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I hereby recommend that the thesis prepared under my supervision by Kenneth M. McMillin

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ABEL SUMMABILITY OF THE CONJUGATE OF THE DOUBLE FOURIER SERIES

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by

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(1.) Let us first consider a single trigonometric series [1]*

(1.1) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The series called conjugate to (1.1) is the series

(1.2) $$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx).$$

The series (1.1) and (1.2) are respectively the real and imaginary parts of a power series

(1.21) $$\sum_{n=0}^{\infty} c_n e^{inx}, \text{ where } c_0 = \frac{a_0}{2}, \text{ and } c_n = a_n - ib_n \text{ if } n > 0.$$

The series (1.1) is the Fourier series of a function $f(x)$, if a_n and b_n are of the form

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt.$$

The summability (A) of the conjugate of the single Fourier series (1.2) has been discussed by Fatou [2], Lichtenstein [3], Plessner [4], Prasad [5], and Misra [6]. If for $0 \leq r < 1$, $V(r,x) = -\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) r^n$

and $\epsilon =$ arc $\sin(1-r)$, then Misra proved that

*Throughout this paper, numbers in brackets refer to the bibliography at the end of the paper.

$$\lim_{\epsilon = \text{arc}\,\sin(1-r)\to 0}\left[V(r,x) - \frac{1}{\pi}\int_{\epsilon}^{\pi} \chi_n(t)\,\frac{\cot\frac{t}{2}}{2}\right]dt = 0,$$

provided that

$$\chi_{n+1}(t) = \frac{1}{\sin t}\int_0^t \chi_n(t)\,dt = o(1) \text{ as } t \to 0,$$

where

$$\chi(t) = f(x+t) - f(x-t), \quad \chi_1(t) = \frac{1}{\sin t}\int_0^t \chi(t)\,dt,$$

and generally

$$\chi_{n+1}(t) = \frac{1}{\sin t}\int_0^t \chi_n(t)\,dt,$$

the integrals being Cauchy integrals.

Misra [6] has further shown that a necessary and sufficient condition for the summability (A) of the conjugate series (1.2) is that the integral

$$-\frac{1}{\pi}\int_0^{\pi} \chi(t)\,\frac{\cot\frac{t}{2}}{2}\,dt$$

exist (C,n) according to the definition of Hardy and Littlewood [7, pg. 214], which integral will then represent the sum of the series,

provided only that

$$\chi_{n+1}(t) = o(1) \text{ as } t \to 0 .$$

Plessner's and Prasad's theorems are respectively the cases $n=0, 1$ of Misra's theorem, and Misra's theorem is the most general of the results so far obtained in this direction for the single series.

This paper concerns itself with the analogous problem for the double series. Therefore, let us now turn to a consideration of the double Fourier series [8]. Let the double Fourier series associated with a function $f(x,y)$ which is integrable (L) in the cell $Q(-\pi,-\pi;\pi,\pi)$ and defined outside of Q by periodicity be

$$(1.3)\qquad \sum_{\substack{m=1\\n=1}}^{\infty\ \infty} \left(\frac{1}{2^{E\left(\frac{1}{m}\right)+E\left(\frac{1}{n}\right)}}\right)\left(\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(U,V)\ P_{mn}(x,y;U,V)\ dU\ dV\right),$$

where $E(Z)$ is used as in the theory of numbers to represent the largest integer contained in Z, and $P_{mn}(x,y;U,V) \equiv \left(\cos\left[(m-1)(U-x)\right]\right)\left(\cos\left[(n-1)(V-y)\right]\right)$.

The conjugate of the double Fourier series is

$$(1.4)\qquad \sum_{\substack{m=1\\n=1}}^{\infty\ \infty} \left(\frac{1}{2^{E\left(\frac{1}{m}\right)+E\left(\frac{1}{n}\right)}}\right)\left(\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(U,V)\ Q_{mn}(x,y;U,V)\ dU\ dV\right)$$

$$= \sum_{\substack{m=2 \\ n=2}}^{\infty} \left(\frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(U,V)\ Q_{mn}(x,y;U,V)\ dU\ dV \right),$$

where $Q_{mn}(x,y;U,V) = \sin\left[(m-1)(U-x)\right] \sin\left[(n-1)(V-y)\right]$.

Let us introduce convergence factors r^m, ρ^n in (1.4), and thereby define the Abel sum $A(x,y;r,\rho)$ as

$$(1.5)\quad A(x,y;r,\rho) = \lim_{\substack{m \\ n}\}\to\infty} \left\{ \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(U,V) \left[\sum_{j=1}^{m} r^j \sin j(U-x)\right] \left[\sum_{k=1}^{n} \rho^k \sin k(V-y)\right] dU\ dV \right\}.$$

$$= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(U,V) \left[\sum_{j=1}^{\infty} r^j \sin j(U-x)\right] \left[\sum_{k=1}^{\infty} \rho^k \sin k(V-y)\right] dU\ dV,$$

where we have interchanged summation and integration. However, this is permissible since for $\left\{\begin{matrix} r < 1 \\ \rho < 1 \end{matrix}\right.$ we have uniform and absolute convergence.

Thus we can write

$$(1.51)\quad A(x,y;r,\rho)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(U,V)\ K_o(r,U-x)\ K_o(\rho,V-y)\ dU\ dV,$$

$$\text{where}\begin{cases} K_o(r,U-x) = \sum_{j=1}^{\infty} r^j \sin j(U-x) = \dfrac{r\sin(U-x)}{1-2r\cos(U-x)+r^2} \\ K_o(\rho,V-y) = \sum_{k=1}^{\infty} \rho^k \sin k(V-y) = \dfrac{\rho\sin(V-y)}{1-2\rho\cos(V-y)+\rho^2} \end{cases}.$$

Let us make the transformation

$U-x=\gamma \qquad U=x+\gamma \qquad dU=d\gamma$

$V-y=\delta \qquad V=y+\delta \qquad dV=d\delta.$

Then we have

$$(1.52)\quad A(x,y;r,\rho)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(U,V)\ K_o(r,U-x)\ K_o(\rho,V-y)\ dU\ dV$$

$$=\frac{1}{\pi^2}\int_{-\pi-x}^{\pi-x}\int_{-\pi-y}^{\pi-y} f(x+\gamma,y+\delta)\ K_o(r,\gamma)\ K_o(\rho,\delta)\ d\gamma\, d\delta$$

$$=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+\gamma,y+\delta)\ K_o(r,\gamma)\ K_o(\rho,\delta)\ d\gamma\, d\delta\ ,$$

since f and K_0 are periodic of period 2π.

Further we have

(1.53) $A(x,y;r,\rho)$

$$= \frac{1}{\pi^2}\left\{\int_0^\pi\int_0^\pi + \int_0^\pi\int_{-\pi}^0 + \int_{-\pi}^0\int_0^\pi + \int_{-\pi}^0\int_{-\pi}^0\right\} f(x+\gamma,y+\delta)\ K_0(r,\gamma)\ K_0(\rho,\delta)\ d\gamma\ d\delta$$

$$= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(x+\gamma,y+\delta)\left(\frac{r\sin\gamma}{1-2r\cos\gamma+r^2}\right)\left(\frac{\rho\sin\delta}{1-2\rho\cos\delta+\rho^2}\right)d\gamma\ d\delta$$

$$-\frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(x+\gamma,y-\delta)\left(\frac{r\sin\gamma}{1-2r\cos\gamma+r^2}\right)\left(\frac{\rho\sin\delta}{1-2\rho\cos\delta+\rho^2}\right)d\gamma\ d\delta$$

$$-\frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(x-\gamma,y+\delta)\left(\frac{r\sin\gamma}{1-2r\cos\gamma+r^2}\right)\left(\frac{\rho\sin\delta}{1-2\rho\cos\delta+\rho^2}\right)d\gamma\ d\delta$$

$$+\frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(x-\gamma,y-\delta)\left(\frac{r\sin\gamma}{1-2r\cos\gamma+r^2}\right)\left(\frac{\rho\sin\delta}{1-2\rho\cos\delta+\rho^2}\right)d\gamma\ d\delta$$

$$= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi\Big[f(x+\gamma,y+\delta)-f(x+\gamma,y-\delta)-f(x-\gamma,y+\delta)+f(x-\gamma,y-\delta)\Big]K_0(r,\gamma)K_0(\rho,\delta)d\gamma\ d\delta$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \chi_0(x,y;\gamma,\delta)\, K_0(r,\gamma)\, K_0(\rho,\delta)\, d\gamma\, d\delta, \text{ thereby defining } \chi_0.$$

The notion of conjugacy for the double series has a slightly different character than in the case of the single series. This follows from the fact that although every series (1.1) is the real part of a power series (1.21), the series conjugate to (1.1) or (1.2) being defined as the imaginary part of (1.21), not every series (1.3) is the real part of a power series of two variables.

The problem of convergence of the conjugate of the double Fourier series (1.4) resolves itself into the convergence of a certain related integral, namely the integral

$$(1.6)\quad \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \chi_0(x,y;\gamma,\delta)\, \frac{\cot\frac{\gamma}{2}}{2}\, \frac{\cot\frac{\delta}{2}}{2}\, d\gamma\, d\delta.$$

We shall say simply that the integral (1.6) exists or converges, if we have the limit of the integral

$$(1.61)\quad \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \chi_0(x,y;\gamma,\delta)\, \frac{\cot\frac{\gamma}{2}}{2}\, \frac{\cot\frac{\delta}{2}}{2}\, d\gamma\, d\delta$$

existing as ϵ and η tend to 0 independently in the manner of the ordinary double limit. On the other hand, following the definition first given by C. N. Moore [9] for what he called restricted convergence, we shall say that the integral (1.6) exists in the restricted sense if the limit of the integral (1.61) exists as ϵ and η tend to 0 in such a way that the ratios $\frac{\epsilon}{\eta}$ and $\frac{\eta}{\epsilon}$ remain bounded.

K. Sokol-Sokolowski [11, pg. 181] has stated without proof that if $f(x,y)$ is integrable (L), then at almost every point (x,y), the difference between the first arithmetic means of the series (1.4) and the integral (1.61) tend to 0 in the restricted sense. K. Sokol-Sokolowski [11] has shown that if $f(x,y)$ belongs to the class L^p $p>1$, the integral (1.6) exists for almost every (x,y). In this paper we shall not be concerned with conditions for the existence of (1.6) at almost every point (x,y) but rather at a particular point (x,y). In doing so, we shall attempt to generalize, insofar as possible, the result of Misra [6] to double series.

(2.) In connection with the existence of an integral, we make the following definition.

(2.1) Definition. Suppose that $G(\gamma,\delta)$ is integrable (L) in

$$\left\{\begin{array}{l} 0<\xi<\gamma<a \\ 0<\nu<\delta<b \end{array}\right\} \text{ for every fixed pair } (\xi,\nu),$$

and that $H(\epsilon,\eta)=\int_\eta^b\int_\epsilon^a G(\gamma,\delta)\,d\gamma\,d\delta$;

then we shall say that the integral

$\int_0^b\int_0^a G(\gamma,\delta)\,d\gamma\,d\delta$ is Cesaro summable in the restricted sense with index n, or more simply exists $(C;n,n)$ in the restricted sense, if

$$\frac{1}{\epsilon\eta}\int_0^{\eta}\int_0^{\epsilon}\frac{1}{\epsilon_1\eta_1}\int_0^{\eta_1}\int_0^{\epsilon_1}\cdots\frac{1}{\epsilon_{n-2}\eta_{n-2}}\int_0^{\eta_{n-2}}\int_0^{\epsilon_{n-2}}\frac{1}{\epsilon_{n-1}\eta_{n-1}}\int_0^{\eta_{n-1}}\int_0^{\epsilon_{n-1}}H(\epsilon_n,\eta_n)\,d\epsilon_n d\eta_n d\epsilon_{n-1}d\eta_{n-1}\cdots d\epsilon_2 d\eta_2 d\epsilon_1 d\eta_1$$

tends to a limit as ϵ and η tend to 0 in the restricted sense.

(2.2) Notation and Definition. We define the following symbolism.

$$\underset{o\ o}{\chi}(x,y;s,t)=f(x+s,y+t)-f(x+s,y-t)-f(x-s,y+t)+f(x-s,y-t)$$

$$\underset{o\ 1}{\chi}(x,y;s,\delta)=\frac{1}{\sin s}\int_0^s \underset{o\ o}{\chi}(x,y;U,\delta)\ dU$$

$$\underset{o\ 2}{\chi}(x,y;s,\delta)=\frac{1}{\sin s}\int_0^s \underset{o\ 1}{\chi}(x,y;U,\delta)\ dU$$

$\vdots$

$$\underset{o\ j}{\chi}(x,y;s,\delta)=\frac{1}{\sin s}\int_0^s \underset{o\ j-1}{\chi}(x,y;U,\delta)\ dU$$

$$\underset{1\ o}{\chi}(x,y;\gamma,t)=\frac{1}{\sin t}\int_0^t \underset{o\ o}{\chi}(x,y;\gamma,V)\ dV$$

$$\underset{1\ 1}{\chi}(x,y;s,t)=\frac{1}{\sin s}\,\frac{1}{\sin t}\int_0^t\int_0^s \underset{o\ o}{\chi}(x,y;U,V)\ dU\ dV$$

$${}_1\chi_2(x,y;s,\delta) = \frac{1}{\sin s}\int_0^s {}_1\chi_1(x,y;U,\delta)\,dU$$

$$\vdots$$

$${}_1\chi_j(x,y;s,\delta) = \frac{1}{\sin s}\int_0^s {}_1\chi_{j-1}(x,y;U,\delta)\,dU$$

$${}_2\chi_0(x,y;\gamma,t) = \frac{1}{\sin t}\int_0^t {}_1\chi_0(x,y;\gamma,V)\,dV$$

$${}_2\chi_1(x,y;\gamma,t) = \frac{1}{\sin t}\int_0^t {}_1\chi_1(x,y;\gamma,V)\,dV$$

$${}_2\chi_2(x,y;s,t) = \frac{1}{\sin s}\,\frac{1}{\sin t}\int_0^t\int_0^s {}_1\chi_1(x,y;U,V)\,dU\,dV$$

$${}_2\chi_3(x,y;s,\delta) = \frac{1}{\sin s}\int_0^s {}_2\chi_2(x,y;U,\delta)\,dU$$

$$\vdots$$

$${}_2\chi_j(x,y;s,\delta) = \frac{1}{\sin s}\int_0^s {}_2\chi_{j-1}(x,y;U,\delta)\,dU$$

$$\vdots$$

$$\chi_{k\,0}(x,y;\gamma,t) = \frac{1}{\sin t}\int_0^t \chi_{k-1\,0}(x,y;\gamma,V)\,dV$$

$$\chi_{k\,1}(x,y;\gamma,t) = \frac{1}{\sin t}\int_0^t \chi_{k-1\,1}(x,y;\gamma,V)\,dV$$

$$\chi_{k\,2}(x,y;\gamma,t) = \frac{1}{\sin t}\int_0^t \chi_{k-1\,2}(x,y;\gamma,V)\,dV$$

$$\vdots$$

$$\chi_{k\,k-1}(x,y;\gamma,t) = \frac{1}{\sin t}\int_0^t \chi_{k-1\,k-1}(x,y;\gamma,V)\,dV$$

$$\chi_{k\,k}(x,y;s,t) = \frac{1}{\sin s}\cdot\frac{1}{\sin t}\int_0^t\int_0^s \chi_{k-1\,k-1}(x,y;U,V)\,dU\,dV$$

$$\chi_{k\,k+1}(x,y;s,\delta) = \frac{1}{\sin s}\int_0^s \chi_{k\,k}(x,y;U,\delta)\,dU$$

$$\vdots$$

$$\chi_{k\,j}(x,y;s,\delta) = \frac{1}{\sin s}\int_0^s \chi_{k\,j-1}(x,y;U,\delta)\,dU.$$

We assume that each of the integrals involved in the definition of the functions above exists in the sense of Harnack Lebesgue [10 pp.657-662] and, moreover, that upon disregarding exterior divisions by the sine function, each iterated integral converges uniformly with respect to the one variable as the other variable tends to 0. To be more explicit, in the case of $\underset{k}{}\chi_{k}(x,y;s,t)$, for example, we require that to any given arbitrary positive number ω, there correspond positive numbers Δ_1, Δ_2 such that for $0 < s < \Delta_1$, we have

$$\left| \int_0^s {}_{k-1}\chi_{k-1}(x,y;U,\delta)\, dU \right| < \omega$$

uniformly for all δ on $0 < \delta < \pi - \beta$ where $\beta > 0$ is small and fixed, and for $0 < t < \Delta_2$, we have

$$\left| \int_0^t {}_{k-1}\chi_{k-1}(x,y;\gamma,V)\, dV \right| < \omega$$

uniformly for all γ on $0 < \gamma < \pi - \alpha$ where $\alpha > 0$ is fixed and small. This will insure the existence and equality of the repeated integrals concerned and moreover give the property that

$$\int_0^t \int_0^s {}_{k-1}\chi_{k-1}(x,y;U,V)\, dU\, dV \to 0 \text{ as } \begin{cases} \left.\begin{matrix} s \\ t \end{matrix}\right\} \to 0 \\ s \to 0 \text{ uniformly on } 0 < t < \pi - \beta \\ t \to 0 \text{ uniformly on } 0 < s < \pi - \alpha \end{cases}$$

(3.) Lemma 1. If $f(x,y)$ is integrable (L) in $Q(-\pi,-\pi;\pi,\pi)$ and periodic of period 2π, then

$$\left(\frac{\cot\frac{\Delta}{2}}{2}\right)\left(\frac{\cot\frac{\tau}{2}}{2}\right)\int_0^{\tau}\int_0^{\Delta}\chi_{kj}(x,y;\gamma,\delta)\,d\gamma\,d\delta\to 0 \text{ as } \begin{cases}\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to\pi\\ \Delta\to\pi \text{ uniformly on } 0<\beta<\tau<\pi\\ \tau\to\pi \text{ uniformly on } 0<\alpha<\Delta<\pi\end{cases}$$

for any k and j where α and β are positive, small and fixed.

Proof of Lemma 1. We will express

$$\int_0^{\tau}\int_0^{\Delta}\chi_{kj}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

in a form so that the existence of the concluded limits will be apparent. We will integrate by parts until the indices k or j become 0. We shall not give a formal mathematical induction proof of each of the relations which follow, but in each case, the fact that such an argument could be made will be apparent.

Without loss of generality, we can assume $k<j$. We break up the domain of integration as follows:

$$(3.01)\quad \int_0^{\tau}\int_0^{\Delta}\chi_{kj}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$=\left\{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\Delta}+\int_{\frac{\pi}{2}}^{\tau}\int_0^{\frac{\pi}{2}}+\int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\Delta}\right\}\chi_{kj}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

Our procedure is a continued integration by parts. We have

$$(3.1)\quad \int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\Delta} \chi_{k\,j}(x,y;\gamma,\delta)\; d\gamma\; d\delta$$

$$= \int_{\frac{\pi}{2}}^{\tau} \left\{ \begin{array}{l} \Big[\chi_{k\,j}(x,y;\gamma,\delta)\ \sin\gamma\ \log\ \tan\frac{\gamma}{2}\Big]_{\frac{\pi}{2}}^{\Delta} \\ -\int_{\frac{\pi}{2}}^{\Delta} \chi_{k\,j-1}(x,y;\gamma,\delta)\ \log\ \tan\frac{\gamma}{2}\; d\gamma \end{array} \right\} d\delta$$

$$= \left(\log\ \tan\frac{\Delta}{2}\right) \int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\,j-1}(x,y;\gamma,\delta)\; d\gamma\; d\delta$$

$$+ \int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\Delta} \chi_{k\,j-1}(x,y;\gamma,\delta) \left(\log \frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right) d\gamma\; d\delta$$

$$= \left(\log\ \tan\frac{\Delta}{2}\right) \int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\,j-1}(x,y;\gamma,\delta)\; d\gamma\; d\delta$$

$$+ \int_{\frac{\pi}{2}}^{\tau} \left\{ \begin{array}{l} \left[-\chi_{k\,j-1}(x,y;\gamma,\delta)(\sin\gamma)\left(\frac{1}{2}\right)\left(\log \frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{2}\right]_{\frac{\pi}{2}}^{\Delta} \\ \int_{\frac{\pi}{2}}^{\Delta} \chi_{k\,j-2}(x,y;\gamma,\delta)\left(\frac{1}{2}\right)\left(\log \frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{2} d\gamma \end{array} \right\} d\delta$$

$$= \left(\log \tan\frac{\Delta}{2}\right)\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\ j-1}(x,y;\gamma,\delta)\ d\gamma\ d\delta$$

$$+ \frac{\left(\log \tan\frac{\Delta}{2}\right)^{2}}{2}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\ j-2}(x,y;\gamma,\delta)\ d\gamma\ d\delta$$

$$+ \int_{\frac{\pi}{2}}^{\tau}\left\{ \begin{array}{l} \left[-\chi_{k\ j-2}(x,y;\gamma,\delta)\ (\sin\gamma)\left(\frac{1}{3!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{3}\right]_{\frac{\pi}{2}}^{\Delta} \\ \int_{\frac{\pi}{2}}^{\Delta}\chi_{k\ j-3}(x,y;\gamma,\delta)\left(\frac{1}{3!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{3} \end{array} \right\} d\delta$$

$$= \left(\log \tan\frac{\Delta}{2}\right)\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\ j-1}(x,y;\gamma,\delta)\ d\gamma\ d\delta$$

$$+ \frac{\left(\log \tan\frac{\Delta}{2}\right)^{2}}{2}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\ j-2}(x,y;\gamma,\delta)\ d\gamma\ d\delta$$

$$+ \frac{\left(\log \tan\frac{\Delta}{2}\right)^{3}}{3!}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\ j-3}(x,y;\gamma,\delta)\ d\gamma\ d\delta + \cdots$$

$$+ \frac{\left(\log \tan\frac{\Delta}{2}\right)^{j-k}}{(j-k)!}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k\ k}(x,y;\gamma,\delta)\ d\gamma\ d\delta$$

$$+ \int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\Delta} \chi_{k\ k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\ d\delta.$$

Let us continue to integrate the last term of (3.1) above by parts. We have

$$(3.2)\quad \int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\Delta} {}_{k}\chi_{k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$= \int_{\frac{\pi}{2}}^{\Delta}\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k}\left\{\begin{array}{l}\left[{}_{k}\chi_{k}(x,y;\gamma,\delta)\ \sin\delta\ \log\ \tan\frac{\delta}{2}\right]_{\frac{\pi}{2}}^{\tau} \\ -\displaystyle\int_{\frac{\pi}{2}}^{\tau}{}_{k-1}\chi_{k}(x,y;\gamma,\delta)\ \log\ \tan\frac{\delta}{2}\ d\delta\end{array}\right\} d\gamma$$

$$= \log\ \tan\frac{\tau}{2}\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\Delta}{}_{k-1}\chi_{k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$+ \int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\Delta}{}_{k-1}\chi_{k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k}\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right) d\gamma\, d\delta$$

$$= \log\ \tan\frac{\tau}{2}\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\Delta}{}_{k-1}\chi_{k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$+ \int_{\frac{\pi}{2}}^{\tau}\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)\left\{\begin{array}{l}\left[-{}_{k-1}\chi_{k}(x,y;\gamma,\delta)\ \sin\gamma\left(\frac{1}{(j-k+1)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+1}\right]_{\frac{\pi}{2}}^{\Delta} \\ \displaystyle\int_{\frac{\pi}{2}}^{\Delta}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\left(\frac{1}{(j-k+1)!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+1} d\gamma\end{array}\right\} d\delta$$

$$= \left(\log\ \tan\frac{\tau}{2}\right)\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{s} {}_{k-1}\chi_{k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$+\left(\frac{1}{(j-k+1)!}\right)\left(\log\ \tan\frac{s}{2}\right)^{j-k+1}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right) d\gamma\, d\delta$$

$$+\int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{s} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\left(\frac{1}{(j-k+1)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+1}\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right) d\gamma\, d\delta$$

$$= \left(\log\ \tan\frac{\tau}{2}\right)\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{s} {}_{k-1}\chi_{k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$+\left(\frac{1}{(j-k+1)!}\right)\left(\log\ \tan\frac{s}{2}\right)^{j-k+1}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right) d\gamma\, d\delta$$

$$+\int_{\frac{\pi}{2}}^{s}\left(\frac{1}{(j-k+1)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+1}\left\{\begin{aligned}&\left[-{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)(\sin\delta)\left(\frac{1}{2}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{2}\right]_{\frac{\pi}{2}}^{\tau}\\ &\int_{\frac{\pi}{2}}^{\tau} {}_{k-2}\chi_{k-1}(x,y;\gamma,\delta)\left(\frac{1}{2}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{2} d\delta\end{aligned}\right\} d\gamma$$

$$= \log \tan\frac{\tau}{2} \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\delta} \chi_{k-1\,k}(x,y;\gamma,\delta) \left(\frac{1}{(j-k)!}\right) \left(\log \frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$+ \left(\frac{1}{(j-k+1)!}\right) \left(\log \tan\frac{\delta}{2}\right)^{j-k+1} \int_{\frac{\pi}{2}}^{\tau} \int_0^{\frac{\pi}{2}} \chi_{k-1\,k-1}(x,y;\gamma,\delta) \left(\log \frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right) d\gamma\, d\delta$$

$$+ \left(\frac{1}{2}\right)\left(\log \tan\frac{\tau}{2}\right)^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\delta} \chi_{k-2\,k-1}(x,y;\gamma,\delta) \left(\frac{1}{(j-k+1)!}\right) \left(\log \frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+1} d\gamma\, d\delta$$

$$+ \int_{\frac{\pi}{2}}^{\tau} \left(\frac{1}{2}\right)\left(\log \frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^2 \left\{ \left[-\chi_{k-2\,k-1}(x,y;\gamma,\delta)(\sin\gamma)\left(\frac{1}{(j-k+2)!}\right)\left(\log \frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+2} \right]_{\frac{\pi}{2}}^{\delta} + \int_{\frac{\pi}{2}}^{\delta} \chi_{k-2\,k-2}(x,y;\gamma,\delta)\left(\frac{1}{(j-k+2)!}\right)\left(\log \frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+2} d\gamma \right\} d\delta .$$

