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GROUPS OF TRANSFORMATIONS
IN A COMPOSITE FUNCTION SPACE

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INTRODUCTION

This paper extends to the composite Euclidean function space (C,E_n), in which a point has as coordinates a real continuous function of a real variable and n real numbers, work which has been done on continuous functional transformations in the Euclidean function space C.

The initiator of the study of continuous functional transformations in C was Kowalewski,[1] who called this space $\mathcal{R}_{\varkappa}$. By means of regular integral power series he introduced the concept of regular infinitesimal transformation groups, and obtained the infinitesimal projective and conformal groups and subgroups of the former. Dines[2] obtained the finite projective group in C, and proved that an infinitesimal projective transformation in C generates a one-parameter family of finite projective transformations in C. Kennison[3] showed that

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[1] G.Kowalewski, Wiener Sitzungsberichte, Band 120 (1911), Abt.IIa, Teil 1,pp.77-109; Teil 2,pp.1435-1472.Teil 1 will be referred to hereafter as Kowalewski I and Teil 2 as Kowalewski II.

[2] L.L.Dines, Transactions of the American Mathematical Society, vol. 20 (1919),pp.45-65, (hereafter referred to as Dines I)

[3] L.S.Kennison, Proceedings of the National Academy of Sciences, vol.16 (1930),pp.607-609.

the generated family is a group.

By introducing homogeneous coordinates Kowalewski[1] simplified the form of the infinitesimal projective and conformal transformations, writing them as linear transformations on a composite range. By the same device Hildebrandt[2] and Barnett[3] succeeded in greatly simplifying Dines' proofs. In another paper Dines[4] employed homogeneous coordinates in making a primary classification of projective transformations in C. The simplest approach to the finite conformal group in C was found by Barnett and Nathan[5] to be through the use of homogeneous sphere coordinates, which led again to linear functional transformations on a composite range. The

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1) G.Kowalewski, Comptes Rendus,vol.153 (1911),pp.1452-1454.

2) T.H.Hildebrandt, Bulletin of the American Mathematical Society, vol. 26 (1919-20), pp.400-405.

3) I.A.Barnett, Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 273-276 (hereafter referred to as Barnett I).

4) L.L.Dines, American Journal of Mathematics, vol.44 (1922) pp. 87-101 (hereafter referred to as Dines II).

5) I.A.Barnett and D.S.Nathan, Proceedings of the National Academy of Sciences, vol.18 (1932),pp.400-403. See also M.J.Delsarte, Comptes Rendus, vol.188 (1929), p.1591.

desirability was thus suggested of developing the theory of linear functional transformations in (C,E_{ω}).

That the investigation of other continuous functional transformations in (C,E_{ω}) was also desirable was suggested by the result, obtained by Barnett[1), that the subgroup of the infinitesimal projective group in $(C,E_{\omega+1})$ taking unit spheres into unit spheres in $(C,E_{\omega+1})$ can by a stereographic projection be converted into the infinitesimal conformal group in C.

In Part I the function-space (C,E_{ω}) is characterized, and a finite continuous transformation group and regular infinitesimal transformation group in (C,E_{ω}) are defined. Part II consists of two lemmas frequently invoked in the ensuing parts. In Part III the group of non-singular finite linear transformations, and the group of infinitesimal linear transformations, in (C,E_{ω}) are obtained, and it is proved that an infinitesimal linear transformation in (C,E_{ω}) generates a one-parameter group of finite linear transformations in (C,E_{ω}). In Part IV the group of finite projective transformations, and the group of regular infinitesimal projective transformations, in (C,E_{ω}) are obtained, and

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1) I.A.Barnett, Proceedings of the National Academy of Sciences, vol.15 (1929),pp.96-98 (hereafter referred to as Barnett II).

it is proved that an infinitesimal projective transformation generates a one-parameter group of finite projective transformations in (C,E_n). Corresponding results for the conformal transformations in (C,E_n) are given in Part V. In Part VI it is proved that the subgroup of the infinitesimal projective group in (C,E_n) taking unit spheres into unit spheres in (C,E_n) can by a stereographic projection be converted into the infinitesimal conformal group in (C,E_{n-1}).

NOTATION

The meanings here attached to our symbols will be applicable whenever no other meanings are expressly indicated.

Any Greek index $\alpha, \beta, \gamma, \ldots$ (except ξ), with or without positive integral indices, ranges over the closed interval I: (0,1) whereas any Latin index $i, j, k, \ldots$ (except n) has the discrete integral range J: 1,2,...,n.

The letters u, v, w, x, y, z, if assigned Greek indices, represent real continuous functions of these indices on I; if assigned positive integral indices, they represent variables which can assume all real values. If they are without indices, they denote sets of $n+1$ ordered elements, e.g.

$$x \equiv (x^\alpha, x^1, \ldots, x^n).$$

Addition of two sets x and y is defined by

$$x + y \equiv (x^\alpha + y^\alpha, x^1 + y^1, \ldots, x^n + y^n).$$

Multiplication of a set x by a real number c is defined by

$$cx \equiv (cx^\alpha, cx^1, \ldots, cx^n).$$

The equation $x = y$ denotes the system of equations

$$x^\alpha = y^\alpha, x^1 = y^1, \ldots, x^n = y^n.$$

The inequality $|x| \leqq c$ denotes the system of inequalities $|x^\alpha| \leqq c,\ |x^1| \leqq c,\ \dots,\ |x^n| \leqq c$.

The letters $K, L, M, \dots$, when given Greek indices, denote real continuous functions of their indices, where each index is on I; in all other cases they denote real constants.

The value of an expression is not affected if simultaneously all superscripts are made subscripts and all subscripts are made superscripts without alteration in the order of the indices; thus

$$x^\alpha \equiv x_\alpha \ , \ x^i \equiv x_i \ , \ K^{\alpha\beta\gamma}_{ij} \equiv K^{ij}_{\alpha\beta\gamma} \ .$$

When the same Greek (or Latin) letter occurs in a term both as superscript and as subscript, Riemann integration over I (or summation over J) with respect to the repeated index will be understood, the repeated index being called a dummy index. Thus the expressions

$$M^{\alpha\beta}_{ij}\, x^{j} x_{\beta} \ , \quad M^{\alpha\beta\gamma} x_{\beta} x_{\gamma}$$

would in ordinary notation appear as

$$\sum_{j=1}^{n} x_j \int_0^1 M_{ij}(\alpha,\beta)\, x(\beta)\, d\beta \qquad (i=1,\dots,n)\ , \quad \int_0^1\!\!\int_0^1 M(\alpha\beta\gamma)\, x(\beta)\, x(\gamma)\, d\beta\, d\gamma$$

respectively. To prevent ambiguity, we will refrain from using as a dummy index a letter already appearing as another index in the same term; thus we will not use the expression $M^{\alpha\alpha}_{ii}\, x^{i} x_{\alpha}$ as the equivalent of $M^{\alpha\beta}_{ij}\, x^{j} x_{\beta}$.

The letters a, b, c, t (with or without positive integral indices) will be used as arbitrary real constants.

When it is desired to indicate that a function is continuous in such parameters, the symbols for them will be inclosed in parentheses and put after the symbol for the function. Thus $x^{\alpha}(t)$ and $K^{\alpha\beta}(a,b)$ will have the same significance respectively as have $x(\alpha;t)$ and $K(\alpha,\beta;a,b)$ in ordinary notation. Differentiation of a set $x(t)\equiv\left(x^{\alpha}(t), x^{1}(t), \ldots, x^{n}(t)\right)$ with respect to t is defined by

$$\frac{\partial x(t)}{\partial t}\equiv\left(\frac{\partial x^{\alpha}(t)}{\partial t}, \frac{\partial x^{1}(t)}{\partial t}, \ldots, \frac{\partial x^{n}(t)}{\partial t}\right).$$

When a positive integer is to serve as an exponent, the quantity whose exponent it is will be inclosed in parentheses or brackets; thus

$$(x^{\alpha})^{2}\equiv x^{\alpha}\cdot x^{\alpha}, \quad [a]^{2}=a\cdot a .$$

The letter ξ , with or without a positive integral index, will denote a definite value in I, while the letter e will denote an arbitrarily small positive quantity.

PART I

TRANSFORMATIONS IN THE COMPOSITE FUNCTION SPACE (C,E_n)

1. The Space (C,E_n).[1] We designate by (C,E_n) the composite Euclidean function-space in which every ordered set x defines a point x. Let us divide the interval I into n parts, each of length $\Delta\alpha$; then as $n \to \infty$, $\Delta\alpha^{\frac{1}{2}} \to 0$. Let ξ_i be a definite value of α in the ith sub-interval. We can regard as the coordinates of the point x the infinite sequence

$$x^{\xi_1}\Delta\alpha^{\frac{1}{2}}, x^{\xi_2}\Delta\alpha^{\frac{1}{2}}, \dots, x^{\xi_n}\Delta\alpha^{\frac{1}{2}}, \dots, x^1, \dots, x^n. \tag{1}$$

Then the coordinates of the point y, defined by the ordered set y, are

$$y^{\xi_1}\Delta\alpha^{\frac{1}{2}}, y^{\xi_2}\Delta\alpha^{\frac{1}{2}}, \dots, y^{\xi_n}\Delta\alpha^{\frac{1}{2}}, \dots, y^1, \dots, y^n.$$

Thus we are led to define the distance between the points x and y as

$$\left[(y^\alpha - x^\alpha)(y_\alpha - x_\alpha) + (y^i - x^i)(y_i - x_i)\right]^{\frac{1}{2}},$$

which in ordinary notation would be

$$\left[\int_0^1 (y(\alpha) - x(\alpha))^2 d\alpha + \sum_{i=1}^{n} (y^i - x^i)^2\right]^{\frac{1}{2}}.$$

For brevity we will represent the distance between x and y by (x, y).

---

[1] The definitions in this section are generalizations of those given by Kowalewski with regard to the space C; see Kowalewski I, p. 77.

The sequence (1) can also be regarded as the components of a vector x which goes from any point z in (C,E_{n}) to the point $z+x$ in (C,E_{n}). If we make the vector start from the origin, i.e. if we choose $z^{\alpha}=z^{1}=\cdots=z^{n}=0$ the components of the vector will become identical with the corresponding coordinates of the end point of the vector. By the norm of the vector x is meant the expression

$$x^{\alpha}x_{\alpha}+x^{i}x_{i}\,.$$

The length of the vector x is defined as the positive square root of the norm. The length of the vector obviously equals the distance between its initial point and its end point. We define the inner product of two vectors x and y to be the expression

$$x^{\alpha}y_{\alpha}+x^{i}y_{i}\,,$$

their outer product to be the set of determinants

$$\begin{vmatrix} x^{\alpha} & x^{\beta} \\ y^{\alpha} & y^{\beta} \end{vmatrix}, \quad \begin{vmatrix} x^{\alpha} & x^{i} \\ y^{\alpha} & y^{i} \end{vmatrix}, \quad \begin{vmatrix} x^{i} & x^{j} \\ y^{i} & y^{j} \end{vmatrix}.$$

If the vectors x and y issue from the same point, the angle θ which they form is defined by

$$\cos\theta = \frac{x^{\alpha}y_{\alpha}+x^{i}y_{i}}{\left[(x^{\gamma}x_{\gamma}+x^{k}x_{k})(y^{\delta}y_{\delta}+y^{l}y_{l})\right]^{\frac{1}{2}}}\,,$$

$$\sin\theta = \frac{\left[(x^{\alpha}x_{\alpha}+x^{i}x_{i})(y^{\beta}y_{\beta}+y^{j}y_{j})-(x^{\alpha}y_{\alpha}+x^{i}y_{i})^{2}\right]^{\frac{1}{2}}}{\left[(x^{\gamma}x_{\gamma}+x^{k}x_{k})(y^{\delta}y_{\delta}+y^{l}y_{l})\right]^{\frac{1}{2}}}\,.$$

2. Operations and Transformations. Certain operations on x, as e.g. $x^\beta x_\beta + x^j x_j$, yield a unique real number for each x. We shall represent n such operations by $F^i(x)$, calling the $F^i(x)$ functionals of x. The $F^i(x)$ are said to be continuous at $x = x_0$ if

$$\left| F^i(x_0+\Delta x) - F^i(x_0) \right| < \varepsilon$$

for maximum $|\Delta x|$ sufficiently small, i.e. for the maximum of the quantities $|\Delta x^\alpha|, |\Delta x^1|, \dots, |\Delta x^n|$ sufficiently small.

An operation on x which yields for each x a unique real continuous function of α defined on I will be represented by $F^\alpha(x)$. An example of such an operation is $K^{\alpha\beta} x_\beta + K^\alpha_j x^j$. When α is fixed, $F^\alpha(x)$ becomes a functional of x.

The simultaneous application of $F^\alpha(x)$ and $F^i(x)$ establishes a correspondence of a point $\bar{x}$ to each point x, the correspondence being expressed by the equation

$$\bar{x} = F(x) \equiv \left(F^\alpha(x), F^1(x), \dots, F^n(x) \right) \tag{2}$$

or by the equivalent system of equations

$$\begin{cases} \bar{x}^\alpha = F^\alpha(x), \\ \bar{x}^i = F^i(x) \end{cases} \tag{2'}$$

The equation (2), or the system (2'), defines a finite point transformation in (C, E_n).

The transformation (2) is called continuous if

$$\left\{\left[F^{\alpha}(y)-F^{\alpha}(x)\right]\left[F_{\alpha}(y)-F_{\alpha}(x)\right]+\left[F^{i}(y)-F^{i}(x)\right]\left[F_{i}(y)-F_{i}(x)\right]\right\}^{\frac{1}{2}}<e$$

for (x , y) sufficiently small.

If the set $F(x)$ of operators involves arbitrary real numbers or arbitrary real continuous functions of one or more real variables, it defines a family of transformations in (C,E_n). An example of such a family is

$$\begin{cases} \bar{x}^{\alpha} = K x^{\alpha} + K^{\alpha\beta} x_{\beta} + K^{\alpha}_{j} x^{j} + x^{\alpha}\left(L^{\beta} x_{\beta} + L^{j} x_{j}\right), \\ \bar{x}^{i} = K x^{i} + K^{i}_{\beta} x^{\beta} + K^{ij} x_{j} + x^{i}\left(L^{\beta} x_{\beta} + L^{j} x_{j}\right). \end{cases}$$

If, for a certain choice of values of the arbitrary quantities, the system of equations defining a family of transformations can be solved uniquely for x^{α}, x^{i}, then the system of equations comprising the solution is called the inverse of the member of the family corresponding to that choice. The transformation resulting from the successive application of two members of a family is called their product. If the product of every two, and the inverse of every one, of the members of a family is also in the family, the family is said to form a group. A subset of a group is called a subgroup of the group if the product of every two, and the inverse of every one, of the members of the subset is also in the subset. The product of a transformation and its inverse is obviously the identity transformation, that is, the transformation taking any point into itself. Conversely, if the product of an inverse-possessing member of the family and of another member of the family is the identity transformation, the second transformation is the inverse of the first.

Let us consider the family of transformations

$$\begin{cases} \bar{x}^{\alpha}(t) = M^{\alpha}(t)\,x^{\alpha} + M^{\alpha\beta}(t)\,x_{\beta} + M^{\alpha}_{j}(t)\,x^{j}, \\ \bar{x}^{i}(t) = \qquad\qquad N^{i}_{\beta}(t)\,x^{\beta} + M^{ij}(t)\,x_{j}, \end{cases}$$

where t is a real variable on any specified interval, finite or infinite. If M^{α}, $M^{\alpha\beta}$, M^{α}_{j}, N^{i}_{β} and M^{ij} are fixed, t alone remains arbitrary. When, as in this instance, the arbitrariness resides in one real quantity alone, a family will be referred to as a one-parameter family.

Let us consider a one-parameter family of transformations

$$\bar{x}(t) = F(x\,|\,t).$$

Let t_1, t_2 be any admissible values of t. The family is said to be continuous if

$$\left\{\left[F^{\alpha}(t_2)-F^{\alpha}(t_1)\right]\left[F_{\alpha}(t_2)-F_{\alpha}(t_1)\right]+\left[F^{i}(t_2)-F^{i}(t_1)\right]\left[F_{i}(t_2)-F_{i}(t_1)\right]\right\}^{\frac{1}{2}}<\epsilon$$

for $|t_2-t_1|$ sufficiently small. If a one-parameter (continuous) family forms a group, the group will be spoken of as a one-parameter (continuous) group.

3. Composite Integral Power Forms. Let

$p, p_1, \dots, p_r, q_1, \dots, q_n$ be equal or distinct positive integers. Using these as exponents, form the function

$$(x^\alpha)^p (x^{\alpha_1})^{p_1} \cdots (x^{\alpha_r})^{p_r} (x^{i_1})^{q_1} \cdots (x^{i_n})^{q_n},$$

which is continuous in the $r+n+1$ arguments $\alpha, \alpha_1, \dots, \alpha_r, x^{i_1}, \dots, x^{i_n}$. Form now the expression

$$(3) \qquad K^{\alpha, \alpha_1, \dots, \alpha_r}_{i_1, \dots, i_n} (x^\alpha)^p (x_{\alpha_1})^{p_1} \cdots (x_{\alpha_r})^{p_r} (x^{i_1})^{q_1} \cdots (x^{i_n})^{q_n},$$

and assume that the indices $\alpha_1, \dots, \alpha_r$ and $i_1, \dots, i_n$ are so arranged that the relations $p_1 \geqq p_2 \geqq \cdots \geqq p_r$ and $q_1 \geqq q_2 \geqq \cdots \geqq q_n$ hold, since such an arrangement is always possible. It is to be observed that (3) is a continuous functional of x when α is fixed and a continuous function of α when x is fixed. Introduce a finite number of such expressions as (3), requiring that for each of them the condition

$$p + p_1 + \cdots + p_r + q_1 + \cdots + q_n = m \qquad (m \leqq r+n)$$

be satisfied, and form their sum. This sum will be called a composite integral power form of the m th order in x, and will be designated by $\mathfrak{A}^\alpha_m(x^\alpha, x^1, \dots, x^n)$ or, more briefly, by $\mathfrak{A}^\alpha_m(x)$ [1]. The most general composite integral power forms of the zero-th, first, second, and third orders are

---

[1] This definition is a generalization of E.Schmidt's definition of an integral power form in x^α ; see E.Schmidt, Mathematische Annalen, vol. 65 (1908), p. 374. See also Kowalewski II, p. 1436.

respectively

$$
(4)\quad \begin{cases}
\mathfrak{A}^{\alpha}_{0}(x) \equiv K^{\alpha}, \\
\mathfrak{A}^{\alpha}_{1}(x) \equiv L^{\alpha}x^{\alpha} + L^{\alpha\beta}x_{\beta} + L^{\alpha}_{j}x^{j}, \\
\mathfrak{A}^{\alpha}_{2}(x) \equiv M^{\alpha}(x^{\alpha})^{2} + M^{\alpha}_{jk}x^{j}x^{k} + x^{\alpha}M^{\alpha\beta}x_{\beta} \\
\qquad + M^{\alpha\beta}_{j}x^{j}x_{\beta} + \hat{M}^{\alpha\beta}(x_{\beta})^{2} \\
\qquad + M^{\alpha\beta\gamma}x_{\beta}x_{\gamma} + M^{\alpha}_{j}x^{j}x^{\alpha}, \\
\mathfrak{A}^{\alpha}_{3}(x) \equiv N^{\alpha}(x^{\alpha})^{3} + N^{\alpha}_{jk\ell}x^{j}x^{k}x^{\ell} + N^{\alpha}_{j}(x^{\alpha})^{2}x^{j} + N^{\alpha}_{jk}x^{\alpha}x^{j}x^{k} \\
\qquad + (x^{\alpha})^{2}N^{\alpha\beta}x_{\beta} + N^{\alpha\beta}_{jk}x^{j}x^{k}x_{\beta} + x^{\alpha}\hat{N}(x_{\beta})^{2} \\
\qquad + N^{\alpha\beta}_{j}x^{j}(x_{\beta})^{2} + x^{\alpha}N^{\alpha\beta\gamma}x_{\beta}x_{\gamma} + N^{\alpha\beta\gamma}_{j}x^{j}x_{\beta}x_{\gamma} \\
\qquad + \hat{\hat{N}}^{\alpha\beta}(x_{\beta})^{3} + \hat{N}^{\alpha\beta\gamma}(x_{\beta})^{2}x_{\gamma} + N^{\alpha\beta\gamma\delta}x_{\beta}x_{\gamma}x_{\delta},
\end{cases}
$$

where the K, L, M, N, called the coefficients of the forms, can without loss of generality be taken symmetric in all Greek indices except α and all Latin indices except i. Thus if the coefficient in the term $M^{\alpha\beta\gamma}x_{\beta}x_{\gamma}$ is not symmetric in β and γ, we can replace this term by the equivalent expression $\frac{1}{2}(M^{\alpha\beta\gamma}+M^{\alpha\gamma\beta})x_{\beta}x_{\gamma}$, whose coefficient is symmetric in β and γ. Or if, say, the coefficient in the term $N^{\alpha}_{jk\ell}x^{j}x^{k}x^{\ell}$ is not symmetric in k and ℓ, we can replace this term by the equivalent expression , equivalent expression $\frac{1}{2}(N^{\alpha}_{jkl} + N^{\alpha}_{jlk})x^{j}x^{k}x^{l}$, whose coefficient is symmetric in k and l.

We shall need also to employ composite integral power forms of a special kind, namely, those that do not involve . We shall represent n such forms of the m th order by $\mathfrak{A}^{i}_{m}(x)$; thus we have

$$
(4')\quad \begin{cases}
\mathfrak{A}^{i}_{0}(x) \equiv \tilde{K}^{i}, \\
\mathfrak{A}^{i}_{1}(x) \equiv \tilde{L}^{i}_{\beta} x^{\beta} + \tilde{L}^{ij} x_{j}, \\
\mathfrak{A}^{i}_{2}(x) \equiv \tilde{M}^{ijk} x_{j} x_{k} + \tilde{M}^{ij}_{\beta} x_{j} x^{\beta} \\
\qquad + \tilde{M}^{i}_{\beta} (x^{\beta})^{2} + \tilde{M}^{i}_{\beta\gamma} x^{\beta} x^{\gamma}, \\
\mathfrak{A}^{i}_{3}(x) \equiv \tilde{N}^{ijkl} x_{j} x_{k} x_{l} + \tilde{N}^{ijk}_{\beta} x_{j} x_{k} x^{\beta} \\
\qquad + \tilde{N}^{ij}_{\beta} x_{j} (x^{\beta})^{2} + \tilde{N}^{ij}_{\beta\gamma} x_{j} x^{\beta} x^{\gamma} \\
\qquad + \tilde{N}^{i}_{\beta} (x^{\beta})^{3} + \tilde{N}^{i}_{\beta\gamma} (x^{\beta})^{2} x^{\gamma} + \tilde{N}^{i}_{\beta\gamma\delta} x^{\beta} x^{\gamma} x^{\delta}.
\end{cases}
$$

4. Differentials of Composite Integral Power Forms.

The first differential, in the increment set y, of $\mathfrak{A}^{\alpha}_{m}(x)$ at the point $x = x_{0}$ is defined as

$$
\begin{aligned}
\mathfrak{A}^{\alpha}_{m}(x_{0}; y) &\equiv \mathfrak{A}^{\alpha}_{m}(x^{\alpha}_{0}, x^{1}_{0}, \ldots, x^{n}_{0}; y^{\alpha}, y^{1}, \ldots, y^{n}) \\
&\equiv \left[\frac{d}{d\epsilon} \mathfrak{A}^{\alpha}_{m}(x^{\alpha}_{0} + \epsilon y^{\alpha}, x^{1}_{0} + \epsilon y^{1}, \ldots, x^{n}_{0} + \epsilon y^{n}) \right]_{\epsilon = 0} \\
&\equiv \left[\frac{d}{d\epsilon} \mathfrak{A}^{\alpha}_{m}(x_{0} + \epsilon y) \right]_{\epsilon = 0}.
\end{aligned}
$$

It is a composite integral power form of the first order in y when x_{0} is fixed, and a composite integral power form of the $(m-1)$st order in x_{0} when y is fixed.

fixed.[1]

The second differential, in the increment sets y and z, of $\mathfrak{A}^{\alpha}_{m}(x)$ at $x = x_0$ is defined as

$$\mathfrak{A}^{\alpha}_{m}(x_0; y; z) \equiv \left[\frac{d^2}{d\epsilon_1 d\epsilon_2} \mathfrak{A}^{\alpha}_{m}(x_0 + \epsilon_1 y + \epsilon_2 z)\right]_{\substack{\epsilon_1 = 0 \\ \epsilon_2 = 0}}$$

When x_0 is fixed, $\mathfrak{A}^{\alpha}_{m}(x_0; y; z)$ is a composite integral power form of the first order in y and of the first order in z. When y and z are fixed, it is a composite integral power form of the ($m-2$)nd order in x_0. From the definition of the second differential we have at once

THEOREM 1.

$$\mathfrak{A}^{\alpha}_{m}(x_0; y; z) \equiv \mathfrak{A}^{\alpha}_{m}(x_0; z; y).$$

The higher differantials are given analogous definitions. The order of forming the successive differentials is always permutable. It is clear from the form of $\mathfrak{A}^{\alpha}_{m}(x)$ that the differentials of all orders exist and that those of higher order than the mth are identically zero.[2]

From the homogeneity of $\mathfrak{A}^{\alpha}_{m}(x)$ we have

---

[1] Cf. I.A.Barnett, American Journal of Mathematics, vol. 44 (1922), p. 186.

[2] If the Frechet definitions of the differentials of continuous functionals is applied to composite integral power forms, we get the same expressions as are obtained from our definitions. See M.Frechet, Transactions of the American Mathematical Society, vol. 15 (1914), p. 139.

$$\mathfrak{A}^{\alpha}_{m}(r x_0) \equiv (r)^{m}\,\mathfrak{A}^{\alpha}_{m}(x_0)\,.$$

Differentiation with respect to r gives

$$\frac{d}{dr}\,\mathfrak{A}^{\alpha}_{m}\Big(x_0 + (r-1)x_0\Big) \equiv m r^{m-1}\,\mathfrak{A}^{\alpha}_{m}(x)\,.$$

Putting $r = 1$ in this identity, we get

$$\mathfrak{A}^{\alpha}_{m}(x_0\,;\,x_0) \equiv m\,\mathfrak{A}^{\alpha}_{m}(x_0)\,.$$

Similarly the formula

$$\mathfrak{A}^{\alpha}_{m}(x_0\,;x_0\,;x_0) \equiv m(m-1)\,\mathfrak{A}^{\alpha}_{m}(x_0)$$

can be established, and continuing the process, we get as the analogue of Euler's formula for homogeneous functions

THEOREM 2.

$$\mathfrak{A}^{\alpha}_{m}(\underbrace{x_0\,;x_0\,;\cdots;x_0}_{s\ \text{times}}) \equiv m(m-1)\cdots(m-s+2)\,\mathfrak{A}^{\alpha}_{m}(x_0)$$
$$(s = 2, 3, \dots, m+1)$$

There is an immediate application of this theorem which will be useful later. If in $\mathfrak{A}^{\alpha}_{m}(x)$ all coefficients are replaced by their absolute values and $x^{\alpha}, x^{1}, \dots, x^{m}$ each by 1, the resulting expression will be a function of α. The maximum of this function is called the height¹⁾ of $\mathfrak{A}^{\alpha}_{m}(x)$. The heights of $\mathfrak{A}^{\alpha}_{m}(x\,;y)$,

---

1) See Kowalewski II, p. 1439.

$\mathfrak{A}^{\alpha}_{m}(x;y;z)$ etc., are similarly defined as the respective maxima of the functions of α which arise when the coefficients of these forms have been replaced by their absolute values and every element of the sets $x, y, z, \cdots$ has been replaced by 1. Let a_m denote the height of $\mathfrak{A}^{\alpha}_{m}(x)$. Then by Theorem 2 the height of $\mathfrak{A}^{\alpha}_{m}(x;y)$ is $m a_m$, the height of $\mathfrak{A}^{\alpha}_{m}(x;y;z)$ is $m(m-1)a_m$ etc.

The extension of Taylor's Theorem to composite integral power forms is readily made. If in $\mathfrak{A}^{\alpha}_{m}(x_0+\epsilon y)$ we consider x_0 and y as fixed, we have a polynomial in ϵ to which we can apply Taylor's Theorem, getting (since all differentials of order higher than m vanish)

$$\mathfrak{A}^{\alpha}_{m}(x_0+\epsilon y)=\mathfrak{A}^{\alpha}_{m}(x_0)+\frac{\epsilon}{1!}\mathfrak{A}^{\alpha}_{m}(x_0;y)+\cdots+\frac{(\epsilon)^m}{m!}\mathfrak{A}^{\alpha}_{m}(x_0;\underbrace{y;y;\cdots;y}_{m\ \text{times}}).$$

Putting $\epsilon=1$ in this identity, we obtain the following theorem, which is the analogue of Taylor's Theorem:

THEOREM 3.

$$\mathfrak{A}^{\alpha}_{m}(x_0+y)=\mathfrak{A}^{\alpha}_{m}(x_0)+\frac{1}{1!}\mathfrak{A}^{\alpha}_{m}(x_0;y)+\cdots+\frac{1}{m!}\mathfrak{A}^{\alpha}_{m}(x_0;\underbrace{y;y;\cdots;y}_{m\ \text{times}}).$$

5. <u>Composite Integral Power Series</u>. An infinite series of the type

$$(5)\qquad \sum_{m=0}^{\infty}\mathfrak{A}^{\alpha}_{m}(x)$$

is called a composite integral power series[1] in x .

Form the power series

$$a_0 + a_1 z + a_2 (z)^2 + \cdots , \tag{6}$$

where a_m is the height of the term $\mathfrak{A}_m^{\alpha}(x)$ in (5) and z is a complex variable. Let R be the radius of convergence of (6). The series (5) is said to converge regularly[2] for

$$|x| \leqq r < R . \tag{7}$$

THEOREM 4. *A composite integral power series* $\sum_{m=0}^{\infty} \mathfrak{A}_m^{\alpha}(x)$ *converges absolutely and uniformly in the subset* (7) *of* (C, E_n) *in which it converges regularly.*

PROOF. From the homogeneity of $\mathfrak{A}_m^{\alpha}(x)$ and from the inequality (7) we infer the inequalities

$$|\mathfrak{A}_p^{\alpha}(x)| \equiv (r)^p \left|\mathfrak{A}_p^{\alpha}\left(\frac{x}{r}\right)\right| \leqq a_p (r)^p \qquad (p=0,1,2,\cdots).$$

Hence in the subset (7), the series (5) is dominated by the convergent series of positive terms

$$a_0 + a_1 r + a_2 (r)^2 + \cdots$$

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[1] This is a generalization of an integral power series as defined by Schmidt. See E. Schmidt, Mathematische Annalen, vol. 65 (1908), p.374; see also Kowalewski II, p.1440.

[2] Cf. E. Schmidt, l.c., p.374.

and is , consequently, absolutely and uniformly convergent in this subset.

From the continuity of $\mathcal{A}_m^\alpha(x)$ in α and from the result in Theorem 4 on the uniform convergence of $\sum\limits_{m=0}^{\infty}\mathcal{A}_m^\alpha(x)$ in α for a fixed x in the subset (7), there follows

THEOREM 5. The series $\sum\limits_{m=0}^{\infty}\mathcal{A}_m^\alpha(x)$ defines a continuous function of α for a fixed x in the subset (7).

In view of this theorem, we shall write

$$\mathcal{P}^\alpha(x) \equiv \sum_{m=0}^{\infty} \mathcal{A}_m^\alpha(x) \qquad (|x| \leqq r < R),$$

and shall speak of $\mathcal{P}^\alpha(x)$ as a regular composite integral power series in x.

THEOREM 6. For a fixed value of α, the regular composite integral power series $\mathcal{P}^\alpha(x)$ is a continuous functional of x.

PROOF. Let us put

$$s_n^\alpha(x) \equiv \sum_{m=0}^{n} \mathcal{A}_m^\alpha(x), \quad r_n^\alpha(x) \equiv \sum_{m=n+1}^{\infty} \mathcal{A}_m^\alpha(x) \qquad (|x| \leqq r < R).$$

Choosing maximum $|\Delta x|$ sufficiently small, we will have $|x+\Delta x| < R$ and consequently

$$\mathcal{P}^\alpha(x+\Delta x) \equiv \sum_{m=0}^{\infty} \mathcal{A}_m^\alpha(x+\Delta x) \qquad (|x| \leqq r < R).$$

We can write

$$|\mathcal{P}^\alpha(x+\Delta x) - \mathcal{P}^\alpha(x)| \leqq |s_n^\alpha(x+\Delta x) - s_n^\alpha(x)| + |r_n^\alpha(x+\Delta x)| + |r_n^\alpha(x)|.$$

Because of the uniform convergence of $\mathcal{P}^\alpha(x)$ and $\mathcal{P}^\alpha(x+\Delta x)$, we can argue that, given an ε, we can find an n_ε such that

$$|r_n^\alpha(x)| < \frac{e}{3}, \quad |r_n^\alpha(x+\Delta x)| < \frac{e}{3} \qquad (n \geqq n_e).$$

Since $s_n^\alpha(x)$, as the sum of a finite number of continuous functionals, is continuous, it follows that for maximum $|\Delta x|$ sufficiently small we will have

$$|s_n^\alpha(x+\Delta x) - s_n^\alpha(x)| < \frac{e}{3}.$$

Hence

$$|\mathcal{P}^\alpha(x+\Delta x) - \mathcal{P}^\alpha(x)| < e$$

for maximum $|\Delta x|$ sufficiently small, which proves the theorem.

6. Differentials of Regular Composite Integral Power Series. By the first differential, in the increment set y, of $\mathcal{P}^\alpha(x)$ at $x = x_0$ we shall mean

$$(8) \qquad \mathcal{P}^\alpha(x_0; y) \equiv \left[\frac{d}{d\epsilon}\mathcal{P}^\alpha(x_0+\epsilon y)\right]_{\epsilon=0} \equiv \left[\frac{d}{d\epsilon}\sum_{m=0}^{\infty}\mathcal{A}_m^\alpha(x_0+\epsilon y)\right]_{\epsilon=0}$$

provided, of course, that (8) exists. The second differential, in the increment sets y and z, of $\mathcal{P}^\alpha(x)$ at $x = x_0$ is defined as

$$(9) \qquad \mathcal{P}^\alpha(x_0; y; z) \equiv \left[\frac{d^2}{d\epsilon_1 d\epsilon_2}\mathcal{P}^\alpha(x_0+\epsilon_1 y+\epsilon_2 z)\right]_{\substack{\epsilon_1=0\\ \epsilon_2=0}} \equiv \left[\frac{d^2}{d\epsilon_1 d\epsilon_2}\sum_{m=0}^{\infty}\mathcal{A}_m^\alpha(x_0+\epsilon_1 y+\epsilon_2 z)\right]_{\substack{\epsilon_1=0\\ \epsilon_2=0}}.$$

provided that (9) exists. Corresponding definitions are given for the higher differentials. We have the following sufficiency theorem:

THEOREM 7. Let x_0 and y be such that the following inequalities hold:

$$(10) \qquad |x_0| \leqq r_1 , \ |y| \leqq r_2 , \ r_1 + r_2 < R .$$

Then (8) exists and can be computed from the formula

$$(11) \qquad \mathfrak{P}^{\alpha}(x_0 ; y) = \sum_{m=0}^{\infty} \mathfrak{A}_m^{\alpha}(x_0 ; y) .$$

PROOF. From the homogeneity of $\mathfrak{A}_m^{\alpha}(x)$, from the properties of its differentials, and from (10), we infer

$$\left|\mathfrak{A}_p^{\alpha}(x_0 ; y)\right| \equiv (r_1)^{p-1} r_2 \left|\mathfrak{A}_p^{\alpha}\left(\frac{x_0}{r} ; \frac{y}{s}\right)\right| \leqq p\, a_p (r_1)^{p-1} r_2$$
$$(p = 1, 2, \cdots) .$$

and similarly

$$\left|\mathfrak{A}_p^{\alpha}(x_0 ; y ; y)\right| \leqq p(p-1)\, a_p (r_1)^{p-2} (r_2)^2 ,$$

$$(10.1) \qquad \Big|\mathfrak{A}_p^{\alpha}(x_0 ; \underbrace{y ; \cdots ; y}_{s \text{ times}})\Big| \leqq p(p-1) \cdots (p-s+1)\, a_p (r_1)^{p-s} (r_2)^s$$
$$(s = 1, 2, \cdots, p \ ; \ p = 1, 2, \cdots) .$$

Differentiating in the formula of Theorem 3 with respect to ϵ , we get

$$\frac{d}{d\epsilon} \mathfrak{A}_p^{\alpha}(x_0 + \epsilon y) = \mathfrak{A}_p^{\alpha}(x_0 ; y) + \frac{\epsilon}{1!} \mathfrak{A}_p^{\alpha}(x_0 ; y ; y)$$
$$+ \frac{(\epsilon)^2}{2!} \mathfrak{A}_p^{\alpha}(x_0 ; y ; y ; y) + \cdots$$
$$+ \frac{(\epsilon)^{p-1}}{(p-1)!} \mathfrak{A}_p^{\alpha}(x_0 ; \underbrace{y ; \cdots ; y}_{p \text{ times}}) ,$$

whence

$$\left|\frac{d}{d\epsilon} \mathfrak{A}_p^{\alpha}(x_0 + \epsilon y)\right| \leqq \left|\mathfrak{A}_p^{\alpha}(x_0 ; y)\right| + \frac{\epsilon}{1!} \left|\mathfrak{A}_p^{\alpha}(x_0 ; y ; y)\right|$$
$$+ \cdots + \frac{(\epsilon)^{p-1}}{(p-1)!} \Big|\mathfrak{A}_p^{\alpha}(x_0 ; \underbrace{y ; \cdots ; y}_{p \text{ times}})\Big|$$

From this inequality there follows by virtue of (10.1)

$$\left|\frac{d}{d\epsilon}\mathfrak{a}_p^\alpha(x_0+\epsilon y)\right| \leqq p\,a_p(r_1)^{p-1}r_2 + \frac{\epsilon}{1!}p(p-1)a_p(r_1)^{p-2}(r_2)^2$$

(10.11)
$$\cdot + \cdot\cdot\cdot + \frac{(\epsilon)^{p-1}}{(p-1)!}p!\,a_p\,(r_2)^p$$

$$= p\,a_p\,(r_1+\epsilon r_2)^{p-1}r_2 \qquad (p=1,2,\ldots)$$

But the series

$$\sum_{m=1}^{\infty} m\,a_m\,(r_1+\epsilon r_2)^{m-1}$$

is readily seen to be convergent, for if we put $\bar{r}=r_1+\epsilon r_2$, the series becomes $\sum_{m=1}^{\infty} m\,a_m\,(\bar{r})^{m-1}$, which converges since it is the series obtained by term-by-term differentiation of the convergent power series $\sum_{m=1}^{\infty} a_m(\bar{r})^m$.

Hence (10.11) tells us that each term of the series

$$\sum_{m=1}^{\infty}\frac{d}{d\epsilon}\mathfrak{a}_m^\alpha(x_0+\epsilon y)$$

is dominated by the corresponding term of the convergent series of positive terms

$$r_2\sum_{m=1}^{\infty} m\,a_m\,(r_1+\epsilon r_2)^{m-1}.$$

Hence the series

$$\sum_{m=0}^{\infty}\frac{d}{d\epsilon}\mathfrak{a}_m^\alpha(x_0+\epsilon y)$$

converges uniformly in ϵ. Consequently the expression $\frac{d}{d\epsilon}\sum_{m=0}^{\infty}\mathfrak{a}_m^\alpha(x_0+\epsilon y)$ exists and can be computed from the formula

$$\frac{d}{d\epsilon}\sum_{m=0}^{\infty}\mathfrak{a}_m^\alpha(x_0+\epsilon y)=\sum_{m=0}^{\infty}\frac{d}{d\epsilon}\mathfrak{a}_m^\alpha(x_0+\epsilon y).$$

Putting $\epsilon = 0$ in this formula, we get (11).

By adding to (10) the further inequalities

$$(\overline{10}) \qquad |z| \leqq r_3 , \quad r_1 + r_2 + r_3 < R ,$$

we can prove in a similar manner that (9) exists and is given by the formula

$$\mathcal{P}^{\alpha}(x_0 ; y ; z) = \sum_{m=0}^{\infty} \mathcal{A}^{\alpha}_{m}(x_0 ; y ; z).$$

The extension of these results to the higher differentials is obvious.

Whenever we employ the successive differentials $\mathcal{P}^{\alpha}(x_0;y)$, $\mathcal{P}^{\alpha}(x_0;y;z)$, ..., we shall assume that the inequalities (10), $(\overline{10})$, and their extensions are satisfied.

By using the method of proof which was used to establish Theorem 6 we can prove

THEOREM 8. <u>The successive differentials</u> $\mathcal{P}^{\alpha}(x_0;y)$, $\mathcal{P}^{\alpha}(x_0;y;z)$, ...<u>are, for a fixed value of</u> α , <u>continuous functionals of the increment sets</u> y, z , ...

