{1+t^2}\right]\left[\frac{(x_0-x)^2+v^2-2\gamma(x_0-x+vt)}{1+t^2}+\gamma^2\right]\Bigg\} exp\left\{-\frac{(x-x_0-vt)^2}{1+t^2}\right\}$$

$$+\frac{\gamma^2}{\sqrt{\pi}\sqrt{1+t^2}}\left[\frac{-2}{1+t^2}+4\left(\frac{x+x_0+vt}{1+t^2}\right)^2\right] exp\left\{-\frac{(x-x_0-vt)^2}{1+t^2}\right\},$$

$$+\frac{2\gamma}{\sqrt{\pi}\sqrt{1+t^2}} exp\left\{-\frac{x^2+(x_0+vt)^2}{1+t^2}\right\}\cdot\left\{sin\left(\frac{-2tx(x_0+vt)}{1+t^2}\right)\left[\frac{2}{1+t^2}\left(\frac{-3x_0t-2vt^2+vt+v}{1+t^2}\right)\right.\right.$$

$$-\left(\frac{(x_0-x)t-v}{1+t^2}\right)\left(\frac{2t(x_0+vt)}{1+t^2}\right)^2 - \frac{4x}{1+t^2}\left(\frac{x_0-x+vt}{1+t^2}-\gamma\right)\left(\frac{2t(x_0+vt)}{1+t^2}\right)$$

$$\left.+\frac{4xt}{(1+t^2)^2}+\left(\frac{2x}{1+t^2}\right)^2\left(\frac{(x_0-x)t-v}{1+t^2}\right)\right]$$

$$+cos\left(\frac{-2tx(x_0+vt)}{1+t^2}\right)\left[\frac{2t}{1+t^2}\left(\frac{2t(x_0+vt)-(x_0-x+vt)}{1+t^2}+\gamma\right)\right.$$

$$-\left(\frac{x_0-x+vt}{1+t^2}-\gamma\right)\left(\frac{2t(x_0+vt)}{1+t^2}\right)^2+\frac{4x}{1+t^2}\left(\frac{(x_0-x)t-v}{1+t^2}\right)\left(\frac{2t(x_0+vt)}{1+t^2}\right)$$

$$\left.\left.+\frac{4x}{(1+t^2)^2}+\left(\frac{2x}{1+t^2}\right)^2\left(\frac{x_0-x+vt}{1+t^2}-\gamma\right)\right]\right\} \tag{57}$$

These functions have been evaluated and tabulated in Table II for the particular case when $x_0 = -4$, $v = 4$, and $\gamma = 4$, all being expressed in atomic units. These values were found convenient because at $t = 0$ the center of the wave packet is at $x = -4$ and diminishes rapidly to either side so that its amplitude is negligible at the origin, and yet is not so far away from the origin that it spreads out too much before reaching the barrier. With the velocity $v = 4$ the packet moves rapidly enough so that it does not spread out too much by the time $t = 2$. At time $t = 1$ the center of the packet is near the origin and at $t = 2$ the reflected part of the packet is back at $x = -4$ and the transmitted part at $x = 4$. By choosing $\gamma = 4$ it turns out that just half the packet is reflected and half transmitted. This is determined by setting $|\psi_L(0,1)|^2 = |\psi_R(0,1)|^2$, the condition for which is that $\gamma = v$.

In Figs. 6, 7, 8 are plotted $|\psi_L|^2$ and U_L for $t = 0, 1, 2$.

Carrying out the same procedure now for the function to the right of the origin, we have for the square of the absolute value of ψ_R :

$$|\psi_R|^2 = \psi_R^* \psi_R = \frac{(x_0 - x)^2 + v^2}{\sqrt{\pi}(1+t^2)^{3/2}} \exp\left\{-\frac{(x - x_0 - vt)^2}{1+t^2}\right\} \qquad (58)$$

The gradient of $|\psi_R|^2$ is:

$$\nabla|\psi_R|^2 = -\frac{2}{\sqrt{\pi}(1+t^2)^{3/2}} \left\{(x_0 - x) + \frac{(x - x_0 - vt)}{1+t^2}\left[(x_0 - x)^2 + v^2\right]\right\} \exp\left\{-\frac{(x - x_0 - vt)^2}{1+t^2}\right\} \qquad (59)$$

TABLE II

$$U_L = \frac{-\nabla^2|\psi_L|}{2|\psi_L|} = \frac{1}{2}\left(\frac{\nabla|\psi_L|^2}{2|\psi_L|^2}\right)^2 - \frac{\nabla^2|\psi_L|^2}{4|\psi_L|^2}$$