Thus we have

$$(3.21)\quad \int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{s} \chi_{k\,k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$= \left(\log\tan\tfrac{\tau}{2}\right)\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{s} \chi_{k-1\,k}(x,y;\gamma,\delta)\left(\frac{1}{(j-k)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k} d\gamma\, d\delta$$

$$+\left(\tfrac{1}{2}\right)\left(\log\tan\tfrac{\tau}{2}\right)^{2}\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{s} \chi_{k-2\,k-1}(x,y;\gamma,\delta)\left(\frac{1}{(j-k+1)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+1} d\gamma\, d\delta$$

$$+\cdots$$

$$+\left(\frac{1}{k!}\right)\left(\log\tan\tfrac{\tau}{2}\right)^{k}\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{s} \chi_{0\,1}(x,y;\gamma,\delta)\left(\frac{1}{(j-1)!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j-1} d\gamma\, d\delta$$

$$+\left(\frac{1}{(j-k+1)!}\right)\left(\log\tan\tfrac{s}{2}\right)^{j-k+1}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k-1\,k-1}(x,y;\gamma,\delta)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right) d\gamma\, d\delta$$

$$+\left(\frac{1}{(j-k+2)!}\right)\left(\log\tan\tfrac{s}{2}\right)^{j-k+2}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{k-2\,k-2}(x,y;\gamma,\delta)\left(\tfrac{1}{2}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{2} d\gamma\, d\delta$$

$$+\cdots$$

$$+\left(\frac{1}{j!}\right)\left(\log\tan\tfrac{s}{2}\right)^{j}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}} \chi_{0\,0}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k} d\gamma\, d\delta$$

$$+\int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{s} \chi_{0\,0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j}\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k} d\gamma\, d\delta\,.$$

For any m and n

$$(3.3)\quad \left(\frac{1}{m!}\right)\left(\log\tan\frac{\pi}{2}\right)^{m}\int_{0}^{\frac{\pi}{2}}\int^{\delta}\chi_{k-m\;k-m}(x,y;\gamma,\delta)\left(\frac{1}{(j-k+m)!}\right)\left(\log\frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+m} d\gamma\, d\delta$$

$$=\left(\frac{1}{m!}\right)\left(\log\tan\frac{\pi}{2}\right)^{m}\int_{0}^{\frac{\pi}{2}}\left\{\left[-\chi_{k-m\;k-m}(x,y;\gamma,\delta)\sin\gamma\left(\frac{1}{(j-k+m+1)!}\right)\left(\log\frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+m+1}\right]_{\frac{\pi}{2}}^{\delta}\right.$$
$$\left.\int_{\frac{\pi}{2}}^{\delta}\chi_{k-m\;k-m-1}(x,y;\gamma,\delta)\left(\frac{1}{(j-k+m+1)!}\right)\left(\log\frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+m+1}d\gamma\right\}d\delta$$

$$=\left(\frac{1}{(j-k+m+1)!}\left(\log\tan\frac{\delta}{2}\right)^{j-k+m+1}\right)\left(\frac{1}{m!}\right)\left(\log\tan\frac{\pi}{2}\right)^{m}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{k-m\;k-m-1}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{m!}\right)\left(\log\tan\frac{\pi}{2}\right)^{m}\int_{0}^{\frac{\pi}{2}}\left\{\left[-\chi_{k-m\;k-m-1}(x,y;\gamma,\delta)\sin\gamma\left(\log\frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+m+2}\right]_{\frac{\pi}{2}}^{\delta}\right.$$
$$\left.+\int_{\frac{\pi}{2}}^{\delta}\chi_{k-m\;k-m-2}(x,y;\gamma,\delta)\left(\frac{1}{(j-k+m+2)!}\right)\left(\log\frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j-k+m+2}d\gamma\right\}d\delta$$

$$=\cdots$$

$$=\left(\frac{1}{m!}\right)\left(\log\tan\frac{\pi}{2}\right)^{m}\sum_{l=1}^{k-m}\left\{\left(\frac{1}{(j-k+m+l)!}\right)\left(\log\tan\frac{\delta}{2}\right)^{j-k+m+l}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{k-m\;k-m-l}(x,y;\gamma,\delta)\,d\gamma\,d\delta\right\}$$

$$+\left(\frac{1}{m!}\right)\left(\log\tan\frac{\pi}{2}\right)^{m}\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\delta}\chi_{k-m\;0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j}d\gamma\,d\delta\,.$$

Similarly, we find for any m and n

$$(3.4)\quad \left(\frac{1}{(j-k+m)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k+m}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}}\chi_{k-n\;k-m}(x,y;\gamma,\delta)\left(\frac{1}{n!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{n}d\gamma\, d\delta$$

$$=\left(\frac{1}{(j-k+m)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k+m}\left(\frac{1}{(n+1)!}\right)\left(\log\tan\frac{\tau}{2}\right)^{n+1}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{k-n-1\;k-m}(x,y;\gamma,\delta)\,d\gamma\, d\delta$$

$$+\left(\frac{1}{(j-k+m)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k+m}\left(\frac{1}{(n+2)!}\right)\left(\log\tan\frac{\tau}{2}\right)^{n+2}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{k-n-2\;k-m}(x,y;\gamma,\delta)\,d\gamma\, d\delta$$

$$+\cdots$$

$$=\left(\frac{1}{(j-k+m)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k+m}\sum_{\ell=1}^{k-n}\left(\frac{1}{(n+\ell)!}\right)\left(\log\tan\frac{\tau}{2}\right)^{n+\ell}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{k-n-\ell\;k-m}(x,y;\gamma,\delta)\,d\gamma\, d\delta$$

$$+\left(\frac{1}{(j-k+m)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k+m}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}}\chi_{0\;k-m}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\, d\delta.$$

By use of the relations (3.1), (3.21), (3.3), (3.4) and (3.01), we can finally express

$$\int_0^{\tau}\int_0^{s}\chi_{k\,j}(x,y;\gamma,\delta)\;d\gamma\,d\delta$$

as follows:

$$(3.7)\qquad \int_0^{\tau}\int_0^{s}\chi_{k\,j}(x,y;\gamma,\delta)\;d\gamma\,d\delta=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k\,j}(x,y;\gamma,\delta)\;d\gamma\,d\delta$$

$$+\sum_{\ell=1}^{j}\left(\frac{1}{\ell!}\right)\left(\log\tan\frac{s}{2}\right)^{\ell}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k\,j-\ell}(x,y;\gamma,\delta)\;d\gamma\,d\delta+\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{s}\chi_{k\,0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{\gamma}{2}}\right)^{j}d\gamma\,d\delta$$

$$+\sum_{\ell=1}^{k}\left(\frac{1}{\ell!}\right)\left(\log\tan\frac{\tau}{2}\right)^{\ell}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k-\ell\,j}(x,y;\gamma,\delta)\;d\gamma\,d\delta+\int_{\frac{\pi}{2}}^{\tau}\int_0^{\frac{\pi}{2}}\chi_{0\,j}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta$$

$$+\log\tan\frac{s}{2}\sum_{\ell=1}^{k}\left(\frac{1}{\ell!}\right)\left(\log\tan\frac{\tau}{2}\right)^{\ell}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k-\ell\,j-1}(x,y;\gamma,\delta)\;d\gamma\,d\delta$$

$$+\log\tan\frac{s}{2}\int_{\frac{\pi}{2}}^{\tau}\int_0^{\frac{\pi}{2}}\chi_{0\,j-1}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta$$

$$+\left(\frac{1}{2}\right)\left(\log\tan\frac{s}{2}\right)^{2}\sum_{\ell=1}^{k}\left(\frac{1}{\ell!}\right)\left(\log\tan\frac{\tau}{2}\right)^{\ell}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k-\ell\,j-2}(x,y;\gamma,\delta)\;d\gamma\,d\delta$$

$$+\left(\frac{1}{2}\right)\left(\log\tan\frac{s}{2}\right)^{2}\int_{\frac{\pi}{2}}^{\tau}\int_0^{\frac{\pi}{2}}\chi_{0\,j-2}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta$$

$$+\cdots$$

$$+\left(\frac{1}{(j-k)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k}\sum_{\ell=1}^{k}\left(\frac{1}{\ell!}\right)\left(\log\tan\frac{\Gamma}{2}\right)^{\ell}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k-\ell,k}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{(j-k)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k}\int_{\frac{\pi}{2}}^{\Gamma}\int_0^{\frac{\pi}{2}}\chi_{0,k}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\Gamma}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta$$

$$+\left(\log\tan\frac{\Gamma}{2}\right)\sum_{\ell=1}^{k}\left(\frac{1}{(j-k+\ell)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k+\ell}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k-1,k-\ell}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\log\tan\frac{\Gamma}{2}\right)\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\Delta}\chi_{k-1,0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j}d\gamma\,d\delta$$

$$+\left(\frac{1}{2}\right)\left(\log\tan\frac{\Gamma}{2}\right)^{2}\sum_{\ell=1}^{k-1}\left(\frac{1}{(j-k+\ell+1)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j-k+\ell+1}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k-2,k-\ell-1}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{2}\right)\left(\log\tan\frac{\Gamma}{2}\right)^{2}\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\Delta}\chi_{k-2,0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j}d\gamma\,d\delta$$

$$+\cdots$$

$$+\left(\frac{1}{(k-1)!}\right)\left(\log\tan\frac{\Gamma}{2}\right)^{k-1}\sum_{\ell=1}^{2}\left(\frac{1}{(j+\ell-2)!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j+\ell-2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{1,2-\ell}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{(k-1)!}\right)\left(\log\tan\frac{\Gamma}{2}\right)^{k-1}\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\Delta}\chi_{1,0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j}d\gamma\,d\delta$$

$$+\left(\frac{1}{k!}\right)\left(\log\tan\frac{\Gamma}{2}\right)^{k}\left(\frac{1}{j!}\right)\left(\log\tan\frac{\Delta}{2}\right)^{j}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{0,0}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{k!}\right)\left(\log\,\tan\frac{\tau}{2}\right)^{k}\int_{0}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\Delta}\chi_{00}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j}d\gamma\,d\delta$$

$$+\left(\frac{1}{(j-k+1)!}\right)\left(\log\,\tan\frac{\Delta}{2}\right)^{j-k+1}\sum_{\ell=1}^{k-1}\left(\frac{1}{(\ell+1)!}\right)\left(\log\,\tan\frac{\tau}{2}\right)^{\ell+1}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{k-\ell-1\,k-1}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{(j-k+1)!}\right)\left(\log\,\tan\frac{\Delta}{2}\right)^{j-k+1}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}}\chi_{0\,k-1}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta$$

$$+\left(\frac{1}{(j-k+2)!}\right)\left(\log\,\tan\frac{\Delta}{2}\right)^{j-k+2}\sum_{\ell=1}^{k-2}\left(\frac{1}{(\ell+2)!}\right)\left(\log\,\tan\frac{\tau}{2}\right)^{\ell+2}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{k-\ell-2\,k-2}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{(j-k+2)!}\right)\left(\log\,\tan\frac{\Delta}{2}\right)^{j-k+2}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}}\chi_{0\,k-2}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta$$

$$+\cdots$$

$$+\left(\frac{1}{(j-1)!}\right)\left(\log\,\tan\frac{\Delta}{2}\right)^{j-1}\left(\frac{1}{k!}\right)\left(\log\,\tan\frac{\tau}{2}\right)^{k}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\chi_{01}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$+\left(\frac{1}{(j-1)!}\right)\left(\log\,\tan\frac{\Delta}{2}\right)^{j-1}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}}\chi_{01}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta$$

$$+\left(\frac{1}{j!}\right)\left(\log\,\tan\frac{\Delta}{2}\right)^{j}\int_{\frac{\pi}{2}}^{\tau}\int_{0}^{\frac{\pi}{2}}\chi_{00}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)d\gamma\,d\delta$$

$$+\int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\Delta}\chi_{00}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\Delta}{2}}{\tan\frac{\gamma}{2}}\right)^{j}\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k}d\gamma\,d\delta.$$

We have the following types of terms appearing in (3.7) above:

$$(3.71)\quad \left(\frac{1}{m!}\right)\left(\log\tan\frac{\sigma}{2}\right)^m\left(\frac{1}{m!}\right)\left(\log\tan\frac{\tau}{2}\right)^n\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\chi_{k-m,\,j-m}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$(3.72)\quad \left(\frac{1}{m!}\right)\left(\log\tan\frac{\tau}{2}\right)^m\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\sigma}\chi_{k-m,\,0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\sigma}{2}}{\tan\frac{\gamma}{2}}\right)^j d\gamma\,d\delta$$

$$(3.73)\quad \left(\frac{1}{m!}\right)\left(\log\tan\frac{\sigma}{2}\right)^m\int_{\frac{\pi}{2}}^{\tau}\int_0^{\frac{\pi}{2}}\chi_{0,\,j-m}(x,y;\gamma,\delta)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^k d\gamma\,d\delta$$

$$(3.74)\quad \int_{\frac{\pi}{2}}^{\tau}\int_{\frac{\pi}{2}}^{\sigma}\chi_{0,0}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\sigma}{2}}{\tan\frac{\gamma}{2}}\right)^j\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^k d\gamma\,d\delta$$

We will show that, when each of the types of terms listed above is divided by $\left(2\tan\frac{\sigma}{2}\right)\left(2\tan\frac{\tau}{2}\right)$, the result tends to 0

$$\text{as}\begin{cases}\left.\begin{matrix}\sigma\\ \tau\end{matrix}\right\}\to\pi\\ \sigma\to\pi \text{ uniformly on } 0<\beta<\tau<\pi\\ \tau\to\pi \text{ uniformly on } 0<\alpha<\sigma<\pi\end{cases}$$

For terms of the type (3.71) in the relation (3.7) above, we have

$$(3.76)\quad \frac{\left(\frac{1}{m!}\right)\left(\log\ \tan\frac{s}{2}\right)^{m}\left(\frac{1}{m!}\right)\left(\log\ \tan\frac{t}{2}\right)^{n}\int_0^{\frac{\alpha}{2}}\int_0^{\frac{\beta}{2}}\chi_{k-m\ j-m}(x,y;\gamma,\delta)\ d\gamma d\delta}{\left(2\tan\frac{s}{2}\right)\left(2\tan\frac{t}{2}\right)}$$

$$\to 0 \text{ as } \begin{cases} \left.\begin{matrix} s \\ t \end{matrix}\right\}\to\pi \\ s\to\pi \text{ uniformly on } 0<\beta<t<\pi \\ t\to\pi \text{ uniformly on } 0<\alpha<s<\pi \end{cases}$$

since

$$\int_0^{\frac{\alpha}{2}}\int_0^{\frac{\beta}{2}}\chi_{k-m\ j-m}(x,y;\gamma,\delta)\ d\gamma\ d\delta \text{ is a constant,}$$

$$\frac{\left(\frac{1}{m!}\right)\left(\log\ \tan\frac{s}{2}\right)^{m}}{\left(2\ \tan\frac{s}{2}\right)} \to 0 \text{ as } s\to\pi \text{ and is bounded on } 0<\alpha<s<\pi,$$

$$\frac{\left(\frac{1}{m!}\right)\left(\log\ \tan\frac{\tau}{2}\right)^{m}}{\left(2\tan\frac{\tau}{2}\right)} \to 0 \text{ as } \tau\to\pi \text{ and is bounded on } 0<\beta<\tau<\pi.$$

For terms of the type (3.72) in the relation (3.7), we have by the second mean value theorem

$$(3.77)\quad \frac{\left(\frac{1}{m!}\right)\left(\log\ \tan\frac{\tau}{2}\right)^m}{\left(2\tan\frac{\sigma}{2}\right)\left(2\tan\frac{\tau}{2}\right)} \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\sigma} \underset{k-m\ 0}{X}(x,y;\gamma,\delta)\left(\frac{1}{j!}\right)\left(\log\frac{\tan\frac{\sigma}{2}}{\tan\frac{\gamma}{2}}\right)^j d\gamma d\delta$$

$$= \frac{\left(\frac{1}{m!}\right)\left(\log\ \tan\frac{\tau}{2}\right)^m\left(\frac{1}{j!}\right)\left(\log\ \tan\frac{\sigma}{2}\right)^j \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\sigma^*} \underset{k-m\ 0}{X}(x,y;\gamma,\delta)\, d\gamma d\delta}{\left(2\tan\frac{\tau}{2}\right)\left(2\tan\frac{\sigma}{2}\right)}$$

where $\frac{\pi}{2} < \sigma^* < \sigma < \pi$

$$\to 0 \text{ as } \begin{cases} \left.\begin{matrix}\sigma\\ \tau\end{matrix}\right\} \to \pi \\ \sigma \to \pi \text{ uniformly on } 0 < \beta < \tau < \pi \\ \tau \to \pi \text{ uniformly on } 0 < \alpha < \sigma < \pi, \end{cases}$$

since

$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\sigma^*} \underset{k-m\ 0}{X}(x,y;\gamma,\delta)\, d\gamma d\delta$ is bounded,

$\dfrac{\left(\frac{1}{m!}\right)\left(\log\ \tan\frac{\tau}{2}\right)^m}{\left(2\ \tan\frac{\tau}{2}\right)} \to 0$ as $\tau \to \pi$ and is bounded on $0 < \beta < \tau < \pi$,

$\dfrac{\left(\frac{1}{j!}\right)\left(\log\ \tan\frac{\sigma}{2}\right)^j}{\left(2\tan\frac{\sigma}{2}\right)} \to 0$ as $\sigma \to \pi$ and is bounded on $0 < \alpha < \sigma < \pi$, For terms of the type (3.73) in the relation (3.7), by an argument analogous to that of (3.77), we have

$$(3.78)\quad \frac{\left(\frac{1}{m!}\right)\left(\log \tan \frac{\sigma}{2}\right)^{m} \int_{\frac{\pi}{2}}^{\tau} \int_{0}^{\frac{\pi}{2}} \chi_{0\,j-m}(x,y;\gamma,\delta) \left(\frac{1}{k!}\right)\left(\log \frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k} d\gamma\, d\delta}{\left(2\tan\frac{\sigma}{2}\right)\left(2\tan\frac{\tau}{2}\right)}$$

$$\to 0 \text{ as } \begin{cases} \left.\begin{matrix}\sigma\\ \tau\end{matrix}\right\} \to \pi \\ \sigma \to \pi \text{ uniformly on } 0 < \beta < \tau < \pi \\ \tau \to \pi \text{ uniformly on } 0 < \alpha < \sigma < \pi . \end{cases}$$

For the term (3.74) in the relation (3.7), we have by the second mean value theorem

$$(3.79)\quad \frac{\int_{\frac{\pi}{2}}^{\tau} \int_{\frac{\pi}{2}}^{\sigma} \chi_{00}(x,y;\gamma,\delta) \left(\frac{1}{j!}\right)\left(\log \frac{\tan\frac{\sigma}{2}}{\tan\frac{\gamma}{2}}\right)^{j} \left(\frac{1}{k!}\right)\left(\log \frac{\tan\frac{\tau}{2}}{\tan\frac{\delta}{2}}\right)^{k} d\gamma\, d\delta}{\left(2\tan\frac{\sigma}{2}\right)\left(2\tan\frac{\tau}{2}\right)}$$

$$= \frac{\left(\frac{1}{j!}\right)\left(\log \tan\frac{\sigma}{2}\right)^{j} \left(\frac{1}{k!}\right)\left(\log \tan\frac{\tau}{2}\right)^{k} \int_{\frac{\pi}{2}}^{\tilde{\tau}} \int_{\frac{\pi}{2}}^{\tilde{\sigma}} \chi_{00}(x,y;\gamma,\delta)\, d\gamma\, d\delta}{\left(2\tan\frac{\sigma}{2}\right)\left(2\tan\frac{\tau}{2}\right)}$$

$$\text{where } \begin{cases} \frac{\pi}{2} < \tilde{\sigma} < \sigma < \pi \\ \frac{\pi}{2} < \tilde{\tau} < \tau < \pi \end{cases}$$

$$\to 0 \text{ as } \begin{cases} \left.\begin{matrix}\sigma\\ \tau\end{matrix}\right\} \to \pi \\ \sigma \to \pi \text{ uniformly on } 0 < \beta < \tau < \pi \\ \tau \to \pi \text{ uniformly on } 0 < \alpha < \sigma < \pi \end{cases}$$

since

$$\int_{\frac{\pi}{2}}^{\tilde{\tau}}\int_{\frac{\pi}{2}}^{\tilde{\sigma}} \chi_{00}(x,y;\gamma,\delta)\, d\gamma\, d\delta \text{ is bounded,}$$

$$\frac{\left(\frac{1}{j!}\right)\left(\log\, \tan \frac{\sigma}{2}\right)^{j}}{\left(2 \tan \frac{\sigma}{2}\right)} \to 0 \text{ as } \sigma \to \pi \text{ and is bounded on } 0 < \alpha < \sigma < \pi,$$

$$\frac{\left(\frac{1}{k!}\right)\left(\log\, \tan \frac{\tau}{2}\right)^{k}}{\left(2 \tan \frac{\tau}{2}\right)} \to 0 \text{ as } \tau \to \pi \text{ and is bounded on } 0 < \beta < \tau < \pi .$$

Hence it follows from (3.7) and from (3.76), (3.77), (3.78), (3.79) that

$$\frac{\int_0^{\tau}\int_0^{\sigma} \chi_{kj}(x,y;\gamma,\delta)\, d\gamma\, d\delta}{\left(2 \tan \frac{\sigma}{2}\right)\left(2 \tan \frac{\tau}{2}\right)} \to 0 \text{ as } \begin{cases} \begin{Bmatrix} \sigma \\ \tau \end{Bmatrix} \to \pi \\ \sigma \to \pi \text{ uniformly on } 0 < \beta < \tau < \pi . \\ \tau \to \pi \text{ uniformly on } 0 < \alpha < \sigma < \pi \end{cases}$$

This proves the lemma.

(4.) Lemma 2. If

$$\Delta(r,\gamma) = 1 - 2r\cos\gamma + r^2,$$

$$K_0(r,\gamma) = \frac{r\sin\gamma}{\Delta(r,\gamma)}, \qquad K_1(r,\gamma) = \sin\gamma\,\frac{dK_0(r,\gamma)}{d\gamma},$$

and generally

$$K_m(r,\gamma) = \sin\gamma\,\frac{dK_{m-1}(r,\gamma)}{d\gamma},$$

then

$$K_m(r,\gamma) = \begin{cases} O(\sin\gamma) \text{ for } \gamma > \xi > 0 \text{ for all } r \text{ on } 0 \leq r \leq 1 \\ O(\frac{1}{\gamma}) \text{ as } \gamma \to 0 \text{ with } \gamma = \text{arc} \sin (1-r) \\ O(1) \text{ in } \gamma \text{ on } 0 \leq \gamma \leq \pi \text{ with } 0 \leq r < 1 \end{cases}$$

moreover

$$K_m(r,0) = K_m(r,\pi) = 0 \text{ for } 0 \leq r < 1.$$

Proof of Lemma 2. By definition $K_0(r,\gamma) = \dfrac{r\sin\gamma}{\Delta(r,\gamma)}$,

where $\Delta(r,\gamma) = 1 - 2r\cos\gamma + r^2$. Hence we can write

(4.11) $$K_0 = \frac{1}{2\Delta}\frac{d\Delta}{d\gamma}, \qquad 2K_0 = \frac{1}{\Delta}\frac{d\Delta}{d\gamma}.$$

Upon taking the derivative, we have

(4.12) $$2\frac{dK_0}{d\gamma} = \frac{-1}{\Delta^2}\left(\frac{d\Delta}{d\gamma}\right)^2 + \frac{1}{\Delta}\frac{d^2\Delta}{d\gamma^2}$$

(4.13) $$\frac{d\Delta}{d\gamma}\frac{dK_0}{d\gamma} = \left(\frac{-1}{2\Delta}\right)\left(\frac{d\Delta}{d\gamma}\right)\left[\frac{1}{\Delta}\left(\frac{d\Delta}{d\gamma}\right)^2 - \frac{d^2\Delta}{d\gamma^2}\right] = -K_0\left[2K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right]$$

$$(4.14)\quad \frac{d\left[2K_0\frac{d\Delta}{d\gamma}-\frac{d^2\Delta}{d\gamma^2}\right]}{d\gamma}=\frac{d\left[\frac{1}{\Delta}\left(\frac{d\Delta}{d\gamma}\right)^2-\frac{d^2\Delta}{d\gamma^2}\right]}{d\gamma}$$

$$=-\frac{1}{\Delta^2}\left(\frac{d\Delta}{d\gamma}\right)^3+\frac{2}{\Delta}\frac{d\Delta}{d\gamma}\,\frac{d^2\Delta}{d\gamma^2}+\frac{d\Delta}{d\gamma}$$

$$(4.15)\quad \left(\frac{d\Delta}{d\gamma}\right)\frac{d\left[2K_0\frac{d\Delta}{d\gamma}-\frac{d^2\Delta}{d\gamma^2}\right]}{d\gamma}=-\left[\frac{1}{\Delta^2}\left(\frac{d\Delta}{d\gamma}\right)^4-\frac{2}{\Delta}\left(\frac{d\Delta}{d\gamma}\right)^2\frac{d^2\Delta}{d\gamma^2}-\left(\frac{d\Delta}{d\gamma}\right)^2\right]$$

$$=-\left[\left(2K_0\frac{d\Delta}{d\gamma}-\frac{d^2\Delta}{d\gamma^2}\right)^2-4r^2\right].$$

Let $W(r,\gamma)=\left(\dfrac{2K_0\frac{d\Delta}{d\gamma}-\frac{d^2\Delta}{d\gamma^2}}{2r}\right)=\dfrac{2r-(1+r^2)\cos\gamma}{1-2r\cos\gamma+r^2}$.

