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I hereby recommend that the thesis prepared under my supervision by Louis F. Doty

entitled Matrix Solution of Linear Differential Equations

be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy

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MATRIX SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

A dissertation submitted to the

Graduate School of Arts and Sciences
of the University of Cincinnati

in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

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		Page
Section A.	Considerations on the Rigid Body Solution	1
A.1.	Summary of Laplace Transform Method for Solving Equations	1
A.2.	Rigid Body Solution for a Sharp Edged Gust by Laplace Transform	10
A.2.1.	Example for Sharp Edged Gust Using One Exponential Term for K(s)	13
A.2.2.	Solution for a Constant Value of K(s)	14
Section B.	The Non Rigid Body Solution without the Use of Normal Modes	17
B.1.	Introduction	17
B.2.	Derivation of the Bending Equations of Motion	17
B.2.1.	Notation	19
B.2.2.	Relation between Displacement and Bending Moment	20
B.2.3.	Relation between Applied Force and Bending Moment	24
B.2.4.	Equation of Motion for Rigid Body Translation	25
B.3.	Equations of Motion for Response to a Sharp Edged Gust Considering Vertical Translation and Wing Bending	26

31.8.51

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Page

Section B.4. Derivation of the Torsional Equations of Motion 37

B.4.1. Notation 38

B.4.2. Relation between Applied Twisting Moment and Angular Displacement 39

B.5. Determination of the Applied Twisting Moment and Normal Force for Bending-Torsion Coupling 42

B.5.1. Determination of Applied Twisting Moment and Normal Force Caused by Air Loads on Wing 44

B.5.2. Determination of Inertia Forces and Moments 48

B.5.3. The Resultant Force and Moment Acting on a Wing Section 49

B.6. The Equations of Motion for the Rigid Body Coordinates for Vertical Translation Combined with Pitching Motion 52

B.7. The Equations of Motion for Response to a Sharp Edged Gust Considering Wing Bending, Wing Torsion, Rigid Body Vertical Translation, and Rigid Body Pitching 59

B.7.1. Definitions and Formulation of the Matrices that are Needed 68

B.7.2. Stresses Caused by the Dynamic Loads 69

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Page

Section C.		The Solution of a System of Linear Integro-Differential Equations having Constant Coefficients	71
	C.1.	Introduction	71
	C.2.	Derivation of the Methods	73
	C.2.1.	Introduction	73
	C.2.2.	Fundamental Concepts of Matrix Algebra	73
	C.2.3.	Method I for Solution of the First Order System	87
	C.2.4.	Method II for Solution of the First Order System	93
	C.2.5.	A Method for the Solution of the Second Order System	100
	C.2.6.	Additional Comments	102
Section D.		Summary	108

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INTRODUCTION

The following discussion of the numerical solution of linear differential systems is based on work done for the U.S. Air Force under contract number AF 33 (038)-169. There, the major problem was to determine the response of an aircraft to gust loads. For the purpose of showing the motivation for the development of such a method, the general equations to be solved will be deduced in terms of the gust problem.

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A. Considerations on the Rigid Body Solution

A.1. Summary of Laplace Transform Method for Solving Equations.

In the following discussion all proofs will be omitted, and only results will be stated for reference. All functions involved are assumed to have the necessary properties that permit the indicated operations. Proofs and limitations may be found in any standard book on the Laplace transform.

The Laplace transform is a method for establishing a correspondence between a function and a new derived function. By its use certain differential and integral equations are converted into algebraic equations.

If we have a given function f(t), its Laplace transform is defined as

$$\overline{f}(m) = L\left\{f(t)\right\} \equiv \int_0^\infty e^{-mt} f(t)\, dt, \qquad (1)$$

where m is a number, real or complex. Thus, in order to find the Laplace transform of a function, it is multiplied by e^{-mt} and the product integrated between zero and infinity. The letter t, being a variable of integration, is eliminated when the limits of integrations are substituted. This leaves a new function of the parameter m. A convenient notation is the bar over the letter to indicate that the transform has been taken. We emphasize, however, that f(t) and $\overline{f}(m)$ are completely different functions, so that it is more than just the arguments that have changed.

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For example, to find the transform of the function $f(t) = 1$, we have, by definition

$$\bar{f}(m) = L(1) \equiv \int_0^{\infty} 1 \cdot e^{-mt}\, dt$$

$$= -\frac{1}{m} e^{-mt}\Big|_0^{\infty} = -\frac{1}{m}(0 - 1) = \frac{1}{m}$$

Similarly, for $f(t) = e^{-at}$,

$$\bar{f}(m) = \int_0^{\infty} e^{-mt}\, e^{-at}\, dt = \frac{1}{m + a}\,.$$

That is, the Laplace transform of e^{-at} is $\frac{1}{m + a}$.

Thus, by direct computation, and also by more advanced methods, tables can be compiled which list the function and its corresponding transform.

The transform also has the property of changing derivatives and certain integrals into algebraic expressions. Thus, the transform of the derivative $\frac{df}{dt}$ is

$$L\left(\frac{df}{dt}\right) = \int_0^{\infty} e^{-mt} \frac{df}{dt}\, dt.$$

An integration by parts gives

$$L\left(\frac{df}{dt}\right) = -f(0) + m \int_0^{\infty} e^{-mt} f(t)\, dt,$$

but the last term on the right is simply the transform of the function itself. Hence,

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$$L(\frac{df}{dt}) = m\ L\left\{f(t)\right\} - f(0),$$

or in the bar notation

$$\overline{\frac{df}{dt}}(m) = m\ \bar{f}(m) - f(0).$$

The following table contains a brief list of functions and their corresponding transforms.

	Transform	Function
1.	$f(m) = L\left\{f(t)\right\} = \int_0^{\infty} e^{-mt}\ f(t)dt$	$f(t)$
2.	$m\ \bar{f}(m) - f(0)$	$\frac{df}{dt}$
3.	$m^2\bar{f}(m) - mf(0) - \frac{df(0)}{dt}$	$\frac{d^2f}{dt^2}$
4.	$\frac{1}{m^n}$	$\frac{t^{n-1}}{(n-1)!}$
5.	$\frac{1}{m}$	1
6.	$\frac{1}{m+a}$	e^{-at}
7.	$\frac{a}{m^2+a^2}$	$\sin at$
8.	$\frac{m}{m^2+a^2}$	$\cos at$
9.	$\bar{f}(m)\ \bar{g}(m)$	$\int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds$

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Item 9, in the table is very useful. It is known as the convolution of f(t) and g(t), and is of use when the transform of a function $\overline{F}(m)$ can be written as the product of two transforms. Thus, in

$$\overline{F}(m) = \overline{f}(m) \cdot \overline{g}(m)$$

the inverse transform F(t) may not be known, but the inverse transforms f(t) and g(t) are known. Then the result can be written down using 9. That is, if a function of m can be factored into a product of two functions, each of which has a known transform, then the inverse transform is known by 9. For example if we have

$$F(m) = \frac{1}{m^2 - a^2},$$

we may write

$$F(m) = \frac{1}{m - a} \cdot \frac{1}{m + a}.$$

Now the inverse transform of $\frac{1}{m - a}$ is e^{at}, and the inverse transform of $\frac{1}{m + a}$ is e^{-at}, by 6. Hence, using 9,

$$F(t) = \int_0^t e^{a(t-s)} e^{-as} ds = \int_0^t e^{a(t-2s)} ds$$

$$= \frac{1}{-2a} (e^{-at} - e^{at}) = \frac{1}{a} \sinh at.$$

The inverse transform in the case of rational functions may be found by the method of partial fractions. Thus, if

$$\overline{f}(m) = \frac{N(m)}{D(m)},$$

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where D(m) and N(m) are polynomials in m, D being of higher degree, f(t) may be found by expressing the rational function as a sum of functions whose inverse transforms are tabulated. The form that the partial fraction expansion takes depends on the character of the zeros of D(m). Use of partial fractions is explained in most calculus books.

As an example, if D(m) has real, simple zeros the expansion is very easy.

Let $$F(m) = \frac{1}{m^2 + m - 2},$$

Then

$$F(m) = \frac{1}{(m+2)(m-1)} = \frac{A}{m+2} + \frac{B}{m-1}$$

To find A we multiply through by (m + 2) and then set m = -2. Thus, $A = -\frac{1}{3}$. Multiplying through by (m - 1) and then setting m = 1 gives $B = \frac{1}{3}$. Then,

$$F(m) = -\frac{1}{3}\frac{1}{m+2} + \frac{1}{3}\frac{1}{m-1}.$$

Using 6 in the table, we have

$$F(t) = \frac{-1}{3} e^{-2t} + \frac{1}{3} e^{t} = \frac{1}{3}(e^{t} - e^{-2t}).$$

Other methods exist for finding inverse transforms; however, the ones described will suffice for the purpose in hand.

We now give serveral examples in which the Laplace transform is used to solve differential equations.

Example 1.

$$\ddot{x} + K^2 x = \sin t; \quad x(0) = 0, \quad \dot{x}(0) = 1$$

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Now x and $\ddot{x}$, although unknown are both functions of t. Taking the Laplace transform of both members, making use of 3. and 7. in the table gives

$$m^2\bar{x}(m) - mx(0) - \dot{x}(0) + K^2\bar{x}(m) = \frac{1}{m^2 + 1} .$$

Solving for $\bar{x}(m)$, and using initial conditions

$$\bar{x}(m) = \frac{1}{m^2 + K^2} + \frac{1}{(m^2 + K^2)(m^2 + 1)}$$

$$= \frac{1}{K}\frac{K}{m^2 + K^2} + \frac{1}{K^2 - 1} \cdot \frac{1}{m^2 + 1} - \frac{1}{(K^2 - 1)} \cdot \frac{1}{K}\frac{K}{(m^2 + K^2)}$$

where the last two terms are obtained by partial fraction expansion. Using 7, the inverse transform is found to be

$$x(t) = \frac{1}{K}\left(\frac{K^2 - 2}{K^2 - 1}\right) \sin Kt + \frac{1}{K^2 - 1} \sin t.$$

Although this simple example could readily be solved by usual methods, it is sufficient to partly show the power of the transform method.

In the usual method we must first solve the reduced equation, then determine a particular solution, and finally evaluate the constants of integration. On the other hand, the transform method gives the final solution at once. The complementary and particular solutions are obtained together, and constants of integration are automatically introduced and evaluated when the transform of the derivative is taken. These advantages grow when a system of equations is considered.

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Example 2.

$$\ddot{x} + x = f(t); \quad x(0) = 0, \quad \dot{x}(0) = 0.$$

Here we have the case of the right member being an arbitrary function, having, of course, sufficient properties to permit operations upon it. This involves considerable difficulty by ordinary methods. The convolution integral offers a simple solution of the problem.

Taking the transform of both sides gives

$$x^2\bar{x}(m) + \bar{x}(m) = \bar{f}(m)$$

or,

$$\bar{x}(m) = \frac{\bar{f}(m)}{m^2 + 1} = \bar{f}(m) \cdot \frac{1}{m^2 + 1}$$

Now the inverse transform of $\bar{f}(m)$ is $f(t)$ by definition, and the inverse transform of $\frac{1}{m^2 + 1}$ is $\sin t$. Hence, using the convolution integral we have

$$x(t) = \int_0^t f(s) \sin(t - s)\, ds.$$

The convolution integral may also be used in a slightly different manner to obtain the solution. It will now be shown that if a solution can be obtained for the case $f(t) = 1$, then the solution for an arbitrary $f(t)$ can be obtained.

In the terminology of mechanics, $f(t)$ is called a forcing function, and the solution of the equation, together with the initial conditions, is called the response. We now first find the response to a unit step function. It is called

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this because it is defined to have the value zero for $t < 0$, and the value 1 for $t \geq 0$.

Hence, we first consider the equation

$$\ddot{x}_o + x_o = 1.; \quad x(0) = 0, \quad \dot{x}(0) = 0$$

Taking the transform of both sides, and solving for $\bar{x}_o(m)$ gives

$$x_o(m) = \frac{1}{m(m^2 + 1)} = \frac{1}{m} - \frac{m}{m^2 + 1}$$

from which

$$x_o(t) = 1 - \cos t.$$

This is the response to a unit step function.

Next, the original equation is

$$\ddot{x} + x = f(t), ; \quad x(0) = 0, \quad \dot{x}(0) = 0$$

and the transform is, as before,

$$\bar{x}(m) = \frac{\bar{f}(m)}{m^2 + 1}.$$

But

$$\bar{x}_o(m) = \frac{1}{m(m^2 + 1)},$$

so that

$$\frac{1}{m^2 + 1} = m\,\bar{x}_o(m),$$

and then we can write

$$\bar{x}(m) = m\,\bar{f}(m)\,\bar{x}_o(m).$$

Now $\bar{f}(m)$ and $\bar{x}_o(m)$ are functions whose inverse transforms are known. They are, respectively, $f(t)$ and $x_o(t)$ which has just been found. Using the convolution integral, we have

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$$L\left\{\int_0^t f(s)\ x_o(t-s)ds\right\} = \overline{f}(m)\ \overline{x}_o(m)$$

It is also known from 2. of the table that

$$L\left\{\frac{d}{dt}\int_0^t f(s)\ x_o(t-s)ds\right\} = p\cdot L\left\{\int_0^t f(s)\ x_o(t-s)ds\right\},$$

for the function, which in this case is the integral, has the value zero at $t = 0$. Now the Laplace transform indicated in the above equation is exactly $\overline{f}(m)\ \overline{x}_o(m)$. Hence,

$$L\left\{\frac{d}{dt}\int_0^t f(s)\ x_o(t-s)ds\right\} = m\ \overline{f}(m)\ \overline{x}_o(m);$$

however, the right member of this equation is $\overline{x}(m)$. From this we evaluate $x(t)$ as

$$\begin{aligned} x(t) &= \frac{d}{dt}\int_0^t f(s)\ x_o(t-s)ds \\ &= \int_0^t f(s)\ x_o'(t-s)ds \\ &= \int_0^t f'(s)\ x_o(t-s)ds \end{aligned}$$

since $x_o(0) = 0$ in the problem being considered.

We have found that $x_o(t) = 1 - \cos t$. Thus,

$$x(t) = \int_0^t f(s)\sin(t-s)ds,$$

the same solution as before.

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Although the first point of view of using the convolution integral seems more direct, the second method was discussed because of its frequent use in the literature of the gust problem. It can be shown by physical interpretation that the first case is equivalent to initially computing the response to a unit impulse rather than a unit step function.

A.2. Rigid Body Solution for a Sharp Edged Gust by Laplace Transform Method.

In the section the work of section 3 will be somewhat duplicated. However, a different approach will be used together with several different approximations for the K function of that section.

The notation of that section will be used except that the positive direction is chozen upward for forces, displacements and accelerations.

The lift after entering the gust is equal to the weight plus the increment in lift caused by the gust, the motion of the airplane, and the apparent mass.

The equation of motion in terms of time is

$M\ddot{z}$ = increment in lift,

and
$$\ddot{z} = z'' \frac{U^2}{t^2} = 4z'' \frac{U^2}{c^2}$$

so that, in terms of half chord distances s, the equation of motion becomes

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$$\frac{4MU^2}{c^2}z'' + \frac{U^2}{c^2}C_{L_\alpha}\frac{\rho}{2}Scz'' = C_{L_\alpha}\frac{\rho}{2}SUV\varphi(s) - C_{L_\alpha}\frac{\rho}{c}SU^2\int_0^s K(s-s_1)z''(s_1)ds_1$$

where $s = \frac{2Ut}{c}$, the distance traveled by the wing in half chords measured to the leading edge of the wing from the gust entry at $s = 0$;

S = wing area,

and all other notation the same as section 3.

We shall introduce the notation

$P = \frac{8M}{C_{L_\alpha}\rho Sc}$, the mass parameter; $p = z''/c$,

the non-dimensional acceleration; and $B = P + 1$. The equation of motion then becomes

$$B\,p(s) + 2\int_0^s K(s-\sigma)\,p(\sigma)d\sigma = v_G\,\varphi(s) \qquad (1)$$

This is a linear integral equation to be solved for $p(s)$.

Taking the Laplace transform of both sides gives

$$B\,\overline{p}(m) + 2\overline{K}(m)\,\overline{p}(m) = v_G\,\overline{\varphi}(m)$$

in which the convolution integral has been used in transforming the second term in the left member.

Thus,

$$p(m) = \frac{v_G\,\overline{\varphi}(m)}{B + 2\overline{K}(m)} \qquad (2)$$

For the deficiency functions we shall use the approximations

$$\varphi(s) = 1 - .5e^{-.13s} - .5e^{-s}$$

$$K(s) = 1 - .41e^{-.3s},$$

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and it will be shown that the use of only one exponential in the expression for K(s), rather than two, leads to the solution of a quadratic equation rather than a cubic as in section 3. Using the expression for K(s), (2) becomes

$$\bar{p}(m) = \frac{v_G \bar{\varphi}(m)}{B + \frac{2}{m} - \frac{.82}{(m + .3)}} = v_G m\bar{\varphi}(m) \frac{(m + .3)}{Bm^2 + (.3B + 1.18)m + .6}$$

Let m_1 and m_2 be the zeros of

$$Bm^2 + (.3B + 1.18)m + .6,$$

and then we may write

$$\bar{p}(m) = \frac{v_G}{B} m\bar{\varphi}(m) \frac{(m + .3)}{(m - m_1)(m - m_2)} .$$

Considering the right member the product of the two functions

$$m\bar{\varphi}(m) \quad \text{and} \quad \frac{m + .3}{(m - m_1)(m - m_2)}$$

and noting that $m\bar{\varphi}(m) = m\bar{\varphi}(m) - \varphi(0)$, since $\varphi(0) = 0$, we have two functions whose inverse transforms are easily obtained. Using item 2 in the table of transforms, it is found that the inverse transform of $m\bar{\varphi}(m)$ $(= m\bar{\varphi}(m) - \varphi(0))$ is $\varphi'(s)$ while the inverse transform of the other function can be found by partial fractions to be

$$\frac{m_1 + .3}{m_1 - m_2} e^{m_1 s} - \frac{m_2 + .3}{m_1 - m_2} e^{m_2 s}$$

Now $\varphi'(s) = .065 e^{-.13s} + .500 e^{-s}$, and using the convolution integral we have

$$p(s) = \frac{v_G}{B(m_1 - m_2)} \int_0^s \left[.065e^{-.13(s-\sigma)} + .5e^{-(s-\sigma)}\right]$$

$$\left[(m_1 + .3)e^{m_1 \sigma_2} - (m_2 + .3)e^{m_2 \sigma}\right] d\sigma$$

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Thus, the general solution is

$$p(s) = \frac{v_G}{B(m_1 - m_2)} \Big\{ C_1(e^{m_1 s} - e^{-.3s}) + C_2(e^{m_1 s} - e^{-s}) + C_3(e^{-13s} - e^{m_2 s}) + C_4(e^{-s} - e^{m_2 s}) \Big\}$$

where

$$C_1 = \frac{.065(m_1 + .3)}{m_1 + .13}, \quad C_2 = \frac{.5(m_1 + .3)}{m_1 + 1}, \quad C_3 = \frac{.065(m_2 + .3)}{m_2 + .13}$$

$$C_4 = \frac{.5(m_2 + .3)}{m_2 + 1},$$

and m_1, m_2 are the roots of the equation

$$Bm^2 + (.3B + 1.18)m + .6 = 0.$$

A2.1. Example for Sharp Edged Gust Using one Exponential Term for K(s).

We shall consider the same example as that used in section 3, remembering that the K function was here approximated by a single exponential term.

For this assumed airplane, B = 234.

To obtain the roots m_1 and m_{21} the quadratic is solved to give $m_1 = -.29652$, $m_2 = -.00865$. From this, the constants are $C_1 = -.001358$, $C_2 = .002473$, $C_3 = .156061$, $C_4 = .146941$, and the final result for the non-dimensional acceleration for this airplane is

$$p(s) = \left[-.0165e^{-.2965s} + 4.498e^{-.00865s} - 2.337e^{-.13s} - 2.140e^{-s}\right]10^{-3}\, v_G.$$

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A plot of this solution shows that p(s) is a maximum at s = 18. At that value of s, $p(18) = 3.63 \times 10^{-3}\ v_G$. Comparing this with the solution of section 3, we find a difference of only 1.09 percent. This indicates that this approximation, requiring the solution of a quadratic rather than a cubic equation, is satisfactory.