x	$\|\psi_L\|^2$	$\nabla\|\psi_L\|^2$	$\nabla^2\|\psi_L\|^2$	$\frac{\nabla^2\|\psi_L\|^2}{4\|\psi_L\|^2}$	$\frac{1}{2}\left(\frac{\nabla\|\psi_L\|^2}{2\|\psi_L\|^2}\right)^2$	U
t = 0						
0				17.8	1.24	-16.6
-1	.0046	-.02	.14	7.6	2.36	- 5.24
-2	.528	-2.03	6.56	3.11	1.85	- 1.26
-3	8.51	-14.95	9.15	.269	.386	+.117
-4	18.06	4.52	-35.00	-.484	.0078	+.492
-5	5.20	11.63	19.95	.96	.626	-.334
-6	.207	.87	3.25	3.92	2.20	-1.72
-7	.0009	.0054	.042	11.66	4.50	-7.16
-3.9	18.3	1.005	-40.1	-.548	.0004	+.548
-3.5	15.9					
-4.5	17.4					
t = 1						
0	6.39	1.6	-5.98	-.234	.0078	+.242
-1	3.03	3.75	1.69	.14	.191	+.051
-2	.54	1.19	2.11	.98	.607	-.37
-3	.038	.117	.332	2.205	1.217	-.988
-4	.0011	.0043	.0161	3.78	2.0	-1.78
-0.5	4.97			5.74	2.98	-2.76
t = 2						
-1.5	1.162	-.563	.770	.166	.029	-.137
-.5	.398	-1.19	2.868	1.80	1.12	-.68
0	.066	.65	6.05	22.9	12.12	-10.8
-1	.48	-1.05	-1.78	-.927	.60	+1.53
-2	1.72	-1.4	2.07	.301	.083	-.218
-3	3.26	-1.3	-1.335	-.102	.020	+.122
-4	4.07	-.0136	-1.62	-.0995	1.4×10^{-6}	+.10
-5	3.28	1.32	-.795	-.061	.0202	+.081
-6	1.85	1.45	.436	.059	.077	+.018
-7	.66	.8	.695	.261	.184	+.077