Then from (4.13), we have

$$(4.16)\quad \sin\gamma\,\frac{dK_0}{d\gamma}=-K_0W$$

and from (4.15) we have

$$(4.17)\quad \sin\gamma\,\frac{dW}{d\gamma}=-\left[W^2-1\right].$$

From the definition of $K_m(r,\gamma)$ for $m=0,1,2,3,4,5,6,$ upon taking the derivatives involved and applying (4.16) and (4.17), we have

$$K_0 = K_0 = K_0 H_0(r,\gamma)$$

$$K_1 = (-1)K_0 W = (-1)K_0 H_1(r,\gamma)$$

$$K_2 = (-1)^2 K_0\left\{2W^2 - 1\right\} = (-1)^2 K_0 H_2(r,\gamma)$$

$$K_3 = (-1)^3 K_0\left\{6W^3 - 5W\right\} = (-1)^3 K_0 H_3(r,\gamma) \tag{4.18}$$

$$K_4 = (-1)^4 K_0\left\{24W^4 - 28W^2 + 5\right\} = (-1)^4 K_0 H_4(r,\gamma)$$

$$K_5 = (-1)^5 K_0\left\{120W^5 - 180W^3 + 61W\right\} = (-1)^5 K_0 H_5(r,\gamma)$$

$$K_6 = (-1)^6 K_0\left\{720W^6 - 1320W + 662W^2 - 61\right\} = (-1)^6 K_0 H_6(r,\gamma)$$

thereby defining $H_m(r,\gamma)$ for $m=0,1,2,3,4,5,6.$

We note that

$$W(r,\gamma) = \frac{2r-(1+r^2)\cos\gamma}{1-2r\cos\gamma + r^2} = \begin{cases} 1 \text{ for } r=1 \text{ with } 0<\gamma\leqq\pi \\ -\cos\gamma \text{ for } r=0 \text{ with } 0\leqq\gamma\leqq\pi \\ 1 \text{ for } \gamma=\pi \text{ with } 0\leqq r\leqq 1. \\ o(1) \text{ as } \gamma = \text{arc sin } (1-r)\to 0 \end{cases} \tag{4.19}$$

We will show by mathematical induction that

$$\left\{\begin{array}{l} K_m(r,\gamma) = (-1)^m K_0(r,\gamma)H_m(r,\gamma) \text{ for all } m=0,1,2,\ldots, \\ \text{where } H_m(r,\gamma) \text{ is a polynomial in } W \text{ of degree } m, \text{ the} \\ \text{sum of whose coefficients is equal to } 1. \end{array}\right\} \tag{4.20}$$

The cases $m=0,1,2,3,4,5,6$ of the property (4.20) follow from (4.18).

Let us assume that

$$(4.22)\quad K_{2k}(r,\gamma)=(-1)^{2k}K_0\left\{\begin{array}{l} a_0W^{2k}+a_1W^{2k-2}+a_2W^{2k-4}+a_3W^{2k-6} \\ +\cdots+a_{k-3}W^6+a_{k-2}W^4+a_{k-1}W^2+a_k\end{array}\right\}$$

$$=(-1)^{2k}K_0H_{2k}(r,\gamma),$$

where

$$a_0+a_1+a_2+a_3+\cdots+a_{k-3}+a_{k-2}+a_{k-1}+a_k=1.$$

By the use of (4.16) and (4.17) we obtain

$$(4.23)\quad K_{2k+1}(r,\gamma)=\sin\gamma\frac{dK_{2k}}{d\gamma}$$

$$=(-1)^{2k+1}K_0\left\{\begin{array}{l} a_0W^{2k+1}+a_1W^{2k-1}+a_2W^{2k-3}+a_3W^{2k-5} \\ +\cdots+a_{k-3}W^7+a_{k-2}W^5+a_{k-1}W^3+a_kW \\ +2ka_0W^{2k+1}+(2k-2)a_1W^{2k-1}+(2k-4)a_2W^{2k-3}+(2k-6)a_3W^{2k-5} \\ +\cdots+6a_{k-3}W^7+4a_{k-2}W^5+2a_{k-1}W^3 \\ -2ka_0W^{2k-1}-(2k-2)a_1W^{2k-3}-(2k-4)a_2W^{2k-5}-(2k-6)a_3W^{2k-7} \\ -\cdots-6a_{k-3}W^5-4a_{k-2}W^3-2a_{k-1}W \end{array}\right\}$$

$$
=(-1)^{2k+1}K_0 \left\{ \begin{aligned}
& [2k+1]\, a_0 W^{2k+1} \\
& + \left[(2k-1)a_1 - (2k)a_0\right] W^{2k-1} \\
& + \left[(2k-3)a_2 - (2k-2)a_1\right] W^{2k-3} \\
& + \left[(2k-5)a_3 - (2k-4)a_2\right] W^{2k-5} \\
& \qquad\qquad - [2k-6]\, a_3 W^{2k-7} \\
& + \cdots \\
& + 7 a_{k-3} W^7 \\
& + \left[5 a_{k-2} - 6 a_{k-3}\right] W^5 \\
& + \left[3 a_{k-1} - 4 a_{k-2}\right] W^3 \\
& + \left[a_k - 2 a_{k-1}\right] W
\end{aligned} \right\} = (-1)^{2k+1} K_0 H_{2k+1}(r,\gamma).
$$

Thus our proposition (4.20) is true for $2k+1$ if it is true for $2k$. We continue with our induction to the next stage. We have

$$
(4.31) \quad K_{2k+2}(r,\gamma) = \sin\gamma \, \frac{dK_{2k+1}(r,\gamma)}{d\gamma}
$$

$$
= (-1)^{2k+2} K_0 \left\{
\begin{aligned}
& (2k+1) a_0 W^{2k+2} + \left[(2k+1) a_1 - 2k a_0\right] W^{2k} \\
& \left[(2k-3) a_2 - (2k-2) a_1\right] W^{2k-2} + \left[(2k-5) a_3 - (2k-4) a_2\right] W^{2k-4} \\
& \qquad\qquad - \left[2k-6\right] a_3 W^{2k-6} \\
& + \cdots + 7 a_{k-3} W^8 + \left[5 a_{k-2} - 6 a_{k-3}\right] W^6 \\
& + \left[3 a_{k-1} - 4 a_{k-2}\right] W^4 + \left[a_k - 2 a_{k-1}\right] W^2 \\
& + \left[(2k+1) a_0\right] (2k+1) W^{2k+2} + \left[(2k-1) a_1 - 2k a_0\right] (2k-1) W^{2k} \\
& + \left[(2k-3) a_2 - (2k-2) a_1\right] (2k-3) W^{2k-2} + \left[(2k-5) a_3 - (2k-4) a_2\right] (2k-5) W^{2k-4} \\
& \qquad\qquad - \left[(2k-6) a_3\right] (2k-7) W^{2k-6} \\
& + \cdots + 49 a_{k-3} W^8 + \left[25 a_{k-2} - 30 a_{k-3}\right] W^6 \\
& + \left[9 a_{k-1} - 12 a_{k-2}\right] W^4 + \left[a_k - 2 a_{k-1}\right] W^2 \\
& - \left[(2k+1) a_0\right] (2k+1) W^{2k} - \left[(2k-1) a_1 - 2k a_0\right] (2k-1) W^{2k-2} \\
& - \left[(2k-3) a_2 - (2k-2) a_1\right] (2k-3) W^{2k-4} - \left[(2k-5) a_3 - (2k-4) a_2\right] (2k-5) W^{2k-6} \\
& \qquad\qquad + \left[(2k-6) a_3\right] (2k-7) W^{2k-8} \\
& - \cdots - 49 a_{k-3} W^6 - \left[25 a_{k-2} - 30 a_{k-3}\right] W^4 \\
& - \left[9 a_{k-1} - 12 a_{k-2}\right] W^2 - \left[a_k - 2 a_{k-1}\right]
\end{aligned}
\right\}
$$

$$
= (-1)^{2k+2} K_0 \left\{
\begin{aligned}
& \Big\{ \big[(2k+1)+(2k+1)(2k+1)\big] a_0 \Big\} W^{2k+2} \\
& + \Big\{ \big[(2k-1)+(2k-1)(2k-1)\big] a_1 - \big[(2k+1)(2k+1)+(2k)(2k)\big] a_0 \Big\} W^{2k} \\
& + \left\{ \begin{aligned} & \big[(2k-3)+(2k-3)(2k-3)\big] a_2 - \big[(2k-1)(2k-1)+(2k-2)(2k-2)\big] a_1 \\ & \qquad - \big[2k-(2k)(2k)\big] a_0 \end{aligned} \right\} W^{2k-2} \\
& + \left\{ \begin{aligned} & \big[(2k-5)+(2k-5)(2k-5)\big] a_3 - \big[(2k-3)(2k-3)+(2k-4)(2k-4)\big] a_2 \\ & \qquad - \big[(2k-2)-(2k-2)(2k-2)\big] a_1 \end{aligned} \right\} W^{2k-4} \\
& + \left\{ \begin{aligned} & - \big[(2k-5)(2k-5)+(2k-6)(2k-6)\big] a_3 \\ & - \big[(2k-4)-(2k-4)(2k-4)\big] a_2 \end{aligned} \right\} W^{2k-6} \\
& + \Big\{ - \big[(2k-6)-(2k-6)(2k-6)\big] a_3 \Big\} W^{2k-8} \\
& + \cdots + 56 a_{k-3} W^8 \\
& + \big[30 a_{k-2} - 85 a_{k-3}\big] W^6 \\
& + \big[12 a_{k-1} - 41 a_{k-2} + 30 a_{k-3}\big] W^4 \\
& + \big[2 a_k - 13 a_{k-1} + 12 a_{k-2}\big] W^2 \\
& - \big[a_k - 2 a_{k-1}\big]
\end{aligned}
\right\}
$$

$$
= (-1)^{2k+2} K_0 H_{2k+2}(r, \gamma).
$$

Thus our proposition (4.20) is true for $2k+1$ and $2k+2$ if it is true for $2k$. Hence, it is true for all k and hence for all $m=0,1,2,\ldots$, we have

$$(4.4)\quad \left\{\begin{array}{l} K_m(r,\gamma)=(-1)^m K_0 H_m(r,\gamma) \text{ where} \\ H_m(r,\gamma) \text{ is a polynomial in } W \text{ of degree } m, \\ \text{the sum of whose coefficients equals } 1. \end{array}\right\}$$

Hence it follows from (4.19) that

$$(4.41)\quad H_m(1,\gamma)=1 \text{ for } 0<\gamma\leq\pi \text{ and } H_m(r,\gamma)=O(1) \text{ with } \begin{cases} 0\leq r\leq 1 \\ 0\leq\gamma\leq\pi \end{cases}$$

Now

$$(4.5)\quad K_0(r,\gamma)=\begin{cases} O(\sin\gamma) \text{ for } \gamma>\xi>0 \text{ for all } r \text{ on } 0\leq r\leq 1 \\ O(\frac{1}{\gamma}) \text{ as } \gamma=\text{arc}\sin(1-r)\to 0 \\ O(1) \text{ in } \gamma \text{ on } 0\leq\gamma\leq\pi \text{ with } 0\leq r<1 \end{cases}$$

and

$K_0(r,0)=K_0(r,\pi)=0$ with $0\leq r<1$.

Hence, by (4.4), (4.41) and (4.5), it follows that

$$(4.6)\quad K_m(r,\gamma)=\begin{cases} O(\sin\gamma) \text{ for } \gamma>\xi>0 \text{ for all } r \text{ on } 0\leq r\leq 1 \\ O(\frac{1}{\gamma}) \text{ as } \gamma=\text{arc}\sin(1-r)\to 0 \\ O(1) \text{ in } \gamma \text{ on } 0\leq\gamma\leq\pi \text{ with } 0\leq r<1 \end{cases}$$

and

$K_m(r,0)=K_m(r,\pi)=0$ for all $m=0,1,2,\ldots$

This proves the lemma.

(5.) Lemma 3.

$$\frac{dK_m(r,\gamma)}{d\gamma} = O\left(\frac{1}{(1-r)^2}\right) \text{ for } 0 < \gamma \leqq \epsilon, \text{ as } \epsilon = \text{arc } \sin(1-r) \to 0.$$

Proof of Lemma 3. From the relations (4.11) and (4.13) of Lemma 2, we have

(5.1) $$\frac{dK_0}{d\gamma} = K_0\left(\frac{-1}{\frac{d\Delta}{d\gamma}}\right)\left[2K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right]$$

$$= \left(\frac{\frac{d\Delta}{d\gamma}}{2\Delta}\right)\left(\frac{-1}{\frac{d\Delta}{d\gamma}}\right)\left[2K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right]$$

$$= \frac{-1}{2\Delta}\left[K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right]$$

$$= \left(\frac{1}{(1-r)^2}\right)\left(\frac{-\frac{1}{2}(1-r)^2}{(1-r)^2+4r\sin^2\frac{\gamma}{2}}\right)\left[2K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right].$$

By (4.19), we have

(5.2) $$\left[2K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right] = O(1) \text{ for } 0<\gamma\leqq\epsilon \text{ as } \epsilon = \text{arc } \sin(1-r)\to 0.$$

Moreover

(5.3) $$\frac{-\frac{1}{2}(1-r)^2}{(1-r)^2+4r\sin^2\frac{\gamma}{2}} = O(1) \text{ for } 0<\gamma\leqq\epsilon \text{ as } \epsilon = \text{arc } \sin(1-r)\to 0,$$

Hence

(5.4) $$\frac{dK_0(r,\gamma)}{d\gamma} = O\left(\frac{1}{(1-r)^2}\right) \text{ for } 0<\gamma\leqq\epsilon \text{ as } \epsilon = \text{arc } \sin(1-r)\to 0.$$

From the relation (4.4) in the proof of Lemma 2, we have

(5.5) $K_m(r,\gamma) = K_0(r,\gamma)\ (-1)^m H_m(r,\gamma),$

where

(5.6) $H(r,\gamma) = O(1)$ for $0<\gamma\leqq\epsilon$ as $\epsilon = \text{arc}\sin(1-r)\to 0.$

Upon taking the derivative of $K_m(r,\gamma)$, we have

(5.7) $$\frac{dK_m(r,\gamma)}{d\gamma} = \frac{dK_0(r,\gamma)}{d\gamma}\ (-1)^m H_m(r,\gamma) + K_0(r,\gamma)\ (-1)^m \frac{dH_m(r,\gamma)}{d\gamma}.$$

By (5.4) and (5.6), we have

(5.71) $$\frac{dK_0(r,\gamma)}{d\gamma}\ (-1)^m H_m(r,\gamma) = O\left(\frac{1}{(1-r)^2}\right) \text{ for } 0<\gamma\leqq\epsilon \text{ as } \epsilon = \text{arc}\sin(1-r)\to 0.$$

By Lemma 2 relation (4.4), $\frac{dH_m(r,\gamma)}{d\gamma}$ is a polynomial in

$\dfrac{d\left[2K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right]}{d\gamma}$ which by (4.15) is equal to

$$-\frac{\left[\left(2K_0\frac{d\Delta}{d\gamma} - \frac{d^2\Delta}{d\gamma^2}\right)^2 - 4r^2\right]}{\frac{d\Delta}{d\gamma}} = O\left(\frac{1}{\sin\gamma}\right) \text{ for } 0<\gamma\leqq\epsilon, \text{ as } \epsilon = \text{arc}\sin(1-r)\to 0.$$

Hence we have

(5.72) $$K_0(r,\gamma)\ \frac{dH_m}{d\gamma} = \left(\frac{r\sin\gamma}{\Delta}\right)\left[O\left(\frac{1}{\sin\gamma}\right)\right] = O\left(\frac{1}{\Delta}\right) = O\left(\frac{1}{(1-r)^2}\right) \text{ for } 0<\gamma\leqq\epsilon \text{ as}$$

$\epsilon = \text{arc}\sin(1-r)\to 0.$

Hence, by (5.7), (5.71), (5.72), we have

$$\frac{dK_m(r,\gamma)}{d\gamma} = O\left(\frac{1}{(1-r)^2}\right) \text{ for } 0<\gamma\leqq\epsilon \text{ as } \epsilon = \text{arc}\sin(1-r)\to 0,$$

This proves the lemma.

(6.) Lemma 4. If

$$q_0(r,\gamma) = \frac{(1-r)^2}{2} \frac{\cot\frac{\gamma}{2}}{\Delta(r,\gamma)},$$

$$q_1(r,\gamma) = \sin\gamma \frac{d[q_0(r,\gamma)]}{d\gamma},$$

$$q_2(r,\gamma) = \sin\gamma \frac{d[q_1(r,\gamma)]}{d\gamma},$$

$$\vdots$$

$$q_m(r,\gamma) = \sin \frac{d[q_{m-1}(r,\gamma)]}{d\gamma},$$

then

$$\frac{q_m(r,\gamma)}{(1-r)^2} = \begin{cases} O(\sin\gamma) \text{ for } \gamma > \xi > 0 \text{ for all } r \text{ on } 0 \leq r \leq 1 \\ O\left(\frac{1}{\gamma^3}\right) \text{as } \gamma = \text{arc } \sin(1-r) \to 0 \end{cases}$$

$$q_m(r,\gamma) = \begin{cases} O\left((1-r)^2(\sin\gamma)\right) \text{ for } \gamma > \xi > 0 \text{ and } 0 \leq r \leq 1 \\ O\left(\frac{(1-r)^2}{\gamma^3}\right) \text{ as } \gamma = \text{arc } \sin(1-r) \to 0 \\ O\left(\frac{1}{\gamma}\right) \text{as } \gamma = \text{arc } \sin(1-r) \to 0 \end{cases}$$

Proof of Lemma 4.

By Lemma 5 and Lemma 6, which are proved independently, we have

(6.1) $q_m(r,\gamma) = J_m(\gamma) - K_m(r,\gamma)$,

(6.2) $$J_m(\gamma) = (-1)^m \frac{\cot\frac{\gamma}{2}}{2} = \left[(-1)^m \frac{\cot\frac{\gamma}{2}}{2}\right] \left[\frac{\gamma^3}{(1-r)^2}\right]\left[\frac{(1-r)^2}{\gamma^3}\right]$$

$$= \left[(-1)^m \frac{\gamma\cot\frac{\gamma}{2}}{2}\right]\left[\frac{\gamma^2}{(1-r)^2}\right]\left[\frac{(1-r)^2}{\gamma^3}\right] = O\left(\frac{(1-r)^2}{\gamma^3}\right) \text{ as } \gamma = \text{arc } \sin(1-r) \to 0.$$

By Lemma 2, we have

(6.3) $$K_m(r,\gamma) = O\left(\frac{1}{\gamma}\right) = O\left(\frac{\gamma^2}{(1-r)^2}\right) O\left(\frac{(1-r)^2}{\gamma^3}\right) = O\left(\frac{(1-r)^2}{\gamma^3}\right) \text{ as } \gamma = \text{arc } \sin(1-r) \to 0.$$

Hence we have from (6.1), (6.2), (6.3),

$$(6.4)\qquad \frac{q_m(r,\gamma)}{(1-r)^2} = O\left(\frac{1}{\gamma^3}\right) \text{ as } \gamma = \arcsin(1-r) \to 0,$$

$$(6.5)\qquad q_m(r,\gamma) = O\left(\frac{(1-r)^2}{\gamma^3}\right) = O\left(\frac{1}{\gamma}\right) \text{ as } \gamma = \arcsin(1-r) \to 0.$$

From (4.4) in the proof of Lemma 2, we have

$$(6.6)\qquad \left\{\begin{array}{l} K_m(r,\gamma) = (-1)^m K_0(r,\gamma) H_m(r,\gamma) \text{ where } H_m(r,\gamma) \text{ is a polynomial} \\ \text{of degree } m \text{ in } W(r,\gamma) = \dfrac{2r-(1+r^2)\cos\gamma}{1-2r\cos\gamma+r^2}, \text{ the sum of whose} \\ \text{coefficients equals } 1 \end{array}\right\},$$

whence from Lemma 5 and Lemma 6, we have

$$\begin{aligned}(6.7)\qquad q_m(r,\gamma) &= J_m(r,\gamma) - K_m(r,\gamma) \\ &= (-1)^m\left[J_0(\gamma) - K_0(r,\gamma)H_m(r,\gamma)\right] \\ &= (-1)^m\left\{J_0(\gamma) - K_0(r,\gamma) + K_0(r,\gamma)\left[1 - H_m(r,\gamma)\right]\right\} \\ &= (-1)^m\left\{q_0(r,\gamma) + K_0(r,\gamma)\left[1 - H_m(r,\gamma)\right]\right\}.\end{aligned}$$

By the factor theorem of algebra if $0 < \gamma \leqq \pi$, we have from (6.6) that

$$(6.71)\qquad \left[1 - H(r,\gamma)\right] = (1-W)R_{m-1}(r,\gamma) = \left\{(1-r)^2\left[\frac{1+\cos\gamma}{1-2r\cos\gamma+r^2}\right]\right\}R_{m-1}(r,\gamma),$$

where $R_{m-1}(r,\gamma)$ is a polynomial in $W(r,\gamma)$ of degree $m-1$ and hence bounded for $\begin{cases} 0 \leqq r \leqq 1 \\ 0 \leqq \gamma \leqq \pi \end{cases}$.

From the definition of $q_0(r,\gamma)$, we have

$$(6.72)\qquad q_0(r,\gamma) = (1-r)^2\left(\frac{J_0(\gamma)}{1-2r\cos\gamma+r^2}\right).$$

Hence from (6.7), (6.71), (6.72), we have

(6.73) $$q_m(r,\gamma) = (1-r)^2 \left\{ \frac{(-1)^m \left[J_0(\gamma) + K_0(r,\gamma)(1+\cos\gamma) R_{m-1}(r,\gamma) \right]}{1-2r\cos\gamma + r^2} \right\}.$$

Since

$$(-1)^m \left\{ \frac{\frac{J_0(\gamma)}{\sin\gamma} + \frac{K_0(r,\gamma)}{\sin\gamma} \left[(1+\cos\gamma) R_{m-1}(r,\gamma) \right]}{1-2r\cos\gamma + r^2} \right\}$$

is bounded for $\begin{cases} 0 < \xi < \gamma < \pi, \\ 0 \leq r \leq 1 \end{cases}$

we have

(6.75) $$\frac{q(r,\gamma)}{(1-r)^2 \sin\gamma} = (-1)^m \left\{ \frac{\frac{J_0(\gamma)}{\sin\gamma} + \frac{K_0(r,\gamma)}{\sin\gamma} \left[(1+\cos\gamma) R_{m-1}(r,\gamma) \right]}{1-2r\cos\gamma + r^2} \right\}$$

bounded for $\begin{cases} 0 < \xi < \gamma < \pi, \\ 0 \leq r \leq 1 \end{cases}$

whence (6.4), (6.5) and (6.75) establish the lemma.

(7.) Lemma 5. If

$$J_0(\gamma) = \tfrac{1}{2}\cot\tfrac{\gamma}{2},$$

$$J_1(\gamma) = \sin\gamma \frac{dJ_0(\gamma)}{d\gamma},$$

$$\vdots$$

$$J_m(\gamma) = \sin\gamma \frac{dJ_{m-1}(\gamma)}{d\gamma},$$

then

$$J_m(\gamma) = (-1)^m \tfrac{1}{2}\cot\tfrac{\gamma}{2}, \qquad m = 0, 1, 2, \ldots .$$

Proof of Lemma 5: The relations follow upon taking successive derivatives.

(8.) Lemma 6.

$$K_m(r,\gamma) = J_m(\gamma) - q_m(r,\gamma), \qquad m = 0, 1, 2, \ldots .$$

Proof of Lemma 6.

$$K_0(r,\gamma) - J_0(\gamma) = \frac{r \sin\gamma}{1-2r\cos\gamma + r^2} - \frac{\sin\gamma}{2(1-\cos\gamma)}$$

$$= -\left(\frac{\frac{1}{2}(1-r)^2 \cot\frac{\gamma}{2}}{(1-r)^2 + 4r\sin^2\frac{\gamma}{2}} \right) = -q_0(r,\gamma)$$

Subsequent relations follow from the definitions of the functions concerned as given in Lemmas 2, 4, 5.

(9.) Lemma 7.

$$K_m(r,\gamma)K_m(\rho,\delta) = J_m(\gamma)J_m(\delta) - \left\{ q_m(r,\gamma)q_m(\rho,\delta) + q_m(r,\gamma)K_m(\rho,\delta) + K_m(r,\gamma)q_m(\rho,\delta) \right\}$$

Proof of Lemma 7.

By Lemma 6, $q_m(r,\gamma) = J_m(\gamma) - K_m(r,\gamma)$.