The result suggests the possibility of using a constant value for the function K(s), for, at least from the present example, the solution seems little affected by the shape of the K(s) curve. To assume that K(s) is a constant is equivalent to assuming damping proportional to the velocity. In the next section this possibility is considered.

A.2.2. Solution for a Constant Value of K(s).

Let K(s) = a, a constant. Then from the general relation already derived,

$$p(m) = \frac{v_G\,\bar{\psi}(m)}{B + 2\bar{K}(m)}$$

$$= \frac{v_G}{B}\, m\, \psi(m) \cdot \frac{1}{m + \frac{2a}{B}}$$

Using the convolution integral we find that, using the present simplification, the non-dimensional acceleration can be written in closed form without the necessity of solving either a cubic or quadratic equation. The solution is found to be

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$$p(s) = \frac{v_G}{B}\left[\left[\frac{.065}{.13 - d} + \frac{.5}{1 - d}\right]e^{-ds} - \frac{.065}{.13 - d}\,e^{-.13s} - \frac{.5}{1 - d}\,e^{-s}\right]$$

where $d = \frac{2a}{B}$.

Evaluating this solution for B = 234, a = 1, it is found that the maximum value for p is

$$p = 3.57\ v_G \times 10^{-3},$$

an error of only 3°/₀ when compared to the solution by the integral equation.

If we choose an average value for K(s) = a, we may expect better results.

For the same airplane, B = 234, with a = 0.7, it is found that the maximum value of p is

$$p = 3.71\ v_G \times 10^{-3}$$

an error of only 1.9°/₀ when compared with the original solution, p being evaluated at s = 18.

Not only is this solution satisfactory for such a high mass parameter, B = 234, but it also gives good results for small values of B.

To compare the result with those of section 3, we choose an extreme case of B = 50. Again using a = 0.7, the closed formula yields

$$p = 13.6\ v_G \times 10^{-3},$$

the maximum value occurring at s = 6.

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The value given by section 3 for this same mass parameter is

$$p = 13.7\ v_G \times 10^{-3}.$$

The error is seen to be less than 1°/o.

Based on these observations, it would appear possible that good results could likewise be obtained for the non-rigid case when K(s) is approximated by a constant.

In section B, the non-rigid solution will be formulated both for K(s) = constant and for K(s) exponentially represented.

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B. The Non-Rigid Body Solution without the use of Normal Modes.

B.1 Introduction

The solution for the non-rigid body has been considered by the method of normal modes. As an alternative approach, the wing may be considered as a system of connected masses. The forced response of this system is then computed directly as a system having a finite number of degrees of freedom. This eliminates the necessity for the computation of the normal modes, and although they do not appear directly, the higher modes that are present in the motion are automatically taken into account.

A system of equations will be derived, and their solution will be considered by a numerical matrix method.

B.2 Derivation of the Bending Equations of Motion

Consider a continuous cantilever beam that has variable stiffness and mass distribution along its length, as replaced by a weightless elastic bar carrying concentrated masses at specified station points. The distance between stations is not necessarily the same all along the beam. This permits the station points to be located at points of high mass concentration such as engines. The distributed mass and any other point masses occurring in any one section between stations are relocated at the two stations at either end of the section as if the two stations in question were the reaction points of a simple beam carrying the same mass. Thus, each internal station will receive

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a contribution from the two sections immediately adjacent the station, one on either side.

The reciprocal of the bending stiffness curve is approximated by a straight line between each two adjacent stations, the value assigned at each station being taken as the true value. This leads to a flexibility curve that has the true value at each station and is a piecewise linear function along the length. These concepts are shown in the following figure.

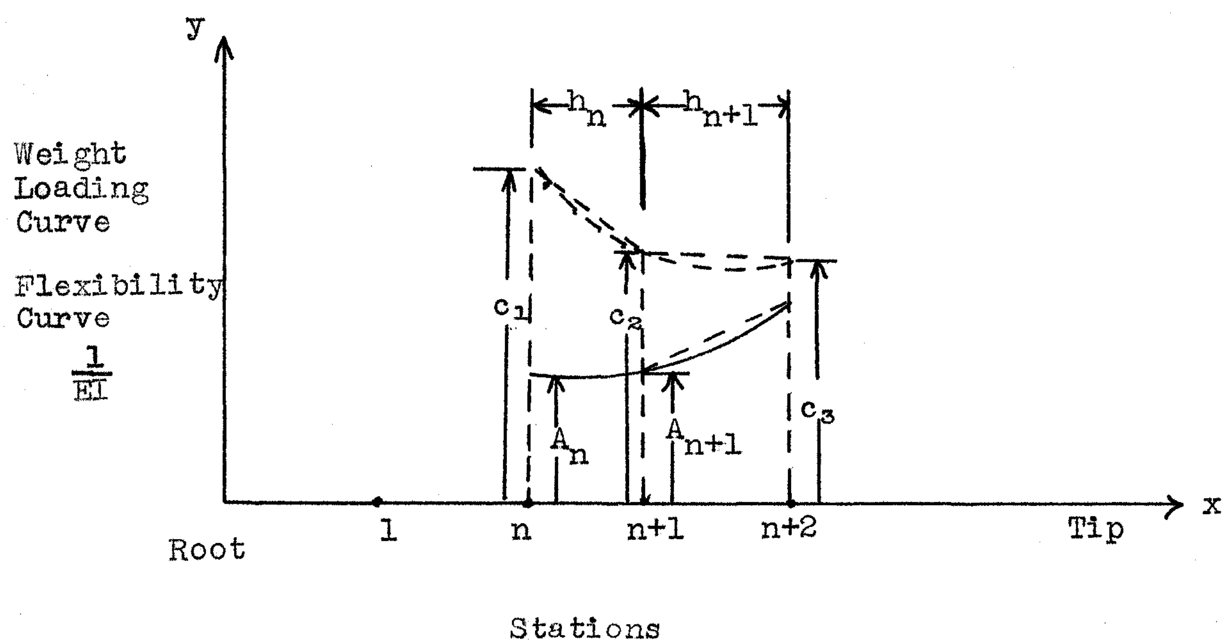

Fig. 1 - Reduction of Physical Wing to Dynamic Model.

Using the weight loading curve in keeping with the discussion and fig. 1, it is seen that, for example, the relocated load acting at station n + 1 is

$$\left(\frac{c_2 h_n}{2} + \frac{c_1 - c_2 h_n}{6}\right) + \left(\frac{c_3\ h_{n+1}}{2} + \frac{c_2 - c_3}{3} h_{n+1}\right).$$

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The use of the segmented flexibility curve is discussed in section B.2.2.

B.2.1 Notation

A coordinate system with origin at the root of the cantilever beam is chosen. The x axis is positive to the right along the undeflected beam. The y axis is positive upward. It may be noted that when the cantilever beam is thought of as an airplane semi-wing, the origin of the coordinate system moves with the wing root, and is, therefore, not a reference frame fixed in space.

m_i concentrated mass acting at station i

y_i deflection of the i th mass, measured positive upward from the x axis

z_i applied force at station i, external force plus inertia force

V_{+i} shear in beam at a position just to the right of the i th mass

V_{-i} shear in beam at a position just to the left of the i th mass

M_i bending moment at the i th mass

E Young's modulus of elasticity

I_i bending moment of inertia at station i

A_i bending flexibility at station i, equal to $\frac{1}{EI_i}$

h_i length between station i and i + 1

r running coordinate, positive in positive x direction with origin at station i, and extending to station i + 1.

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The positive directions for shear, bending moment, and load are shown in fig. 2

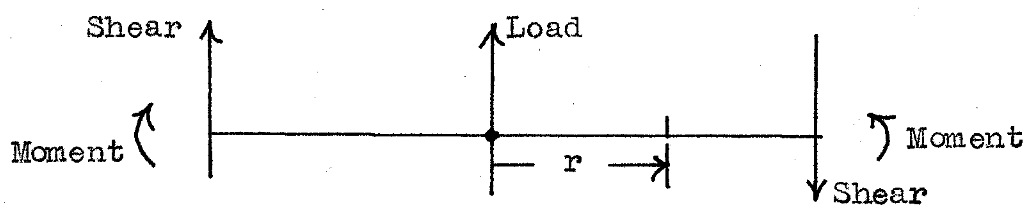

Fig. 2 - Positive Sign Conventions

B.2.2 Relation between Displacement and Bending Moment

Fig. 3 shows a free body diagram of two adjacent spans of the beam.

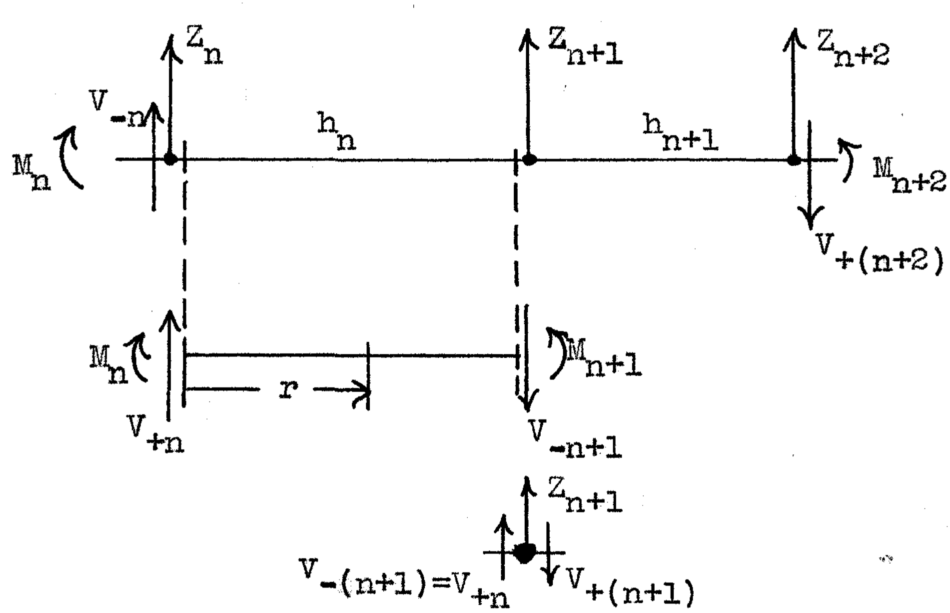

Fig. 3 - Free Body Diagrams of Two Adjacent Sections of Beam

Let r be a variable of integration denoting an arbitrary position in any one section, measured from the left

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end of that section, and let primes denote differentiation with respect to x or r. Since x = constant + r for an arbitrary position on the beam, we have

$$y'' = \frac{d^2y}{dx^2} = \frac{d^2y}{dr^2} .$$

Thus, at any position r, the relation between deflection and bending moment is

$$y''(r) = A(r)\ M(r),$$

in which the equation is applied to any one section at a time. That is, r runs from r = 0 to r = h_n in the n th section.

Under the assumption that the flexibility curve A(r) can be approximated by a single straight line in any one section, we have for the n th section

$$A(r) = \frac{\Delta A_n}{h_n} r + A_n,$$

where $\Delta A_n = A_{n+1} - A_n$, and from the free body diagram in fig. 3,

$$M(r) = v_{+n}\ r + M_n.$$

Thus,

$$y''(r) = (\frac{\Delta A_n}{h_n} r + A_n)(v_{+n}\ r + M_n)$$

$$= \frac{\Delta A_n}{h_n} v_{+n}\ r^2 + (\frac{\Delta A_n}{h_n} M_n + A_n\ v_{+n})r + A_n\ M_n \qquad (1)$$

Integrating twice gives, successively,

$$y'(r) - y'_n = \frac{\Delta A_n}{h_n} v_{+n} \frac{r^3}{3} + (\frac{\Delta A_n}{h_n} M_n + A_n\ v_{+n}) \frac{r^2}{2} + A_n\ M_n\ r \quad (2)$$

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$$y(r) - y_n - y'_n\, r = \frac{\Delta A_n}{h_n} v_{+n} \frac{r^4}{12} + \left(\frac{\Delta A_n}{h_n} M_n + A_n v_{+n}\right) \frac{r^3}{6} + A_n M_n \frac{r^2}{2} \tag{3}$$

where y'_n and y_n are, respectively, the slope and deflection at station n.

Upon evaluating (2) and (3) at $r = h_n$ we have

$$y'_{n+1} - y'_n = \Delta A_n v_{+n} \frac{h^2_n}{3} + (\Delta A_n M_n + A_n h_n v_{+n}) \frac{h_n}{2} + A_n M_n h_n \tag{4}$$

and

$$\frac{y_{n+1}}{h_n} - \frac{y_n}{h_n} - y'_n = \Delta A_n v_{+n} \frac{h^2_n}{12} + (\Delta A_n M_n + A_n h_n v_{+n}) \frac{h_n}{6} + A_n \frac{M_n h_n}{2} \tag{5}$$

An equation similar to (5) can be derived for the adjacen (n + 1) st section. By changing the subscript from n to (n + 1) it is seen to be

$$\frac{y_{n+2}}{h_{n+1}} - \frac{y_{n+1}}{h_{n+1}} - y'_{n+1} = \Delta A_{n+1} v_{+n+1} \frac{h^2_{n+1}}{12}$$

$$+ (\Delta A_{n+1} M_{n+1} + A_{n+1} h_{n+1} v_{+n+1}) \frac{h_{n+1}}{6} + A_{n+1} M_{n+1} \frac{h_{n+1}}{2} \tag{6}$$

The slopes y'_n and y'_{n+1} can be eliminated in the following manner. The subtraction of (6) from (5) leads to a term $(y'_{n+1} - y'_n)$ which can be eliminated by using its value as given by (4). Further, v_{+n} and v_{n+1} can be eliminated, for with reference to fig. 3 it is seen that

$$v_{n+1} h_{n+1} = M_{n+2} - M_{n+1}$$

$$v_n h_n = M_{n+1} - M_n .$$

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Upon performing the indicated manipulation we have the recurrence relation

$$\frac{y_{n+2}}{h_{n+1}} - \left(\frac{1}{h_{n+1}} - \frac{1}{h_n}\right) y_{n+1} + \frac{y_n}{h_n} = \left[\Delta A_{n+1} \frac{h_{n+1}}{12} + A_{n+1} \frac{h_{n+1}}{6}\right] M_{n+2} +$$

$$+ \left[\Delta A_{n+1} \frac{h_{n+1}}{12} + \Delta A_n \frac{h_n}{4} + A_{n+1} \frac{h_{n+1}}{3} + A_n \frac{h_n}{3}\right] M_{n+1} +$$

$$+ \left[\Delta A_n \frac{h_n}{12} + A_n \frac{h_n}{6}\right] M_n \qquad (7)$$

which gives the relation between displacement and moment for sections along the span. If there are K sections, the recurrence formula (7) may be applied once for every station, $n = 0, 1, 2, \ldots, (k - 2)$ giving $(K - 1)$ equations. The relation (7) may not, however, be applied to station $n = (K - 1)$, for in the derivation two adjacent sections were required for the elimination of the slopes. Thus, we have one less equation than the number of unknown displacements y_i. An additional equation may be obtained by consideration of the first section only between stations 0 and 1. Since station 0 is taken at the origin of coordinates at which, for a cantilever beam, the displacement and slope are zero, the slope does not appear when (5) is applied for $n = 0$. Eliminating the shear, as before, gives the additional equation required. Thus, from (5)

$$\frac{y_1}{h_o} = \left(\Delta A_o \frac{h_o}{12} + A_o \frac{h_o}{6}\right) M_1 + \left(\Delta A_o \frac{h_o}{12} + A_o \frac{h_o}{3}\right) M_o \qquad (8)$$

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B.2.3 Relation between Applied Force and Bending Moment

With reference to fig. 3, we may write

$$M_{n+1} - M_n = v_{+n} h_n, \tag{9}$$

and for the next adjacent span,

$$M_{n+2} - M_{n+1} = v_{+n+1} h_{n+1} \tag{10}$$

Subtracting (9) from (10) and using the relation

$$v_{+n+1} - v_n = z_{n+1}$$

gives

$$\frac{M_{n+2}}{h_{n+1}} - \left(\frac{1}{h_{n+1}} + \frac{1}{h_n}\right) M_{n+1} + \frac{M_n}{h_n} = z_{n+1}, \tag{11}$$

a recurrence relation which may be applied along the span allowing n to take all values necessary to account for all the loads z_i. Thus, $n = 0, 1, \cdots, K - 1$, while $M_k = M_{k+1} = 0$.

Hence, equations (7), (8), and (11) yield a system of equations for determining the forced motion of a cantilever beam with reference to a fixed root section. However, when applied to an airplane semi-wing, the root section, being fixed in the fuselage, is not a fixed frame of reference. Hence it is necessary to establish a fixed reference line from which to measure deflections. We shall take this reference line to be the position of the x axis before the onset of the gust. This is shown in fig. 4. Further, since we are considering symmetrical gusts, it is sufficient to consider only half the airplane mass.

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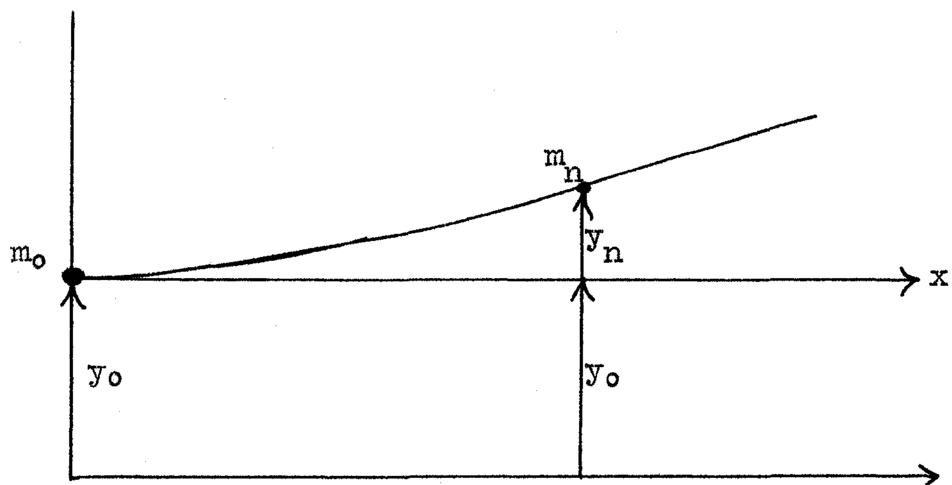


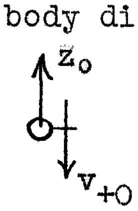
Original Position of x axis before Gust

Fig. 4 - Coordinate System for an Airplane in Gust

It may be seen that y_o is the coordinate representing the rigid body displacement of the airplane, and that y_o is measured from a fixed reference line while y_1, y_2, ••• y_n are measured from the instantaneous position of the x axis.

B.2.4 Equation of Motion for the Rigid Body Coordinate for Vertical Translation Only.

The introduction of y_o requires an additional equation of motion. We may obtain this by observing that the shear in the wing just to the right of the fuselage is v_{+0}. Then, the appropriate free body diagram is



from which we have the equation

$$z_o - v_{+0} = 0$$

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From fig. 3,

$$v_{+0} = \frac{M_1}{h_o} - \frac{M_o}{h_o},$$

so that the equation of motion becomes

$$\frac{M_1}{h_o} - \frac{M_o}{h_o} = z_o \qquad (12)$$

In forming (12) it may be noted that the weight of the fuselage was neglected. This follows from the fact that the wing weights will also be neglected in forming the z forces that appear in (11). These two weight forces are balanced by the lift before gust entry. Hence, during the gust we need only consider the increment in the lift forces and the inertia forces when forming the equations of motion. Upon solving these equations we obtain the dynamic bending moments. Then the total bending moment, acting at any station, during the gust is the sum of the dynamic bending moment and the static bending moment. The latter takes into account the wing weight and the static lift before gust entry.

B.3 The Equations of Motion for Response to a Sharp Edged Gust Considering Vertical Translation and Wing Bending.

We shall apply equations (7), (8), (11), and (12) to determine the equations of motion for this case. Since we are considering only vertical motion and wing bending, the airplane is replaced by a dynamic model consisting of a cantilever beam whose root is attached to a rigid mass moving through space as shown in fig. 4.

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The wing is divided into five sections, and for the purpose of analysis the wing is considered to extend to the center line. The concentrated masses are m_0, at the center line; m_1, m_2, m_3, m_4, placed along the wing; and m_5 at the tip.

In applying equation (7) we write it, successively, for two sections at a time; that is, for sections 1 and 2, 2 and 3, 3 and 4, 4 and 5. This corresponds to allowing the index n to range for n = 0, 1, 2, 3. Equation (8) is then written. Next (11) is applied for n = 0, 1, 2, 3, 4, and finally the necessary number of equations are obtained by writing Eq. (12).