The extension of Taylor's Theorem to regular composite integral power series will now be made. For fixed x_0 and y , $\mathcal{P}^{\alpha}(x_0+\epsilon y)$ is a uniformly convergent series in ϵ whose terms are continuous functions of ϵ and is therefore itself a continuous function of ϵ . To this function of ϵ we can apply Taylor's Theorem since by Theorem 7 and its extensions $\mathcal{P}^{\alpha}(x)$ has differentials of all orders at $x=x_0$ in a suitably restricted increment set y. We thus get

$$\mathcal{P}^{\alpha}(x_0+\epsilon y)=\mathcal{P}^{\alpha}(x_0)+\frac{\epsilon}{1!}\mathcal{P}^{\alpha}(x_0;y)+\frac{(\epsilon)^2}{2!}\mathcal{P}^{\alpha}(x_0;y;y)+\cdots$$

which for $\epsilon=1$ yields

THEOREM 9.

$$\mathcal{P}^{\alpha}(x_0+y)=\mathcal{P}^{\alpha}(x_0)+\frac{1}{1!}\mathcal{P}^{\alpha}(x_0;y)+\frac{1}{2!}\mathcal{P}^{\alpha}(x_0;y;y)+\cdots$$

Let $x^{\alpha}(t)$, $x^{i}(t)$ be functions of the parameter t with first and second partial derivatives with respect to t. To an increment Δt corresponds an increment Δx. Let us form

$$\frac{\mathcal{P}^{\alpha}(x+\Delta x)-\mathcal{P}^{\alpha}(x)}{\Delta t}.$$

Expanding $\mathcal{P}^{\alpha}(x+\Delta x)$ by Theorem 9, we get

$$\frac{\mathcal{P}^{\alpha}(x+\Delta x)-\mathcal{P}^{\alpha}(x)}{\Delta t}=\frac{\mathcal{P}^{\alpha}(x;\Delta x)+\frac{1}{2!}\mathcal{P}^{\alpha}(x;\Delta x;\Delta x)+\cdots}{\Delta t}$$

Because of the properties of the differentials, this yields

$$(12)\quad \frac{\mathcal{P}^{\alpha}(x+\Delta x)-\mathcal{P}^{\alpha}(x)}{\Delta t}=\mathcal{P}^{\alpha}\left(x;\frac{\Delta x}{\Delta t}\right)+\frac{1}{2!}\mathcal{P}^{\alpha}\left(x;\frac{\Delta x}{\Delta t};\Delta x\right)+\cdots$$

It follows from the continuity of the successive differentials of $\mathcal{P}^{\alpha}(x)$ in the increment sets (Theorem 8) that as $\Delta t\to 0$ the right-hand side of (12) approaches the definite limit $\mathcal{P}^{\alpha}\left(x;\frac{\partial x}{\partial t}\right)$, since $\frac{\Delta x}{\Delta t}\to\frac{\partial x}{\partial t}$ and $\Delta x\to 0$ as $\Delta t\to 0$. We thus have

THEOREM 10.

$$\frac{\partial \mathcal{P}^{\alpha}(x)}{\partial t} = \mathcal{P}^{\alpha}\left(x\,;\,\frac{\partial x}{\partial t}\right).$$

In a similar manner we now prove

THEOREM 10′.

$$\frac{\partial^2 \mathcal{P}^{\alpha}(x)}{\partial t^2} = \mathcal{P}^{\alpha}\left(x\,;\,\frac{\partial^2 x}{\partial t^2}\right) + \mathcal{P}^{\alpha}\left(x\,;\,\frac{\partial x}{\partial t}\,;\,\frac{\partial x}{\partial t}\right).$$

PROOF. By Theorem 10 we have

$$\frac{\partial^2}{\partial t^2}\mathcal{P}^{\alpha}(x) = \frac{\partial}{\partial t}\mathcal{P}^{\alpha}\left(x\,;\,\frac{\partial x}{\partial t}\right).$$

Putting $y = \frac{\partial x}{\partial t}$, we then have

$$\frac{\partial^2}{\partial t^2}\mathcal{P}^{\alpha}(x) = \frac{\partial}{\partial t}\mathcal{P}^{\alpha}(x\,;\,y)$$

$$(13) \qquad = \lim_{\Delta t \to 0} \frac{\mathcal{P}^{\alpha}(x+\Delta x\,;\,y+\Delta y) - \mathcal{P}^{\alpha}(x\,;\,y)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\mathcal{P}^{\alpha}(x+\Delta x\,;\,y+\Delta y) - \mathcal{P}^{\alpha}(x\,;\,y+\Delta y)}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathcal{P}^{\alpha}(x\,;\,y+\Delta y) - \mathcal{P}^{\alpha}(x\,;\,y)}{\Delta t}.$$

Since first differentials are linear in the increment set, we have the identity

$$\frac{\mathcal{P}^{\alpha}(x\,;\,y+\Delta y) - \mathcal{P}^{\alpha}(x\,;\,y)}{\Delta t} \equiv \mathcal{P}^{\alpha}\left(x\,;\,\frac{\Delta y}{\Delta t}\right),$$

so that, in view of Theorem 8, the second term in the right-hand member of (13) is

$$\mathcal{P}^{\alpha}\left(x\,;\,\frac{\partial y}{\partial t}\right) \equiv \mathcal{P}^{\alpha}\left(x\,;\,\frac{\partial^2 x}{\partial t^2}\right).$$

For a fixed $y + \Delta y$, $\mathcal{P}^{\alpha}(x\,;\,y+\Delta y)$ is a regular composite integral power series in x. Put

$$\tilde{\mathcal{P}}^{\alpha}(x) = \mathcal{P}^{\alpha}(x\,;\,y+\Delta y).$$

Expanding $\tilde{\mathcal{P}}^{\alpha}(x+\Delta x)$ by Theorem 9, we get

$$\tilde{\mathcal{P}}^{\alpha}(x+\Delta x)-\tilde{\mathcal{P}}^{\alpha}(x)=\frac{1}{1!}\tilde{\mathcal{P}}^{\alpha}(x\,;\Delta x)+\frac{1}{2!}\tilde{\mathcal{P}}^{\alpha}(x\,;\Delta x\,;\Delta x)+\cdots,$$

so that

$$\frac{\tilde{\mathcal{P}}^{\alpha}(x+\Delta x)-\tilde{\mathcal{P}}^{\alpha}(x)}{\Delta t}=\tilde{\mathcal{P}}^{\alpha}\Big(x\,;\frac{\Delta x}{\Delta t}\Big)+\frac{1}{2!}\tilde{\mathcal{P}}^{\alpha}\Big(x\,;\frac{\Delta x}{\Delta t}\,;\Delta x\Big)+\frac{1}{3!}\tilde{\mathcal{P}}^{\alpha}\Big(x\,;\frac{\Delta x}{\Delta t}\,;\Delta x\,;\Delta x\Big) + \cdots,$$

i.e.

$$\frac{\mathcal{P}^{\alpha}(x+\Delta x\,;y+\Delta y)-\mathcal{P}^{\alpha}(x\,;y+\Delta y)}{\Delta t}=\mathcal{P}^{\alpha}\Big(x\,;y+\Delta y\,;\frac{\Delta x}{\Delta t}\Big)$$

$$+\frac{1}{2!}\mathcal{P}^{\alpha}\Big(x\,;y+\Delta y\,;\frac{\Delta x}{\Delta t}\,;\Delta x\Big)+\frac{1}{3!}\mathcal{P}^{\alpha}\Big(x\,;y+\Delta y\,;\frac{\Delta x}{\Delta t}\,;\Delta x\,;\Delta x\Big)$$

$$+ \cdots$$

Passing to the limit in this last identity as $\Delta t \to 0$, we see, in view of Theorem 8, that the first term of the right-hand member of (13) becomes

$$\mathcal{P}^{\alpha}\Big(x\,;y\,;\frac{\partial x}{\partial t}\Big)\equiv\mathcal{P}^{\alpha}\Big(x\,;\frac{\partial x}{\partial t}\,;\frac{\partial x}{\partial t}\Big).$$

Thus the theorem is proved.

7. Regular Infinitesimal Transformations. A regular infinitesimal transformation in $(C,\underset{\sim}{E})$ is defined by the system of integro-differential equations

$$(14)\qquad\begin{cases}\dfrac{dx^{\alpha}}{dt}=\mathcal{P}^{\alpha}(x)\\[2ex]\dfrac{dx^{i}}{dt}=\mathcal{P}^{i}(x)\end{cases}$$

where the $\mathcal{P}^{i}(x)$ are regular composite integral power series not involving α, i.e.

$$\xi^i(x) = \sum_{m=0}^{\infty} \mathfrak{A}^i_m(x).$$

For a fixed x , $\xi^\alpha(x)$ is a real continuous function of α and the $\xi^i(x)$ are real constants, by Theorem 5. We introduce the symbol

$$\xi(x) \equiv (\xi^\alpha(x), \xi^1(x), \ldots, \xi^n(x)).$$

Let (14) and

$$(\overline{14}) \qquad \begin{cases} \dfrac{dx^\alpha}{dt} = \bar{\xi}^\alpha(x), \\ \dfrac{dx^i}{dt} = \bar{\xi}^i(x) \end{cases}$$

be two regular infinitesimal transformations. By Theorem 7 we can impose suitable restrictions on $\xi(x)$ to secure the existence of $\bar{\xi}^\alpha(x_0; \xi(x_0))$, and suitable restrictions on $\bar{\xi}(x)$ to secure the existence of $\xi^\alpha(x_0; \bar{\xi}(x_0))$. The expression

$$(\xi\bar{\xi})^\alpha(x) \equiv \bar{\xi}^\alpha(x; \xi(x)) - \xi^\alpha(x; \bar{\xi}(x))$$

is a regular composite integral power series in x , while the expressions

$$(\xi\bar{\xi})^i(x) \equiv \bar{\xi}^i(x; \xi(x)) - \xi^i(x; \bar{\xi}(x))$$

are regular composite integral power series in x which do not involve α . Hence the system of integro-differential equations

$$(15)\quad \begin{cases} \dfrac{dx^\alpha}{dt} = (\mathfrak{p}\bar{\mathfrak{p}})^\alpha(x) \\[2ex] \dfrac{dx^i}{dt} = (\mathfrak{p}\bar{\mathfrak{p}})^i(x) \end{cases}$$

defines a regular infinitesimal transformation in $(C, \underset{\sim}{E})$. We call (15) the commutator 1) of (14) and $(\overline{14})$.

A set of regular infinitesimal transformations is called a group if the commutator of every pair of transformations in the set is itself in the set. A subset of a group of regular infinitesimal transformations is called a subgroup of the group if the commutator of every pair of transformations in the subset is itself in the subset.

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1) This is a generalization of Kowalewski's definition of the commutator of two regular infinitesimal transformations in C; see Kowalewski II, p. 1444.

PART II

LEMMAS

Lemma 1. If the equation

$$(16)\qquad K^{\alpha\beta\gamma}x^{\alpha} + L^{\alpha\beta\gamma}x^{\beta} + M^{\alpha\beta\gamma}x^{\gamma} + K^{\alpha\beta\gamma\delta}x_{\delta} + K^{\alpha\beta}_{j}x^{j} = 0$$

is an identity in x, α, β, and γ, the coefficients are all identically zero.

PROOF. Let ξ_1, ξ_2, and ξ_3 be any three distinct values in I. Let x be defined as follows:

$$x^{\alpha} = \begin{cases} a & (\xi_1 - e \leqq \alpha \leqq \xi_1 + e) \\ b & (\xi_2 - e \leqq \alpha \leqq \xi_2 + e) \\ c & (\xi_3 - e \leqq \alpha \leqq \xi_3 + e) \end{cases},$$

and elsewhere in I let x^{α} lie between definite bounds independent of e but remain continuous; further, let

$$x^{i} = a^{i}.$$

Here, as throughout our treatment, the a, b, c, a^{i} represent arbitrary constants. Putting in (16) the x thus defined, we have

$$K^{\alpha\beta\gamma}a + L^{\alpha\beta\gamma}b + M^{\alpha\beta\gamma}c + K^{\alpha\beta\gamma\delta}x_{\delta} + K^{\alpha\beta}_{j}a^{j} \equiv 0 \quad \begin{pmatrix} \xi_1 - e \leqq \alpha \leqq \xi_1 + e \\ \xi_2 - e \leqq \beta \leqq \xi_2 + e \\ \xi_3 - e \leqq \gamma \leqq \xi_3 + e \end{pmatrix}.$$

Now passing to the limit as $e \to 0$, we get, because of the continuity of the coefficients,

$$K^{\xi_1\xi_2\xi_3}a + L^{\xi_1\xi_2\xi_3}b + M^{\xi_1\xi_2\xi_3}c + K^{\xi_1\xi_2\xi_3\delta}x_{\delta} + K^{\xi_1\xi_2}_{i}a^{j} = 0$$

from which, because of the arbitrariness of a, b, c, a^i, we infer

(16.1) $$K^{\xi_1\xi_2\xi_3} = L^{\xi_1\xi_2\xi_3} = M^{\xi_1\xi_2\xi_3} = 0,$$

(16.2) $$K^{\xi_1\xi_2\xi_3\delta} x_\delta = 0,$$

(16.3) $$K_j^{\xi_1\xi_2} = 0.$$

Choosing $x_\delta = K_{\xi_1\xi_2\xi_3\delta}$ in (16.2), we obtain

$$K^{\xi_1\xi_2\xi_3\delta} K_{\xi_1\xi_2\xi_3\delta} \equiv 0,$$

from which follows

(16.21) $$K^{\xi_1\xi_2\xi_3\delta} \equiv 0.$$

Since ξ_1, ξ_2, and ξ_3 were any values in I, it follows from (16.1), (16.21), and (16.3) that the coefficients of (16) are identically zero.

LEMMA 2. If the equation

(17) $$K^\alpha (x^\alpha)^2 + x^\alpha K^{\alpha\beta} x_\beta + x^\alpha K_j^\alpha x^j + L^{\alpha\beta}(x_\beta)^2 + K^{\alpha\beta\gamma} x_\beta x_\gamma + L_j^{\alpha\beta} x^j x_\beta + K_{jk}^\alpha x^j x^k \equiv 0$$

is an identity in x and α, then

$$K^\alpha \equiv K^{\alpha\beta} \equiv K_i^\alpha \equiv L^{\alpha\beta} \equiv L_i^{\alpha\beta} \equiv K^{\alpha\beta\gamma} + K^{\alpha\gamma\beta} \equiv K_{jk}^\alpha + K_{kj}^\alpha \equiv 0.$$

PROOF. Let x be defined as follows:

$$x^\alpha = a \qquad (\xi - e \leqq \alpha \leqq \xi + e)$$

and elsewhere in I let x^α lie between definite bounds independent of e but remain continuous; further, let

$$x^i = b^i .$$

Putting in (17) the x thus defined, we get

$$K^\alpha a^2 + a K^{\alpha\beta} x_\beta + a K^\alpha_j b^j + L^{\alpha\beta}(x_\beta)^2 + K^{\alpha\beta\gamma} x_\beta x_\gamma$$
$$+ L^{\alpha\beta}_j x_\beta b^j + K^\alpha_{jk} b^j b^k \equiv 0 \qquad (\xi - e \leqq \alpha \leqq \xi + e).$$

Passing to the limit as $e \to 0$, we get, because of the continuity of the coefficients,

$$K^\xi a^2 + a K^{\xi\beta} x_\beta + a K^\xi_j b^j + L^{\xi\beta}(x_\beta)^2 + K^{\xi\beta\gamma} x_\beta x_\gamma$$
$$+ L^{\xi\beta}_j x_\beta b^j + K^\xi_{jk} b^j b^k = 0 .$$

Since a, b^i are arbitrary, we infer

(17.1) $$K^\xi = K^\xi_j = 0,$$

(17.2) $$K^{\xi\beta} x_\beta = 0,$$

(17.3) $$L^{\xi\beta}_j x_\beta = 0,$$

(17.4) $$K^\xi_{jk} + K^\xi_{kj} = 0,$$

(17.5) $$L^{\xi\beta}(x_\beta)^2 + K^{\xi\beta\gamma} x_\beta x_\gamma = 0 .$$

From (17.1) there follows, since ξ was any value in I,

$$K^{\alpha} \equiv K^{\alpha}_{j} \equiv 0 .$$

Applying Lemma 1 to (17.2), we get

$$K^{\xi\beta} \equiv 0 ,$$

from which we deduce

$$K^{\alpha\beta} \equiv 0 .$$

By Lemma 1, (17.3) yields

$$L^{\xi\beta}_{j} \equiv 0 ,$$

whence

$$L^{\alpha\beta}_{j} \equiv 0 .$$

From (17.4) we get

$$K^{\alpha}_{jk} + K^{\alpha}_{kj} \equiv 0 .$$

Putting in (17.5)

$$x^{\alpha} = y^{\alpha} + z^{\alpha},$$

we get

(17.51) $$L^{\xi\beta}(y_{\beta} + z_{\beta})^{2} + K^{\xi\beta\gamma}(y_{\beta} + z_{\beta})(y_{\gamma} + z_{\gamma}) = 0 .$$

For the choice

$$x^{\alpha} = y^{\alpha} - z^{\alpha}$$

(17.5) becomes

(17.52) $$L^{\xi\beta}(y_{\beta} - z_{\beta})^{2} + K^{\xi\beta\gamma}(y_{\beta} - z_{\beta})(y_{\gamma} - z_{\gamma}) = 0 .$$

Subtracting (17.52) from (17.51), we get

$$\left[2 L^{\xi\beta} y^{\beta} + (K^{\xi\beta\gamma} + K^{\xi\gamma\beta}) y_{\gamma}\right] z_{\beta} = 0$$

Regarding this equation as an identity in z and applying Lemma 1, we obtain

$$2L^{\xi\beta}y^{\beta}+\left(K^{\xi\beta\gamma}+K^{\xi\gamma\beta}\right)y_{\gamma}\equiv 0,$$

which is an identity in y, β, so that Lemma 1 is again applicable, yielding

$$L^{\xi\beta}\equiv K^{\xi\beta\gamma}+K^{\xi\gamma\beta}\equiv 0,$$

and from this we infer

$$L^{\alpha\beta}\equiv K^{\alpha\beta\gamma}+K^{\alpha\gamma\beta}\equiv 0 .$$

COROLLARY. *If to the hypothesis of the lemma is added* $K^{\alpha\beta\gamma}\equiv K^{\alpha\gamma\beta}$ *or* ${K^{\alpha}}_{jk}\equiv {K^{\alpha}}_{kj}$ *or both relations, then we have the further conclusions for these respective cases* $K^{\alpha\beta\gamma}\equiv 0$ *or* ${K^{\alpha}}_{jk}\equiv 0$ *or* $K^{\alpha\beta\gamma}\equiv {K^{\alpha}}_{jk}\equiv 0$.

PART III

LINEAR TRANSFORMATIONS IN (C,E_n)

1. Finite Linear Transformations. A finite linear transformation 1) in (C,E_n) is a transformation defined by a system of linear homogeneous integral equations

$$(18) \quad \begin{cases} \bar{x}^{\alpha} = K^{\alpha} x^{\alpha} + K^{\alpha\beta} x_{\beta} + K^{\alpha}_{j} x^{j} & (K^{\alpha} \neq 0), \\ \bar{x}^{i} = L^{i}_{\beta} x^{\beta} + K^{ij} x_{j} . \end{cases}$$

Let the linear transformations (18) and

$$(\overline{18}) \quad \begin{cases} \bar{\bar{x}}^{\alpha} = \bar{K}^{\alpha} \bar{x}^{\alpha} + \bar{K}^{\alpha\beta} \bar{x}_{\beta} + \bar{K}^{\alpha}_{j} \bar{x}^{j} & (\bar{K}^{\alpha} \neq 0), \\ \bar{\bar{x}}^{i} = \bar{L}^{i}_{\beta} \bar{x}^{\beta} + \bar{K}^{ij} \bar{x}_{j} \end{cases}$$

be successively applied. The product transformation, obtained by elimination of $\bar{x}$, is found to be

$$(19) \quad \begin{cases} \bar{\bar{x}}^{\alpha} = \bar{\bar{K}}^{\alpha} x^{\alpha} + \bar{\bar{K}}^{\alpha\beta} x_{\beta} + \bar{\bar{K}}^{\alpha}_{j} x^{j} , \\ \bar{\bar{x}}^{i} = \bar{\bar{L}}^{i}_{\beta} x^{\beta} + \bar{\bar{K}}^{ij} x_{j} , \end{cases}$$

---

1) Cf. A.D.Michal and L.S.Kennison, Proceedings of the National Academy of Sciences, vol. 16 (1930), p. 617.

where

$$(19')\quad \begin{cases} \bar{\bar{K}}^{\alpha} \equiv \bar{K}^{\alpha} K^{\alpha} \neq 0, \\ \bar{\bar{K}}^{\alpha\beta} \equiv \bar{K}^{\alpha} K^{\alpha\beta} + \bar{K}^{\alpha\beta} K^{\beta} + \bar{K}_{\alpha\gamma} K^{\gamma\beta} + \bar{K}^{\alpha}_{j} L^{j}_{\beta}, \\ \bar{\bar{K}}^{\alpha}_{j} \equiv \bar{K}^{\alpha} K^{\alpha}_{j} + \bar{K}^{\alpha\beta} K^{j}_{\beta} + \bar{K}^{\alpha}_{k} K^{kj}, \\ \bar{\bar{L}}^{i}_{\beta} \equiv \bar{L}^{i}_{\beta} K_{\beta} + \bar{L}^{i}_{\gamma} K^{\gamma\beta} + \bar{K}^{ij} L^{\beta}_{j}, \\ \bar{\bar{K}}^{ij} \equiv \bar{L}^{\beta}_{i} K^{j}_{\beta} + \bar{K}_{ik} K^{kj}. \end{cases}$$

We thus have

THEOREM 11. *The product of any two linear transformations* (18) *and* ($\overline{18}$) *in* (C,E_n) *is the linear transformation in* (C,E_n) *defined by* (19) *and* (19′).

Since $K^{\alpha} \neq 0$ in (18), we can rewrite (18) in the form

$$(18')\quad \begin{cases} \frac{1}{K^{\alpha}} \bar{x}^{\alpha} = x^{\alpha} + \frac{K^{\alpha\beta}}{K^{\alpha}} x_{\beta} + \frac{K^{\alpha}_{j}}{K^{\alpha}} x^{j}, \\ \bar{x}^{i} = L^{i}_{\beta} x^{\beta} + K^{ij} x_{j}. \end{cases}$$

The system (18′) is equivalent to the single Fredholm equation[1)]

$$(18'')\qquad L^{\alpha} \bar{y}^{\alpha} = y^{\alpha} + L^{\alpha\beta} y_{\beta} \qquad (0 \leqq \alpha \leqq n+1),$$

---

1) Cf. T.H.Hildebrandt, l.c., p.400; cf. also Dines II, p.89.

where the integration with respect to β is from 0 to $n+1$ and where

$$y^{\alpha} = \begin{cases} x^{\alpha} & (0 \leqq \alpha \leqq 1) \\ x^{i} & (i < \alpha \leqq i+1) \end{cases}$$

$$\bar{y}^{\alpha} = \begin{cases} \bar{x}^{\alpha} & (0 \leqq \alpha \leqq 1) \\ \bar{x}^{i} & (i < \alpha \leqq i+1) \end{cases}$$

$$L^{\alpha} = \begin{cases} \frac{1}{K^{\alpha}} & (0 \leqq \alpha \leqq 1) \\ 1 & (i < \alpha \leqq i+1) \end{cases}$$

$$L^{\alpha\beta} = \begin{cases} \frac{K^{\alpha\beta}}{K^{\alpha}} & (0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1) \\ \frac{K_{j}^{\alpha}}{K^{\alpha}} & (0 \leqq \alpha \leqq 1, j < \beta \leqq j+1) \\ L_{i}^{\beta} & (i < \alpha \leqq i+1, 0 \leqq \beta \leqq 1) \\ K^{ij} & (i < \alpha \leqq i+1, j < \beta \leqq j+1, i \neq j) \\ K^{ij}-1 & (i < \alpha \leqq i+1, j < \beta \leqq j+1, i = j) \end{cases}$$

The Fredholm determinant of (18″) is found on calculation to be 1)

$$(20) \qquad D = \left| K_{ij} \right| + \sum_{m=1}^{\infty} \frac{1}{m!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} \frac{K^{\alpha_s \alpha_t}}{K^{\alpha_s}} & \frac{K_j^{\alpha_s}}{K^{\alpha_s}} \\ L_i^{\alpha_t} & K_{ij} \end{vmatrix} d\alpha_1 \cdots d\alpha_m$$

$$(s, t = 1, \cdots, m).$$

---

1) This determinant is given, but in slightly erroneous form, both by Hildebrandt, l.c. , p. 404, and by Michal and Kennison, l.c., p.618.

We shall call $\mathcal{D}$ the Fredholm determinant of (18).

From the well-known theorem that the determinant of the product of two Fredholm transformations is equal to the product of the determinants of the transformations, there follows at once

THEOREM 12. *The determinant of the product of two linear transformations in $(C;E_m)$ is equal to the product of the determinants of the transformations.*

2. *Non-singular Transformations.* A linear transformation (18) whose Fredholm determinant is different from zero will be termed non-singular.

If (18) is non-singular, (18'') has a unique solution linear in $\bar{y}^\alpha$; hence (18), when non-singular, has a unique inverse linear in $\bar{x}$. Let this inverse be

$$(18.1)\quad \begin{cases} x^\alpha = \bar{K}^\alpha \bar{x}^\alpha + \bar{K}^{\alpha\beta} \bar{x}_\beta + \bar{K}^\alpha_{\dot{j}} \bar{x}^{\dot{j}} & (\bar{K}^\alpha \neq 0), \\ x^{\dot{i}} = \bar{L}^{\dot{i}}_\beta \bar{x}^\beta + \bar{K}^{\dot{i}\dot{j}} \bar{x}_{\dot{j}} . \end{cases}$$

We wish to show that (18.1) is non-singular. By Theorem 11 the product of (18) and (18.1) is

$$(19.1)\quad \begin{cases} x^\alpha = \bar{\bar{K}}^\alpha x^\alpha + \bar{\bar{K}}^{\alpha\beta} x_\beta + \bar{\bar{K}}^\alpha_{\dot{j}} x^{\dot{j}}, \\ x^{\dot{i}} = \bar{\bar{L}}^{\dot{i}}_\beta x^\beta + \bar{\bar{K}}^{\dot{i}\dot{j}} x_{\dot{j}} \end{cases}$$

where $\overline{\overline{K}}^{\alpha}_{\alpha}$, $\overline{\overline{K}}^{\alpha\beta}_{\alpha\beta}$, $\overline{\overline{K}}^{\alpha}_{\alpha}$, $\overline{\overline{L}}^{i}_{i}$, $\overline{\overline{K}}^{ij}$ have the values

where $\overline{\overline{K}}^{\alpha}$, $\overline{\overline{K}}^{\alpha\beta}$, $\overline{\overline{K}}^{\alpha}_{j}$, $\overline{\overline{L}}^{i}_{\beta}$, $\overline{\overline{K}}^{ij}$ have the values given by (19.1). But (19.1), being the product of a transformation and its inverse, is the identity transformation. Hence we have $\overline{\overline{K}}^{\alpha} = \overline{\overline{K}}^{ij} = 1$ $(i = j)$ and all other coefficients in (19.1) identically zero. The Fredholm determinant of (19.1) is consequently equal to unity. Representing the Fredholm determinant of (18.1) by $\overline{D}$, we have $D\overline{D} = 1$ by Theorem 12, whence $\overline{D} \neq 0$. We have, then, proved

THEOREM 13. *The inverse of a non-singular linear transformation in $(C, E_{\sim})$ exists and is itself a non-singular linear transformation in $(C, E_{\sim})$.*

From Theorems 11 and 13 follows

THEOREM 14. *The totality of non-singular linear transformations in $(C, E_{\sim})$ constitute a group.*

3. *Infinitesimal Linear Transformations.* An infinitesimal linear transformation in $(C, E_{\sim})$ is a transformation defined by a system of linear homogeneous integro-differential equations

$$(21)\quad \begin{cases} \dfrac{dx^{\alpha}}{dt} = \varphi^{\alpha}(x) \equiv K^{\alpha} x^{\alpha} + K^{\alpha\beta} x_{\beta} + K^{\alpha}_{j} x^{j} \\[2ex] \dfrac{dx^{i}}{dt} = \varphi^{i}(x) \equiv \qquad L^{i}_{\beta} x^{\beta} + K^{ij} x_{j} \end{cases}$$

Let (21) and

$$(\overline{21})\quad\begin{cases}\dfrac{dx^\alpha}{dt} = \bar{\varphi}^\alpha(x) \equiv \bar{K}^\alpha x^\alpha + \bar{K}^{\alpha\beta} x_\beta + \bar{K}^\alpha_j x^j, \\[2ex] \dfrac{dx^i}{dt} = \bar{\varphi}^i(x) \equiv \qquad \bar{L}^i_\beta x^\beta + \bar{K}^{ij} x_j\end{cases}$$

be any two infinitesimal linear transformations in (C, E_n). To show that the group property is present, we compute their commutator. Applying

$$\varphi^\alpha(x) = K^\alpha x^\alpha, \ \varphi^i(x) = 0, \bar{\varphi}^\alpha(x) = \bar{K}^\alpha x^\alpha, \bar{\varphi}^i(x) = 0,$$

we get

$$(\varphi\bar{\varphi})^\alpha(x) = 0, \quad (\varphi\bar{\varphi})^i(x) = 0 .$$

Applying

$$\varphi^\alpha(x) = K^\alpha x^\alpha, \varphi^i(x) = 0, \bar{\varphi}^\alpha(x) = \bar{K}^{\alpha\beta} x_\beta, \bar{\varphi}^i(x) = 0,$$

we get

$$(\varphi\bar{\varphi})^\alpha(x) = \bar{\bar{K}}^{\alpha\beta} x_\beta, \ (\varphi\bar{\varphi})^i(x) = 0,$$

where

$$\bar{\bar{K}}^{\alpha\beta} = \bar{K}^{\alpha\beta}(K^\beta - K^\alpha).$$

Applying

$$\varphi^\alpha(x) = K^\alpha x^\alpha, \ \varphi^i(x) = 0, \bar{\varphi}^\alpha(x) = \bar{K}^\alpha_j x^j, \bar{\varphi}^i(x) = 0,$$

we get

$$(\varphi\bar{\varphi})^\alpha(x) = \bar{\bar{K}}^\alpha_j x^j, \ (\varphi\bar{\varphi})^i(x) = 0,$$

where

$$\bar{\bar{K}}^\alpha_j = -K^\alpha \bar{K}^\alpha_j .$$

Applying

$$\varphi^\alpha(x) = K^\alpha x^\alpha, \varphi^i(x) = 0, \bar{\varphi}^\alpha(x) = 0, \bar{\varphi}^i(x) = \bar{L}^i_\beta x^\beta,$$

we get
$$(\varphi\bar{\varphi}^{\alpha})(x)=0,\ (\varphi\bar{\varphi})^{i}(x)=\bar{\bar{L}}^{i}_{\beta}x^{\beta},$$

where
$$\bar{\bar{L}}^{i}_{\beta}=\bar{L}^{i}_{\beta}K_{\beta}\,.$$

Applying
$$\varphi^{\alpha}(x)=K^{\alpha}x^{\alpha},\ \varphi^{i}(x)=0,\ \bar{\varphi}^{\alpha}(x)=0,\ \bar{\varphi}^{i}(x)=\bar{K}^{ij}x_{j},$$

we get
$$(\varphi\bar{\varphi})^{\alpha}(x)=0,\ (\varphi\bar{\varphi})^{i}(x)=0\,.$$

Applying
$$\varphi^{\alpha}(x)=K^{\alpha\beta}x_{\beta},\ \varphi^{i}(x)=0,\ \bar{\varphi}^{\alpha}(x)=\bar{K}^{\alpha\beta}x_{\beta},\ \bar{\varphi}^{i}(x)=0,$$

we get
$$(\varphi\bar{\varphi})^{\alpha}(x)=\bar{\bar{K}}^{\alpha\beta}x_{\beta},\ (\varphi\bar{\varphi})^{i}(x)=0,$$

where
$$\bar{\bar{K}}^{\alpha\beta}=\bar{K}_{\alpha\gamma}K^{\gamma\beta}-K_{\alpha\gamma}\bar{K}^{\gamma\beta}.$$

Applying
$$\varphi^{\alpha}(x)=K^{\alpha\beta}x_{\beta},\ \varphi^{i}(x)=0,\ \bar{\varphi}^{\alpha}(x)=\bar{K}^{\alpha}_{j}x^{j},\ \bar{\varphi}^{i}(x)=0,$$

we get
$$(\varphi\bar{\varphi})^{\alpha}(x)=\bar{\bar{K}}^{\alpha}_{j}x^{j},\ (\varphi\bar{\varphi})^{i}(x)=0,$$

where
$$\bar{\bar{K}}^{\alpha}_{j}=-K_{\alpha\beta}\bar{K}^{\beta}_{j}\,.$$

Applying
$$\varphi^{\alpha}(x)=K^{\alpha\beta}x_{\beta},\ \varphi^{i}(x)=0,\ \bar{\varphi}^{\alpha}(x)=0,\ \bar{\varphi}^{i}(x)=\bar{L}^{i}_{\beta}x^{\beta},$$

we get
$$(\varphi\bar{\varphi})^{\alpha}(x)=0,\ (\varphi\bar{\varphi})^{i}(x)=\bar{\bar{L}}^{i}_{\beta}x^{\beta},$$

where
$$\bar{\bar{L}}^{i}_{\beta}=\bar{L}^{\gamma}_{i}K_{\gamma\beta}\,.$$

Applying

$$\mathfrak{p}^{\alpha}(x) = K^{\alpha\beta}x_{\beta}, \ \mathfrak{p}^{i}(x) = 0, \ \bar{\mathfrak{p}}^{\alpha}(x) = 0, \ \bar{\mathfrak{p}}^{i}(x) = \bar{K}^{ij}x_{j},$$

we get

$$(\mathfrak{p}\bar{\mathfrak{p}})^{\alpha}(x) = 0, \ (\mathfrak{p}\bar{\mathfrak{p}})^{i}(x) = 0.$$

Applying

$$\mathfrak{p}^{\alpha}(x) = K^{\alpha}_{j}x^{j}, \ \mathfrak{p}^{i}(x) = 0, \ \bar{\mathfrak{p}}^{\alpha}(x) = \bar{K}^{\alpha}_{j}x^{j}, \ \bar{\mathfrak{p}}^{i}(x) = 0,$$

we get

$$(\mathfrak{p}\bar{\mathfrak{p}})^{\alpha}(x) = 0, \ (\mathfrak{p}\bar{\mathfrak{p}})^{i}(x) = 0.$$

Applying

$$\mathfrak{p}^{\alpha}(x) = K^{\alpha}_{j}x^{j}, \ \mathfrak{p}^{i}(x) = 0, \ \bar{\mathfrak{p}}^{\alpha}(x) = 0, \ \bar{\mathfrak{p}}^{i}(x) = \bar{L}^{i}_{\beta}x^{\beta},$$

we get

$$(\mathfrak{p}\bar{\mathfrak{p}})^{\alpha}(x) = \bar{\bar{\bar{K}}}^{\alpha\beta}x_{\beta}, \ (\mathfrak{p}\bar{\mathfrak{p}})^{i}(x) = \bar{\bar{K}}^{ij}x_{j},$$

where

$$\bar{\bar{\bar{K}}}^{\alpha\beta} = -K^{j}_{\alpha}\bar{L}^{\beta}_{j}, \ \bar{\bar{K}}^{ij} = \bar{L}^{\beta}_{i}K^{j}_{\beta}.$$

Applying

$$\mathfrak{p}^{\alpha}(x) = K^{\alpha}_{j}x^{j}, \ \mathfrak{p}^{i}(x) = 0, \ \bar{\mathfrak{p}}^{\alpha}(x) = 0, \ \bar{\mathfrak{p}}^{i}(x) = \bar{K}^{ij}x_{j},$$

we get

$$(\mathfrak{p}\bar{\mathfrak{p}})^{\alpha}(x) = \bar{\bar{\bar{K}}}^{\alpha}_{j}x^{j}, \quad (\mathfrak{p}\bar{\mathfrak{p}})^{i}(x) = 0,$$

where

$$\bar{\bar{\bar{K}}}^{\alpha}_{j} = -K^{\alpha}_{k}\bar{K}^{kj}.$$

Applying

$$\mathfrak{p}^{\alpha}(x) = 0, \ \mathfrak{p}^{i}(x) = L^{i}_{\beta}x^{\beta}, \ \bar{\mathfrak{p}}^{\alpha}(x) = 0, \ \bar{\mathfrak{p}}^{i}(x) = \bar{L}^{i}_{\beta}x^{\beta},$$

we get

$$(\mathfrak{p}\bar{\mathfrak{p}})^{\alpha}(x) = 0, \ (\mathfrak{p}\bar{\mathfrak{p}})^{i}(x) = 0.$$

Applying

$$\mathfrak{p}^{\alpha}(x) = 0,\ \mathfrak{p}^{i}(x) = L_{\beta}^{\ i} x^{\beta},\ \bar{\mathfrak{p}}^{\alpha}(x) = 0,\ \bar{\mathfrak{p}}^{i}(x) = \bar{K}^{ij} x_{j},$$

we get

$$(\mathfrak{p}\bar{\mathfrak{p}})^{\alpha}(x) = 0,\ (\mathfrak{p}\bar{\mathfrak{p}})^{i}(x) = \overset{\equiv}{L}_{\beta}^{\ i} x^{\beta},$$

where

$$\overset{\equiv}{L}_{\beta}^{\ i} = \bar{K}_{ij} L_{\beta}^{\ j}.$$

Applying

$$\mathfrak{p}^{\alpha}(x) = 0,\ \mathfrak{p}^{i}(x) = K^{ij} x_{j},\ \bar{\mathfrak{p}}^{\alpha}(x) = 0,\ \bar{\mathfrak{p}}^{i}(x) = \bar{K}^{ij} x_{j},$$

we get

$$(\mathfrak{p}\bar{\mathfrak{p}})^{\alpha}(x) = 0,\ (\mathfrak{p}\bar{\mathfrak{p}})^{i}(x) = \overset{\equiv}{K}^{ij} x_{j},$$

where

$$\overset{\equiv}{K}^{ij} = \bar{K}_{ik} K^{kj} - K_{ik} \bar{K}^{kj}.$$

From these computations we see that the commutator of any two infinitesimal linear transformations in $(C,\underset{\sim}{E})$ is an infinitesimal linear transformation in $(C,\underset{\sim}{E})$. We thus have

THEOREM 15. *The totality of infinitesimal linear transformations in* $(C,\underset{\sim}{E})$ *constitute a group*.

4. *The Finite Transformations Generated by an Infinitesimal Linear Transformation*. Let the system of integro-differential equations

$$(22)\quad \begin{cases} \dfrac{\partial \bar{x}^{\alpha}(t)}{\partial t} = K^{\alpha} \bar{x}^{\alpha}(t) + K^{\alpha\beta} \bar{x}_{\beta}(t) + K_{j}^{\ \alpha} \bar{x}^{j}(t), \\[2ex] \dfrac{\partial \bar{x}^{i}(t)}{\partial t} = \qquad\qquad L_{\beta}^{\ i} \bar{x}^{\beta}(t) + K^{ij} \bar{x}_{j}(t), \end{cases}$$

with the boundary conditions

(22′) $\bar{x}^{\alpha}(0) = x^{\alpha}, \quad \bar{x}^{i}(0) = x^{i},$

define a given infinitesimal linear transformation in $(C, \underset{\sim}{E})$. We shall show that this system has a unique solution

$$(23)\quad \begin{cases} \bar{x}^{\alpha}(t) = M^{\alpha}(t)\,x^{\alpha} + M^{\alpha\beta}(t)\,x_{\beta} + M^{\alpha}_{j}(t)\,x^{j}, \\ \bar{x}^{i}(t) = \qquad\qquad N^{i}_{\beta}(t)\,x^{\beta} + M^{ij}(t)\,x_{j}, \end{cases}$$

where the coefficients are to be determined in terms of the coefficients of (22).[1]

Let us assume that (23) satisfies (22). Differentiating equations (23) with respect to t, we get

$$(23.1)\quad \begin{cases} \dfrac{\partial \bar{x}^{\alpha}(t)}{\partial t} = \dfrac{\partial M^{\alpha}(t)}{\partial t}\,x^{\alpha} + \dfrac{\partial M^{\alpha\beta}(t)}{\partial t}\,x_{\beta} + \dfrac{\partial M^{\alpha}_{j}(t)}{\partial t}\,x^{j}, \\ \dfrac{\partial \bar{x}^{i}(t)}{\partial t} = \qquad\qquad \dfrac{\partial N^{i}_{\beta}(t)}{\partial t}\,x^{\beta} + \dfrac{\partial M^{ij}(t)}{\partial t}\,x_{j}. \end{cases}$$

Substitution in (23.1) of the values of $\dfrac{\partial \bar{x}^{\alpha}(t)}{\partial t}$ and $\dfrac{\partial \bar{x}^{i}(t)}{\partial t}$ from (22) gives

$$(24)\quad \begin{cases} K^{\alpha}\bar{x}^{\alpha}(t) + K^{\alpha\beta}\bar{x}_{\beta}(t) + K^{\alpha}_{j}\bar{x}^{j}(t) = \dfrac{\partial M^{\alpha}(t)}{\partial t}\,x^{\alpha} + \\ \qquad\qquad + \dfrac{\partial M^{\alpha\beta}(t)}{\partial t}\,x_{\beta} + \dfrac{\partial M^{\alpha}_{j}(t)}{\partial t}\,x^{j}, \\ L^{i}_{\beta}\bar{x}^{\beta}(t) + K^{ij}\bar{x}_{j}(t) = \dfrac{\partial N^{i}_{\beta}(t)}{\partial t}\,x^{\beta} + \dfrac{\partial M^{ij}(t)}{\partial t}\,x_{j}. \end{cases}$$

---

1) Cf. Barnett I, p. 273.