Hence

$$J_m(\gamma)J_m(\delta) - \left\{ q_m(r,\gamma)q_m(\rho,\delta) + q_m(r,\gamma)K_m(\rho,\delta) + K_m(r,\gamma)q_m(\rho,\delta) \right\}$$

$$= J_m(\gamma)J_m(\delta) - \left[J_m(\gamma) - K_m(r,\gamma)\right]\left[J_m(\delta) - K_m(\rho,\delta)\right] - \left[J_m(\gamma) - K_m(r,\gamma)\right]\left[K_m(\rho,\delta)\right]$$

$$-\left[K_m(r,\gamma)\right]\left[J_m(\delta) - K_m(\rho,\delta)\right] = J_m(\gamma)J_m(\delta) - K_m(r,\gamma)K_m(\rho,\delta) - J_m(\gamma)J_m(\delta)$$

$$+ J_m(\gamma)K_m(\rho,\delta) + J_m(\delta)K_m(r,\gamma) + K_m(r,\gamma)K_m(\rho,\delta) - J_m(\gamma)K_m(\rho,\delta) - J_m(\delta)K_m(r,\gamma)$$

$$+ K_m(r,\gamma)K_m(\rho,\delta)$$

$$= K_m(r,\gamma)K_m(\rho,\delta) .$$

(10.) Lemma 8: If $f(x,y)$ is integrable (L) in $Q(-\pi,-\pi;\pi,\pi)$ and periodic of period 2π and ${}_k\chi_k(x,y;U,V)$, $J_k(U)$, $J_k(V)$ are as defined, $k=0,1,2,\cdots,m+1$, then $\int_\eta^\pi\int_\epsilon^\pi {}_0\chi_0(x,y;U,V)\, J_0(U)\, J_0(V)\, dU\, dV$

$$= \sin\epsilon\ \sin\eta\Big[J_0(\epsilon)J_0(\eta)\,{}_1\chi_1(x,y;\epsilon,\eta)+\cdots+J_m(\epsilon)J_m(\eta)\,{}_{m+1}\chi_{m+1}(x,y;\epsilon,\eta)\Big]$$

$$+\sin\epsilon\int_\eta^\pi\Big[J_0(\epsilon)\,{}_1\chi_1(x,y;\epsilon,V)\,J_1(V)+\cdots+J_m(\epsilon)\,{}_{m+1}\chi_{m+1}(x,y;\epsilon,V)J_{m+1}(V)\Big]\,dV$$

$$+\sin\eta\int_\epsilon^\pi\Big[J_0(\eta)\,{}_1\chi_1(x,y;U,\eta)J_1(U)+\cdots+J_m(\eta)\,{}_{m+1}\chi_{m+1}(x,y;U,\eta)J_{m+1}(U)\Big]\,dU$$

$$+\int_\eta^\pi\int_\epsilon^\pi {}_{m+1}\chi_{m+1}(x,y;U,V)J_{m+1}(U)J_{m+1}(V)\,dU\,dV\,,$$

each of the integrals above existing for fixed positive ϵ and η and for any given $m=0,1,2,\cdots$.

Proof of Lemma 8. We use mathematical induction. We first develop a recurrence relation by an integration by parts. Consider the integration:

$$(10.01)\quad \int_\eta^\tau\int_\epsilon^s {}_k\chi_k(x,y;U,V)\,J_k(U)\,J_k(V)\,dU\,dV$$

$$=\int_\eta^\tau J_k(V)\left\{\begin{aligned}&\Big[{}_k\chi_{k+1}(x,y;U,V)\sin U\,J_k(U)\Big]_\epsilon^s\\&-\int_\epsilon^s {}_k\chi_{k+1}(x,y;U,V)\,J_{k+1}(U)\,dU\end{aligned}\right\}dV$$

$$=\sin s\,J_k(s)\left\{\begin{aligned}&\Big[{}_{k+1}\chi_{k+1}(x,y;s,V)\sin V\,J_k(V)\Big]_\eta^\tau\\&-\int_\eta^\tau {}_{k+1}\chi_{k+1}(x,y;s,V)\,J_{k+1}(V)\,dV\end{aligned}\right\}$$

$$-\sin\epsilon\; J_k(\epsilon)\left\{\left[{}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\,\sin V\; J_k(V)\right]_\eta^\tau - \int_\eta^\tau {}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\; J_{k+1}(V)\; dV\right\}$$

$$-\int_\epsilon^s J_{k+1}(U)\left\{\left[{}_{k+1}\chi_{k+1}(x,y;U,V)\,\sin V\; J_k(V)\right]_\eta^\tau - \int_\eta^\tau {}_{k+1}\chi_{k+1}(x,y;U,V)\; J_{k+1}(V)\; dV\right\} dU$$

$$= \sin\epsilon\; J_k(\epsilon)\;\sin\eta\; J_k(\eta)\;{}_{k+1}\chi_{k+1}(x,y;\epsilon,\eta)$$

$$+\sin\epsilon\; J_k(\epsilon)\int_\eta^\tau {}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\; J_{k+1}(V)\; dV$$

$$+\sin\eta\; J_k(\eta)\int_\epsilon^s {}_{k+1}\chi_{k+1}(x,y;U,\eta)\; J_{k+1}(U)\; dU$$

$$+\int_\eta^\tau\int_\epsilon^s {}_{k+1}\chi_{k+1}(x,y;U,V)\; J_{k+1}(U)\; J_{k+1}(V)\; dU\; dV$$

$$+\sin s\; J_k(s)\;\sin t\; J_k(t)\;{}_{k+1}\chi_{k+1}(x,y;s,t)$$

$$-\sin s\; J_k(s)\;\sin\eta\; J_k(\eta)\;{}_{k+1}\chi_{k+1}(x,y;s,\eta)$$

$$-\sin\epsilon\; J_k(\epsilon)\;\sin t\; J_k(t)\;{}_{k+1}\chi_{k+1}(x,y;\epsilon,t)$$

$$-\sin s\; J_k(s)\int_\eta^\tau {}_{k+1}\chi_{k+1}(x,y;s,V)\; J_{k+1}(V)\; dV$$

$$-\sin t\; J_k(t)\int_\epsilon^s {}_{k+1}\chi_{k+1}(x,y;U,t)\; J_{k+1}(U)\; dU.$$

For the case $k=0$, all the integrals above exist, and the last five terms disappear as λ and τ are replaced by π since ${}_0\chi_0(x,y;U,V)$ is integrable (L) by hypothesis and $\begin{cases}\epsilon \neq 0\\ \eta \neq 0.\end{cases}$

Thus our lemma is true for $k=0$, and we have

$$(10.02)\quad \int_\eta^\pi\int_\epsilon^\pi {}_0\chi_0(x,y;U,V)\, J_0(U)\, J_0(V)\, dU\, dV$$

$$= \sin\epsilon\ J_0(\epsilon)\ \sin\eta\ J_0(\eta)\ {}_1\chi_1(x,y;\epsilon,\eta) + \sin\epsilon\ J_0(\epsilon)\int_\eta^\pi {}_1\chi_1(x,y;\epsilon,V)J_1(V)dV$$

$$+ \sin\eta\ J_0(\eta)\int_\epsilon^\pi {}_1\chi_1(x,y;U,\eta)\ J_1(U)\ dU + \int_\eta^\pi\int_\epsilon^\pi {}_1\chi_1(x,y;U,V)J_1(U)J_1(V)\ dU\ dV.$$

Suppose the lemma is true for k. In particular, suppose that

$$(10.1)\quad \int_\eta^\pi\int_\epsilon^\pi {}_0\chi_0(x,y;U,V)\ J_0(U)\ J_0(V)\ dU\ dV$$

$$= \sin\epsilon\ \sin\eta\ \left[J_0(\epsilon)J_0(\eta)\,{}_1\chi_1(x,y;\epsilon,\eta) + \cdots + J_{k-1}(\epsilon)J_{k-1}(\eta)\,{}_k\chi_k(x,y;\epsilon,\eta)\right]$$

$$+ \sin\epsilon\int_\eta^\pi\left[{}_1\chi_1(x,y;\epsilon,V)\ J_0(\epsilon)J_1(V) + \cdots + {}_k\chi_k(x,y;\epsilon,V)J_{k-1}(\epsilon)J_k(V)\right] dV$$

$$+ \sin\eta\int_\epsilon^\pi\left[{}_1\chi_1(x,y;U,\eta)\ J_1(U)J_0(\eta) + \cdots + {}_k\chi_k(x,y;U,\eta)J_k(U)J_{k-1}(\eta)\right] dU$$

$$+ \int_\eta^\pi\int_\epsilon^\pi {}_k\chi_k(x,y;U,V)\ J_k(U)J_k(V)\ dU\ dV,$$

with each of the integrals existing.

We will show that then

$$(10.11)\quad \int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \chi_{k\,k}(x,y;U,V)\, J_k(U)\, J_k(V)\, dU\, dV$$

$$= \sin\epsilon \sin\eta\, J_k(\epsilon)\, J_k(\eta)\, \chi_{k+1\,k+1}(x,y;\epsilon,\eta)$$

$$+ \sin\epsilon \int_{\eta}^{\pi} \chi_{k+1\,k+1}(x,y;\epsilon,V)\, J_k(\epsilon)\, J_{k+1}(V)\, dV$$

$$+ \sin\eta \int_{\epsilon}^{\pi} \chi_{k+1\,k+1}(x,y;U,\eta)\, J_{k+1}(U)\, J_k(\eta)\, dU$$

$$+ \int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \chi_{k+1\,k+1}(x,y;U,V)\, J_{k+1}(U)\, J_{k+1}(V)\, dU\, dV,$$

each one of the integrals existing which will then prove the lemma.

To do this, we refer to the recurrence relation (10.01) and show that each of the integrals

$$(10.20)\quad \sin\epsilon\, J_k(\epsilon)\, \sin\eta\, J_k(\eta)\, \chi_{k+1\,k+1}(x,y;\epsilon,\eta),$$

$$(10.30)\quad \sin\epsilon\, J_k(\epsilon) \int_{\eta}^{\tau} \chi_{k+1\,k+1}(x,y;\epsilon,V)\, J_{k+1}(V)\, dV,$$

$$(10.40)\quad \sin\eta\, J_k(\eta) \int_{\epsilon}^{\Delta} \chi_{k+1\,k+1}(x,y;U,\eta)\, J_{k+1}(U)\, dU,$$

exist as $\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to\pi$, and that each of the integrals

$$(10.50)\quad \sin\Delta\, J_k(\Delta)\, \sin\tau\, J_k(\tau)\, \chi_{k+1\,k+1}(x,y;\Delta,\tau),$$

$$(10.60) \quad -\sin \mathcal{L}\, J_k(\mathcal{L}) \sin \eta\, J_k(\eta)\, {}_{k+1}\chi_{k+1}(x,y;\mathcal{L},\eta),$$

$$(10.70) \quad -\sin \epsilon\, J_k(\epsilon) \sin \tau\, J_k(\tau)\, {}_{k+1}\chi_{k+1}(x,y;\epsilon,\tau),$$

$$(10.80) \quad -\sin \mathcal{L}\, J_k(\mathcal{L}) \int_{\eta}^{\tau} {}_{k+1}\chi_{k+1}(x,y;\mathcal{L},V)\, J_{k+1}(V)\, dV,$$

$$(10.90) \quad -\sin \tau\, J_k(\tau) \int_{\epsilon}^{\mathcal{L}} {}_{k+1}\chi_{k+1}(x,y;U,\tau)\, J_{k+1}(U)\, dU,$$

tend to 0 as $\left.\begin{matrix}\mathcal{L}\\ \tau\end{matrix}\right\} \to \pi$, and hence that the asserted relation (10.11) is true, thus proving the lemma.

Since $\left\{\begin{matrix}\epsilon \neq 0\\ \eta \neq 0\end{matrix}\right.$ are fixed throughout the lemma, the integral (10.20) already exists as $\left.\begin{matrix}\mathcal{L}\\ \tau\end{matrix}\right\} \to \pi$.

For the existence of the integral

$$(10.30) \quad \sin \epsilon\, J_k(\epsilon) \int_{\eta}^{\pi} {}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\, J_{k+1}(V)\, dV$$

$$= -\sin \epsilon\, \frac{\cot \epsilon}{2} \int_{\eta}^{\pi} {}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\, \frac{\cot V}{2}\, dV,$$

for fixed $\left\{\begin{matrix}\epsilon \neq 0\\ \eta \neq 0\end{matrix}\right.$ as $\tau \to \pi$, we again integrate by parts as follows:

$$(10.31)\quad -\sin\epsilon\ \frac{\cot\frac{\epsilon}{2}}{2}\int_{\eta}^{\tau} {}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\ \frac{\cot\frac{V}{2}}{2}\,dV$$

$$= -\sin\epsilon\ \frac{\cot\frac{\epsilon}{2}}{2}\left\{\left[-{}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\ \sin V\ \frac{\cot\frac{V}{2}}{2}\right]_{\eta}^{\tau} + \int_{\eta}^{\pi} {}_{k}\chi_{k+1}(x,y;\epsilon,V)\ \frac{\cot\frac{V}{2}}{2}\,dV\right\}$$

$$= \frac{\cot\frac{\epsilon}{2}}{2}\ \frac{\cot\frac{\tau}{2}}{2}\int_{0}^{\tau}\int_{0}^{\epsilon} {}_{k}\chi_{k}(x,y;U,V)\,dU\,dV$$

$$- \frac{\cot\frac{\epsilon}{2}}{2}\ \frac{\cot\frac{\eta}{2}}{2}\int_{0}^{\eta}\int_{0}^{\epsilon} {}_{k}\chi_{k}(x,y;U,V)\,dU\,dV$$

$$- \sin\epsilon\ \frac{\cot\frac{\epsilon}{2}}{2}\int_{\eta}^{\pi} {}_{k}\chi_{k+1}(x,y;\epsilon,V)\ \frac{\cot\frac{V}{2}}{2}\,dV$$

$$= \cdots$$

$$= \frac{\cot\frac{\epsilon}{2}}{2}\ \frac{\cot\frac{\tau}{2}}{2}\int_{0}^{\tau}\int_{0}^{\epsilon}\left[{}_{k}\chi_{k}(x,y;U,V) + \cdots + {}_{0}\chi_{k}(x,y;U,V)\right]dU\,dV$$

$$- \frac{\cot\frac{\epsilon}{2}}{2}\ \frac{\cot\frac{\eta}{2}}{2}\int_{0}^{\eta}\int_{0}^{\epsilon}\left[{}_{k}\chi_{k}(x,y;U,V) + \cdots + {}_{0}\chi_{k}(x,y;U,V)\right]dU\,dV$$

$$- \sin\epsilon\ \frac{\cot\frac{\epsilon}{2}}{2}\int_{\eta}^{\tau} {}_{0}\chi_{k+1}(x,y;\ ,V)\ \frac{\cot\frac{V}{2}}{2}\,dV,$$

and we have

$$(10.32)\quad \frac{\cot\frac{\epsilon}{2}}{2}\ \frac{\cot\frac{\tau}{2}}{2}\int_{0}^{\tau}\int_{0}^{\epsilon}\left[{}_{k}\chi_{k}(x,y;U,V) + \cdots + {}_{0}\chi_{k}(x,y;U,V)\right]dU\,dV$$

$\to 0$ as $\tau \to \pi$ uniformly on $0 < \alpha < \epsilon < \pi$ by Lemma 1,

$$(10.33)\quad -\cot\frac{\epsilon}{2}\ \cot\frac{\eta}{2}\int_0^{\eta}\!\!\int_0^{\epsilon}\left[\chi_k(x,y;U,V)+\cdots+{}_0\chi_k(x,y;U,V)\right]dU\,dV$$

existing for fixed $\begin{cases}\epsilon\\ \eta\end{cases}$ such that $\begin{cases}0<\alpha<\epsilon<\pi\\ 0<\beta<\eta<\pi\end{cases}$ by the definition given

in (2.2), while

$$(10.34)\quad -\sin\epsilon\ \cot\frac{\epsilon}{2}\int_{\eta}^{\tau}{}_0\chi_{k+1}(x,y;\epsilon,V)\ \cot\frac{V}{2}\ dV$$

$$\rightarrow -\sin\epsilon\ \cot\frac{\epsilon}{2}\int_{\eta}^{\pi}{}_0\chi_{k+1}(x,y;\epsilon,V)\ \cot\frac{V}{2}\ dV$$

as $\tau\to\pi$ since ${}_0\chi_0(x,y;U,V)$ is integrable (L), and hence the integrand in (10.34) above is integrable in V all the way up to π for fixed $\begin{cases}\epsilon\neq 0\\ \eta\neq 0\end{cases}$, whence from (10.31), (10.32), (10.33), (10.34) the integral (10.30) exists as $\tau\to\pi$ for fixed $\begin{cases}\epsilon\neq 0\\ \eta\neq 0\end{cases}$.

Similarly, the integral (10.40) exists as $\lambda\to\pi$ for fixed $\begin{cases}\epsilon\neq 0\\ \eta\neq 0\end{cases}$.

We now proceed to show that the integrals (10.50), (10.60), (10.70), (10.80), (10.90) all tend to 0 as $\begin{matrix}\lambda\\ \tau\end{matrix}\Big\}\to\pi$ for fixed $\begin{cases}\epsilon\neq 0\\ \eta\neq 0\end{cases}$.

We have

$$(10.50)\quad \sin s \; J_k(s) \; \sin\tau \; J_k(\tau) \; {}_{k+1}\chi_{k+1}(x,y;s,\tau)$$

$$= \frac{\int_0^{\tau}\int_0^{s} {}_k\chi_k(x,y;U,V)\, dU\, dV}{\left(2\tan\frac{s}{2}\right)\left(2\tan\frac{\tau}{2}\right)} \to 0 \text{ as } \begin{cases} \left.\begin{matrix} s \\ \tau \end{matrix}\right\} \to \pi \\ s \to \pi \text{ uniformly on } 0<\beta<\tau<\pi \\ \tau \to \pi \text{ uniformly on } 0<\alpha<s<\pi \end{cases}$$

by Lemma 1,

$$(10.60)\quad -\sin s \; J_k(s) \; \sin\eta \; J_k(\eta) \; {}_{k+1}\chi_{k+1}(x,y;s,\eta)$$

$$= -\frac{\int_0^{\eta}\int_0^{s} {}_k\chi_k(x,y;U,V)\, dU\, dV}{\left(2\tan\frac{s}{2}\right)\left(2\tan\frac{\eta}{2}\right)} \to 0 \text{ as } s \to \pi \text{ uniformly on } 0<\beta<\eta<\pi$$

by Lemma 1,

$$(10.70)\quad -\sin\epsilon \; J_k(\epsilon) \; \sin\tau \; J_k(\tau) \; {}_{k+1}\chi_{k+1}(x,y;\epsilon,\tau)$$

$$= -\frac{\int_0^{\tau}\int_0^{\epsilon} {}_k\chi_k(x,y;U,V)\, dU\, dV}{\left(2\tan\frac{\epsilon}{2}\right)\left(2\tan\frac{\tau}{2}\right)} \to 0 \text{ as } \tau \to \pi \text{ uniformly on } 0<\alpha<\epsilon<\pi$$

by Lemma 1.

For (10.80) an integration by parts exactly like (10.31) leads to the relation

$$(10.81)\quad -\sin s\ J_k(s)\int_\eta^\tau \underset{k+1}{X}_{k+1}(x,y;s,V)\ J_{k+1}(V)\ dV$$

$$=\sin s\ \frac{\cot\frac{s}{2}}{2}\int_\eta^\pi \underset{k+1}{X}_{k+1}(x,y;\ ,V)\ \frac{\cot\frac{V}{2}}{2}\,dV$$

$$= -\frac{\cot\frac{s}{2}}{2}\ \frac{\cot\frac{\tau}{2}}{2}\int_0^\tau\int_0^s\Big[\underset{k}{X}_k(x,y;U,V)+\cdots+\underset{0}{X}_k(x,y;U,V)\Big]\,dU\,dV$$

$$+\frac{\cot\frac{s}{2}}{2}\ \frac{\cot\frac{\eta}{2}}{2}\int_0^\eta\int_0^s\Big[\underset{k}{X}_k(x,y;U,V)+\cdots+\underset{0}{X}_k(x,y;U,V)\Big]\,dU\,dV$$

$$+\sin s\ \frac{\cot\frac{s}{2}}{2}\int_\eta^\tau \underset{0}{X}_{k+1}(x,y;s,V)\ \frac{\cot\frac{V}{2}}{2}\ dV.$$

From Lemma 1, we have

$$(10.82)\quad -\frac{\cot\frac{s}{2}}{2}\ \frac{\cot\frac{\tau}{2}}{2}\int_0^\tau\int_0^s\Big[\underset{k}{X}_k(x,y;U,V)+\cdots+\underset{0}{X}_k(x,y;U,V)\Big]\,dU\,dV$$

$$\to 0\ \text{ as }\begin{cases}\left.\begin{matrix}s\\ \tau\end{matrix}\right\}\to\pi\\ s\to\pi\ \text{uniformly on } 0<\beta<\tau<\pi,\\ \tau\to\pi\ \text{uniformly on } 0<\alpha<s<\pi\end{cases}$$

Also, by Lemma 1, we have

$$(10.83)\quad \frac{\cot\frac{s}{2}}{2}\ \frac{\cot\frac{\eta}{2}}{2}\int_0^\eta\int_0^s\Big[\underset{k}{X}_k(x,y;U,V)+\cdots+\underset{0}{X}_k(x,y;U,V)\Big]\,dU\,dV$$

$$\to 0\ \text{ as } s\to\pi\ \text{uniformly on } 0<\beta<\eta<\pi.$$

We will further integrate the last term of the right member of (10.81) above by parts as follows:

$$(10.84)\quad \sin s \; \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau} {}_{0}\chi_{k+1}(x,y;s,V)\, \frac{\cot \frac{V}{2}}{2}\, dV$$

$$= \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau}\int_{0}^{s} {}_{0}\chi_{k}(x,y;U,V)\, \frac{\cot \frac{V}{2}}{2}\, dU\, dV$$

$$= \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}} {}_{0}\chi_{k}(x,y;U,V)\, \frac{\cot \frac{V}{2}}{2}\, dU\, dV$$

$$+ \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau}\int_{\frac{\pi}{2}}^{s} {}_{0}\chi_{k}(x,y;U,V)\, \frac{\cot \frac{V}{2}}{2}\, dU\, dV$$

$$= \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}} {}_{0}\chi_{k}(x,y;U,V)\, \frac{\cot \frac{V}{2}}{2}\, dU\, dV$$

$$+ \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau} \frac{\cot \frac{V}{2}}{2} \left\{ \left[{}_{0}\chi_{k}(x,y;U,V) \sin U \log \tan \frac{U}{2} \right]_{\frac{\pi}{2}}^{s} - \int_{\frac{\pi}{2}}^{s} {}_{0}\chi_{k}(x,y;U,V) \log \tan \frac{U}{2}\, dU \right\} dV$$

$$= \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}} {}_{0}\chi_{k}(x,y;U,V)\, \frac{\cot \frac{V}{2}}{2}\, dU\, dV$$

$$+ \frac{\cot \frac{s}{2}}{2} \log \tan \frac{s}{2} \int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}} {}_{0}\chi_{k-1}(x,y;U,V)\, \frac{\cot \frac{V}{2}}{2}\, dU dV$$

$$+ \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau}\int_{\frac{\pi}{2}}^{s} {}_{0}\chi_{k-1}(x,y;U,V) \left(\log \frac{\tan \frac{s}{2}}{\tan \frac{U}{2}} \right) \frac{\cot \frac{V}{2}}{2}\, dU\, dV$$

$$= \frac{\cot \frac{s}{2}}{2} \int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}} {}_{0}\chi_{k}(x,y;U,V)\, \frac{\cot \frac{V}{2}}{2}\, dU\, dV$$

$$+\frac{\cot\frac{s}{2}}{2}\log\tan\frac{s}{2}\int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}}{}_{0}\chi_{k-1}(x,y;U,V)\,\frac{\cot\frac{V}{2}}{2}\,dU\,dV$$

$$+\frac{\cot\frac{s}{2}}{2}\int_{\eta}^{\tau}\frac{\cot\frac{V}{2}}{2}\left\{\begin{array}{l}\left[-{}_{0}\chi_{k-1}(x,y;U,V)\sin U\left(\frac{1}{2}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{U}{2}}\right)^{2}\right]_{\frac{\pi}{2}}^{s}\\ \displaystyle\int_{\frac{\pi}{2}}^{s}{}_{0}\chi_{k-2}(x,y;U,V)\left(\frac{1}{2}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{U}{2}}\right)^{2}dU\end{array}\right\}dV$$

$$=\cdots$$

$$=\frac{\cot\frac{s}{2}}{2}\int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}}{}_{0}\chi_{k}(x,y;U,V)\,\frac{\cot\frac{V}{2}}{2}\,dV$$

$$+\frac{\cot\frac{s}{2}}{2}\log\tan\frac{s}{2}\int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}}{}_{0}\chi_{k-1}(x,y;U,V)\,\frac{\cot\frac{V}{2}}{2}\,dU\,dV$$

$$+\frac{\cot\frac{s}{2}}{2}\,\frac{\left(\log\tan\frac{s}{2}\right)^{2}}{2}\int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}}{}_{0}\chi_{k-2}(x,y;U,V)\,\frac{\cot\frac{V}{2}}{2}\,dU\,dV$$

$$+\cdots$$

$$+\frac{\cot\frac{s}{2}}{2}\,\frac{\left(\log\tan\frac{s}{2}\right)^{k}}{k!}\int_{\eta}^{\tau}\int_{0}^{\frac{\pi}{2}}{}_{0}\chi_{0}(x,y;U,V)\,\frac{\cot\frac{V}{2}}{2}\,dU\,dV$$

$$+\frac{\cot\frac{s}{2}}{2}\int_{\eta}^{\tau}\int_{\frac{\pi}{2}}^{s}{}_{0}\chi_{0}(x,y;U,V)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{U}{2}}\right)^{k}dU\,dV\,.$$

Each of the terms in (10.84) above, which are of the form

$$(10.85)\quad \frac{\cot\frac{s}{2}}{2}\ \frac{\left(\log\ \tan\frac{s}{2}\right)^m}{m!}\int_\eta^\tau\int_0^{\frac{\pi}{2}} \chi_{k-m}^{}{}_0(x,y;U,V)\ \frac{\cot\frac{V}{2}}{2}\ dU\ dV$$

$\rightarrow 0$ as $s \rightarrow \pi$ uniformly with respect to τ on $0 < \eta < \tau \leqq \pi$ for fixed $\eta \neq 0$, since

$$\frac{\cot\frac{s}{2}}{2}\ \frac{\left(\log\ \tan\frac{s}{2}\right)^m}{m!} \rightarrow 0 \text{ as } s \rightarrow \pi \text{ and}$$

$$\int_\eta^\pi\int_0^{\frac{\pi}{2}} \chi_{k-m}^{}{}_0(x,y;U,V)\ \frac{\cot\frac{V}{2}}{2}\ dU\ dV \text{ exists for } \eta \neq 0 .$$

For the last term in (10.84) above, we have by the second mean value theorem

$$(10.86)\quad \frac{\cot\frac{s}{2}}{2}\int_\eta^\tau\int_{\frac{\pi}{2}}^{s} \chi_{0}{}_0(x,y;U,V)\left(\frac{1}{k!}\right)\left(\log\frac{\tan\frac{s}{2}}{\tan\frac{U}{2}}\right)^k \frac{\cot\frac{V}{2}}{2}\ dU\ dV$$

$$= \frac{\cot\frac{s}{2}}{2}\ \frac{\left(\log\ \tan\frac{s}{2}\right)^k}{k!}\int_\eta^\tau\int_{\frac{\pi}{2}}^{s^*} \chi_{0}{}_0(x,y;U,V)\ \frac{\cot\frac{V}{2}}{2}\ dU\ dV,$$

where $\frac{\pi}{2} < s^* < s \leqq \pi$,

$\rightarrow 0$ as $s \rightarrow \pi$ uniformly with respect to τ on $0 < \eta < \tau < \pi$ for fixed $\eta \neq 0$, since

$$\frac{\cot\frac{s}{2}}{2}\ \frac{\left(\log\ \tan\frac{s}{2}\right)^k}{k!}\to 0 \text{ as } s\to\pi \text{ and } \int_\eta^\tau\int_{\frac{\pi}{2}}^{s^*}{}_0\chi_0(x,y;U,V)\ \frac{\cot\frac{V}{2}}{2}\ dU\ dV$$

is bounded for fixed $\eta\neq 0$.