As applied to a given airplane, the dynamic model would first be constructed as indicated in sections B.2 and B.2.2. This would yield the values to be used for the constants m_i, h_i, A_i, and ΔA_i, for i = 0, 1, 2, 3, 4, and the tip mass m_5. Thus, from Eq. (7) for n = 0, 1, 2, 3 successively,

$$\begin{aligned} a_{01} + a_{02}\,y_2 &= b_{00}\,M_0 + b_{01}\,M_1 + b_{02}\,M_2 \\ a_{11}\,y_1 + a_{12}\,y_2 + a_{13}\,y_3 &= b_{11}\,M_1 + b_{12}\,M_2 + b_{13}\,M_3 \\ a_{22}\,y_2 + a_{23}\,y_3 + a_{24}\,y_4 &= b_{22}\,M_2 + b_{23}\,M_3 + b_{24}\,M_4 \\ a_{33}\,y_3 + a_{34}\,y_4 + a_{35}\,y_5 &= b_{33}\,M_3 + b_{34}\,M_4 \end{aligned} \qquad (13)$$

from Eq. (8),

$$a_1 y_1 = b_0 M_0 + b_1 M_1 \qquad (14)$$

from Eq. (11) for n = 0, 1, 2, 3, 4 successively.

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$$z_1 = d_{00}\, M_0 + d_{01}\, M_1 + d_{02}\, M_2$$
$$z_2 = d_{11}\, M_1 + d_{12}\, M_2 + d_{13}\, M_3$$
$$z_3 = d_{22}\, M_2 + d_{23}\, M_3 + d_{24}\, M_4 \qquad (15)$$
$$z_4 = d_{33}\, M_3 + d_{34}\, M_4$$
$$z_5 = d_{44}\, M_4,$$

and from Eq. (12),

$$z_0 = d_0\, M_0 + d_1\, M_1. \qquad (16)$$

The constants a_{ij}, b_{ij}, d_{ij}, b_0, b_1, d_0, and d_1 are easily determined from the equations from which the above relations were written. For example, in Eq. (13) a_{23} is coefficient of y_{n+1} in recurrence relation (7) for n = 2. Thus, referring to (7), we have

$$a_{23} = -\left(\frac{1}{h_3} + \frac{1}{h_2}\right)$$

Likewise,

$$a_{35} = \frac{1}{h_4}\,.$$

The b_{ij} are also found from (7). Thus,

$$b_{00} = \Delta A_0\, \frac{h_0}{12} + A_0\, \frac{h_0}{6},$$

$$b_{24} = \text{coefficient of } M_4 \text{ when } n = 2,$$

$$= \Delta A_3\, \frac{h_3}{12} + A_3\, \frac{h_3}{6}\,.$$

The coefficients d_{ij} are found from (11). For example, d_{12}, the coefficient of M_2 when n = 1 is

$$d_{12} = -\left(\frac{1}{h_2} + \frac{1}{h_1}\right),$$

and the coefficients d_0, d_1 are $d_0 = -\frac{1}{h_0}$; $d_1 = \frac{1}{h_0}$.

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The above equations can be expressed more conveniently in matrix form. (In this connection, the reader not familiar with matrix operations can consult, "Elementary Matrices", by Frazer, Duncan, and Collar, Macmillan, 1947. A brief discussion is also given in section C.2.1 of this report.)

Combining equations (13) and (14), and (15) and (16), we have

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_{01} & a_{02} & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{33} & a_{34} & a_{35} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & 0 & 0 & 0 \\ b_{00} & b_{01} & b_{02} & 0 & 0 \\ 0 & b_{11} & b_{12} & b_{13} & 0 \\ 0 & 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & b_{33} & b_{34} \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \quad \begin{bmatrix} d_0 & d_1 & 0 & 0 & 0 \\ d_{00} & d_{01} & d_{02} & 0 & 0 \\ 0 & d_{11} & d_{12} & d_{13} & 0 \\ 0 & 0 & d_{22} & d_{23} & d_{24} \\ 0 & 0 & 0 & d_{33} & d_{34} \\ 0 & 0 & 0 & 0 & d_{44} \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} \tag{18}$$

The forces z_i consist of the inertia force acting at station i and the increment in the airforce acting at station i. It will be recalled that the static weight and airforce before gust entry are neglected in these computations, for, as mentioned before, they are taken care of separately in computing the final shears and moments.

At station i the inertia force is

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$$-m_i \frac{d^2}{dt^2}(y_o + y_i) = -m_i \ddot{y}_o - m_i \ddot{y}_i$$

since $y_o + y_i$ is the distance to station i from the fixed reference line.

The increment in the air forces arise from the effect of the gust, the effect of changing angle of attack caused by the upward speed of the airfoil, and the apparent mass effect. With reference to section 2, these are seen to be, for a sharp edged gust, increment in airforce at station i =

$$= C_{L\alpha} \frac{\rho}{2} S_i \, UV\psi(s) - C_{L\alpha} \frac{\rho}{2} S_i \, U \int_0^t \left[\ddot{y}_i(t_1) + \ddot{y}_o(t_1)\right] K(s - s_1)dt_1 +$$

$$- \pi \rho S_i \frac{C_i}{4} (\ddot{y}_i + \ddot{y}_o), \quad \text{for} \quad (i = 1, 2, 3, 4, 5).$$

where S_i is the area of the wing associated with station i, C_i is the wing chord at station i, t_1 is a variable of integration, and $S = \frac{2U}{C_o} t$, C_o being the root chord of the wing. All other symbols have the same meaning as that previously used.

Under the assumption that the wing extends into the centerline, there will be a lift acting at station 0. Since the coordinate y_o is measured from a fixed reference, the previous equation simplifies to

increment in airforce at station 0 =

$$= C_{L\alpha} \frac{\rho}{2} S_o \, UV\psi(s) - C_{L\alpha} \frac{\rho}{2} S_o \, U \int_0^t \left[y_o(t_1)\right] K(s - s_1)dt_1 +$$

$$- \pi \quad S_o \frac{C_o}{4} \ddot{y}_o,$$

where S_o is the wing area associated with station 0.

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The proper values for C_o, S_o, and C_i, S_i are shown in fig. 5.

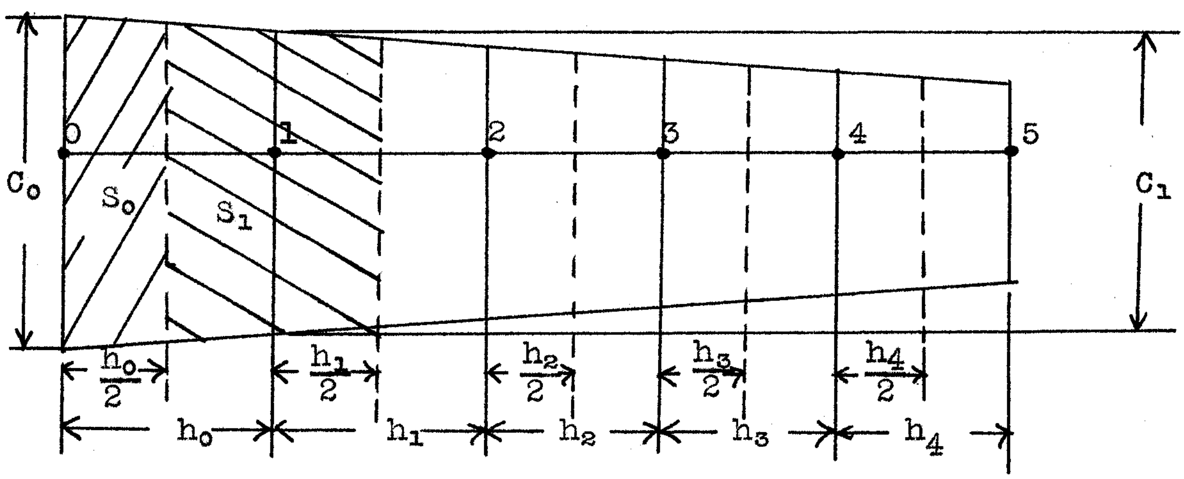

Fig. 5 - Plan Form of Wing Showing Reference Lengths and Areas.

Thus, the wing area is divided into sections, extending from each station to half the distance to the adjacent station (2).

We may change the independent variable from t to s by using the definition of s. Thus

$$dt = \frac{C_o}{2U}\,ds, \qquad \ddot{y} = y''\,\frac{4U^2}{C_o^{\,2}}, \qquad \dot{y}\,dt = y''\,\frac{2U}{C_o}$$

In terms of s, the lift force at station i is

$$C_{L\alpha}\frac{\rho}{2}S_i\,UV\psi(s) - C_{L\alpha}\frac{\rho}{C_o}S_i\,U^2\int_0^s\left[y''_i(\sigma) + y_o''(\sigma)\right]K(s-\sigma)\,d\sigma\ +$$

$$-\ \pi\rho S_i\,C_i\,\frac{U^2}{\quad}\,(y''_i + y_o''), \qquad (i = 1,\ 2,\ 3,\ 4,\ 5)$$

and the lift force at station 0 is obtained by setting $y''_i = 0$.

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Now in the discussion of the rigid body case it was shown that good results could be obtained, for a large range of mass parameters, by replacing the deficiency function K(s) by the constant K = 0.7. This amounts to assuming that the aerodynamic damping is proportional to the speed. Although such simplification is not necessary for the solution of the problem, the degree of uncertainness in the basic data, together with the good results obtained for the rigid body case, seems to make such an assumption justified. Further, from the point of view of computation, such an assumption converts integro-differential equations into differential equations. This simplification, however, should receive further study in the light of test results. For the more general case of coupled motion, we shall set up the equations using an exponential deficiency function. This is done in sections B.5 and following.

The inertia force acting at station i is

$$- m_i \, (\ddot{y}_i + \ddot{y}_o), \qquad (i = 1, 2, 3, 4, 5)$$

or in terms of s,

$$- m_i \, \frac{4U^2}{C_o^2} \, (y_i'' + y_o'').$$

Using the constant value K = 0.7 for the deficiency function, and noting that $y_i'(0) = y_o'(0) = 0$, we have

$$z_o = \alpha_o \, y_o'' + \beta_o \, y_o' - \gamma_o \, \psi(s), \qquad \text{and}$$

$$z_i = + \alpha_i \, y_o'' + \alpha_i \, y_i'' + \beta_i \, y_o' + \beta_i \, y_i' - \gamma_i \, \psi(s), \qquad (19)$$

for i = 1, 2, 3, 4, 5.

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where the constants α_i, β_i, γ_i, $i = 0, 1, 2, 3, 4, 5$ are given by

$$\alpha_i = -\left(\frac{4U^2}{C_o^{\,2}} m_i + \pi \rho S_i \, C_i \frac{U^2}{C_o^{\,2}}\right) \tag{20}$$

$$\beta_i = -\left(.7\, C_{L\alpha} \frac{\rho}{C_o} S_i \, U^2\right) \tag{21}$$

$$\gamma_i = -\left(C_{L\alpha} \frac{\rho}{2} S_i \, UV\right) \tag{22}$$

Using (19), the column matrix involving the z's may be written

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} + \alpha_0 y_0'' + \beta_0 y_0' - \gamma_0 \psi(s) \\ + \alpha_1 y_0'' + \alpha_1 y_1'' + \beta_1 y_0' + \beta_1 y_1' - \gamma_1 \psi(s) \\ + \alpha_2 y_0'' + \alpha_2 y_2'' + \beta_2 y_0' + \beta_2 y_2' - \gamma_2 \psi(s) \\ + \alpha_3 y_0'' + \alpha_3 y_3'' + \beta_3 y_0' + \beta_3 y_3' - \gamma_3 \psi(s) \\ + \alpha_4 y_0'' + \alpha_4 y_4'' + \beta_4 y_0' + \beta_4 y_4' - \gamma_4 \psi(s) \\ + \alpha_5 y_0'' + \alpha_5 y_5'' + \beta_5 y_0' + \beta_5 y_5' - \gamma_5 \psi(s) \end{bmatrix}, \tag{23}$$

noting that each row in the right hand matrix is but a single element, thus giving a (6 x 1) matrix.

Since like matrices may be added, element by element, we may apply the concept and the concept of matrix multiplication to the right member of (23) and obtain

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & \alpha_3 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 & \alpha_4 & 0 \\ \alpha_5 & 0 & 0 & 0 & 0 & \alpha_5 \end{bmatrix} \begin{bmatrix} y_0'' \\ y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \\ y_5'' \end{bmatrix} + \begin{bmatrix} \beta_0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & \beta_2 & 0 & 0 & 0 \\ \beta_3 & 0 & 0 & \beta_3 & 0 & 0 \\ \beta_4 & 0 & 0 & 0 & \beta_4 & 0 \\ \beta_5 & 0 & 0 & 0 & 0 & \beta_5 \end{bmatrix} \begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} +$$

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$$- \begin{bmatrix} \gamma_0 \ \Psi(s) \\ \gamma_1 \ \Psi(s) \\ \gamma_2 \ \Psi(s) \\ \gamma_3 \ \Psi(s) \\ \gamma_4 \ \Psi(s) \\ \gamma_5 \ \Psi(s) \end{bmatrix}, \tag{24}$$

We shall now adopt the following notation.

The (5 x 5) matrix of a's denote by $[a]$. (see eq. (17))

The (5 x 5) matrix of t's denote by $[b]$. (see eq. (17))

The (6 x 5) matrix of d's denote by $[d]$. (see eq. (18))

The (6 x 6) matrix of α's denote by $[\alpha]$. (see eq. (24))

The (6 x 6) matrix of β's denote by $[\beta]$. (see eq. (24))

With this (17) may be **written**

$$[a] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = [b] \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix}, \tag{25}$$

while if the expression for the matrix of z's in (24) is substituted for the same expression in (18) we have,

$$[\alpha] \begin{bmatrix} y_0'' \\ y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \\ y_5'' \end{bmatrix} + [\beta] \begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} - \begin{bmatrix} \gamma_0 \ \Psi(s) \\ \gamma_1 \ \Psi(s) \\ \gamma_2 \ \Psi(s) \\ \gamma_3 \ \Psi(s) \\ \gamma_4 \ \Psi(s) \\ \gamma_5 \ \Psi(s) \end{bmatrix} = [d] \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ \end{bmatrix}. \tag{26}$$

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The matrix equation of motion is obtained by solving (25) for the matrix of M's and substituting in (26). This is done by premultiplying both sides of (25) by the reciprocal of $[b]$. Thus, after rearranging, the equation is

$$[\alpha] \begin{bmatrix} y_0'' \\ y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \\ y_5'' \end{bmatrix} + [\beta] \begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} - [d][b]^{-1}[a] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} y_0 \ \Psi(s) \\ y_1 \ \Psi(s) \\ y_2 \ \Psi(s) \\ y_3 \ \Psi(s) \\ y_4 \ \Psi(s) \\ y_5 \ \Psi(s) \end{bmatrix}, \qquad (27)$$

where $[b]^{-1}$ is the reciprocal of $[b]$.

We note that every term of the equation is a (6 x 1) matrix. In particular, for the third term on the left, using the usual matrix notation for order,

$$[(6 \times 5) \cdot (5 \times 5)] \cdot [(5 \times 5) \cdot (5 \times 1)] =$$
$$= (6 \times 5) \cdot (5 \times 1) = (6 \times 1).$$

It may also be observed that y_0 does not appear explicitly in (27); however to make the equations more symmetrical, and for ease in subsequent numerical solution, we may write the matrix involving the y's in a somewhat different manner.

Let the (6 x 5) matrix $[d][b]^{-1}[a]$ be written $[H]$, and border the $[H]$ matrix by a first column of zeros, thus,

$$\begin{bmatrix} 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \end{bmatrix} = \big[0, \ [H]\big], \qquad (28)$$

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using the notation for a partitioned (1 x 2) matrix.

Let

$$[Y] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

and form the partitioned (2 x 1) matrix

$$\begin{bmatrix} y_0 \\ [Y] \end{bmatrix}$$

Then

$$\begin{bmatrix} 0, & [H] \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ [Y] \end{bmatrix} = \begin{bmatrix} [H] \cdot [Y] \end{bmatrix},$$

so that we may write

$$[H] \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \end{bmatrix} = [R] \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

where $[R]$ is the $[H]$ matrix bordered by a first column of zeros as indicated in (28).

Thus, the final equation of motion may be written

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$$[\alpha]\begin{bmatrix} y_0'' \\ y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \\ y_5'' \end{bmatrix} + [\beta]\begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} - [R]\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} \gamma_0\,\Psi(s) \\ \gamma_1\,\Psi(s) \\ \gamma_1\,\Psi(s) \\ \gamma_3\,\Psi(s) \\ \gamma_4\,\Psi(s) \\ \gamma_5\,\Psi(s) \end{bmatrix} \tag{29}$$

In Eq. (29) it must be remembered that while y_0 is measured from a fixed reference line, y_1, y_2, y_3, y_4, and y_5 are measured relative to the wing root. Thus, the vertical acceleration at station i, (i = 1, 2, 3, 4, 5), is $\ddot{y}_0 + \ddot{y}_i$. It is possible, of course, to write the equations of motion in terms of displacements, all of which are measured from a fixed reference.

We shall next consider the more general case of wing bending, torsion, rigid body vertical translation, and pitching before the solution of (29) is discussed.

B.4. Derivation of the Torsional Equations of Motion

In deriving the equations of motion, we will first determine the relation between applied twisting moment and angular displacement for a cantilever beam such as that shown in fig. 5. It will be assumed that the applied twisting moment, like the normal force, acts at concentrated mass points along the beam. It is necessary to introduce new symbols and sign conventions regarding torsion. With reference to figs. 5, 6, and 7, we shall use the following.

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B.4.1 Notation

φ_i angular displacement of wing chord at station i, measured relative to the root chord at station 0, positive when leading edge is raised relative to trailing edge.

Q_i applied twisting moment, including inertia moment, about elastic axis, positive in sense of positive φ.

T_{+i} torque in beam just to right (outboard) of station i

T_{-i} torque in beam just to left (inboard) of station i

h_i length of beam between station i and i + 1

JG torsional rigidity, variable along beam

C_i torsional flexibility at station i; $C_i = \frac{1}{(JG)}$

$\overline{C}_i$ average value of C between station i and i + 1;

$$\overline{C}_i = \frac{C_i + C_{i+1}}{2}$$

r running coordinate, positive in outboard direction, with origin at station i and running to station i + 1.

The positive sense for angular displacements and moments are shown in fig. 6.

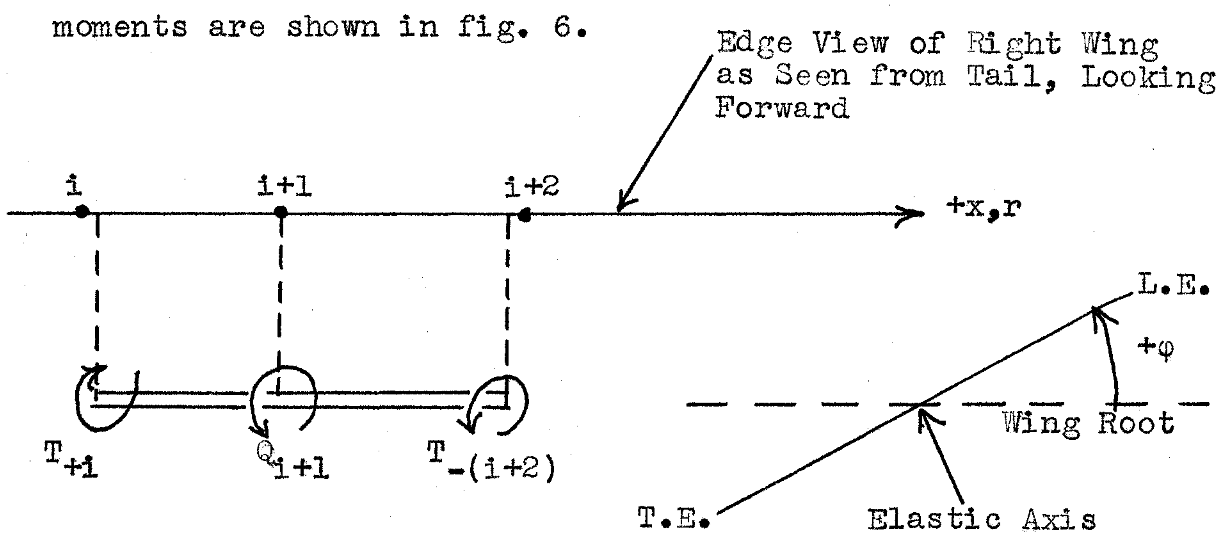

Fig. 6. Positive Conventions For Torsion.