Replacing $\bar{x}^\alpha(t)$, $\bar{x}^i(t)$ in (24) by their values from (23), we obtain the following system of identities in x and α :

$$\left\{\begin{aligned}
&\left[K^\alpha M^\alpha(t) - \frac{\partial M^\alpha(t)}{\partial t}\right]x^\alpha + \Big[K^\alpha M^{\alpha\beta}(t) + K^{\alpha\beta}M^\beta(t) \\
&\qquad + K_{\alpha\gamma}M^{\gamma\beta}(t) + K^{j}_{\alpha}N^{\beta}_{j}(t) - \frac{\partial M^{\alpha\beta}(t)}{\partial t}\Big]x_\beta \\
&\qquad + \Big[K^\alpha M^{\alpha}_{j}(t) + K_{\alpha\beta}M^{\beta}_{j}(t) + K^{k}_{\alpha}M_{kj}(t) \\
&\qquad - \frac{\partial M^{\alpha}_{j}(t)}{\partial t}\Big]x^j = 0, \\
&\Big[L^{i}_{\beta}M_{\beta}(t) + L^{\gamma}_{i}M_{\gamma\beta}(t) + K_{ij}N^{j}_{\beta}(t) - \frac{\partial N^{i}_{\beta}(t)}{\partial t}\Big]x^\beta \\
&\qquad + \Big[L^{\beta}_{i}M^{j}_{\beta}(t) + K_{ik}M^{kj}(t) - \frac{\partial M^{ij}(t)}{\partial t}\Big]x_j = 0 .
\end{aligned}\right. \tag{24.1}$$

By Lemma 1 we deduce from (24.1) the system of linear homogeneous integro-differential equations

$$\left\{\begin{aligned}
&\frac{\partial M^\alpha(t)}{\partial t} = K^\alpha M^\alpha(t), \\
&\frac{\partial M^{\alpha\beta}(t)}{\partial t} = K^\alpha M^{\alpha\beta}(t) + K^{\alpha\beta}M^\beta(t) + K_{\alpha\gamma}M^{\gamma\beta}(t) + K^{j}_{\alpha}N^{\beta}_{j}(t), \\
&\frac{\partial M^{\alpha}_{j}(t)}{\partial t} = K^\alpha M^{\alpha}_{j}(t) + K_{\alpha\beta}M^{\beta}_{j}(t) + K^{k}_{\alpha}M_{kj}(t), \\
&\frac{\partial N^{i}_{\beta}(t)}{\partial t} = L^{i}_{\beta}M_{\beta}(t) + L^{\gamma}_{i}M_{\gamma\beta}(t) + K_{ij}N^{j}_{\beta}(t), \\
&\frac{\partial M^{ij}(t)}{\partial t} = L^{\beta}_{i}M^{j}_{\beta}(t) + K_{ik}M^{kj}(t)
\end{aligned}\right. \tag{24.11}$$

with the boundary conditions

$$(24.11') \quad M^{\alpha}(0)=1,\ M^{\alpha\beta}(0)=M^{\alpha}_{j}(0)=N^{i}_{\beta}(0)=0,\ M^{ij}(0)=\begin{cases}1 & (i=j)\\ 0 & (i\neq j).\end{cases}$$

Let us assume that the system (24.11) with the boundary conditions (24.11) is satisfied by a set of functions analytic in t and regular at $t=0$.[1)]

Representing

$$\left.\frac{\partial^{m} M^{\alpha}(t)}{\partial t^{m}}\right]_{t=0},\quad \left.\frac{\partial^{m} M^{\alpha\beta}(t)}{\partial t^{m}}\right]_{t=0},\quad \left.\frac{\partial^{m} M^{\alpha}_{j}(t)}{\partial t^{m}}\right]_{t=0},$$

$$\left.\frac{\partial^{m} N^{i}_{\beta}(t)}{\partial t^{m}}\right]_{t=0},\quad \left.\frac{\partial^{m} M^{ij}(t)}{\partial t^{m}}\right]_{t=0}$$

respectively by

$$M^{\alpha}[m],\ M^{\alpha\beta}[m],\ M^{\alpha}_{j}[m],\ N^{i}_{\beta}[m],\ M^{ij}[m],$$

the assumed solution of (24.11) is of the form

$$(25)\quad \begin{cases} M^{\alpha}(t) = \sum\limits_{m=0}^{\infty} \dfrac{M^{\alpha}[m]}{m!}\, t^{m}, \\ M^{\alpha\beta}(t) = \sum\limits_{m=0}^{\infty} \dfrac{M^{\alpha\beta}[m]}{m!}\, t^{m}, \\ M^{\alpha}_{j}(t) = \sum\limits_{m=0}^{\infty} \dfrac{M^{\alpha}_{j}[m]}{m!}\, t^{m}, \\ N^{i}_{\beta}(t) = \sum\limits_{m=0}^{\infty} \dfrac{N^{i}_{\beta}[m]}{m!}\, t^{m}, \\ M^{ij}(t) = \sum\limits_{m=0}^{\infty} \dfrac{M^{ij}[m]}{m!}\, t^{m}. \end{cases}$$

---

1) Cf. Dines I, p. 59.

The first term in each of the series expansions (25) is given by (24.11$'$). We thus have

(25.0) $$M^{\alpha}[0]=1,\ M^{\alpha\beta}[0]=M^{\alpha}_{j}[0]=N^{i}_{\beta}[0]=0,\ M^{ij}[0]=\begin{cases}1\ (i=j)\\ 0\ (i\neq j).\end{cases}$$

The coefficients of all subsequent terms are uniquely determined by the following recursion formulas, obtained by repeated differentiation of equations (24.11):

(25.1) $$M^{\alpha}[m+1]=K^{\alpha}M^{\alpha}[m],$$

(25.2) $$\begin{aligned}M^{\alpha\beta}[m+1]&=K^{\alpha}M^{\alpha\beta}[m]+K^{\alpha\beta}M^{\beta}[m]\\&\quad+K_{\alpha\gamma}M^{\gamma\beta}[m]+K^{j}_{\alpha}N^{\beta}_{j}[m],\end{aligned}$$

(25.3) $$M^{\alpha}_{j}[m+1]=K^{\alpha}M^{\alpha}_{j}[m]+K_{\alpha\beta}M^{\beta}_{j}[m]+K^{k}_{\alpha}M_{kj}[m],$$

(25.4) $$N^{i}_{\beta}[m+1]=L^{i}_{\beta}M_{\beta}[m]+L^{\gamma}_{i}M_{\gamma\beta}[m]+K_{ij}N^{j}_{\beta}[m],$$

(25.5) $$M^{ij}[m+1]=L^{\beta}_{i}N^{j}_{\beta}[m]+K_{ik}M^{kj}[m].$$

It remains to establish the convergence of the series (25). Let P, Q, R_{j}, S^{i} be the maxima of the absolute values of K^{α}, $K^{\alpha\beta}$, K^{α}_{j}, L^{i}_{β}

respectively. Put

$$T = P + Q + \sum_{j=1}^{n} R_j + \sum_{i=1}^{n} S^i + \sum_{i,j=1}^{n} K^{ij},$$

and introduce the series

$$(26) \qquad \sum_{m=0}^{\infty} \frac{(T)^m}{m!}(t)^m .$$

Each coefficient numerator in (25) is dominated by the corresponding coefficient numerator $(T)^m$ in (26). Consequently, since (26) converges for all finite values of t, each of the series (25) converges for all finite values of t and, for a fixed t, uniformly in α and β. Hence, for any fixed t, the coefficients $M^\alpha(t)$, $M^{\alpha\beta}(t)$, $M^\alpha_j(t)$, $N^i_\beta(t)$ are continuous functions of their Greek indices. When use is made of (25.1), $M^\alpha(t)$ is seen to have the expansion

$$M^\alpha(t) = e^{M^\alpha(t)}$$

where e is the base of natural logarithms. Hence $M^\alpha(t)$ never vanishes.

We have, then,

THEOREM 16. <u>An infinitesimal linear transformation</u> (22), (22*) <u>generates a one-parameter continuous family of finite linear transformations</u> (23). <u>For any value of the parameter</u> t, <u>the coefficient functions of the generated transformation are given by</u>

(25),(25.0),(25.1),(25.2),(25.3),(25.4), and (25.5).

We next prove

THEOREM 17. *The product of any two of the generated transformations* (23) *corresponding respectively to parameter values* t_1 *and* t_2 *is the member of the family* (23) *corresponding to the parameter value* t_1+t_2.[1)]

PROOF. Let T_{t_1} and T_{t_2} be the members of the family (23) for which $t=t_1$ and $t=t_2$ respectively. The product of T_{t_1} and T_{t_2} is, by Theorem 11,

$$T_{t_1}T_{t_2}:\begin{cases}\bar{x}^{\alpha}(t_2)=\bar{M}^{\alpha}(t_1,t_2)x^{\alpha}+\bar{M}^{\alpha\beta}(t_1,t_2)x_{\beta}+\bar{M}_{j}^{\alpha}(t_1,t_2)x^{j},\\ \bar{x}^{i}(t_2)=\qquad\qquad\bar{N}_{\beta}^{i}(t_1,t_2)x_{\beta}+\bar{M}^{ij}(t_1,t_2)x^{j}\end{cases}$$

where

$$(27)\begin{cases}\bar{M}^{\alpha}(t_1,t_2)=M^{\alpha}(t_2)M^{\alpha}(t_1),\\ \bar{M}^{\alpha\beta}(t_1,t_2)=M^{\alpha}(t_2)M^{\alpha\beta}(t_1)+M^{\alpha\beta}(t_2)M^{\beta}(t_1)\\ \qquad\qquad+M_{\alpha\gamma}(t_2)M^{\gamma\beta}(t_1)+M_{\alpha}^{j}(t_2)N_{j}^{\beta}(t_1),\\ \bar{M}_{j}^{\alpha}(t_1,t_2)=M^{\alpha}(t_2)M_{j}^{\alpha}(t_1)+M_{\alpha\beta}(t_2)M_{j}^{\beta}(t_1)\\ \qquad\qquad+M_{\alpha}^{k}(t_2)M_{kj}(t_1),\\ \bar{N}_{\beta}^{i}(t_1,t_2)=N_{\beta}^{i}(t_2)M_{\beta}(t_1)+N_{i}^{\gamma}(t_2)M_{\gamma\beta}(t_1)\\ \qquad\qquad+M_{ij}(t_2)N_{\beta}^{j}(t_1),\\ M^{ij}(t_1,t_2)=N_{i}^{\beta}(t_2)M_{\beta}^{j}(t_1)+M_{ik}(t_2)M^{kj}(t_1).\end{cases}$$

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1) Cf. L.S.Kennison, l.c., [illegible].

We have to prove the relation

$$T_{t_1} T_{t_2} = T_{t_1+t_2} . \tag{28}$$

In the proof of (28) we shall require the following recursion formulas:

$$M^{\alpha}[p] = M^{\alpha}[m] M^{\alpha}[p-m] , \tag{28.1}$$

$$\begin{aligned} M^{\alpha\beta}[p] = {} & M^{\alpha}[m] M^{\alpha\beta}[p-m] \\ & + M^{\alpha\beta}[m] M^{\beta}[p-m] \\ & + M_{\alpha\gamma}[m] M^{\gamma\beta}[p-m] \\ & + M_{\alpha}^{j}[m] N_{j}^{\beta}[p-m] , \end{aligned} \tag{28.2}$$

$$\begin{aligned} M_{j}^{\alpha}[p] = {} & M^{\alpha}[m] M_{j}^{\alpha}[p-m] + M_{\alpha\beta}[m] M_{j}^{\beta}[p-m] \\ & + M_{\alpha}^{k}[m] M_{kj}[p-m] , \end{aligned} \tag{28.3}$$

$$\begin{aligned} N_{\beta}^{i}[p] = {} & N_{\beta}^{i}[m] M_{\beta}[p-m] + N_{i}^{\gamma}[m] M_{\gamma\beta}[p-m] \\ & + M_{ij}[m] N_{\beta}^{j}[p-m] , \end{aligned} \tag{28.4}$$

$$M^{ij}[p] = N_{i}^{\beta}[m] M_{\beta}^{j}[p-m] + M_{ij}[m] M^{kj}[p-m] , \tag{28.5}$$

where $m \leqq p$.

These formulas are readily verified for $p = 0$, $m = 0$, for $p = 1$, $m = 0$, and for $p = 1$, $m = 1$. Their general validity is established by induction on p. We shall give the details only for (28.1). Assuming that (28.1) is true for $p = k$, where k is any positive integer, and using (25.1), we get

$$M^{\alpha}[k+1] = K^{\alpha} M^{\alpha}[k] = K^{\alpha} M^{\alpha}[m] M^{\alpha}[k-m]$$

$$= M^{\alpha}[m] M^{\alpha}[k+1-m].$$

Hence (28.1) is true.

When use is made, in the right-hand members of (27), of the series expansions (25) and the recursion formulas (28.1), (28.2), (28.3), (28.4), and (28.5), we find

$$\bar{M}^{\alpha}(t_1, t_2) = \sum_{m=0}^{\infty} \frac{(t_1 + t_2)^m}{m!} M^{\alpha}[m] = M^{\alpha}(t_1 + t_2),$$

$$\bar{M}^{\alpha\beta}(t_1, t_2) = \sum_{m=0}^{\infty} \frac{(t_1 + t_2)^m}{m!} M^{\alpha\beta}[m] = M^{\alpha\beta}(t_1 + t_2),$$

$$\bar{M}^{\alpha}_{j}(t_1,t_2)=\sum_{m=0}^{\infty}\frac{(t_1+t_2)^m}{m!}M^{\alpha}_{j}[m]=M^{\alpha}_{j}(t_1+t_2),$$

$$\bar{N}^{i}_{\beta}(t_1,t_2)=\sum_{m=0}^{\infty}\frac{(t_1+t_2)^m}{m!}N^{i}_{\beta}[m]=N^{i}_{\beta}(t_1+t_2),$$

$$\bar{M}^{ij}(t_1,t_2)=\sum_{m=0}^{\infty}\frac{(t_1+t_2)^m}{m!}M^{ij}[m]=M^{ij}(t_1+t_2).$$

Hence (28) is proved and the theorem established.

We now prove

THEOREM 18. The generated transformations (23) are non-singular.

PROOF. Corresponding to the choice Δt for the parameter, the generated transformation is

$$T_{\Delta t}:\begin{cases}\bar{\bar{x}}^{\alpha}(\Delta t)=\bar{M}^{\alpha}(\Delta t)\bar{x}^{\alpha}+\bar{M}^{\alpha\beta}(\Delta t)\bar{x}_{\beta}+\bar{M}^{\alpha}_{j}(\Delta t)\bar{x}^{j}, \\ \bar{\bar{x}}^{i}(\Delta t)= \qquad\qquad \bar{N}^{i}_{\beta}(\Delta t)\bar{x}^{\beta}+\bar{M}^{ij}(\Delta t)\bar{x}_{j},\end{cases}$$

where

$$\bar{M}^{\alpha}(\Delta t) = 1 + K^{\alpha}\Delta t + ((\Delta t^{2})),$$

$$\bar{M}^{\alpha\beta}(\Delta t) = K^{\alpha\beta}\Delta t + ((\Delta t^{2})),$$

$$\bar{M}^{\alpha}_{\cdot j}(\Delta t) = K^{\alpha}_{\cdot j}\Delta t + ((\Delta t^{2})),$$

$$\bar{N}^{i}_{\beta}(\Delta t) = L^{i}_{\beta}\Delta t + ((\Delta t^{2})),$$

$$\bar{M}^{ij}(\Delta t) = 1 + K^{ij}\Delta t + ((\Delta t^{2})), \qquad (i=j),$$

$$\bar{M}^{ij}(\Delta t) = K^{ij}\Delta t + ((\Delta t^{2})) \qquad (i \neq j),$$

the symbol $((\Delta t^{2}))$ standing for terms having as a factor Δt raised to a power not lower than the second. Let $\bar{\mathfrak{D}}(\Delta t)$ denote the Fredholm determinant of $T_{\Delta t}$, and $\mathfrak{D}(t)$ the Fredholm determinant of

$$T_{t} : \begin{cases} \bar{x}^{\alpha}(t) = M^{\alpha}(t)x^{\alpha} + M^{\alpha\beta}(t)x_{\beta} + M^{\alpha}_{\cdot j}(t)x^{j}, \\ \bar{x}^{i}(t) = \qquad N^{\beta}_{\cdot i}(t)x_{\beta} + M_{\cdot ij}(t)x^{j}. \end{cases}$$

By Theorem 17 we have

$$T_{t+\Delta t} = T_{t}\,T_{\Delta t} \,,$$

whence

$$\mathfrak{D}(t+\Delta t) = \text{the Fredholm determinant of } T_{t}\,T_{\Delta t}.$$

But by Theorem 12 the Fredholm determinant of $T_t \, T_{\Delta t}$ equals $\mathcal{D}(t)\bar{\mathcal{D}}(\Delta t)$. Hence we get

$$(29) \qquad \mathcal{D}(t+\Delta t) = \mathcal{D}(t)\,\bar{\mathcal{D}}(\Delta t).$$

Computing $\bar{\mathcal{D}}(\Delta t)$ from (20), we find

$$\bar{\mathcal{D}}(\Delta t) = \left| \bar{M}_{ij}(\Delta t) \right| + \sum_{m=1}^{\infty} \frac{1}{m!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} \dfrac{\bar{M}^{\alpha_s \alpha_t}(\Delta t)}{\bar{M}^{\alpha_s}(\Delta t)} & \dfrac{\bar{M}_j^{\alpha_s}(\Delta t)}{\bar{M}^{\alpha_s}(\Delta t)} \\ \bar{N}_i^{\alpha_t}(\Delta t) & \bar{M}_{ij}(\Delta t) \end{vmatrix} d\alpha_1 \cdots d\alpha_m$$

$$(s, t = 1, \ldots, m)$$

$$= \begin{vmatrix} 1+K_{11}\Delta t+((\Delta t^2)) & K_{12}\Delta t+((\Delta t^2)) \cdots & K_{1m}\Delta t+((\Delta t^2)) \\ K_{21}\Delta t+((\Delta t^2)) & 1+K_{22}\Delta t+((\Delta t^2)) \cdots & K_{2m}\Delta t+((\Delta t^2)) \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ K_{m1}\Delta t+((\Delta t^2)) & K_{m2}\Delta t+((\Delta t^2)) \cdots & 1+K_{mm}\Delta t+((\Delta t^2)) \end{vmatrix}$$

$$+\int_0^1 \begin{vmatrix} \dfrac{K^{\alpha_1\alpha_1}\Delta t+((\Delta t^2))}{1+K^{\alpha_1}\Delta t+((\Delta t^2))} & \dfrac{K_1^{\alpha_1}\Delta t+((\Delta t^2))}{1+K^{\alpha_1}\Delta t+((\Delta t^2))} & \dfrac{K_2^{\alpha_1}\Delta t+((\Delta t^2))}{1+K^{\alpha_1}\Delta t+((\Delta t^2))} \cdots & \dfrac{K_m^{\alpha_1}\Delta t+((\Delta t^2))}{1+K^{\alpha_1}\Delta t+((\Delta t^2))} \\ L_1^{\alpha_1}\Delta t+((\Delta t^2)) & 1+K_{11}\Delta t+((\Delta t^2)) & K_{12}\Delta t+((\Delta t^2)) \cdots & K_{1m}\Delta t+((\Delta t^2)) \\ L_2^{\alpha_1}\Delta t+((\Delta t^2)) & K_{21}\Delta t+((\Delta t^2)) & 1+K_{22}\Delta t+((\Delta t^2)) \cdots & K_{2m}\Delta t+((\Delta t^2)) \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\ L_m^{\alpha_1}\Delta t+((\Delta t^2)) & K_{m1}\Delta t+((\Delta t^2)) & K_{m2}\Delta t+((\Delta t^2)) \cdots & 1+K_{mm}\Delta t+((\Delta t^2)) \end{vmatrix}$$

$$d\alpha_1 + ((\Delta t^2))$$

$$= 1 + \left[K_{11} + K_{22} + \cdots + K_{mm} + \int_0^1 K^{\alpha_1 \alpha_1} d\alpha_1\right] \Delta t + ((\Delta t^2)).$$

Then in view of (29) we have

$$D(t+\Delta t) = D(t)\left[1 + \left(K_{11} + K_{22} + \cdots + K_{mm} + \int_0^1 K^{\alpha_1 \alpha_1} d\alpha_1\right)\Delta t + ((\Delta t^2))\right],$$

or

$$(29.1) \quad \frac{D(t+\Delta t) - D(t)}{\Delta t} = D(t)\left[K_{11} + K_{22} + \cdots + K_{mm} + \int_0^1 K^{\alpha_1 \alpha_1} d\alpha_1 + ((\Delta t))\right],$$

where $((\Delta t))$ denotes terms having as a factor Δt to some power not lower than the first. As $\Delta t \to 0$, the right-hand side of (29.1) approaches a definite limit. We therefore have

$$\frac{1}{D(t)} \frac{dD(t)}{dt} = K_{11} + K_{22} + \cdots + K_{mm} + \int_0^1 K^{\alpha_1 \alpha_1} d\alpha_1 .$$

Integrating, we get

$$\log D(t) = \int_0^t \left[K_{11} + K_{22} + \cdots + K_{mm} + \int_0^1 K^{\alpha_1 \alpha_1} d\alpha_1\right] dt,$$

the constant of integration being zero since $D(0) = 1$. Then

$$D(t) = e^{\int_0^t \left[K_{11} + K_{22} + \cdots + K_{mm} + \int_0^1 K^{\alpha_1 \alpha_1} d\alpha_1\right] dt},$$

where e is the base of natural logarithms; hence $D(t)$ is different from zero for all values of t, and the theorem is proved.

From Theorems 13 and 18 we have

THEOREM 19. *The inverse of every member of the family* (23) *exists and is a non-singular linear transformation in* (C, E_m).

Finally we prove

THEOREM 20. *The inverse of any member* T_{t_1} *of the family* (23) *is a member of the family, namely,* T_{-t_1}.

PROOF. By Theorem 17 we have

$$T_{t_1} T_{-t_1} = T_0 .$$

But T_0 is the identity transformation, as is seen from (25) and (25.0). Hence the inverse of T_{t_1} is T_{-t_1}.

Theorems 16, 17, 18, 19, and 20 are combined in

THEOREM 21. *An infinitesimal linear transformation* (22), (22′) *generates a one-parameter continuous group of non-singular finite linear transformations* (23). *For any value of the parameter* t, *the coefficients of the generated transformation are given by* (25), (25.0), (25.1), (25.2), (25.3), (25.4), and (25.5).

PART IV

PROJECTIVE GROUP IN (C, E_n)

1. <u>Homogeneous</u> <u>Coordinates</u>. Homogeneous coordinates of a point y are any $n+2$ ordered elements

$$x^{\alpha},\ x^{1},\ \dots,\ x^{n+1} \qquad (x^{n+1} \neq 0)$$

such that the following relations hold:

$$\frac{x^{\alpha}}{x^{n+1}} = y^{\alpha}, \quad \frac{x^{i}}{x^{n+1}} = y^{i}. \tag{30}$$

Then the $n+2$ ordered elements

$$r x^{\alpha},\ r x^{1},\ \dots,\ r x^{n+1} \qquad (r \neq 0)$$

are homogeneous coordinates of the same point y.

Conversely, any $n+2$ ordered elements

$$r x^{\alpha},\ r x^{1},\ \dots,\ r x^{n+1} \qquad (x^{n+1} \neq 0, r \neq 0)$$

are homogeneous coordinates of some point

$$y = \left(\frac{x^{\alpha}}{x^{n+1}}, \frac{x^{1}}{x^{n+1}}, \dots, \frac{x^{n}}{x^{n+1}} \right). \tag{31}$$

A transformation of the form

$$\left\{ \begin{aligned} r\bar{x}^{\alpha} &= K^{\alpha} x^{\alpha} + K^{\alpha\beta} x_{\beta} + K^{\alpha}_{j} x^{j} + K^{\alpha}_{n+1} x^{n+1}, \\ r\bar{x}^{i} &= \qquad L^{i}_{\beta} x^{\beta} + K^{ij} x_{j} + K^{i\,n+1} x_{n+1}, \\ r\bar{x}^{n+1} &= \qquad L^{n+1}_{\beta} x^{\beta} + K^{n+1\,j} x_{j} + K^{n+1\,n+1} x_{n+1}, \\ & r \neq 0,\ K^{\alpha} \neq 0,\ x^{n+1} \neq 0,\ \bar{x}^{n+1} \neq 0. \end{aligned} \right. \tag{32}$$

can be regarded as a finite transformation in (C,E_n) in homogeneous coordinates taking any point (31) into a point

$$\bar{y} = \left(\frac{\bar{x}^{\alpha}}{\bar{x}^{n+1}}, \frac{\bar{x}^{1}}{\bar{x}^{n+1}}, \dots, \frac{\bar{x}^{n}}{\bar{x}^{n+1}} \right).$$

Since the transformations (32) are linear, the results in Part III are applicable. Hence those transformations (32) which are non-singular, i.e., whose Fredholm determinant

$$(33) \quad D = \begin{vmatrix} K_{ij} & K_{i\,n+1} \\ K_{n+1\,j} & K_{n+1\,n+1} \end{vmatrix} + \sum_{m=1}^{\infty} \int_0^1 \cdots \int_0^1 \begin{vmatrix} \frac{K^{\alpha_s \alpha_t}}{K^{\alpha_s}} & \frac{K_j^{\alpha_s}}{K^{\alpha_s}} & \frac{K_{n+1}^{\alpha_s}}{K^{\alpha_s}} \\ L_i^{\alpha_t} & K_{ij} & K_{i\,n+1} \\ L_{n+1}^{\alpha_t} & K_{n+1\,j} & K_{n+1\,n+1} \end{vmatrix}$$

$$d\alpha_1 \cdots d\alpha_m \qquad (s, t = 1, \dots, m).$$

is different from zero, form a group.

2. Finite Projective Group. A finite projective transformation in (C,E_n) in homogeneous coordinates is defined as the most general non-singular transformation (32) taking a straight line into a straight line in (C,E_n).

Let

$$(34) \qquad x^{\alpha}, \; x^{1}, \; \dots, \; x^{n+1}$$

and

$$(34') \qquad z^{\alpha}, z^{1}, \ldots, z^{n+1}$$

be respective homogeneous coordinates of any two points in (C, E_n). An arbitrary point of the line joining these two points has the homogeneous coordinates

$$ax^{\alpha} + bz^{\alpha},\ ax^{1} + bz^{1}, \ldots, ax^{n+1} + bz^{n+1}.$$

Let (32) be applied to (34) and let the corresponding transformation

$$(32')\left\{\begin{aligned} &r\bar{z}^{\alpha} = K^{\alpha} z^{\alpha} + K^{\alpha\beta} z_{\beta} + K^{\alpha}_{j} z^{j} + K^{\alpha}_{n+1} z^{n+1}, \\ &r\bar{z}^{i} = \quad L^{i}_{\beta} z^{\beta} + K^{ij} z_{j} + K^{i\,n+1} z_{n+1}, \\ &r\bar{z}^{n+1} = \quad L^{n+1}_{\beta} z^{\beta} + K^{n+1\,j} z_{j} + K^{n+1\,n+1} z_{n+1}, \\ &\qquad r \neq 0,\ K^{\alpha} \neq 0,\ z^{n+1} \neq 0,\ \bar{z}^{n+1} \neq 0, \end{aligned}\right.$$

be applied to (34'). Multiplying the equations in (32) by a and the equations in (32') by b, and adding the corresponding equations, we get

$$(34.1)\left\{\begin{aligned} &r(a\bar{x}^{\alpha} + b\bar{z}^{\alpha}) = K^{\alpha}(ax^{\alpha} + bz^{\alpha}) + K^{\alpha\beta}(ax_{\beta} + bz_{\beta}) \\ &\qquad + K^{\alpha}_{j}(ax^{j} + bz^{j}) + K^{\alpha}_{n+1}(ax^{n+1} + bz^{n+1}), \\ &r(a\bar{x}^{i} + b\bar{z}^{i}) = L^{i}_{\beta}(ax^{\beta} + bz^{\beta}) + K^{ij}(ax_{j} + bz_{j}) \\ &\qquad + K^{i\,n+1}(ax_{n+1} + bz_{n+1}), \end{aligned}\right.$$

$$\left\{\begin{aligned} r(a\bar{x}^{n+1}+b\bar{z}^{n+1}) = {} & L_{\beta}^{n+1}(ax^{\beta}+bz^{\beta}) \\ & + K^{n+1j}(ax_j+bz_j)+K^{n+1\,n+1}(ax_{n+1}+bz_{n+1}). \end{aligned}\right.$$

Equations(34.1)tell us that the point with homogeneous coordinates

$$ax^{\alpha}+bz^{\alpha},\ ax^{1}+bz^{1},\ldots,ax^{n+1}+bz^{n+1}$$

is taken into the point with homogeneous coordinates

$$a\bar{x}^{\alpha}+b\bar{z}^{\alpha},\ a\bar{x}^{1}+b\bar{z}^{1},\ldots,a\bar{x}^{n+1}+b\bar{z}^{n+1}.$$

In other words, an arbitrary point of the line joining the points (34) and (34′) is taken into a point of the line joining $\bar{x}^{\alpha}, \bar{x}^{1},\ldots,\bar{x}^{n+1}$ and $\bar{z}^{\alpha}, \bar{z}^{1},\ldots,\bar{z}^{n+1}$. Combining this result with the fact that the non-singular transformations (32) form a group, we have

THEOREM 22. <u>The totality of non-singular transformations (32) constitute the group of finite projective transformations in (C,E_n) in homogeneous coordinates.</u>

A second characterization of the finite projective transformations in (C,E_n) in homogeneous coordinates is that they are the most general non-singular transformations (32) taking a lineoid 1) into a lineoid in (C,E_n). That this definition is the equivalent of the one already given is seen from

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1) Cf. Dines II, p. 89.

THEOREM 23. *The non-singular transformations* (32) *take a lineoid in* (C,E_n)

(35) $$M^{\beta}\bar{x}_{\beta} + M^{j}\bar{x}_{j} + M^{n+1}\bar{x}_{n+1} = 0 \qquad (\bar{x}^{n+1} \neq 0)$$

into a lineoid in (C,E_n).

PROOF. Replacing $\bar{x}_{\beta}$, $\bar{x}_{j}$, and $\bar{x}_{n+1}$ in (35) by their values from (32), we get the equation of a lineoid

$$\bar{M}^{\beta}x_{\beta} + \bar{M}^{j}x_{j} + \bar{M}^{n+1}x_{n+1} = 0,$$

where

$$\bar{M}^{\beta} = M^{\beta}K^{\beta} + M_{\gamma}K^{\gamma\beta} + M^{j}L_{j}^{\beta} + M^{n+1}L_{n+1}^{\beta},$$

$$\bar{M}^{j} = M^{\beta}K_{\beta}^{j} + M_{k}K^{kj} + M_{n+1}K^{n+1\,j},$$

$$\bar{M}^{n+1} = M^{\beta}K_{\beta}^{n+1} + M_{k}K^{k\,n+1} + M_{n+1}K^{n+1\,n+1}.$$

The finite projective group in (C,E_n) in non-homogeneous coordinates, obtained by making in (32) the substitutions (30) and

$(\overline{30})$ $$\frac{\bar{x}^{\alpha}}{\bar{x}^{n+1}} = \bar{y}^{\alpha}, \quad \frac{\bar{x}^{i}}{\bar{x}^{n+1}} = \bar{y}^{i} \qquad (\bar{x}^{n+1} \neq 0)$$

consists of all transformations

$$\begin{cases} \bar{y}^{\alpha} = \dfrac{K^{\alpha}y^{\alpha} + K^{\alpha\beta}y_{\beta} + K_{j}^{\alpha}y^{j} + K_{n+1}^{\alpha}}{L_{\gamma}^{n+1}y^{\gamma} + K^{n+1\,k}y_{k} + K^{n+1\,n+1}} \qquad (K^{\alpha} \neq 0), \\[2ex] \bar{y}^{i} = \dfrac{L_{\beta}^{i}y^{\beta} + K^{ij}y_{j} + K^{i\,n+1}}{L_{\gamma}^{n+1}y^{\gamma} + K^{n+1\,k}y_{k} + K^{n+1\,n+1}}. \end{cases}$$

for which (33) is different from zero.

3. Infinitesimal Projective Group.1) An infinitesimal projective transformation in (C,E_n) is defined as the most general regular infinitesimal transformation in (C,E_n) which takes a straight line into a straight line in (C,E_n).

The parametric equations of a straight line in (C,E_n) are

$$(35)\quad \begin{cases} x^\alpha = u^\alpha + a v^\alpha, \\ x^i = u^i + a v^i. \end{cases}$$

Introduce a new parameter b. If the equations

$$x^\alpha = x^\alpha(a),$$
$$x^i = x^i(a)$$

are to represent a straight line in (C,E_n), the following conditions must be satisfied:

$$\frac{\partial^2 x^\alpha}{\partial b^2} = \frac{\partial^2 x^\alpha}{\partial a^2}\left(\frac{da}{db}\right)^2 + \frac{\partial x^\alpha}{\partial a}\frac{d^2 a}{db^2} = 0,$$

$$\frac{\partial^2 x^i}{\partial b^2} = \frac{\partial^2 x^i}{\partial a^2}\left(\frac{da}{db}\right)^2 + \frac{\partial x^i}{\partial a}\frac{d^2 a}{db^2} = 0.$$

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1) In this section we generalize Kowalewski's treatment of the infinitesimal projective group in C . See Kowalewski II, p.1452.

These conditions in turn imply

$$(36)\qquad \frac{\dfrac{\partial^2 x^\alpha}{\partial a^2}}{\dfrac{\partial x^\alpha}{\partial a}} = \frac{\dfrac{\partial^2 x^i}{\partial a^2}}{\dfrac{\partial x^i}{\partial a}} .$$

Let us assume that the regular infinitesimal transformation (14) takes the points of the straight line (35) in a suitably restricted subset of $(C,E_{\underset{\sim}{m}})$ into points of a straight line in $(C,E_{\underset{\sim}{m}})$. Then conditions (36) and consequently the relations

$$\frac{\dfrac{\partial^2 \left(x^\alpha + p^\alpha(x)\right)}{\partial a^2}}{\dfrac{\partial \left(x^\alpha + p^\alpha(x)\right)}{\partial a}} = \frac{\dfrac{\partial^2 \left(x^i + p^i(x)\right)}{\partial a^2}}{\dfrac{\partial \left(x^i + p^i(x)\right)}{\partial a}}$$

must hold, and the latter imply

$$(36.1)\qquad \frac{\dfrac{\partial^2 p^\alpha(x)}{\partial a^2}}{\dfrac{\partial x^\alpha}{\partial a}} = \frac{\dfrac{\partial^2 p^i(x)}{\partial a^2}}{\dfrac{\partial x^i}{\partial a}} .$$

By Theorem $10'$ we have, since

$$(37)\quad \begin{cases} \dfrac{\partial^2 p^\alpha(x)}{\partial a^2} = \underset{2}{\mathfrak{A}}^\alpha\left(x ; \dfrac{\partial x}{\partial a} ; \dfrac{\partial x}{\partial a}\right) + \underset{3}{\mathfrak{A}}^\alpha\left(x ; \dfrac{\partial x}{\partial a} ; \dfrac{\partial x}{\partial a}\right) \\ \qquad + \cdots \\ \dfrac{\partial^2 p^i(x)}{\partial a^2} = \underset{2}{\mathfrak{A}}^i\left(x ; \dfrac{\partial x}{\partial a} ; \dfrac{\partial x}{\partial a}\right) + \underset{3}{\mathfrak{A}}^i\left(x ; \dfrac{\partial x}{\partial a} ; \dfrac{\partial x}{\partial a}\right) \\ \qquad + \cdots \end{cases}$$

From (35) we get

$$(35.1) \qquad \frac{\partial x^\alpha}{\partial a} = v^\alpha, \quad \frac{\partial x^i}{\partial a} = v^i .$$

Substituting in (36.1) the values of $\frac{\partial^2 p^\alpha(x)}{\partial a^2}, \frac{\partial^2 p^i(x)}{\partial a^2}$, $\frac{\partial x^\alpha}{\partial a}, \frac{\partial x^i}{\partial a}$ given by (37) and (35.1), we obtain

$$(38) \qquad \frac{\mathfrak{A}^\alpha_2(x;v;v)}{v^\alpha} + \frac{\mathfrak{A}^\alpha_3(x;v;v)}{v^\alpha} + \cdots = \frac{\mathfrak{A}^i_2(x;v;v)}{v^i} + \frac{\mathfrak{A}^i_3(x;v;v)}{v^i} + \cdots .$$

Since every term of $\mathfrak{A}^\alpha_m(x;v;v)$ and every term of $\mathfrak{A}^i_m(x;v;v)$ is of the m th degree in x and v, we get, upon replacing in (38) x^α by cx^α and x^i by cx^i,

$$(39) \qquad \frac{\mathfrak{A}^\alpha_m(x;v;v)}{v^\alpha} = \frac{\mathfrak{A}^i_m(x;v;v)}{v^i} \qquad (m=2,3,\ldots)$$

Let us first consider the case $m=2$. Computing $\mathfrak{A}^\alpha_2(x;v;v)$, $\mathfrak{A}^i_2(x;v;v)$ and putting their values in (39), we get the following identity in v:

$$(39.1) \qquad \begin{aligned} & v^i M^\alpha (v^\alpha)^2 + v^i M^\alpha_{.jk} v^j v^k + v^i v^\alpha M^{\alpha\beta} v_\beta \\ & + v^i v^j M^{\alpha\beta}_{j} v_\beta + v^i \hat{M}^{\alpha\beta} (v_\beta)^2 \\ & + v^i M^{\alpha\beta\gamma} v_\beta v_\gamma + v^i M^\alpha_{.j} v^j v^\alpha \\ & - v^\alpha \tilde{M}^{ijk} v_j v_k - v^\alpha v_j \tilde{M}^{ij}_\beta v^\beta \\ & - v^\alpha \tilde{M}^i_\beta (v^\beta)^2 - v^\alpha \tilde{M}^i_{\beta\gamma} v^\beta v^\gamma = 0 . \end{aligned}$$

Let us put in (39.1) the particular v defined as follows:

$$v^{\alpha} = a \qquad (\xi - e \leqq \alpha \leqq \xi + e),$$

and elsewhere in I let v^{α} lie between definite bounds independent of e but remain continuous; also let

$$v^{i} = b^{i}.$$

Passing to the limit as $e \to 0$, we get, because of the continuity of the coefficients,

$$\begin{aligned}(a)^2 M^{\xi} b^{i} + b^{i} M^{\xi}_{jk} b^{j} b^{k} + a b^{i} M^{\xi\beta} v_{\beta} \\ + b^{i} b^{j} M^{\xi\beta}_{\cdot j} v_{\beta} + b^{i} \hat{M}^{\xi\beta} (v_{\beta})^2 + b^{i} M^{\xi\beta\gamma} v_{\beta} v_{\gamma} \\ + a b^{i} M^{\xi}_{\cdot j} b^{j} - a \tilde{M}_{ijk} b^{j} b^{k} - a b^{j} \tilde{M}^{\beta}_{ij} v_{\beta} \\ - a \tilde{M}^{\beta}_{i} (v_{\beta})^2 - a \tilde{M}^{\beta\gamma}_{i} v_{\beta} v_{\gamma} = 0 .\end{aligned}$$

Because of the arbitrariness of a, b^{i}, we deduce

(39.11) $$M^{\xi} \equiv M^{\xi}_{jk} \equiv \tilde{M}_{ijk} \equiv 0 \qquad (i \neq k),$$

(39.12) $$M^{\xi}_{\cdot j} - \tilde{M}_{ijk} \equiv 0 \qquad (i = k)$$

(39.13) $$\tilde{M}^{\beta}_{ij} v_{\beta} = 0 \qquad (i \neq j),$$

(39.14) $$(M^{\xi\beta} - \tilde{M}^{\beta}_{ij}) v_{\beta} = 0 \qquad (i = j),$$

(39.15) $$\hat{M}^{\xi\beta} (v_{\beta})^2 + M^{\xi\beta\gamma} v_{\beta} v_{\gamma} = 0,$$

(39.16) $$\tilde{M}^{\beta}_{\cdot i}(v_{\beta})^{2} + \tilde{M}^{\beta\gamma}_{\cdot i} v_{\beta} v_{\gamma} = 0,$$

(39.17) $$M^{\xi\beta}_{\cdot j} v_{\beta} = 0 .$$

From (39.11) we get, since ξ was any value in I,

$$M^{\alpha} \equiv M^{\alpha}_{\cdot jk} \equiv 0 .$$

Similarly from (39.12) we infer

$$M^{\alpha}_{\cdot j} \equiv \tilde{M}_{ijk} \qquad (i = k),$$

so that $M^{\alpha}_{\cdot j}$ and $\tilde{M}_{iji}$ can both be replaced by a new symbol P_{j}. Applying Lemma 1 to (39.13), we get

$$\tilde{M}^{\beta}_{ij} \equiv 0 \qquad (i \neq j).$$

From (39.14) we have by Lemma 1

$$M^{\xi\beta} \equiv \tilde{M}^{\beta}_{ij} \qquad (i = j)$$

and consequently

$$M^{\alpha\beta} \equiv \tilde{M}^{\beta}_{ij} \qquad (i = j),$$

from which it follows that both $M^{\alpha\beta}$ and $\tilde{M}^{\beta}_{ij}$ $(i = j)$ can be replaced by a new symbol P^{β}. From (39.15) we get by Lemma 2 and its corollary

$$\hat{M}^{\xi\beta} \equiv M^{\xi\beta\gamma} \equiv 0,$$

whence

$$\hat{M}^{\alpha\beta} \equiv M^{\alpha\beta\gamma} \equiv 0 .$$

From (39.16) we have by Lemma 2 and its corollary

$$\tilde{M}^{\beta}_{i} \equiv \tilde{M}^{\beta\gamma}_{\cdot i} \equiv 0 .$$

From (39.17) we have by Lemma 1

$$M^{\xi\beta}_{j} \equiv 0,$$

whence

$$M^{\alpha\beta}_{j} \equiv 0 .$$

We are thus able to conclude that if (14) takes (35) into a straight line in $(C, \underset{n}{E})$, then $\underset{2}{\mathfrak{A}}^{\alpha}(x)$, $\underset{2}{\mathfrak{A}}^{i}(x)$ must be of the form

$$(40) \quad \begin{cases} \underset{2}{\mathfrak{A}}^{\alpha}(x) = x^{\alpha}\left(p^{\beta}x_{\beta} + p^{j}x_{j}\right), \\ \underset{2}{\mathfrak{A}}^{i}(x) = x^{i}\left(p^{\beta}x_{\beta} + p^{j}x_{j}\right). \end{cases}$$

The investigation for $m > 2$ is facilitated by the observation that (39) implies

$$(39.2) \quad \frac{\underset{m}{\mathfrak{A}}^{\alpha}(x; w; v; v)}{v^{\alpha}} = \frac{\underset{m}{\mathfrak{A}}^{i}(x; w; v; v)}{v^{i}} \qquad (m = 3, 4, \cdots).$$

For $m = 3$, (39.2) becomes

$$\frac{\underset{3}{\mathfrak{A}}^{\alpha}(x; w; v; v)}{v^{\alpha}} = \frac{\underset{3}{\mathfrak{A}}^{i}(x; w; v; v)}{v^{i}},$$

which tells us that $\underset{3}{\mathfrak{A}}^{\alpha}(x; w)$ and $\underset{3}{\mathfrak{A}}^{i}(x; w)$, which are composite integral power forms of the second order in x, are related in the same way as are $\underset{2}{\mathfrak{A}}^{\alpha}(x)$ and $\underset{2}{\mathfrak{A}}^{i}(x)$ in (39). Hence $\underset{3}{\mathfrak{A}}^{\alpha}(x; w)$, $\underset{3}{\mathfrak{A}}^{i}(x; w)$ must be of the same form (as composite integral power forms in x)as $\underset{2}{\mathfrak{A}}^{\alpha}(x)$, $\underset{2}{\mathfrak{A}}^{i}(x)$ respectively .