It follows from (10.81), (10.82), (10.83), (10.84, (10.85),

(10.86) that (10.80) $\to 0$ as $\begin{cases}\left.\begin{matrix}s\\ \tau\end{matrix}\right\}\to\pi\\ s\to\pi \text{ uniformly on } 0<\beta<\tau<\pi.\\ \tau\to\pi \text{ uniformly on } 0<\alpha<s<\pi\end{cases}$

A similar result follows for (10.90) by analogy with (10.80).

This proves the lemma.

(11.) Lemma 9. If $f(x,y)$ is integrable (L) in $Q(-\pi,-\pi;\pi,\pi)$ and

periodic of period 2π, and if ${}_k\chi_k(x,y;\gamma,\delta)$, $K_k(r,\gamma)$, $K_k(\rho,\delta)$

are as defined, $k=0,1,2,\cdots, m, m+1$,

then

$$\int_0^\pi\int_0^\pi {}_0\chi_0(x,y;\gamma,\delta)\ K_0(r,\gamma)\ K_0(\rho,\delta)\ d\gamma\ d\delta$$

$$=\int_0^\pi\int_0^\pi {}_1\chi_1(x,y;\gamma,\delta)\ K_1(r,\gamma)\ K_1(\rho,\delta)\ d\gamma\ d\delta$$

$$=\cdots=\int_0^\pi\int_0^\pi {}_m\chi_m(x,y;\gamma,\delta)\ K_m(r,\gamma)\ K_m(\rho,\delta)\ d\gamma\ d\delta$$

$$=\int_0^\pi\int_0^\pi {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\ K_{m+1}(r,\gamma)\ K_{m+1}(\rho,\delta)\ d\gamma\ d\delta \text{ for } \begin{cases} r<1,\\ \rho<1,\end{cases}$$

each one of the integrals existing.

Proof of Lemma 9. The proof is entirely analogous to the proof of Lemma 8. The integration properties of $K_m(r,\gamma)$ are exactly like those of $J_m(\gamma)$ for $0<\xi<\gamma<\pi$. Thus we have immediately from Lemma 9 for any $m=0,1,2,\ldots$

$$(11.1)\quad \int_\eta^\pi\int_\epsilon^\pi {}_0\chi_0(x,y;U,V)\; K_0(r,U)\; K_0(\rho,V)\; dU\; dV$$

$$= \sin\epsilon\;\sin\eta\;\left[K_0(r,\epsilon)K_0(\rho,\eta)\;{}_1\chi_1(x,y;\epsilon,\eta)+\cdots+K_m(r,\epsilon)K_m(\rho,\eta)\;{}_{m+1}\chi_{m+1}(x,y;\epsilon,\eta)\right]$$

$$+\sin\epsilon\int_\eta^\pi\left[K_0(r,\epsilon)\;{}_1\chi(x,y;\epsilon,V)K_1(\rho,V)+\cdots+K_m(r,\epsilon)\;{}_{m+1}\chi_{m+1}(x,y;\epsilon,V)K_{m+1}(\rho,V)\right]dV$$

$$+\sin\eta\int_\epsilon^\pi\left[K_0(\rho,\eta)\;{}_1\chi(x,y;U,\eta)K_1(r,U)+\cdots+K_m(\rho,\eta)\;{}_{m+1}\chi_{m+1}(x,y;U,\eta)K_{m+1}(r,U)\right]dU$$

$$+\int_\eta^\pi\int_\epsilon^\pi {}_{m+1}\chi_{m+1}(x,y;U,V)K_{m+1}(r,U)K_{m+1}(\rho,V)\; dU\; dV.$$

We will show that

$$(11.2)\quad \sin\epsilon\;\sin\eta\;K_k(r,\epsilon)K_k(\rho,\eta)\;{}_{k+1}\chi_{k+1}(x,y;\epsilon,\eta)=o(1)$$

$$\text{for}\begin{cases} r<1\\ \rho<1\end{cases}\qquad \text{as}\begin{cases}\left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\}\to 0\\ \epsilon\to 0 \text{ uniformly on } 0<\eta<\pi,\\ \eta\to 0 \text{ uniformly on } 0<\epsilon<\pi\end{cases}$$

$$(11.3)\quad \sin\epsilon \int_{\eta}^{\pi} K_k(r,\epsilon) \underset{k+1\,k+1}{\chi}(x,y;\epsilon,V)\, K_{k+1}(\rho,V)\, dV = o(1)$$

for $\begin{cases} r<1 \\ \rho<1 \end{cases}$ as $\begin{cases} \begin{Bmatrix}\epsilon\\ \eta\end{Bmatrix} \to 0 \\ \epsilon \to 0 \text{ uniformly on } 0<\eta<\pi \end{cases}$,

$$(11.4)\quad \sin\eta \int_{\epsilon}^{\pi} K_k(\rho,\eta) \underset{k+1\,k+1}{\chi}(x,y;U,\eta)\, K_{k+1}(r,U)\, dU = o(1)$$

for $\begin{cases} r<1 \\ \rho<1 \end{cases}$ as $\begin{cases} \begin{Bmatrix}\epsilon\\ \eta\end{Bmatrix} \to 0 \\ \eta \to 0 \text{ uniformly on } 0<\epsilon<\pi \end{cases}$,

which will then prove the lemma.

Since $K_k(r,0) = K_k(\rho,0) = K_k(r,\pi) = K_k(\rho,\pi) = 0$,

$\dfrac{K_k(r,\gamma)}{\cot\frac{\gamma}{2}} = O(1)$, $\dfrac{K_k(\rho,\delta)}{\cot\frac{\delta}{2}} = O(1)$ by Lemma 2,

and $\displaystyle\int_0^{\eta}\int_0^{\epsilon} \underset{k\,k}{\chi}(x,y;U,V)\, dU\, dV = o(1)$ as $\begin{cases} \begin{Bmatrix}\epsilon\\ \eta\end{Bmatrix} \to 0 \\ \epsilon \to 0 \text{ uniformly on } 0<\eta<\pi-\beta \\ \eta \to 0 \text{ uniformly on } 0<\epsilon<\pi-\alpha \end{cases}$

by the definition given in (2.2), we have, using Lemma 1

$$(11.2)\quad \sin\epsilon\ \sin\eta\ K_k(r,\epsilon)\, K_k(\rho,\eta) \underset{k+1\,k+1}{\chi}(x,y;U,V)\, dU\, dV$$

$$= K_k(r,\epsilon)\, K_k(\rho,\eta) \int_0^{\eta}\int_0^{\epsilon} \underset{k\,k}{\chi}(x,y;U,V)\, dU\, dV = o(1) \quad \text{as} \begin{cases} \begin{Bmatrix}\epsilon\\ \eta\end{Bmatrix} \to 0 \\ \epsilon \to 0 \text{ uniformly on } 0<\eta<\pi. \\ \eta \to 0 \text{ uniformly on } 0<\epsilon<\pi \end{cases}$$

For (11.3), we integrate by parts as follows:

$$(11.31)\quad \sin\epsilon\; K_k(r,\epsilon)\int_\eta^\tau {}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\; K_{k+1}(\rho,V)\; dV$$

$$= \sin\epsilon\; K_k(r,\epsilon)\left\{\Big[{}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\; \sin V\; K_k(\rho,V)\Big]_\eta^\tau - \int_\eta^\tau {}_k\chi_{k+1}(x,y;\;,V)\; K_k(\rho,V)\; dV\right\}$$

$$= \sin\epsilon\; K_k(r,\epsilon)\Big[{}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\; \sin V\; K_k(\rho,\; V)\Big]_\eta^\tau$$

$$- \sin\epsilon\; K_k(r,\epsilon)\left\{\Big[{}_k\chi_{k+1}(x,y;\epsilon,V)\; \sin V\; K_{k-1}(\rho,V)\Big]_\eta^\tau - \int_\eta^\tau {}_{k-1}\chi_{k+1}(x,y;\epsilon,V)\; K_{k-1}(\rho,V)\; dV\right\}$$

$$= \cdots$$

$$= \sin\epsilon\; K_k(r,\epsilon)\Big[{}_{k+1}\chi_{k+1}(x,y;\epsilon,V)\; \sin V\; K_k(\rho,V) - {}_k\chi_{k+1}(x,y;\epsilon,V)\sin V\; K_{k-1}(\rho,V) + \cdots + (-1)^k\, {}_1\chi_{k+1}(x,y;\epsilon,V)\; \sin V\; K_0(\rho,V)\Big]_\eta^\tau$$

$$+ (-1)^{k+1}\sin\epsilon\; K_k(r,\epsilon)\int_\eta^t {}_0\chi_{k+1}(x,y;\epsilon,V)\; K_1(\rho,V)\; dV$$

$$= \left\{ \begin{array}{l} K_k(r,\epsilon)\, K_k(\rho,\tau) \displaystyle\int_0^{\tau}\!\!\int_0^{\epsilon} {}_k\chi_k(x,y;U,V)\, dU\, dV \\ - K_k(r,\epsilon)\, K_k(\rho,\eta) \displaystyle\int_0^{\eta}\!\!\int_0^{\epsilon} {}_k\chi_k(x,y;U,V)\, dU\, dV \end{array} \right\}$$

$$- \left\{ \begin{array}{l} K_k(r,\epsilon)\, K_{k-1}(\rho,\tau) \displaystyle\int_0^{\tau}\!\!\int_0^{\epsilon} {}_{k-1}\chi_k(x,y;U,V)\, dU\, dV \\ - K_k(r,\epsilon)\, K_{k-1}(\rho,\eta) \displaystyle\int_0^{\eta}\!\!\int_0^{\epsilon} {}_{k-1}\chi_k(x,y;U,V)\, dU\, dV \end{array} \right\}$$

$$+ \cdots$$

$$+ (-1)^k \left\{ \begin{array}{l} K_k(r,\epsilon)\, K_0(\rho,\tau) \displaystyle\int_0^{\tau}\!\!\int_0^{\epsilon} {}_0\chi_k(x,y;U,V)\, dU\, dV \\ - K(r,\epsilon)\, K(\rho,\eta) \displaystyle\int_0^{\eta}\!\!\int_0^{\epsilon} {}_0\chi_k(x,y;U,V)\, dU\, dV \end{array} \right\}$$

$$+ (-1)^{k+1} K_k(r,\epsilon) \int_{\eta}^{\tau}\!\!\int_0^{\epsilon} {}_0\chi_k(x,y;U,V)\, K_0(\rho,V)\, dU\, dV .$$

The terms in the expression (11.31) above are of the following forms:

$$(11.32) \quad K_k(r,\epsilon)\, K_m(\rho,\tau) \int_0^{\tau}\!\!\int_0^{\epsilon} {}_m\chi_k(x,y;U,V)\, dU\, dV,$$

$$(11.35) \quad K_k(r,\epsilon)\, K_m(\rho,\eta) \int_0^{\eta}\!\!\int_0^{\epsilon} {}_m\chi_k(x,y;U,V)\, dU\, dV,$$

$$(11.38) \quad K_k(r,\epsilon) \int_{\eta}^{\tau}\!\!\int_0^{\epsilon} {}_0\chi_k(x,y;U,V)\, K_0(\rho,\, V)\, dU\, dV .$$

By the definition given in (2.2), Lemma 1 and Lemma 2, we have

$$(11.32)\quad K_k(r,\epsilon)\ \frac{K_m(\rho,\tau)}{\frac{\cot\frac{\tau}{2}}{2}}\ \frac{\cot\frac{\tau}{2}}{2}\int_0^\tau\int_0^\epsilon {}_m\chi_k(x,y;U,V)\,dU\,dV$$

$$\text{as}\begin{cases}\tau\to\pi \text{ uniformly on } 0<\epsilon<\pi\\ \epsilon\to 0 \text{ uniformly on } 0<\tau<\pi\\ \left.\begin{matrix}\tau\to\pi\\ \epsilon\to 0\end{matrix}\right\}\text{ simultaneously and independently}\end{cases}$$

$$(11.35)\quad K_k(r,\epsilon)\ K_m(\rho,\eta)\int_0^\eta\int_0^\epsilon {}_m\chi_k(x,y;U,V)\,dU\,dV$$

$$\text{as}\begin{cases}\left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\}\to 0\\ \epsilon\to 0 \text{ uniformly on } 0<\eta<\pi,\\ \eta\to 0 \text{ uniformly on } 0<\epsilon<\pi\end{cases}$$

and since ${}_0\chi_0(x,y;U,V)$ is integrable (L) in V on $0<V<\pi$, it also follows that

$$(11.39)\quad K_k(r,\epsilon)\int_0^\pi\int_0^\epsilon {}_0\chi_k(x,y;\epsilon,V)\ K(\rho,V)\,dV$$

exists and tends to 0 as $\epsilon\to 0$.

(11.3) follows from (11.31), (11.35), (11.38), (11.39).

(11.4) follows by analogy with (11.3).

This proves the lemma.

(12.) Lemma 10. If $f(x,y)$ is integrable (L) in $Q(-\pi,-\pi;\pi,\pi)$ and periodic of period 2π, then

the existence in the restricted sense of Cesaro (2.1) of any one of the integrals

$$\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} {}_0\chi_0(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta\ (C;m,m),$$

$$\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} {}_1\chi_1(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta\ (C;m-1,m-1),$$

(12.01)
$$\vdots$$

$$\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} {}_{m-1}\chi_{m-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta\ (C;1,1),$$

$$\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} {}_m\chi_m(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta\ (C;0,0),$$

to the order indicated implies that of the remainder, and further that the integrals are all summable in the order indicated to the same value.

Proof of Lemma 10. As a consequence of Lemma 8, we can let

(12.11) $$J(\epsilon,\eta)=\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta,$$

(12.12) $$K(\epsilon,\eta)=\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} {}_k\chi_k(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta,$$

(12.13) $${}_1J_1(\epsilon,\eta;\lambda,\tau)=\frac{1}{\pi^2}\int_{\tau}^{\eta}\int_{\lambda}^{\epsilon} J(U,V)\ dU\, dV.$$

We take $\begin{cases} 0 < s < \epsilon \\ 0 < t < \eta \end{cases}$ and proceed to integrate (12.13) above by parts.

$$(12.13)\quad {}_{1}J_{1}(\epsilon,\eta;s,t) = \frac{1}{\pi^2}\int_{\tau}^{\eta}\int_{s}^{\epsilon}\left\{\int_{V}^{\pi}\int_{U}^{\pi} \underset{k-1\;k-1}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta\right\} dU\, dV$$

$$= \frac{1}{\pi^2}\int_{\tau}^{\eta}\left\{\begin{array}{l}\left[U\left(\displaystyle\int_{V}^{\pi}\int_{U}^{\pi} \underset{k-1\;k-1}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta\right)\right]_{\epsilon}^{s} \\ + \displaystyle\int_{s}^{\epsilon} U\left(\int_{V}^{\pi} \underset{k-1\;k-1}{\chi}(x,y;U,\delta)\ \frac{\cot\frac{U}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\delta\right) dU\end{array}\right\}$$

We continue with an integration by parts as follows:

$$(12.13)\quad {}_{1}J_{1}(\epsilon,\eta;s,\tau)$$

$$= \frac{\epsilon}{\pi^2}\int_{\tau}^{\eta}\left\{\int_{V}^{\pi}\int_{\epsilon}^{\pi} \underset{k-1\;k-1}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta\right\} dV$$

$$- \frac{s}{\pi^2}\int_{\tau}^{\eta}\left\{\int_{V}^{\pi}\int_{s}^{\pi} \underset{k-1\;k-1}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta\right\} dV$$

$$+ \frac{1}{\pi^2}\int_{t}^{\eta}\int_{s}^{\epsilon}\left\{U\left(\int_{V}^{\pi} \underset{k-1\;k-1}{\chi}(x,y;U,\delta)\ \frac{\cot\frac{U}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\delta\right)\right\} dU\, dV$$

$$
= \frac{\epsilon}{\pi^2}\left\{ \begin{aligned} &\left[V\left(\int_V^{\pi}\int_{\epsilon}^{\pi} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\, \cot\frac{\gamma}{2}\, \cot\frac{\delta}{2}\, d\gamma\, d\delta \right) \right]_{\tau}^{\eta} \\ &+ \int_{\tau}^{\eta} V\left(\int_{\epsilon}^{\pi} {}_{k-1}\chi_{k-1}(x,y;\gamma,V)\, \cot\frac{\gamma}{2}\, \cot\frac{V}{2}\, d\gamma \right) dV \end{aligned} \right\}
$$

$$
- \frac{s}{\pi^2}\left\{ \begin{aligned} &\left[V\left(\int_V^{\pi}\int_{s}^{\pi} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\, \cot\frac{\gamma}{2}\, \cot\frac{\delta}{2}\, d\gamma\, d\delta \right) \right]_{\tau}^{\eta} \\ &+ \int_{\tau}^{\eta} V\left(\int_{s}^{\pi} {}_{k-1}\chi_{k-1}(x,y;\gamma,V)\, \cot\frac{\gamma}{2}\, \cot\frac{V}{2}\, d\gamma \right) dV \end{aligned} \right\}
$$

$$
+ \frac{1}{\pi^2}\int_{s}^{\epsilon} U \cot\frac{U}{2} \left\{ \begin{aligned} &\left[V\left(\int_V^{\pi} {}_{k-1}\chi_{k-1}(x,y;U,\delta)\, \cot\frac{\delta}{2}\, d\delta \right) \right]_{\tau}^{\eta} \\ &+ \int_{\tau}^{\eta} V\, {}_{k-1}\chi_{k-1}(x,y;U,V)\, \cot\frac{V}{2}\, dV \end{aligned} \right\} dU,
$$

(12.13) ${}_1J_1(\epsilon,\eta;s,\tau)$

$$
= \frac{\epsilon\eta}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\, \cot\frac{\gamma}{2}\, \cot\frac{\delta}{2}\, d\gamma\, d\delta
$$

$$+\frac{\epsilon}{\pi^2}\int_{\tau}^{\eta}\int_{\epsilon}^{\pi}\underset{k-1\,k-1}{\chi}(x,y;\gamma,V)\left(\cot\frac{\gamma}{2}\right)\left(V\cot\frac{V}{2}\right)\,d\gamma\,dV$$

$$+\frac{\eta}{\pi^2}\int_{\eta}^{\pi}\int_{s}^{\epsilon}\underset{k-1\,k-1}{\chi}(x,y;U,\delta)\left(U\cot\frac{U}{2}\right)\left(\cot\frac{\delta}{2}\right)\,dU\,d\delta$$

$$+\frac{1}{\pi^2}\int_{\tau}^{\eta}\int_{s}^{\epsilon}\underset{k-1\,k-1}{\chi}(x,y;U,V)\left(U\cot\frac{U}{2}\right)\left(V\cot\frac{V}{2}\right)\,dU\,dV$$

$$-\frac{\epsilon\tau}{\pi^2}\int_{\tau}^{\pi}\int_{\epsilon}^{\pi}\underset{k-1\,k-1}{\chi}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta$$

$$-\frac{s\eta}{\pi^2}\int_{\eta}^{\pi}\int_{s}^{\pi}\underset{k-1\,k-1}{\chi}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta$$

$$+\frac{s\tau}{\pi^2}\int_{\tau}^{\pi}\int_{s}^{\pi}\underset{k-1\,k-1}{\chi}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta$$

$$-\frac{s}{\pi^2}\int_{\tau}^{\eta}\int_{s}^{\pi}\underset{k-1\,k-1}{\chi}(x,y;\;,V)\left(\cot\frac{\gamma}{2}\right)\left(V\cot\frac{V}{2}\right)\,d\gamma\,dV$$

$$-\frac{\tau}{\pi^2}\int_{\tau}^{\pi}\int_{s}^{\epsilon}\underset{k-1\,k-1}{\chi}(x,y;U,\delta)\left(U\cot\frac{U}{2}\right)\left(\cot\frac{\delta}{2}\right)\,dU\,d\delta.$$

In reference to the right member of (12.13), we will show that each of the following integrals tend to 0 as $\left.\begin{matrix} s \\ \tau \end{matrix}\right\} \to 0$.

$$(12.14)\quad \frac{\epsilon\tau}{\pi^2}\int_{\tau}^{\pi}\int_{\epsilon}^{\pi} \chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta,$$

$$(12.15)\quad \frac{s\eta}{\pi^2}\int_{\eta}^{\pi}\int_{s}^{\pi} \chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta,$$

$$(12.16)\quad \frac{s\tau}{\pi^2}\int_{\tau}^{\pi}\int_{s}^{\pi} \chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma\, d\delta,$$

$$(12.17)\quad \frac{s}{\pi^2}\int_{\tau}^{\eta}\int_{s}^{\pi} \chi_{k-1,k-1}(x,y;\gamma,V)\left(\cot\frac{\gamma}{2}\right)\left(V\ \cot\frac{V}{2}\right)\ d\gamma\, dV,$$

$$(12.18)\quad \frac{\tau}{\pi^2}\int_{\tau}^{\pi}\int_{s}^{\epsilon} \chi_{k-1,k-1}(x,y;U,\delta)\left(U\ \cot\frac{U}{2}\right)\left(\cot\frac{\delta}{2}\right)\ dU\, d\delta.$$

We have by the second mean value theorem

$$(12.14)\quad \frac{\epsilon\tau}{\pi^2}\int_{\tau}^{\pi}\int_{\epsilon}^{\pi} {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta$$

$$=\frac{\epsilon\tau}{\pi^2}\left\{\int_{\tau}^{\nu}\int_{\epsilon}^{\pi}+\int_{\nu}^{\pi}\int_{\epsilon}^{\pi}\right\}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta$$

$$=\frac{\epsilon\tau\cot\frac{\tau}{2}}{\pi^2}\int_{\tau}^{\nu^*}\int_{\epsilon}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\,d\gamma\,d\delta$$

$$+\frac{\epsilon\tau}{\pi^2}\int_{\nu}^{\pi}\int_{\epsilon}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta,$$

where $\begin{cases}0<\epsilon<\pi\\ 0<\tau<\nu^*<\nu<\pi\end{cases}$.

It follows from the definition given in (2.2) and Lemma 8 that

${}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}$ is integrable with respect to (γ,δ)

for $\begin{cases}0<\epsilon<\gamma<\pi\\ 0<\delta<\nu^*<\nu<\pi\end{cases}$,

and we can choose ν so small that

$$(12.15)\quad \frac{\epsilon\tau\cot\frac{\tau}{2}}{\pi^2}\int_{\tau}^{\nu^*}\int_{\epsilon}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\,d\gamma\,d\delta \text{ becomes as small as you please.}$$

The integral $\int_{\nu}^{\pi}\int_{\epsilon}^{\pi}\chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma d\delta$ exists for given ϵ and ν by Lemma 8.