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B.4.2. Relation between Applied Twisting Moment and Angular Displacement

From the usual strength of materials formula for twisting about the elastic axis, we have

$$\frac{d\varphi}{dr} = + C(r) \cdot T(r) \tag{30}$$

in which the + sign is used in keeping with the sign convention adopted.

Since no torque is applied between stations, consideration of fig. 6 shows that the torque between stations is constant. Thus, between stations n and n + 1 the torque T is

$$T = T_{+n} = T_{-(n+1)}.$$

If it is assumed that the torsional flexibility curve C(r) can be approximated by a straight line between stations, then between stations n and n + 1,

$$C(r) = \frac{C_{n+1} - C_n}{h_n} r + C_n.$$

Using this relation and integrating between $r = 0$, and $r = h_n$, corresponding to $\varphi = \varphi_n$, and $\varphi = \varphi_{n+1}$, gives

$$\varphi_{n+1} - \varphi_n = \frac{C_n + C_{n+1}}{2} h_n T_{+n},$$

which we will write

$$\varphi_{n+1} - \varphi_n = \overline{C}_n h_n T_{+n}, \tag{31}$$

where $\overline{C}_n = \frac{C_n + C_{n+1}}{2}$.

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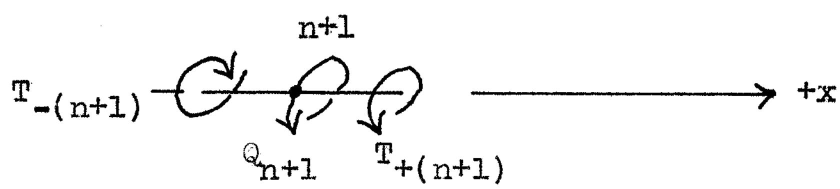
Consideration of a free body diagram of a section of the beam at station n + 1,



for dynamic equilibrium, leads to the relation

$$Q_{n+1} + T_{+(n+1)} - T_{-(n+1)} = 0$$

However, $T_{-(n+1)} = T_{+n}$, so that we have

$$Q_{n+1} = T_{+n} - T_{+(n+1)}. \tag{32}$$

Shifting the index in (31) gives

$$\varphi_{n+2} - \varphi_{n+1} = \bar{C}_{n+1}\, h_{n+1}\, T_{+(n+1)},$$

which when subtracted from (31) leads to the recurrence relation

$$T_{+n} - T_{+(n+1)} = -\frac{\varphi_n}{\bar{C}_n\, h_n} + \left(\frac{1}{\bar{C}_n\, h_n} + \frac{1}{\bar{C}_{n+1}\, h_{n+1}}\right)\varphi_{n+1} - \frac{\varphi_{n+2}}{\bar{C}_{n+1}\, h_{n+1}} \tag{33}$$

Upon eliminating the difference in section torques by the use of (31), there results

$$T_{+n} - T_{+(n+1)} = Q_{n+1} = e_{nn}\, \varphi_n + e_{n(n+1)}\, \varphi_{n+1} + e_{n(n+2)}\, \varphi_{n+2}, \tag{34}$$

where

e_{nn} = coefficient of φ_n when n = n

$e_{n(n+1)}$ = coefficient of φ_{n+1} when n = n

$e_{n(n+2)}$ = coefficient of φ_{n+2} when n = n.

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Since the derivation of (34) required the consideration of two adjacent spans, it may not be applied to the last span. Thus, for the case in which the last span is h_4, (34) may be applied only for n = 0, 1, 2, 3. An additional equation is obtained by observing that the boundary conditions require that $T_{+5} = 0$. Hence, from (32), using n = 4,

$$Q_5 = T_{+4}$$

and from (31), for n = 4, we have

$$\varphi_5 - \varphi_4 = \overline{C}_4\, h_4\, T_{+4},$$

so that

$$Q_5 = -\frac{\varphi_4}{\overline{C}_4\, h_4} + \frac{\varphi_5}{\overline{C}_4\, h_4},$$

which we shall write

$$Q_5 = e_4\, \varphi_4 + e_5\, \varphi_5, \qquad (35)$$

where

$$e_4 = -\frac{1}{\overline{C}_4\, h_4}, \qquad e_5 = \frac{1}{\overline{C}_4\, h_4}$$

For a cantilever beam divided into five sections as shown in fig. 5, we may apply (34) for n = 0, 1, 2, 3, (remembering that $\varphi = 0$ at the root), which together with (35) gives

$$\begin{aligned} n = 0&: \quad Q_1 = e_{01}\, \varphi_1 + e_{02}\, \varphi_2 \\ n = 1&: \quad Q_2 = e_{11}\, \varphi_1 + e_{12}\, \varphi_2 + e_{13}\, \varphi_3 \\ n = 2&: \quad Q_3 = e_{22}\, \varphi_2 + e_{23}\, \varphi_3 + e_{24}\, \varphi_4 \\ n = 3&: \quad Q_4 = e_{33}\, \varphi_3 + e_{34}\, \varphi_4 + e_{35}\, \varphi_5, \end{aligned} \qquad (36)$$

and

$$Q_5 = e_4\, \varphi_4 + e_5\, \varphi_5.$$

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The coefficients e_{ij} are obtained from (33). For example,

$$e_{23} = \text{coefficient of } \varphi_3 \text{ when } n = 2$$

$$= + \left(\frac{1}{\overline{C}_2\, h_2} + \frac{1}{\overline{C}_3\, h_3}\right),$$

and

$$e_{11} = \text{coefficient of } \varphi_1 \text{ when } n = 1$$

$$= - \frac{1}{\overline{C}_1\, h_1}$$

The coefficients e_4 and e_5 have been defined by (35).

We may write (36) in matrix form. Thus,

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} e_{01} & e_{02} & 0 & 0 & 0 \\ e_{11} & e_{12} & e_{13} & 0 & 0 \\ 0 & e_{22} & e_{23} & e_{24} & 0 \\ 0 & 0 & e_{33} & e_{34} & e_{35} \\ 0 & 0 & 0 & e_4 & e_5 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{bmatrix} \qquad (37)$$

B.5. Determination of the Applied Twisting Moment and Normal Force at a Given Station for Bending-Torsion Coupling.

With reference to fig. 7 we shall use the following notation.

E denotes elastic axis position on wing section

G denotes center of gravity position on wing section

a distance from elastic axis to center of gravity, positive when G is aft E.

L total applied force at a given station, including air and inertia effects, positive upward

L_a applied force at a given station, caused by air forces, positive upward

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Q total applied twisting moment about E, at a given station, positive when it tends to raise leading edge.

Q_a applied twisting moment about E, at a given station, positive when it tends to raise leading edge, caused by air forces

z_G vertical distance from fixed reference to G, positive upward

$z = y + y_o$ vertical distance from fixed reference to E, positive upward

m mass concentrated at particular station at G.

θ angle measured from fixed reference line to wing chord

K_G radius of gyration of wing mass associated with a given station, about G.

K radius of gyration of wing mass, associated with a given station, about E.

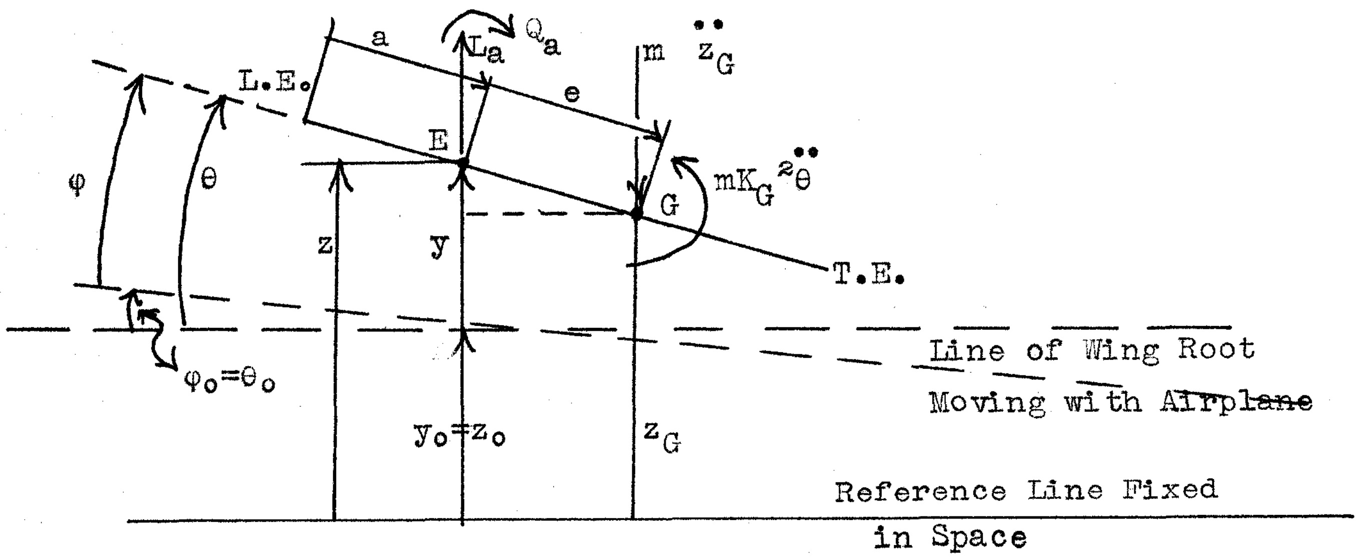

Fig. 7. Section of Wing at a Given Station Showing Applied Forces, Applied Moments, and Distances.

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B.5.1. Determination of the Applied Twisting Moment and Normal Force Caused by Air Loads on Wing

In accord with two-dimensional theory as discussed in section 2, the air forces and couples act as shown in fig. 8.

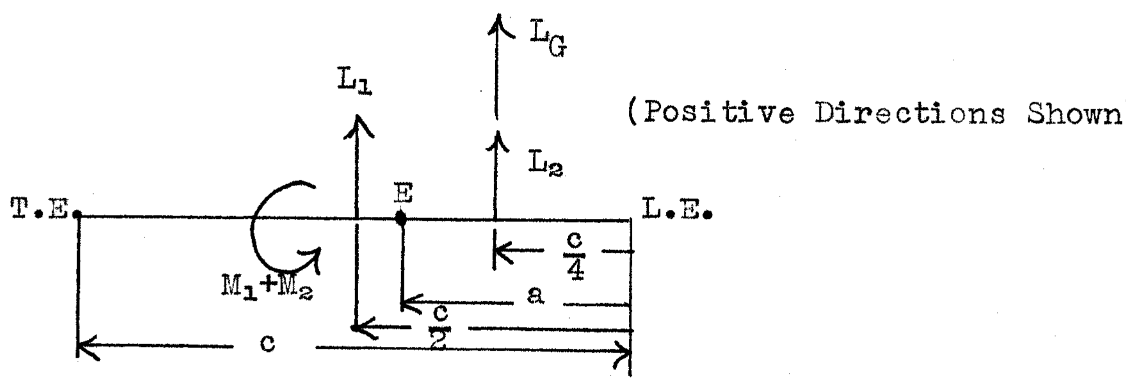

Fig. 8. View of Wing Section Showing Air Loads.

The loading shown in fig. 8. and the following expressions are taken from section 2. Before writing down these expressions, however, it must be remembered that φ_i and y_i (i = 1, 2, 3, 4, 5; see figs. 5 and 7) are not measured relative to a fixed reference line, but are measured, instead, relative to the root chord which is attached to the moving airplane. In the expressions for the air forces and moments, the true velocities and accelerations are needed.

Let y_o be the distance, from a fixed reference line, to the elastic axis position at the wing root; and let φ_o be the angular displacement, from a fixed reference line, to the wing root, both as shown in fig. 7.

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Then, for any wing station, 1, 2, 3, 4, 5 (see fig. 5), the true velocity and acceleration, of the elastic axis point E, are respectively,

$$\dot{y} + \dot{y}_o; \qquad \ddot{y} + \ddot{y}_o,$$

and the true angular velocity and acceleration are, respectively,

$$\dot{\varphi} + \dot{\varphi}_o; \qquad \ddot{\varphi} + \ddot{\varphi}_o$$

We now define

$$z = y + y_o \tag{40}$$

$$\theta = \varphi + \varphi_o, \tag{41}$$

for stations 1, 2, 3, 4, 5; z and θ being measured from fixed references as shown in fig. 7.

At the root station 0,

$$z_o = y_o$$

$$\theta_o = \varphi_o \tag{42}$$

Using expressions (40), (41), and (42), the expressions of section 2, which are written for force and moment per unit length, may be written for an area S associated with each station as shown in fig. 5.

$$L_G = C_{L\alpha} \frac{\rho}{2} SUV \, \Psi(s) \tag{43}$$

$$L_2 = C_{L\alpha} \frac{\rho}{C_o} SU^2 \int_0^s \left[-z'' + \frac{C}{4} \theta'' \right] K(s - \sigma) d\sigma +$$

$$+ C_{L\alpha} \frac{\rho}{2} SU^2 \int_0^s \theta' \, K(s - \sigma) d\sigma. \tag{44}$$

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$$L_1 = C_{L\alpha}\rho \frac{C}{4} S\left(\frac{U^2\theta'}{C_o} - \frac{2U^2}{C_o^2} z''\right) \tag{45}$$

$$M_1 = -\frac{\pi \rho S C^3}{32} \frac{U^2}{C_o^2} \theta'' \tag{46}$$

$$M_2 = -\frac{\pi \rho S C^2}{8} \overline{U^2} \theta' \tag{47}$$

In the above equations, (see fig. 5)

S = wing area associated with given station

C = wing chord at given station

C_o = wing chord at root (station zero),

and the L's and M's are increments in lift and moment, respectively, that act at a particular station. The primes refer to differentiation with respect to s, and the change in independent variable from t to s is made by the substitutions

$$s = \frac{2Ut}{C_o}, \qquad dt = \frac{C_o}{2U} ds$$

$$\dot{z} = z' \frac{2U}{C_o}, \qquad \dot{\theta} = \theta' \frac{2U}{C_o} \tag{48}$$

$$\ddot{z} = z'' \frac{4U^2}{C_o^2}, \qquad \ddot{\theta} = \theta'' \frac{4U^2}{C_o^2} .$$

Upon taking moments about the elastic axis, with reference to fig. 8, we have

$$Q_a = (a - \frac{C}{4}) L_2 + (a - \frac{C}{4}) L_G - (\frac{C}{2} - a) L_1 + M_1 + M_2. \tag{49}$$

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It will be convenient to express L_2 in a different form. Integrating by parts gives

$$L_2 = C_{L\alpha}\rho\frac{SU^2}{C_o}\left[\left[-z' + \frac{C}{4}\theta'\right]K(0) + \left[-z + \frac{C}{4}\theta\right]K'(0) + \right.$$

$$\left. + \int_0^s \left[-z + \frac{C}{4}\theta\right] K''(s - \sigma)d\sigma\right] +$$

$$+ C_{L\alpha}\rho\frac{SU^2}{2}\left[\theta K(0) + \int_0^s \theta K'(z - 0)d0\right], \tag{50}$$

where $K'(s - \sigma) = \dfrac{dK}{d(s - \sigma)}$; $K''(s - \sigma) = \dfrac{d^2K}{d(s - \sigma)^2}$,

so that $K'(s - \sigma) = -\dfrac{\partial K}{\partial\sigma}$; $K''(s - \sigma) = \dfrac{\partial^2 K}{\partial\sigma^2}$.

The primes appearing on z and θ refer to differentiation with respect to s.

Introducing the notation, $\delta = C_{L\alpha}\ SU^2$, and approximating the K function by

$$K(s) = 1 - \lambda e^{-\alpha s} \tag{51}$$

we have

$$L_G = \frac{\delta}{2} v_G\ \psi(s); \qquad v_G = \frac{V}{U} \tag{51}$$

$$L_2 = -\frac{\delta}{C_o}(1 - \lambda)z' + \frac{\delta}{C_o}(1 - \lambda)\frac{C}{4}\theta' - \frac{\delta}{C_o}(\alpha\lambda)z + \frac{\delta}{C_o}(\alpha\lambda)\frac{C}{4}\theta +$$

$$+ \frac{\delta}{2}(1 - \lambda)\theta + \frac{\delta\alpha^2}{C_o}\int_0^s ze^{-\alpha(s-\sigma)}d\sigma + \frac{\delta}{2}\alpha\lambda\int_0^s \theta e^{-\alpha(s-\sigma)}d\sigma +$$

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$$- \frac{\delta a^2 \lambda}{C_o} \cdot \frac{c}{4} \int_0^s \theta e^{-a(s-\sigma)} d\sigma \tag{52}$$

$$L_1 = \frac{\delta}{C_o} \frac{c}{4} \theta' - \frac{\delta}{C_o^2} \frac{c}{2} z'' \tag{53}$$

$$M_1 = - \frac{\delta}{64} \frac{c^3}{C_o^2} \theta'' \tag{54}$$

$$M_2 = - \frac{\delta}{16} \frac{c^2}{C_o} \theta' , \tag{55}$$

where the theoretical slope of the lift curve, 2π, has been replaced by $C_{L\alpha}$ in (54) and (55).

B.5.2. Determination of Inertia Forces and Moments Acting on Wing Section

Fig. 7 shows applied forces and moments acting on a wing section. As is usual in plane motion problems, the inertia effects have been placed at the center of gravity. It will be more convenient, however, to express the moments about the elastic axis E. Thus,

$$z_G = z - e\theta, \qquad (\sin \theta \approx \theta) \tag{56}$$

so that

$$\ddot{z}_G = \ddot{z} - e\ddot{\theta} .$$

Hence, the normal inertia force is

$$- m\ddot{z}_G = - m\ddot{z} + me\ddot{\theta},$$

which, in terms of s is, using (48),

$$- m\ddot{z}_G = - \frac{4U^2}{C_o^2} mz'' + \frac{4U^2}{C_o^2} me\theta'' \tag{57}$$

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Taking moments about E gives, using (56), (48), and $\cos\theta \approx 1$,

$$em\ddot{z}_G - mK^2_G\ddot{\theta} = \frac{4U^2}{C_o^2}\, em(z'' - e\theta'') - \frac{4U^2}{C_o^2}\, mK^2_G\theta'',$$

or upon using the moment of inertia transfer theorem,

$$mK^2_G = mK^2 - me^2,$$

$$em\ddot{z}_G - mK^2_G\ddot{\theta} = \frac{4U^2}{C_o^2}\, emz'' - \frac{4U^2}{C_o^2}\, mK^2\theta'', \tag{58}$$

where m is the concentrated mass acting at the given station.

B.5.3. The Resultant Force and Moment Acting on Wing Section

By combining (57) with (51), (52), and (53) we obtain the total applied force, acting at the elastic axis of the wing section. Thus,

$$L = L_a - \frac{4U^2}{C_o^2}\, mz'' + \frac{4U^2}{C_o^2}\, me\theta''. \tag{59}$$

By combining (58) with (49) we obtain the total applied moment, acting about the elastic axis, of the wing section. Thus,

$$Q = Q_a + \frac{4U^2}{C_o^2}\, emz'' - \frac{4U^2}{C_o^2}\, mK^2\theta''. \tag{60}$$

Written out in full, (59) and (60) become

$$L_i = \varepsilon_{1_i} z_i'' + \varepsilon_{2_i} \theta_i'' + \varepsilon_{3_i} z_i' + \varepsilon_{4_i} \theta_i' + \varepsilon_{5_i} z_i + \varepsilon_{6_i} \theta_i +$$

$$+ \varepsilon_{7_i} \int_0^s z_i e^{-\alpha(s-\sigma)} d\sigma + \varepsilon_{8_i} \int_0^s \theta_i e^{-\alpha(s-\sigma)} d\sigma - \varepsilon_{9_i} \Psi(s) \tag{61}$$

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$$Q_i = \delta_{1_i} z_i'' + \delta_{2_i} \theta_i'' + \delta_{3_i} z_i' + \delta_{4_i} \theta_i' + \delta_{5_i} z_i + \delta_{6_i} \theta_i + \tag{62}$$

$$+ \delta_{7_i} \int_0^s z_i \; e^{-\alpha(s-\sigma)} d\sigma + \delta_{8_i} \int_0^s \theta_i \; e^{-\alpha(s-\sigma)} d\sigma - \delta_{9_i} \psi(s),$$

where the δ's and ε's are constants, and the subscript i has been added to emphasize that L, Q, and δ's, the ε's, z, θ, and their derivatives are different at each station. These formulas, (61) and (62), apply to the wing stations, and for the number of stations being considered here, they apply for i = 1,2,3,4,5. In the next section they will be interpreted to apply to station zero also.