Making use of the symmetry of the coefficients of $\underset{3}{\mathfrak{A}}^{\alpha}(x;w)$, $\underset{3}{\mathfrak{A}}^{i}(x;w)$ in β , γ , δ and in j , k , l , we can write these composite integral power forms as follows:

$$
(41)\quad \left\{
\begin{aligned}
\underset{3}{\mathfrak{A}}^{\alpha}(x;w) &= \left(3N^{\alpha}w^{\alpha} + N^{\cdot\,\alpha}_{j}w^{j} + N^{\alpha\beta}w_{\beta}\right)(x^{\alpha})^{2} \\
&\quad + 2x^{\alpha}\left(N^{\alpha\beta}w^{\alpha} + \hat{N}^{\alpha\beta}w^{\beta} + N^{\alpha\beta\gamma}w_{\gamma}\right)x_{\beta} \\
&\quad + 2x^{\alpha}\left(N^{\cdot\,\alpha}_{j}w^{\alpha} + N^{\cdot\,\alpha}_{jk}w^{k}\right)x^{j} \\
&\quad + \left(\hat{N}^{\alpha\beta}w^{\alpha} + N^{\alpha\beta}_{j}w^{j} + 3\widehat{N}^{\alpha\beta}w^{\beta} + \hat{N}^{\alpha\beta\gamma}w_{\gamma}\right)(x_{\beta})^{2} \\
&\quad + \left(N^{\alpha\beta\gamma}w^{\alpha} + N^{\alpha\beta\gamma}_{j}w^{j} + 2\hat{N}^{\alpha\beta\gamma}w^{\beta} + 3N^{\alpha\beta\gamma\delta}w_{\delta}\right)x_{\beta}x_{\gamma} \\
&\quad + 2\left(N^{\cdot\,\alpha\beta}_{j}w^{\beta} + N^{\alpha\beta\gamma}_{j}w_{\gamma}\right)x^{j}x_{\beta} \\
&\quad + \left(3N^{\alpha}_{jkl}w^{l} + N^{\cdot\,\alpha}_{jk}w^{\alpha} + N^{\alpha\beta}_{jk}w_{\beta}\right)x^{j}x^{k}, \\
\underset{3}{\mathfrak{A}}^{i}(x;w) &= \left(\tilde{N}^{ij}_{\beta}w_{j} + 3\tilde{N}^{i}_{\beta}w_{\beta} + \tilde{N}^{i}_{\beta\gamma}w^{\gamma}\right)(x^{\beta})^{2} \\
&\quad + \left(\tilde{N}^{ij}_{\beta\gamma}w_{j} + 2\tilde{N}^{i}_{\beta\gamma}w_{\beta} + 3\tilde{N}^{i}_{\beta\gamma\delta}w^{\delta}\right)x^{\beta}x^{\gamma} \\
&\quad + 2\left(\tilde{N}^{ijk}_{\beta}w_{k} + \tilde{N}^{ij}_{\beta}w_{\beta} + \tilde{N}^{ij}_{\beta\gamma}w^{\gamma}\right)x_{j}x^{\beta} \\
&\quad + \left(3\tilde{N}^{ijkl}w_{l} + \tilde{N}^{ijk}_{\beta}w^{\beta}\right)x_{j}x_{k}.
\end{aligned}
\right.
$$

Since each composite integral power form in (41) must be of the same form as the corresponding composite integral power form in (40), we can infer the following identities in x :

$$
\begin{aligned}
(41.1)\quad & \left(3N^{\alpha} w^{\alpha} + N^{\alpha}_{.j} w^{j} + N^{\alpha\beta} w_{\beta}\right)(x^{\alpha})^{2} \\
& + \left(\hat{N}^{\alpha\beta} w^{\alpha} + N^{\alpha\beta}_{.j} w^{j} + 3\hat{N}^{\alpha\beta} w^{\beta} + \hat{N}^{\alpha\beta\gamma} w_{\gamma}\right)(x_{\beta})^{2} \\
& + \left(N^{\alpha\beta\gamma} w^{\alpha} + N^{\alpha\beta\gamma}_{.j} w^{j} + 2\hat{N}^{\alpha\beta\gamma} w^{\beta} + 3N^{\alpha\beta\gamma\delta} w_{\delta}\right) x_{\beta} x_{\gamma} \\
& + 2\left(N^{\alpha\beta}_{.j} w^{\beta} + N^{\alpha\beta\gamma}_{.j} w_{\gamma}\right) x^{j} x_{\beta} \\
& + \left(3N^{\alpha}_{.jkl} w^{l} + N^{\alpha}_{.jk} w^{\alpha} + N^{\alpha\beta}_{.jk} w_{\beta}\right) x^{j} x^{k} = 0,
\end{aligned}
$$

$$
\begin{aligned}
(41.2)\quad & \left(\tilde{N}^{ij}_{\beta} w_{j} + 3\tilde{N}^{i}_{\beta} w_{\beta} + \tilde{N}^{i}_{\beta\gamma} w^{\gamma}\right)(x^{\beta})^{2} \\
& + \left(\tilde{N}^{ij}_{\beta\gamma} w_{j} + 2\tilde{N}^{i}_{\beta\gamma} w_{\beta} + 3\tilde{N}^{i}_{\beta\gamma\delta} w^{\delta}\right) x^{\beta} x^{\gamma} \\
& + 2\left(\tilde{N}^{ij'k}_{\beta} w_{k} + \tilde{N}^{ij'}_{\beta} w_{\beta} + \tilde{N}^{ij'}_{\beta\gamma} w^{\gamma}\right) x_{j'} x^{\beta} \\
& + \left(3\tilde{N}^{ij'k'l} w_{l} + \tilde{N}^{ij'k'}_{\beta} w^{\beta}\right) x_{j'} x_{k'} = 0,
\end{aligned}
$$

$$
(41.3)\quad 2x^{i}\left(3\tilde{N}^{ijil} w_{l} + \tilde{N}^{iji}_{\beta} w^{\beta}\right) x_{j} = x^{i} P^{j} x_{j},
$$

where a prime after an index j or k signifies that the summation with respect to that index is from

to and from to .

1 to $i-1$ and from $i+1$ to n.[1] From (41.1) and (41.2) we get by Lemma 2 and its corollary the following identities in w:

$$3N^{\alpha}w^{\alpha}+N^{\alpha}_{.j}w^{j}+N^{\alpha\beta}w_{\beta}=0,$$

$$\hat{N}^{\alpha\beta}w^{\alpha}+N^{\alpha\beta}_{.j}w^{j}+3\hat{\hat{N}}^{\alpha\beta}w^{\beta}+\hat{N}^{\alpha\beta\gamma}w_{\gamma}=0,$$

$$N^{\alpha\beta\gamma}w^{\alpha}+N^{\alpha\beta\gamma}_{.j}w^{j}+2\hat{N}^{\alpha\beta\gamma}w^{\beta}+3N^{\alpha\beta\gamma\delta}w_{\delta}=0,$$

$$N^{\alpha\beta}_{.j}w^{\beta}+N^{\alpha\beta\gamma}_{.j}w_{\gamma}=0,$$

$$3N^{\alpha}_{.jkl}w^{l}+N^{\alpha}_{.jk}w^{\alpha}+N^{\alpha\beta}_{jk}w_{\beta}=0,$$

$$\tilde{N}^{\beta}_{ij}w^{j}+3\tilde{N}^{\beta}_{i}w^{\beta}+\tilde{N}^{\beta\gamma}_{i}w_{\gamma}=0,$$

---

[1] As an aid to inferring (41.2) and (41.3), we write

$$\begin{aligned}\left(3\tilde{N}^{ijkl}w_{l}+\tilde{N}^{ijk}_{\beta}w^{\beta}\right)x_{j}x_{k}&=x^{i}\left(3\tilde{N}^{ijil}w_{l}+\tilde{N}^{iji}_{\beta}w^{\beta}\right)x_{j}\\&\quad+x^{i}\left(3\tilde{N}^{iikl}w_{l}+\tilde{N}^{iik}_{\beta}w^{\beta}\right)x_{k}\\&\quad+\left(3\tilde{N}^{ij'k'l}w_{l}+\tilde{N}^{ij'k'}_{\beta}w^{\beta}\right)x_{j'}x_{k'}\\&=2x^{i}\left(3\tilde{N}^{ijil}w_{l}+\tilde{N}^{iji}_{\beta}w^{\beta}\right)x_{j}\\&\quad+\left(3\tilde{N}^{ij'k'l}w_{l}+\tilde{N}^{ij'k'}_{\beta}w^{\beta}\right)x_{j'}x_{k'}\end{aligned}$$

$$\tilde{N}^{\beta\gamma}_{.ij} w^j + 2\tilde{N}^{\beta\gamma}_{i} w^\beta + 3\tilde{N}^{\beta\gamma\delta}_{i} w_\delta = 0,$$

$$\tilde{N}^{\beta}_{ijk} w^k + \tilde{N}^{\beta}_{.ij} w^\beta + \tilde{N}^{\beta\gamma}_{ij} w_\gamma = 0 \qquad (i \neq j),$$

$$3\tilde{N}_{ijkl} w^l + \tilde{N}^{\beta}_{ijk} w_\beta = 0 \qquad (i \neq k).$$

Applying Lemma 1 to each of these identities, we find that all the coefficients of $\underset{3}{\mathfrak{A}}^{\alpha}(x)$ and $\underset{3}{\mathfrak{A}}^{i}(x)$ vanish except possibly $\tilde{N}^{\beta}_{iik}$, $\tilde{N}_{ijil}$, $\tilde{N}_{iikl}$. From (41.3), however, we have

$$3\tilde{N}_{ijil} w^l + \tilde{N}^{\beta}_{iji} w_\beta = \text{a constant}. \tag{41.31}$$

Let us put in (41.31) $w^\alpha = 0$, $w^i = a^i$. (41.31) then yields

$$3\tilde{N}_{ijil} a^l = \text{a constant},$$

whence, because of the arbitrariness of a^i,

$$\tilde{N}_{ijil} \equiv 0. \tag{41.311}$$

Thus (41.31) reduces to

$$\tilde{N}^{\beta}_{iji} w_\beta = \text{a constant}.$$

Choosing $w^\alpha = b$ in this last relation, we get

$$\tilde{N}^{\beta}_{iji}\, b = \text{a constant},$$

and from this, since b is arbitrary, there follows

$$\tilde{N}^{\beta}_{iji} \equiv 0,$$

whence, since $\tilde{N}^{\beta}_{iji} = \tilde{N}^{\beta}_{iij}$,

$$\tilde{N}^{\beta}_{iik} \equiv 0 .$$

From (41.311), since $\tilde{N}_{ijil} = \tilde{N}_{iijl}$, we get

$$\tilde{N}_{iikl} \equiv 0 .$$

If, then, (14) takes (35) into a straight line in (C,E_n), we must have

$$\mathfrak{A}^{\alpha}_{3}(x) \equiv \mathfrak{A}^{i}_{3}(x) \equiv 0 .$$

It remains to determine the composite integral power forms of higher order. Since (39) implies (39.2), it implies successively for $m > 3$

$$(39.21)\quad \left\{ \begin{array}{l} \dfrac{\mathfrak{A}^{\alpha}_{m}(x; w; w; v; v)}{v^{\alpha}} = \dfrac{\mathfrak{A}^{i}_{m}(x; w; w; v; v)}{v^{i}} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\mathfrak{A}^{\alpha}_{m}(x; \overbrace{w; \cdots; w}^{m-3 \text{ times}}; v; v)}{v^{\alpha}} = \dfrac{\mathfrak{A}^{i}_{m}(x; \overbrace{w; \cdots; w}^{m-3 \text{ times}}; v; v)}{v^{i}} . \end{array} \right.$$

Putting

$$\tilde{\mathfrak{A}}^{\alpha}_{3}(x) = \mathfrak{A}^{\alpha}_{m}(x; \underbrace{w; \cdots; w}_{m-3 \text{ times}}), \quad \tilde{\mathfrak{A}}^{i}_{3}(x) = \mathfrak{A}^{i}_{m}(x; \underbrace{w; \cdots; w}_{m-3 \text{ times}}) ,$$

the last relation of (39.21) becomes

$$\frac{\tilde{\mathfrak{A}}^{\alpha}_{3}(x; v; v)}{v^{\alpha}} = \frac{\tilde{\mathfrak{A}}^{i}_{3}(x; v; v)}{v^{i}} .$$

In other words, $\tilde{\mathfrak{A}}^{\alpha}_{3}(x)$ and $\tilde{\mathfrak{A}}^{i}_{3}(x)$, which are composite integral power forms of the third order in x, must satisfy (39) for $m = 3$. Hence $\tilde{\mathfrak{A}}^{\alpha}_{3}(x)$, $\tilde{\mathfrak{A}}^{i}_{3}(x)$ must be of the same form respectively as $\mathfrak{A}^{\alpha}_{3}(x)$, $\mathfrak{A}^{i}_{3}(x)$. We thus have

$$\tilde{\mathfrak{A}}^{\alpha}_{3}(x) = \tilde{\mathfrak{A}}^{i}_{3}(x) = 0,$$

i.e.

$$\mathfrak{A}^{\alpha}_{m}(x;\underbrace{w;\cdots;w}_{m-3\text{ times}}) = \mathfrak{A}^{i}_{m}(x;\underbrace{w;\cdots;w}_{m-3\text{ times}}) = 0$$

Choosing $w = x$, we get from this

$$\mathfrak{A}^{\alpha}_{m}(\underbrace{x;\cdots;x}_{m-2\text{ times}}) = \mathfrak{A}^{i}_{m}(\underbrace{x;\cdots;x}_{m-2\text{ times}}) = 0 .$$

But by Theorem 2 we have

$$\mathfrak{A}^{\alpha}_{m}(\underbrace{x;\cdots;x}_{m-2\text{ times}}) = m(m-1)\cdots(4)\,\mathfrak{A}^{\alpha}_{m}(x),$$

$$\mathfrak{A}^{i}_{m}(\underbrace{x;\cdots;x}_{m-2\text{ times}}) = m(m-1)\cdots(4)\,\mathfrak{A}^{i}_{m}(x).$$

Hence

$$\mathfrak{A}^{\alpha}_{m}(x) = \mathfrak{A}^{i}_{m}(x) = 0 \qquad (m > 3).$$

Noting finally that no restrictions have arisen on $\mathfrak{A}^{\alpha}_{0}(x)$, $\mathfrak{A}^{i}_{0}(x)$, $\mathfrak{A}^{\alpha}_{1}(x)$, $\mathfrak{A}^{i}_{1}(x)$, and replacing $L^{\alpha\beta}$, L^{α}_{j}, P^{β}, P^{j}, $\tilde{K}^{i}$, $\tilde{L}^{i}_{\beta}$, $\tilde{L}^{ij}$ respectively by $K^{\alpha\beta}$, K^{α}_{j}, M^{β}, M^{j}, K^{i}, L^{i}_{β}, K^{ij}, we have

THEOREM 24. *A necessary condition that a regular infinitesimal transformation in* (C,E_n) *take a straight line* (35) *into a straight line in* (C,E_n) *is that the transformation be of the form*

$$(42)\quad \begin{cases} \dfrac{dx^{\alpha}}{dt} = K^{\alpha} + L^{\alpha}x^{\alpha} + K^{\alpha\beta}x_{\beta} + K^{\alpha}_{\cdot j}x^{j} + x^{\alpha}\left(M^{\beta}x_{\beta} + M^{j}x_{j}\right), \\ \dfrac{dx^{i}}{dt} = K^{i} + L^{i}_{\beta}x^{\beta} + K^{ij}x_{j} + x^{i}\left(M^{\beta}x_{\beta} + M^{j}x_{j}\right). \end{cases}$$

As an aid to proving that (42) actually takes a straight line into a straight line in (C,E_n),[1] we first establish the following

LEMMA. *Let* x , $x+y$, $x+z$ *be any three points in* (C,E_n). *A necessary and sufficient condition that they be collinear is that the following relations hold:*

$$\Delta^{\alpha\beta} \equiv \begin{vmatrix} y^{\alpha} & y^{\beta} \\ z^{\alpha} & z^{\beta} \end{vmatrix} = 0, \qquad \Delta^{\alpha i} \equiv \begin{vmatrix} y^{\alpha} & y^{i} \\ z^{\alpha} & z^{i} \end{vmatrix} = 0 .$$

PROOF. Let x , $x+y$, $x+z$ lie on a straight line in (C,E_n). Regarding each of these points in turn as the variable point, the parametric equations of the line can be given in the three forms

$$(35)\qquad x^{\alpha} = u^{\alpha} + a v^{\alpha}, \qquad x^{i} = u^{i} + a v^{i},$$

$$(35.1)\qquad x^{\alpha} + y^{\alpha} = u^{\alpha} + b v^{\alpha}, \qquad x^{i} + y^{i} = u^{i} + b v^{i},$$

$$(35.2)\qquad x^{\alpha} + z^{\alpha} = u^{\alpha} + c v^{\alpha}, \qquad x^{i} + z^{i} = u^{i} + c v^{i}.$$

---

[1] The proof which we shall give is simpler than that given by Kowalewski for the space C ; cf. Kowalewski II, p. 1459.

Substituting from the first equation of (35) in the first equations of (35.1) and (35.2), we get the two equations

$$y^{\alpha} = v^{\alpha}(b-a), \quad z^{\alpha} = v^{\alpha}(c-a),$$

whence

(35.3) $$y^{\alpha} = \frac{b-a}{c-a} z^{\alpha},$$

so that y^{α} and z^{α} are linearly dependent and their Gramian determinant consequently vanishes, i.e.

(35.31) $$\begin{vmatrix} y^{\alpha} y_{\alpha} & y^{\alpha} z_{\alpha} \\ z^{\beta} y_{\beta} & z^{\beta} z_{\beta} \end{vmatrix} = 0 .$$

Since we can write

(43) $$\begin{vmatrix} y^{\alpha} y_{\alpha} & y^{\alpha} z_{\alpha} \\ z^{\beta} y_{\beta} & z^{\beta} z_{\beta} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} y^{\alpha} & y^{\beta} \\ z^{\alpha} & z^{\beta} \end{vmatrix} \begin{vmatrix} y_{\alpha} & y_{\beta} \\ z_{\alpha} & z_{\beta} \end{vmatrix},$$

(35.31) yields

$$\Delta^{\alpha\beta} = 0 .$$

Similarly, we find

(35.4) $$y^{i} = \frac{b-a}{c-a} z^{i},$$

and from (35.3) and (35.4) we get

$$\Delta^{\alpha i} = 0 .$$

To prove the sufficiency of the condition, let our hypothesis be

(44)
$$\Delta^{\alpha\beta} = \Delta^{\alpha i} = 0 ,$$

and let (35) be the parametric equations of the straight line whose variable point is x . Then because of (43) and (44) we have

$$\begin{vmatrix} y^{\alpha} y_{\alpha} & y^{\alpha} z_{\alpha} \\ z^{\beta} y_{\beta} & z^{\beta} z_{\beta} \end{vmatrix} = 0 .$$

Hence y^{α} and z^{α} are linearly dependent. Let this dependence be expressed by the relation

(45)
$$y^{\alpha} = \frac{b-a}{c-a} z^{\alpha} .$$

Then, since $\frac{y^{\alpha}}{z^{\alpha}} = \frac{y^{i}}{z^{i}}$ by (44), we have the further relations

(45′)
$$y^{i} = \frac{b-a}{c-a} z^{i} .$$

In view of (45) and (45′) we can put

$$y^{\alpha} = (b-a)v^{\alpha}, \quad y^{i} = (b-a)v^{i},$$
$$z^{\alpha} = (c-a)v^{\alpha}, \quad z^{i} = (c-a)v^{i},$$

whence

(46)
$$\begin{cases} u^{\alpha} + a v^{\alpha} + y^{\alpha} = u^{\alpha} + b v^{\alpha}, \\ u^{i} + a v^{i} + y^{i} = u^{i} + b v^{i}, \\ u^{\alpha} + a v^{\alpha} + z^{\alpha} = u^{\alpha} + c v^{\alpha}, \\ u^{i} + a v^{i} + z^{i} = u^{i} + c v^{i}. \end{cases}$$

Substituting in (46) from (35), we get

$$x^\alpha + y^\alpha = u^\alpha + b v^\alpha, \quad x^i + y^i = u^i + b v^i,$$

$$x^\alpha + z^\alpha = u^\alpha + c v^\alpha, \quad x^i + z^i = u^i + c v^i.$$

Hence the points x, $x+y$, and $x+z$ are collinear.

THEOREM 25. The transformation (42) takes a straight line into a straight line in (C, E_n).

PROOF. Let the points x, $x+y$, and $x+z$ be collinear. By the preceding lemma we then have

$$\Delta^{\alpha\beta} = \Delta^{\alpha i} = 0, \tag{47}$$

and from (47) we get

$$\Delta^{ij} = \begin{vmatrix} y^i & y^j \\ z^i & z^j \end{vmatrix} = 0. \tag{47.1}$$

If, when we apply (42), we should get

$$\frac{d\Delta^{\alpha\beta}}{dt} = \frac{d\Delta^{\alpha i}}{dt} = 0,$$

it will follow that $\Delta^{\alpha\beta}$ and $\Delta^{\alpha i}$ are invariant under (42) and therefore, by the preceding lemma, that three collinear points in (C, E_n) are taken by (42) into three collinear points in (C, E_n), so that the theorem will be proved.

We seek, therefore, to show that

$$\frac{d\Delta^{\alpha\beta}}{dt} \equiv \begin{vmatrix} \dfrac{dy^\alpha}{dt} & y^\beta \\ \dfrac{dz^\alpha}{dt} & z^\beta \end{vmatrix} + \begin{vmatrix} y^\alpha & \dfrac{dy^\beta}{dt} \\ z^\alpha & \dfrac{dz^\beta}{dt} \end{vmatrix}$$

and

$$\frac{d\Delta^{\alpha i}}{dt} \equiv \begin{vmatrix} \frac{dy^{\alpha}}{dt} & y^{i} \\ \frac{dz^{\alpha}}{dt} & z^{i} \end{vmatrix} + \begin{vmatrix} y^{\alpha} & \frac{dy^{i}}{dt} \\ z^{\alpha} & \frac{dz^{i}}{dt} \end{vmatrix}$$

vanish when (42) is applied. Applying (42), we have

$$(42.1)\begin{cases} \frac{dx^{\alpha}}{dt} + \frac{dy^{\alpha}}{dt} = \frac{d}{dt}(x^{\alpha}+y^{\alpha}) \\ \qquad = K^{\alpha} + L^{\alpha}(x^{\alpha}+y^{\alpha}) + K^{\alpha\beta}(x_{\beta}+y_{\beta}) \\ \qquad + K^{\alpha}_{j}(x^{j}+y^{j}) \\ \qquad + [x^{\alpha}+y^{\alpha}][M^{\beta}(x_{\beta}+y_{\beta}) + M^{j}(x_{j}+y_{j})], \\ \frac{dx^{i}}{dt} + \frac{dy^{i}}{dt} = \frac{d}{dt}(x^{i}+y^{i}) = K^{i} + L^{i}_{\beta}(x^{\beta}+y^{\beta}) + K^{ij}(x_{j}+y_{j}) \\ \qquad + [x^{i}+y^{i}][M^{\beta}(x_{\beta}+y_{\beta}) + M^{j}(x_{j}+y_{j})]. \end{cases}$$

Subtracting the first equation of (42) from the first equation of (42.1) and subtracting each of the other equations of (42) from the corresponding equation of (42.1), we get

$$\frac{dy^{\alpha}}{dt} = L^{\alpha}y^{\alpha} + K^{\alpha\beta}y_{\beta} + K^{\alpha}_{j}y^{j} + x^{\alpha}M^{\beta}y_{\beta} + y^{\alpha}M^{\beta}x_{\beta}$$
$$+ y^{\alpha}M^{\beta}y_{\beta} + x^{\alpha}M^{j}y_{j} + y^{\alpha}M^{j}x_{j} + y^{\alpha}M^{j}y_{j},$$

$$\frac{dy^{i}}{dt} = L^{i}_{\beta}y^{\beta} + K^{ij}y_{j} + x^{i}M^{\beta}y_{\beta} + y^{i}M^{\beta}x_{\beta}$$
$$+ y^{i}M^{\beta}y_{\beta} + x^{i}M^{j}y_{j} + y^{i}M^{j}x_{j} + y^{i}M^{j}y_{j}.$$

Similarly, we have

$$\frac{dz^{\alpha}}{dt} = L^{\alpha} z^{\alpha} + K^{\alpha\beta} z_{\beta} + K^{\alpha}_{\cdot j} z^{j} + x^{\alpha} M^{\beta} z_{\beta} + z^{\alpha} M^{\beta} x_{\beta}$$
$$+ z^{\alpha} M^{\beta} z_{\beta} + x^{\alpha} M^{j} z_{j} + z^{\alpha} M^{j} x_{j} + z^{\alpha} M^{j} z_{j},$$

$$\frac{dz^{i}}{dt} = L^{i}_{\beta} z^{\beta} + K^{ij} z_{j} + x^{i} M^{\beta} z_{\beta} + z^{i} M^{\beta} x_{\beta}$$
$$+ z^{i} M^{\beta} z_{\beta} + x^{i} M^{j} z_{j} + z^{i} M^{j} x_{j} + z^{i} M^{j} z_{j} .$$

Applying

$$\varphi^{\alpha}(x) = K^{\alpha}, \quad \varphi^{i}(x) = 0,$$

we get
$$\frac{d\Delta^{\alpha\beta}}{dt} = 0, \quad \frac{d\Delta^{\alpha i}}{dt} = 0 .$$

Applying

$$\varphi^{\alpha}(x) = L^{\alpha} x^{\alpha}, \quad \varphi^{i}(x) = 0,$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = \begin{vmatrix} L^{\alpha} y^{\alpha} & y^{\beta} \\ L^{\alpha} z^{\alpha} & z^{\beta} \end{vmatrix} + \begin{vmatrix} y^{\alpha} & L^{\beta} y^{\beta} \\ z^{\alpha} & L^{\beta} z^{\beta} \end{vmatrix} = (L^{\alpha} + L^{\beta}) \Delta^{\alpha\beta},$$

$$\frac{d\Delta^{\alpha i}}{dt} = \begin{vmatrix} L^{\alpha} y^{\alpha} & y^{i} \\ L^{\alpha} z^{\alpha} & z^{i} \end{vmatrix} = L^{\alpha} \Delta^{\alpha i} .$$

Applying

$$p^\alpha(x) = K^{\alpha\beta} x_\beta \,, \quad p^i(x) = 0 \,,$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = \begin{vmatrix} K_{\alpha\gamma} y^\gamma & y^\beta \\ K_{\alpha\gamma} z^\gamma & z^\beta \end{vmatrix} + \begin{vmatrix} y^\alpha & K_{\beta\gamma} y^\gamma \\ z^\alpha & K_{\beta\gamma} z^\gamma \end{vmatrix} = K_{\alpha\gamma} \Delta^{\gamma\beta} + K_{\beta\gamma} \Delta^{\alpha\gamma} \,,$$

$$\frac{d\Delta^{\alpha i}}{dt} = \begin{vmatrix} K_{\alpha\gamma} y^\gamma & y^i \\ K_{\alpha\gamma} z^\gamma & z^i \end{vmatrix} = K_{\alpha\gamma} \Delta^{\gamma i} .$$

Applying

$$p^\alpha(x) = K^\alpha_j x^j, \quad p^i(x) = 0 \,,$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = \begin{vmatrix} K^\alpha_k y^k & y^\beta \\ K^\alpha_k z^k & z^\beta \end{vmatrix} + \begin{vmatrix} y^\alpha & K^\beta_k y^k \\ z^\alpha & K^\beta_k z^k \end{vmatrix} = -K^\alpha_k \Delta^{\beta k} + K^\beta_k \Delta^{\alpha k} ,$$

$$\frac{d\Delta^{\alpha i}}{dt} = \begin{vmatrix} L^\alpha_k y^k & y^i \\ L^\alpha_k z^k & z^i \end{vmatrix} = L^\alpha_k \Delta^{ki} .$$

Applying

$$p^\alpha(x) = x^\alpha M^\beta x_\beta \,, \quad p^i(x) = x^i M^\beta x_\beta \,,$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = \begin{vmatrix} x^\alpha M_\gamma y^\gamma + y^\alpha M_\gamma x^\gamma + y^\alpha M_\gamma y^\gamma & y^\beta \\ x^\alpha M_\gamma z^\gamma + z^\alpha M_\gamma x^\gamma + z^\alpha M_\gamma z^\gamma & z^\beta \end{vmatrix}$$

$$+ \begin{vmatrix} y^\alpha & x^\beta M_\gamma y^\gamma + y^\beta M_\gamma x^\gamma + y^\beta M_\gamma y^\gamma \\ z^\alpha & x^\beta M_\gamma z^\gamma + z^\beta M_\gamma x^\gamma + z^\beta M_\gamma z^\gamma \end{vmatrix}$$

$$= x^\alpha M_\gamma \Delta^{\gamma\beta} + 2 M_\gamma x^\gamma \Delta^{\alpha\beta} + x^\beta M_\gamma \Delta^{\alpha\gamma}$$

$$+ M_\gamma (y^\gamma + z^\gamma) \Delta^{\alpha\beta},$$

$$\frac{d\Delta^{\alpha i}}{dt} = \begin{vmatrix} x^\alpha M_\gamma y^\gamma + y^\alpha M_\gamma x^\gamma + y^\alpha M_\gamma y^\gamma & y^i \\ x^\alpha M_\gamma z^\gamma + z^\alpha M_\gamma x^\gamma + z^\alpha M_\gamma z^\gamma & z^i \end{vmatrix}$$

$$+ \begin{vmatrix} y^\alpha & x^i M_\gamma y^\gamma + y^i M_\gamma x^\gamma + y^i M_\gamma y^\gamma \\ z^\alpha & x^i M_\gamma z^\gamma + z^i M_\gamma x^\gamma + z^i M_\gamma z^\gamma \end{vmatrix}$$

$$= x^\alpha M_\gamma \Delta^{\gamma i} + 2 M_\gamma x^\gamma \Delta^{\alpha i} + x^i M_\gamma \Delta^{\alpha\gamma} + M_\gamma (y^\gamma + z^\gamma) \Delta^{\alpha i}.$$

Applying

$$p^\alpha(x) = x^\alpha M^j x_j \,, \quad p^i(x) = x^i M^j x_j \,,$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = \begin{vmatrix} x^\alpha M_k y^k + y^\alpha M_k x^k + y^\alpha M_k y^k & y^\beta \\ x^\alpha M_k z^k + z^\alpha M_k x^k + z^\alpha M_k z^k & z^\beta \end{vmatrix}$$

$$+ \begin{vmatrix} y^\alpha & x^\beta M_k y^k + y^\beta M_k x^k + y^\beta M_k y^k \\ z^\alpha & x^\beta M_k z^k + z^\beta M_k x^k + z^\beta M_k z^k \end{vmatrix}$$

$$= -x^\alpha M_k \Delta^{\beta k} + 2 M_k x^k \Delta^{\alpha\beta} + x^\beta M_k \Delta^{\alpha k}$$

$$+ M_k (y^k + z^k) \Delta^{\alpha\beta},$$

$$\frac{d\Delta^{\alpha i}}{dt} = \begin{vmatrix} x^\alpha M_k y^k + y^\alpha M_k x^k + y^\alpha M_k y^k & y^i \\ x^\alpha M_k z^k + z^\alpha M_k x^k + z^\alpha M_k z^k & z^i \end{vmatrix}$$

$$+ \begin{vmatrix} y^\alpha & x^i M_k y^k + y^i M_k x^k + y^i M_k y^k \\ z^\alpha & x^i M_k z^k + z^i M_k x^k + z^i M_k z^k \end{vmatrix}$$

$$= x^\alpha M_k \Delta^{ki} + 2 M_k x^k \Delta^{\alpha i} + x^i M_k \Delta^{\alpha k} + M_k (y^k + z^k) \Delta^{\alpha i}.$$

Applying

$$p^\alpha(x) = 0, \quad p^i(x) = K^i,$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = 0, \quad \frac{d\Delta^{\alpha i}}{dt} = 0.$$

Applying

$$p^{\alpha}(x) = 0, \quad p^{i}(x) = L^{i}_{\beta}\, x^{\beta},$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = 0, \quad \frac{d\Delta^{\alpha i}}{dt} = \begin{vmatrix} y^{\alpha} & L^{i}_{\gamma}\, y^{\gamma} \\ z^{\alpha} & L^{i}_{\gamma}\, z^{\gamma} \end{vmatrix} = L^{i}_{\gamma}\, \Delta^{\alpha\gamma}.$$

Applying

$$p^{\alpha}(x) = 0, \quad p^{i}(x) = K^{ij} x_{j},$$

we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = 0, \quad \frac{d\Delta^{\alpha i}}{dt} = \begin{vmatrix} y^{\alpha} & K_{ik}\, y^{k} \\ z^{\alpha} & K_{ik}\, z^{k} \end{vmatrix} = K_{ik}\, \Delta^{\alpha k}.$$

These computations inform us, in view of (47) and (47.1) that when (42) is applied we get

$$\frac{d\Delta^{\alpha\beta}}{dt} = \frac{d\Delta^{\alpha i}}{dt} = 0.$$

Hence the theorem is proved.

From Theorems 24 and 25 we have

THEOREM 26. <u>An infinitesimal projective transformation in</u> (C, E_n) <u>has the form</u> (42).

Computing the commutator of (42) and

$$(42) \quad \begin{cases} \dfrac{dx^{\alpha}}{dt} = \bar{K}^{\alpha} + \bar{L}^{\alpha} x^{\alpha} + \bar{K}^{\alpha\beta} x_{\beta} + \bar{K}^{\alpha}_{\dot{j}}\, x^{\dot{j}} + x^{\alpha}\left(\bar{M}^{\beta} x_{\beta} + \bar{M}^{\dot{j}} x_{\dot{j}}\right), \\ \dfrac{dx^{i}}{dt} = \bar{K}^{i} + \bar{L}^{i}_{\beta}\, x^{\beta} + \bar{K}^{ij} x_{j} + x^{i}\left(\bar{M}^{\beta} x_{\beta} + \bar{M}^{\dot{j}} x_{\dot{j}}\right), \end{cases}$$

we get

THEOREM 27. The totality of infinitesimal projective transformations in (C, E_n) constitute a group.

Putting

$$x^\alpha = \frac{y^\alpha}{y^{n+1}}, \quad x^i = \frac{y^i}{y^{n+1}} \qquad (y^{n+1} \neq 0)$$

in the infinitesimal projective transformation (42), we obtain as the equations of an infinitesimal projective transformation in (C, E_n) in homogeneous coordinates

$$\frac{dy^\alpha}{dt} = L^\alpha y^\alpha + K^{\alpha\beta} y_\beta + K^\alpha_j y^j + K^\alpha y^{n+1},$$

$$\frac{dy^i}{dt} = L^i_\beta y^\beta + K^{ij} y_j + K^i y^{n+1},$$

$$\frac{dy^{n+1}}{dt} = -M_\beta y^\beta - M^j y_j .$$

4. The Finite Transformations Generated by an Infinitesimal Projective Transformation. Let the equations of a given infinitesimal projective transformation in (C, E_n) in homogeneous coordinates be

$$(48) \quad \begin{cases} \dfrac{\partial \bar{x}^\alpha(t)}{\partial t} = K^\alpha \bar{x}^\alpha(t) + K^{\alpha\beta} \bar{x}_\beta(t) + K^\alpha_j \bar{x}^j(t) + K^\alpha_{n+1} \bar{x}^{n+1}(t), \\[2ex] \dfrac{\partial \bar{x}^i(t)}{\partial t} = L^i_\beta \bar{x}^\beta(t) + K^{ij} \bar{x}_j(t) + K^{i\,n+1} \bar{x}^{n+1}(t), \\[2ex] \dfrac{\partial \bar{x}^{n+1}(t)}{\partial t} = L^{n+1}_\beta \bar{x}^\beta(t) + K^{n+1\,j} \bar{x}_j(t) \end{cases}$$

with the boundary conditions

$$\bar{x}^{\alpha}(0) = x^{\alpha}, \quad \bar{x}^{i}(0) = x^{i}, \quad \bar{x}^{n+1}(0) = x^{n+1}. \tag{48'}$$

Since the transformation (48) is linear in $\bar{x}^{\alpha}(t), \bar{x}^{i}(t), \ldots, \bar{x}^{n+1}(t)$, we can apply Theorem 21, stated for the case of an infinitesimal linear transformation in (C, E_{n+1}) with $K_{n+1\,n+1} = 0$. We thus obtain

THEOREM 28. *An infinitesimal projective transformation in* (C, E_n) *whose equations in homogeneous coordinates are given by* (48) *and* (48') *generates a one-parameter continuous group of finite projective transformations in* (C, E_n) *whose equations in homogeneous coordinates are*

$$(49)\quad \begin{cases} r\bar{x}^{\alpha}(t) = M^{\alpha}(t)x^{\alpha} + M^{\alpha\beta}(t)x_{\beta} + M^{\alpha}_{j}(t)x^{j} + M^{\alpha}_{n+1}(t)x^{n+1}, \\ r\bar{x}^{i}(t) = N^{i}_{\beta}(t)x^{\beta} + M^{ij}(t)x_{j} + M^{i\,n+1}(t)x^{n+1}, \\ r\bar{x}^{n+1}(t) = N^{n+1}_{\beta}(t)x^{\beta} + M^{n+1\,j}(t)x_{j} + M^{n+1\,n+1}(t)x^{n+1}, \\ \qquad r \neq 0, \quad M^{\alpha}(t) \neq 0. \end{cases}$$

For any value of the parameter t, *the coefficients of* (49) *are given by*

$$(49.1)\qquad \left\{\begin{aligned}
M^{\alpha}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha}[m]}{m!} t^{m}, \\
M^{\alpha\beta}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha\beta}[m]}{m!} t^{m}, \\
M^{\alpha}_{j}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha}_{j}[m]}{m!} t^{m}, \\
M^{\alpha}_{n+1}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha}_{n+1}[m]}{m!} t^{m}, \\
N^{i}_{\beta}(t) &= \sum_{m=0}^{\infty} \frac{N^{i}_{\beta}[m]}{m!} t^{m} \\
M^{ij}(t) &= \sum_{m=0}^{\infty} \frac{M^{ij}[m]}{m!} t^{m}, \\
M^{i\,n+1}(t) &= \sum_{m=0}^{\infty} \frac{M^{i\,n+1}[m]}{m!} t^{m}, \\
N^{\beta}_{n+1}(t) &= \sum_{m=0}^{\infty} \frac{N^{\beta}_{n+1}[m]}{m!} t^{m}, \\
M^{n+1\,j}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+1\,j}[m]}{m!} t^{m}, \\
M^{n+1\,n+1}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+1\,n+1}[m]}{m!} t^{m},
\end{aligned}\right.$$

$$\begin{aligned}
(49.2)\qquad & M^{\alpha}[0] = 1,\ M^{\alpha\beta}[0] = M^{\alpha}_{j}[0] = M^{\alpha}_{n+1}[0] \\
& = N^{i}_{\beta}[0] = M^{i\,n+1}[0] = N^{\beta}_{n+1}[0] = M^{n+1\,j}[0] = 0, \\
& M^{ij}[0] = 1,\ (i=j),\ M^{ij}[0] = 0\ (i \neq j),\ M^{n+1\,n+1}[0] = 1,
\end{aligned}$$

and

$$
\left\{
\begin{aligned}
&M^{\alpha}[m+1] = K^{\alpha} M^{\alpha}[m],\\
&M^{\alpha\beta}[m+1] = K^{\alpha} M^{\alpha\beta}[m] + K^{\alpha\beta} M^{\beta}[m]\\
&\qquad + K_{\alpha\gamma} M^{\gamma\beta}[m] + K^{j}_{\alpha} N^{\beta}_{j}[m] + K^{\alpha}_{n+1} N^{\beta}_{n+1}[m],\\
&M^{\alpha}_{j}[m+1] = K^{\alpha} M^{\alpha}_{j}[m] + K_{\alpha\beta} M^{\beta}_{j}[m]\\
&\qquad + K^{k}_{\alpha} M_{kj}[m] + K^{\alpha}_{n+1} M_{n+1\,j}[m],\\
&M^{\alpha}_{n+1}[m+1] = K^{\alpha} M^{\alpha}_{n+1}[m] + K_{\alpha\beta} M^{\beta}_{n+1}[m]\\
&\qquad + K^{k}_{\alpha} M_{k\,n+1}[m] + K^{\alpha}_{n+1} M_{n+1\,n+1}[m],\\
&N^{i}_{\beta}[m+1] = L^{i}_{\beta} M_{\beta}[m] + L^{i}_{\gamma} M^{\gamma\beta}[m]\\
&\qquad + K_{ij} N^{j}_{\beta}[m] + K_{i\,n+1} N^{n+1}_{\beta}[m],\\
&N^{n+1}_{\beta}[m+1] = L^{n+1}_{\beta} M_{\beta}[m] + L^{n+1}_{\gamma} M^{\gamma\beta}[m] + K^{n+1\,j} N^{\beta}_{j}[m],\\
&M^{ij}[m+1] = L^{\beta}_{i} N^{j}_{\beta}[m] + K_{ik} M^{kj}[m] + K_{i\,n+1} M^{n+1\,j}[m],\\
&M^{n+1\,j}[m+1] = L^{\beta}_{n+1} N^{j}_{\beta}[m] + K_{n+1\,k} M^{kj}[m],\\
&M^{i\,n+1}[m+1] = L^{i}_{\beta} N^{\beta}_{n+1}[m] + K^{ik} M_{k\,n+1}[m]\\
&\qquad + K^{i\,n+1} M_{n+1\,n+1}[m],\\
&M^{n+1\,n+1}[m+1] = L^{n+1}_{\beta} N^{\beta}_{n+1}[m] + K^{n+1\,k} M_{k\,n+1}[m].
\end{aligned}
\right.
\tag{49.3}
$$

PART V

CONFORMAL GROUP IN (C,E_n)

<u>Sphere Geometry</u>.[1] A sphere in (C,E_n) is defined by the equation

$$(50)\qquad x^{n+1}(y^\beta y_\beta + y^j y_j) - 2(x^\beta y_\beta + x^j y_j) + x^{n+2} = 0 ,$$

where y is the variable point and $x^\alpha, x^1, \dots, x^{n+2}$ are considered fixed for the moment. By writing (50) in the form

$$\left(y^\beta - \frac{x^\beta}{x^{n+1}}\right)\left(y_\beta - \frac{x_\beta}{x^{n+1}}\right) + \left(y^j - \frac{x^j}{x^{n+1}}\right)\left(y_j - \frac{x_j}{x^{n+1}}\right)$$

$$= \frac{x^\beta x_\beta + x^j x_j - x^{n+1} x^{n+2}}{(x^{n+1})^2} ,$$

we see that the distance from the fixed point $\frac{x}{x^{n+1}}$ to the variable point y is constant. We call the point $\frac{x}{x^{n+1}}$ the center of the sphere and define the radius R by

$$(R)^2 = \frac{x^\beta x_\beta + x^j x_j - x^{n+1} x^{n+2}}{(x^{n+1})^2} .$$

The angle θ between two spheres (50) and

$$(\widetilde{50})\qquad \tilde{x}^{n+1}(y^\beta y_\beta + y^j y_j) - 2(\tilde{x}^\beta y_\beta + \tilde{x}^j y_j) + \tilde{x}^{n+2} = 0$$

---

[1] In this section we generalize the sphere geometry of C ; see Barnett and Nathan, l.c.

is defined by

$$\cos\theta = \frac{(R)^2 + (\tilde{R})^2 - (d)^2}{2R\tilde{R}},$$

where R and $\tilde{R}$ are the radii of the spheres (50) and $\widetilde{(50)}$ respectively, and d is the distance between the centers. The two spheres are said to be orthogonal if and only if $\cos\theta = 0$. By an easy calculation we get

THEOREM 29. <u>A necessary and sufficient condition that two spheres (50) and $\widetilde{(50)}$ be orthogonal is that the coefficients satisfy the relation</u>

$$2\left(x^\beta \tilde{x}_\beta + x^j \tilde{x}_j\right) - x^{n+1}\tilde{x}^{n+2} - \tilde{x}^{n+1}x^{n+2} = 0 \tag{51}$$

The left-hand member of (51) is called the polar of the quadratic functional

$$x^\beta x_\beta + x^j x_j - x^{n+1}x^{n+2}.$$

A sphere (50) is completely determined when its set of coefficients $(x^\alpha, x', \cdots, x^{n+2})$ is given, and the same sphere is determined by all the ordered sets

$$(\rho x^\alpha, \rho x', \cdots, \rho x^{n+2}) \qquad (\rho \neq 0), \tag{52}$$

where $x^\alpha, x', \cdots, x^{n+2}$ are the coefficients of (50). The homogeneous coordinates of a sphere (50) are accordingly defined as any one of the ordered sets (52). We thus obtain a geometry in which the sphere in (C, E_n) is the fundamental element and which is the analogue of the elementary sphere geometry of n-space. We shall call this geometry the sphere geometry of (C,E_n).