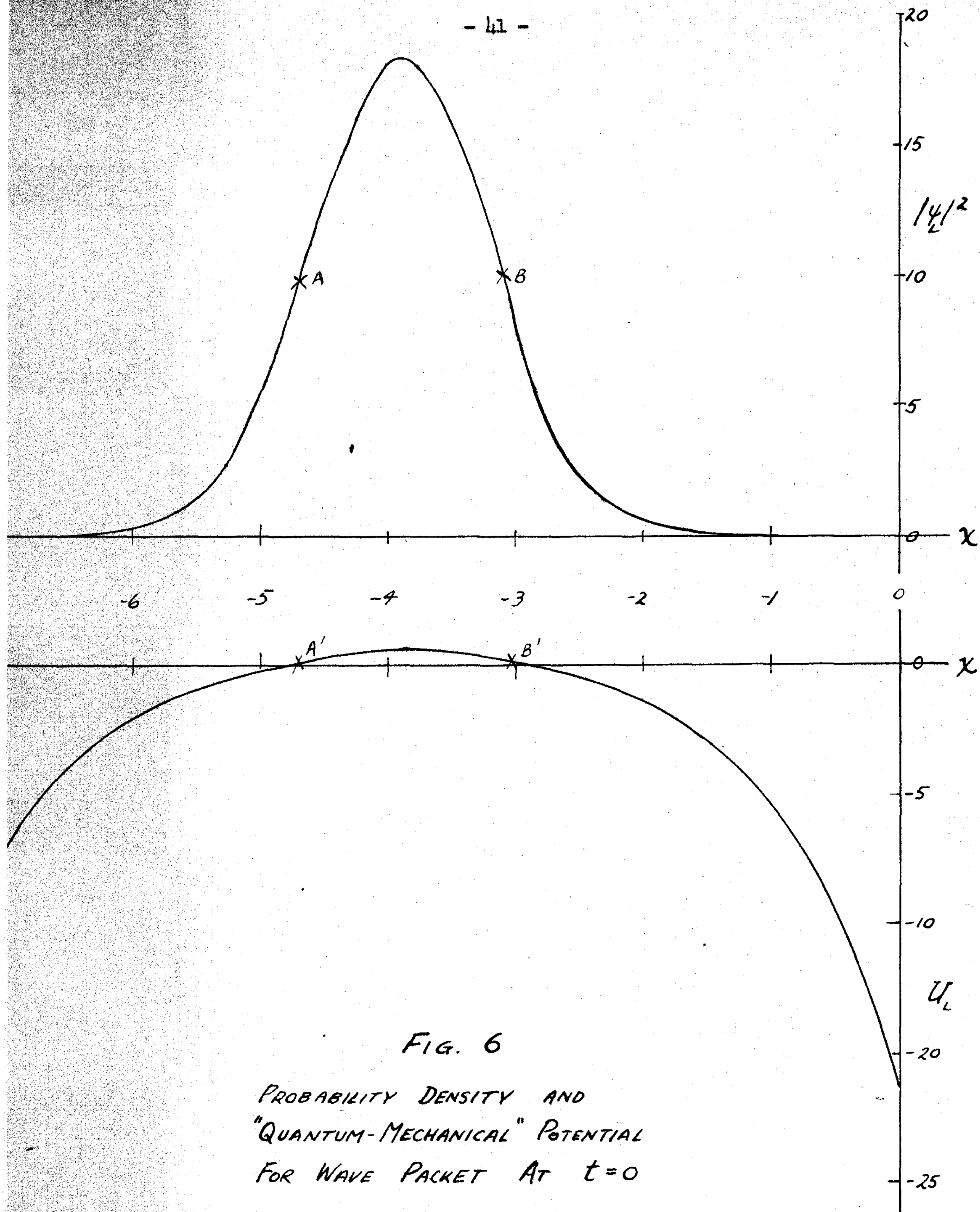

FIG. 6

PROBABILITY DENSITY AND "QUANTUM-MECHANICAL" POTENTIAL FOR WAVE PACKET AT $t=0$

PARAMETERS $x_0=-4$, $\nu=4$, $\gamma=4$

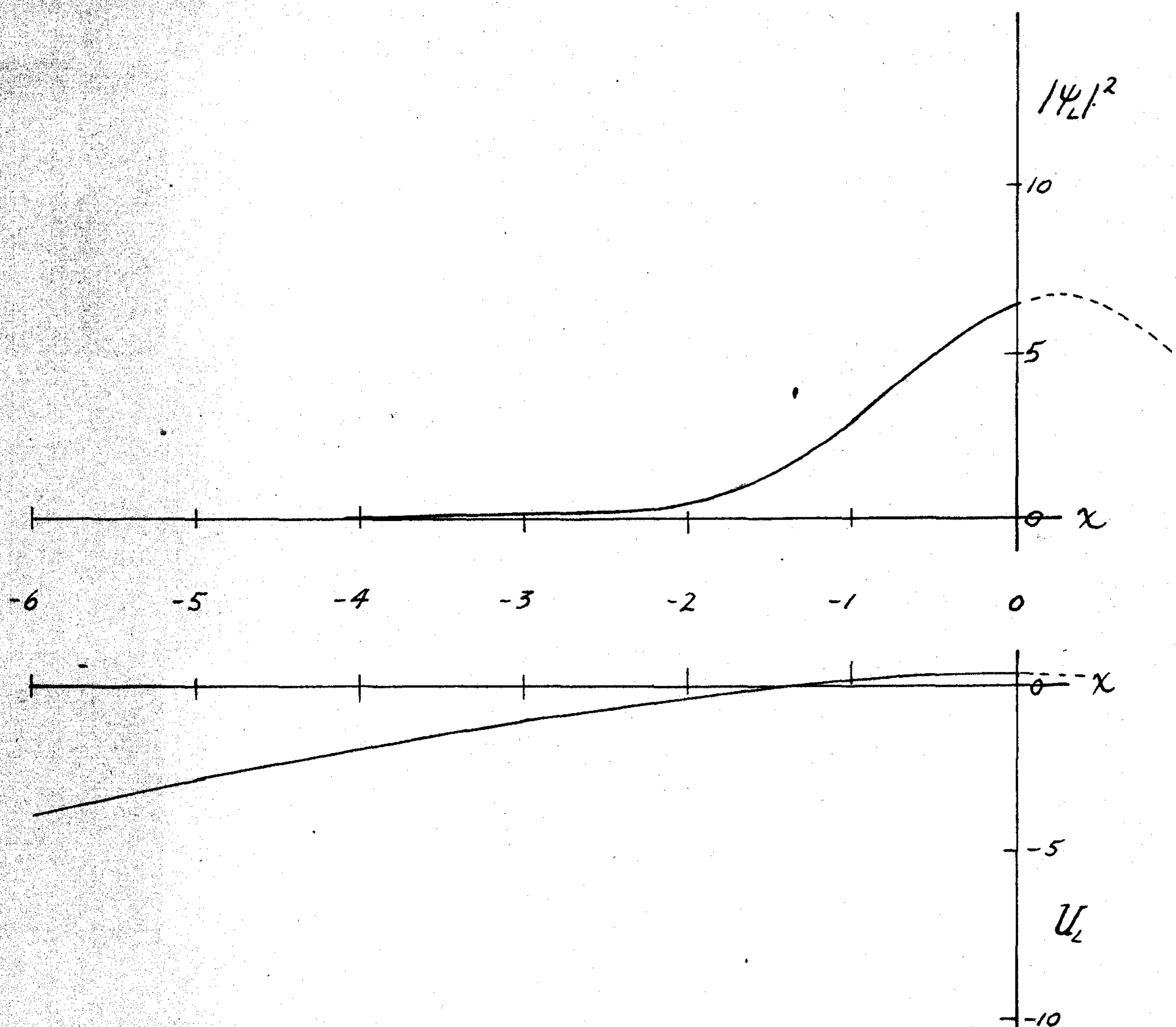

FIG. 7

PROBABILITY DENSITY AND

"QUANTUM-MECHANICAL" POTENTIAL

FOR WAVE PACKET AT $t = 1$

PARAMETERS $x_0 = -4$, $v = 4$, $\gamma = 4$

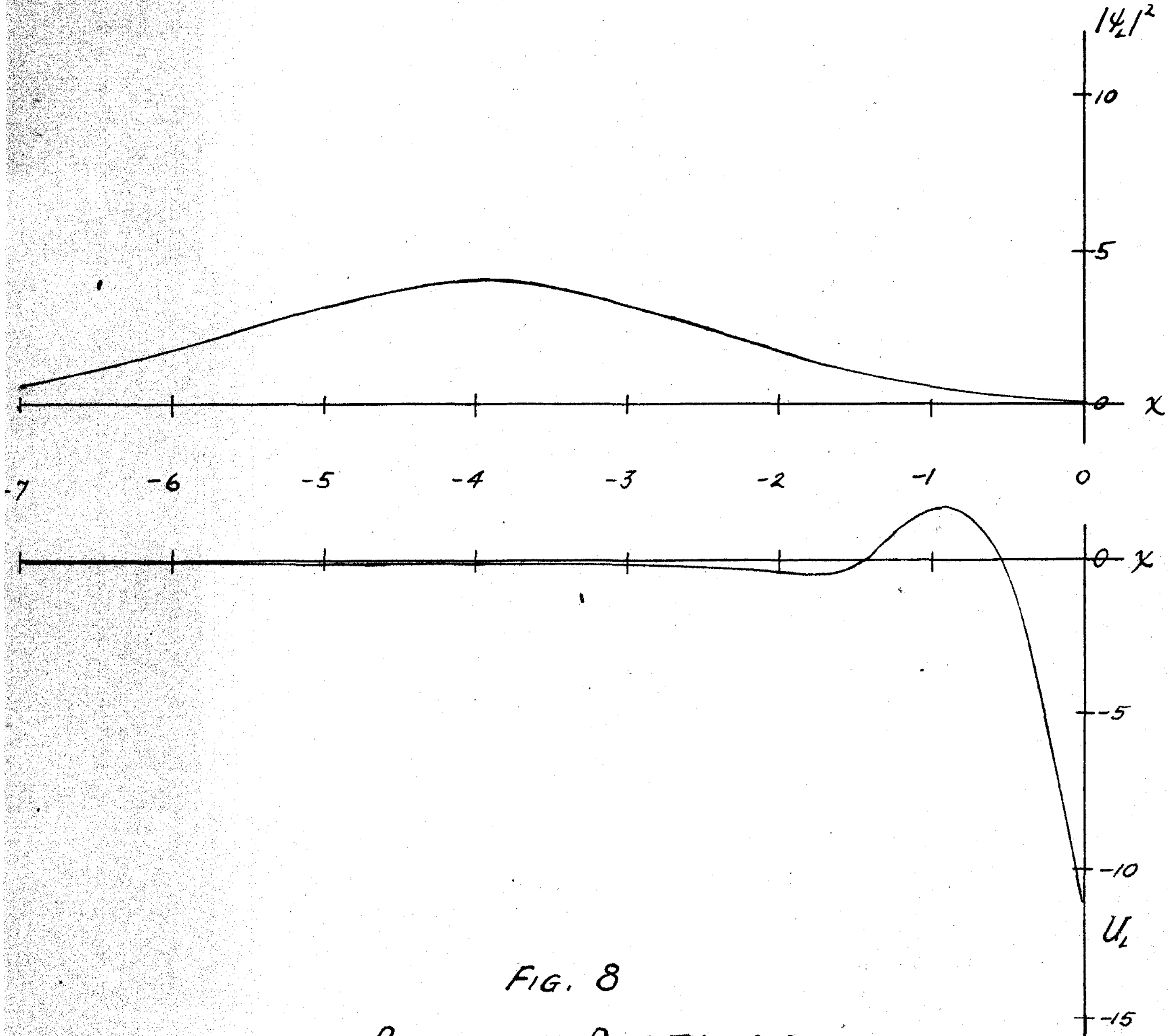

FIG. 8

PROBABILITY DENSITY AND
"QUANTUM-MECHANICAL POTENTIAL
FOR WAVE PACKET AT $t=2$
PARAMETERS $x_0=-4$, $v=4$, $\gamma=4$

The Laplacian of $|\psi_R|^2$ is:

$$\nabla^2|\psi_R|^2 = -\frac{2}{\sqrt{\pi}(1+t^2)^{3/2}}\Big\{-1+\frac{(x_0-x)^2+v^2}{1+t^2}-\frac{4(x_0-x)}{1+t^2}(x-x_0-vt)$$

$$-2\Big[\frac{(x-x_0-vt)}{1+t^2}\Big]^2\Big[(x_0-x)^2+v^2\Big]\Big\}\exp\Big\{-\frac{(x-x_0-vt)^2}{1+t^2}\Big\} \qquad (60)$$

These functions have been evaluated and tabulated in Table III for the same conditions as before, namely $x_0 = -4$, $v = 4$, $\gamma = 4$. In Figs. 9, 10 are plotted $|\psi_R|^2$ and U_R for t = 1,2.

From these figures of the probability density, $|\psi|^2$, and the "quantum-mechanical" potential we see that the wave packet spreads out as it travels along, as expected. We also see that the "quantum-mechanical" potential has a maximum value at the same point as the probability density, and drops off on either side. The shape of the "quantum-mechanical" potential moves along with the wave packet, and becomes flatter as the wave packet spreads out. But we do not yet have any information about the "quantum-mechanical" potential at the origin.

Our problem is to find the value of $U = -\frac{\nabla^2|\psi|}{|\psi|}$ at x = 0, where ψ is defined by equation (45). We shall make use of the boundary conditions as expressed by (47a and b). The derivative of the absolute value of a complex function is

$$\frac{d}{dx}|\psi| = \frac{d}{dx}(\psi^*\psi)^{1/2} = \frac{1}{2}\,\frac{\psi^{*\prime}\psi+\psi^*\psi'}{|\psi|} \,.$$