The constants are given by

$$\varepsilon_{1_i} = \left[- \left(\frac{4U^2 em}{C_o^2}\right) - \left(\frac{\delta}{C_o} \; \frac{C}{2}\right) \right]_i$$

$$\varepsilon_{2_i} = \left[+ \left(\frac{4U^2 em}{C_o^2}\right) \right]_i$$

$$\varepsilon_{3_i} = \left[- \left(\frac{\delta(1 - \lambda)}{C_o}\right) \right]_i \tag{63}$$

$$\varepsilon_{4_i} = \left[+ \left(\frac{\delta(1 - \lambda)C}{4C_o}\right) + \left(\frac{\delta C}{4C_o}\right) \right]_i$$

$$\varepsilon_{5_i} = \left[- \left(\frac{\delta\alpha\lambda}{C_o}\right) \right]_i$$

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$$\varepsilon_{6_i} = \left[+ \left(\frac{\delta\alpha\ C}{4C_o}\right) + \left(\frac{\delta(1-\lambda)}{2}\right) \right]_i$$

$$\varepsilon_{7_i} = \left[+ \left(\frac{\delta\alpha^2\lambda}{C_o}\right) \right]_i$$

$$\varepsilon_{8_i} = \left[- \left(\frac{\delta\alpha^2\ C}{4C_o}\right) + \left(\frac{\delta\alpha\lambda}{2}\right) \right]_i \qquad (63)$$

$$\varepsilon_{9_i} = \left[- \left(\frac{\delta}{2}\right) v_G \right]_i ,$$

where the subscript i has been added to emphasize that the quantities may be different at each station. That is, when i = 1, for example, the constants are to be evaluated using data at station 1.

Likewise, for the δ_{ji},

$$\delta_{1_i} = \left[\frac{4U^2 em}{C_o^2} + \left(\frac{C}{2} - a\right) \frac{\delta C}{2C_o^2} \right]_i$$

$$\delta_{2_i} = \left[- \frac{4U^2 mK^2}{C_o^2} \quad \frac{\delta C^3}{64C_o^2} \right]_i$$

$$\delta_{3_i} = \left[- \left(a - \frac{C}{4}\right) \cdot \frac{\delta(1-\lambda)}{C_o} \right]_i \qquad (64)$$

$$\delta_{4_i} = \left[- \left(\frac{C}{2} - a\right) \frac{\delta C}{4C_o} - \frac{\delta C^2}{16C_o} + \left(a - \frac{C}{4}\right) \frac{\delta(1-\lambda)C}{4C_o} \right]_i$$

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$$\delta_{5_i} = \left[- (a - \frac{C}{4}) \frac{\delta\alpha\lambda}{C_o}\right]_i$$

$$\delta_{6_i} = \left[(a - \frac{C}{4})(\frac{\delta\alpha\lambda C}{4C_o} + \frac{\delta - \delta\lambda}{2})\right]_i$$

$$\delta_{7_i} = \left[(a - \frac{C}{4}) \frac{\delta\alpha^2\lambda}{C_o}\right]_i \qquad (64)$$

$$\delta_{8_i} = \left[(a - \frac{C}{4})(\frac{\delta\alpha\lambda}{2} \quad \frac{\delta\alpha^2\lambda C}{4C_o})\right]_i$$

$$\delta_{9_i} = \left[- (a - \frac{C}{4})(\frac{\delta}{2} v_G)\right]_i,$$

where, for reference, $\delta = (C_{L\alpha} \ SU^2)_i$ computed at each station, for $C_{L\alpha}$ and S may vary from station to station.

B.6. The Equations of Motion for the Rigid Body Coordinates for Vertical Translation Combined with Pitching Motion.

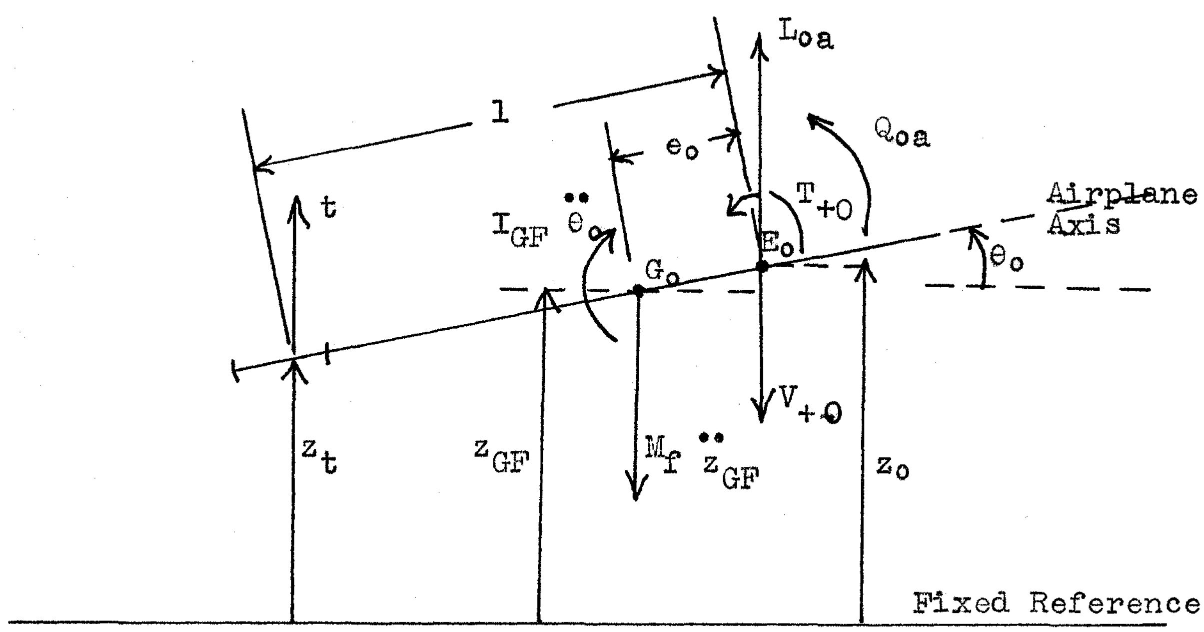

Fig. 9 Free Body Diagram of Fuselage, Wing Removed

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With reference to Fig. 9 we shall use the additional

Definitions

L_{oa} applied force on fuselage (station 0), caused by airforces.

Q_{oa} applied moment on fuselage (station 0), caused by airforces.

z_o distance to elastic axis position from fixed reference.

z_{GF} distance to center of gravity of fuselage from fixed reference.

t one half airforce on tail, assumed to act at tail quarter chord point.

M_f one half mass of fuselage. The word fuselage is intended to mean the entire airplane less wings and objects thereon attached, as represented by the dynamic model discussed in section B.1.

I_{GF} mass moment of inertia of one half fuselage about an axis through G_o perpendicular to plane of symmetry. $I_{GF} = M_f K^2_{GF}$

z_t distance to 1/4 chord position on tail from fixed reference

l distance from elastic axis position on fuselage to 1/4 chord position on tail.

T_{+0} and v_{+0} are, respectively, the twisting moment and shear in the wing just to the right of station 0. That is, in the present free body diagram, they are the twisting moment and shear transferred from the wing to the fuselage. On Fig. 9 all applied forces and moments are shown in the positive direction and sense. The reactions, T_{+0} and v_{+0} are shown in their positive

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sense and direction in keeping with the sign conventions adopted for internal torque and shear.

With reference to Fig. 9, a summation of vertical forces gives

$$- v_{+0} - M_f \ddot{z}_{GF} + L_{oa} + t = 0 \qquad (65)$$

But

$$z_{GF} = z_o - e_o\,\theta_o, \qquad (\sin\theta_o \approx \theta_o)$$

so that

$$M_f \ddot{z}_{GF} = M_f \ddot{z}_o - M_f e_o \ddot{\theta}_o$$

$$= M_f z_o'' \frac{4U^2}{C_o^2} - M_f e_o \theta_o'' \frac{4U^2}{C_o^2},$$

where (48) has been used in writing the last line above. Using this, (65) becomes

$$\left[L_{oa} - M_f \frac{4U^2}{C_o^2} z_o'' + M_f e_o \frac{4U^2}{C_o^2}\right] + t = v_{+0} \qquad (66)$$

Upon taking moments about E,

$$T_{+0} + Q_{oa} - I_{GF} \ddot{\theta}_o - tl + M_f \ddot{z}_{GF} e_o = 0, \qquad (67)$$

but $I_{GF} = M_f K^2_{GF} = M_f K_o^2 - e_o^2 M_f$,

where K_o is the radius of gyration of M_f about E.

Using this result in (67) gives

$$\left[Q_{oa} + \frac{4U^2}{C_o^2} e_o M_f z_o'' - \frac{4U^2}{C_o^2} M_f K_o^2 \theta_o''\right] - tl = - T_{+0} \qquad (68)$$

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We observe that the expressions in brackets in (66) and (68) have exactly the same form as (59) and (60). Hence, we may refer to the bracketed expressions as L_o and Q_o, respectively, and write

$$L_o + t = \varepsilon_{10} z_o'' + \varepsilon_{20} \theta_o'' + \varepsilon_{30} z_o' + \varepsilon_{40} \theta_o' + \varepsilon_{50} z_o + \varepsilon_{60} \theta_o + \\ + \varepsilon_{70} \int_0^s z_o e^{-\alpha(s-\sigma)} d\sigma + \varepsilon_{80} \int_0^s \theta_o e^{-\alpha(s-\sigma)} d\sigma - \varepsilon_{90} \Psi(s) + t = v_{+0} \tag{69}$$

where the ε's may be computed by (63) for $i = 0$, noting that $m_o = M_f$ and S_o is the wing area as shown in fig. 5.

In the same manner we may write (68) as

$$Q_o - tl = \delta_{10} z_o'' + \delta_{20} \theta_o'' + \delta_{30} z_o' + \delta_{40} \theta_o' + \delta_{50} z_o + \delta_{60} \theta_o + \\ + \delta_{70} \int_0^s z_o e^{-\alpha(s-\sigma)} d\sigma + \delta_{80} \int_0^s \theta_o e^{-\alpha(s-\sigma)} d\sigma - \delta_{90} \Psi(s) - tl = -T_{+0} \tag{70}$$

where the δ's may be computed by (64) for $i = 0$.

Determination of the Force on the Tail

Since the static weight is balanced by the static lift and tail force we need only consider the increment in lift on the tail caused by the motion of the airplane and the gust. Thus, we may compute the lift on the tail from the unsteady lift formulas (51), (52), and (53). In computing tail moments we will neglect the aerodynamic couples on the tail as small compared with the term tl.

Considering the fuselage as a rigid body the appropriate values of displacement, velocity and acceleration to be

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used are those corresponding to the 1/4 chord position on the tail. Thus, with reference to fig. 9,

$$z_t = z_o - l\theta_o \qquad (\sin\theta_o \approx \theta_o)$$

$$z_t' = z_o' - l\theta_o'$$

$$z_t'' = z_o'' - l\theta_o'',$$

and with the substitution of these values in (52) and (53), collecting like terms, and using (51) we have

$$t = \left[-\frac{\delta C}{2C_o^2}\right]_t z_o'' + \left[\frac{l\delta C}{2C_o^2}\right]_t \theta_o'' + \left[-\frac{\delta(1-\lambda)}{C_o}\right]_t z_o' +$$

$$+ \left[\frac{l\delta(1-\lambda)}{C_o} + \frac{\delta C + \delta(1-\lambda)C}{4C_o}\right] \theta_o' + \left[-\frac{\delta a\lambda}{C_o}\right]_t z_o +$$

$$+ \left[\frac{l\delta\lambda a}{C_o} + \frac{\delta a\lambda C}{4C_o} + \frac{\delta(1-\lambda)}{2}\right]_t \theta_o + \left[\frac{\delta a^2}{C_o}\right]_t \int_0^s z_o e^{-a(s-\sigma)} d\sigma +$$

$$+ \left[-\frac{l\delta a^2\lambda}{C_o} + \frac{\delta a\lambda}{2} - \frac{\delta a^2\lambda C}{4C_o}\right]_t \int_0^s \theta_o e^{-a(s-\sigma)} d\sigma +$$

$$+ \left[\frac{\delta}{2} v_G\right]_t \Psi(s - d_t) \cdot H(s - d_t), \qquad (71)$$

where the subscript t on the bracketed coefficients indicates that these coefficients are to be computed for the tail. In particular, the value of δ to be used in $\delta = C_{La}\rho SU^2$, where S is one half the horizontal tail area, and C_{La} is the slope of the tail lift curve. The value of C is the mean chord of the tail, but C_o is the root chord of the wing as before. This

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follows from the fact that C_o is only an arbitrary dimension used to change from the independent variable t to s.

It should be noted, also, that when the wing leading edge first strikes the gust, the tail has not yet reached the gust boundary. Hence, the term ψ does not enter the equation until the leading edge of the tail first penetrates the gust. Further, after penetration of the tail, the deficiency function ψ depends on the distance of penetration from gust boundary to the leading edge of the tail. This distance is, of course, not s, for s measures the penetration of the wing leading edge. The distance that the tail has entered the gust is $(s - d_t)$, where d_t is the distance, expressed as a non-dimensional distance in half-chords, $\frac{C_o}{2}$, that the leading edge of the tail is behind the leading edge of the wing.

Both these ideas are expressed by the function

$$\psi(s - d_t) \cdot H(s - d_t),$$

where $(s - d_t)$ is the argument of ψ and H, and H is the unit step function defined by

$$H(s) = 0, \qquad s < 0$$

$$H(s) = 1, \qquad s \geq 0,$$

so that $H(s - d_t)$ has the value 0 (zero) for $s < d_t$ and the value 1 (one) for $s \geq d_t$.

The moment caused by the tail force is found by multiplying equation (71) throughout by l.

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Returning now to Eqs. (69) and (70) we see that the coefficients in (71) can be combined with the ε's and δ's so that the final equations for the rigid body coordinates may be written

$$L_0 + t\ x_1 z_0'' + x_2\theta_0'' + x_3 z_0' + x_4\theta_0' + x_5 z_0 + x_6\theta_0 +$$

$$+ x_7 \int_0^s z_0 e^{-\alpha(s-\sigma)} d\sigma + x_8 \int_0^s \theta_0 e^{-\alpha(s-\sigma)} d\sigma - g_1(s) = v_{+0}, \quad (72)$$

and

$$Q_0 - tl = u_1 z_0'' + u_2\theta_0'' + u_3 z_0' + u_4\theta_0' + u_5 z_0 + u_6\theta_0 +$$

$$+ u_7 \int_0^s z_0 e^{-\alpha(s-\sigma)} d\sigma + u_8 \int_0^s \theta_0 e^{-\alpha(s-\sigma)} d\sigma - g_2(s) = -T_{+0} \quad (73)$$

where the constant u_i, x_i, and the functions $g_1(s)$ and $g_2(s)$ are given by the combinations of (71) and (69), and (71) x 1 and (70). Thus,

$$x_1 = \varepsilon_{10} + \left[\frac{-\delta C}{2C_0^2}\right]_t;\ x_2 = \varepsilon_{20} + \left[\frac{1sC}{2C_0^2}\right]_t;\ x_3 = \varepsilon_{30} + \left[\frac{-\delta(1-\lambda)}{C_0}\right]_t;$$

$$x_4 = \varepsilon_{40} + \left[\frac{1\delta(1-\lambda)}{C_0} + \frac{\delta C + \delta(1-\lambda)C}{4C_0}\right]_t;\ x_5 = \varepsilon_{50} + \left[\frac{-\delta\lambda\alpha}{C_0}\right]_t;$$

$$(74)$$

$$x_6 = \varepsilon_{60} + \left[\frac{1\delta\lambda\alpha}{C_0} + \frac{\delta\alpha\lambda C}{4C_0} + \frac{\delta(1-\lambda)}{2}\right]_t;\ x_7 = \varepsilon_{70} + \left[\frac{\delta\alpha^2\lambda}{C_0}\right]_t;$$

$$x_8 = \varepsilon_{80} + \left[-\frac{1\delta\alpha^2\lambda}{C_0} + \frac{\delta\alpha\lambda}{2} - \frac{\delta\alpha^2\lambda C}{4C_0}\right]$$

$$g_1(s) = +\ \varepsilon_{90}\ \psi(s) + \left[\frac{-\delta}{\ }\ v_G\right]_t \psi(s - d_t) \cdot H(s - d_t).$$

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$$u_1 = \delta_{10} - 1 \left[\frac{-\delta C}{2C_o^2}\right]_t ; \; u_2 = \delta_{20} - 1 \left[\frac{1sC}{2C_o^2}\right]_t ; \tag{75}$$

$$u_3 = \delta_{20} - 1 \left[\frac{-\delta(1-\lambda)}{C_o}\right]_t ; \; u_4 = \delta_{40} - 1 \left[\frac{1\delta(1-\lambda)}{C_o} + \frac{\delta C + \delta(1-\lambda)C}{4C_o}\right]_t ;$$

$$u_5 = \delta_{50} - 1 \left[\frac{-\delta\lambda\alpha}{C_o}\right]_t ; \; u_6 = \delta_{60} - 1 \left[\frac{1\delta\lambda\alpha}{C_o} + \frac{\delta\alpha\, C}{4C_o} + \frac{\delta(1-\lambda)}{2}\right]_t ;$$

$$u_7 = \delta_{70} - 1 \left[\frac{\delta\alpha^2\lambda}{C_o}\right]_t ; \; u_8 = \delta_{80} - 1 \left[- \frac{1\delta\alpha^2\lambda}{C_o} + \frac{\delta\alpha\lambda}{2} - \frac{\delta\alpha^2\lambda C}{4C_o}\right]$$

$$g_2(s) = + \delta_{90}\, \Psi(s) - 1 \left[\frac{-\delta}{2} v_G\right]_t \Psi(s - d_t) \cdot H(s - d_t)$$

It now remains to evaluate v_{+0} and T_{+0} appearing in (72) and (73) in terms of known quantities.

Returning to fig. 3, of section B.2.2 it is seen that

$$v_{+0} = \frac{M_1}{h_o} - \frac{M_o}{h_o} \tag{76}$$

while from Eq. (31) of section B.4.2, it is seen that

$$T_{+0} = \frac{\varphi_1}{\overline{C}_o\, h_o} \tag{77}$$

remembering that $\varphi = 0$ at the wing root (station zero).

B.7. The Equations of Motion for Response to a Sharp Edged Gust Considering Wing Bending, Wing Torsion, Rigid Body Vertical Translation, and Rigid Body Pitching.

The various equations that have been derived will now be collected and expressed in matrix form so that they may be solved by the methods presented in section C.

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In the following discussion reference will be made to certain needed equations by number. It will be understood that all equation numbers refer to equations of section B. Reference should be made to the stated reference equation for the definition and meaning of the symbols used.

We shall form the equations on the basis of a dynamic model whose semi-wing is divided into six stations, 0, 1, 2, 3, 4, 5. If it is desired to write the equations for more wing stations, it may easily be done by applying the basic recurrence relations derived for bending and torsion. Abbreviated notation will be used for the matrices. The complete form will be presented at the end of this section.

For the bending equations, Eqs. (17) and (18) are rewritten with the change that the z's of those equations are called L's to conform with the notation used in the general case. Thus,

$$\left[a \right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \left[b \right] \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} \tag{78}$$

$$\begin{bmatrix} L_0 + \tau \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix} = \left[d \right] \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ \end{bmatrix} \tag{79}$$

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For the torsion equations we combine Eq. (37) with the rigid body pitching equation (68) and Eq. (77) to give

$$\begin{bmatrix} Q_0 - tl \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} e \end{bmatrix} \begin{bmatrix} \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{bmatrix} \tag{80}$$

Premultiply (78) by $\begin{bmatrix} b \end{bmatrix}^{-1}$ to solve for the column of M's, and substitute this result in (79) to give

$$\begin{bmatrix} L_0 + t \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix} = \begin{bmatrix} d \end{bmatrix} \begin{bmatrix} b \end{bmatrix}^{-1} \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}, \tag{81}$$

where the $\begin{bmatrix} R \end{bmatrix}$ matrix is that discussed in connection with Eqs. (28) and (29).