A transformation of the form

$$(53)\quad \begin{cases} r\bar{x}^{\alpha} = K^{\alpha}x^{\alpha} + K^{\alpha\beta}x_{\beta} + K^{\alpha}_{j}x^{j} + K^{\alpha}_{n+1}x^{n+1} + K^{\alpha}_{n+2}x^{n+2}, \\ r\bar{x}^{i} = L^{i}_{\beta}x^{\beta} + K^{ij}x_{j} + K^{i\,n+1}x_{n+1} + K^{i\,n+2}x_{n+2}, \\ r\bar{x}^{n+1} = L^{n+1}_{\beta}x^{\beta} + K^{n+1\,j}x_{j} + K^{n+1\,n+1}x_{n+1} + K^{n+1\,n+2}x_{n+2}, \\ r\bar{x}^{n+2} = L^{n+2}_{\beta}x^{\beta} + K^{n+2\,j}x_{j} + K^{n+2\,n+1}x_{n+1} + K^{n+2\,n+2}x_{n+2}, \\ r \neq 0,\ K^{\alpha} \neq 0 \end{cases}$$

can be regarded as a transformation in (C,E_n) in homogeneous sphere coordinates taking a sphere (50) into a sphere with homogeneous coordinates $r\bar{x}^{\alpha}, r\bar{x}^{1}, \dots, r\bar{x}^{n+2}$. Since the transformations (53) are linear, the results in Part III apply.

2. <u>Finite Conformal Group</u>.1) A finite conformal transformation in (C,E_n) in homogeneous sphere coordinates is defined as the most general non-singular transformation (53) leaving unchanged the quadratic functional equation

$$(54)\qquad x^{\beta}x_{\beta} + x^{j}x_{j} - x^{n+1}x^{n+2} = 0,$$

i.e. taking spheres of zero radius into spheres of zero

---

1) In this section we generalize the work on the finite conformal group in C done by Barnett and Nathan, l.c..

radius in (C,E_n).

Let (54) be invariant under (53). Then if the inverse of (53) is applied to

$$(\overline{54})\qquad \overline{x}^{\beta}\overline{x}_{\beta} + \overline{x}^{j}\overline{x}_{j} - \overline{x}^{n+1}\,\overline{x}^{n+2} = 0,$$

we will have

$$(r)^2\left(\overline{x}^{\beta}\overline{x}_{\beta} + \overline{x}^{j}\overline{x}_{j} - \overline{x}^{n+1}\overline{x}^{n+2}\right) = (M)^2\left(x^{\beta}x_{\beta} + x^{j}x_{j} - x^{n+1}x^{n+2}\right),$$

which, when we replace $r\overline{x}^{\alpha}, r\overline{x}^{\prime}, \ldots, r\overline{x}^{n+2}$ by their values from (53), becomes the identity in $x^{\alpha}, x^{\prime}, \ldots, x^{n+2}$

$$\left[(K^{\beta})^2 - (M)^2\right]x^{\beta}x_{\beta} + \left(K^{\beta\gamma}K_{\beta\delta} + 2K^{\gamma}K_{\gamma\delta} + L^{\gamma}_{j}L^{j}_{\delta}\right.$$

$$\left. - L^{\gamma}_{n+2}L^{n+1}_{\delta}\right)x_{\gamma}x^{\delta} + \left(2K_{\gamma}K^{\gamma}_{k} + 2K_{\beta\gamma}K^{\beta}_{k} + 2L^{j}_{\gamma}K_{jk}\right.$$

$$\left. - K_{n+1\,k}L^{n+2}_{\gamma} - L^{n+1}_{\gamma}K_{n+2\,k}\right)x^{k}x^{\gamma} + \left(K^{\beta}_{k}K^{l}_{\beta} + K_{jk}K^{jl}\right.$$

$$\left. - K_{n+2\,k}K^{n+1\,l} - M_{kl}M^{kl}\right)x^{k}x_{l} + \left(2K^{\gamma}K^{\gamma}_{n+1}\right.$$

$$\left. + 2K^{\beta\gamma}K^{n+1}_{\beta} + 2L^{\gamma}_{j}K^{j\,n+1} - K_{n+1\,n+1}L^{\gamma}_{n+2} - L^{\gamma}_{n+1}K_{n+2\,n+1}\right)x^{n+1}x_{\gamma}$$

$$+ \left(2K^{\beta}_{k}K^{n+1}_{\beta} + 2K_{jk}K^{j\,n+1} - K_{n+1\,n+1}K^{n+2\,k} - K_{n+1\,k}K^{n+2\,n+1}\right)x^{n+1}x^{k}$$

$$+ \left(2K^{\gamma}K^{\gamma}_{n+2} + 2K^{\beta\gamma}K^{n+2}_{\beta} + 2L^{\gamma}_{j}K^{j\,n+2} - K_{n+1\,n+2}L^{\gamma}_{n+2}\right.$$

$$-L^{\gamma}_{n+1}K_{n+2\,n+2}\Big)x_{n+2}x_{\gamma}+\Big(2K^{\beta}_{k}K^{n+2}_{\beta}+2K_{jk}K^{jn+2}$$

$$-K_{n+2\,k}K^{n+1\,n+2}-K_{n+1\,k}K^{n+2\,n+2}\Big)x^{n+2}x^{k}$$

$$+\Big(K^{\beta}_{n+1}K^{n+1}_{\beta}+K_{jn+1}K^{jn+1}-K_{n+1\,n+1}K^{n+2\,n+1}\Big)\Big(x^{n+1}\Big)^{2}$$

$$+\Big(K^{\beta}_{n+2}K^{n+2}_{\beta}+K_{jn+2}K^{jn+2}-K_{n+1\,n+2}K^{n+2\,n+2}\Big)\Big(x^{n+2}\Big)^{2}$$

$$+\Big[2K^{\beta}_{n+1}K^{n+2}_{\beta}+2K_{jn+1}K^{jn+2}-K_{n+1\,n+2}K^{n+2\,n+1}$$

$$-K_{n+1\,n+1}K^{n+2\,n+2}+(M)^{2}\Big]x^{n+1}x^{n+2}=0,$$

where

$$M_{ij}=\begin{cases}M & (i=j)\\ 0 & (i\neq j).\end{cases}$$

Putting in this identity $x^{n+1}=a$, $x^{n+2}=b$ and then arguing on the basis of the arbitrariness of a and b, we infer the following:

$$K^{\beta}_{n+1}K^{n+1}_{\beta}+K_{jn+1}K^{jn+1}-K_{n+1\,n+1}K^{n+2\,n+1}=0, \tag{55.1}$$

$$K^{\beta}_{n+2}K^{n+2}_{\beta}+K_{jn+2}K^{jn+2}-K_{n+1\,n+2}K^{n+2\,n+2}=0, \tag{55.2}$$

$$2K^{\beta}_{n+1}K_{\beta}^{n+2}+2K_{j\,n+1}K^{j\,n+2}-K_{n+1\,n+2}K^{n+2\,n+1}-K_{n+1\,n+1}K^{n+2\,n+2}+(M)^2=0, \tag{55.3}$$

$$\begin{aligned}&\left(2K^{\gamma}K^{\gamma}_{n+1}+2K^{\beta\gamma}K_{\beta}^{n+1}+2L^{\gamma}_{j}K^{j\,n+1}-K_{n+1\,n+1}L^{\gamma}_{n+2}\right.\\ &\left.-L^{\gamma}_{n+1}K_{n+2\,n+1}\right)x_{\gamma}+\left(2K^{\beta}_{k}K_{\beta}^{n+1}+2K_{jk}K^{j\,n+1}\right.\\ &\left.-K_{n+2\,k+1}K^{n+2\,n+1}-K_{n+1\,k}K^{n+2\,n+1}\right)x^{k}=0,\end{aligned} \tag{55.4}$$

$$\begin{aligned}&\left(2K^{\gamma}K^{\gamma}_{n+2}+2K^{\beta\gamma}K_{\beta}^{n+2}+2L^{\gamma}_{j}K^{j\,n+2}-K_{n+1\,n+2}L^{\gamma}_{n+2}\right.\\ &\left.-L^{\gamma}_{n+1}K_{n+2\,n+2}\right)x_{\gamma}+\left(2K^{\beta}_{k}K_{\beta}^{n+2}+2K_{jk}K^{j\,n+2}\right.\\ &\left.-K_{n+2\,k}K^{n+1\,n+2}-K_{n+1\,k}K^{n+2\,n+2}\right)x^{k}=0,\end{aligned} \tag{55.5}$$

$$\begin{aligned}&\left[(K^{\beta})^2-(M)^2\right]x^{\beta}x_{\beta}+\left(K^{\beta\gamma}K_{\beta\delta}+2K^{\gamma}K_{\gamma\delta}+L^{\gamma}_{j}L^{j}_{\delta}\right.\\ &\left.-L^{\gamma}_{n+2}L^{n+1}_{\delta}\right)x_{\gamma}x^{\delta}+\left(2K_{\gamma}K^{\gamma}_{k}+2K_{\beta\gamma}K^{\beta}_{k}\right.\\ &\left.+2L^{j}_{\gamma}K_{jk}-K_{n+1\,k}L^{n+2}_{\gamma}-L^{n+1}_{\gamma}K_{n+2\,k}\right)x^{k}x^{\gamma}\\ &+\left(K^{\beta}_{k}K^{l}_{\beta}+K_{jk}K^{jl}-K_{n+2\,k}K^{n+1\,l}\right.\\ &\left.-M_{kl}M^{kl}\right)x^{k}x_{l}=0.\end{aligned} \tag{55.6}$$

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From (55.4) we get by Lemma 1

(55.41)
$$\begin{aligned}&2K^{\alpha}K^{\alpha}_{n+1} + 2K^{\beta\alpha}K_{\beta}{}^{n+1} + 2L^{\alpha}_{j}K^{j\,n+1}\\ &\qquad - K_{n+1\,n+1}L^{\alpha}_{n+2} - K_{n+2\,n+1}L^{\alpha}_{n+1} = 0,\end{aligned}$$

(55.42)
$$2K^{\beta}_{i}K_{\beta}{}^{n+1} + 2K_{ji}K^{j\,n+1} - K_{n+1\,n+1}K_{n+2\,i} - K_{n+2\,n+1}K_{n+1\,i} = 0,$$

From (55.5) we get by Lemma 1

(55.51)
$$\begin{aligned}&2K^{\alpha}K^{\alpha}_{n+2} + 2K^{\beta\alpha}K_{\beta}{}^{n+2} + 2L^{\alpha}_{j}K^{j\,n+2}\\ &\qquad - K_{n+1\,n+2}L^{\alpha}_{n+2} - K_{n+2\,n+2}L^{\alpha}_{n+1} = 0,\end{aligned}$$

(55.52)
$$2K^{\beta}_{i}K_{\beta}{}^{n+2} + 2K_{ji}K^{j\,n+2} - K_{n+1\,n+2}K^{n+2\,i} - K_{n+2\,n+2}K^{n+1\,i} = 0.$$

From (55.6) we get by Lemma 2

(55.61)
$$(K^{\alpha})^{2} - (M)^{2} = 0,$$

(55.62)
$$\begin{aligned}&2K^{\gamma\alpha}K_{\gamma\beta} + 2L^{\alpha}_{j}L^{j}_{\beta} + 2K^{\alpha}K^{\alpha\beta} + 2K^{\beta}K^{\beta\alpha}\\ &\qquad - L^{\alpha}_{n+1}L^{\beta}_{n+2} - L^{\beta}_{n+1}L^{\alpha}_{n+2} = 0,\end{aligned}$$

(55.63)
$$\begin{aligned}&2K^{\alpha}K^{\alpha}_{i} + 2K^{\beta\alpha}K^{i}_{\beta} + 2K^{ji}L^{\alpha}_{j} - K^{n+1\,i}L^{\alpha}_{n+2}\\ &\qquad - K^{n+2\,i}L^{\alpha}_{n+1} = 0,\end{aligned}$$

(55.64)
$$\begin{aligned}&2K^{\beta}_{i}K^{j}_{\beta} + 2K^{ki}K_{kj} - 2(M^{ij})^{2} - K^{n+2\,i}K^{n+1\,j}\\ &\qquad - K^{n+2\,j}K^{n+1\,i} = 0,\end{aligned}$$

where

$$M^{ij} = \begin{cases} M & (i = j) \\ 0 & (i \neq j). \end{cases}$$

We thus have

THEOREM 30. A necessary condition that (53) leave (54) invariant is that coefficients of (53) satisfy the relations (55.1), (55.2), (55.3), (55.41), (55.42), (55.51), (55.52), (55.61), (55.62), (55.63), (55.64).[1]

---

[1] These relations will be collectively referred to as (55).

We now show that the transformation defined by (53) and (55) leaves (54) invariant. Let the inverse of this transformation be applied to $(\overline{54})$. We then have

$$\begin{aligned}
(r)^2(\bar{x}^\beta\bar{x}_\beta + \bar{x}^j\bar{x}_j - \bar{x}^{n+1}\bar{x}^{n+2}) \\
= (K^\beta)^2(x_\beta)^2 + \big(K^{\beta\gamma}K_{\beta\delta} + 2K^\gamma K_{\gamma\delta} + L^\gamma_j L^j_\delta \\
- L^\gamma_{n+2} L^{n+1}_\delta\big) x_\gamma x^\delta + \big(2K_\gamma K^\gamma_k + 2K_\beta K^\beta_{\gamma k} \\
+ 2L^j_\gamma K_{jk} - K_{n+1\,k} L^{n+2}_\gamma - L^{n+1}_\gamma K_{n+2\,k}\big) x^k x^\gamma \\
+ \big(K^\beta_k K^l_\beta + K_{jk} K^{jl} - K_{n+2\,k} K^{n+1\,l}\big) x^k x_l \\
+ \big(2K^\gamma K^\gamma_{n+1} + 2K^{\beta\gamma} K^{n+1}_\beta + 2L^\gamma_j K^{j\,n+1} \\
- K_{n+1\,n+1} L^\gamma_{n+2} - L^\gamma_{n+1} K_{n+2\,n+1}\big) x_{n+1} x_\gamma
\end{aligned} \tag{56}$$

$$+\Big(2K_k^{\beta}K_{\beta}^{n+1}+2K_{jk}K^{jn+1}-K_{n+1\,n+1}K_{n+2\,k}$$

$$-K_{n+2\,n+1}K_{n+1\,k}\Big)x^{n+1}x^{k}+\Big(2K^{\gamma}K_{n+2}^{\gamma}+2K^{\beta\gamma}K_{\beta}^{n+2}$$

$$+2L_j^{\gamma}K^{jn+2}-K_{n+1\,n+2}L_{n+2}^{\gamma}-L_{n+1}^{\gamma}K_{n+2\,n+2}\Big)x_{n+2}x_{\gamma}$$

$$+\Big(2K_k^{\beta}K_{\beta}^{n+2}+2K_{jk}K^{jn+2}-K_{n+1\,n+2}K_{n+2\,k}-K_{n+2\,n+2}K_{n+1\,k}\Big)x^{n+2}x^{k}$$

$$+\Big(K_{n+1}^{\beta}K_{\beta}^{n+1}+K_{jn+1}K^{jn+1}-K_{n+1\,n+1}K_{n+2\,n+1}\Big)\left(x^{n+1}\right)^{2}$$

$$+\Big(K_{n+2}^{\beta}K_{\beta}^{n+2}+K_{jn+2}K^{jn+2}-K_{n+1\,n+2}K_{n+2\,n+2}\Big)\left(x^{n+2}\right)^{2}$$

$$+\Big(2K_{n+1}^{\beta}K_{\beta}^{n+2}+2K_{jn+1}K^{jn+2}-K_{n+1\,n+2}K_{n+2\,n+1}$$

$$-K_{n+1\,n+1}K_{n+2\,n+2}\Big)x^{n+1}x^{n+2}.$$

But we find

$$\Big(K^{\beta\gamma}K_{\beta\delta}+2K^{\gamma}K_{\gamma\delta}+L_j^{\gamma}L_{\delta}^{j}-L_{n+2}^{\gamma}L_{\delta}^{n+1}\Big)x_{\gamma}x^{\delta}=0$$[1]

because of (55.62),

$$\Big(K_k^{\beta}K_{\beta}^{l}+K_{jk}K^{jl}-K_{n+2\,k}K^{n+1\,l}\Big)x^{k}x_{l}=\begin{cases}0 & (k\neq l)\\ (M)^{2}x^{j}x_{j} & (k=l)\end{cases}$$

in view of (55.64),

---

[1] We have here made use of the fact that if $P^{\beta\gamma}+P^{\gamma\beta}\equiv 0$, then $P^{\beta\gamma}x_{\beta}x_{\gamma}=0$.

$$(K^\beta)(x_\beta)^2 = (M)^2 x^\beta x_\beta$$

in view of (55.61), and

$$\left(2K^\beta_{n+1}K_\beta^{n+2} + 2K_{j\,n+1}K^{j\,n+2} - K_{n+1\,n+2}K_{n+2\,n+1}\right.$$

$$\left. - K_{n+1\,n+1}K_{n+2\,n+2}\right)x^{n+1}x^{n+2} = -(M)^2 x^{n+1}x^{n+2}$$

in view of (55.3). All other terms of the right-hand side of (56) vanish because of appropriate relations from (55). Thus (56) reduces to

$$(\lambda)^2\left(\bar{x}^\beta\bar{x}_\beta + \bar{x}^j\bar{x}_j - \bar{x}^{n+1}\bar{x}^{n+2}\right) = (M)^2\left(x^\beta x_\beta + x^j x_j - x^{n+1}x^{n+2}\right).$$

Hence (54) is invariant under (53) and (55), and we have

THEOREM 31. *A sufficient condition that* (53) *leave* (54) *invariant is that the relations* (55) *hold.*

From Theorems 30 and 31 we have

THEOREM 32. *The finite conformal transformations in* (C,E_n) *in homogeneous sphere coordinates are those non-singular transformations* (53) *for which the relations* (55) *hold.*

A second characterization of the finite conformal transformations in (C,E_n) in homogeneous sphere coordinates is that they are the most general non-singular transformations (53) taking orthogonal spheres into orthogonal spheres in (C,E_n). The equivalence of this definition with the one previously given is seen from with the one previously given is evident from

THEOREM 33. A necessary and sufficient condition that a non-singular transformation (53) take orthogonal spheres into orthogonal spheres in (C, E_n) is that the coefficients of the transformation satisfy (55).

To prove that the condition stated in the theorem is necessary, we assume that a non-singular transformation (53) takes orthogonal spheres into orthogonal spheres in (C, E_n). Let $(r_1 y^\alpha, r_1 y^1, \dots, r_1 y^{n+2})$ and $(r_2 z^\alpha, r_2 z^1, \dots, r_2 z^{n+2})$ be sets of homogeneous coordinates of two spheres, and let the homogeneous coordinates of the transformed spheres be respectively $(r_1 \bar{y}^\alpha, r_1 \bar{y}^1, \dots, r_1 \bar{y}^{n+2})$ and $(r_2 \bar{z}^\alpha, r_2 \bar{z}^1, \dots, r_2 \bar{z}^{n+2})$, $r_1, r_2 \neq 0$. When the inverse transformation is applied to the latter pair of spheres, the following relation must hold because of Theorem 29:

$$r_1 r_2 \left[2(\bar{y}^\beta \bar{z}_\beta + \bar{y}^j \bar{z}_j) - \bar{y}^{n+1} \bar{z}^{n+2} - \bar{z}^{n+1} \bar{y}^{n+2} \right]$$

$$= [M]^2 \left[2(y^\beta z_\beta + y^j z_j) - y^{n+1} z^{n+2} - z^{n+1} y^{n+2} \right].$$

Replacing $r_1 \bar{y}^\alpha, r_1 \bar{y}^1, \dots, r_1 \bar{y}^{n+2}, r_2 \bar{z}^\alpha, r_2 \bar{z}^1, \dots, r_2 \bar{z}^{n+2}$ in this relation by their values as given by the equations of the transformations, we get the following identity in $y^\alpha, y^1, \dots, y^{n+2}, z^\alpha, z^1, \dots, z^{n+2}$:

$$
(57)\left\{\begin{aligned}
&2\left[(K^{\beta})^{2}-(M)^{2}\right] y^{\beta} z_{\beta}+\Big(2K_{\delta}K^{\delta\gamma}+2K^{\gamma}K_{\gamma\delta}+2K^{\beta\gamma}K_{\beta\delta}\\
&\quad+2L^{\gamma}_{j}L^{j}_{\delta}-L^{\gamma}_{n+1}L^{n+2}_{\delta}-L^{\gamma}_{n+2}L^{n+1}_{\delta}\Big) y_{\gamma} z^{\delta}\\
&\quad+\Big(2K_{k}{}^{\beta}K_{\beta}{}^{l}+2K_{jk}K^{jl}-K_{n+1\,k}K^{n+2\,l}-K_{n+2\,k}K^{n+1\,l}\\
&\quad-2M^{kl}M_{kl}\Big) y^{k} z_{l}+\Big(2K^{\gamma}K_{\gamma}{}^{k}+2K^{\beta\gamma}K_{\beta}{}^{k}\\
&\quad+2L^{\gamma}_{j}K^{jk}-K^{n+1\,k}L^{\gamma}_{n+2}-K^{n+2\,k}L^{\gamma}_{n+1}\Big) y_{k} z_{\gamma}\\
&\quad+\Big(2K^{\gamma}K_{\gamma}{}^{k}+2K^{\beta\gamma}K_{\beta}{}^{k}+2L^{\gamma}_{j}K^{jk}-L^{\gamma}_{n+1}K^{n+2\,k}\\
&\quad-K^{n+1\,k}L^{\gamma}_{n+2}\Big) z_{k} y_{\gamma}+y^{n+1}\Big(2K^{\gamma}K_{n+1}{}^{\gamma}+2K^{\beta\gamma}K_{\beta}{}^{n+1}\\
&\quad+2L^{\gamma}_{j}K^{j\,n+1}-K_{n+1\,n+1}L^{\gamma}_{n+2}-K_{n+2\,n+1}L^{\gamma}_{n+1}\Big) z_{\gamma}\\
&\quad+y^{n+1}\Big(2K_{n+1}{}^{\beta}K_{\beta}{}^{k}+2K_{j\,n+1}K^{jk}-K_{n+1\,n+1}K^{n+2\,k}\\
&\quad-K_{n+2\,n+1}K^{n+1\,k}\Big) z_{k}+y^{n+2}\Big(2K^{\gamma}K_{n+2}{}^{\gamma}\\
&\quad+2K^{\beta\gamma}K_{\beta}{}^{n+2}+2L^{\gamma}_{j}K^{j\,n+2}-K_{n+1\,n+2}L^{\gamma}_{n+2}\\
&\quad-K_{n+2\,n+2}L^{\gamma}_{n+1}\Big) z_{\gamma}+y^{n+2}\Big(2K_{n+2}{}^{\beta}K_{\beta}{}^{k}+2K_{j\,n+2}K^{jk}
\end{aligned}\right.
$$

$$\begin{aligned}
&- K_{n+1\,n+2} K^{n+2\,k} - K_{n+2\,n+2} K^{n+1\,k} \Big) z_k + z^{n+1} \Big(2K^{\gamma} K^{\gamma}_{n+1} \\
&+ 2K^{\beta}_{n+1} K_{\beta\gamma} + 2L^{\gamma}_{j} K^{j\,n+1} - K_{n+2\,n+1} L^{\gamma}_{n+1} \\
&- K_{n+1\,n+1} L^{\gamma}_{n+2} \Big) y_{\gamma} + z^{n+1} \Big(2K^{\beta}_{n+1} K^{k}_{\beta} + 2K^{jk} K_{j\,n+1} \\
&- K_{n+2\,n+1} K^{n+1\,k} - K_{n+1\,n+1} K^{n+2\,k} \Big) y_k + z^{n+2} \Big(2K^{\gamma} K^{\gamma}_{n+2} \\
&+ 2K^{n+2}_{\beta} K^{\beta\gamma} + 2L^{\gamma}_{j} K^{j\,n+2} - K_{n+2\,n+2} L^{\gamma}_{n+1} \\
&- K_{n+1\,n+2} L^{\gamma}_{n+2} \Big) y_{\gamma} + z^{n+2} \Big(2K^{\beta}_{n+2} K^{k}_{\beta} + 2K_{j\,n+2} K^{jk} \\
&- K_{n+2\,n+2} K^{n+1\,k} - K_{n+1\,n+2} K^{n+2\,k} \Big) y_k + \Big(2K^{\beta}_{n+1} K^{n+1}_{\beta} \\
&+ 2K_{j\,n+1} K^{j\,n+1} - K_{n+1\,n+1} K_{n+2\,n+1} - K_{n+1\,n+1} K_{n+2\,n+1} \Big) y^{n+1} z^{n+1} \\
&+ \Big[2K^{\beta}_{n+1} K^{n+2}_{\beta} + 2K_{j\,n+1} K^{j\,n+2} - K_{n+1\,n+1} K_{n+2\,n+2} \\
&- K_{n+1\,n+2} K_{n+2\,n+1} + (M)^2 \Big] y^{n+1} z^{n+2} + \Big[2K^{\beta}_{n+1} K^{n+2}_{\beta} \\
&+ 2K_{j\,n+2} K^{j\,n+1} - K_{n+1\,n+2} K_{n+2\,n+1} - K_{n+1\,n+1} K_{n+2\,n+2} \\
&+ (M)^2 \Big] y^{n+2} z^{n+1} + \Big(2K^{\beta}_{n+2} K^{n+2}_{\beta} + 2K_{j\,n+2} K^{j\,n+2} \\
&- K_{n+1\,n+2} K_{n+2\,n+2} - K_{n+1\,n+2} K_{n+2\,n+2} \Big) y^{n+2} z^{n+2} = 0,
\end{aligned}$$

where

$$M_{kl} = \begin{cases} M & (k=l) \\ 0 & (k \neq l) \end{cases}.$$

By choosing in this identity

$$y^{n+1} = a^1,\quad y^{n+2} = a^2,\quad z^{n+1} = b^1,\quad z^{n+2} = b^2,$$

we can infer the following identities in y and z :

$$\begin{aligned}
(57.1)\qquad & 2\left[(K^{\beta})^2 - (M)^2\right] y^{\beta} z_{\beta} + 2\Big(K^{\gamma} K_{\gamma\delta} + K^{\beta\gamma} K_{\beta\delta} \\
& + L^{\gamma}_{j} L^{j}_{\delta} - L^{\gamma}_{n+1} L^{n+2}_{\delta}\Big) y_{\gamma} z^{\delta} + 2\Big(K^{\beta}_{k} K^{l}_{\beta} \\
& + K_{jk} K^{jl} - K_{n+1\,k} K^{n+2\,l} - M^{kl} M_{kl}\Big) y^{k} z_{l} \\
& + \Big(2K^{\gamma} K^{k}_{\gamma} + 2K^{\beta\gamma} K^{k}_{\beta} + 2L^{\gamma}_{j} K^{jk} \\
& - K^{n+1\,k} L^{\gamma}_{n+2} - K^{n+2\,k} L^{\gamma}_{n+1}\Big) y_{k} z_{\gamma} + \Big(2K^{\gamma} K^{k}_{\gamma} \\
& + 2K^{\beta\gamma} K^{k}_{\beta} + 2L^{\gamma}_{j} K^{jk} - L^{\gamma}_{n+1} K^{n+2\,k} \\
& - K^{n+1\,k} L^{\gamma}_{n+2}\Big) z_{k} y_{\gamma} = 0,
\end{aligned}$$

$$\begin{aligned}
(57.2)\qquad & \Big(2K^{\gamma} K^{\gamma}_{n+1} + 2K^{\beta\gamma} K^{n+1}_{\beta} + 2L^{\gamma}_{j} K^{j\,n+1} - K_{n+1\,n+1} L^{\gamma}_{n+2} \\
& - K_{n+2\,n+1} L^{\gamma}_{n+1}\Big) z_{\gamma} + \Big(2K^{\beta}_{n+1} K^{k}_{\beta} + 2K_{j\,n+1} K^{jk} \\
& - K_{n+1\,n+1} K^{n+2\,k} - K_{n+2\,n+1} K^{n+1\,k}\Big) z_{k} = 0,
\end{aligned}$$

$$2K^{\gamma}K^{\gamma}_{n+2} + 2K^{\beta\gamma}K^{n+2}_{\beta} + 2L^{\gamma}_{j}K^{j\,n+2} - K_{n+1\,n+2}L^{\gamma}_{n+2} - K_{n+2\,n+2}L^{\gamma}_{n+1}\Big)z_{\gamma} + \Big(2K^{\beta}_{n+2}K^{k}_{\beta} + 2K_{j\,n+2}K^{jk} - K_{n+1\,n+2}K^{n+2\,k} - K_{n+2\,n+2}K^{n+1\,k}\Big)z_{k} = 0, \tag{57.3}$$

$$K^{\beta}_{n+1}K^{n+1}_{\beta} + K_{j\,n+1}K^{j\,n+1} - K_{n+1\,n+1}K_{n+2\,n+1} = 0, \tag{57.4}$$

$$2K^{\beta}_{n+1}K^{n+2}_{\beta} + 2K_{j\,n+1}K^{j\,n+2} - K_{n+1\,n+1}K_{n+2\,n+2} - K_{n+1\,n+2}K_{n+2\,n+1} + (M)^{2} = 0, \tag{57.5}$$

$$K^{\beta}_{n+2}K^{n+2}_{\beta} + K_{j\,n+2}K^{j\,n+2} - K_{n+1\,n+2}K_{n+2\,n+2} = 0, \tag{57.6}$$

In (57.1) let us choose $y^{\alpha} = z^{\alpha}$; then by Lemma 2 we have (55.61), (55.62), (55.63) and (55.64). By Lemma 1 we infer (55.41) and (55.42) from (57.2), and (55.51) and (55.52) from (57.3), while (57.4), (57.5) and (57.6) are precisely (55.1), (55.3) and (55.2) respectively.

It remains to show that the condition is also sufficient. Let two spheres with respective sets of homogeneous coordinates

$$(y^{\alpha}, y^{1}, \dots, y^{n+2}), \; (z^{\alpha}, z^{1}, \dots, z^{n+2})$$

be transformed by a non-singular transformation (53) whose coefficients satisfy (55). Then, applying the inverse to the transformed spheres $(r_1\bar{y}^{\alpha}, r_1\bar{y}^{1}, \dots, r_1\bar{y}^{n+2})$ and $(r_2\bar{z}^{\alpha}, r_2\bar{z}^{1}, \dots, r_2\bar{z}^{n+2})$, we find that the expression

$$r_1 r_2\left[2(\bar{y}^{\beta}\bar{z}_{\beta} + \bar{y}^{j}\bar{z}_{j}) - \bar{y}^{n+1}\bar{z}^{n+2} - \bar{z}^{n+1}\bar{y}^{n+2}\right]$$

is equal to the expression which the right-hand side of (57) reduces to when the terms involving M are omitted, and because of relations (55) the latter expression reduces to

$$[M]^2\left[2(y^{\beta}z_{\beta} + y^{j}z_{j}) - y^{n+1}z^{n+2} - z^{n+1}y^{n+2}\right].$$

From this and Theorem 29 the sufficiency of the condition follows.

Let us apply successively the conformal transformation (53), (55) and the conformal transformation

$$(53)\quad \begin{cases} \bar{r}\,\bar{\bar{x}}^{\alpha} = \bar{K}^{\alpha}x^{\alpha} + \bar{K}^{\alpha\beta}\bar{x}_{\beta} + \bar{K}_{j}^{\alpha}\bar{x}^{j} + \bar{K}_{n+1}^{\alpha}\bar{x}^{n+1} + \bar{K}_{n+2}^{\alpha}\bar{x}^{n+2}, \\ \bar{r}\,\bar{\bar{x}}^{i} = \bar{L}_{\beta}^{i}\bar{x}^{\beta} + \bar{K}^{ij}\bar{x}_{j} + \bar{K}^{i\,n+1}\bar{x}_{n+1} \\ \qquad\qquad + \bar{K}^{i\,n+2}\bar{x}_{n+2} \\ \bar{r}\,\bar{\bar{x}}^{n+1} = \bar{L}_{\beta}^{n+1}\bar{x}^{\beta} + \bar{K}^{n+1\,j}\bar{x}_{j} + \bar{K}^{n+1\,n+1}\bar{x}_{n+1} + \bar{K}^{n+1\,n+2}\bar{x}_{n+2}, \\ \bar{r}\,\bar{\bar{x}}^{n+2} = \bar{L}_{\beta}^{n+2}\bar{x}^{\beta} + \bar{K}^{n+2\,j}\bar{x}_{j} + \bar{K}^{n+2\,n+1}\bar{x}_{n+1} + \bar{K}^{n+2\,n+2}\bar{x}_{n+2}, \\ r \neq 0, \quad \bar{K}^{\alpha} \neq 0, \end{cases}$$

$(\overline{55})$, where $(\overline{55})$ represents conditions satisfied by the coefficients of $(\overline{53})$ corresponding to (55). By Theorem 11 the product is a linear transformation

$$(\overline{\overline{53}})\quad\begin{cases}\bar{\bar{r}}\,\bar{\bar{x}}^{\alpha}=\bar{\bar{K}}^{\alpha}x^{\alpha}+\bar{\bar{K}}^{\alpha\beta}x_{\beta}+\bar{\bar{K}}^{\alpha}_{j}x^{j}+\bar{\bar{K}}^{\alpha}_{n+1}x^{n+1}\\ \qquad\qquad+\bar{\bar{K}}^{\alpha}_{n+2}x^{n+2},\\ \bar{\bar{r}}\,\bar{\bar{x}}^{i}=\bar{\bar{L}}^{i}_{\beta}x^{\beta}+\bar{\bar{K}}^{ij}x_{j}+\bar{\bar{K}}^{i\,n+1}x_{n+1}+\bar{\bar{K}}^{i\,n+2}x_{n+2},\\ \bar{\bar{r}}\,\bar{\bar{x}}^{n+1}=\bar{\bar{L}}^{n+1}_{\beta}x^{\beta}+\bar{\bar{K}}^{n+1\,j}x_{j}+\bar{\bar{K}}^{n+1\,n+1}x_{n+1}+\bar{\bar{K}}^{n+1\,n+2}x_{n+2},\\ \bar{\bar{r}}\,\bar{\bar{x}}^{n+2}=\bar{\bar{L}}^{n+2}_{\beta}x^{\beta}+\bar{\bar{K}}^{n+2\,j}x_{j}+\bar{\bar{K}}^{n+2\,n+1}x_{n+1}+\bar{\bar{K}}^{n+2\,n+2}x_{n+2},\end{cases}$$

where

$$\bar{\bar{r}}=\bar{r}r\neq 0,\quad \bar{\bar{K}}^{\alpha}=\bar{K}^{\alpha}K^{\alpha}\neq 0.$$

Since the transformations which we have applied are conformal, we have

$$(r)^{2}(\bar{x}^{\beta}\bar{x}_{\beta}+\bar{x}^{j}\bar{x}_{j}-\bar{x}^{n+1}\bar{x}^{n+2})=(M)^{2}(x^{\beta}x_{\beta}+x^{j}x_{j}-x^{n+1}x^{n+2}),$$

$$(\bar{r})^{2}(\bar{\bar{x}}^{\beta}\bar{\bar{x}}_{\beta}+\bar{\bar{x}}^{j}\bar{\bar{x}}_{j}-\bar{\bar{x}}^{n+1}\bar{\bar{x}}^{n+2})=(\bar{M})^{2}(\bar{x}^{\beta}\bar{x}_{\beta}+\bar{x}^{j}\bar{x}_{j}-\bar{x}^{n+1}\bar{x}^{n+2}).$$

From these two relations we get

$$(\bar{\bar{r}})^{2}(\bar{\bar{x}}^{\beta}\bar{\bar{x}}_{\beta}+\bar{\bar{x}}^{j}\bar{\bar{x}}_{j}-\bar{\bar{x}}^{n+1}\bar{\bar{x}}^{n+2})=(\bar{\bar{M}})^{2}(x^{\beta}x_{\beta}+x^{j}x_{j}-x^{n+1}x^{n+2}).$$

Hence $(\overline{\overline{53}})$ takes spheres of zero radius into spheres of zero radius in (C,E_n). We thus have

THEOREM 34. *The product of two finite conformal transformations in* (C,E_n) *is a finite conformal trans-*

formation in (C, E_n).

By Theorem 13 the conformal transformation defined by (53) and (55) has a non-singular inverse

$$(58)\quad \begin{cases} \bar{r} x^\alpha = \bar{K}^\alpha \bar{x}^\alpha + \bar{K}^{\alpha\beta} \bar{x}_\beta + \bar{K}^\alpha_j \bar{x}^j + \bar{K}^\alpha_{n+1} \bar{x}^{n+1} + \bar{K}^\alpha_{n+2} \bar{x}^{n+2}, \\ \bar{r} x^i = \qquad \bar{L}^i_\beta \bar{x}^\beta + \bar{K}^{ij} \bar{x}_j - \bar{K}^{i\,n+1} \bar{x}_{n+1} + \bar{K}^{i\,n+2} \bar{x}_{n+2}, \\ \bar{r} x^{n+1} = \qquad \bar{L}^{n+1}_\beta \bar{x}^\beta + \bar{K}^{n+1\,j} \bar{x}_j + \bar{K}^{n+1\,n+1} x_{n+1} + \bar{K}^{n+1\,n+2} \bar{x}_{n+2}, \\ \bar{r} x^{n+2} = \qquad \bar{L}^{n+2}_\beta \bar{x}^\beta + \bar{K}^{n+2\,j} \bar{x}_j + \bar{K}^{n+2\,n+1} \bar{x}_{n+1} + \bar{K}^{n+2\,n+2} \bar{x}_{n+2}, \\ \quad \bar{r} \neq 0, \quad \bar{K}^\alpha \neq 0. \end{cases}$$

Since the transformation defined by (53) and (55) is conformal, we have

$$(59)\qquad (r)^2\left(\bar{x}^\beta \bar{x}_\beta + \bar{x}^j \bar{x}_j - \bar{x}^{n+1} \bar{x}^{n+2}\right) = (M)^2\left(x^\beta x_\beta + x^j x_j - x^{n+1} x^{n+2}\right).$$

From (58) we have

$$(58')\qquad (\bar{r})^2\left(x^\beta x_\beta + x^j x_j - x^{n+1} x^{n+2}\right) = (\overline{56}),$$

where $(\overline{56})$ denotes the expression which the right-hand side of (56) becomes when bars are placed above the coefficients and above $x^\alpha, x^1, \cdots, x^{n+2}$. Replacing in (58′) the quadratic functional

$$x^\beta x_\beta + x^j x_j - x^{n+1} x^{n+2}$$

by its value from (58), we get the identity

$$(60)\qquad \frac{(\bar{r})^2(r)^2}{(M)^2}\left(\bar{x}^\beta \bar{x}_\beta + \bar{x}^j \bar{x}_j - \bar{x}^{n+1}\bar{x}^{n+2}\right) = (\overline{56}).$$

By first putting in (60) $\bar{x}^{n+1} = a$, $\bar{x}^{n+2} = b$, then invoking the arbitrariness of a and b , and then using Lemmas 1 and 2, we obtain from (60) relations ($\overline{55}$), where

$$\bar{M} = \frac{\bar{r}r}{M}, \quad \bar{M}^{ij} = \begin{cases} \bar{M} & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Hence (58) is conformal, and we have

THEOREM 35. *The inverse of a finite conformal transformation in* (C,E_n) *exists and is itself a finite conformal transformation in* (C,E_n).