TABLE III

$$U_R = -\frac{\nabla^2|\psi_R|}{2|\psi_R|} = \frac{1}{2}\left(\frac{\nabla|\psi_R|^2}{2|\psi_R|^2}\right)^2 - \frac{\nabla^2|\psi_R|^2}{4|\psi_R|^2}$$

t = 1

x	$\|\psi_R\|^2$	$\nabla\|\psi_R\|^2$	$\nabla^2\|\psi_R\|^2$	$\frac{\nabla^2\|\psi_R\|^2}{4\|\psi_R\|^2}$	$\frac{1}{2}\left(\frac{\nabla\|\psi_R\|^2}{2\|\psi_R\|^2}\right)^2$	U_R
0	6.39	1.6	-5.98	-.234	.0078	+ .242
1	4.96	-3.75	-2.18	-.110	.0714	+ .181
2	1.41	-2.48	2.97	.527	.386	-.141
3	.144	- .432	.97	1.68	1.13	-.550
4	.0053	- .0202	.0715	3.36	1.81	-1.55
5				4.97	2.90	-2.07

t = 2

x	$\|\psi_R\|^2$	$\nabla\|\psi_R\|^2$	$\nabla^2\|\psi_R\|^2$	$\frac{\nabla^2\|\psi_R\|^2}{4\|\psi_R\|^2}$	$\frac{1}{2}\left(\frac{\nabla\|\psi_R\|^2}{2\|\psi_R\|^2}\right)^2$	U_R
0	.066	.122	.200	.758	.427	-.331
1	.342	.494	.523	.382	.261	-.121
2	1.18	1.22	.765	.162	.134	-.028
3	2.69	1.65	- .099	-.0092	.047	+.056
4	4.04	.808	1.52	-.094	.005	+.099
5	4.01	- .86	1.48	-.092	.0058	+.098
6	2.63	-1.65	.049	-.0047	.0492	+.054
7	1.14	-1.19	.766	.168	.136	-.032
8	.330	- .479	.558	.423	.264	-.159
9	.063	- .117	.186	.738	.431	-.307