The $\begin{bmatrix} Q \end{bmatrix}$ matrix in (80) with the aid of Eqs. (62) and (73) may be written out in full. Thus,

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$$\begin{bmatrix} Q_0 - tl \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} u_1 z_0'' + u_2\theta_0'' + u_3 z_0' + u_4\theta_0' + u_5 z_0 + u_6\theta_0 + u_7\int_0^s z_0 e^{-\alpha(s-\sigma)}d\sigma + u_8\int_0^s \theta_0 e^{-\alpha(s-\sigma)}d\sigma - g_2(s) \\ \delta_{11}z_1'' + \delta_{21}\theta_1'' + \delta_{31}z_1' + \delta_{41}\theta_1' + \delta_{51}z_1 + \delta_{61}\theta_1 + \delta_{71}\int_0^s z_1 e^{-\alpha(s-\sigma)}d\sigma + \delta_{81}\int_0^s \theta_1 e^{-\alpha(s-\sigma)}d\sigma - \delta_{91}\Psi(s) \\ \delta_{12}z_2'' + \delta_{22}\theta_2'' + \delta_{32}z_2' + \delta_{42}\theta_2' + \delta_{52}z_2 + \delta_{62}\theta_2 + \delta_{72}\int_0^s z_2 e^{-\alpha(s-\sigma)}d\sigma + \delta_{82}\int_0^s \theta_2 e^{-\alpha(s-\sigma)}d\sigma - \delta_{92}\Psi(s) \\ \delta_{13}z_3'' + \delta_{23}\theta_3'' + \delta_{33}z_3' + \delta_{43}\theta_3' + \delta_{53}z_3 + \delta_{63}\theta_3 + \delta_{73}\int_0^s z_3 e^{-\alpha(s-\sigma)}d\sigma + \delta_{83}\int_0^s \theta_3 e^{-\alpha(s-\sigma)}d\sigma - \delta_{93}\Psi(s) \\ \delta_{14}z_4'' + \delta_{24}\theta_4'' + \delta_{34}z_4' + \delta_{44}\theta_4' + \delta_{54}z_4 + \delta_{64}\theta_4 + \delta_{74}\int_0^s z_4 e^{-\alpha(s-\sigma)}d\sigma + \delta_{84}\int_0^s \theta_4 e^{-\alpha(s-\sigma)}d\sigma - \delta_{94}\Psi(s) \\ \delta_{15}z_5'' + \delta_{25}\theta_5'' + \delta_{35}z_5' + \delta_{45}\theta_5' + \delta_{55}z_5 + \delta_{65}\theta_5 + \delta_{75}\int_0^s z_5 e^{-\alpha(s-\sigma)}d\sigma + \delta_{85}\int_0^s \theta_5 e^{-\alpha(s-\sigma)}d\sigma - \delta_{95}\Psi(s) \end{bmatrix} \tag{82}$$

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Using the properties of matrices as discussed in section C of this report (82) may be written

$$[Q] = D_1Z'' + D_2\theta'' + D_3Z' + D_4\theta' + D_5Z + D_6\theta + D_7\int_0^s Ze^{-\alpha(s-\sigma)}d\sigma +$$

$$+ D_8\int_0^s \theta e^{-\alpha(s-\sigma)}d\sigma - G_2(s), \qquad (83)$$

where $[Q]$, Z, θ, and $G_2(s)$ are column matrices, and D_i ($i = 1, \cdots, 8$) are square matrices, the extended form of these symbols to be given presently.

In a similar manner, the $[L]$ matrix in (81) with the aid of Eqs. (61) and (72) can be written out in full corresponding to Eq. (82) from which it can then be condensed into the form

$$[L] = E_1Z'' + E_2\theta'' + E_3Z' + E_4\theta' + E_5Z + E_6\theta + E_7\int_0^s Ze^{-\alpha(s-\sigma)}d\sigma +$$

$$+ E_8\int_0^s \theta e^{-\alpha(s-\sigma)}d\sigma - G_1(s), \qquad (84)$$

where E_i are square matrices, and $[L]$ and $G_1(s)$ are column matrices. Z and θ are the same matrices as in (83).

We now wish to eliminate the matrix of φ's and the matrix of y's in (80) and (81). To do this, we use Eqs. (40), (41), and (42) to give

$$\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\\ \varphi_5\end{bmatrix} = \begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{bmatrix} - \begin{bmatrix}\theta_0\\ \theta_0\\ \theta_0\\ \theta_0\\ \theta_0\end{bmatrix}, \qquad (85)$$

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and

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} - \begin{bmatrix} 0 \\ z_0 \\ z_0 \\ z_0 \\ z_0 \\ z_0 \end{bmatrix} \tag{86}$$

We may now write (85) and (86) in a more useful form. We shall illustrate with the (85).

Now the right member of (80) was obtained from (37) combined with (68) and (77). Written out in full, we have

$$[e]\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{bmatrix} = \begin{bmatrix} e_1 & 0 & 0 & 0 & 0 \\ e_{01} & e_{02} & 0 & 0 & 0 \\ e_{11} & e_{12} & e_{13} & 0 & 0 \\ 0 & e_{22} & e_{23} & e_{24} & 0 \\ 0 & 0 & 0 & e_{34} & e_{35} \\ 0 & 0 & 0 & e_4 & e_5 \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{bmatrix}, \tag{87}$$

where $e_1 = -\dfrac{1}{\overline{C}_0\ h_0}$ from (77), and all others are previously defined.

Using (85) it can be seen by matrix multiplication that (87) is equal to

$$[e]\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{bmatrix} = \left[\begin{array}{l|ccccc} -(e_1) & e_1 & 0 & 0 & 0 & 0 \\ -(e_{01}+e_{02}) & e_{01} & e_{02} & 0 & 0 & 0 \\ -(e_{11}+e_{12}+e_{13} & e_{11} & e_{12} & e_{13} & 0 & 0 \\ -(e_{22}+e_{23}+e_{24} & 0 & e_{22} & e_{23} & e_{24} & 0 \\ -(e_{33}+e_{34}+e_{35} & 0 & 0 & e_{33} & e_{34} & e_{35} \\ -(e_4+e_5) & 0 & 0 & 0 & e_4 & e_5 \end{array}\right]\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}, \tag{88}$$

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for upon multiplying out the right member of (88) we have

$$\begin{bmatrix} -e_1\theta_0 + e_1\theta_1 \\ -(e_{01} + e_{02})\theta_0 + e_{01}\theta_1 + e_{02}\theta_2 \\ -(e_{11} + e_{12} + e_{13})\theta_0 + e_{11}\theta_1 + e_{12}\theta_2 + e_{13}\theta_3 \\ -(e_{22} + e_{23} + e_{24})\theta_0 + e_{22}\theta_2 + e_{23}\theta_3 + e_{24}\theta_4 \\ -(e_{33} + e_{34} + e_{35})\theta_0 + e_{33}\theta_3 + e_{34}\theta_2 + e_{35}\theta_5 \\ -(e_4 + e_5)\theta_0 + e_4\theta_4 + e_5\theta_5 \end{bmatrix}$$

$$= + \begin{bmatrix} e_1 & 0 & 0 & 0 & 0 \\ e_{01} & e_{02} & 0 & 0 & 0 \\ e_{11} & e_{12} & e_{13} & 0 & 0 \\ 0 & e_{22} & e_{23} & e_{24} & 0 \\ 0 & 0 & e_{33} & e_{34} & e_{35} \\ 0 & 0 & 0 & e_4 & e_5 \end{bmatrix} \left\{ + \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} - \begin{bmatrix} \theta_0 \\ \theta_0 \\ \theta_0 \\ \theta_0 \\ \theta_0 \end{bmatrix} \right\},$$

which is exactly (87) when the substitution (85) is made in (87).

From this, it is seen that (80) and (81) can be replaced by

$$[Q] = [\bar{e}] \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}, \qquad (89)$$

and

$$[L] = [R] \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}, \qquad (90)$$

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where the $[\bar{e}]$ is derived from the $[e]$ matrix by bordering the $[e]$ matrix by a first column. Each element in this first column is the negative sum of all elements in the corresponding row of the $[e]$ matrix. This is clearly illustrated by comparing (87) and (88). The same rule applies to the $[R]$ matrix, except that the $[R]$ matrix already has a bordering column of zeros. These are replaced by the new elements formed according to the above rule.

We may now substitute the expressions for $[Q]$ and $[L]$ given by (89) and (90) into (83) and (84) and obtain

$$D_1Z'' + D_2\theta'' + D_3Z' + D_4\theta' + D_5Z + (D_6 - [\bar{e}])\theta + $$
$$+ D_7\int_0^s Ze^{-\alpha(s-\sigma)}d\sigma + D_8\int_0^s \theta e^{-\alpha(s-\sigma)}d\sigma = G_2(s) \qquad (91)$$

and

$$E_1Z_0 + E_2\theta'' + E_3Z' + E_4\theta' + (E_5 - [\bar{R}])Z + E_6\theta + $$
$$+ E_7\int_0^s Ze^{-\alpha(s-\sigma)}d\sigma + E_8\int_0^s \theta e^{-\alpha(s-\sigma)}d\sigma = G_1(s) \qquad (92)$$

We may next combine (91) and (92) in the following way.

Define the partitioned matrices

$$W = \begin{bmatrix} Z \\ \theta \end{bmatrix}, \qquad F(s) = \begin{bmatrix} G_2(s) \\ G_1(s) \end{bmatrix} \qquad (93)$$

Then using properties set forth in section C we may write

$$A_1W'' + A_2W' + A_3W + A4\int_0^s We^{-\alpha(s-\sigma)}d\sigma = F(s). \qquad (94)$$

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We may now solve this equation by the method given in section C, after multiplying through by A^{-1}.

Alternately, we could introduce another partitioned matrix and reduce (94) to a first order system. To do this introduce the new dependent variable defined by

$$X = W' \tag{95}$$

Substituting in (95) we form the new system

$$A_1X' + A_2X + A_3X + A_4 \int_0^s We^{-\alpha(s-\sigma)}d\sigma = F(s)$$

(96)

$$W' - X = 0$$

Then let $Y = \begin{bmatrix} X \\ W \end{bmatrix}$, and then (96) becomes

$$\begin{bmatrix} A_1 & 0 \\ O & I \end{bmatrix}\begin{bmatrix} X' \\ W' \end{bmatrix} \quad \begin{bmatrix} A_2 & A_3 \\ -I & 0 \end{bmatrix}\begin{bmatrix} X \\ W \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \int_0^s \begin{bmatrix} X \\ W \end{bmatrix} e^{-\alpha(s-0)}d0 = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}, \tag{97}$$

which, as a first order system, is

$$C_1Y' + C_2Y + C_3 \int_0^s Ye^{-\alpha(s-\sigma)}d\sigma = P(s), \tag{98}$$

where $P(s) = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$, and the C's are defined by correspondence with Eq. (97).

Methods for solving (98) are presented in section C.

We shall now give explicit statements for the formation of the matrices that lead up to Eq. (94).

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B.7.1. Definitions and Formulation of the Matrices that are Needed.

Eq. (94):

$$A_1 = \begin{bmatrix} D_1 & D_2 \\ E_1 & E_2 \end{bmatrix}; \quad A_2 = \begin{bmatrix} D_3 & D_4 \\ E_3 & E_4 \end{bmatrix}; \quad A_3 = \begin{bmatrix} D_5 & (D_6 - [\bar{e}] \\ (E_5 - [\bar{R}] & E_6 \end{bmatrix}; \quad A_4 = \begin{bmatrix} D_7 & D_8 \\ E_7 & E_8 \end{bmatrix};$$

F(s) defined by Eq. (93). W defined by Eq. (93)

Eqs. (91) and (92):

$$z = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}; \quad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}; \quad G_2(s) = \begin{bmatrix} g_2(s) \\ \delta_{91}\ \Psi(s) \\ \delta_{92}\ \Psi(s) \\ \delta_{93}\ \Psi(s) \\ \delta_{94}\ \Psi(s) \\ \delta_{95}\ \Psi(s) \end{bmatrix}; \quad G_1(s) = \begin{bmatrix} g_1(s) \\ \varepsilon_{91}\ \Psi(s) \\ \varepsilon_{92}\ \Psi(s) \\ \varepsilon_{93}\ \Psi(s) \\ \varepsilon_{94}\ \Psi(s) \\ \varepsilon_{95}\ \Psi(s) \end{bmatrix}$$

$$D_1 = \begin{bmatrix} u_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{15} \end{bmatrix}; \quad D_2 = \begin{bmatrix} u_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{25} \end{bmatrix},$$

and in general, with the aid of (82) we may write

$$D_i = \begin{bmatrix} u_i & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_{i_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_{i_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{i_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_{i4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{i5} \end{bmatrix}$$

for i = (1, 2, 3, 4, 5, 6, 7, 8).

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By comparing (60) with (61) and (72) with (73) we see that

$$E_i \begin{bmatrix} x_i & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{i_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{i_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{i_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{i_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_{i_5} \end{bmatrix}$$

$\left[\bar{e}\right]$, $\left[\bar{R}\right]$ defined by discussion following Eq. (90).

$g_1(s)$ defined by Eq. (74)

$g_2(s)$ defined by Eq. (75)

x_i defined by (74)

u_i defined by (75)

δ_{ij} defined by (64), $\delta = C_{L\alpha} \rho S U^2$

ε_{ij} defined by (63)

B.7.2. Stresses Caused by the Dynamic Loads.

After the matrix equation has been solved numerically as outlined in section C, the determination of the internal wing bending moments, shears, and torques is a simple manner.

Matrix equations have been developed which determine the torques and bending moments in terms of the displacements φ_i and y_i. These are easily computed, once the z_i and θ_i values have been obtained from the solution of the matrix equation, by the simple relations

$$\varphi_i = \theta_i - \theta_o$$

$$y_i = z_i - z_o$$

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The shears may then be computed with the knowledge of the bending moments, for we have developed the relation

$$V_{+n} = \frac{M_{n+1} - M_n}{h_n}.$$

With the knowledge of the bending moments, shears, and torques, the stresses can be computed by the appropriate strength-of-materials formula.

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C. The Solution of a System of Linear Integro-Differential Equations having Constant Coefficients.

C.1. Introduction

All the equations thus far derived are either a system of differential equations or a system of integro-differential equations. In the gust problem, we have seen that the former result when the deficiency function K(s) is taken equal to a constant, and that the latter occur when K(s) is a function of s. The solution of both these cases will be considered.

The most obvious solution is direct application of the Laplace transform. This analytical solution has several disadvantages. While superior to the classical method in that a particular solution need not be found, and also that arbitrary constants of integration need not be evaluated, it is still necessary to solve a system of linear algebraic equations containing a literal parameter. The length of time required to solve such a system rapidly increases with the number of degrees of freedom of the original dynamic problem. After the algebraic equations are solved for each of the variables in terms of the parameter, it is necessary to determine the roots of an algebraic equation in order to find the inverse transform. The labor, again, increases rapidly with increase in degrees of freedom. Another computational problem arises when numerical values for the dependent variables are wanted. Thus, actually evaluating

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the analytical solution for a sequence of values of the independent variable likewise is very time consuming for a system of many degrees of freedom.

In order to obviate some of these difficulties, a solution may be obtained by numerical methods. These, too, require much computation. They do, however, have the obvious advantage directly from the solution, so that it is not necessary to evaluate a complicated analytical expression for every value of the dependent variable that is desired.

There are many methods for the numerical solution of differential equations. Two good references for these are: "Numerical Mathematical Analysis" by Scarborough, Johns Hopkins University Press, 1949. The usual methods described, however, apply equally well to linear and non-linear equations and are usually discussed in terms of a system of low order.

In the following pages method will be developed which are designed especially for linear systems having constant coefficients; and by the use of matrix algebra, a system of very high order can easily be solved. Of course, as the number of degrees of freedom increases, the setting up of the equations takes a correspondingly longer time; but in the actual numerical solution, the labor increases only slightly as degrees of freedom are added.

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The methods given here are applicable to many linear systems having constant coefficients, and, as such, are applicable to many problems arising in aircraft dynamics. It is only incidental that they are considered here in connection with the gust problem.

C.2. Derivation of the Methods.

C.2.1. Introduction

The integro-differential system that has been developed for the gust problem may be written in either of two forms

$$Y' + aY + b\int_0^s e^{-\alpha(s-\sigma)}\, Y(\sigma)d\sigma = g(s), \qquad (1)$$

$$Z'' + JZ' + MZ + N\int_0^s e^{-\alpha(s-\sigma)}\, Z(\sigma)d\sigma = r(s), \qquad (1a)$$

where Z, Y, g, and r are column matrices and a, b, J, M, and N are square matrices. α, s, σ, and e are scalars. We shall consider methods for solving both these equations.

In the numerical methods presented, use will be made of certain elementary properties of matrix algebra, and for the benefit of the reader not familiar with these, a short discussion of those properties that are needed will be given.

C.2.2. Fundamental Concepts of Matrix Algebra.

We shall introduce the idea of a matrix by means of a system of linear algebraic equations, and in so doing attempt to show the motivation for the definitions that are given and the operations that are used. For brevity, the remarks will be confined to two equations.

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Consider the system of algebraic equations

$$a_{11}x_1 + a_{12}x_2 = d_1$$
$$a_{21}x_1 + a_{22}x_2 = d_2 \qquad (2)$$

in which a_{ij}, and d_i are constants, and x_i are to be found.

Rather than solve the system in the usual algebraic manner, it is possible to consider the coefficients and the unknowns separately. Although this is certainly unnecessary for a second order system, it becomes very convenient for systems of high order. First, however, we will define a matrix.

A matrix is a rectangular array of numbers, or, more generally, a rectangular array of elements, taken in a definite order, arranged in rows and columns. Thus

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$$

is a set of four numbers taken in the order a_{11}, a_{12} in the first row, and a_{21}, a_{22} in the second row. Alternately, a_{11}, a_{21} appear in the first column while a_{12}, a_{22} appear in the second column. Thus, the set being considered has two rows and two columns. The uppermost row is referred to a the first row, and the column fartherest to the left is called the first column. When literal numbers are used it is convenient to let the subscripts indicate the position of the element with respect to the rows and columns. That is, a_{12} is the number appearing in the first row, second column. For an array of m rows and n columns the element in the i th row and j th column would be

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designated a_{ij}, the first subscript referring to the row, the second subscript referring to the column.

A set or array of elements arranged in m rows and n columns is called a matrix of order m by n, and the order is indicated by the symbol, (m x n).

It is conventional to enclose the array of numbers so arranged in brackets, so that the matrix of the four numbers indicated above would be written

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix} = A, \quad \text{where } \begin{bmatrix} a_{ij} \end{bmatrix} \text{ and}$$

A are merely shorthand symbols to indicate the more extended form on the left.

It must be emphasized that a matrix is not the same as a determinate; for while the determinate

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

has a definite numberical (scalar) value so that it represents, when expanded, a single number, the matrix is not a single number, but a collection of numbers that are, for convenience, being considered all at one time. Hence, an expression such as, "The value or expansion of a matrix", has no meaning. A matrix may be thought of as an operator that acquires a meaning when applied to some operation in much the same way that the symbol $\int$ has in itself no meaning, but when we write

$$\int f(x)dx$$

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we know that the function f(x) is to be operated upon by a process called integration from which a new function is derived. Similarly, the symbol + has a meaning only as an operator, but if we write a + b, the plus sign is interpreted to mean a process called addition in which a is added to be to give a new number.

Thus, a matrix is a separate entity that obeys certain laws of manipulation.

We shall make several definitions and see that these definitions are not arbitrarily made, but are designed to be of use in linear systems, and so that most of the algebra of numbers is retained in form when applied to matrices.

Definitions

Equality of Matrices

Two matrices are said to be equal if every element in one is equal to the corresponding element in the other. Thus, if A = B, then $a_{ij} = b_{ij}$ for every i and every j. From this, it follows that only matrices of the same order can be equal. The idea of corresponding elements is important, for while

$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$$

even though both matrices have the same elements.

Addition of Matrices.

Two matrices of the same order are added by adding the elements of the first to the corresponding elements of the second

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to produce a new matrix of the same order. Thus, given A and B, we obtain the sum C. C = A + B.

In symbols

$$[c_{ij}] = [a_{ij}] + [b_{ij}] - [a_{ij} + b_{ij}]$$

for all i and j. The word addition means algebraic addition and therefore includes subtraction. As an example,

if $$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 6 \\ 7 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 6 & 3 \end{bmatrix},$$

and $$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}.$$

Since corresponding scalar elements are added, and the commutative and associative laws hold for numbers, the commutative and associative laws hold for matrix addition. Thus, A + B = B + A, and (A + B) + C = A + (B + C).