Theorems 32, 34 and 35 together give

THEOREM 36. *The totality of non-singular transformations defined by (53) and (55) constitute the group of finite conformal transformations in* (C,E_n) *in homogeneous sphere coordinates*.

3. *Infinitesimal Conformal Group*. For a fixed a the set

$$x(a) \equiv \left(x^\alpha(a), x^1(a), \dots, x^n(a)\right)$$

defines a point in (C,E_n). As a varies in any interval $a_1 \leqq a \leqq a_2$, the point describes a curve in (C,E_n). Hence

$$(61)\qquad x(a) \qquad\qquad (a_1 \leqq a \leqq a_2)$$

defines a curve in (C, E_n). The components of the tangent to the curve are

$$\mu^\alpha \equiv \frac{dx^\alpha}{da}, \mu^1 \equiv \frac{dx^1}{da}, \dots, \mu^n \equiv \frac{dx^n}{da},$$

so that the tangent is the vector

$$\mu \equiv \frac{dx}{da} \equiv (\mu^\alpha, \mu^1, \dots, \mu^n).$$

Together the point x and the direction μ define a lineal element in (C, E_n), namely, the vector μ which starts from x.

If in a suitably restricted subset of (C, E_n) a regular infinitesimal transformation (14) is applied to the curve (61), there will be a corresponding regular infinitesimal transformation of the direction, for by Theorem 10 we will have

$$\frac{\partial \mu^\alpha}{\partial t} = \frac{\partial^2 x^\alpha}{\partial a \partial t} = \frac{\partial}{\partial a} p^\alpha(x) = p^\alpha\left(x; \frac{\partial x}{\partial a}\right) = p^\alpha(x; \mu),$$

$$\frac{\partial \mu^i}{\partial t} = \frac{\partial^2 x^i}{\partial a \partial t} = \frac{\partial}{\partial a} p^i(x) = p^i\left(x; \frac{\partial x}{\partial a}\right) = p^i(x; \mu).$$

The regular infinitesimal transformations (14) and

$$(62) \quad \begin{cases} \dfrac{d\mu^\alpha}{dt} = p^\alpha(x; \mu), \\ \dfrac{d\mu^i}{dt} = p^i(x; \mu) \end{cases}$$

together define a regular infinitesimal transformation of lineal elements, which is called the first extension of (14).

An infinitesimal conformal transformation in (C, E_n) is defined as the most general regular infinitesimal transformation in (C, E_n) which leaves invariant the angle between two lineal elements having a common origin.

Two lineal elements with a common origin are represented by $\{x, \mu\}$ and $\{x, \nu\}$. The angle between them is the angle between the vectors μ and ν, and the cosine of this angle is

$$(63) \qquad \frac{\mu^\beta \nu_\beta + \mu^j \nu_j}{\left[(\mu^\gamma \mu_\gamma + \mu^k \mu_k)(\nu^\delta \nu_\delta + \nu^l \nu_l)\right]^{\frac{1}{2}}} .$$

To obtain a necessary condition that (14) be conformal we assume that (63) is invariant under (14) in a suitably restricted subset of (C, E_n). We then have

$$\frac{d}{dt}\left\{ \frac{\mu^\beta \nu_\beta + \mu^j \nu_j}{\left[(\mu^\gamma \mu_\gamma + \mu^k \mu_k)(\nu^\delta \nu_\delta + \nu^l \nu_l)\right]^{\frac{1}{2}}} \right\} = 0,$$

whence

$$\mu^\beta \frac{d\nu_\beta}{dt} + \nu^\beta \frac{d\mu_\beta}{dt} + \mu^j \frac{d\nu_j}{dt} + \nu^j \frac{d\mu_j}{dt} - \left(\mu^\beta \nu_\beta + \mu^j \nu_j\right)\left(\frac{\nu^\gamma \frac{d\nu_\gamma}{dt} + \nu^k \frac{d\nu_k}{dt}}{\nu^\delta \nu_\delta + \nu^l \nu_l} + \frac{\mu^\gamma \frac{d\mu_\gamma}{dt} + \mu^k \frac{d\mu_k}{dt}}{\mu^\delta \mu_\delta + \mu^l \mu_l} \right) = 0 .$$

In particular, let the lineal elements be orthogonal, so that we have

$$(64) \qquad \mu^\beta \nu_\beta + \mu^j \nu_j = 0 .$$

Then the following equation must hold for every choice of μ and ν satisfying (64):

$$\mu_\beta \frac{d\nu^\beta}{dt} + \nu_\beta \frac{d\mu^\beta}{dt} + \mu_j \frac{d\nu^j}{dt} + \nu_j \frac{d\mu^j}{dt} = 0 ,$$

or, in view of (62),

$$\mu_\beta \mathcal{P}^\beta(x;\nu) + \nu_\beta \mathcal{P}^\beta(x;\mu) + \mu_j \mathcal{P}^j(x;\nu) + \nu_j \mathcal{P}^j(x;\mu) = 0.$$

Replacing μ, ν by $c\mu$, $c\nu$ in this equation, we obtain the equations

$$(65) \qquad \begin{aligned} &\mu_\beta \mathfrak{A}^\beta_p(x;\nu) + \nu_\beta \mathfrak{A}^\beta_p(x;\mu) + \mu_j \mathfrak{A}^j_p(x;\nu) \\ &\quad + \nu_j \mathfrak{A}^j_p(x;\mu) = 0 \qquad (p = 1, 2, 3, \cdots), \end{aligned}$$

which must hold for every μ and ν satisfying (64).

For $p = 1$, (65) becomes

$$(65.1) \qquad \begin{aligned} &2L^\beta \mu_\beta \nu_\beta + (L^{\beta\gamma} + L^{\gamma\beta})\mu_\beta \nu_\gamma + (\tilde{L}^{jk} + \tilde{L}^{kj})\mu_j \nu_k \\ &\quad + (L^\beta_j + \tilde{L}^\beta_j)\nu_\beta \mu^j + (L^\beta_j + \tilde{L}^\beta_j)\mu_\beta \nu^j = 0 . \end{aligned}$$

Taking ξ_1 and ξ_2 as any distinct values in I, let us define μ and ν as follows:

$$\mu^{\alpha} = 0 \qquad (0 \leqq \alpha < \xi_1 - \varepsilon,\ \xi_1 + \varepsilon < \alpha \leqq 1),$$

$$\mu^{\alpha} > 0 \qquad (\xi_1 - \varepsilon \leqq \alpha \leqq \xi_1 + \varepsilon),$$

$$\nu^{\alpha} = 0 \qquad (0 \leqq \alpha < \xi_2 - \varepsilon,\ \xi_2 + \varepsilon < \alpha \leqq 1),$$

$$\nu^{\alpha} > 0 \qquad (\xi_2 - \varepsilon \leqq \alpha \leqq \xi_2 + \varepsilon),$$

$$\mu^{i} = \nu^{i} = 0 .$$

The μ and ν thus defined satisfy (64). Putting them in (65.1), we get

$$\int_{\xi_1-\varepsilon}^{\xi_1+\varepsilon}\int_{\xi_2-\varepsilon}^{\xi_2+\varepsilon}(L^{\beta\gamma} + L^{\gamma\beta})\mu^{\beta}\nu^{\gamma}\,d\beta\, d\gamma = 0, \tag{65.11}$$

and, since μ^{α} and ν^{α} are positive on the ranges of integration here involved, (65.11) implies

$$L^{\beta\gamma} + L^{\gamma\beta} = 0 \qquad (\xi_1 - \varepsilon \leqq \beta \leqq \xi_1 + \varepsilon,\ \xi_2 - \varepsilon \leqq \gamma \leqq \xi_2 + \varepsilon).$$

Passing to the limit as $\varepsilon \to 0$, we get from this relation

$$L^{\xi_1\xi_2} + L^{\xi_2\xi_1} = 0,$$

and, since ξ_1 and ξ_2 were any distinct values in I, we have finally

$$L^{\alpha\beta} + L^{\beta\alpha} \equiv 0 .$$

Thus (65.1) reduces to

$$\begin{aligned} 2L^{\beta}\mu_{\beta}\nu_{\beta} + (\tilde{L}^{jk} + \tilde{L}^{kj})\mu_{j}\nu_{k} + (L^{\beta}{}_{j} + \tilde{L}^{\beta}{}_{j})\nu_{\beta}\mu^{j} \\ + (L^{\beta}{}_{j} + \tilde{L}^{\beta}{}_{j})\mu_{\beta}\nu^{j} = 0 . \end{aligned} \tag{65.12}$$

Let us now redefine μ and ν as follows:

$$\mu^\alpha = b^\alpha,$$

$$\nu^\alpha = 0 \quad (0 \leqq \alpha < \xi_1 - e,\ \xi_1 + e < \alpha < \xi_2 - e,\ \xi_2 + e < \alpha \leqq 1),$$

$$\nu^{\xi_1 + t} = -\nu^{\xi_2 + t} > 0 \quad (-e \leqq t \leqq e),$$

$$\mu^i = \nu^i = 0 .$$

Since (64) is again satisfied, we can use in (65.12) the μ and ν just defined, getting

(65.121) $$\int_{\xi_1 - e}^{\xi_1 + e} L^\beta \nu^\beta d\beta + \int_{\xi_2 - e}^{\xi_2 + e} L^\beta \nu^\beta d\beta = 0 .$$

If we make in the second term of (65.121) the change of variable

$$\tilde{\beta} = \beta + \xi_1 - \xi_2 ,$$

that term becomes

$$\int_{\xi_1 - e}^{\xi_1 + e} L^{\tilde{\beta} - \xi_1 + \xi_2} \nu^{\tilde{\beta} - \xi_1 + \xi_2} d\tilde{\beta},$$

or

$$\int_{\xi_1 - e}^{\xi_1 + e} L^{\beta - \xi_1 + \xi_2} \nu^{\beta - \xi_1 + \xi_2} d\beta,$$

which, in view of the definition of ν, is equivalent to

$$- \int_{\xi_1 - e}^{\xi_1 + e} L^{\beta - \xi_1 + \xi_2} \nu^\beta d\beta .$$

(65.121) thus can be written in the form

$$\int_{\xi_1 - e}^{\xi_1 + e} \left(L^\beta - L^{\beta - \xi_1 + \xi_2} \right) \nu^\beta d\beta = 0,$$

which, since v is positive in $(\xi_1-\epsilon, \xi_1+\epsilon)$, implies

$$L^{\beta} = L^{\beta-\xi_1+\xi_2} \qquad (\xi_1-\epsilon \leqq \beta \leqq \xi_1+\epsilon).$$

Passing to the limit as $\epsilon \to 0$, we get

$$L^{\xi_1} = L^{\xi_2},$$

whence it follows that L^{α} is a constant, say K.

(65.12) now reduces to

$$\begin{aligned} &2K\mu^{\beta}v_{\beta} + (\tilde{L}^{jk}+\tilde{L}^{kj})\mu_j v_k \\ (65.122)\qquad &+(L^{\beta}_{\cdot j}+\tilde{L}^{\beta}_{\cdot j})v_{\beta}\mu^{j}+(L^{\beta}_{\cdot j}+\tilde{L}^{\beta}_{\cdot j})\mu_{\beta}v^{j}=0 \end{aligned}$$

We make now the following choice of μ and v:

$$\mu^{\alpha}=0,\ \mu^{i}=b^{i},\ v^{i}=0,\quad v^{\alpha}\ \text{unrestricted.}$$

When this choice, which satisfies (64), is applied in (65.122), we get

$$(L^{\beta}_{\cdot j}+\tilde{L}^{\beta}_{\cdot j})v_{\beta}b^{j}=0,$$

whence, because of the arbitrariness of the b^{i}, we infer

$$(L^{\beta}_{\cdot j}+\tilde{L}^{\beta}_{\cdot j})v_{\beta}=0,$$

and by Lemma 1 this identity in v^{α} yields

$$L^{\alpha}_{\cdot i}+\tilde{L}^{\alpha}_{\cdot i}\equiv 0 .$$

(65.122) thus reduces to

$$2K\mu^{\beta}v_{\beta}+(\tilde{L}^{jk}+\tilde{L}^{kj})\mu_j v_k = 0,$$

which, in view of (64), can be written as

$$-2K u^j v_j + (\tilde{L}^{jk} + \tilde{L}^{kj}) u_j v_k = 0,$$

whence

$$\tilde{L}^{ij} \equiv K \qquad (i=j),$$

$$\tilde{L}^{ij} + \tilde{L}^{ji} \equiv 0 \qquad (i \neq j).$$

We conclude, then, that if (63) is invariant under (14) $\mathfrak{A}_1^\alpha(x)$ and $\mathfrak{A}_1^i(x)$ must be of the form

$$(66) \quad \begin{cases} \mathfrak{A}_1^\alpha(x) = K x^\alpha + L^{\alpha\beta} x_\beta + L^{\alpha}_{\cdot j} x^j, \\ \mathfrak{A}_1^i(x) = K x^i - L^i_{\beta} x^\beta + L^{ij} x_j, \end{cases}$$

where

$$L^{\alpha\beta} + L^{\beta\alpha} \equiv 0, \quad L^{ij} + L^{ji} \equiv 0.$$

In seeking to determine what the composite integral power forms of higher order than the first must be, we make use of the fact that (65) implies

$$(65.2) \quad u_\beta \mathfrak{A}_p^\beta(x;w;v) + v_\beta \mathfrak{A}_p^\beta(x;w;u) + u_j \mathfrak{A}_p^j(x;w;v) + v_j \mathfrak{A}_p^j(x;w;u) = 0 \qquad (p=2,3,\dots).$$

Putting

$$\tilde{\mathfrak{A}}_{p-1}^\alpha(x) = \mathfrak{A}_p^\alpha(x;w), \quad \tilde{\mathfrak{A}}_{p-1}^i(x) = \mathfrak{A}_p^i(x;w) \quad (p=2,3,\dots),$$

(65.2) becomes

$$(65.2') \quad \mu_{\beta}\,\tilde{\mathfrak{A}}^{\beta}_{p-1}(x;v)+v_{\beta}\,\tilde{\mathfrak{A}}^{\beta}_{p-1}(x;\mu)+\mu_{j}\,\tilde{\mathfrak{A}}^{j}_{p-1}(x;v)$$
$$+v_{j}\,\tilde{\mathfrak{A}}^{j}_{p-1}(x;\mu)=0 \qquad (p=2,3,\cdots).$$

A comparison of (65.2') with (65) shows that $\tilde{\mathfrak{A}}^{\alpha}_{p-1}(x)$ and $\tilde{\mathfrak{A}}^{i}_{p-1}(x)$ must satisfy the same relation as exists between $\mathfrak{A}^{\alpha}_{p-1}(x)$ and $\mathfrak{A}^{i}_{p-1}(x)$ and must therefore be of the same form respectively as $\mathfrak{A}^{\alpha}_{p-1}(x)$ and $\mathfrak{A}^{i}_{p-1}(x)$, as composite integral power forms in x.

In particular, $\tilde{\mathfrak{A}}^{\alpha}_{1}(x)$ and $\tilde{\mathfrak{A}}^{i}_{1}(x)$ must be of the same form respectively as $\mathfrak{A}^{\alpha}_{1}(x)$ and $\mathfrak{A}^{i}_{1}(x)$. When, in computing $\tilde{\mathfrak{A}}^{\alpha}_{1}(x)$ and $\tilde{\mathfrak{A}}^{i}_{1}(x)$, use is made of the symmetry of the coefficients in all indices other than α and i, we get

$$(67)\quad \left\{\begin{aligned}
\tilde{\mathfrak{A}}^{\alpha}_{1}(x) &= \mathfrak{A}^{\alpha}_{2}(x;w) = \Big(2M^{\alpha}w^{\alpha}+M^{\alpha\beta}w_{\beta} \\
&\quad + M^{\alpha}_{j}w^{j}\Big)x^{\alpha}+\Big(M^{\alpha\beta}w^{\alpha}+M^{\alpha\beta}_{j}w^{j} \\
&\quad + 2\hat{M}^{\alpha\beta}w^{\beta}+2M^{\alpha\beta\gamma}w_{\gamma}\Big)x_{\beta} \\
&\quad +\Big(2M^{\alpha}_{jk}w^{k}+M^{\alpha\beta}_{j}w_{\beta}+M^{\alpha}_{j}w^{\alpha}\Big)x^{j}, \\
\tilde{\mathfrak{A}}^{i}_{1}(x) &= \mathfrak{A}^{i}_{2}(x;w) = \Big(\tilde{M}^{ij}_{\beta}w_{j}+2\tilde{M}^{i}_{\beta}w_{\beta} \\
&\quad + 2\tilde{M}^{i}_{\beta\gamma}w^{\gamma}\Big)x^{\beta}+\Big(2\tilde{M}^{ijk}w_{k}+\tilde{M}^{ij}_{\beta}w^{\beta}\Big)x_{j}.
\end{aligned}\right.$$

Since the right-hand members of (67) must be of the same form (as composite integral power series in x)as the corresponding right-hand members of (66), we can infer the following identities in w :

(67.1) $$2M^{\alpha}w^{\alpha}+\left(M^{\alpha\beta}-\tilde{M}^{\beta}_{ii}\right)w_{\beta}+\left(M^{\alpha}_{k}-2\tilde{M}_{iik}\right)w^{k}=0,$$

(67.2) $$\left(M^{\alpha\beta}+2\hat{M}^{\beta\alpha}\right)w^{\alpha}+\left(M^{\beta\alpha}+2\hat{M}^{\alpha\beta}\right)w^{\beta}$$
$$+\left(M^{\alpha\beta}_{j}+M^{\beta\alpha}_{j}\right)w^{j}+2\left(M^{\alpha\beta\gamma}+M^{\beta\alpha\gamma}\right)w_{\gamma}=0,$$

(67.3) $$\left(2M^{\alpha}_{ij}+\tilde{M}^{\alpha}_{ij}\right)w^{j}+\left(M^{\alpha}_{i}+2\tilde{M}^{\alpha}_{i}\right)w^{\alpha}$$
$$+\left(M^{\alpha\beta}_{i}+2\tilde{M}^{\alpha\beta}_{i}\right)w_{\beta}=0,$$

(67.4) $$2\left(\tilde{M}^{ijk}+\tilde{M}^{jik}\right)w_{k}+\left(\tilde{M}^{ij}_{\beta}+\tilde{M}^{ji}_{\beta}\right)w^{\beta}=0 \quad (i\neq j).$$

From (67.1) we get by Lemma 1

(67.11) $$M^{\alpha}\equiv 0,$$

(67.12) $$M^{\alpha\beta}-\tilde{M}^{\beta}_{ii}\equiv 0,$$

(67.13) $$M^{\alpha}_{k}-2\tilde{M}_{iik}\equiv 0.$$

From (67.12) and (67.13) respectively it follows that, introducing new symbols $\hat{M}^{\beta}$ and $\hat{M}_{k}$, we can put

(67.121) $$M^{\alpha\beta}\equiv\tilde{M}^{\beta}_{ii}=2\hat{M}^{\beta},$$

(67.131) $M^{\alpha}_{k} \equiv 2\tilde{M}_{iik} = 2\hat{M}_{k}$.

From (67.131) and the relation $\tilde{M}_{ijk} = \tilde{M}_{ikj}$ it follows that

(67.132) $\tilde{M}_{iji} \equiv \tilde{M}_{iij} \equiv \hat{M}_{j}$.

Applying Lemma 1 to (67.2), we get

(67.21) $M^{\beta\alpha} + 2\hat{M}^{\alpha\beta} \equiv 0,$

(67.22) $M^{\alpha\beta}_{j} + M^{\beta\alpha}_{j} \equiv 0,$

(67.23) $M^{\alpha\beta\gamma} + M^{\beta\alpha\gamma} \equiv 0$.

From (67.121) and (67.21) there follows

(67.211) $\hat{M}^{\alpha\beta} \equiv -\hat{M}^{\alpha}$.

From (67.23) and the symmetry of $M^{\alpha\beta\gamma}$ in β and γ, we find

$$M^{\alpha\beta\gamma} \equiv -M^{\beta\alpha\gamma} \equiv -M^{\beta\gamma\alpha} \equiv M^{\gamma\beta\alpha} \equiv M^{\gamma\alpha\beta} \equiv -M^{\alpha\gamma\beta} \equiv -M^{\alpha\beta\gamma},$$

whence

(67.231) $M^{\alpha\beta\gamma} \equiv 0$.

Applying Lemma 1 to (67.3), we get

(67.31) $$2M^{\alpha}_{ij} + \tilde{M}^{\alpha}_{ij} \equiv 0,$$

(67.32) $$M^{\alpha}_{i} + 2\tilde{M}^{\alpha}_{i} \equiv 0,$$

(67.33) $$M^{\alpha\beta}_{i} + 2\tilde{M}^{\alpha\beta}_{i} \equiv 0\,.$$

From (67.31) and (67.121) there follows

(67.311) $$M^{\alpha}_{ii} \equiv -\hat{M}^{\alpha}$$

From (67.32) and (67.131) there follows

(67.321) $$\tilde{M}^{\alpha}_{i} \equiv -\hat{M}_{i}\,.$$

From (67.33) and (67.22) we get

$$\tilde{M}^{\alpha\beta}_{i} + \tilde{M}^{\beta\alpha}_{i} \equiv 0,$$

but $\tilde{M}^{\alpha\beta}_{i} \equiv \tilde{M}^{\beta\alpha}_{i}$; hence

(67.331) $$\tilde{M}^{\alpha\beta}_{i} \equiv 0,$$

and in view also of (67.33) we get

(67.332) $$M^{\alpha\beta}_{i} \equiv 0\,.$$

Applying Lemma 1 to (67.4), we get

(67.41) $$\tilde{M}_{ijk} + \tilde{M}_{jik} \equiv 0 \qquad (i \neq j),$$

(67.42) $$\tilde{M}^{\beta}_{ij\cdot} + \tilde{M}^{\beta}_{ji\cdot} \equiv 0 \qquad (i \neq j).$$

From (67.42) and (67.31) we find

$$M^{\alpha}_{ij\cdot} + M^{\alpha}_{ji\cdot} \equiv 0 \qquad (i \neq j),$$

and since, further, $M^{\alpha}_{ij\cdot} \equiv M^{\alpha}_{ji\cdot}$,it follows that

(67.312) $$M^{\alpha}_{ij\cdot} \equiv 0 \qquad (i \neq j),$$

and consequently, in view of (67.31),

(67.313) $$\tilde{M}^{\alpha}_{ij\cdot} \equiv 0 \qquad (i \neq j).$$

From (67.41) and the relation $\tilde{M}_{ijk} \equiv \tilde{M}_{ikj}$ we find

$$\tilde{M}_{ijk} \equiv -\tilde{M}_{jik} \equiv -\tilde{M}_{jki} \equiv \tilde{M}_{kji} \equiv \tilde{M}_{kij} \equiv -\tilde{M}_{ikj} \equiv -\tilde{M}_{ijk} \quad (i \neq j \neq k),$$

whence

(67.411) $$\tilde{M}_{ijk} \equiv 0 \qquad (i \neq j \neq k).$$

From (67.41) we find

$$\tilde{M}_{ijk} \equiv \tilde{M}_{\cdot ijj} \equiv -\tilde{M}_{\cdot jij} \qquad (j = k \neq i),$$

or, since $\tilde{M}_{jij} \equiv \hat{M}_{i}$ by (67.132),

(67.412) $$\tilde{M}_{ijk} \equiv -\hat{M}_{i} \qquad (j = k \neq i).$$

We are now in a position to state what form $\underset{2}{\mathfrak{A}}^{\alpha}(x)$ and $\underset{2}{\mathfrak{A}}^{i}(x)$ will have to take. Since, however, the terms to which $\tilde{M}_{ijk}x^{j}x^{k}$ must reduce require some care in determining, we present the details. All possible cases are included without repetition in the following list:

$$\tilde{M}^{ijk} = \hat{M}^{k} \qquad (i = j \neq k) \qquad \text{by (67.131),}$$

$$\tilde{M}^{ijk} = \hat{M}^{j} \qquad (i = k \neq j) \qquad \text{by (67.132),}$$

$$\tilde{M}^{ijk} = \hat{M}^{i} \qquad (i = j = k) \qquad \text{by (67.131),}$$

$$\tilde{M}^{ijk} = -\hat{M}^{i} \qquad (j = k \neq i) \qquad \text{by (67.412),}$$

$$\tilde{M}^{ijk} = 0 \qquad (i \neq j \neq k) \qquad \text{by (67.411).}$$

Hence we must have

$$\tilde{M}^{ijk} x_j x_k = x^i \hat{M}^{k'} x_{k'} + x^i \hat{M}^{j'} x_{j'} + \hat{M}^i (x^i)^2 - \hat{M}^i x^{j'} x_{j'} ,$$

where a prime after an index j or k indicates that the summation is from 1 to $i-1$ and from $i+1$ to n. We can rewrite this expression in the simpler form without primes

$$(68) \qquad \tilde{M}^{ijk} x_j x_k = 2x^i \hat{M}^j x_j - \hat{M}^i x^j x_j .$$

Using (67.1), (67.12), (67.13), (67.211), (67.231), (67.311), (67.321), (67.331), (67.332), (67.312), (67.313) and (68), we have the result that, if (63) is invariant under (14), $\underset{2}{a}^{\alpha}(x)$ and $\underset{2}{a}^{i}(x)$ must be of the form

$$(69) \begin{cases} \underset{2}{a}^{\alpha}(x) = 2x^{\alpha}\left(\hat{M}^{\beta} x_{\beta} + \hat{M}^{j} x_{j}\right) - \hat{M}^{\alpha}\left(x^{\beta} x_{\beta} + x^{j} x_{j}\right), \\ \underset{2}{a}^{i}(x) = 2x^{i}\left(\hat{M}^{\beta} x_{\beta} + \hat{M}^{j} x_{j}\right) - \hat{M}^{i}\left(x^{\beta} x_{\beta} + x^{j} x_{j}\right). \end{cases}$$

$\tilde{\mathfrak{A}}^{\alpha}_{1}(x)$ and $\tilde{\mathfrak{A}}^{i}_{1}(x)$ must be of the same form respectively as $\mathfrak{A}^{\alpha}_{1}(x)$ and $\mathfrak{A}^{i}_{1}(x)$, as composite integral power series in x. Then the right-hand members of (41) must be of the same form as the corresponding right-hand members of (69), as composite integral power series in x. Hence the following identities in w must hold:

$$(70.1)\quad \left(3N^{\alpha}w^{\alpha}+N^{\alpha}_{\cdot j}w^{j}+N^{\alpha\beta}w_{\beta}\right)\left(x^{\alpha}\right)^{2}+\left(N^{\alpha\beta\gamma}w^{\alpha}+N^{\alpha\beta\gamma}_{\cdot j}w^{j}\right.$$
$$\left.+2\hat{N}^{\alpha\beta\gamma}w^{\beta}+3N^{\alpha\beta\gamma\delta}w_{\delta}\right)x_{\beta}x_{\gamma}+2\left(N^{\alpha\beta}_{\cdot j}w^{\beta}\right.$$
$$\left.+N^{\alpha\beta\gamma}_{j}w_{\gamma}\right)x^{j}x_{\beta}+\left(3N^{\alpha}_{jkl}w^{l}+N^{\alpha}_{jk}w^{\alpha}\right.$$
$$\left.+N^{\alpha\beta}_{jk}w_{\beta}\right)x^{j}x^{k}=0\qquad(j\neq k),$$

$$(70.2)\quad \left(\tilde{N}^{ij}_{\beta\gamma}w_{j}+2\tilde{N}^{i}_{\beta\gamma}w_{\beta}+3\tilde{N}^{i}_{\beta\gamma\delta}w^{\delta}\right)x^{\beta}x^{\gamma}$$
$$+2\left(\tilde{N}^{ij'k}_{\beta}w_{k}+\tilde{N}^{ij'}_{\beta}w_{\beta}+\tilde{N}^{ij'}_{\beta\gamma}w^{\gamma}\right)x_{j'}x^{\beta}$$
$$+\left(3\tilde{N}^{ij'k'l}w_{l}+\tilde{N}^{ij'k'}_{\beta}w^{\beta}\right)x_{j'}x_{k'}=0\quad(j\neq k),$$

where a prime after a dummy index has the same significance as before. Likewise the following identities in w must hold:

$$(70.3)\quad N^{\alpha\beta}w^{\alpha}+\left(\hat{N}^{\alpha\beta}-\tilde{N}^{\beta}_{ii}\right)w^{\beta}+\left(N^{\alpha\beta\gamma}-\tilde{N}^{\beta\gamma}_{ii}\right)w_{\gamma}-\tilde{N}^{\beta}_{iik}w^{k}=0,$$

(70.4) $$(\hat N^{\alpha\beta}+\hat N^{\beta\alpha})w^{\alpha}+N^{\alpha\beta}_{\cdot j}w^{j}+(3\hat{\hat N}^{\alpha\beta}+N^{\beta\alpha})w^{\beta}$$
$$+(\hat N^{\alpha\beta\gamma}+N^{\beta\alpha\gamma})w_{\gamma}=0,$$

(70.5) $$(\hat N^{\alpha\beta}-N^{\alpha}_{jj})w^{\alpha}+(N^{\alpha\beta}_{l}-3N^{\alpha}_{jjl})w^{l}+3\hat{\hat N}^{\alpha\beta}w^{\beta}$$
$$+(\hat N^{\alpha\beta\gamma}-N^{\alpha\gamma}_{\cdot jj})w_{\gamma}=0,$$

(70.6) $$(\tilde N^{\beta}_{il}-3\tilde N_{ijjl})w^{l}+3\tilde N^{\beta}_{i}w^{\beta}+(\tilde N^{\beta\gamma}_{i}-\tilde N^{\gamma}_{ijj})w_{\gamma}=0,$$

(70.7) $$N^{\alpha}_{\cdot j}w^{\alpha}+(N^{\alpha}_{\cdot jl}-3\tilde N_{ijil})w^{l}-\tilde N^{\beta}_{iji}w_{\beta}=0.$$

By Lemma 2 we infer from (70.1) and (70.2)

(70.11) $$3N^{\alpha}w^{\alpha}+N^{\alpha}_{\cdot j}w^{j}+N^{\alpha\beta}w_{\beta}=0,$$

(70.12) $$N^{\alpha\beta\gamma}w^{\alpha}+N^{\alpha\beta\gamma}_{\cdot j}w^{j}+\hat N^{\alpha\beta\gamma}w^{\beta}+3N^{\alpha\beta\gamma\delta}w_{\delta}+\hat N^{\alpha\beta\gamma}w^{\gamma}=0,$$

(70.13) $$N^{\alpha\beta}_{\cdot j}w^{\beta}+N^{\alpha\beta\gamma}_{\cdot j}w_{\gamma}=0,$$

(70.14) $$3N^{\alpha}_{jkl}w^{l}+N^{\alpha}_{jk}w^{\alpha}+N^{\alpha\beta}_{jk}w_{\beta}=0\quad(j\neq k),$$

(70.21) $$\tilde N^{ij}_{\beta\gamma}w_{j}+\tilde N^{i}_{\beta\gamma}w_{\beta}+3\tilde N^{i}_{\beta\gamma\delta}w^{\delta}+\tilde N^{i}_{\beta\gamma}w_{\gamma}=0,$$

(70.22) $$\tilde N^{ijk}_{\beta}w_{k}+\tilde N^{ij}_{\beta}w_{\beta}+\tilde N^{ij}_{\beta\gamma}w^{\gamma}=0\quad(i\neq j),$$

(70.23) $$3\,\tilde{N}^{ijkl}\,w_{l} + \tilde{N}^{ijk}_{\beta}\,w^{\beta} = 0 \qquad (i \neq j \neq k).$$

By Lemma 1 we infer from (70.11)

(70.111) $$N^{\alpha} \equiv N^{\alpha}_{\dot{j}} \equiv N^{\alpha\beta} \equiv 0\ ;$$

from (70.12)

(70.121) $$N^{\alpha\beta\gamma} \equiv N^{\alpha\beta\gamma}_{\dot{j}} \equiv \hat{N}^{\alpha\beta\gamma} \equiv N^{\alpha\beta\gamma\delta} \equiv 0\ ;$$

from (70.13)

(70.131) $$N^{\alpha\beta}_{\dot{j}} \equiv 0\ ;$$

from (70.14)

(70.141) $$N^{\alpha}_{jkl} \equiv N^{\alpha}_{\dot{j}k} \equiv N^{\alpha\beta}_{jk} \equiv 0 \qquad (j \neq k);$$

from (70.21)

(70.211) $$\tilde{N}^{ij}_{\beta\gamma} \equiv \tilde{N}^{i}_{\beta\gamma} \equiv \tilde{N}^{i}_{\beta\gamma\delta} \equiv 0\ ;$$

from (70.22)

(70.221) $$\tilde{N}^{ijk}_{\beta} \equiv \tilde{N}^{ij}_{\beta} \equiv 0 \qquad (i \neq j);$$

from (70.23)

(70.231) $$\tilde{N}^{ijkl} \equiv 0 \qquad (i \neq j \neq k)$$

In view of (70.111), (70.121) and (70.311), we find that (70.3) reduces to

$$\left(\hat{N}^{\alpha\beta} - \tilde{N}^{\beta}_{ii}\right) w^{\beta} - \tilde{N}^{\beta}_{iik} w^{k} = 0,$$

and by Lemma 1 we conclude from this identity in w

(70.31) $$\hat{N}^{\alpha\beta} \equiv \tilde{N}^{\beta}_{ii},$$

(70.32) $$\tilde{N}^{ijk}_{\beta} \equiv 0 \quad (i = j).$$

Because of (70.111), (70.121) and (70.131) we see that (70.4) reduces to

$$\left(\hat{N}^{\alpha\beta} + \hat{N}^{\beta\alpha}\right) w^{\alpha} + 3\hat{\hat{N}}^{\alpha\beta} w^{\beta} = 0,$$

from which we get by Lemma 1

(70.41) $$\hat{N}^{\alpha\beta} + \hat{N}^{\beta\alpha} \equiv 0,$$

(70.42) $$\hat{\hat{N}}^{\alpha\beta} \equiv 0.$$

Because of (70.31) we can put

$$\hat{N}^{\alpha\beta} = \hat{N}^{\beta};$$

then (70.41) becomes

$$\hat{N}^{\beta} + \hat{N}^{\alpha} \equiv 0,$$

whence

(70.411) $$\hat{N}^{\alpha\beta} \equiv 0,$$

and, on account of (70.31),

(70.412) $$\tilde{N}^{ij}_{\beta} \equiv 0 \quad (i = j).$$

Because of (70.411), (70.131), (70.42) and (70.121), we find that (70.5) reduces to

$$N^{\alpha}_{jj} w^{\alpha} + 3 N^{\alpha}_{jjl} w^{l} + N^{\alpha\beta}_{\cdot\, jj} w_{\beta} = 0,$$

and by Lemma 1 we get from this

(70.51) $$N^{\alpha}_{\cdot\, jk} \equiv 0 \qquad (j = k),$$

(70.52) $$N^{\alpha}_{jkl} \equiv 0 \qquad (j = k),$$

(70.53) $$N^{\alpha\beta}_{\cdot\, jk} \equiv 0 \qquad (j = k).$$

In view of (70.221), (70.412), (70.211), (70.221) and (70.32), we observe that (70.6) reduces to

$$\tilde{N}_{ijjl} w^{l} - \tilde{N}^{\beta}_{i} w^{\beta} = 0,$$

whence by Lemma 1

(70.61) $$\tilde{N}_{ijkl} \equiv 0 \quad (j = k),$$

(70.62) $$\tilde{N}^{\beta}_{i} \equiv 0 \; .$$

Finally, because of (70.111), (70.141), (70.51), (70.221) and (70.32), we find that (70.7) reduces to

$$\tilde{N}_{ijil} w^{l} = 0,$$

whence by Lemma 1

(70.71) $$\tilde{N}_{ijkl} \equiv 0 \qquad (i = k).$$

Since $\tilde{N}_{ijil} = \tilde{N}_{iijl}$, it follows from (70.71) that

(70.72) $$\tilde{N}_{ijkl} \equiv 0 \qquad (i=j).$$

Thus we have found that if (63) is invariant under (14) all the coefficients of $\mathfrak{A}^\alpha_3(x)$ and $\mathfrak{A}^i_3(x)$ must vanish, i.e.

$$\mathfrak{A}^\alpha_3(x) = \mathfrak{A}^i_3(x) \equiv 0 .$$

Just as (65) implies (65.2), it implies for $p > 2$

(65.3) $$u_\beta \mathfrak{A}^\beta_p(x;\underbrace{w;\cdots;w}_{p-3\text{ times}};v) + v_\beta \mathfrak{A}^\beta_p(x;\underbrace{w;\cdots;w}_{p-3\text{ times}};u) + u_j \mathfrak{A}^j_p(x;\underbrace{w;\cdots;w}_{p-3\text{ times}};v) + v_j \mathfrak{A}^j_p(x;\underbrace{w;\cdots;w}_{p-3\text{ times}};u) = 0.$$

Let us put

$$\tilde{\tilde{\mathfrak{A}}}^\alpha_3(x) = \mathfrak{A}^\alpha_p(x;\underbrace{w;\cdots;w}_{p-3\text{ times}}), \quad \tilde{\tilde{\mathfrak{A}}}^i_3(x) = \mathfrak{A}^i_p(x;\underbrace{w;\cdots;w}_{p-3\text{ times}}).$$

Then (65.3) becomes

(65.3′) $$u_\beta \tilde{\tilde{\mathfrak{A}}}^\beta_3(x;v) + v_\beta \tilde{\tilde{\mathfrak{A}}}^\beta_3(x;u) + u_j \tilde{\tilde{\mathfrak{A}}}^j_3(x;v) + v_j \tilde{\tilde{\mathfrak{A}}}^j_3(x;u) = 0.$$

The relation (65.3′) between $\tilde{\tilde{\mathfrak{A}}}^\alpha_3(x)$ and $\tilde{\tilde{\mathfrak{A}}}^i_3(x)$ is the same as the relation (65) between $\mathfrak{A}^\alpha_3(x)$ and $\mathfrak{A}^i_3(x)$. Hence $\tilde{\tilde{\mathfrak{A}}}^\alpha_3(x)$ and $\tilde{\tilde{\mathfrak{A}}}^i_3(x)$ must be of the same form (as composite integral power series in x) as $\mathfrak{A}^\alpha_3(x)$ and $\mathfrak{A}^i_3(x)$ respectively. Hence

$$\tilde{\tilde{\mathfrak{A}}}^\alpha_3(x) = \tilde{\tilde{\mathfrak{A}}}^i_3(x) = 0 ;$$

i.e.

$$\mathfrak{A}^{\alpha}_{p}(x;\underbrace{w;\cdots;w}_{p-3\text{ times}}) = \mathfrak{A}^{i}_{p}(x;\underbrace{w;\cdots;w}_{p-3\text{ times}}) = 0\,,$$

and, choosing $w = x$, we get

$$\mathfrak{A}^{\alpha}_{p}(\underbrace{x;\cdots;x}_{p-2\text{ times}}) = \mathfrak{A}^{i}_{p}(\underbrace{x;\cdots;x}_{p-2\text{ times}}) = 0\,.$$

But by Theorem 2 we have

$$\mathfrak{A}^{\alpha}_{p}(\underbrace{x;\cdots;x}_{p-2\text{ times}}) = p(p-1)\cdots 4\,\mathfrak{A}^{\alpha}_{p}(x)\,,$$

$$\mathfrak{A}^{i}_{p}(\underbrace{x;\cdots;x}_{p-2\text{ times}}) = p(p-1)\cdots 4\,\mathfrak{A}^{i}_{p}(x)\,.$$

Hence

$$\mathfrak{A}^{\alpha}_{p}(x) = \mathfrak{A}^{i}_{p}(x) = 0 \qquad (p>2)\,.$$

Noting finally that no restrictions have arisen on $\mathfrak{A}^{\alpha}_{0}(x)$ and $\mathfrak{A}^{i}_{0}(x)$, and replacing $L^{\alpha\beta}$, L^{α}_{j}, $\hat{M}^{\alpha}$, $\hat{M}^{i}$, L^{i}_{β}, L^{ij} respectively by $K^{\alpha\beta}$, K^{α}_{j}, L^{α}, L^{i}, K^{i}_{β}, K^{ij}, we have