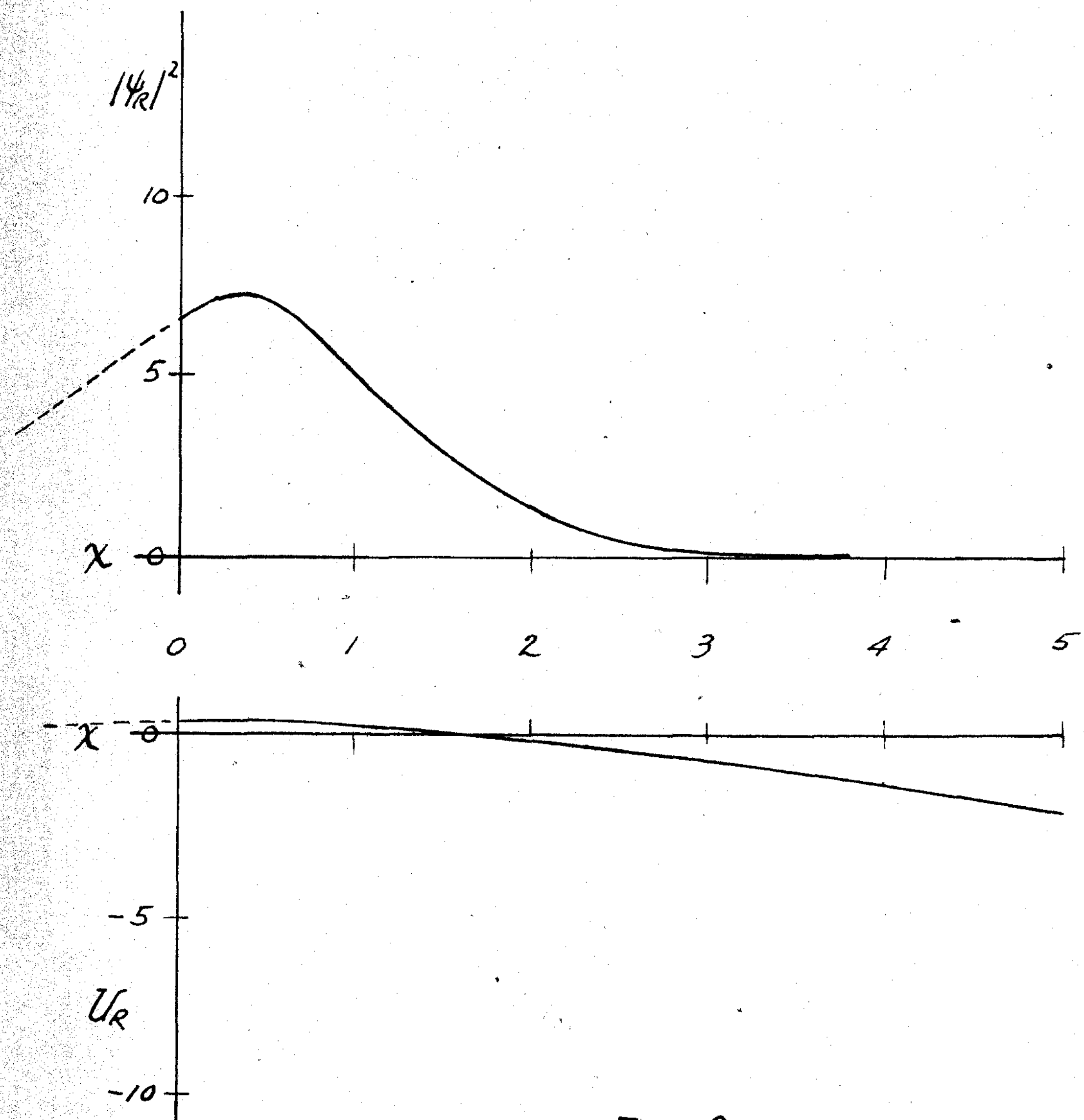

FIG. 9

PROBABILITY DENSITY AND "QUANTUM-MECHANICAL" POTENTIAL FOR WAVE PACKET AT $t = 1$

PARAMETERS $x_0 = -4$, $v = 4$, $\gamma = 4$

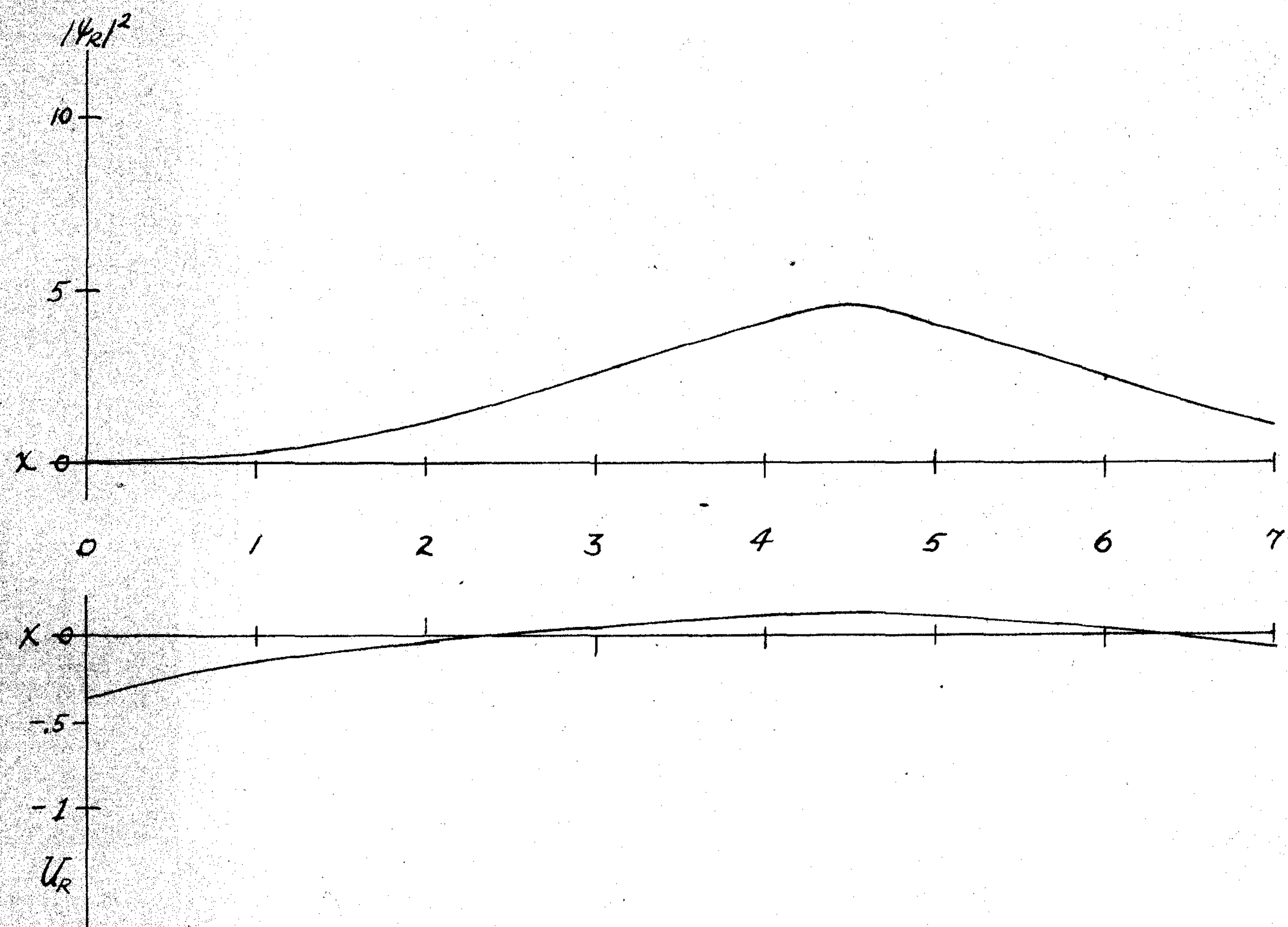

FIG. 10

PROBABILITY DENSITY AND "QUANTUM-MECHANICAL POTENTIAL" FOR WAVE PACKET AT $t = 2$

PARAMETERS $x_0 = -4$, $v = 4$, $\gamma = 4$

Applying this formula to the ψ-functions for $x<0$ and $x>0$, we get

$$|\psi_L|' = \frac{1}{2}\,\frac{(\psi_1^*+\psi_2^*)'(\psi_1+\psi_2)+(\psi_1^*+\psi_2^*)(\psi_1+\psi_2)'}{|\psi|}$$

$$|\psi_R|' = \frac{1}{2}\,\frac{\psi_1^{*\prime}\psi_1+\psi_1^*\psi_1'}{|\psi_1|} \; .$$