Hence (12.14) tends to 0 as $\left.\begin{matrix}\delta\\ \tau\end{matrix}\right\}\to 0$.

By a similar discussion, it follows that (12.15) tends to 0 as $\left.\begin{matrix}\delta\\ \tau\end{matrix}\right\}\to 0$.

Consider that, by the second mean value theorem

$$(12.16)\quad \frac{4\tau}{\pi^2}\int_{\tau}^{\pi}\int_{\delta}^{\pi}\chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma d\delta$$

$$=\frac{4\tau}{\pi^2}\left\{\int_{\tau}^{\nu}\int_{\xi}^{\pi}+\int_{\nu}^{\pi}\int_{\delta}^{\xi}+\int_{\tau}^{\nu}\int_{\delta}^{\xi}+\int_{\nu}^{\pi}\int_{\xi}^{\pi}\right\}\chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ \cot\frac{\delta}{2}\ d\gamma d\delta$$

$$=\frac{4\tau}{\pi^2}\cot\frac{\xi}{2}\int_{\tau}^{\nu^*}\int_{\xi}^{\pi}\chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\delta}{2}\ d\gamma d\delta$$

$$+\frac{4\tau}{\pi^2}\cot\frac{\nu}{2}\int_{\nu}^{\pi}\int_{\delta}^{\xi^*}\chi_{k-1,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ d\gamma d\delta$$

$$+\frac{4\tau}{\pi^2}\cot\frac{\nu}{2}\ \cot\frac{\xi}{2}\int_{\tau}^{\phi}\int_{\delta}^{\theta}\chi_{k-1,k-1}(x,y;\gamma,\delta)\ d\gamma d\delta$$

$$+ \frac{\Delta\tau}{\pi^2}\int_{\nu}^{\pi}\int_{\xi}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma d\delta,$$

where

$$\begin{cases} 0<\Delta<\xi^{*}<\xi<\pi \\ 0<\tau<\nu^{*}<\nu<\pi \\ 0<\Delta<\theta<\xi<\pi \\ 0<\tau<\phi<\nu<\pi \end{cases}.$$

Now each of the integrands in (12.16) above remains integrable as $\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to 0$ by our definition of integrability in (2.2) and Lemma 8. Moreover, we can choose $\left\{\begin{matrix}\xi\\ \nu\end{matrix}\right.$ so small that

$$\int_{\tau}^{\phi}\int_{\Delta}^{\theta}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\ d\gamma d\delta$$

becomes as small as you please. Hence (12.16) tends to 0 as $\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to 0.$

Consider that, by the second mean value theorem

$$(12.17)\quad \frac{\Delta}{\pi^2}\int_{\tau}^{\eta}\int_{\Delta}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,V)\left(\frac{\cot\frac{\gamma}{2}}{2}\right)\left(V\ \frac{\cot\frac{V}{2}}{2}\right)\ d\gamma dV$$

$$=\frac{\Delta}{\pi^2}\left\{\int_{\tau}^{\eta}\int_{\Delta}^{\xi}+\int_{\tau}^{\eta}\int_{\xi}^{\pi}\right\}{}_{k-1}\chi_{k-1}(x,y;\gamma,V)\left(\frac{\cot\frac{\gamma}{2}}{2}\right)\left(V\ \frac{\cot\frac{V}{2}}{2}\right)\ d\gamma dV$$

$$= \frac{\Delta}{\pi^2}\frac{\cot\frac{\Delta}{2}}{2}\int_{\tau}^{\eta}\int_{\Delta}^{\xi^*}{}_{k-1}\chi_{k-1}(x,y;\gamma,V)\left(V\,\frac{\cot\frac{V}{2}}{2}\right)d\gamma dV$$

$$+\frac{\Delta}{\pi^2}\int_{\Delta}^{\eta}\int_{\xi}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,V)\left(\frac{\cot\frac{\gamma}{2}}{2}\right)\left(\frac{V\cot\frac{V}{2}}{2}\right)d\gamma dV,$$

where $0<\Delta<\xi^*<\xi<\pi$.

Each of the integrands in (12.17) above remains integrable as $\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to 0$ by the definition given in (2.2) and Lemma 8, and we can choose $\xi>0$ so small that

$$\int_{\tau}^{\eta}\int_{\Delta}^{\xi^*}{}_{k-1}\chi_{k-1}(x,y;\gamma,V)\left(V\,\frac{\cot\frac{V}{2}}{2}\right)\ d\gamma dV$$

becomes as small as you please. Hence (12.17) tends to 0 as $\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to 0$.

(12.18) tends to 0 as $\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to 0$ by analogy with (12.17).

Each of the remaining integrals in (12.13) above has its integrand remaining integrable as $\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to 0$, by the definition given in (2.2) and Lemma 8. Hence we can write

(12.18) $\lim\limits_{\left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\}\to 0} {}_1J_1(\epsilon,\eta;\Delta,\tau) = {}_1J_1(\epsilon,\eta)$,

thereby defining ${}_1J_1(\epsilon,\eta)$ as follows:

$$(12.19)\quad {}_1J_1(\epsilon,\eta) = \frac{1}{\pi^2}\int_0^{\eta}\int_0^{\epsilon}\left\{\int_V^{\pi}\int_U^{\pi}\chi_{k-1\,k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta\right\}dU\,dV$$

$$= \frac{\epsilon\eta}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}\chi_{k-1\,k-1}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta$$

$$+\frac{\epsilon}{\pi^2}\int_0^{\eta}\int_{\epsilon}^{\pi}\chi_{k-1\,k-1}(x,y;\gamma,V)\left(\frac{\cot\frac{\gamma}{2}}{2}\right)\left(\frac{V\cot\frac{V}{2}}{2}\right)d\gamma\,dV$$

$$+\frac{\eta}{\pi^2}\int_{\eta}^{\pi}\int_0^{\epsilon}\chi_{k-1\,k-1}(x,y;U,\delta)\left(\frac{U\cot\frac{U}{2}}{2}\right)\left(\frac{\cot\frac{\delta}{2}}{2}\right)dU\,d\delta$$

$$+\frac{1}{\pi^2}\int_0^{\eta}\int_0^{\epsilon}\chi_{k-1\,k-1}(x,y;U,V)\left(\frac{U\cot\frac{U}{2}}{2}\right)\left(\frac{V\cot\frac{V}{2}}{2}\right)dU\,dV$$

Let us now consider $K(\epsilon,\eta)$ as defined by (12.12) and integrate by parts. We have, making use of Lemma 1,

$$(12.20)\quad K(\epsilon,\eta) = \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}\chi_{k\,k}(x,y;\gamma,\delta)\frac{\cot\frac{\gamma}{2}}{2}\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\,d\delta$$

$$= \frac{1}{\pi^2}\int_{\eta}^{\pi}\frac{\cot\frac{\delta}{2}}{2}\left\{\int_{\epsilon}^{\pi}\chi_{k\,k}(x,y;\gamma,\delta)\sin\gamma\,\frac{\csc^2\frac{\gamma}{2}}{4}\,d\gamma\right\}d\delta$$

$$= \frac{1}{\pi^2}\int_{\eta}^{\pi} \frac{\cot\frac{\delta}{2}}{2} \left\{ \left[-\underset{k}{\chi}_{k}(x,y;\gamma,\delta)\ \sin\gamma\ \frac{\cot\frac{\gamma}{2}}{2} \right]_{\epsilon}^{\pi} + \int_{\epsilon}^{\pi} \underset{k}{\chi}_{k-1}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\, d\gamma \right\} d\delta$$

$$= \frac{\sin\epsilon}{\pi^2}\ \frac{\cot\frac{\epsilon}{2}}{2} \int_{\eta}^{\pi} \underset{k}{\chi}_{k}(x,y;\epsilon,\delta)\ \sin\delta\ \frac{\csc^2\frac{\delta}{2}}{4}\ d\delta$$

$$+ \frac{1}{\pi^2}\int_{\epsilon}^{\pi} \frac{\cot\frac{\gamma}{2}}{2} \left\{ \int_{\eta}^{\pi} \underset{k}{\chi}_{k-1}(x,y;\gamma,\delta)\ \sin\delta\ \frac{\csc^2\frac{\delta}{2}}{4}\ d\delta \right\} d\gamma,$$

$$(12.20)\quad K(\epsilon,\eta) = \frac{\sin\epsilon}{\pi^2}\ \frac{\cot\frac{\epsilon}{2}}{2} \left[-\underset{k}{\chi}_{k}(x,y;\epsilon,\delta)\ \sin\delta\ \frac{\cot\frac{\delta}{2}}{2} \right]_{\eta}^{\pi}$$

$$+ \frac{\sin\epsilon}{\pi^2}\ \frac{\cot\frac{\epsilon}{2}}{2} \int_{\eta}^{\pi} \underset{k-1}{\chi}_{k}(x,y;\epsilon,\delta)\ \frac{\cot\frac{\delta}{2}}{2}\ d\delta$$

$$+ \frac{1}{\pi^2}\int_{\epsilon}^{\pi} \frac{\cot\frac{\gamma}{2}}{2} \left\{ \left[-\underset{k}{\chi}_{k-1}(x,y;\gamma,\delta)\ \sin\delta\ \frac{\cot\frac{\delta}{2}}{2} \right]_{\eta}^{\pi} + \int_{\eta}^{\pi} \underset{k-1}{\chi}_{k-1}(x,y;\gamma,\delta)\ \frac{\cot\frac{\delta}{2}}{2}\ d\delta \right\} d\gamma$$

$$= \frac{\sin\epsilon}{\pi^2}\,\frac{\cot\frac{\epsilon}{2}}{2}\,\sin\eta\,\frac{\cot\frac{\eta}{2}}{2}\;{}_{k}\chi_{k}(x,y;\epsilon,\eta)$$

$$+\frac{\sin\epsilon}{\pi^2}\,\frac{\cot\frac{\epsilon}{2}}{2}\int_{\eta}^{\pi}{}_{k-1}\chi_{k}(x,y;\epsilon,\delta)\,\frac{\cot\frac{\delta}{2}}{2}\,d\delta$$

$$+\frac{\sin\eta}{\pi^2}\,\frac{\cot\frac{\eta}{2}}{2}\int_{\epsilon}^{\pi}{}_{k}\chi_{k-1}(x,y;\gamma,\eta)\,\frac{\cot\frac{\gamma}{2}}{2}\,d\gamma$$

$$+\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\, d\delta\,.$$

Thus we can write (12.20) as follows:

$$(12.20)\quad K(\epsilon,\eta) = \left(\frac{\cot\frac{\epsilon}{2}\;\cot\frac{\eta}{2}}{4\pi^2}\right)\int_{0}^{\eta}\int_{0}^{\epsilon}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,d\gamma\, d\delta$$

$$+\frac{\cot\frac{\epsilon}{2}}{2\pi^2}\int_{\eta}^{\pi}\int_{0}^{\epsilon}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\, d\delta$$

$$+\frac{\cot\frac{\eta}{2}}{2\pi^2}\int_{0}^{\eta}\int_{\epsilon}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,d\gamma\, d\delta$$

$$+\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\, d\delta,$$

which we shall compare with $\dfrac{{}_{1}J_{1}(\epsilon,\eta)}{\epsilon\eta}$ from (12.19).

We have

$$(12.21)\quad \frac{{}_1J_1(\epsilon,\eta)}{\epsilon\eta} = \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\,d\delta$$

$$+\frac{\frac{1}{\eta}}{\pi^2}\int_0^\eta\int_\epsilon^\pi {}_{k-1}\chi_{k-1}(x,y;\gamma,V)\left(\frac{\cot\frac{\gamma}{2}}{2}\right)\left(V\,\frac{\cot\frac{V}{2}}{2}\right)d\;dV$$

$$+\frac{\frac{1}{\epsilon}}{\pi^2}\int_\eta^\pi\int_0^\epsilon {}_{k-1}\chi_{k-1}(x,y;U,\delta)\left(U\cot\frac{U}{2}\right)\left(\frac{\cot\frac{\delta}{2}}{2}\right)dU\,d\delta$$

$$+\frac{\frac{1}{\epsilon\eta}}{\pi^2}\int_0^\eta\int_0^\epsilon {}_{k-1}\chi_{k-1}(x,y;U,V)\left(U\,\frac{\cot\frac{U}{2}}{2}\right)\left(V\,\frac{\cot\frac{V}{2}}{2}\right)dU\,dV.$$

Upon combining (12.20) with (12.21), we obtain

$$(12.22)\quad K(\epsilon,\eta) = \frac{{}_1J_1(\epsilon,\eta)}{\epsilon\eta}$$

$$+\left(\frac{\cot\frac{\epsilon}{2}}{2}\,\frac{\cot\frac{\eta}{2}}{2}\right)\left(\frac{1}{\pi^2}\right)\int_0^\eta\int_0^\epsilon {}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,d\gamma\,d\delta$$

$$-\left(\frac{1}{\epsilon\eta}\right)\left(\frac{1}{\pi^2}\right)\int_0^\eta\int_0^\epsilon {}_{k-1}\chi_{k-1}(x,y;U,V)\left(U\,\frac{\cot\frac{U}{2}}{2}\right)\left(V\,\frac{\cot\frac{V}{2}}{2}\right)dU\,dV$$

$$+\left(\frac{\cot\frac{\epsilon}{2}}{2}\right)\left(\frac{1}{\pi^2}\right)\int_{\eta}^{\pi}\int_{0}^{\epsilon}\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;\gamma,\delta)\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta$$

$$-\left(\frac{1}{\epsilon}\right)\left(\frac{1}{\pi^2}\right)\int_{\eta}^{\pi}\int_{0}^{\epsilon}\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;U,\delta)\left(U\ \frac{\cot\frac{U}{2}}{2}\right)\left(\frac{\cot\frac{\delta}{2}}{2}\right)\ dU\ d\delta$$

$$+\left(\frac{\cot\frac{\eta}{2}}{2}\right)\left(\frac{1}{\pi^2}\right)\int_{0}^{\eta}\int_{\epsilon}^{\pi}\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ d\gamma\, d\delta$$

$$-\left(\frac{1}{\eta}\right)\left(\frac{1}{\pi^2}\right)\int_{0}^{\eta}\int_{\epsilon}^{\pi}\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;\gamma,V)\left(\frac{\cot\frac{\gamma}{2}}{2}\right)\left(V\ \frac{\cot\frac{V}{2}}{2}\right)\ d\gamma\, dV.$$

Now, (12.22) can be rewritten as follows:

$$(12.23)\quad K(\epsilon,\eta)=\frac{J_1(\epsilon,\eta)}{\epsilon\eta}$$

$$+\left(\frac{1}{4\pi^2}\right)\left(\cot\frac{\epsilon}{2}\ \cot\frac{\eta}{2}-\frac{4}{\epsilon\eta}\right)\int_{0}^{\eta}\int_{0}^{\epsilon}\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;\gamma,\delta)\ d\gamma\, d\delta$$

$$+\left(\frac{1}{4\pi^2}\right)\left(\frac{4}{\epsilon\eta}\right)\int_{0}^{\eta}\int_{0}^{\epsilon}\left(1-\frac{UV}{4}\cot\frac{U}{2}\cot\frac{V}{2}\right)\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;U,V)\ dU\ dV$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\cot\frac{\epsilon}{2}-\frac{2}{\epsilon}\right)\int_{\eta}^{\pi}\int_{0}^{\epsilon}\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;\gamma,\delta)\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\frac{2}{\epsilon}\right)\int_{\eta}^{\pi}\int_{0}^{\epsilon}\left(1-\frac{U}{2}\cot\frac{U}{2}\right)\underset{k-1}{\mathcal{X}}\underset{k-1}{}(x,y;U,\delta)\ \frac{\cot\frac{\delta}{2}}{2}\ dU\ d\delta$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\cot\frac{\eta}{2}-\frac{2}{\eta}\right)\int_0^{\eta}\int_{\epsilon}^{\pi}\underset{k-1\ k-1}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ d\gamma d\delta$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\frac{2}{\eta}\right)\int_0^{\eta}\int_{\epsilon}^{\pi}\left(1-\frac{V}{2}\cot\frac{V}{2}\right)\underset{k-1\ k-1}{\chi}(x,y;\gamma,V)\ \frac{\cot\frac{\gamma}{2}}{2}\ d\gamma dV .$$

Now, $1-\frac{UV}{4}\cot\frac{U}{2}\cot\frac{V}{2}$ is monotone increasing on $\begin{cases} 0\leq U<\epsilon<\frac{\pi}{2} \\ 0\leq V<\eta<\frac{\pi}{2}\end{cases}$,

$1-\frac{U}{2}\cot\frac{U}{2}$ is monotone increasing on $0\leq U<\epsilon<\frac{\pi}{2}$,

$1-\frac{V}{2}\cot\frac{V}{2}$ is monotone increasing on $0\leq V<\eta<\frac{\pi}{2}$.

Hence, upon applying the second mean value theorem, (12.23) above becomes

(12.24) $$K(\epsilon,\eta)=\frac{{}_1J_1(\epsilon,\eta)}{\epsilon\eta}$$

$$+\frac{1}{4\pi^2}\left(\cot\frac{\epsilon}{2}\cot\frac{\eta}{2}-\frac{4}{\epsilon\eta}\right)\int_0^{\eta}\int_0^{\epsilon}\underset{k-1\ k-1}{\chi}(x,y;\gamma,\delta)\ d\gamma d\delta$$

$$+\frac{1}{4\pi^2}\left(\frac{4}{\epsilon\eta}-\cot\frac{\epsilon}{2}\cot\frac{\eta}{2}\right)\int_{\phi\eta}^{\eta}\int_{\theta\epsilon}^{\epsilon}\underset{k-1\ k-1}{\chi}(x,y;\gamma,\delta)\ d\gamma d\delta$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\cot\frac{\epsilon}{2}-\frac{2}{\epsilon}\right)\int_{\eta}^{\pi}\int_{0}^{\epsilon}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\,d\delta$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\frac{2}{\epsilon}-\cot\frac{\epsilon}{2}\right)\int_{\eta}^{\pi}\int_{\theta_1\epsilon}^{\epsilon}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\,d\delta$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\cot\frac{\eta}{2}-\frac{2}{\eta}\right)\int_{0}^{\eta}\int_{\epsilon}^{\pi}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,d\gamma\,d\delta$$

$$+\left(\frac{1}{2\pi^2}\right)\left(\frac{2}{\eta}-\cot\frac{\eta}{2}\right)\int_{\phi_1\eta}^{\eta}\int_{\epsilon}^{\pi}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,d\gamma\,d\delta,$$

where $\begin{cases}0<\theta<1\\0<\phi<1\end{cases}$ and $\begin{cases}0<\theta_1<1\\0<\phi_1<1\end{cases}$.

We can rewrite (12.24) above as follows:

$$(12.24)\quad K(\epsilon,\eta)=\frac{J_1(\epsilon,\eta)}{\epsilon\eta}$$

$$+\frac{1}{4\pi^2}\left[\left(\cot\frac{\epsilon}{2}\cot\frac{\eta}{2}-\frac{4}{\epsilon\eta}\right)\left\{\int_0^{\phi\eta}\int_0^{\theta\epsilon}+\int_{\phi\eta}^{\eta}\int_0^{\theta\epsilon}+\int_0^{\phi\eta}\int_{\theta\epsilon}^{\epsilon}\right\}\chi_{k-1}(x,y;\gamma,\delta)\,d\gamma\,d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\epsilon}{2}-\frac{2}{\epsilon}\right)\int_{\eta}^{\pi}\int_{0}^{\theta\epsilon}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\,d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\eta}{2}-\frac{2}{\eta}\right)\int_{0}^{\phi\eta}\int_{\epsilon}^{\pi}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,d\gamma\,d\delta\right]$$

We have, by the second mean value theorem

$$(12.25)\quad K(\epsilon,\eta) - \frac{J_1(\epsilon,\eta)}{\epsilon\eta} =$$

$$\frac{1}{4\pi^2}\left[\left(\cot\frac{\epsilon}{2}\cot\frac{\eta}{2} - \frac{4}{\epsilon\eta}\right)\left\{\int_0^{\phi\eta}\int_0^{\theta\epsilon} + \int_{\phi\eta}^{\eta}\int_0^{\theta\epsilon} + \int_0^{\phi\eta}\int_{\theta\epsilon}^{\epsilon}\right\}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\, d\gamma d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\epsilon}{2} - \frac{2}{\epsilon}\right)\left\{\int_\eta^{\nu}\int_0^{\theta\epsilon} + \int_\nu^{\pi}\int_0^{\theta\epsilon}\right\}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\delta}{2}}{2}\, d\gamma d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\eta}{2} - \frac{2}{\eta}\right)\left\{\int_0^{\phi\eta}\int_\epsilon^{\xi} + \int_0^{\phi\eta}\int_\xi^{\pi}\right\}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\, d\gamma d\delta\right]$$

$$= \frac{1}{4\pi^2}\left[\left(\cot\frac{\epsilon}{2}\cot\frac{\eta}{2} - \frac{4}{\epsilon\eta}\right)\left\{\int_0^{\phi\eta}\int_0^{\theta\epsilon} + \int_{\phi\eta}^{\eta}\int_0^{\theta\epsilon} + \int_0^{\phi\eta}\int_{\theta\epsilon}^{\epsilon}\right\}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\, d\gamma d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\epsilon}{2} - \frac{2}{\epsilon}\right)\left(\frac{\cot\frac{\eta}{2}}{2}\right)\int_\eta^{\nu^*}\int_0^{\theta\epsilon}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\, d\gamma d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\epsilon}{2} - \frac{2}{\epsilon}\right)\int_\nu^{\pi}\int_0^{\theta\epsilon}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\,\frac{\cot\frac{\delta}{2}}{2}\, d\gamma d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\eta}{2} - \frac{2}{\eta}\right)\left(\frac{\cot\frac{\epsilon}{2}}{2}\right)\int_0^{\phi\eta}\int_\epsilon^{\xi^*}{}_{k-1}\chi_{k-1}(x,y;\gamma,\delta)\, d\gamma d\delta\right]$$

$$+\frac{1}{2\pi^2}\left[\left(\cot\frac{\eta}{2}-\frac{2}{\eta}\right)\int_0^{\phi}\int_{\xi}^{\pi} \chi_{k-1\,k-1}(x,y;\gamma,\delta)\ \cot\frac{\gamma}{2}\ d\gamma d\delta\right],$$

where $\begin{cases} 0<\epsilon<\xi^*<\xi<\pi \\ 0<\eta<\nu^*<\nu<\pi \end{cases}.$

Each of the integrands in the right member of (12.25) remains integrable as $\left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\}\to 0$, by the definition given in (2.2) and Lemma 8, and each of

$$(12.26)\quad \left\{\begin{matrix} \left(\cot\frac{\epsilon}{2}\cot\frac{\eta}{2}-\frac{4}{\epsilon\eta}\right) \\ \left(\cot\frac{\epsilon}{2}-\frac{2}{\epsilon}\right)\left(\frac{\cot\frac{\eta}{2}}{2}\right) \\ \left(\cot\frac{\eta}{2}-\frac{2}{\eta}\right)\left(\frac{\cot\frac{\epsilon}{2}}{2}\right) \end{matrix}\right\}\text{ remains bounded}$$

as $\left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\}\to 0$ if the ratios $\frac{\epsilon}{\eta}$ and $\frac{\eta}{\epsilon}$ remain bounded.

Hence, each of the terms in the right member of (12.25) above tends to 0 by the definition given in (2.2) and Lemma 8 as $\left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\}\to 0$ in the restricted sense.

Hence, if we use restricted convergence

$$K(\epsilon,\eta) - \frac{{}_1J_1(\epsilon,\eta)}{\epsilon\eta} \to 0 \quad \text{as} \quad \left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\} \to 0$$

in the restricted sense; or we can write

$$(12.30) \quad K(\epsilon,\eta) = \frac{{}_1J_1(\epsilon,\eta)}{\epsilon\eta} + o(1)$$

as $\left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\} \to 0$ in the restricted sense.

We note that, by (12.19) we have

$$(12.31) \quad \frac{{}_1J_1(\epsilon,\eta)}{\epsilon\eta} = \frac{1}{\epsilon\eta}\int_0^\eta\int_0^\epsilon\left\{\frac{1}{\pi^2}\int_{\sqrt{U}}^{\pi}\int^{\pi}\chi_{k-1\,k-1}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\right\} d\gamma\, d\delta\ \ dU\ dV\,.$$

Hence, if we consider restricted Cesaro summability in the sense defined by (2.1), we can show the equivalence of the integrals (12.01) by taking the appropriate number of integral means of both sides of (12.30) for given values of k.