THEOREM 37. _A necessary condition that_ (63) _be invariant under_ (14) _is that_ (14) _be of the form_

$$(71)\quad \begin{cases} \dfrac{dx^{\alpha}}{dt} = K^{\alpha} + K x^{\alpha} + K^{\alpha\beta} x_{\beta} + K^{\alpha}_{j} x^{j} \\ \qquad\qquad + 2x^{\alpha}(L^{\beta} x_{\beta} + L^{j} x_{j}) - L^{\alpha}(x^{\beta} x_{\beta} + x^{j} x_{j}), \\ \dfrac{dx^{i}}{dt} = K^{i} + K x^{i} - K^{i}_{\beta} x^{\beta} + K^{ij} x_{j} \\ \qquad\qquad + 2x^{i}(L^{\beta} x_{\beta} + L^{j} x_{j}) - L^{i}(x^{\beta} x_{\beta} + x^{j} x_{j}), \end{cases}$$

where

(71′) $K^{\alpha\beta}+K^{\beta\alpha}\equiv 0, \quad K^{ij}+K^{ji}\equiv 0.$

We now prove that the transformation defined by (71) and (71′) actually leaves (63) invariant. The derivative of (63) with respect to t is

$$\frac{A}{(\mu^{\alpha}\mu_{\alpha}+\mu^{i}\mu_{i})^{3/2}(\nu^{\beta}\nu_{\beta}+\nu^{j}\nu_{j})^{3/2}},$$

where

$$\begin{aligned}
A = {} & \Big[\nu_{\gamma}(\mu^{\delta}\mu_{\delta}+\mu^{k}\mu_{k})(\nu^{\epsilon}\nu_{\epsilon}+\nu^{l}\nu_{l}) \\
& \quad -\mu_{\gamma}(\mu^{\delta}\nu_{\delta}+\mu^{k}\nu_{k})(\nu^{\epsilon}\nu_{\epsilon}+\nu^{l}\nu_{l})\Big]\frac{d\mu^{\gamma}}{dt} \\
& +\Big[\mu_{\gamma}(\mu^{\delta}\mu_{\delta}+\mu^{k}\mu_{k})(\nu^{\epsilon}\nu_{\epsilon}+\nu^{l}\nu_{l}) \\
& \quad -\nu_{\gamma}(\mu^{\delta}\nu_{\delta}+\mu^{k}\nu_{k})(\mu^{\epsilon}\mu_{\epsilon}+\mu^{l}\mu_{l})\Big]\frac{d\nu^{\gamma}}{dt} \\
& +\Big[\nu_{k}(\mu^{\gamma}\mu_{\gamma}+\mu^{l}\mu_{l})(\nu^{\delta}\nu_{\delta}+\nu^{m}\nu_{m}) \\
& \quad -\mu_{k}(\mu^{\gamma}\nu_{\gamma}+\mu^{l}\nu_{l})(\nu^{\delta}\nu_{\delta}+\nu^{m}\nu_{m})\Big]\frac{d\mu^{k}}{dt} \\
& +\Big[\mu_{k}(\mu^{\gamma}\mu_{\gamma}+\mu^{l}\mu_{l})(\nu^{\delta}\nu_{\delta}+\nu^{m}\nu_{m}) \\
& \quad -\nu_{k}(\mu^{\gamma}\nu_{\gamma}+\mu^{l}\nu_{l})(\mu^{\delta}\mu_{\delta}+\mu^{m}\mu_{m})\Big]\frac{d\nu^{k}}{dt}.
\end{aligned}$$

The invariance of (63) will be established if we verify that A vanishes when in the expression for A we substitute the values of $\frac{du^\alpha}{dt}$, $\frac{du^i}{dt}$, $\frac{dv^\alpha}{dt}$, $\frac{dv^i}{dt}$ given by the equations

$$\begin{aligned}\frac{du^\alpha}{dt} = \psi^\alpha(x;u) &= K u^\alpha + K^{\alpha\beta} u_\beta + K^\alpha_{\ j} u^j \\ &\quad + 2u^\alpha L^\beta x_\beta + 2x^\alpha L^\beta u_\beta + 2u^\alpha L^j x_j \\ &\quad + 2x^\alpha L^j u_j - 2L^\alpha \left(x^\beta u_\beta + x^j u_j \right),\end{aligned}$$

$$\begin{aligned}\frac{du^i}{dt} = \psi^i(x;u) &= K u^i - K^i_{\ \beta} u^\beta + K^{ij} u_j + 2u^i L^\beta x_\beta \\ &\quad + 2x^i L^\beta u_\beta + 2u^i L^j x_j + 2x^i L^j u_j \\ &\quad - 2L^i \left(x^\beta u_\beta + x^j u_j \right),\end{aligned}$$

$$\begin{aligned}\frac{dv^\alpha}{dt} = \psi^\alpha(x;v) &= K v^\alpha + K^{\alpha\beta} v_\beta + K^\alpha_{\ j} v^j + 2v^\alpha L^\beta x_\beta \\ &\quad + 2x^\alpha L^\beta v_\beta + 2v^\alpha L^j x_j + 2x^\alpha L^j v_j \\ &\quad - 2L^\alpha \left(x^\beta v_\beta + x^j v_j \right),\end{aligned}$$

$$\begin{aligned}\frac{dv^i}{dt} = \psi^i(x;v) &= K v^i - K^i_{\ \beta} v^\beta + K^{ij} v_j \\ &\quad + 2v^i L^\beta x_\beta + 2x^i L^\beta v_\beta + 2v^i L^j x_j \\ &\quad + 2x^i L^j v_j - 2L^i \left(x^\beta v_\beta + x^j v_j \right).\end{aligned}$$

When we apply

$$p^\alpha(x) = K^\alpha, \quad p^i(x) = 0,$$

A vanishes. When we apply

$$p^\alpha(x) = K x^\alpha, \quad p^i(x) = K x^i,$$

A becomes

$$\begin{aligned}
&K u^\gamma \Big[v_\gamma (u^\delta u_\delta + u^k u_k)(v^\epsilon v_\epsilon + v^l v_l) \\
&\quad - u_\gamma (u^\delta v_\delta + u^k v_k)(v^\epsilon v_\epsilon + v^l v_l) \Big] \\
&+ K v^\gamma \Big[u_\gamma (u^\delta u_\delta + u^k u_k)(v^\epsilon v_\epsilon + v^l v_l) \\
&\quad - v_\gamma (u^\delta v_\delta + u^k v_k)(u^\epsilon u_\epsilon + u^l u_l) \Big] \\
&+ K u^k \Big[v_k (u^\gamma u_\gamma + u^l u_l)(v^\delta v_\delta + v^m v_m) \\
&\quad - u_k (u^\gamma v_\gamma + u^l v_l)(v^\delta v_\delta + v^m v_m) \Big] \\
&+ K v^k \Big[u_k (u^\gamma u_\gamma + u^l u_l)(v^\delta v_\delta + v^m v_m) \\
&\quad - v_k (u^\gamma v_\gamma + u^l v_l)(u^\delta u_\delta + u^m u_m) \Big],
\end{aligned}$$

which reduces to zero. When we apply

$$p^\alpha(x) = K^{\alpha\beta} x_\beta, \quad p^i(x) = 0,$$

A becomes

$$(K^{\gamma\delta} + K^{\delta\gamma}) u_\delta v_\gamma \big(u^\epsilon u_\epsilon v^\eta v_\eta + u^k u_k v^\epsilon v_\epsilon$$

$$+u^{\epsilon}u_{\epsilon}v^{k}v_{k}+u^{k}u_{k}v^{l}v_{l}\Big)-K^{\gamma\delta}u_{\gamma}u_{\delta}\Big(u^{\epsilon}v_{\epsilon}v^{\eta}v_{\eta}$$

$$+u^{k}v_{k}v^{\epsilon}v_{\epsilon}+u^{\epsilon}v_{\epsilon}v^{k}v_{k}+u^{k}v_{k}v^{l}v_{l}\Big)$$

$$-K^{\gamma\delta}v_{\gamma}v_{\delta}\Big(u^{\epsilon}v_{\epsilon}u^{\eta}u_{\eta}+u^{k}v_{k}u^{\epsilon}u_{\epsilon}$$

$$+u^{\epsilon}v_{\epsilon}u^{k}u_{k}+u^{k}v_{k}u^{l}u_{l}\Big),$$

which vanishes because of (71$'$). When we apply

$$p^{\alpha}(x)=K^{\alpha}_{j}x^{j},\quad p^{i}(x)=-K^{i}_{\beta}x^{\beta},$$

A becomes

$$K^{\gamma}_{k}u^{k}\Big[v_{\gamma}\Big(u^{\delta}u_{\delta}v^{\epsilon}v_{\epsilon}+u^{l}u_{l}v^{\delta}v_{\delta}+u^{\delta}u_{\delta}v^{l}v_{l}$$

$$+u^{l}u_{l}v^{m}v_{m}\Big)-u_{\gamma}\Big(u^{\delta}v_{\delta}v^{\epsilon}v_{\epsilon}+u^{l}v_{l}v^{\delta}v_{\delta}$$

$$+u^{\delta}v_{\delta}v^{l}v_{l}+u^{l}v_{l}v^{m}v_{m}\Big)\Big]$$

$$+K^{\gamma}_{k}v^{k}\Big[u_{\gamma}\Big(u^{\delta}u_{\delta}v^{\epsilon}v_{\epsilon}+u^{l}u_{l}v^{\delta}v_{\delta}$$

$$+u^{\delta}u_{\delta}v^{l}v_{l}+u^{l}u_{l}v^{m}v_{m}\Big)-v_{\gamma}\Big(u^{\delta}v_{\delta}u^{\epsilon}u_{\epsilon}$$

$$+u^{l}v_{l}u^{\delta}u_{\delta}+u^{\delta}v_{\delta}u^{l}u_{l}+u^{l}v_{l}u^{m}u_{m}\Big)\Big]$$

$$-K^{\gamma}_{k}u_{\gamma}\Big[v^{k}\Big(u^{\delta}u_{\delta}v^{\epsilon}v_{\epsilon}+u^{l}u_{l}v^{\delta}v_{\delta}+u^{\delta}u_{\delta}v^{l}v_{l}$$

$$+u^{l}u_{l}v^{m}v_{m})-u^{k}(u^{\delta}v_{\delta}v^{\epsilon}v_{\epsilon}+u^{l}v_{l}v^{\delta}v_{\delta}+u^{\delta}v_{\delta}v^{l}v_{l}$$
$$+u^{l}v_{l}v^{m}v_{m})]-K^{\gamma}_{k}v_{\gamma}[u^{k}(u^{\delta}u_{\delta}v^{\epsilon}v_{\epsilon}+u^{l}u_{l}v^{\delta}v_{\delta}$$
$$+u^{\delta}u_{\delta}v^{l}v_{l}+u^{l}u_{l}v^{m}v_{m})-v^{k}(u^{\delta}v_{\delta}u^{\epsilon}u_{\epsilon}$$
$$+u^{l}v_{l}u^{\delta}u_{\delta}+u^{\delta}v_{\delta}u^{l}u_{l}+u^{l}v_{l}u^{m}u_{m})],$$

which reduces to zero. When we apply

$$\mathcal{P}^{\alpha}(x)=2x^{\alpha}L^{\beta}x_{\beta}-L^{\alpha}(x^{\beta}x_{\beta}+x^{j}x_{j}),\ \mathcal{P}^{i}(x)=2x^{i}L^{\beta}x_{\beta},$$

A becomes

$$2[u^{\gamma}L^{\delta}x_{\delta}+x^{\gamma}L^{\delta}u_{\delta}-L^{\gamma}x^{\delta}u_{\delta}-L^{\gamma}x^{k}u_{k}]$$
$$[v_{\gamma}(u^{\epsilon}u_{\epsilon}v^{\eta}v_{\eta}+u^{l}u_{l}v^{\epsilon}v_{\epsilon}+u^{\epsilon}u_{\epsilon}v^{l}v_{l}$$
$$+u^{l}u_{l}v^{m}v_{m})-u_{\gamma}(u^{\epsilon}v_{\epsilon}v^{\eta}v_{\eta}+u^{l}v_{l}v^{\epsilon}v_{\epsilon}$$
$$+u^{\epsilon}v_{\epsilon}v^{l}v_{l}+u^{l}v_{l}v^{m}v_{m})]+2[v^{\gamma}L^{\delta}x_{\delta}$$
$$+x^{\gamma}L^{\delta}v_{\delta}-L^{\gamma}x^{\delta}v_{\delta}-L^{\gamma}x^{k}v_{k}][u_{\gamma}(u^{\epsilon}u_{\epsilon}v^{\eta}v_{\eta}$$
$$+u^{l}u_{l}v^{\epsilon}v_{\epsilon}+u^{\epsilon}u_{\epsilon}v^{l}v_{l}+u^{l}u_{l}v^{m}v_{m})$$
$$-v_{\gamma}(u^{\epsilon}v_{\epsilon}u^{\eta}u_{\eta}+u^{l}v_{l}u^{\epsilon}u_{\epsilon}+u^{\epsilon}v_{\epsilon}u^{l}u_{l}$$
$$+u^{l}v_{l}u^{m}u_{m})]+2[u^{k}L^{\gamma}x_{\gamma}+x^{k}L^{\gamma}u_{\gamma}]$$

$$\Big[v_k\big(u^\delta u_\delta v^\epsilon v_\epsilon + u^l u_l v^\delta v_\delta + u^\delta u_\delta v^l v_l$$
$$+ u^l u_l v^m v_m\big) - u_k\big(u^\delta v_\delta v^\epsilon v_\epsilon + u^l v_l v^\delta v_\delta$$
$$+ u^\delta v_\delta v^l v_l + u^l v_l v^m v_m\big)\Big] + 2\Big[v^k L^\gamma x_\gamma$$
$$+ x^k L^\gamma v_\gamma\Big]\Big[u_k\big(u^\delta u_\delta v^\epsilon v_\epsilon + u^l u_l v^\delta v_\delta$$
$$+ u^\delta u_\delta v^l v_l + u^l u_l v^m v_m\big) - v_k\big(u^\delta v_\delta u^\epsilon u_\epsilon$$
$$+ u^l v_l u^\delta u_\delta + u^\delta v_\delta u^l u_l + u^l v_l u^m u_m\big)\Big],$$

which reduces to zero. When we apply

$$p^\alpha(x) = 2x^\alpha L^{\dot{j}} x_{j}, \quad p^i(x) = 2x^i L^{j} x_{\dot{j}} - L^i\big(x^\beta x_\beta + x^{\dot{j}} x_{\dot{j}}\big),$$

A becomes

$$2\Big[u^\gamma L^k x_k + x^\gamma L^k u_k\Big]\Big[v_\gamma\big(u^\delta u_\delta v^\epsilon v_\epsilon + u^l u_l v^\delta v_\delta$$
$$+ u^\delta u_\delta v^l v_l + u^l u_l v^m v_m\big) - u_\gamma\big(u^\delta v_\delta v^\epsilon v_\epsilon$$
$$+ u^l v_l v^\delta v_\delta + u^\delta v_\delta v^l v_l + u^l v_l v^m v_m\big)\Big]$$
$$+ 2\Big[v^\gamma L^k x_k + x^\gamma L^k v_k\Big]\Big[u_\gamma\big(u^\delta u_\delta v^\epsilon v_\epsilon$$
$$+ u^l u_l v^\delta v_\delta + u^\delta u_\delta v^l v_l + u^l u_l v^m v_m\big)$$

$$-v_\gamma\left(u^\delta v_\delta u^\epsilon u_\epsilon + u^l v_l u^\delta u_\delta + u^\delta v_\delta u^l u_l\right.$$

$$\left.\left. + u^l v_l u^m u_m\right)\right] + 2\left[u^k L^l x_l + x^k L^l u_l - L^k x^\gamma u_\gamma\right.$$

$$\left. - L^k x^l u_l\right]\left[v_k\left(u^\delta u_\delta v^\epsilon v_\epsilon + u^m u_m v^\delta v_\delta + u^\delta u_\delta v^m v_m\right.\right.$$

$$\left. + u^m u_m v^p v_p\right) - u_k\left(u^\delta v_\delta v^\epsilon v_\epsilon + u^m v_m v^\delta v_\delta\right.$$

$$\left.\left. + u^\delta v_\delta v^m v_m + u^m v_m v^p v_p\right)\right] + 2\left[v^k L^l x_l\right.$$

$$\left. + x^k L^l v_l - L^k x^\gamma v_\gamma - L^k x^l v_l\right]\left[u_k\left(u^\delta u_\delta v^\epsilon v_\epsilon\right.\right.$$

$$\left. + u^m u_m v^\delta v_\delta + u^\delta u_\delta v^m v_m + u^m u_m v^p v_p\right)$$

$$- v_k\left(u^\delta v_\delta u^\epsilon u_\epsilon + u^m v_m u^\delta u_\delta\right.$$

$$\left.\left. + u^\delta v_\delta u^m u_m + u^m v_m u^p u_p\right)\right],$$

which reduces to zero. When we apply

$$\mathcal{P}^\alpha(x) = 0, \quad \mathcal{P}^i(x) = K^i,$$

A vanishes. When we apply

$$\mathcal{P}^\alpha(x) = 0, \quad \mathcal{P}^i(x) = K^{ij} x_j,$$

A becomes

$$(K_{kl}+K_{lk})u^k v^l(u^\gamma u_\gamma v^\delta v_\delta + u^m u_m v^\gamma v_\gamma$$
$$+u^\gamma u_\gamma v^m v_m + u^m u_m v^p v_p)$$
$$-K_{kl}u^k u^l(u^\gamma v_\gamma v^\delta v_\delta + u^m v_m v^\gamma v_\gamma + u^\gamma v_\gamma v^m v_m$$
$$+u^m v_m v^p v_p) - K_{kl}v^k v^l(u^\gamma v_\gamma u^\delta u_\delta$$
$$+u^m v_m u^\gamma u_\gamma + u^\gamma v_\gamma u^m u_m + u^m v_m u^p u_p),$$

which vanishes because of (71′).

We thus have

THEOREM 38. <u>The transformation defined by</u> (71) <u>and</u> (71′) <u>leaves</u> (63) <u>invariant</u>.

Theorems 37 and 38 together give

THEOREM 39. <u>The infinitesimal conformal transformations in</u> (C,E_n) <u>are the transformations defined by</u> (71) <u>and</u> (71′).

If we form the commutator of (71), (71′) and

$$(\overline{71})\quad\begin{cases}\dfrac{dx^\alpha}{dt} = \bar K^\alpha + \bar K x^\alpha + \bar K^{\alpha\beta}x_\beta + \bar K^\alpha_j x^j + 2x^\alpha(\bar L^\beta x_\beta \\ \qquad + \bar L^j x_j) - \bar L^\alpha(x^\beta x_\beta + x^j x_j), \\ \dfrac{dx^i}{dt} = \bar K^i + \bar K x^i - \bar K^i_\beta x^\beta + \bar K^{ij}x_j + 2x^i(\bar L^\beta x_\beta \\ \qquad + \bar L^j x_j) - \bar L^i(x^\beta x_\beta + x^j x_j),\end{cases}$$

(71) $$\bar{K}^{\alpha\beta}+\bar{K}^{\beta\alpha}\equiv 0,\quad \bar{K}^{ij}+\bar{K}^{ji}\equiv 0,$$

we find it to be a transformation of the same kind. Hence we have

THEOREM 40. <u>The totality of infinitesimal conformal transformations in (C,E_n) constitute a group.</u>

The infinitesimal conformal group in (C,E_n) in homogeneous sphere coordinates can be obtained by putting

(72) $$x^\alpha=\frac{y^\alpha}{y^{n+2}},\quad x^i=\frac{y^i}{y^{n+2}},\quad y^{n+2}\left(x^\beta x_\beta+x^j x_j\right)=y^{n+1}$$
$$(y^{n+2}\neq 0)$$

in the transformation defined by (71) and (71′). We thus get

(71.1) $$\frac{dy^\alpha}{dt}-x^\alpha\frac{dy^{n+2}}{dt}=K^\alpha y^{n+2}+Ky^\alpha+K^{\alpha\beta}y_\beta+K^{\alpha}_{\cdot j}y^j+2x^\alpha\left(L^\beta y_\beta+L^j y_j\right)-L^\alpha y^{n+1},$$

(71.2) $$\frac{dy^i}{dt}-x^i\frac{dy^{n+2}}{dt}=K^i y^{n+2}+Ky^i-K_\beta^{\cdot i}y^\beta+K^{ij}y_j+2x^i\left(L^\beta y_\beta+L^j y_j\right)-L^i y^{n+1}.$$

Equating coefficients of x^α in (71.1) and equating coefficients of x^i in (71.2), we get

$$(71.3)\begin{cases} \dfrac{dy^{\alpha}}{dt} = Ky^{\alpha} + K^{\alpha\beta}y_{\beta} + K^{\alpha}_{\cdot j}y^{j} - L^{\alpha}y^{n+1} + K^{\alpha}y^{n+2}, \\ \dfrac{dy^{i}}{dt} = Ky^{i} - K^{i}_{\beta}y^{\beta} + K^{ij}y_{j} - L^{i}y^{n+1} + K^{i}y^{n+2}, \\ \dfrac{dy^{n+2}}{dt} = -2\left(L^{\beta}y_{\beta} + L^{j}y_{j}\right). \end{cases}$$

From (72) we have

$$(72') \quad y^{\beta}y_{\beta} + y^{j}y_{j} = y^{n+2}y^{n+1}.$$

Differentiating (72′) with respect to t, and using (71.3) and (71′), we find

$$\frac{dy^{n+1}}{dt} = 2\left(K^{\beta}y_{\beta} + K^{j}y_{j}\right) + 2Ky^{n+1}.$$

An infinitesimal conformal transformation in (C, E_n) in homogeneous sphere coordinates is, then, of the form

$$\begin{cases} \dfrac{dy^{\alpha}}{dt} = Ky^{\alpha} + K^{\alpha\beta}y_{\beta} + K^{\alpha}_{j}y^{j} - L^{\alpha}y^{n+1} + K^{\alpha}y^{n+2}, \\ \dfrac{dy^{i}}{dt} = Ky^{i} - K^{i}_{\beta}y^{\beta} + K^{ij}y_{j} - L^{i}y^{n+1} + K^{i}y^{n+2}, \\ \dfrac{dy^{n+1}}{dt} = 2\left(K^{\beta}y_{\beta} + K^{j}y_{j}\right) + 2Ky^{n+1}, \\ \dfrac{dy^{n+2}}{dt} = -2\left(L^{\beta}y_{\beta} + L^{j}y_{j}\right), \end{cases}$$

where

$$K^{\alpha\beta} + K^{\beta\alpha} \equiv 0, \quad K^{ij} + K^{ji} \equiv 0.$$

4. <u>The Finite Transformations Generated by an Infinitesimal Conformal Transformation</u>. Let the equations of a given infinitesimal conformal transformation in (C, E_n) in homogeneous sphere coordinates be

$$(73)\begin{cases} \dfrac{\partial \bar{x}^{\alpha}(t)}{\partial t} = K\,\bar{x}^{\alpha}(t) + K^{\alpha\beta}\bar{x}_{\beta}(t) + K_{j}^{\alpha}\,\bar{x}^{j}(t) \\ \qquad\qquad + K_{n+1}^{\alpha}\,\bar{x}^{n+1}(t) + K_{n+2}^{\alpha}\,\bar{x}^{n+2}(t), \\ \dfrac{\partial \bar{x}^{i}(t)}{\partial t} = K\bar{x}^{i}(t) - K_{\beta}^{i}\,\bar{x}^{\beta}(t) + K^{ij}\bar{x}_{j}(t) \\ \qquad\qquad + K^{i\,n+1}\bar{x}_{n+1}(t) + K^{i\,n+2}\bar{x}_{n+2}(t), \\ \dfrac{\partial \bar{x}^{n+1}(t)}{\partial t} = 2K_{\beta}^{n+2}\bar{x}^{\beta}(t) + 2K^{j\,n+2}\bar{x}_{j}(t) + 2K\bar{x}_{n+1}(t), \\ \dfrac{\partial \bar{x}^{n+2}(t)}{\partial t} = 2K_{\beta}^{n+1}\bar{x}^{\beta}(t) + 2K^{j\,n+1}\bar{x}_{j}(t), \end{cases}$$

where

$$(73') \qquad K^{\alpha\beta} + K^{\beta\alpha} \equiv 0, \quad K^{ij} + K^{ji} \equiv 0,$$

with the boundary conditions

$$(73'') \qquad \bar{x}^{\alpha}(0) = x^{\alpha}, \quad \bar{x}^{i}(0) = x^{i}, \quad \bar{x}^{n+1}(0) = x^{n+1}, \quad \bar{x}^{n+2}(0) = x^{n+2}.$$

Since the transformation (73) is linear, we can apply Theorem 21, stated for the case of an infinitesimal linear transformation in (C, E_{n+2}). We thus obtain the result that the given transformation generates a one-parameter continuous group of finite transformations, which in homogeneous sphere coordinates are expressed as the non-singular linear transformations

$$(74)\quad \left\{ \begin{aligned} r\bar{x}^{\alpha}(t) &= M^{\alpha}(t)x^{\alpha} + M^{\alpha\beta}(t)x_{\beta} + M^{\alpha}_{j}(t)x^{j} \\ &\quad + M^{\alpha}_{n+1}(t)x^{n+1} + M^{\alpha}_{n+2}(t)x^{n+2}, \\ r\bar{x}^{i}(t) &= N^{i}_{\beta}(t)x^{\beta} + M^{ij}(t)x_{j} + M^{i}_{n+1}(t)x_{n+1} \\ &\quad + M^{i\,n+2}(t)x_{n+2}, \\ r\bar{x}^{n+1}(t) &= N^{n+1}_{\beta}(t)x^{\beta} + M^{n+1\,j}(t)x_{j} \\ &\quad + M^{n+1\,n+1}(t)x_{n+1} + M^{n+1\,n+2}(t)x_{n+2}, \\ r\bar{x}^{n+2}(t) &= N^{n+2}_{\beta}(t)x^{\beta} + M^{n+2\,j}(t)x_{j} + \\ &\quad + M^{n+2\,n+1}(t)x_{n+1} + M^{n+2\,n+2}(t)x_{n+2}, \\ r &\neq 0, \quad M^{\alpha}(t) \neq 0. \end{aligned} \right.$$

where, for any value of the parameter t, the coefficients are given by

$$(74.1)\quad \left\{ \begin{aligned} M^{\alpha}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha}[m]}{m!}t^{m}, \\ M^{\alpha\beta}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha\beta}[m]}{m!}t^{m}, \\ M^{\alpha}_{j}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha}_{j}[m]}{m!}t^{m}, \\ M^{\alpha}_{n+1}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha}_{n+1}[m]}{m!}t^{m}, \\ M^{\alpha}_{n+2}(t) &= \sum_{m=0}^{\infty} \frac{M^{\alpha}_{n+2}[m]}{m!}t^{m}, \\ N^{i}_{\beta}(t) &= \sum_{m=0}^{\infty} \frac{N^{i}_{\beta}[m]}{m!}t^{m}, \end{aligned} \right.$$

$$
\left[
\begin{aligned}
M^{ij}(t) &= \sum_{m=0}^{\infty} \frac{M^{ij}[m]}{m!} t^{m}, \\
M^{i\,n+1}(t) &= \sum_{m=0}^{\infty} \frac{M^{i\,n+1}[m]}{m!} t^{m}, \\
M^{i\,n+2}(t) &= \sum_{m=0}^{\infty} \frac{M^{i\,n+2}[m]}{m!} t^{m}, \\
N_{\beta}^{n+1}(t) &= \sum_{m=0}^{\infty} \frac{N_{\beta}^{n+1}[m]}{m!} t^{m}, \\
M^{n+1\,j}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+1\,j}[m]}{m!} t^{m}, \\
M^{n+1\,n+1}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+1\,n+1}[m]}{m!} t^{m}, \\
M^{n+1\,n+2}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+1\,n+2}[m]}{m!} t^{m}, \\
N_{\beta}^{n+2}(t) &= \sum_{m=0}^{\infty} \frac{N_{\beta}^{n+2}[m]}{m!} t^{m}, \\
M^{n+2\,j}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+2\,j}[m]}{m!} t^{m}, \\
M^{n+2\,n+1}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+2\,n+1}[m]}{m!} t^{m}, \\
M^{n+2\,n+2}(t) &= \sum_{m=0}^{\infty} \frac{M^{n+2\,n+2}[m]}{m!} t^{m},
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
& M^{\alpha}[0] = 1, \; M^{\alpha\beta}[0] = M^{\alpha}_{j}[0] = M^{\alpha}_{n+1}[0] \\
& \quad = M^{\alpha}_{n+2}[0] = N_{\beta}^{\ i}[0] = 0, \; M^{ij}[0] = 1 \quad (i = j), \\
& M^{ij}[0] = 0 \quad (i \neq j), \; M^{i\,n+1}[0] = M^{i\,n+2}[0] \\
& = N_{\beta}^{\ n+1}[0] = M^{n+1\,j}[0] = 0, \; M^{n+1\,n+1}[0] = 1, \\
& M^{n+1\,n+2}[0] = N_{\beta}^{\ n+2}[0] = M^{n+2\,j}[0] \\
& = M^{n+2\,n+1}[0] = 0, \; M^{n+2\,n+2}[0] = 1,
\end{aligned}
\right.
\tag{74.2}
$$

and

$$
\left\{
\begin{aligned}
& M^{\alpha}[m+1] = K M^{\alpha}[m], \\
& M^{\alpha\beta}[m+1] = K M^{\alpha\beta}[m] + K^{\alpha\beta} M^{\beta}[m] \\
& \qquad + K_{\alpha\gamma} M^{\gamma\beta}[m] + K_{\alpha}^{\ j} N^{\beta}_{\ j}[m] + K_{\alpha}^{\ n+1} N^{\beta}_{\ n+1}[m] + K_{\alpha}^{\ n+2} N^{\beta}_{\ n+2}[m], \\
& M^{\alpha}_{j}[m+1] = K M^{\alpha}_{j}[m] + K_{\alpha\beta} M^{\beta}_{j}[m] + K_{\alpha}^{\ k} M_{kj}[m] \\
& \qquad + K_{\alpha}^{\ n+1} M_{n+1\,j}[m] + K_{\alpha}^{\ n+2} M_{n+2\,j}[m], \\
& M^{\alpha}_{n+1}[m+1] = K M^{\alpha}_{n+1}[m] + K^{\alpha\beta} M_{\beta}^{\ n+1}[m] + K_{j}^{\ \alpha} M^{j\,n+1}[m] \\
& \qquad + K^{\alpha}_{n+1} M^{n+1\,n+1}[m] + K^{\alpha}_{n+2} M^{n+2\,n+1}[m],
\end{aligned}
\right.
\tag{74.3}
$$

$$M^{\alpha}_{n+2}[m+1] = K M^{\alpha}_{n+2}[m] + K^{\alpha\beta} M_{\beta}^{n+2}[m]$$

$$+ K_{j}^{\alpha} M^{j\, n+2}[m] + K_{n+1}^{\alpha} M^{n+1\, n+2}[m] + K_{n+2}^{\alpha} M^{n+2\, n+2}[m],$$

$$N_{\beta}^{i}[m+1] = K N_{\beta}^{i}[m] - K_{\beta}^{i} M_{\beta}[m] - K_{i}^{\gamma} M_{\gamma\beta}[m]$$

$$+ K_{ij} N_{\beta}^{j}[m] + K^{i\, n+1} N_{\beta}^{n+1}[m] + K^{i\, n+2} N_{\beta}^{n+2}[m],$$

$$N_{\beta}^{n+1}[m+1] = 2K_{\beta}^{n+2} M_{\beta}[m] + 2K_{n+2}^{\gamma} M_{\gamma\beta}[m]$$

$$+ 2K_{j\, n+2} N_{\beta}^{j}[m] + 2K N_{\beta}^{n+1}[m],$$

$$N_{\beta}^{n+2}[m+1] = 2K_{\beta}^{n+1} M_{\beta}[m] + 2K_{n+1}^{\gamma} M_{\gamma\beta}[m]$$

$$+ 2K_{j\, n+1} N_{\beta}^{j}[m],$$

$$M^{ij}[m+1] = K M^{ij}[m] - K_{i}^{\beta} N_{\beta}^{j}[m] + K_{ik} M^{kj}[m]$$

$$+ K^{i\, n+1} M^{n+1\, j}[m] + K^{i\, n+2} M^{n+2\, j}[m],$$

$$M^{n+1\, j}[m+1] = 2K_{n+2}^{\beta} N_{\beta}^{j}[m] + 2K_{k\, n+2} M^{kj}[m]$$

$$+ 2K M^{n+1\, j}[m],$$

$$M^{n+2\, j}[m+1] = 2K_{n+1}^{\beta} N_{\beta}^{j}[m] + 2K_{k\, n+1} M^{kj}[m],$$

$$\left\{\begin{aligned}
M^{i\,n+1}[m+1] &= K M^{i\,n+1}[m] - K^{i}_{\beta} N^{\beta}_{n+1}[m] + K^{ij} M_{j\,n+1}[m] \\
&\quad + K^{i\,n+1} M_{n+1\,n+1}[m] + K^{i\,n+2} M_{n+2\,n+1}[m], \\
M^{i\,n+2}[m+1] &= K M^{i\,n+2}[m] - K^{i}_{\beta} N^{\beta}_{n+2}[m] \\
&\quad + K^{ij} M_{j\,n+2}[m] + K^{i\,n+1} M_{n+1\,n+2}[m] \\
&\quad + K^{i\,n+2} M_{n+2\,n+2}[m], \\
M^{n+1\,n+1}[m+1] &= 2K^{n+2}_{\beta} N^{\beta}_{n+1}[m] + 2K^{j\,n+2} M_{j\,n+1}[m] + 2K M^{n+1\,n+1}[m], \\
M^{n+2\,n+1}[m+1] &= 2K^{n+1}_{\beta} N^{\beta}_{n+1}[m] + 2K^{j\,n+1} M_{j\,n+1}[m], \\
M^{n+1\,n+2}[m+1] &= 2K^{n+2}_{\beta} N^{\beta}_{n+2}[m] + 2K^{j\,n+2} M_{j\,n+2}[m] + 2K M^{n+1\,n+2}[m], \\
M^{n+2\,n+2}[m+1] &= 2K^{n+1}_{\beta} N^{\beta}_{n+2}[m] + 2K^{k\,n+1} M_{k\,n+2}[m].
\end{aligned}\right.$$

We shall now show that the transformations defined by (74), (74.1), (74.2), and (74.3) are conformal, by verifying that the coefficients satisfy the following relations; to be referred to collectively as (75):

(75.1) $M^{\beta}_{n+1}(t)\, M^{n+1}_{\beta}(t) + M_{j\,n+1}(t)\, M^{j\,n+1}(t) - M_{n+1\,n+1}(t)\, M^{n+2\,n+1}(t) = 0,$

(75.2) $M^{\beta}_{n+2}(t)\, M^{n+2}_{\beta}(t) + M_{j\,n+2}(t)\, M^{j\,n+2}(t) - M_{n+1\,n+2}(t)\, M^{n+2\,n+2}(t) = 0,$

(75.3)

(75.3) $$2M_{n+1}^{\beta}(t)M_{\beta}^{n+2}(t)+2M_{j\,n+1}(t)M^{j\,n+2}(t)$$
$$-M_{n+1\,n+2}(t)M^{n+2\,n+1}(t)-M_{n+1\,n+1}(t)M^{n+2\,n+2}(t)$$
$$+\left[M(t)\right]^{2}=0,$$

(75.4) $$2M^{\alpha}(t)M_{n+1}^{\alpha}(t)+2M^{\beta\alpha}(t)M_{\beta}^{n+1}(t)+2N_{j}^{\alpha}(t)M^{j\,n+1}(t)$$
$$-M_{n+1\,n+1}(t)N_{n+2}^{\alpha}(t)-M_{n+2\,n+1}(t)N_{n+1}^{\alpha}(t)=0,$$

(75.5) $$2M_{\beta}^{i}(t)M_{n+1}^{\beta}(t)+2M^{ji}(t)M_{j\,n+1}(t)-M^{n+1\,n+1}(t)M^{n+2\,i}(t)$$
$$-M^{n+2\,n+1}(t)M^{n+1\,i}(t)=0,$$

(75.6) $$2M^{\alpha}(t)M_{n+2}^{\alpha}(t)+2M^{\beta\alpha}(t)M_{\beta}^{n+2}(t)+2N_{j}^{\alpha}(t)M^{j\,n+2}(t)$$
$$-M^{n+1\,n+2}(t)N_{n+2}^{\alpha}(t)-M^{n+2\,n+2}(t)N_{n+1}^{\alpha}(t)=0,$$

(75.7) $$2M_{\beta}^{i}(t)M_{n+2}^{\beta}(t)+2M^{ji}(t)M_{j\,n+2}(t)-M^{n+1\,n+2}(t)M^{n+2\,i}(t)$$
$$-M^{n+2\,n+2}(t)M^{n+1\,i}(t)=0,$$

(75.8) $$\left[M^{\alpha}(t)\right]^{2}-\left[M(t)\right]^{2}=0,$$

(75.9) $$2M^{\gamma\alpha}(t)M_{\gamma\beta}(t)+2N_{j}^{\alpha}(t)N_{\beta}^{j}(t)+2M^{\alpha}(t)M^{\alpha\beta}(t)$$
$$+2M^{\beta}(t)M^{\beta\alpha}(t)-N_{n+1}^{\alpha}(t)N_{n+2}^{\beta}(t)-N_{n+1}^{\beta}(t)N_{n+2}^{\alpha}(t)=0,$$

$$(75.10)\quad 2M^{\alpha}(t)M_{i}^{\alpha}(t)+2M^{\beta\alpha}(t)M_{\beta}^{i}(t)+2M^{ji}(t)N_{j}^{\alpha}(t)$$
$$-M^{n+1\,i}(t)N_{n+2}^{\alpha}(t)-M^{n+2\,i}(t)N_{n+1}^{\alpha}(t)=0,$$

$$(75.11)\quad 2M_{\beta}^{i}(t)M_{j}^{\beta}(t)+2M^{ki}(t)M_{kj}(t)-2\left[M^{ij}(t)\right]^{2}$$
$$-M^{n+2\,i}(t)M^{n+1\,j}(t)-M^{n+1\,i}(t)M^{n+2\,j}(t)=0,$$

where

$$M^{ij}(t)=M(t)\quad (i=j),\qquad M^{ij}(t)=0\quad (i\neq j).$$

The coefficients of the generated transformations mentioned in Theorem 21 were found as the unique solution of the system of integro-differential equations (24.11) with the boundary conditions (24.11$'$). Applying this knowledge to the present case, we see that the coefficients of (74) are obtained as the unique solution of the following system of integro-differential equations[1]:

$$(76)\quad \begin{cases} {}'M^{\alpha}(t)=KM^{\alpha}(t), \\ {}'M^{\alpha\beta}(t)=KM^{\alpha\beta}(t)+K^{\alpha\beta}M^{\beta}(t)+K_{\alpha\gamma}M^{\gamma\beta}(t) \end{cases}$$

---

[1] The prime before a function denotes differentiation of the function with respect to t .

$$
\left\{
\begin{aligned}
&+ K_{\alpha}{}^{j} N_{j}{}^{\beta}(t) + K_{\alpha}{}^{n+1} N_{n+1}{}^{\beta}(t) + K_{\alpha}{}^{n+2} N_{n+2}{}^{\beta}(t),\\
&{}^{\prime}M^{\alpha}{}_{j}(t) = K M^{\alpha}_{\cdot j}(t) + K_{\alpha\beta} M^{\beta}_{\cdot j}(t) + K_{\alpha}{}^{k} M^{\cdot}_{kj}(t) + K_{\alpha}{}^{n+1} M_{n+1\,j}(t)\\
&\qquad + K_{\alpha}{}^{n+2} M_{n+2\,j}(t),\\
&{}^{\prime}M^{\alpha}{}_{n+1}(t) = K M^{\alpha}{}_{n+1}(t) + K^{\alpha\beta} M_{\beta}{}^{\cdot\, n+1}(t) + K_{j}^{\cdot\,\alpha} M^{j\, n+1}(t)\\
&\qquad + K_{n+1}{}^{\alpha} M^{n+1\, n+1}(t) + K_{n+2}{}^{\alpha} M^{n+2\, n+1}(t),\\
&{}^{\prime}M^{\alpha}{}_{n+2}(t) = K M^{\alpha}{}_{n+2}(t) + K^{\alpha\beta} M_{\beta}{}^{n+2}(t) + K_{j}^{\cdot\,\alpha} M^{j\, n+2}(t)\\
&\qquad + K_{n+1}{}^{\alpha} M^{n+1\, n+2}(t) + K_{n+2}{}^{\alpha} M^{n+2\, n+2}(t),\\
&{}^{\prime}N_{\beta}{}^{i}(t) = K N_{\beta}{}^{i}(t) - K_{\beta}{}^{i} M_{\beta}(t) - K_{i}{}^{\gamma} M_{\gamma\beta}(t)\\
&\qquad + K_{ij} N_{\beta}{}^{j}(t) + K^{i\, n+1} N_{\beta}{}^{n+1}(t) + K^{i\, n+2} N_{\beta}{}^{n+2}(t),\\
&{}^{\prime}N_{\beta}{}^{n+1}(t) = 2 K_{\beta}{}^{n+2} M_{\beta}(t) + 2 K_{n+2}{}^{\gamma} M_{\gamma\beta}(t) + 2 K_{j\, n+2} N_{\beta}{}^{j}(t)\\
&\qquad + 2 K N_{\beta}{}^{n+1}(t),\\
&{}^{\prime}N_{\beta}{}^{n+2}(t) = 2 K_{\beta}{}^{n+1} M_{\beta}(t) + 2 K_{n+1}{}^{\gamma} M_{\gamma\beta}(t)\\
&\qquad + 2 K_{j\, n+1} N_{\beta}{}^{j}(t),
\end{aligned}
\right.
$$

$$\left[\begin{aligned}
{}'M^{ij}(t) &= KM^{ij}(t) - K^{\beta}_{i}\,N^{j}_{\beta}(t) + K_{ik}\,M^{kj}(t) \\
&\quad + K^{i\,n+1}\,M^{n+1\,j}(t) + K^{i\,n+2}\,M^{n+2\,j}(t), \\
{}'M^{n+1\,j}(t) &= 2K^{\beta}_{n+2}\,N^{j}_{\beta}(t) + 2K_{k\,n+2}\,M^{kj}(t) \\
&\quad + 2KM^{n+1\,j}(t), \\
{}'M^{n+2\,j}(t) &= 2K^{\beta}_{n+1}\,N^{j}_{\beta}(t) + 2K_{k\,n+1}\,M^{kj}(t), \\
{}'M^{i\,n+1}(t) &= KM^{i\,n+1}(t) - K^{i}_{\beta}\,N^{\beta}_{n+1}(t) + K^{ij}\,M_{j\,n+1}(t) \\
&\quad + K^{i\,n+1}\,M_{n+1\,n+1}(t) + K^{i\,n+2}\,M_{n+2\,n+1}(t), \\
{}'M^{i\,n+2}(t) &= KM^{i\,n+2}(t) - K^{i}_{\beta}\,N^{\beta}_{n+2}(t) + K^{ij}\,M_{j\,n+2}(t) \\
&\quad + K^{i\,n+1}\,M_{n+1\,n+2}(t) + K^{i\,n+2}\,M_{n+2\,n+2}(t), \\
{}'M^{n+1\,n+1}(t) &= 2K^{n+2}_{\beta}\,N^{\beta}_{n+1}(t) + 2K^{j\,n+2}\,M_{j\,n+1}(t) + 2KM^{n+1\,n+1}(t), \\
{}'M^{n+2\,n+1}(t) &= 2K^{n+1}_{\beta}\,N^{\beta}_{n+1}(t) + 2K^{j\,n+1}\,M_{j\,n+1}(t), \\
{}'M^{n+1\,n+2}(t) &= 2K^{n+2}_{\beta}\,N^{\beta}_{n+2}(t) + 2K^{j\,n+2}\,M_{j\,n+2}(t) + 2KM^{n+1\,n+2}(t), \\
{}'M^{n+2\,n+2}(t) &= 2K^{n+1}_{\beta}\,N^{\beta}_{n+2}(t) + 2K^{k\,n+1}\,M_{k\,n+2}(t),
\end{aligned}\right.$$

where

$$M^{\alpha}(0) = 1,\ M^{\alpha\beta}(0) = M^{\alpha}_{j}(0) = M^{\alpha}_{n+1}(0) = M^{\alpha}_{n+2}(0)$$
$$= N^{i}_{\beta}(0) = 0,\ M^{ij}(0) = 1\ (i = j),$$
$$M^{ij}(0) = 0\ (i \neq j),\ M^{i\,n+1}(0) = M^{i\,n+2}(0)$$