At $x = 0$ these become

$$|\psi_L(0,t)|' = \frac{1}{2}\,\frac{(\psi_1^{*\prime}-2\gamma\psi_1^*)\psi_1+\psi_1^*(\psi_1'-2\gamma\psi_1)}{|\psi_1|} = \frac{1}{2}\,\frac{\psi_1^{*\prime}\psi_1+\psi_1^*\psi_1'-4\gamma\psi_1^*\psi_1}{|\psi_1|}$$

$$|\psi_R(0,t)|' = \frac{1}{2}\,\frac{\psi_1^{*\prime}\psi_1+\psi_1^*\psi_1'}{|\psi_1|} \; .$$

The discontinuity in the derivative of $|\psi|$ at the origin is thus given by

$$|\psi_R(0,t)|' - |\psi_L(0,t)|' = 2\gamma|\psi_1(0,t)| \; . \tag{61}$$

The derivative of $|\psi|$ may be written in the following form:

$$|\psi|' = \begin{cases} g(x,t) - \gamma|\psi_1(0,t)| \; , & x<0 \\ g(x,t) + \gamma|\psi_1(0,t)| \; , & x>0 \end{cases} \tag{62}$$

where $g(x,t)$ and $g'(x,t)$ are continuous functions of x. It is readily seen that (62) satisfies the condition that $|\psi|'$ is continuous everywhere except at the origin where it has a discontinuity as given by (61).

The second derivative of $|\psi|$, or, what is the same thing, the Laplacian, may thus be written

$$\nabla^2|\psi| = g'(x,t) + 2\gamma|\psi_1(0,t)|\,\delta(x) \; . \tag{63}$$

That this is valid can be shown by integrating from 0- to 0+:

$$\int_{0-}^{0+} \nabla^2 |\psi| \, dx = \int_{0-}^{0+} g'(x,t)\, dx + 2\gamma |\psi(0,t)| \int_{0-}^{0+} \delta(x)\, dx$$

$$|\psi(0+)|' - |\psi(0-)|' = g(0+) - g(0-) + 2\gamma |\psi(0,t)| = 2\gamma |\psi(0,t)|$$

since g(0+) = g(0-) = g(0). We see thus that (63) leads to the proper discontinuity in $|\psi|'$.

Now

$$U = -\frac{\nabla^2 |\psi|}{|\psi|} = -\frac{g'(x,t) + 2\gamma |\psi(0,t)| \delta(x)}{2|\psi(x,t)|} .$$

When $x \neq 0$

$$U(x,t) = -\frac{g'(x,t)}{2|\psi(x,t)|}$$

which is the quantity that has already been calculated and plotted in Figs. 6, 7, 8, 9, and 10. When x = 0

$$U(0,t) = -\frac{g'(0,t) + 2\gamma |\psi(0,t)| \delta(0)}{2|\psi(0,t)|}$$

$$= -\frac{g'(0,t)}{2|\psi(0,t)|} - \gamma \delta(0) .$$

Thus, for all x,

$$U(x,t) = -\frac{g'(x,t)}{2|\psi(x,t)|} - \gamma \delta(x) . \tag{64}$$

Recalling that the classical potential V = $\gamma\delta(x)$, we see that the net potential is

$$V + U = -\frac{g(x,t)}{2/\psi(x,t)/} .$$

The classical potential is completely cancelled at all times by the "quantum-mechanical" potential and the only potential affecting the particles is that shown in the figures.

The question now is how to account for the spreading and reflection of the wave packet in terms of the net potential acting on the particles. An analysis of the figures 6 to 10 will give us a qualitative understanding of the process. The function $/\psi/^2$ represents the distribution in x of an ensemble of particles at an instant of time. There is a corresponding distribution in velocities which can be found by a Fourier analysis of the ψ-function. As a convenient approximation let us find the Fourier analysis at t = 0 of

$$\psi = \frac{1}{\pi^{1/4}\sqrt{1+it}} \exp\left\{-\frac{(x-x_0-v_0t)^2}{2(1+it)} + i(x-x_0)v_0 - i\frac{v_0^2}{2}t\right\} .$$

Thus

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} C(v)\, e^{ixv}\, dv \qquad (65)$$

is the Fourier analysis. The Fourier transform gives

$$C(v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(x,0)\, e^{-ixv}\, dx$$

$$= \frac{\pi^{-3/4}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2} + i(x-x_0)v_0 - ivx}\, dx .$$

Let $x - x_0 = u$. Then

$$C(v) = \frac{\pi^{-3/4}}{\sqrt{2}} e^{-i x_0 v} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2} + i u (v_0 - v)} du$$

A convenient formula for this integral is found in Rojansky's Introductory Quantum Mechanics[10].

We obtain then

$$C(v) = \frac{1}{\pi^{1/4}} e^{-\frac{(v_0 - v)^2}{2} - i x_0 v} \tag{66}$$

for the coefficients of the Fourier analysis. The distribution in velocity is thus given by

$$|C(v)|^2 = \frac{1}{\sqrt{\pi}} e^{-(v_0 - v)^2}, \tag{67}$$

which is a normal distribution about v_0, the velocity of the wave packet.

We have, therefore, an ensemble of particles with a distribution in x as given by $|\psi(x)|^2$ and a non-correlated distribution in v given by $|C(v)|^2$. Let us consider what happens to particles at A and B in Fig. 6. First we observe that there are an equal number of particles at A and B, since they are symmetrical points on the distribution curve. The particles at A have a distribution in velocities exactly similar to those at B. To be specific, half the particles at A or B are going faster than the velocity of the packet envelope, v_0, and half are going slower.