For $k=1$, upon taking $m-1$ integral means of both sides of (12.30), we have the equivalence in the restricted sense of

$$\begin{cases} K(\epsilon,\eta) \to A & (C;\ m-1,m-1) \\ \dfrac{{}_1J_1(\epsilon,\eta)}{\epsilon\eta} \to A & (C;\ m-1,m-1), \end{cases}$$

or from (10.31)

$$\begin{cases} K(\epsilon,\eta) \to A & (C;\ m-1,m-1) \\ J(\epsilon,\eta) \to A & (C;\ m,m), \end{cases}$$

or the equivalence in the restricted sense of

$$\begin{cases} \dfrac{1}{\pi^2}\displaystyle\int_0^\pi\int_0^\pi {}_1\chi_1(x,y;\gamma,\delta)\, \frac{\cot\frac{\gamma}{2}}{2}\, \frac{\cot\frac{\delta}{2}}{2}\, d\gamma d\delta & (C;\ m-1,m-1) \\ \dfrac{1}{\pi^2}\displaystyle\int_0^\pi\int_0^\pi {}_0\chi_0(x,y;\gamma,\delta)\, \frac{\cot\frac{\gamma}{2}}{2}\, \frac{\cot\frac{\delta}{2}}{2}\, d\gamma d\delta & (C;m,m). \end{cases}$$

For $k=2$, upon taking $m-2$ integral means of (12.30), we have by means of (12.31), the equivalence in the restricted sense of

$$\begin{cases} \dfrac{1}{\pi^2}\displaystyle\int_0^\pi\int_0^\pi {}_2\chi_2(x,y;\gamma,\delta)\, \frac{\cot\frac{\gamma}{2}}{2}\, \frac{\cot\frac{\delta}{2}}{2}\, d\gamma d\delta & (C;\ m-2,m-2) \\ \dfrac{1}{\pi^2}\displaystyle\int_0^\pi\int_0^\pi {}_1\chi_1(x,y;\gamma,\delta)\, \frac{\cot\frac{\gamma}{2}}{2}\, \frac{\cot\frac{\delta}{2}}{2}\, d\gamma d\delta & (C;\ m-1,m-1). \end{cases}$$

Continuing in this way, we finally obtain for $k=m$, upon taking the limit of (12.30) in the restricted sense just as it stands, the equivalence in the restricted sense of

$$\begin{cases} \dfrac{1}{\pi^2}\displaystyle\int_0^{\pi}\!\!\int_0^{\pi}{}_{m\;m}\chi(x,y;\gamma,\delta)\;\frac{\cot\frac{\gamma}{2}}{2}\;\frac{\cot\frac{\delta}{2}}{2}\;d\gamma\,d\delta & (C;0,0) \\[2ex] \dfrac{1}{\pi^2}\displaystyle\int_0^{\pi}\!\!\int_0^{\pi}{}_{m-1\;m-1}\chi(x,y;\gamma,\delta)\;\frac{\cot\frac{\gamma}{2}}{2}\;\frac{\cot\frac{\delta}{2}}{2}\;d\gamma\,d\delta & (C;1,1). \end{cases}$$

This proves the lemma.

(13.) Lemma 11. If

$$
{}_{m+1}\chi_{m+1}(x,y;\lambda,\tau) = \begin{cases} o(1) \text{ as } \left.\begin{matrix}\lambda\\ \tau\end{matrix}\right\} \to 0 \\ o\left(\dfrac{1}{\sin\tau}\right) \text{ as } \lambda \to 0 \text{ uniformly on } 0 < \tau < \pi , \\ o\left(\dfrac{1}{\sin\lambda}\right) \text{ as } \tau \to 0 \text{ uniformly on } 0 < \lambda < \pi \end{cases}
$$

then for fixed ξ such that $0 < \xi < \pi$,

(13.1) $$\frac{q_{m+1}(\rho,\delta)}{(1-\rho)^2}\int_{\epsilon}^{\xi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,K_{m+1}(r,\gamma)\,d\gamma = O(1) \text{ as } \left\{\begin{matrix}\epsilon = \text{arc sin } (1-r)\\ \eta = \text{arc sin } (1-\rho)\end{matrix}\right\} \to 0$$

for $0 < \nu < \delta < \pi$,

(13.2) $$\frac{q_{m+1}(\rho,\delta)}{(1-\rho)^2}\int_{\epsilon}^{\xi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,d\gamma = O(1) \text{ as } \left\{\begin{matrix}\epsilon = \text{arc sin } (1-r)\\ \eta = \text{arc sin } (1-\rho)\end{matrix}\right\} \to 0$$

for $0 < \nu < \delta < \pi$,

and for fixed ν such that $0 < \nu < \pi$,

(13.5) $$\frac{q_{m+1}(r,\gamma)}{(1-r)^2}\int_{\eta}^{\nu} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,K_{m+1}(\rho,\delta)\,d\delta = O(1) \text{ as } \left\{\begin{matrix}\epsilon = \text{arc sin } (1-r)\\ \eta = \text{arc sin } (1-\rho)\end{matrix}\right\} \to 0$$

for $0 < \xi < \gamma < \pi$,

(13.6) $$\frac{q_{m+1}(r,\gamma)}{(1-r)^2}\int_{\eta}^{\nu} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(\rho,\delta)\,d\delta = O(1) \text{ as } \left\{\begin{matrix}\epsilon = \text{arc sin } (1-r)\\ \eta = \text{arc sin } (1-\rho)\end{matrix}\right\} \to 0$$

for $0 < \xi < \pi$.

Proof of Lemma 11. In consideration of (13.1), we have,

$$\frac{q_{m+1}(\rho,\delta)}{(1-\rho)^2\sin\delta} = O(1) \quad \text{as } \eta = \text{arc } \sin(1-\rho)\to 0 \text{ for } 0<\nu<\delta<\pi$$

by Lemma 4.

$\sin\delta \underset{m+1\ m+1}{\chi}(x,y;\gamma,\delta) = o(1)$ as $\gamma\to 0$ uniformly on $0<\delta<\pi$

by hypothesis,

$$\frac{dK_m(r,\gamma)}{d\gamma} = O\left(\frac{1}{(1-r)^2}\right) \quad \text{as } \epsilon = \text{arc } \sin(1-r)\to 0 \quad \text{for } 0<\gamma\leq\epsilon$$

by Lemma 3, and hence

by Lemma 3, and hence

$$(13.12)\quad \frac{q_{m+1}(\rho,\delta)}{(1-\rho)^2}\int_0^{\epsilon}\underset{m+1\ m+1}{\chi}(x,y;\gamma,\delta)\, K_{m+1}(r,\gamma)\, d\gamma$$

$$= \left(\frac{q_{m+1}(\rho,\delta)}{(1-\rho)^2\sin\delta}\right)\sin\delta\int_0^{\epsilon}\underset{m+1\ m+1}{\chi}(x,y;\gamma,\delta)\sin\gamma\ \frac{dK_m(r,\gamma)}{d\gamma}\quad d\gamma$$

$$= o\left\{\frac{1}{(1-r)^2}\int_0^{\epsilon}\gamma\, d\gamma\right\} = o(1)$$

as ϵ = arc sin $(1-r)\to 0$ uniformly on $0<\nu<\delta<\pi$ from which (13.1) follows.

In consideration of (13.2), we have

$$\sin\delta \, \chi_{m+1\,m+1}(x, y; \gamma, \delta) = o(1) \text{ as } \gamma \to 0 \text{ uniformly on } 0 < \delta < \pi$$

by hypothesis,

$$\frac{q_{m+1}(\rho, \delta)}{(1-\rho)^2 \sin\delta} = O(1) \text{ as } \eta = \text{arc sin } (1-\rho) \to 0 \text{ for } 0 < \nu < \delta < \pi$$

$$\text{and } q_{m+1}(r, \gamma) = O\left(\frac{(1-r)^2}{\gamma^3}\right) \text{ as } \gamma = \text{arc sin } (1-r) \to 0$$

by Lemma 4, and hence given any arbitrary $\omega > 0$, there exists a corresponding $\Delta > 0$ such that

$$(13.22) \qquad \left| \frac{q_{m+1}(\rho, \delta)}{(1-\rho)^2} \int_{\epsilon}^{h} \chi_{m+1\,m+1}(x, y; \gamma, \delta) \, q_{m+1}(r, \gamma) \, d\gamma \right|$$

$$= \left| \frac{q_{m+1}(\rho, \delta)}{(1-\rho)^2 \sin\delta} \int_{\epsilon}^{h} (\sin\delta) \, \chi_{m+1\,m+1}(x, y; \gamma, \delta) \, q_{m+1}(r, \gamma) \, d\gamma \right| \leqq \left| (1-r)^2 \omega \int_{\epsilon}^{h} \frac{d\gamma}{\gamma^3} \right| < \omega,$$

$$\text{whenever } \begin{cases} 0 < \epsilon = \text{arc sin } (1-r) < h < \Delta \\ 0 < \eta = \text{arc sin } (1-\rho) < h < \Delta \end{cases},$$

uniformly on $0 < \nu < \delta < \pi$.

Hence, (13.2) follows. (13.5) and (13.6) follow by analogy with (13.1) and (13.2).

This proves the lemma.

(14.) Lemma 12. If

$$ {}_{m+1}\chi_{m+1}(x,y;\Delta,\tau) = \begin{cases} o(1) \text{ as } \left.\begin{matrix}\Delta\\ \tau\end{matrix}\right\} \to 0 \\ o\left(\dfrac{1}{\sin\tau}\right) \text{ as } \Delta \to 0 \text{ uniformly on } 0 < \tau < \pi , \\ o\left(\dfrac{1}{\sin\Delta}\right) \text{ as } \tau \to 0 \text{ uniformly on } 0 < \Delta < \pi \end{cases} $$

then

$$ (14.1) \quad \int_{\epsilon}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\eta)\, K_{m+1}(r,\gamma)\, d\gamma = o(1) \text{ as } \left.\begin{cases} \epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho) \end{cases}\right\} \to 0 , $$

$$ (14.2) \quad \int_{\epsilon}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\eta)\, q_{m+1}(r,\gamma)\, d\gamma = o(1) \text{ as } \left.\begin{cases} \epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho) \end{cases}\right\} \to 0 , $$

$$ (14.3) \quad \int_{\epsilon}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\eta)\, \cot\frac{\gamma}{2}\, d\gamma = o(1) \text{ as } \left.\begin{cases} \epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho) \end{cases}\right\} \to 0 , $$

$$ (14.4) \quad \int_{\eta}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\epsilon,\delta)\, K_{m+1}(\rho,\delta)\, d\delta = o(1) \text{ as } \left.\begin{cases} \epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho) \end{cases}\right\} \to 0 , $$

$$ (14.5) \quad \int_{\eta}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\epsilon,\delta)\, q_{m+1}(\rho,\delta)\, d\delta = o(1) \text{ as } \left.\begin{cases} \epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho) \end{cases}\right\} \to 0 , $$

$$ (14.6) \quad \int_{\eta}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\epsilon,\delta)\, \cot\frac{\delta}{2}\, d\delta = o(1) \text{ as } \left.\begin{cases} \epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho) \end{cases}\right\} \to 0 . $$

Proof of Lemma 12. In consideration of (14.1), we have

$$\frac{dK_m(r,\gamma)}{d\gamma} = O\left(\frac{1}{(1-r)^2}\right) \text{ as } \epsilon = \text{arc sin }(1-r) \to 0 \text{ for } 0 < \gamma \leqq \epsilon$$

by Lemma 3,

$$\underset{m+1}{\chi}(x, y; \gamma, \xi) = o(1) \text{ as } \left.\begin{matrix}\gamma\\ \eta\end{matrix}\right\} \to 0 \text{ by hypothesis,}$$

and hence

$$(14.11) \quad \int_0^\epsilon \underset{m+1}{\chi}(x,y;\gamma,\eta) K_{m+1}(r,\gamma)\, d\gamma = \int_0^\epsilon \underset{m+1}{\chi}(x,y;\gamma,\eta)\sin\gamma \; \frac{dK_m(r,\gamma)}{d\gamma} \; d\gamma$$

$$= o\left\{\frac{1}{(1-r)^2}\int_0^\epsilon \gamma\, d\gamma\right\} = o(1) \text{ as } \left.\begin{cases}\epsilon = \text{arc sin }(1-r)\\ \eta = \text{arc sin }(1-\rho)\end{cases}\right\} \to 0 .$$

We have further

$$\sin\gamma \underset{m+1}{\chi}(x,y;\gamma,\eta) = o(1) \text{ as } \eta \to 0 \text{ uniformly on } 0 < \gamma < \pi$$

by hypothesis, and for arbitrary fixed α such that $0 < \alpha < \pi$,

$$\frac{K_{m+1}(r,\gamma)}{\sin\gamma} = O(1) \text{ for } 0 < \alpha < \gamma < \pi \text{ for all r on } 0 \leqq r \leqq 1$$

by Lemma 2, and hence

(14.12) $$\int_{\alpha}^{\pi} \chi_{m+1}(x,y;\gamma,\eta) K_{m+1}(r,\gamma)\, d\gamma = \int_{\alpha}^{\pi} \chi_{m+1}(x,y;\gamma,\eta) \sin\gamma \frac{K_{m+1}(r,\gamma)}{\sin\gamma}\, d\gamma$$

$\rightrightarrows o(1)$ as $\eta = \text{arc sin } (1-\rho) \to 0$ regardless of r on $0 \leq r \leq 1$.

Now, given any arbitrary small positive number $\omega > 0$, we can first choose $\xi > 0$ less than ω and so small that

(14.14) $$\left| \int_{0}^{\xi} \chi_{m+1}(x,y;\gamma,\eta) K_{m+1}(r,\gamma)\, d\gamma \right| < \frac{\omega}{3} \quad \text{by (14.11)}$$

with $0 < \eta = \text{arc sin } (1-\rho) \leq \xi$,

and then we can find Δ less than ξ and so small that we have simultaneously

(14.15) $$\left| \int_{\xi}^{\pi} \chi_{m+1}(x,y;\gamma,\eta) K_{m+1}(r,\gamma)\, d\gamma \right| < \frac{\omega}{3} \quad \text{by (14.12)}$$

provided only that $0 < \eta = \text{arc sin } (1-\rho) < \Delta < \xi$,

and

(14.16) $$\left| \int_{0}^{\epsilon} \chi_{m+1}(x,y;\gamma,\eta) K_{m+1}(r,\gamma)\, d\gamma \right| < \frac{\omega}{3} \quad \text{by (14.11)}$$

whenever $\begin{cases} 0 < \epsilon = \text{arc sin } (1-r) < \Delta < \xi \\ 0 < \eta = \text{arc sin } (1-\rho) < \Delta < \xi \end{cases}$, and hence

$$(14.17)\quad \left| \int_{\epsilon}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\eta)\,K_{m+1}(r,\gamma)\,d\gamma \right|$$

$$= \left| \left\{ \int_{0}^{\xi} + \int_{\xi}^{\pi} - \int_{0}^{\epsilon} \right\} {}_{m+1}\chi_{m+1}(x,y;\gamma,\eta)\,K_{m+1}(r,\gamma)\,d\gamma \right| < \frac{\omega}{3} + \frac{\omega}{3} + \frac{\omega}{3} = \omega$$

provided that $\begin{cases} 0<\epsilon = \text{arc}\sin(1-r) < \Delta \\ 0<\eta = \text{arc}\sin(1-\rho) < \Delta \end{cases}$, thereby establishing (14.1).

In consideration of (14.2), we have

${}_{m+1}\chi_{m+1}(x,y;\gamma,\eta) = o(1)$ as $\left.\begin{matrix}\gamma\\ \eta\end{matrix}\right\}\to 0$ by hypothesis,

$q_{m+1}(r,\gamma) = O\left(\dfrac{(1-r)^2}{\gamma^3}\right)$ as $\gamma = \text{arc}\sin(1-r)\to 0$ by Lemma 4,

and hence, given any arbitrary $\omega>0$, there exists a corresponding $\Delta_1>0$, such that

$$(14.21)\quad \left| \int_{\epsilon}^{h} {}_{m+1}\chi_{m+1}(x,y;\gamma,\eta)\,q_{m+1}(r,\gamma)\,d\gamma \right| \leqq \left| (1-r)^2\,\omega \int_{\epsilon}^{h} \frac{d\gamma}{\gamma^3} \right|$$

whenever $\begin{cases} 0<\epsilon = \text{arc}\sin(1-r) < h < \Delta_1 \\ 0<\eta = \text{arc}\sin(1-\rho) < h < \Delta_1 \end{cases}$

We have further

$$\sin\gamma \underset{m+1\,m+1}{\chi}(x,y;\gamma,\eta) = o(1) \text{ as } \eta = \text{arc sin } (1-r) \rightarrow 0$$

uniformly on $0<\gamma<\pi$ by hypothesis, and for arbitrary fixed α such that $0<\alpha<\pi$,

$$\frac{q_{m+1}(r,\gamma)}{\sin\gamma\,(1-r)^2} = O(1) \text{ as } \epsilon = \text{arc sin } (1-r) \rightarrow 0 \text{ with } 0<\alpha<\gamma<\pi$$

by Lemma 4, and hence for given α,

$$(14.22)\quad \int_{\alpha}^{\pi} \underset{m+1\,m+1}{\chi}(x,y;\gamma,\eta)\, q_{m+1}(r,\gamma)\, d\gamma = (1-r)^2 \int_{\alpha}^{\pi} \underset{m+1\,m+1}{\chi}(x,y;\gamma,\eta) \sin\gamma \frac{q_{m+1}(r,\gamma)}{\sin\gamma\,(1-r)^2}\, d\gamma$$

$$= o(1) \text{ as } \left\{ \begin{matrix} \epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho) \end{matrix} \right\} \rightarrow 0.$$

Therefore, we can first choose $\xi>0$ less than ω so small that

$$\left| \int_{\epsilon}^{\xi} \underset{m+1\,m+1}{\chi}(x,y;\gamma,\eta)\, q_{m+1}(r,\gamma)\, d\gamma \right| < \frac{\omega}{2} \text{ by } (14.21)$$

$$\text{with} \left\{ \begin{matrix} 0<\epsilon = \text{arc sin } (1-r) \leqq \xi \\ 0<\eta = \text{arc sin } (1-\rho) \leqq \xi \end{matrix} \right. ,$$

and then we can find $\Delta > 0$ less than ξ and so small that

$$\left| \int_{\xi}^{\pi} \underset{m+1}{\chi}(x, y; \gamma, \eta) \underset{m+1}{q}(r, \gamma)\, d\gamma \right| < \frac{\omega}{2} \text{ by } (14.22)$$

whenever $\begin{cases} 0 < \epsilon = \text{arc} \sin (1-r) < \Delta < \xi, \\ 0 < \eta = \text{arc} \sin (1-\rho) < \Delta < \xi \end{cases}$

and hence

$$\left| \int_{\epsilon}^{\pi} \underset{m+1}{\chi}(x, y; \gamma, \eta) \underset{m+1}{q}(r, \gamma)\, d\gamma \right| < \frac{\omega}{2} + \frac{\omega}{2} = \omega$$

whenever $\begin{cases} 0 < \epsilon \ \text{arc} \sin (1-r) < \Delta \\ 0 < \eta = \text{arc} \sin (1-\rho) < \Delta \end{cases}$, thereby establishing (14.2).

Since $\underset{m+1}{q}(r, \gamma) = \underset{m+1}{J}(\gamma) - \underset{m+1}{K}(r, \gamma)$ by Lemma 6,

(14.3) follows from (14.1) and (14.2). (14.4), (14.5), (14.6) follow by anaology with (14.1), (14.2) and (14.3).

(15.) Theorem 1. Let $f(x,y)$ be integrable (L) in $Q(-\pi,-\pi;\pi,\pi)$ and defined outside Q by periodicity.

If $A(x,y;r,\rho)$ is the Abel sum for the conjugate of the double Fourier series, then

$$\lim_{\substack{\epsilon = \text{arc sin}\,(1-r) \\ \eta = \text{arc sin}\,(1-\rho)}\to 0} \left\{ A(x,y;r,\rho) - \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \underset{m\ m}{\chi}(x,y;\gamma,\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\,d\delta \right\} = 0,$$

provided that

$$\underset{m+1\ m+1}{\chi}(x,y;\lambda,\tau) = \begin{cases} o(1) \text{ as } \left.\begin{matrix}\lambda\\ \tau\end{matrix}\right\}\to 0 \\ o\left(\dfrac{1}{\sin\tau}\right) \text{ as } \lambda\to 0 \text{ uniformly on } 0<\tau<\pi \\ o\left(\dfrac{1}{\sin\lambda}\right) \text{ as } \tau\to 0 \text{ uniformly on } 0<\lambda<\pi \end{cases}.$$

Proof of Theorem 1.

From (1.53), we have for the Abel sum,

$$A(x,y;r,\rho) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \underset{0\ 0}{\chi}(x,y;\gamma,\delta)K_0(r,\gamma)K_0(\rho,\delta)\,d\gamma\,d\delta.$$

By Lemma 9, we have

(15.01) $$A(x,y;r,\rho) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \underset{0\ 0}{\chi}(x,y;\gamma,\delta)K_0(r,\gamma)K_0(\rho,\delta)\,d\gamma\,d\delta$$

$$= \dots = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \underset{m\ m}{\chi}(x,y;\gamma,\delta)K_m(r,\gamma)K_m(\rho,\delta)\,d\gamma\,d\delta$$

$$= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \underset{m+1\ m+1}{\chi}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)K_{m+1}(\rho,\delta)\,d\gamma\,d\delta.$$

By Lemma 7, we have

$$K_{m+1}(r,\gamma)K_{m+1}(\rho,\delta) = J_{m+1}(\gamma)\,J_{m+1}(\delta) - q_{m+1}(r,\gamma)q_{m+1}(\rho,\delta) - K_{m+1}(r,\gamma)q_{m+1}(\rho,\delta) - q_{m+1}(r,\gamma)K_{m+1}(\rho,\delta),$$

Thus we can write

$$(15.02)\quad A(x,y;r,\rho) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \underset{m+1\;m+1}{X}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)K_{m+1}(\rho,\delta)\;d\gamma d\delta$$

$$= \frac{1}{\pi^2}\left\{\int_0^{\eta}\int_0^{\epsilon} + \int_0^{\eta}\int_{\epsilon}^{\pi} + \int_{\eta}^{\pi}\int_0^{\epsilon} + \int_{\eta}^{\pi}\int_{\epsilon}^{\pi}\right\} \underset{m+1\;m+1}{X}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)K_{m+1}(\rho,\delta)\;d\gamma d\delta$$

$$= \frac{1}{\pi^2}\left\{\int_0^{\eta}\int_0^{\epsilon} + \int_0^{\eta}\int_{\epsilon}^{\pi} + \int_{\eta}^{\pi}\int_0^{\epsilon}\right\} \underset{m+1\;m+1}{X}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)K_{m+1}(\rho,\delta)\;d\gamma d\delta$$

$$- \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \underset{m+1\;m+1}{X}(x,y;\gamma,\delta)\Big[q_{m+1}(r,\gamma)q_{m+1}(\rho,\delta) + K_{m+1}(r,\gamma)q_{m+1}(\rho,\delta) + q_{m+1}(r,\gamma)K_{m+1}(\rho,\delta)\Big]\;d\gamma d\delta$$

$$+ \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \underset{m+1\;m+1}{X}(x,y;\gamma,\delta)\;J_{m+1}(\gamma)J_{m+1}(\delta)\;d\gamma d\delta$$

Consider that, upon integrating by parts, we have

$$(15.03)\quad \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \underset{m\;m}{X}(x,y;\gamma,\delta)\;J_m(\gamma)J_m(\delta)\;d\gamma d\delta$$

$$= \frac{1}{\pi^2}\int_{\eta}^{\pi} J_m(\delta)\left\{ \Big[\underset{m\;m+1}{X}(x,y;\gamma,\delta)\sin\gamma\;J_m(\gamma)\Big]_{\epsilon}^{\pi} - \int_{\epsilon}^{\pi} \underset{m\;m+1}{X}(x,y;\gamma,\delta)\sin\gamma\;\frac{dJ_m(\gamma)}{d\gamma}\,d\gamma \right\} d\delta$$

$$= -\frac{\sin\epsilon\; J_m(\epsilon)}{\pi^2}\int_\eta^\pi \underset{m\,m+1}{\chi}(x,y;\epsilon,\delta)\; J_m(\delta)\; d\delta$$

$$-\frac{1}{\pi^2}\int_\epsilon^\pi \sin\gamma\; \frac{dJ_m(\gamma)}{d\gamma}\left\{\begin{array}{l}\left[\underset{m+1\,m+1}{\chi}(x,y;\gamma,\delta)\;\sin\delta\; J_m(\delta)\right]_\eta^\pi \\ -\displaystyle\int_\eta^\pi \underset{m+1\,m+1}{\chi}(x,y;\gamma,\delta)\;\sin\delta\;\frac{dJ_m(\delta)}{d\delta}\,d\delta\end{array}\right\} d\gamma$$

$$= -\frac{\sin\epsilon\; J_m(\epsilon)}{\pi^2}\left\{\left[\underset{m+1\,m+1}{\chi}(x,y;\gamma,\delta)\;\sin\delta\; J_m(\delta)\right]_\eta^\pi - \int_\eta^\pi \underset{m+1\,m+1}{\chi}(x,y;\epsilon,\delta)\;\sin\delta\;\frac{dJ_m(\delta)}{d\delta}\;d\delta\right\}$$

$$+\frac{\sin\eta\; J_m(\eta)}{\pi^2}\int_\epsilon^\pi \underset{m+1\,m+1}{\chi}(x,y;\gamma,\eta)\;\sin\gamma\;\frac{dJ_m(\gamma)}{d\gamma}\quad d\gamma$$

$$+\frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \underset{m+1\,m+1}{\chi}(x,y;\gamma,\delta)\;\sin\gamma\;\frac{dJ_m(\gamma)}{d\gamma}\;\sin\delta\;\frac{dJ_m(\delta)}{d\delta}\quad d\gamma\, d\delta,$$

whence we have

$$(15.04)\qquad \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \underset{m+1\,m+1}{\chi}(x,y;\gamma,\delta)\; J_{m+1}(\gamma) J_{m+1}(\delta)\quad d\gamma\, d\delta$$

$$= -\frac{\sin\epsilon\; J_m(\epsilon)}{\pi^2}\int_\eta^\pi \underset{m+1\,m+1}{\chi}(x,y;\epsilon,\delta)\; J_{m+1}(\delta)\; d\delta - \frac{\sin\eta\; J_m(\eta)}{\pi^2}\int_\epsilon^\pi \underset{m+1\,m+1}{\chi}(x,y;\gamma,\eta) J_{m+1}(\gamma)\; d\gamma$$

$$-\frac{\sin\epsilon\; J_m(\epsilon)\;\sin\eta\; J_m(\eta)}{\pi^2}\underset{m+1\,m+1}{\chi}(x,y;\epsilon,\eta) + \frac{1}{\pi^2}\int_\eta\int_\epsilon \underset{m\,m}{\chi}(x,y;\gamma,\delta) J_m(\gamma) J_m(\delta)\quad d\gamma\, d\delta.$$