(76′)
$$= N^{n+1}_{\beta}(0) = M^{n+1\,j}(0) = 0,\ M^{n+1\,n+1}(0) = 1,$$
$$M^{n+1\,n+2}(0) = N^{n+2}_{\beta}(0) = M^{n+2\,j}(0)$$
$$= M^{n+2\,n+1}(0) = 0,\ M^{n+2\,n+2}(0) = 1.$$

We note first that the boundary conditions (76′) are satisfied in (75). Next, differentiating relations (75) with respect to t, we get the following equations, to be referred to collectively as (77):

(77.1)
$$2M^{\beta}_{n+1}(t)\,{}'M^{n+1}_{\beta}(t) + 2M_{j\,n+1}(t)\,{}'M^{j\,n+1}(t)$$
$$- M_{n+1\,n+1}(t)\,{}'M^{n+2\,n+1}(t) - M^{n+2\,n+1}(t)\,{}'M_{n+1\,n+1}(t) = 0,$$

(77.2)
$$2M^{\beta}_{n+2}(t)\,{}'M^{n+2}_{\beta}(t) + 2M_{j\,n+2}(t)\,{}'M^{j\,n+2}(t)$$
$$- M_{n+1\,n+2}(t)\,{}'M^{n+2\,n+2}(t) - M^{n+2\,n+2}(t)\,{}'M_{n+1\,n+2}(t)$$
$$= 0,$$

$$\begin{aligned}
&2M^{\beta}_{n+1}(t)\,{}'M^{n+2}_{\beta}(t) + 2M^{n+2}_{\beta}(t)\,{}'M^{\beta}_{n+1}(t)\\
&\quad + 2M_{j\,n+1}(t)\,{}'M^{j\,n+2}(t) + 2M^{j\,n+2}(t)\,{}'M_{j\,n+1}(t)\\
&\quad - M_{n+1\,n+2}(t)\,{}'M^{n+2\,n+1}(t) - M_{n+2\,n+1}(t)\,{}'M^{n+1\,n+2}(t)\\
&\quad - M_{n+1\,n+1}(t)\,{}'M^{n+2\,n+2}(t) - M_{n+2\,n+2}(t)\,{}'M^{n+1\,n+1}(t)\\
&\quad + 2M(t)\,{}'M(t) = 0,
\end{aligned}\tag{77.3}$$

$$\begin{aligned}
&2M^{\alpha}(t)\,{}'M^{\alpha}_{n+1}(t) + 2M^{\alpha}_{n+1}(t)\,{}'M^{\alpha}(t) + 2M^{\beta\alpha}(t)\,{}'M^{n+1}_{\beta}(t)\\
&\quad + 2M^{n+1}_{\beta}(t)\,{}'M^{\beta\alpha}(t) + 2N^{\alpha}_{j}(t)\,{}'M^{j\,n+1}(t)\\
&\quad + 2M^{j\,n+1}(t)\,{}'N^{\alpha}_{j}(t) - M^{n+1\,n+1}(t)\,{}'N^{\alpha}_{n+2}(t)\\
&\quad - N^{\alpha}_{n+2}(t)\,{}'M^{n+1\,n+1}(t) - M^{n+2\,n+1}(t)\,{}'N^{\alpha}_{n+1}(t)\\
&\quad - N^{\alpha}_{n+1}\,{}'M^{n+2\,n+1}(t) = 0,
\end{aligned}\tag{77.4}$$

$$\begin{aligned}
&2M^{i}_{\beta}(t)\,{}'M^{\beta}_{n+1}(t) + 2M^{\beta}_{n+1}(t)\,{}'M^{i}_{\beta}(t) + 2M_{ji}(t)\,{}'M^{j\,n+1}(t)\\
&\quad + 2M^{j\,n+1}(t)\,{}'M_{ji}(t) - M^{n+1\,n+1}(t)\,{}'M^{n+2\,i}(t)\\
&\quad - M^{n+2\,i}(t)\,{}'M^{n+1\,n+1}(t) - M^{n+2\,n+1}(t)\,{}'M^{n+1\,i}(t)\\
&\quad - M^{n+1\,i}(t)\,{}'M^{n+2\,n+1}(t) = 0,
\end{aligned}\tag{77.5}$$

(77.6)
$$\begin{aligned}&2M^{\alpha}(t)\,{}'M^{\alpha}_{n+2}(t) + 2M^{\alpha}_{n+2}(t)\,{}'M^{\alpha}(t) + 2M^{\beta\alpha}(t)\,{}'M^{n+2}_{\beta}(t)\\&+ 2M^{n+2}_{\beta}(t)\,{}'M^{\beta\alpha}(t) + 2N^{\alpha}_{j}(t)\,{}'M^{j\,n+2}(t)\\&+ 2M^{j\,n+2}(t)\,{}'N^{\alpha}_{j}(t) - M^{n+1\,n+2}(t)\,{}'N^{\alpha}_{n+2}(t)\\&- N^{\alpha}_{n+2}(t)\,{}'M^{n+1\,n+2}(t) - M^{n+2\,n+2}(t)\,{}'N^{\alpha}_{n+1}(t)\\&- N^{\alpha}_{n+1}(t)\,{}'M^{n+2\,n+2}(t) = 0,\end{aligned}$$

(77.7)
$$\begin{aligned}&2M^{i}_{\beta}(t)\,{}'M^{\beta}_{n+2}(t) + 2M^{\beta}_{n+2}(t)\,{}'M^{i}_{\beta}(t)\\&+ 2M^{ji}(t)\,{}'M_{j\,n+2}(t) + 2M_{j\,n+2}(t)\,{}'M^{ji}(t)\\&- M^{n+1\,n+2}(t)\,{}'M^{n+2\,i}(t) - M^{n+2\,i}(t)\,{}'M^{n+1\,n+2}(t)\\&- M^{n+2\,n+2}(t)\,{}'M^{n+1\,i}(t) - M^{n+1\,i}(t)\,{}'M^{n+2\,n+2}(t) = 0,\end{aligned}$$

(77.8)
$$M^{\alpha}(t)\,{}'M^{\alpha}(t) - M(t)\,{}'M(t) = 0,$$

(77.9)
$$\begin{aligned}&2M^{\gamma\alpha}(t)\,{}'M_{\gamma\beta}(t) + 2M_{\gamma\beta}(t)\,{}'M^{\gamma\alpha}(t) + 2N^{\alpha}_{j}(t)\,{}'N^{j}_{\beta}(t)\\&+ 2N^{j}_{\beta}(t)\,{}'N^{\alpha}_{j}(t) + 2M^{\alpha}(t)\,{}'M^{\alpha\beta}(t)\\&+ 2M^{\alpha\beta}(t)\,\overline{{}'M^{\alpha}}(t) + 2M^{\beta}(t)\,{}'M^{\beta\alpha}(t)\end{aligned}$$

$$+ 2M^{\beta\alpha}(t)\,{}'M^{\beta}(t) - N^{\alpha}_{n+1}(t)\,{}'N^{\beta}_{n+2}(t)$$

$$- N^{\beta}_{n+2}(t)\,{}'N^{\alpha}_{n+1}(t) - N^{\beta}_{n+1}(t)\,{}'N^{\alpha}_{n+2}(t) - N^{\alpha}_{n+2}(t)\,{}'N^{\beta}_{n+1}(t) = 0,$$

(77.10) $$2M^{\alpha}(t)\,{}'M^{\alpha}_{i}(t) + 2M^{\alpha}_{i}(t)\,{}'M^{\alpha}(t) + 2M^{\beta\alpha}(t)\,{}'M^{i}_{\beta}(t)$$

$$+ 2M^{i}_{\beta}(t)\,{}'M^{\beta\alpha}(t) + 2M^{ji}(t)\,{}'N^{\alpha}_{j}(t)$$

$$+ 2N^{\alpha}_{j}(t)\,{}'M^{ji}(t) - M^{n+1\,i}(t)\,{}'N^{\alpha}_{n+2}(t)$$

$$- N^{\alpha}_{n+2}(t)\,{}'M^{n+1\,i}(t) - M^{n+2\,i}(t)\,{}'N^{\alpha}_{n+1}(t)$$

$$- N^{\alpha}_{n+1}(t)\,{}'M^{n+2\,i}(t) = 0,$$

(77.11) $$2M^{i}_{\beta}(t)\,{}'M^{\beta}_{j}(t) + 2M^{\beta}_{j}(t)\,{}'M^{i}_{\beta}(t) + 2M^{ki}(t)\,{}'M_{kj}(t)$$

$$+ 2M^{kj}(t)\,{}'M_{ki}(t) - 4M^{ij}(t)\,{}'M^{ij}(t)$$

$$- M^{n+2\,i}(t)\,{}'M^{n+1\,j}(t) - M^{n+1\,j}(t)\,{}'M^{n+2\,i}(t)$$

$$- M^{n+2\,j}(t)\,{}'M^{n+1\,i}(t) - M^{n+1\,i}(t)\,{}'M^{n+2\,j}(t) = 0,$$

where

$$M^{ij}(t) = \begin{cases} M(t) & (i = j) \\ 0 & (i \neq j). \end{cases}$$

If we substitute from (76) in the left-hand member of each equation of (77), we shall find that the resulting expressions all reduce to zero. Thus the left-hand member of (77.1) becomes

$$2K\left[M^{\beta}_{n+1}(t)M^{n+1}_{\beta}(t)+M_{j\,n+1}(t)M^{j\,n+1}(t)-M_{n+1\,n+1}(t)M^{n+2\,n+1}(t)\right]$$
$$+2K^{\beta\gamma}M^{n+1}_{\beta}(t)M^{n+1}_{\gamma}(t)+2K_{jk}M^{j\,n+1}(t)M^{k\,n+1}(t),$$

which reduces to zero since the expression in brackets vanishes by (75.1) and the other terms vanish by (73′). The left-hand member of (77.2) becomes

$$2K\left[M^{\beta}_{n+2}(t)M^{n+2}_{\beta}(t)+M_{j\,n+2}(t)M^{j\,n+2}(t)-M_{n+1\,n+2}(t)M^{n+2\,n+2}(t)\right]$$
$$+2K^{\beta\gamma}M^{n+2}_{\beta}(t)M^{n+2}_{\gamma}(t)+2K_{jk}M^{j\,n+2}(t)M^{k\,n+2}(t),$$

which reduces to zero since the expression in brackets vanishes by (75.2) and the other terms vanish by (73′). When use is made of (75.8), the left-hand member of (77.3) becomes

$$2K\Big[2M^{\beta}_{n+1}(t)M^{n+2}_{\beta}(t)+2M_{j\,n+1}(t)M^{j\,n+2}(t)-M_{n+1\,n+2}(t)M^{n+2\,n+1}(t)$$
$$-M_{n+1\,n+1}(t)M^{n+2\,n+2}(t)+\left(M(t)\right)^{2}\Big]+2\left[K^{\beta\gamma}+K^{\gamma\beta}\right]$$
$$\left[M^{n+1}_{\beta}(t)M^{n+2}_{\gamma}(t)\right]+2\left[K_{jk}+K_{kj}\right]\left[M^{j\,n+1}(t)\,M^{k\,n+2}(t)\right],$$

which reduces to zero since the first expression in brackets vanishes by (75.3) and the other terms vanish by (73′). The left-hand member of (77.4) becomes

$$2K\Big[2M^{\alpha}(t)M^{\alpha}_{n+1}(t)+2M^{\beta\alpha}(t)M^{n+1}_{\beta}(t)+2N^{\alpha}_{j}(t)M^{j\,n+1}(t)$$
$$-M_{n+1\,n+1}(t)N^{\alpha}_{n+2}(t)-M_{n+2\,n+1}(t)N^{\alpha}_{n+1}(t)\Big]+2\left[K^{\alpha\beta}+K^{\beta\alpha}\right]M^{\alpha}(t)M^{n+1}_{\beta}(t)$$

$$+2\left[K_{\beta\gamma}+K_{\gamma\beta}\right]M^{\beta\alpha}(t)\,M^{\gamma}_{n+1}(t)+2\left[K^{jk}+K^{kj}\right]N^{\alpha}_{j}(t)\,M_{k\,n+1}(t),$$

which reduces to zero since the first expression in brackets vanishes by (75.4) and the other terms vanish by (73′). The left-hand member of (77.5) becomes

$$\begin{aligned}2K\Big[&2M^{i}_{\beta}(t)\,M^{\beta}_{n+1}(t)+2M^{ji}(t)\,M_{j\,n+1}(t)-M^{n+1\,n+1}(t)\,M^{n+2\,i}(t)\\&-M^{n+2\,n+1}(t)\,M^{n+1\,i}(t)\Big]+2\left[K^{\beta\gamma}+K^{\gamma\beta}\right]M^{i}_{\beta}(t)\,M^{n+1}_{\gamma}(t)\\&+2\left[K_{jk}+K_{kj}\right]M^{ji}(t)\,M^{k\,n+1}(t),\end{aligned}$$

which reduces to zero since the first expression in brackets vanishes by (75.5) and the other terms vanish by (73′). The left-hand member of (77.6) becomes

$$\begin{aligned}2K\Big[&2M^{\alpha}(t)\,M^{\alpha}_{n+2}(t)+2M^{\beta\alpha}(t)\,M^{n+2}_{\beta}(t)+2N^{\alpha}_{j}(t)\,M^{j\,n+2}(t)\\&-M^{n+1\,n+2}(t)\,N^{\alpha}_{n+2}(t)-M^{n+2\,n+2}(t)\,N^{\alpha}_{n+1}(t)\Big]\\&+2\left[K^{\alpha\beta}+K^{\beta\alpha}\right]M^{\alpha}(t)\,M^{n+2}_{\beta}(t)+2\left[K_{\beta\gamma}+K_{\gamma\beta}\right]M^{\beta\alpha}(t)\,M^{\gamma}_{n+2}(t)\\&+2\left[K^{jk}+K^{kj}\right]N^{\alpha}_{j}(t)\,M_{k\,n+2}(t),\end{aligned}$$

which reduces to zero since the first expression in brackets vanishes by (75.6) and the other terms vanish by (73′). The left-hand member of (77.7) becomes

$$
\begin{aligned}
2K\Big[&2M^{i}_{\beta}(t)\,M^{\beta}_{n+2}(t)+2M^{ji}(t)\,M_{j\,n+2}(t)\\
&-M^{n+1\,n+2}(t)\,M^{n+2\,i}(t)-M^{n+2\,n+2}(t)\,M^{n+1\,i}(t)\Big]\\
&+2\big[K^{\beta\gamma}+K^{\gamma\beta}\big]M^{i}_{\beta}M^{n+2}_{\gamma}(t)+2\big[K_{jk}+K_{kj}\big]M^{ji}(t)\,M^{k\,n+2}(t),
\end{aligned}
$$

which reduces to zero since the first expression in brackets vanishes by (75.7) and the other terms vanish by (73'). The left-hand member of (77.8) vanishes by (75.8). The left-hand member of (77.9) becomes

$$
\begin{aligned}
2K\Big[&2M^{\gamma\alpha}(t)\,M_{\gamma\beta}(t)+2N^{\alpha}_{j}(t)\,N^{j}_{\beta}(t)+2M^{\alpha}(t)\,M^{\alpha\beta}(t)\\
&+2M^{\beta}(t)\,M^{\beta\alpha}(t)-N^{\alpha}_{n+1}(t)\,N^{\beta}_{n+2}(t)-N^{\beta}_{n+1}(t)\,N^{\alpha}_{n+2}(t)\Big]\\
&+2\big[K^{\alpha\beta}+K^{\beta\alpha}\big]M^{\alpha}(t)\,M^{\beta}(t)+2\big[K_{\beta\gamma}+K_{\gamma\beta}\big]M^{\gamma\alpha}(t)\,M_{\beta}(t)\\
&+2\big[K^{\alpha\gamma}+K^{\gamma\alpha}\big]M^{\alpha}(t)\,M_{\gamma\beta}(t)+2\big[K_{\gamma\delta}+K_{\delta\gamma}\big]M^{\gamma\alpha}(t)\,M^{\delta\beta}(t)\\
&+2\big[K^{jk}+K^{kj}\big]N^{\alpha}_{j}(t)\,N^{\beta}_{k}(t),
\end{aligned}
$$

which reduces to zero since the first expression in brackets vanishes by (75.9) and the other terms vanish by (73'). The left-hand member of (77.10) becomes

$$
2K\Big[2M^{\alpha}(t)\,M^{\alpha}_{i}(t)+2M^{\beta\alpha}(t)\,M^{i}_{\beta}(t)+2M^{ji}(t)\,N^{\alpha}_{j}(t)
$$

$$
\begin{aligned}
&- M^{n+1\,i}(t)\, N^{\alpha}_{n+2}(t) - M^{n+2\,i}(t)\, N^{\alpha}_{n+1}(t)\Big] + 2\left[K^{\alpha\beta} + K^{\beta\alpha}\right] M^{\alpha}(t)\, M_{\beta}^{\ i}(t) \\
&+ 2\left[K_{\beta\gamma} + K_{\gamma\beta}\right] M^{\beta\alpha}(t)\, M_{i}^{\ \gamma}(t) + 2\left[K^{jk} + K^{kj}\right] N_{j}^{\ \alpha}(t)\, M_{ki}(t), \\
&+ 2\left[K_{\beta\gamma} + K_{\gamma\beta}\right] M^{\beta\alpha}(t)\, M_{i}^{\ \gamma}(t) + 2\left[K^{jk} + K^{kj}\right] N_{j}^{\ \alpha}(t)\, M_{ki}(t),
\end{aligned}
$$

which reduces to zero since the first expression in brackets vanishes by (75.10) and the other terms vanish by (73′). Using the first equation of (76), we find

$$M^{ij}(t)\,{}'M^{ij}(t) = K\left[M^{ij}(t)\right]^{2},$$

so that the left-hand member of (77.11) becomes

$$
\begin{aligned}
&2K\Big[2M_{\beta}^{\ i}(t)\, M_{j}^{\ \beta}(t) + 2M^{ki}(t)\, M_{kj}(t) - 2\left(M^{ij}(t)\right)^{2} \\
&\quad - M^{n+2\,i}(t)\, M^{n+1\,j}(t) - M^{n+1\,i}(t)\, M^{n+2\,j}(t)\Big] \\
&+ 2\left[K^{\beta\gamma} + K^{\gamma\beta}\right] M_{\beta}^{\ i}(t)\, M_{\gamma}^{\ j}(t) + 2\left[K_{kl} + K_{lk}\right] M^{ki}(t)\, M^{lj}(t),
\end{aligned}
$$

which reduces to zero since the first expression in brackets vanishes by (75.11) and the other terms vanish by (73′).

Thus we have shown that the coefficients of the generated transformations (74) satisfy (75). Hence the generated transformations are conformal.

We have now completed the proof of

THEOREM 41. <u>A given infinitesimal conformal trans-</u>

formation in (C,E_n), *which in homogeneous sphere coordinates is defined by* (73), $(73')$ *and* $(73'')$, *generates a one-parameter continuous group of finite conformal transformations in* (C,E_n), *which in homogeneous sphere coordinates are given by* (74), (74.1), (74.2) *and* (74.3).

PART VI

A SUBGROUP OF THE PROJECTIVE GROUP IN (C,E_n)[1]

1. **The Subgroup of the Infinitesimal Projective Group in (C,E_n) Taking Unit Spheres into Unit Spheres in (C,E_n).** We seek to determine the form of those infinitesimal projective transformations (42) which take the unit sphere in (C,E_n)

$$x^{\beta}x_{\beta} + x^{\dot{j}}x_{\dot{j}} = 1 \tag{78}$$

into a unit sphere in (C,E_n), i.e. which leave (78) invariant.

First let us assume that (78) is invariant under (42). Then for every x satisfying (78) we have

$$\frac{d}{dt}\left(x^{\beta}x_{\beta} + x^{\dot{j}}x_{\dot{j}}\right) = 0,$$

or

$$x_{\beta}\frac{dx^{\beta}}{dt} + x_{\dot{j}}\frac{dx^{\dot{j}}}{dt} = 0,$$

which, when use is made of (42) and (78), becomes

$$\begin{aligned}(K^{\beta}+M^{\beta})x_{\beta} + (K_{\dot{j}}+M_{\dot{j}})x^{\dot{j}} + L^{\beta}(x_{\beta})^{2} \\ + K^{\beta\gamma}x_{\beta}x_{\gamma} + (K_{\dot{j}}^{\beta}+L_{\dot{j}}^{\beta})x_{\beta}x^{\dot{j}} + K_{\dot{j}k}x^{\dot{j}}x^{k} = 0.\end{aligned} \tag{79}$$

---

[1] We here generalize results obtained by I.A.Barnett, Proceedings of the National Academy of Sciences, vol. 15 (1929), p.96.

The choice

$$x = \frac{y}{\left[y^{\beta}y_{\beta} + y^{j}y_{j}\right]^{\frac{1}{2}}}$$

satisfies (78) for every y. Putting in (79) the x thus defined, we get the following identity in y :

$$(79.1)\quad \begin{aligned} &\left(y^{\beta}y_{\beta} + y^{j}y_{j}\right)^{\frac{1}{2}}\left(K^{\gamma}+M^{\gamma}\right)y_{\gamma} \\ &+ \left(y^{\beta}y_{\beta} + y^{j}y_{j}\right)^{\frac{1}{2}}\left(K_{k}+M_{k}\right)y^{k} + L^{\beta}\left(y_{\beta}\right)^{2} \\ &+ K^{\beta\gamma}y_{\beta}y_{\gamma} + \left(K^{\beta}_{j} + L^{\beta}_{j}\right)y_{\beta}y^{j} + K_{jk}y^{j}y^{k} = 0 . \end{aligned}$$

Let us use in (79.1) the y defined as follows:

$$y^{\alpha} > 0 \qquad \left(\xi_1 - e \leqq \alpha \leqq \xi_1 + e\right),$$

$$y^{\alpha} = 0 \qquad \left(0 \leqq \alpha < \xi_1 - e,\ \xi_1 + e < \alpha \leqq 1\right),$$

$$y^{i} = 0 .$$

We thus obtain an identity which, after we divide through by $\int_{\xi_1-e}^{\xi_1+e}\left(y^{\beta}\right)^{2}d\beta$, becomes

$$(79.11)\quad \frac{\int_{\xi_1-e}^{\xi_1+e}\left(K^{\beta}+M^{\beta}\right)y^{\beta}d\beta}{\left[\int_{\xi_1-e}^{\xi_1+e}\left(y^{\beta}\right)^{2}d\beta\right]^{\frac{1}{2}}} + \frac{\int_{\xi_1-e}^{\xi_1+e}L^{\beta}\left(y^{\beta}\right)^{2}d\beta}{\int_{\xi_1-e}^{\xi_1+e}\left(y^{\beta}\right)^{2}d\beta} + \frac{\int_{\xi_1-e}^{\xi_1+e}\int_{\xi_1-e}^{\xi_1+e}K^{\beta\gamma}y^{\beta}y^{\gamma}d\beta\, d\gamma}{\int_{\xi_1-e}^{\xi_1+e}\left(y^{\beta}\right)^{2}d\beta} = 0,$$

Applying the Mean Value Theorem of Definite Integrals, we find that (79.11) becomes

$$(79.111)\quad \left(K^{\xi_2}+M^{\xi_2}\right)\frac{\int_{\xi_1-e}^{\xi_1+e} y'^{\beta}\,d\beta}{\left[\int_{\xi_1-e}^{\xi_1+e}(y^{\beta})^2\,d\beta\right]^{\frac{1}{2}}}+L^{\xi_3}+K^{\xi_4\xi_5}\frac{\left[\int_{\xi_1-e}^{\xi_1+e} y'^{\beta}\,d\beta\right]^2}{\int_{\xi_1-e}^{\xi_1+e}(y^{\beta})^2\,d\beta}=0,$$

where ξ_2 , ξ_3 , ξ_4 , ξ_5 are certain intermediate values in the interval (ξ_1-e, ξ_1+e) . Applying Schwarz's Inequality, we have

$$\left(\int_{\xi_1-e}^{\xi_1+e} y'^{\beta}\,d\beta\right)^2 \leqq 2e\int_{\xi_1-e}^{\xi_1+e}(y^{\beta})^2\,d\beta,$$

whence

$$\left[K^{\xi_2}+M^{\xi_2}\right]\frac{\int_{\xi_1-e}^{\xi_1+e} y'^{\beta}\,d\beta}{\left[\int_{\xi_1-e}^{\xi_1+e}(y^{\beta})^2\,d\beta\right]^{\frac{1}{2}}}+K^{\xi_4\xi_5}\frac{\left[\int_{\xi_1-e}^{\xi_1+e} y'^{\beta}\,d\beta\right]^2}{\int_{\xi_1-e}^{\xi_1+e}(y^{\beta})^2\,d\beta}$$

$$\leqq \left(K^{\xi_2}+M^{\xi_2}\right)(2e)^{\frac{1}{2}}+K^{\xi_4\xi_5}(2e).$$

Passing to the limit in this inequality as $e \to 0$, we see that the right-hand side, and therefore the left-hand side, approaches zero. Since, also, $L^{\xi_3} \to L^{\xi_1}$ as $e \to 0$, (79.111) reduces to

$$L^{\xi_1} = 0$$

when we pass to the limit in (79.111) as $e \to 0$. But since ξ_1 was any value in I , we conclude

(79.1111) $$L^\alpha \equiv 0$$

(79.11) thus reduces to

$$\int_{\xi_1-e}^{\xi_1+e}(K^\beta+M^\beta)y^\beta d\beta + \frac{\int_{\xi_1-e}^{\xi_1+e}\int_{\xi_1-e}^{\xi_1+e}K^{\beta\gamma}y^\beta y^\gamma d\beta d\gamma}{\left[\int_{\xi_1-e}^{\xi_1+e}(y^\beta)^2 d\beta\right]^{\frac{1}{2}}} = 0,$$

which, when divided through by $\int_{\xi_1-e}^{\xi_1+e} y^\beta d\beta$, becomes

$$\frac{\int_{\xi_1-e}^{\xi_1+e}(K^\beta+M^\beta)y^\beta d\beta}{\int_{\xi_1-e}^{\xi_1+e}y^\beta d\beta} + \frac{\int_{\xi_1-e}^{\xi_1+e}\int_{\xi_1-e}^{\xi_1+e}K^{\beta\gamma}y^\beta y^\gamma d\beta d\gamma}{\int_{\xi_1-e}^{\xi_1+e}y^\beta d\beta\left[\int_{\xi_1-e}^{\xi_1+e}(y^\beta)^2 d\beta\right]^{\frac{1}{2}}} = 0 .$$

Again applying the Mean Value Theorem, we have

(79.112) $$K^{\xi_2}+M^{\xi_2}+K^{\xi_4\xi_5}\frac{\int_{\xi_1-e}^{\xi_1+e}y^\beta d\beta}{\left[\int_{\xi_1-e}^{\xi_1+e}(y^\beta)^2 d\beta\right]^{\frac{1}{2}}} = 0 .$$

By Schwarz's Inequality we have

$$\frac{\int_{\xi_1-e}^{\xi_1+e}y^\beta d\beta}{\left[\int_{\xi_1-e}^{\xi_1+e}(y^\beta)^2 d\beta\right]^{\frac{1}{2}}} \leqq (2e)^{\frac{1}{2}} .$$

Since the right-hand side of this inequality approaches zero as $e \to 0$, so does the left-hand side. But

$K^{\xi_2}+M^{\xi_2}\to K^{\xi_1}+M^{\xi_1}$ as $e\to 0$. Hence, passing to the limit in (79.112) as $e\to 0$, we obtain

$$K^{\xi_1}+M^{\xi_1}=0,$$

whence

(79.1121) $$K^{\alpha}+M^{\alpha}\equiv 0\,.$$

(79.1) thus reduces to

(79.12) $$\left(y^{\beta}y_{\beta}+y^{j}y_{j}\right)^{\frac{1}{2}}\left(K_{k}+M_{k}\right)y^{k}+K^{\beta\gamma}y_{\beta}y_{\gamma}+\left(K^{\beta}_{\cdot j}+L^{\beta}_{\cdot j}\right)y_{\beta}y^{j}+K_{jk}y^{j}y^{k}=0.$$

Putting $y^{\alpha}=0$, $y^{i}=a^{i}$ in (79.12), we get

$$a^{j}a_{j}\left(K_{k}+M_{k}\right)a^{k}+K_{jk}a^{j}a^{k}=0,$$

from which we infer

(79.121) $$K_{k}+M_{k}\equiv 0,\quad K_{jk}+K_{kj}\equiv 0\,.$$

(79.12) thus reduces to

(79.122) $$K^{\beta\gamma}y_{\beta}y_{\gamma}+\left(K^{\beta}_{\cdot j}+L^{\beta}_{\cdot j}\right)y_{\beta}y^{j}=0,$$

from which we get by Lemma 2

(79.1221) $$K^{\beta\gamma}+K^{\gamma\beta}\equiv 0,\quad K^{\beta}_{\cdot j}+L^{\beta}_{\cdot j}\equiv 0\,.$$

From (79.1111), (79.1121), (79.121) and (79.1221), we have

THEOREM 42. *A necessary condition that* (42) *leave* (78) *invariant is that* (42) *be of the form*

$$(80)\quad \begin{cases} \dfrac{dx^\alpha}{dt} = K^\alpha + K^{\alpha\beta} x_\beta + K^\alpha_j x^j - x^\alpha K^\beta x_\beta + x^\alpha M^j x_j\,, \\[2ex] \dfrac{dx^i}{dt} = -M^i - K^i_\beta x^\beta + K^{ij} x_j - x^i K^\beta x_\beta + x^i M^j x_j\,, \end{cases}$$

where

$$(80')\qquad K^{\alpha\beta} + K^{\beta\alpha} \equiv 0,\; K^{ij} + K^{ji} \equiv 0.$$

We readily establish the sufficiency of the condition stated as necessary in Theorem 42. Applying the transformation we find by (80) and (80′), we have

$$\begin{aligned} \frac{d}{dt}\left(x^\beta x_\beta + x^j x_j\right) &= 2\left(x_\beta \frac{dx^\beta}{dt} + x_j \frac{dx^j}{dt}\right) \\ &= 2\Big[x_\beta \left(K^\beta + K^{\beta\gamma} x_\gamma + K^\beta_j x^j - x^\beta K^\gamma x_\gamma + x^\beta M^j x_j\right) \\ &\quad + x_j \left(-M^j - K^j_\beta x^\beta + K^{jk} x_k - x^j K^\beta x_\beta + x^j M^k x_k\right)\Big] \\ &= 2\Big[K^\gamma x_\gamma \left(1 - x_\beta x^\beta - x_j x^j\right) + M^k x_k \left(x_\beta x^\beta + x_j x^j - 1\right) \\ &\quad + x_\beta K^{\beta\gamma} x_\gamma + x_j K^{jk} x_k\Big], \end{aligned}$$

which is zero for any point x satisfying (78), in view of (80′). We thus have

THEOREM 43. <u>The transformation defined by</u> (80) <u>and</u> (80′) <u>leaves</u> (78) <u>invariant</u>.

Theorems 42 and 43 together give

THEOREM 44. *The most general infinitesimal projective transformation in* (C, E_n) *leaving* (78) *invariant is defined by* (80) *and* (80′).

Computation of the commutator of the transformation (80), (80′) and the transformation

$$\begin{cases} \dfrac{dx^\alpha}{dt} = \bar{K}^\alpha + \bar{K}^{\alpha\beta} x_\beta + \bar{K}_j{}^\alpha x^j - x^\alpha \bar{K}^\beta x_\beta + x^\alpha \bar{M}^j x_j , \\ \dfrac{dx^i}{dt} = -\bar{M}^i - \bar{K}_\beta{}^i x^\beta + \bar{K}^{ij} x_j - x^i \bar{K}^\beta x_\beta + x^i \bar{M}^j x_j , \\ \bar{K}^{\alpha\beta} + \bar{K}^{\beta\alpha} \equiv 0, \quad \bar{K}^{ij} + \bar{K}^{ji} \equiv 0 \end{cases}$$

gives

THEOREM 45. *The totality of transformations of the kind defined by* (80) *and* (80′) *constitute a subgroup of the infinitesimal projective group in* (C, E_n).

2. *A Relation with the Infinitesimal Conformal Group in* (C, E_{n-1}). By analogy with the terminology of n-space, we say that a point x is projected stereographically with respect to the sphere (78) on the linear spread $x^n = -1$ when x is transformed by the formula

$$(81) \quad \begin{cases} \bar{x}^\alpha = \dfrac{2x^\alpha}{1 - x^n} , \\ \bar{x}^i = \dfrac{2x^i}{1 - x^n} \end{cases} \qquad (i = 1, 2, \cdots, n-1).$$

We proceed to transform (80), (80′) stereographically by means of (81). Differentiating (81) with respect to t, we have

$$\frac{d\bar{x}^{\alpha}}{dt} = \frac{2(1-x^{n})\frac{dx^{\alpha}}{dt} + 2x^{\alpha}\frac{dx^{n}}{dt}}{(1-x^{n})^{2}},$$

$$\frac{d\bar{x}^{i}}{dt} = \frac{2(1-x^{n})\frac{dx^{i}}{dt} + 2x^{i}\frac{dx^{n}}{dt}}{(1-x^{n})^{2}} \qquad (i=1,2,\dots,n-1),$$

and replacing $\frac{dx^{\alpha}}{dt}$, $\frac{dx^{i}}{dt}$, $\frac{dx^{n}}{dt}$ by their values from (80), (80'), we get

(81.1) $$\frac{d\bar{x}^{\alpha}}{dt} = 2\left\{\frac{K^{\alpha}+K^{\alpha\beta}x_{\beta}+K_{j}^{\alpha}x^{j}+K_{n}^{\alpha}x^{n}-M^{n}x^{\alpha}}{1-x^{n}} + \frac{x^{\alpha}}{[1-x^{n}]^{2}}\left[-(K_{n}^{\beta}+K^{\beta})x_{\beta}+M^{j}x_{j}+K^{nj}x_{j}\right]\right\},$$

(81.2) $$\frac{d\bar{x}^{i}}{dt} = 2\left\{\frac{-M^{i}-K_{\beta}^{i}x^{\beta}+K^{ij}x_{j}+K^{in}x_{n}-M^{n}x^{i}}{1-x^{n}} + \frac{x^{i}}{[1-x^{n}]^{2}}\left[-(K_{n}^{\beta}+K^{\beta})x_{\beta}+M^{j}x_{j}+K^{nj}x_{j}\right]\right\},$$

where $K^{\alpha\beta}+K^{\beta\alpha}\equiv 0,\ K^{ij}+K^{ji}\equiv 0,\ K^{in}+K^{ni}\equiv 0 \quad (i,j=1,2,\dots,n-1).$

But from (81) we have

$$(81') \quad \begin{cases} x^\alpha = \dfrac{(1-x^n)\,\bar{x}^\alpha}{2}, \\[2ex] x^i = \dfrac{(1-x^n)\,\bar{x}^i}{2} \qquad (i=1,2,\cdots,n-1). \end{cases}$$

Substituting in (78) the values of x^α, x^i, given by (81'), we have

$$(78') \qquad \bar{x}^\beta \bar{x}_\beta + \bar{x}^j \bar{x}_j = 4\left(\frac{1+x^n}{1-x^n}\right) \qquad (j=1,2,\cdots,n-1).$$

Replacing in (81.1) x^α, x^i by their values from (81'), we get

$$(81.1') \qquad \frac{d\bar{x}^\alpha}{dt} = K^{\alpha\beta}\bar{x}_\beta + K_j^\alpha \bar{x}^j - M^n \bar{x}^\alpha + \frac{\bar{x}^\alpha}{2}\Big[-(K_n^\beta + K^\beta)\bar{x}_\beta + M^j \bar{x}_j + K^{nj}\bar{x}_j\Big] + 2\left(\frac{K^\alpha + K_n^\alpha x^n}{1-x^n}\right)$$
$$(j=1,2,\cdots,n-1),$$

where

$$K^{\alpha\beta} + K^{\beta\alpha} \equiv 0, \quad K^{nj} + K^{jn} \equiv 0 .$$

But using (78') we find

$$2\left(\frac{K^\alpha + K_n^\alpha x^n}{1-x^n}\right) = \frac{1}{4}\left(K^\alpha + K_n^\alpha\right)\left(\bar{x}^\beta \bar{x}_\beta + \bar{x}^j \bar{x}_j\right) + K^\alpha - K_n^\alpha \qquad (j=1,2,\cdots,n-1),$$

so that (81.1') becomes

$$\frac{d\bar{x}^\alpha}{dt} = K^\alpha - K_n^\alpha - M^n \bar{x}^\alpha + K^{\alpha\beta}\bar{x}_\beta + K_j^\alpha \bar{x}^j$$

$$+\frac{\bar{x}^{\alpha}}{2}\left[-\left(K_{n}^{\beta}+K^{\beta}\right)\bar{x}_{\beta}+\left(M^{j}+K^{nj}\right)\bar{x}_{j}\right]$$

$$+\frac{1}{4}\left(K^{\alpha}+K_{n}^{\alpha}\right)\left(\bar{x}^{\beta}\bar{x}_{\beta}+\bar{x}^{j}\bar{x}_{j}\right) \qquad (j=1,2,\cdots,n-1),$$

where

$$K^{\alpha\beta}+K^{\beta\alpha}\equiv 0, \quad K^{nj}+K^{jn}\equiv 0.$$

Replacing x^{α}, x^{i} in (81.2) by their values from (81$'$), we get

$$(81.2') \qquad \frac{d\bar{x}^{i}}{dt}=-K_{\beta}^{i}\bar{x}^{\beta}+K^{ij}\bar{x}_{j}-M^{n}\bar{x}^{i}$$

$$+\frac{\bar{x}^{i}}{2}\left[-\left(K_{n}^{\beta}+K^{\beta}\right)\bar{x}_{\beta}+M^{j}\bar{x}_{j}+K^{nj}\bar{x}_{j}\right]$$

$$-2\left(\frac{M^{i}-K^{in}x_{n}}{1-x^{n}}\right) \qquad (i,j=1,2,\cdots,n-1),$$

where

$$K^{ij}+K^{ji}\equiv 0, \quad K^{in}+K^{ni}\equiv 0.$$

Using (78$'$), we find

$$2\left(\frac{M^{i}-K^{in}x_{n}}{1-x^{n}}\right)=\frac{1}{4}\left(M^{i}-K^{in}\right)\left(\bar{x}^{\beta}\bar{x}_{\beta}+\bar{x}^{j}\bar{x}_{j}\right)+M^{i}+K^{in} \quad (i,j=1,2,\cdots,n-1),$$

so that (81.2$'$) becomes

$$\frac{d\bar{x}^{i}}{dt}=-\left(M^{i}+K^{in}\right)-M^{n}\bar{x}^{i}-K_{\beta}^{i}\bar{x}^{\beta}+K^{ij}\bar{x}_{j}$$

$$+\frac{\bar{x}^{i}}{2}\left[-\left(K_{n}^{\beta}+K^{\beta}\right)\bar{x}_{\beta}+M^{j}\bar{x}_{j}+K^{nj}\bar{x}_{j}\right]$$

$$-\frac{1}{4}(M^i-K^{in})(\bar{x}^\beta\bar{x}_\beta+\bar{x}^j\bar{x}_j) \qquad (i,j=1,2,\dots,n-1),$$

where

$$K^{ij}+K^{ji}\equiv 0,\quad K^{in}+K^{ni}\equiv 0.$$

Putting

$$K^\alpha-K_n^\alpha=\bar{K}^\alpha,\quad M^n=-K,\quad -(K_n^\alpha+K^\alpha)=4L^\alpha,$$

$$M^i-K^{in}=4L^i,\quad M^i+K^{in}=-K^i,$$

we have finally

$$(82)\quad \begin{cases} \dfrac{d\bar{x}^\alpha}{dt}=\bar{K}^\alpha+K\bar{x}^\alpha+K^{\alpha\beta}\bar{x}_\beta+K^\alpha_j\bar{x}^j \\ \qquad\qquad +2\bar{x}^\alpha(L^\beta\bar{x}_\beta+L^j\bar{x}_j)-L^\alpha(\bar{x}^\beta\bar{x}_\beta+\bar{x}^j\bar{x}_j), \\ \dfrac{d\bar{x}^i}{dt}=K^i+K\bar{x}^i-K^i_\beta\bar{x}^\beta+K^{ij}\bar{x}_j \\ \qquad\qquad +2\bar{x}^i(L^\beta\bar{x}_\beta+L^j\bar{x}_j)-L^i(\bar{x}^\beta\bar{x}_\beta+\bar{x}^j\bar{x}_j) \\ \qquad\qquad (i,j=1,2,\dots,n-1), \end{cases}$$

where

$$(80')\qquad K^{\alpha\beta}+K^{\beta\alpha}\equiv 0,\quad K^{ij}+K^{ji}\equiv 0.$$

The totality of transformations of the kind defined by (82),(82$'$) comprise, however, the infinitesimal conformal group in (C,E_{n-1}). Hence we have

THEOREM 46. *The subgroup* (80), (80$'$) *of the group of infinitesimal projective transformations in* (C,E_n) *is transformed by a stereographic projection* (81) *into the infinitesimal conformal group in* (C,E_{n-1}).

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