Multiplication by a Scalar

To multiply a matrix by a number means to multiply every element of the matrix by that number. In symbols

$$K[a_{ij}] = [Ka_{ij}] = [a_{ij}]K.$$

For example

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 9 & 6 \\ 0 & -3 \end{bmatrix}.$$

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Multiplication of a Matrix by Another Matrix

Given a matrix A of order (m x n) and a matrix B of order (n x p) the product AB is a matrix of order (m x p) whose element in the i th row and j th column is formed by multiplying the elements of the i th row of A into the corresponding elements of the j th column of B, and summing the products thus obtained. If $C = [c_{ij}]$ is the product matrix, C = AB, or

$$[c_{ij}] = [a_{ij}][b_{ij}].$$

The above rule may be written, for each element of C,

$$c_{ij} = \sum_{k=1}^{n} a_{ij} b_{kj}$$

where k is the summation index.

Thus, matrix multiplication is defined only for matrices in which the number of columns in the first is equal to the number of rows of the second. From this, it follows that although the product AB exists, the product BA may not exist.

Now suppose A is of order (m x n) and B is of order (n x m). Then, in symbols, for AB we have (m x n)(n x m), so the resulting product is an (m x m) matrix. While for BA, we have (n x m)(m x n), the result is an (n x n) matrix. Hence AB $\neq$ BA, since the order of the two is not even the same.

In the case that both matrices are square of order (n x n), then AB exists and is of order (n x n). Likewise BA exists and is of order (n x n). Even in this case, however,

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the general rule is that AB $\neq$ BA, for

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

and

$$d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}.$$

For equality, $c_{ij} = d_{ij}$ for all i and j. For arbitrary elements, however, the above rule shows that this is not true. In special cases, of course, it could be true.

The general situation is summarized by saying that matrix multiplication is non-commutative, or AB $\neq$ BA. Thus, a distinction between pre-multiplication and post-multiplication must be made. For example, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then

$$AB = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

then

$$CD = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \quad DC = \begin{bmatrix} 1 & 5 \\ 1 & 0 \end{bmatrix},$$

and CD $\neq$ DC.

$$DE = \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

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while ED corresponds to (2 x 1) • (2 x 2); since there is one column in first and two rows in second, ED is undefined.

Although non-commutative, matrices can be shown to obey the associative and distributative laws of multiplication. That is

$$A(BC) = (AB)C,$$

and
$$A(B + C) = AB + AC,$$

noting that the order of the products is maintained.

We shall now consider matrices in which the elements are not ordinary numbers, but are themselves matrices. Such matrices are said to be partitioned.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 6 & 2 & 0 \end{bmatrix},$$

and draw horizontal and vertical lines between the rows and columns in any manner. For example, consider the partitioning, indicated by dotted lines,

$$A = \left[\begin{array}{c:cc} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 6 & 2 & 0 \end{array}\right]$$

and it is seen that the given matrix is divided into matrices.

Let $B = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$, and let $C = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ 2 & 0 \end{bmatrix}$, then A may be written

$$A = \begin{bmatrix} B & C \end{bmatrix},$$

a (1 x 2) matrix. Much use is made of this concept.

With certain restrictions, two partitioned matrices may be added or multiplied by the usual rules.

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For addition, it is necessary that the two original matrices be of the same order, and that the partition lines be drawn in a corresponding manner in each. Thus,

$$[A, B] + [C, D] = [A + C, B + D],$$

the commas being inserted for clarity. Obviously, A and C, and B and D must be of the same order.

For multiplication, the original two matrices must have a product, and in addition, the indicated products of the submatrices must exist. In the product

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix},$$

the first requirement is met, for considering submatrices as elements, we have (2 x 2)(2 x 1), a possible product. Carrying out the indicated multiplication gives

$$\begin{bmatrix} (AE + BF) \\ (CE + DF) \end{bmatrix},$$

a (2 x 1) matrix. Of course, for this to exist the individual products AE, BF, CE, and DF must exist. In terms of the original two matrices, this corresponds to the requirement that for every partitioning line between columns of the matrix on the left, there must be a partitioning line between the corresponding rows of the matrix on the right. As an example, consider

$$R = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 2 & 1 \\ 5 & 9 \end{bmatrix}$$

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Let the (3 x 3) matrix have a partitioning line between the second and third columns so that the (3 x 2) matrix on the left must have a partitioning line between the second and third rows.

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 \end{bmatrix}$

Then

$$R = \left[\begin{array}{cc:c} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{array}\right] \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \\ \hdashline 1 & 3 \end{array}\right] = \begin{bmatrix} A, & B \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$R = \begin{bmatrix} AC + BD \end{bmatrix}$, a (1 x 1) matrix in terms of submatrices. Note that since AC and BD are connected by an addition sign, (AC + BD) consists of but one element in terms of submatrices.

From the definitions of A, B, C, D, we have

$$AC = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}$$

$$BD = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 0 & 0 \\ 2 & 6 \end{bmatrix},$$

and, therefore, using matrix addition,

$$R = \begin{bmatrix} AC + BD \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 2 & 1 \\ 5 & 9 \end{bmatrix},$$

the same result as before.

Matrices may be partitioned in many ways, but when multiplication is contemplated, the horizontal lines on the right must correspond to the vertical lines on the left.

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<u>The Reciprocal Matrix</u>

If a determinate is formed from the elements of a square matrix, taken in the same order, the resulting determinate is a scalar number either zero or not zero. If zero, the parent matrix is said to be <u>singular</u>. If the determinate is not zero, the parent matrix is <u>non-singular</u>.

It can be shown that any non-singular square matrix has a reciprocal. The reciprocal is not defined for a non-square matrix.

If a square matrix is denoted by A, its reciprocal is denoted by A^{-1}, and is defined by the equation

$$AA^{-1} = I = A^{-1}A,$$

where I is the unit matrix, all of whose elements are zero except those along the principal diagonal where they are all one. For example, for the third order

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the equation

$$Ax = B,$$

where A, X, and B are matrices. Premultiply both sides of the equation by A^{-1}, which, by definition, gives

$$A^{-1}AX = A^{-1}B,$$

but $A^{-1}A = I$, and using the rules of matrix multiplication it is seen that IX = X. Hence, we have

$$X = A^{-1}B,$$

the solution of the matrix equation.

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The classical definition of the reciprocal matrix (not given here) is of little value in the actual computation of the reciprocal of a numerical matrix. That is, a matrix whose elements are number, rather than literal letters. For computational purposes, it is best thought of as a square matrix, which, when multiplied into a given matrix of the same order, produces the unit matrix.

Various methods for the computation of the reciprocal matrix exist and are discussed at length in the book by Frazer, Duncan, and Collar, "Elementary Matrices". All these require a fair amount of explanation, and will not be discussed, except to remark that in the gust problem many of the matrix elements are zero, thus making the task of inversion much easier than usual.

An Example in the Use of Matrices

Returning to Eq. (2), we may first write the set of equations in matrix form using the definition of equality of matrices, Thus,

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \qquad (3)$$

each a (2 x 1) matrix. Note that on the left, there is but one matrix element in each row. Next using the definition of matrix multiplication, (3) may be written

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

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where, on the left, we have a (2 x 2)(2 x 1) = (2 x 1) matrix. Thus, matrix multiplication and equality have been defined such that the coefficients appearing in (2) can be written in detached form. Then, using the concept of the reciprocal matrix,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

Carrying out the indicated multiplication on the right produces a (2 x 2) • (2 x 1) = (2 x 1) matrix, say

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad \text{so that} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Finally, again using matrix equality, we have

$$x_1 = p_1, \quad \text{and} \quad x_2 = p_2.$$

A Second Example in the Use of Matrices.

In this example we shall introduce and define the concepts of integration and differentiation of a matrix of functions, and show how they are used in reducing a system of differential equations to matrix form. For brevity, a system of two equations will be used.

Consider the system of equations,

$$\begin{aligned} a_{11}y_1' + a_{12}y_2' + b_{11}y_1 + b_{12}y_2 &= f_1(s) \\ a_{21}y_1' + a_{22}y_2' + b_{21}y_1 + b_{22}y_2 &= f_2(s) \end{aligned} \qquad (4)$$

where a_{ij}, b_{ij} are constants and $y' = \frac{dy}{ds}$.
Let the initial conditions be $y_1 = y(0)$, $y_2 = y_2(0)$ at $s = 0$. Integrate both equations from $s = 0$ to $s = s$, using the initial conditions to give

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$$a_{11}y_1 - a_{11}y_1(0) + a_{12}y_2 - a_{12}y_2(0) + b_{11}\int_0^s y_1 ds + b_{12}\int_0^s y_2 ds = \int_0^s f_1 ds$$

$$(5)$$

$$a_{21}y_1 - a_{21}y_1(0) + a_{22}y_2 - a_{22}y_2(0) + b_{21}\int_0^s y_1 ds + b_{22}\int_0^s y_2 ds = \int_0^s f_1 ds$$

Now let us write (4) in matrix form using the properties of matrix equality and matrix multiplication, as already explained, to give

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \qquad (6)$$

Let the matrix of a's be denoted by A, the matrix of b's by B, and let

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \qquad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

We now define the derivative of a matrix to be equal to the matrix of the derivatives, so that

$$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = (Y)' = Y'.$$

Hence, (6), and therefore (4), becomes

$$AY' + BY = F \qquad (7)$$

Now integrate (7) formally as if all the quantities were ordinary scalars, constants for A and B and variables for Y', Y, and F, to give

$$AY - AY(0) + B\int_0^s Y\, ds = \int_0^s F\, ds \qquad (8)$$

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We now define the integral of a matrix to be equal to the matrix of the integrals, so that by definition,

$$\int_0^s Y\,ds = \int_0^s \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} ds = \begin{bmatrix} \int_0^s y_1 ds \\ \int_0^s y_2 ds \end{bmatrix}$$

Write (8) in extended matrix form,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \int_0^s \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} ds = \int_0^s \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} ds, \tag{10}$$

which, upon using the definition (9), gives

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \int_0^s y_1 ds \\ \int_0^s y_2 ds \end{bmatrix} = \begin{bmatrix} \int_0^s f_1 ds \\ \int_0^s f_2 ds \end{bmatrix}$$

Carrying out the indicated matrix multiplication, and using the definition of equality of matrices, we see that (11) is exactly equivalent to (5).

Thus, using the definitions of matrix multiplication and equality, and the definitions of integration and differentiation of a matrix, we see that we may treat the matrix equation (7) formally as if the matrices were scalars, and arrive at the correct result. This is of far reaching importance in the numerical solution of high order linear systems.

C.2.3. Method I for the Solution of the First Order System

Returning now to Eq. (1) of section C.2.1., we shall derive a numerical method of solution that depends upon

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eliminating the integral term containing the independent variable s.

For convenience, (1) is repeated

$$Y' + aY + b \int_0^s e^{-\alpha(s-\sigma)} Y(\sigma)d\sigma = g(s). \qquad (1)$$

Differentiate (1) with respect to s, and divide the result by α. Thus,

$$\frac{Y''}{\alpha} + \frac{a}{\alpha} Y' - b \int_0^s e^{-\alpha(s-\sigma)} Y(\sigma)d\sigma + \frac{b}{\alpha} Y = \frac{g'(s)}{\alpha}, \qquad (12)$$

and upon adding (12) and (12), we have

$$y'' + (a + \alpha I)Y' + (\alpha a + b)Y = \alpha g(s) + g'(s), \qquad (13)$$

where I is the unit matrix of order (n x n).

We now reduce (13) to a first order system by the introduction of the new dependent variable defined by $u = Y'$, so that $u' = Y''$, and the new system may be written

$$\begin{aligned} u' + Au + BY &= G(s) \\ Y' - u &= 0, \end{aligned} \qquad (14)$$

where $A = (a + \alpha I)$, $B = (\alpha a + b)$, $G(s) = \alpha g(s) + g'(s)$. Write this system in matrix form, using partitioned matrices,

$$\begin{bmatrix} I & O \\ O & I \end{bmatrix} \begin{bmatrix} u' \\ Y' \end{bmatrix} + \begin{bmatrix} A & B \\ -I & O \end{bmatrix} \begin{bmatrix} u \\ Y \end{bmatrix} = \begin{bmatrix} G(s) \\ O \end{bmatrix}, \qquad (15)$$

and introduce single letter notations for the partitioned matrices in (15). Thus,

$$w = \begin{bmatrix} u \\ Y \end{bmatrix}, \quad f = \begin{bmatrix} G \\ O \end{bmatrix}, \quad c = \begin{bmatrix} A & B \\ -I & O \end{bmatrix},$$

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so that $\begin{bmatrix} u' \\ Y' \end{bmatrix} = \begin{bmatrix} u \\ Y \end{bmatrix}' = w'$, and (15) may be written

$$w' + cw = f. \tag{16}$$

We wish to determine w for equally spaced values of the independent variable s. This will be done by applying the idea of numerical integration.

Let w_n be the value of w at $s = s_n$. Then, integrating (16) gives

$$w - w_n + c \int_{s_n}^{s} w \, ds = \int_{s_n}^{s} f \, ds. \tag{17}$$

The integral on the right contains known functions of a and may be integrated. Calling this result h(s), and applying Simpson's rule to the integral on the left, taking the upper limit $s = s_{n+2} = s_n + 2K$, yields the recurrence relation $w_{n+2} - w_n + \frac{K}{3} c(w_n + 4w_{n+1} + w_{n+2}) = h(s_{n+2}) - h(s_n)$.

(18)

Collecting like terms,

$$(\tfrac{K}{3} c + I)\, w_{n+2} + \tfrac{4K}{3} c\, w_{n+1} + (\tfrac{K}{3} c - I)\, w_n = h(s_{n+2}) - h(s_n), \tag{19}$$

which may be written

$$w_{n+2} = pd_n + qw_{n+1} + rw_n, \quad (n = 0, 1, 2, \cdots\cdots). \tag{20}$$

where $d_n = h(s_{n+2}) - h(s_n)$, and p, q, and r are square matrices obtained by multiplying, in (19), the coefficients of d_n, w_{n+1}, and w_n by the reciprocal matrix

$$(\tfrac{K}{3} c + I)^{-1}.$$

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Thus, if w_0 and w_1 are known, w_2 can be found by taking n = 0; Next, taking n = 1, w_3 can be found by using w_1 and w_2 just found. Continuing in this way, a set of values, w_2, w_3, w_4, ..., w_n, ... can be found from the recurrence relation (20).

The value w_0 is obtained from the initial conditions of the problem. However, w_1, as is usual in most numerical methods, requires special attention. Since w_{-1} is not known, the recurrence is needed. Returning to (16) and integrating from $s = 0$ to $s = \frac{s_1}{2} = \frac{K}{2}$, we have

$$w_{\frac{1}{2}} - w_0 + c\int_0^{K/2} w\,ds = \int_0^{K/2} f\,ds \qquad (21)$$

By methods given in Milne's, "Numerical Calculus", Princeton University Press, various integration formula can be constructed. For the purpose in hand, we shall use the approximation

$$\int_0^{K/2} w\,ds = \frac{K}{12}\left[6w_0 + \frac{3K}{2}w_0' + \frac{K^2}{4}w_0''\right], \qquad (22)$$

where the primes indicate the values of the first and second derivatives of w with respect to s, evaluated at s = 0, and $w_{\frac{1}{2}}$ is the value of w at $s = s_{\frac{1}{2}} = \frac{K}{2}$, the middle point of the first standard interval. The Eq. (22) when substituted in (21) permits the calculation of $w_{\frac{1}{2}}$. The values w_0' and w_0'' are

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obtained by differentiation of (16), followed by evaluation at $s = 0$. Thus,

$$w_{\frac{1}{2}} - w_0 + c\,\frac{K}{12}\left[6w_0 + \frac{3K}{2}\,w_0' + \frac{K^2}{4}\,w_0''\right] = \int_0^{K/2} f(s)ds, \qquad (23)$$

from which $w_{\frac{1}{2}}$ is determined.

Eq. (16) is now integrated from $s = 0$ to $s = K$, making use of the value $w_{\frac{1}{2}}$ just computed, with the aid of the formula

$$\int_0^K w\,ds = \frac{K}{12}\left[-12w_0 + 24w_{\frac{1}{2}} - 6Kw_0' - K^2w_0''\right], \qquad (24)$$

we give

$$w_1 - w_0 + c\,\frac{K}{12}\left[-12w_0 + 24w_{\frac{1}{2}} - 6Kw_0' - K^2w_0''\right] = \int_0^K f(s)ds, \qquad (25)$$

from which w_1 is determined.

Knowing w_0 and w_1 the solution may be continued by the recurrence relation (20).

It would seem that an equation like (23) could be applied to obtain w_1 directly without the intermediate step (25). However, in order to keep the accuracy of (23) comparable to Simpson's rule, it is necessary to use only half an interval to start the solution, and when that is done, the starting formulas (23) and (25) are, in fact, more accurate than Simpson's rule.

In any step-by-step numerical solution, small errors tend to accumulate so that it is necessary to take the interval K small enough to insure the correctness of a specified decimal

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place in the result, for the range of independent variable being considered.

Although the recurrence relation (20) has been based on Simpson's rule, it is possible to use a recurrence relation based on integration formulas of much greater accuracy; however, increasing the number of terms leads to longer computations. It is felt that (20) represents a good compromise. For checking purposes, however, it is well to have a more accurate formula. Instead of integrating (16) by Simpson's rule, a three ordinate rule, we shall indicate the integration by a seven ordinate rule. Thus,

$$w_{n+6} - w_n + c \cdot \frac{6K}{840}\Big[41w_n + 216w_{n+1} + 27w_{n+2} + 272w_{n+3} + + 27w_{n+4} + 216w_{n+5} + 41w_{n+6}\Big] = \int_{s_n}^{s_{n+6}} f\, ds \qquad (26)$$

In order to use (26), take seven adjacent values, already computed, and substitute them into the left member. This should equal the integral on the right, except for the allowable error in a specified decimal place. If equality is not attained (in the sense used above), the interval K should be chosen smaller.

It will be recalled that the independent variable s was defined by the equation

$$s = \frac{2Ut}{c_o},$$

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for the gust problem, where U is the forward speed of the airplane in feet per second, t is the time in seconds, and c_o is a reference chord length in feet. Evidently the higher the frequency of wing vibration, the smaller the increment K need be chosen in order to follow the wing movement. In terms of increments, we have

$$K = \Delta s = \frac{2U}{c_o} \Delta t.$$

It is felt that if Δt is chosen to be between $\frac{1}{20}$ and $\frac{1}{30}$ of the period of the fundamental mode of the wing, the corresponding $\Delta s = K$ will be satisfactory for obtaining the first seven values. These should be checked by Eq. (26), decreasing K if necessary, and increasing K, if possible. That is, if the check is satisfactory, the interval could possibly be lengthened.

C.2.4. Method II for the Solution of the First Order System. In this section we shall consider a method for solving Eq. (1) without first differentiating, thus raising the order.

Multiply (1) through by $e^{\alpha s}$ and integrate from s_n to s to give

$$\int_{s_n}^{s} e^{\alpha\sigma} Y'(\sigma)\,d\sigma + a\int_{s_n}^{s} e^{\alpha\sigma} Y(\sigma)\,d\sigma + b\int_{s_n}^{s}\int_{0}^{r} e^{\alpha\sigma} Y(\sigma)\,d\sigma dr = \int_{s_n}^{s} e^{\alpha s} g(s) \tag{27}$$

Upon integrating the first integral by parts and collecting like terms, we have

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$$e^{\alpha s} Y(s) - e^{\alpha s_n} Y(s_n) + (a - I\alpha) \int_{s_n}^{s} e^{\alpha\sigma} Y(\sigma) d\sigma +$$

$$+ b \int_{s_n}^{s} \int_{0}^{r} e^{\alpha\sigma} Y(\sigma) d\sigma\, dr = \int_{s_n}^{s} e^{\alpha\sigma} g(\sigma) d\sigma, \tag{28}$$

the function $e^{\alpha s}\, Y(s)$ appearing in every term on the left.

Introduce the new variable defined by

$$X(s) = e^{\alpha s}\, Y(s), \tag{29}$$

which, $e^{\alpha s}$ being a scalar, is a matrix of the same order as $Y(s)$.