Three situations may arise at A. (1) If a particle is going slower than v_0 it will tend to go even slower and eventually go in the negative x direction because it is on a portion of the potential curve, A', with positive slope. (2) If a particle is going only slightly faster than v_o so that it is not able to surmount the slight potential hill to the right of A', it will slow down and turn back, and eventually go off to the left. (3) If a particle is going faster than v_o so that it can surmount the potential hill it will then enter a region where the potential curve has a negative slope and it will then accelerate to the right. A corresponding argument holds for particles at point B. Clearly no particle will maintain the same position over a period of time relative to the $|\psi(x,o)|^2$ curve except a particle having exactly the velocity v_o, and, of course, there are no such particles.

This discussion accounts for the spreading out of the density distribution, and it also shows that as time increases more and more particles will have negative velocities, thus making up a "reflected" packet. If the potential function, U, is symmetrical then it appears that just half the particles will be "transmitted", and half "reflected". The words transmitted and reflected are put in quotation marks because they certainly no longer have their original meaning in terms of the result of an interaction of a wave packet with a potential barrier.

VII. Conclusions

We have examined a number of quantum mechanical problems by Bohm's interpretation. The process consisted of finding the wave-function or ψ-field, as it should now be called, from it a "quantum-mechanical" potential and with the aid of this potential treating the problem in a classical manner.

In order to carry out this process with a reasonable facility we have seen that the ψ-function must be of a fairly simple form. Any complexity in the ψ-function is magnified many times by the process of taking its absolute value and finding the Laplacian, as was seen in Case 3 of the potential barrier problem. The labor involved in most problems would get completely out of hand. And those problems for which an exact ψ-function cannot be found cannot be treated by this method at all.

Of course, it is not necessary to find the "quantum-mechanical" potential when solving a problem for all physically varifiable information comes from the probability density $|\psi|^2$. But if we are to have a better understanding of this "quantum-mechanical" potential and the classical behaviour of a system with this potential taken into account, and if we are to be able to make a judgment as to the reality of this potential, then we must gain some experience in actually working with this potential. That has been the purpose of this work.

We have obtained some interesting and sometimes unexpected results. In the case of stationary states we have seen that the "quantum-mechanical" potential is always composed of two parts, one of which is just equal and opposite to the classical potential, and the other is the negative of the kinetic energy. The force derived from this second part is equal and opposite to the inertial forces of the system. In the case of the ψ_{211} state of the hydrogen atom it was found that the plane of the orbit of the electron did not in general include the nucleus. This is something entirely new in our experience and certainly leads to conceptual difficulties.

In working on the potential barrier problems the fact is brought out that one cannot consider the behaviour of only a single particle for if we attempt to do so we find that the particle is influenced by the potential at an infinite distance. But it is not surprising that we cannot consider a single particle because the potential function being derived from the ψ-function holds only for a continuous stream of particles, as does the ψ-function. Thus we must always deal with an ensemble of particles whether we use the usual interpretation or Bohm's interpretation, even if we suppose that the initial conditions are known. The implication of Bohm's interpretation is that if we could know the initial conditions of a particle we could determine its trajectory by means of classical equations of motion including the "quantum-mechanical" potential. But since we cannot write a ψ-function for a single particle, we also cannot find a "quantum-mechanical" potential applying to a single particle, and so we cannot find the motion of a single particle.

It seems therefore, that we are even in Bohm's interpretation inherently, not merely by practical necessity, restricted to the behaviour of ensembles of particles. This being the case it would not seem that this new interpretation really offers a much broader conceptual framework as Bohm claims.

We must now look into the questions of the reality of the "quantum-mechanical" potential and the validity of Bohm's interpretation. There undoubtedly are numerous tests that must be applied to Bohm's new interpretation of quantum theory before it will be generally accepted as valid. It seems to this writer that a necessary, although not sufficient, condition for the validity of Bohm's interpretation is that this "quantum-mechanical" potential have an objective reality. Bohm claims that it does but he does not define what he means by "objective reality", and he does not substantiate his claim.

Of course, "objective reality" is a difficult concept to define. We think we have some intuitive feeling about what is and is not objectively real, but this is not dependable. Bohm's "quantum-mechanical" potential is reminiscent of another device invented to account for observed facts, and that is the FitzGerald-Lorentz contraction. The assumption that a body moving with a velocity v contracted a fractional amount $\sqrt{1-v^2/c^2}$ of its length in the direction of motion exactly accounted for the observed facts. There is no physical experiment that can prove this assumption wrong, and yet it is doubtful if any physicists today consider the FitzGerald-

Lorentz contraction an objective reality. The reason for the abandonment of the FitzGerald-Lorentz contraction assumption is that it is sterile—it does not lead to any new or deeper understanding of nature.

The writer believes that this "quantum-mechanical" potential is a device of the same nature. We can say that <u>if</u> there were such a potential, then observed facts could be accounted for. But there is no physical experiment that can prove or disprove the existence of this potential. Does this assumed potential lead to any new or deeper understanding of nature? We believe not and so we believe that it does not have an objective reality.

According to our premise, then, we must conclude that Bohm's new interpretation is not a valid one.

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