Before a recurrence relation can be obtained from (28), it is necessary to eliminate the lower limit 0 (zero) in the double integral. To do this, we consider the integral

$$I(n,\ n + 2) = \int_{s}^{s_{n+2}} \int_{0}^{r} X(\sigma) d\sigma\, dr.$$

Let $\int X(\sigma) d\sigma = \varphi(\sigma)$, so that

$$X(\sigma) = \varphi'(\sigma), \quad \text{and} \quad \int_{0}^{r} X(\sigma) d\sigma = \varphi(r) - \varphi(0). \tag{30}$$

Then,

$$I(n,\ n + 2) = \int_{s_n}^{s_{n+2}} [\varphi(r) - \varphi(0)] dr = \int_{s_n}^{s_{n+2}} r' [\varphi(r) - \varphi(0)] dr$$

$$= r [\varphi(r) - \varphi(0)] \Big|_{s_n}^{s_{n+2}} - \int_{s_n}^{s_{n+2}} r\, \varphi'(r) dr$$

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$$= s_{n+2}\,[\varphi(s_{n+2}) - \varphi(0)] - s_n[\varphi(s_n) - \varphi(0)] - \int_{s_n}^{s_{n+2}} r\,X(r)\,dr$$

$$= s_{n+2}\int_0^{s_{n+2}} X\,dr - s_n\int_0^{s_n} X\,dr - \int_{s_n}^{s_{n+2}} r\,X\,dr, \qquad (31)$$

where Eq. (30) has been used in the last line above.

Now $s_{n+2} = 2K + s_n$, so that (31) may be written

$$I(n,\ n+2) = s_n\int_{s_n}^{s_{n+2}} X\,dr + 2K\int_0^{s_n} X\,dr + 2K\int_{s_n}^{s_{n+2}} X\,dr - \int_{s_n}^{s_{n+2}} r\,X\,dr \qquad (32)$$

Now define $I(n+2,\ n+4) = \int_{s_{n+2}}^{s_{n+4}}\int_0^{r} X(\sigma)\,d\sigma,$

and following the same steps as before gives

$$I(n+2,\ n+4) = s_{n+2}\int_{s_{n+2}}^{s_{n+4}} X\,dr + 2K\int_0^{s_{n+2}} X\,dr + 2K\int_{s_n}^{s_{n+2}} X\,dr - \int_{s_n}^{s_{n+2}} r\,X\,dr =$$

$$= s_{n+2}\int_{s_{n+2}}^{s_{n+4}} X\,dr + 2K\int_0^{s_n} X\,dr + 2K\int_{s_n}^{s_{n+2}} X\,dr + 2K\int_{s_n}^{s_{n+2}} X\,dr - \int_{s_n}^{s_{n+2}} r\,X\,dr. \qquad (33)$$

From this,

$I(n+2,\ n+4) - I(n,\ n+2) =$

$$= s_{n+2}\int_{s_{n+2}}^{s_{n+4}} X\,dr - s_n\int_{s_n}^{s_{n+2}} X\,dr + 2K\int_{s_{n+2}}^{s_{n+4}} X\,dr - \int_{s_{n+2}}^{s_{n+4}} rX\,dr + \int_{s_n}^{s_{n+2}} r\,X\,dr,$$

or combining terms

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$$I(n+2,\ n+4) - I(n,\ n+2) = \int_{s_{n+2}}^{s_{n+4}} (s_{n+4} - r)X\,dr - \int_{s_n}^{s_{n+2}} (s_n - r)X\,dr.$$

Now evaluate (28) for $s = s_{n+2}$; evaluate the same equation again, but for $s = s_{n+4}$ using s_{n+2} as lower limit, and subtract, with the aid of (34), to give

$$X_{n+4} - 2X_{n+2} + X_n + (a - I\alpha)\left[\int_{s_{n+2}}^{s_{n+4}} X\,d\theta - \int_{s_n}^{s_{n+2}} X\,d\theta\right] +$$

$$+ b\left[\int_{s_{n+2}}^{s_{n+4}} (s_{n+4} - \sigma)X\,d\sigma - \int_{s_n}^{s_{n+2}} (s_n - \sigma)X\,d\sigma =\right.$$

$$= \int_{s_{n+2}}^{s_{n+4}} e^{\alpha\theta}\,g(\theta)d\theta - \int_{s_n}^{s_{n+2}} e^{\alpha\theta}\,g(\theta)d\theta \qquad (35)$$

Evaluating the integrals by Simpson's rule, and collecting terms gives the recurrence relation,

$$A_4X_{n+4} + A_3X_{n+3} + A_2X_{n+2} + A_1X_{n+1} + A_oX_o = H(s_n,\ s_{n+4}),$$

where the A's are constant matrices given by

$$A_4 = I + \frac{K}{3}(a - I\alpha); \qquad A_3 = \frac{4K}{3}(a - I\alpha) + \frac{4K^2b}{3}$$

$$A_2 = \frac{4K^2b}{3} - 2I; \qquad A_1 = \frac{4K^2b}{3} - \frac{4K}{3}(a - I\alpha)$$

$$A_o = I - \frac{K}{3}(a - I\alpha), \quad \text{and}$$

$$H(s_n,\ s_{n+4}) = \int_{s_{n+2}}^{s_{n+4}} e^{\alpha\sigma}\,g(\sigma) - \int_{s_n}^{s_{n+2}} e^{\alpha\sigma}\,g(\sigma)d\sigma.$$

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To obtain the final form of (36), the reciprocal A_4^{-1} is found, and then pre-multiplied through the entire equation.

To use (34) for n = 0 to find X_4, the four previous values X_3, X_2, X_1, X_0 are needed.
X_0 is given by the initial conditions, and X_1, X_2, and X_3 are found by applying various integration formula to (28), with the aid of equations of the type (32).

In (28) choose $s_n = 0$, using (29), to give

$$X - X_0 + (a - I\alpha)\int_0^s X(\sigma)d\sigma + b\int_0^s\int_0^r X(\sigma)d\sigma\, dr = \int_0^s e^{\alpha\sigma}g(\sigma)d\sigma. \tag{37}$$

Using (32) with $s_n = 0$, s_{n+2} set equal to s, we obtain

$$\int_0^s\int_0^r X(\sigma)d\sigma\, dr = \int_0^s (s - \sigma)\ X(\sigma)d\sigma, \tag{38}$$

which when substituted in (37) gives

$$X - X_0 + (a - I\alpha)\int_0^s X(\sigma)d\sigma + b\int_0^s (s - \sigma)\ X(\sigma)d\sigma = \int_0^s e^{\alpha\sigma}g(\sigma)d\sigma \tag{39}$$

We first obtain an intermediate value, $X_{\frac{1}{2}} = X(\frac{K}{2})$ by using (22). Thus, with $s = s_{\frac{1}{2}} = K/2$,

$$X_{\frac{1}{2}} - X_0 + (a - I\alpha)\frac{K}{12}\left[6X_0 + \frac{3K}{2}X_0' + \frac{K^2}{4}X_0''\right] +$$

$$+ \frac{bK}{12}\left[6(\frac{K}{2}X_0) + \frac{3K}{2}(\frac{K}{2}X_0' - X_0) + \frac{K^2}{4}(\frac{K}{2}X_0'' - 2X_0')\right] = \int_0^{K/2} e^{\alpha\sigma}g(\sigma)d\sigma, \tag{40}$$

where X_0' and X_0'' are found by successive differentiation of (39), evaluated at s = 0, or from the original Eq. (1).

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The value X_1 is found from (39) with the aid of (24). Thus,

$$X_1 - X_0 + (a - I\alpha)\frac{K}{12}\left[-12X_0 + 24X_{\frac{1}{2}} - 6Kx_0' - K^2x_0''\right] +$$

$$+ b\frac{K}{12}\left[-12KX_0 + 24\frac{K}{2}X_{\frac{1}{2}} - 6K(KX_0' - X_0) - K^2(KX_0'' - 2X_0')\right] =$$

$$= \int_0^K e^{\alpha\sigma}g(\sigma)d\sigma. \qquad (41)$$

Having X_1 from (39), X_2 is found by applying Simpson's rule to (39) to give

$$X_2 - X_0 + (a - I\alpha)\frac{K}{3}\left[X_0 + 4X_1 + X_2\right] + \frac{bK}{3}\left[2KX_0 + 4KX_1\right] =$$

$$= \int_0^{2K} e^{\alpha\sigma}g(\sigma)d\sigma \qquad (42)$$

In solving (42) for X_2, it is noted that the coefficient of X_2 is $\left[I + (a - I\alpha)\frac{K}{3}\right]$. The reciprocal of this matrix must be found. It, however, is needed later in the recurrence relation (36), for it is A_4^{-1} there defined.

X_3 is next found from (39) by integrating from 0 to $s = s_3 = 3K$. It would, at first, appear logical to do this by the common $\frac{3}{8}$ rule of Simpson. For reference, this rule is

$$\int_0^{3K} p(x)dx = \frac{3K}{8}(p_0 + 3p_1 + 3p_2 + p_3).$$

If (39) is integrated by this rule, it will be seen upon inspection that, in order to solve for X_3, it is necessary

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to invert the matrix $\left[I + (a - I\alpha)\frac{3K}{8}\right]$, the coefficient of X_3. This is a new matrix, and its reciprocal is needed only once. Hence, it would be better to avoid the use of a formula requiring this. Instead, we use an open end formula, so that X_3 does not appear in the evaluation of the integral. A formula whose accuracy is of the same order as those already used is

$$\int_0^{3K} p(x)dx = \frac{K}{16}\left[39p_0 - 36p_1 + 45p_2 + 18hp_0'\right], \tag{43}$$

where $p_0' = \frac{dp}{dx}$ evaluated at $x = 0$, and it is seen that p_3 does not appear in (43). Applying (43) to (39) gives

$$X_3 - X_0 + (a - I\alpha)\frac{K}{16}\left[39X_0 - 36X_1 + 45X_2 + 18KX_0'\right] +$$

$$+ b\frac{K}{16}\left[39(3KX_0) - 36(2KX_1) + 45(KX_2) + 18K(3KX_0' - X_0)\right] =$$

$$= \int_0^{3K} e^{\alpha\sigma} g(\sigma)d\sigma; \tag{44}$$

from which X_3 can be determined without the necessity of forming the reciprocal matrix. This concludes the calculation of the individual values X_i from separate formulas. From here onward, X_4, X_5, •••, the recurrence relation (36) can be used. After the matrices X_i are computed, the Y_i matrices can be determined from the equation (29). Thus,

$$Y_i = X_i e^{-\alpha s_i} \; (i = 1, 2, \cdots\cdot). \tag{45}$$

In method II, we have considered a solution of the original problem which involves the use of smaller matrices;

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however, less information is attained than in method I using the w matrix. Further, the recurrence relation depends upon four previous values, while (20) depends only upon two.

Method II has been given for reference only, as the idea behind its derivation may be useful; however, it is the writer's opinion that method I is superior for computational purposes. It has already been pointed out that the matrices, although of high order, contain many zeros and ones, a fact which leads itself to easy inversion and subsequent computation.

C.2.5. A Method for the Solution of the Second Order System

Eq. (1a), repeated here for reference,

$$Z'' + JZ' + MZ + N \int_0^s e^{-\alpha(s-\sigma)} Z(\sigma)\,d\sigma = r(s) \qquad (1a)$$

is the equation that arises naturally in the gust problem rather than the first order equation (1). Throughout part C of this report, the equations were considered only from the point of view of solving them, rather than their physical interpretation. The present section will continue this plan, the physical interpretation being discussed in a separate section.

To solve Eq. (1a) we may either first differentiate as in C.2.3., or we may first integrate as in C.2.4. It will be recalled that the former method lead to higher order matrices, but that the resulting recurrence relation was shorter and easier to apply. The corresponding situation is also true with

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Eq. (1a). Only the method of first differentiating (1a) will be discussed in this report, as it is felt that this method leads to an easier and quicker solution. The interested reader will have no difficulty in formulating the other method by applying the same ideas as those in section C.2.4.

Upon differentiating (1a) with respect to s, we have

$$Z''' + JZ'' + MZ' - N\alpha \int_0^s e^{-\alpha(s-\sigma)} Z(\sigma)d\sigma + NZ = r'(s), \qquad (46)$$

which, when divided by the scalar α and added to (1a), gives

$$Z''' + (J + I\alpha)Z'' + (M + J\alpha)Z' + (N + M\alpha)Z = ar + r'(s). \qquad (47)$$

This may be reduced to a first order system by the introduction of the new variables defined by

$$Z' = v, \qquad v' = p, \qquad (48)$$

so that $p = Z''$.

Combining (48) and (47) the system is

$$\begin{aligned} p' + (J + I\alpha)p + (M + J\alpha)v + (N + M\alpha)Z &= R \\ v' - p &= 0 \\ Z' - v &= 0. \end{aligned} \qquad (49)$$

In matrix form the system becomes, using partitioned matrices,

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} p' \\ v' \\ Z' \end{bmatrix} + \begin{bmatrix} (J+I\alpha) & (M+J\alpha) & (N+M\alpha) \\ -I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} p \\ v \\ Z \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \qquad (50)$$

Define the new matrices

$$u = \begin{bmatrix} p \\ v \\ Z \end{bmatrix}, \qquad F = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}, \text{ and let} \qquad (51)$$

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E be the square matrix that premultiplies u in (50). Since, by definition, the derivative of the matrix is the matrix of the derivatives, (50) may be written as the first order system,

$$u' + Eu = F, \tag{52}$$

which, in form, is identical with (16), and may be solved by the same formulas with only a change in notation.

C.2.6. Physical Interpretation of the Solutions and Additional Comments.

It remains to compare the two methods of elimination of the integral term by differentiation as illustrated in sections C.2.5 and C.2.3.

As mentioned before, (1) is obtained from (1a), the difference being the elimination of the integral term either before or after the reduction to a first order system. In the case discussed in section C.2.3, it may be seen that the order of the matrices involved is four times the number of degrees of freedom in the system. Physically, the matrices contain the displacement, the velocity twice, and the acceleration. The method of section C.2.5, on the other hand, leads to a matrix of order three times the number of degrees of freedom. This corresponds to matrix elements for the displacement, the velocity, and the acceleration. The former method has the advantage that it leads to square matrices of order 2m.

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This symmetry is of some slight advantage and provides for a check on the velocity terms.

The integration was carried out by means of Simpson's rule. This, as was shown required the inversion of a matrix. It is possible to avoid this by using open integration formula. These, of course, are less accurate for the same number of ordinates considered, and it is felt that this disadvantage is great enough to avoid their use in the recurrence relations.

C.2.7. A Worked Example

The ideas discussed in this section will now be applied to a sample problem. A simple example is chosen for easy reference; however, all steps necessary for the solution of the gust equations are contained in this problem.

The problem to be solved is

$$x_1'' + x_2 = 1$$

$$x_2'' + x_1 = 0$$

together with the initial conditions, $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$.

In matrix form we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Define the matrix $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and use the fact that the derivative of a matrix is the matrix of the derivatives, to obtain

$$X'' + AX = B,$$

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where $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

To reduce this to a first order system we let

$$Y = X'$$

so that the second order system becomes

$$Y' + AX = B$$

$$X' - Y = 0$$

Again, define a new matrix $W = \begin{bmatrix} Y \\ X \end{bmatrix}$, and we have

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Y' \\ X' \end{bmatrix} + \begin{bmatrix} 0 & A \\ -I & 0 \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

or

$$W' + CW = D,$$

the standard form for the first order system.

When written out in extended form we have

$$W = \begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} X' \\ X \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_1 \\ x_2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & A \\ -I & 0 \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hdashline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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Integrating the standard equation from the origin gives

$$W - W_o + C \int_0^s W \, ds = Ds,$$

and by use of the initial conditions, $W_o = 0$.

To start the solution, integration is carried out to the mid point of the first interval. Thus, with $s = \frac{K}{2}$

$$W_{\frac{1}{2}} + C \frac{K}{12} \left[6W_o + \frac{3K}{2} W_o' + \frac{K^2}{4} W_o'' \right] = D \frac{K}{2}$$

Now from the matrix equation,

$$W' = D - CW,$$

so that,

$$W'' = -CW'.$$

From this,

$$W_o' = D - CW_o = D \qquad (W_o = 0)$$

and

$$W_o'' = -CW_o' = -CD.$$

Thus,

$$W_{\frac{1}{2}} = D \frac{K}{2} - C \frac{K}{12} \left[\frac{3K}{2} D - \frac{K^2}{4} CD \right]$$

with K = 0.1, we have

$$W_{\frac{1}{2}} = \begin{bmatrix} x_1' \\ x_2' \\ x_1 \\ x_2 \end{bmatrix}_{\frac{1}{2}} = \begin{bmatrix} .05 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & (.00833) \\ 0 & 0 & (.00833) & 0 \\ (-.00833) & 0 & 0 & 0 \\ 0 & (-.00833) & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} .15 \\ 0 \\ 0.0025 \\ 0 \end{bmatrix} = \begin{bmatrix} .05 \\ -.00002 \\ .00120 \\ 0 \end{bmatrix}$$

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We next compute w_1. Using the formula that was discussed in the previous section, we have

$$W_1 + \frac{CK}{12}\left[-12W_0 + 24W_{\frac{1}{2}} - 6KW_0' - K^2W_0''\right] = DK,$$

or

$$W_1 = DK + \frac{CK}{12}\left[-24W_{\frac{1}{2}} + 6KD - K^2CD\right]$$

$$W_1 = \begin{bmatrix} x_1' \\ x_2' \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -.00015 \\ .005 \\ 0 \end{bmatrix} = \begin{bmatrix} .1 \\ -.00015 \\ .005 \\ 0 \end{bmatrix}$$

where the previous expression for $\frac{CK}{12}$ has been used.

Now having W_1 (and W_0) the recurrence relation derived from Simpson's rule may be used.

Integrating the first order system and applying Simpson's rule, we obtain

$$W_{n+2} - W_n + \frac{CK}{3}\left[W_n + 4W_{n+1} + W_{n+2}\right] = D\,2K$$

or

$$W_{n+2} = \left(\frac{CK}{3} + I\right)^{-1} D2K - \left(\frac{CK}{3} + I\right)^{-1} \frac{4CK}{3} W_{n-1} - \left(\frac{CK}{3} + I\right)^{-1} \left(\frac{CK}{3} - I\right)W_n$$

using K = .1, and computing the matrices, we obtain

$$W_{n+2} = \begin{bmatrix} .2 \\ -.00022 \\ .00666 \\ -.00001 \end{bmatrix} - P\begin{bmatrix} x_1' \\ x_2' \\ x_1 \\ x_2 \end{bmatrix}_{n-1} - S\begin{bmatrix} x_1' \\ x_2' \\ x_1 \\ x_2 \end{bmatrix}_n$$

where

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$$P = \begin{bmatrix} 0 & .00444 & -.00015 & .13333 \\ .00444 & 0 & .13333 & -.00015 \\ -.13333 & .00015 & 0 & .00444 \\ .00015 & -.13333 & .00444 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & .00222 & -.00007 & .06666 \\ .00222 & -1 & .06666 & -.00007 \\ -.06666 & .00007 & -1 & .00222 \\ .00007 & -.06666 & .00222 & -1 \end{bmatrix}$$

It is to be noted that in applying the recurrence relation, the P and the S matrices are constants. Hence, the only labor of successive computation is multiplying a column matrix by a square matrix, an operation which may be done very rapidly. Setting $n = 0, 1, 2, \cdots$, we obtain W_2, W_3, W_4, $\cdots$.

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D. Summary

In section A the rigid body case is briefly considered, and it is shown that good results can be obtained by taking Wagner's deficiency function equal to a constant.

Following this idea in section B, the equations of motion are developed for the case of wing bending and vertical translation for a constant deficiency function. The equations of motion are derived by considering the system to be discrete masses having elastic connections. No appeal is made to the use of normal modes.

The more general case of wing bending, wing torsion, and rigid body pitching and translation is next considered. The dynamic model is again a system of connected masses, and general recurrence relations are developed which permit the division of the wing into any convenient number of stations. The equations in this part express the deficiency function as an exponential as is the usual representation. The resulting equations then become integro-differential equations. These are expressed in matrix form and eventually reduced to a first order system.

In section C a detailed study is made of the matrix equations, and it is shown how to obtain the solution in several different ways, all being variations of numerical integration methods.

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While this report was being written there appeared John C. Houbolt's NACA TN 2060, "A Recurrence Matrix Solution for the Dynamic Response of Aircraft in Gusts". Although there are some variations in derivation, a matrix presentation is used not too different from the formulation in the present text. The method of solution is, however, quite different, being based on polynomial representation of the derivatives.

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