Upon substituting (15.04) for the last term of the right member of (15.02) and making use of (15.01), we have

$$(15.05)\quad A(x,y;r,\rho) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \underset{m}{\chi}(x,y;\gamma,\delta)\underset{m}{K}(r,\gamma)\underset{m}{K}(\rho,\delta)\;d\gamma\,d\delta$$

$$= \frac{1}{\pi^2}\left\{\int_0^{\eta}\int_0^{\epsilon} + \int_0^{\eta}\int_{\epsilon}^{\pi} + \int_{\eta}^{\pi}\int_0^{\epsilon}\right\}\underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{K}(\rho,\delta)\;d\gamma\,d\delta$$

$$-\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\delta)\left[\underset{m+1}{q}(r,\gamma)\underset{m+1}{q}(\rho,\delta)+\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\delta)+\underset{m+1}{q}(r,\gamma)\underset{m+1}{K}(\rho,\delta)\right]d\gamma\,d\delta$$

$$-\frac{\sin\epsilon\;\underset{m}{J}(\epsilon)}{\pi^2}\int_{\eta}^{\pi}\underset{m+1}{\chi}(x,y;\epsilon,\delta)\;\underset{m+1}{J}(\delta)\;d\delta-\frac{\sin\eta\;\underset{m}{J}(\eta)}{\pi^2}\int_{\epsilon}^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\eta)\;\underset{m+1}{J}(\gamma)\;d\gamma$$

$$-\frac{\sin\epsilon\;\underset{m}{J}(\epsilon)\;\sin\eta\;\underset{m}{J}(\eta)\;\underset{m+1}{\chi}(x,y;\epsilon,\eta)}{\pi^2}+\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}\underset{m}{\chi}(x,y;\gamma,\delta)\;\underset{m}{J}(\gamma)\underset{m}{J}(\delta)\;d\gamma\,d\delta.$$

We will show that each of the integrals

$$(15.1)\quad \frac{1}{\pi^2}\int_0^{\eta}\int_0^{\epsilon}\underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{K}(\rho,\delta)\;d\gamma\,d\delta,$$

$$(15.2)\quad \frac{1}{\pi^2}\int_0^{\eta}\int_0^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{K}(\rho,\delta)\;d\gamma\,d\delta,$$

$$(15.3)\quad \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_0^{\epsilon}\underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{K}(\rho,\delta)\;d\gamma\,d\delta,$$

$$(15.4)\quad \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \chi_{m+1}(x,y;\gamma,\delta)\, q_{m+1}(r,\gamma)\, q_{m+1}(\rho,\delta)\ d\gamma d\delta,$$

$$(15.5)\quad \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \chi_{m+1}(x,y;\gamma,\delta)\, K_{m+1}(r,\gamma)\, q_{m+1}(\rho,\delta)\ d\gamma d\delta,$$

$$(15.6)\quad \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \chi_{m+1}(x,y;\gamma,\delta)\, q_{m+1}(r,\gamma)\, K_{m+1}(\rho,\delta)\ d\gamma d\delta,$$

$$(15.7)\quad \frac{\sin\epsilon\ J_m(\epsilon)}{\pi^2}\int_\eta^\pi \chi_{m+1}(x,y;\epsilon,\delta)\, J_{m+1}(\delta)\ d\delta,$$

$$(15.8)\quad \frac{\sin\eta\ J_m(\eta)}{\pi^2}\int_\epsilon^\pi \chi_{m+1}(x,y;\gamma,\eta)\, J_{m+1}(\gamma)\ d\gamma,$$

$$(15.9)\quad \frac{\sin\epsilon\ J_m(\epsilon)\ \sin\eta\ J_m(\eta)\ \chi_{m+1}(x,y;\epsilon,\eta)}{\pi^2},$$

tends to 0 as $\left\{\begin{array}{l}\epsilon = \text{arc sin }(1-r)\\ \eta = \text{arc sin }(1-\rho)\end{array}\right\} \to 0,$

and hence from (15.05) that

$$A(x,y;r,\rho) - \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \chi_m(x,y;\gamma,\delta)\, J_m(\gamma)\, J_m(\delta)\ d\gamma d\delta$$

tends to 0 as $\left\{\begin{array}{l}\epsilon = \text{arc sin }(1-r)\\ \eta = \text{arc sin }(1-\rho)\end{array}\right\} \to 0.$

We proceed to consider them one by one.

We have:

$$(15.1)\quad \frac{1}{\pi^2}\int_0^{\eta}\int_0^{\epsilon} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\, K_{m+1}(r,\gamma) K_{m+1}(\rho,\delta)\, d\gamma\, d\delta$$

$$=\frac{1}{\pi^2}\int_0^{\eta}\int_0^{\epsilon} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,\sin\gamma\,\frac{dK_m(r,\gamma)}{d\gamma}\,\sin\delta\,\frac{dK_m(\rho,\delta)}{d\delta}\,d\gamma\,d\delta$$

$$=o\left\{\left(\frac{1}{(1-r)^2}\right)\left(\frac{1}{(1-\rho)^2}\right)\int_0^{\eta}\int_0^{\epsilon}\gamma\delta\,d\gamma\,d\delta\right\}=o(1)\text{ as }\left\{\begin{matrix}\epsilon=\text{arc sin }(1-r)\\ \eta=\text{arc sin }(1-\rho)\end{matrix}\right\}\to 0,$$

since

$${}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)=o(1)\text{ as }\left.\begin{matrix}\gamma\\ \delta\end{matrix}\right\}\to 0\quad\text{by hypothesis,}$$

$$\frac{dK_m(r,\gamma)}{d\gamma}=O\left(\frac{1}{(1-r)^2}\right)\text{ for } 0<\gamma\leq\epsilon \text{ as } \epsilon=\text{arc sin }(1-r)\to 0 \text{ by Lemma 3.}$$

We have

$$(15.2)\quad \frac{1}{\pi^2}\int_0^{\eta}\int_{\epsilon}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\, K_{m+1}(r,\gamma) K_{m+1}(\rho,\delta)\, d\gamma\, d\delta$$

$$=\frac{1}{\pi^2}\int_0^{\eta}\left\{\int_{\epsilon}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\, K_{m+1}(r,\gamma)\, d\gamma\right\}\sin\delta\,\frac{dK_m(\rho,\delta)}{d\delta}\,d\delta$$

$$=o\left\{\frac{1}{(1-\rho)^2}\int_0^{\eta}\delta\,d\delta\right\}=o(1)\text{ as }\left\{\begin{matrix}\epsilon=\text{arc sin }(1-r)\\ \eta=\text{arc sin }(1-\rho)\end{matrix}\right\}\to 0,\text{ since}$$

$$\int_{\epsilon}^{\pi} {}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\, K_{m+1}(r,\gamma)\, d\gamma=o(1)\text{ as }\left\{\begin{matrix}\epsilon=\text{arc sin }(1-r)\\ \delta=\text{arc sin }(1-\rho)\end{matrix}\right\}\to 0 \text{ by Lemma 12.}$$

$\frac{dK_m(\rho,\delta)}{d\delta} = O\left(\frac{1}{(1-\rho)^2}\right)$ for $0<\delta\leqq\eta$ as $\eta = \text{arc sin }(1-\rho)\to 0$ by Lemma 3.

By analogy with (15.2), we have (15.3)$\to 0$ as $\left\{\begin{matrix}\text{arc sin }(1-r)\\ \text{arc sin }(1-\rho)\end{matrix}\right\}\to 0.$

We have

$$(15.4)\quad \frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\quad d\gamma d\delta$$

$$=\frac{1}{\pi^2}\left\{\int_\eta^\nu\int_\epsilon^\pi+\int_\nu^\pi\int_\epsilon^\xi+\int_\nu^\pi\int_\xi^\pi\right\}\chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\quad d\gamma d\delta.$$

We will consider separately each of the integrals

$$(15.41)\quad \frac{1}{\pi^2}\int_\eta^\nu\int_\epsilon^\pi \chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\quad d\gamma d\delta,$$

$$(15.44)\quad \frac{1}{\pi^2}\int_\nu^\pi\int_\epsilon^\xi \chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\quad d\gamma d\delta,$$

$$(15.47)\quad \frac{1}{\pi^2}\int_\nu^\pi\int_\xi^\pi \chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\quad d\gamma d\delta.$$

Let any arbitrary positive number ω be assigned. In consideration of

(15.41), we have

$$\int_{\epsilon}^{\pi} \underset{m+1}{\chi}_{m+1}(x,y;\gamma,\varsigma)\underset{m+1}{q}(r,\gamma)\, d\gamma = o(1) \text{ as } \left\{\begin{array}{l}\epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\rho)\end{array}\right\} \to 0 \text{ by Lemma 12,}$$

$$\underset{m+1}{q}(\rho,\delta) = O\left(\frac{(1-\rho)^2}{\delta^3}\right) \text{ as } \delta = \text{arc sin } (1-\rho) \to 0 \text{ by Lemma 4,}$$

and hence we can choose $\Delta_1 > 0$ so small that

$$(15.42) \quad \left|\frac{1}{\pi^2}\int_{\eta}^{\Delta_1}\int_{\epsilon}^{\pi} \underset{m+1}{\chi}_{m+1}(x,y;\gamma,\delta)\underset{m+1}{q}(r,\gamma)\underset{m+1}{q}(\rho,\delta)\, d\gamma d\delta\right| \leq \left|(1-\rho)^2 \omega \int_{\eta}^{\Delta_1}\frac{d\delta}{\delta^3}\right| < \omega,$$

$$\text{with } \left\{\begin{array}{l}0<\epsilon = \text{arc sin } (1-r) < \Delta_1 \\ 0<\eta = \text{arc sin } (1-\rho) < \Delta_1\end{array}\right. .$$

In consideration of (15.44), we have

$$\frac{\underset{m+1}{q}(\varsigma,\delta)}{(1-\varsigma)^2} \quad \underset{m+1}{\chi}_{m+1}(x,y;\gamma,\delta)\underset{m+1}{q}(r,\gamma)\, d\gamma = O(1)$$

$$\text{as } \left\{\begin{array}{l}\epsilon = \text{arc sin } (1-r) \\ \eta = \text{arc sin } (1-\varsigma)\end{array}\right\} \to 0 \text{ for } 0 < \nu < \delta < \pi$$

with fixed ξ such that $0 < \xi < \pi$ by Lemma 11, and hence for given $\begin{cases}\xi \\ \nu\end{cases}$, we can choose $\Delta_2 > 0$ so small that

$$(15.45)\quad \left| \frac{1}{\pi^2} \int_{\nu}^{\pi} \int_{\epsilon}^{\xi} \underset{m+1}{X}(x,y;\gamma,\delta) \underset{m+1}{q}(r,\gamma) \underset{m+1}{q}(\rho,\delta)\, d\gamma\, d\delta \right|$$

$$= \left| \frac{(1-\rho)^2}{\pi^2} \int_{\nu}^{\pi} \left\{ \frac{\underset{m+1}{q}(\rho,\delta)}{(1-\rho)^2} \int_{\epsilon}^{\xi} \underset{m+1}{X}(x,y;\gamma,\delta) \underset{m+1}{q}(r,\gamma)\, d\gamma \right\} d\delta \right| < \omega$$

provided only that $\begin{cases} 0 < \epsilon = \text{arc sin }(1-r) < \Delta_2 \\ 0 < \eta = \text{arc sin }(1-\rho) < \Delta_2 \end{cases}$.

In consideration of (15.47), for given $\begin{cases}\xi \neq 0 \\ \nu \neq 0\end{cases}$, we have

$\int_{\nu}^{\pi} \int_{\xi}^{\pi} \underset{m+1}{X}(x,y;\gamma,\delta) \underset{m+1}{q}(r,\gamma) \underset{m+1}{q}(\rho,\delta)\, d\gamma\, d\delta$ existing by Lemmas 8 and 9. Now

$\underset{m+1}{q}(r,\gamma)$ contains a factor of $(1-r)^2$ not involving γ from the definition of

q_{m+1} given in Lemma 4, and hence for given $\begin{cases}\xi \neq 0 \\ \nu \neq 0\end{cases}$

$\int_{\nu}^{\pi} \int_{\xi}^{\pi} \underset{m+1}{X}(x,y;\gamma,\delta) \frac{\underset{m+1}{q}(r,\gamma)}{(1-r)^2} \frac{\underset{m+1}{q}(\rho,\delta)}{(1-\rho)^2}\, d\gamma\, d\delta$ exists for all $\begin{cases} r \\ \rho \end{cases}$ such that $\begin{cases} 0 \leq r \leq 1 \\ 0 \leq \rho \leq 1 \end{cases}$,

whence for given $\begin{cases}\xi \neq 0 \\ \nu \neq 0\end{cases}$, we can choose $\Delta_3 > 0$ so small that

$$(15.48)\quad \left|\frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\xi}^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{q}(r,\gamma)\underset{m+1}{q}(\rho,\delta)\,d\gamma d\delta\right|$$

$$=\left|\frac{(1-r)^2(1-\rho)^2}{\pi^2}\int_{\nu}^{\pi}\int_{\xi}^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\delta)\frac{\underset{m+1}{q}(r,\gamma)}{(1-r)^2}\frac{\underset{m+1}{q}(\rho,\delta)}{(1-\rho)^2}\,d\gamma d\delta\right|<\omega,$$

provided only that $\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\Delta_3 \\ 0<\eta=\text{arc sin }(1-\rho)<\Delta_3 \end{cases}$.

Therefore, in consideration of (15.4), we can first choose $\nu>0$ so small that

$$\left|\frac{1}{\pi^2}\int_{\eta}^{\nu}\int_{\epsilon}^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{q}(r,\gamma)\underset{m+1}{q}(\rho,\delta)\,d\gamma d\delta\right|<\frac{\omega}{3}\text{ with }\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\nu \\ 0<\eta=\text{arc sin }(1-\rho)<\nu \end{cases}\text{ by (15.42),}$$

and then we can find $\Delta>0$ less than ν and so small that we have simultaneously

$$\left|\frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\epsilon}^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{q}(r,\gamma)\underset{m+1}{q}(\rho,\delta)\,d\gamma d\delta\right|<\frac{\omega}{3}$$

provided only that $\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\Delta \\ 0<\eta=\text{arc sin }(1-\rho)<\Delta \end{cases}$ by (15.45), and

$$\left|\frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\xi}^{\pi}{}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\,d\gamma\,d\delta\right| < \frac{\omega}{3}$$

provided only that $\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\Delta \\ 0<\eta=\text{arc sin }(1-\rho)<\Delta \end{cases}$ by (15.48),

and hence

$$\left|\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}{}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,q_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\,d\gamma\,d\delta\right| < \frac{\omega}{3}+\frac{\omega}{3}+\frac{\omega}{3}=\omega$$

provided only that $\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\Delta \\ 0<\eta=\text{arc sin }(1-\rho)<\Delta \end{cases}$,

whence (15.4) $\to 0$ as $\left.\begin{cases} \epsilon=\text{arc sin }(1-r) \\ \eta=\text{arc sin }(1-\rho) \end{cases}\right\}\to 0$.

We have

$$(15.5)\qquad \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}{}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,K_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\,d\gamma\,d\delta$$

$$=\frac{1}{\pi^2}\left\{\int_{\eta}^{\nu}\int_{\epsilon}^{\pi}+\int_{\nu}^{\pi}\int_{\epsilon}^{\xi}+\int_{\nu}^{\pi}\int_{\xi}^{\pi}\right\}{}_{m+1}\chi_{m+1}(x,y;\gamma,\delta)\,K_{m+1}(r,\gamma)\,q_{m+1}(\rho,\delta)\,d\gamma\,d\delta\,.$$

We will consider separately each of the integrals

$$(15.51)\quad \frac{1}{\pi^2}\int_{\eta}^{\nu}\int_{\epsilon}^{\pi} \underset{m+1}{\chi}(x,y;\gamma,\zeta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\zeta)\,d\gamma\,d\zeta,$$

$$(15.54)\quad \frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\epsilon}^{\xi} \underset{m+1}{\chi}(x,y;\gamma,\zeta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\zeta)\,d\gamma\,d\zeta,$$

$$(15.57)\quad \frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\xi}^{\pi} \underset{m+1}{\chi}(x,y;\gamma,\zeta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\zeta)\,d\gamma\,d\zeta.$$

Let any arbitrary positive number ω be assigned. In consideration of (15.51), we have

$$\int_{\epsilon}^{\pi}\underset{m+1}{\chi}(x,y;\gamma,\zeta)\underset{m+1}{K}(r,\gamma)\,d\gamma = o(1) \text{ as } \left\{\begin{matrix}\epsilon = \text{arc sin } (1-r)\\ \eta = \text{arc sin } (1-\rho)\end{matrix}\right\} \to 0 \text{ by Lemma 12,}$$

$$\underset{m+1}{q}(\rho,\zeta) = O\left(\frac{(1-\rho)^2}{\zeta^3}\right) \text{ as } \zeta = \text{arc sin } (1-\rho) \to 0 \text{ by Lemma 4,}$$

and hence we can choose $\Delta_1 > 0$ so small that

$$(15.52)\quad \left|\frac{1}{\pi^2}\int_{\eta}^{\Delta_1}\int_{\epsilon}^{\pi} \underset{m+1}{\chi}(x,y;\gamma,\zeta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\zeta)\,d\gamma\,d\zeta\right| \leqq \left|(1-\rho)^2\omega\int_{\eta}^{\Delta_1}\frac{d\zeta}{\zeta^3}\right| < \omega$$

$$\text{with}\left\{\begin{matrix}0 < \epsilon = \text{arc sin } (1-r) < \Delta_1\\ 0 < \eta = \text{arc sin } (1-\rho) < \Delta_1\end{matrix}\right. .$$

In consideration of (15.54), we have

$$\frac{q_{m+1}(\rho,\delta)}{(1-\rho)^2}\int_{\epsilon}^{\xi}\chi_{m+1}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)\,d\gamma = O(1)$$

as $\left\{\begin{matrix}\epsilon = \text{arc sin }(1-r)\\ \eta = \text{arc sin }(1-\rho)\end{matrix}\right\} \to 0$ for $0<\nu<\delta<\pi$ with fixed ξ such that $0<\xi<\pi$ by Lemma 11, and hence for given $\left\{\begin{matrix}\xi\\ \nu\end{matrix}\right.$, we can choose $\Delta_2>0$ so small that

$$(15.55)\quad \left|\frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\epsilon}^{\xi}\chi_{m+1}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)q_{m+1}(\rho,\delta)\,d\gamma d\delta\right|$$

$$=\left|\frac{(1-\rho)^2}{\pi^2}\int_{\nu}^{\pi}\left\{\frac{q_{m+1}(\rho,\delta)}{(1-\rho)^2}\int_{\epsilon}^{\xi}\chi_{m+1}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)d\gamma\right\}d\delta\right| < \omega$$

provided only that $\left\{\begin{matrix}0<\epsilon = \text{arc sin }(1-r) < \Delta_2\\ 0<\eta = \text{arc sin }(1-\rho) < \Delta_2\end{matrix}\right.$.

In consideration of (15.57) for given $\left\{\begin{matrix}\xi\neq 0\\ \nu\neq 0\end{matrix}\right.$, we have

$\int_{\nu}^{\pi}\int_{\xi}^{\pi}\chi_{m+1}(x,y;\gamma,\delta)K_{m+1}(r,\gamma)q_{m+1}(\rho,\delta)\,d\gamma d\delta$ existing by Lemmas 8 and 9. Now $q_{m+1}(\rho,\delta)$ contains a factor of $(1-\rho)^2$ not involving δ from the definition of q_{m+1} given in Lemma 4 and $K_{m+1}(r,\gamma) = O(\sin\gamma)$ for $0<\xi<\gamma<\pi$ for all r on $0\leqq r\leqq 1$ by Lemma 2.

Hence, for given $\begin{cases}\xi \neq 0\\ \nu \neq 0\end{cases}$

$$\int_{\nu}^{\pi}\int_{\xi}^{\pi} \underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\frac{\underset{m+1}{q}(\rho,\delta)}{(1-\rho)^2}\,d\gamma d\delta \text{ exists for all } \begin{cases} r\\ \rho\end{cases} \text{ such that } \begin{matrix}0 \leq r \leq 1,\\ 0 \leq \rho \leq 1\end{matrix},$$

whence for given $\begin{cases}\xi \neq 0\\ \nu \neq 0\end{cases}$, we can choose $\Delta_3 > 0$ so small that

$$(15.58)\quad \left|\frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\xi}^{\pi} \underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\delta)\,d\gamma d\delta\right|$$

$$= \left|\frac{(1-\rho)^2}{\pi^2}\int_{\nu}^{\pi}\int_{\xi}^{\pi} \underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\frac{\underset{m+1}{q}(\rho,\delta)}{(1-\rho)^2}\,d\gamma d\delta\right| < \omega$$

provided only that $\begin{cases} 0 < \epsilon = \text{arc sin } (1-r) < \Delta_3\\ 0 < \eta = \text{arc sin } (1-\rho) < \Delta_3\end{cases}$.

Therefore, in consideration of (15.5), we can first choose $\nu > 0$ so small that

$$\left|\frac{1}{\pi^2}\int_{\eta}^{\nu}\int_{\epsilon}^{\pi} \underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\delta)\,d\gamma d\delta\right| < \frac{\omega}{3}$$

with $\begin{cases} 0 < \epsilon = \text{arc sin } (1-r) < \nu\\ 0 < \eta = \text{arc sin } (1-\rho) < \nu\end{cases}$ by (15.52)

and then we can find $\Delta > 0$ less than ν and so small that we have simultaneously

$$\left|\frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\epsilon}^{\xi} \underset{m+1}{\chi}(x,y;\gamma,\delta)\underset{m+1}{K}(r,\gamma)\underset{m+1}{q}(\rho,\delta)\,d\gamma d\delta\right| < \frac{\omega}{3}$$

provided only that $\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\Delta \\ 0<\eta=\text{arc sin }(1-\rho)<\Delta \end{cases}$ by (15.55)

and

$$\left|\frac{1}{\pi^2}\int_{\nu}^{\pi}\int_{\zeta}^{\pi}\chi_{m+1}(x,y;\gamma,\zeta)K_{m+1}(r,\gamma)q_{m+1}(\rho,\zeta)\,d\gamma\,d\zeta\right|<\frac{\omega}{3}$$

provided only that $\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\Delta \\ 0<\eta=\text{arc sin }(1-\rho)<\Delta \end{cases}$ by (15.58),

and hence

$$\left|\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}\chi_{m+1}(x,y;\gamma,\zeta)K_{m+1}(r,\gamma)q_{m+1}(\rho,\zeta)\,d\gamma\,d\zeta\right|<\frac{\omega}{3}+\frac{\omega}{3}+\frac{\omega}{3}=\omega$$

provided only that $\begin{cases} 0<\epsilon=\text{arc sin }(1-r)<\Delta \\ 0<\eta=\text{arc sin }(1-\rho)<\Delta \end{cases}$;

whence (15.5)$\to 0$ as $\left.\begin{cases} \epsilon=\text{arc sin }(1-r) \\ \eta=\text{arc sin }(1-\rho) \end{cases}\right\}\to 0$.

By analogy with (15.5), we have (15.6)$\to 0$ as $\left.\begin{cases} \epsilon=\text{arc sin }(1-r) \\ \eta=\text{arc sin }(1-\rho) \end{cases}\right\}\to 0$, (15.7) and (15.8) tend to 0 as $\left.\begin{cases} \epsilon=\text{arc sin}(1-r) \\ \eta=\text{arc sin}(1-\rho) \end{cases}\right\}\to 0$ by Lemma 12, while (15.9) tends to 0 as $\left.\begin{cases} \epsilon=\text{arc sin }(1-r) \\ \eta=\text{arc sin }(1-\rho) \end{cases}\right\}\to 0$ from our hypothesis.

This proves the theorem.

(16.) Theorem 2. Let $f(x,y)$ be integrable (L) in $Q(-\pi, -\pi; \pi, \pi)$ and defined outside Q by periodicity.

Then, at any point where

$$\underset{m+1}{\chi}(x,y;\underset{m+1}{\Delta},\tau)=\begin{cases} o(1) \text{ as } \begin{matrix}\Delta\\ \tau\end{matrix}\Big\}\to 0 \\ o\left(\frac{1}{\sin\tau}\right) \text{ as } \Delta\to 0 \text{ uniformly on } 0<\tau<\pi, \\ o\left(\frac{1}{\sin\Delta}\right) \text{ as } \tau\to 0 \text{ uniformly on } 0<\Delta<\pi \end{cases}$$

a necessary and sufficient condition for the summability (A) of the conjugate of the double Fourier series is that the integral

$$\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\underset{m}{\chi}(x,y;\underset{m}{\gamma},\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\, d\delta$$

converge which will then represent the sum of the series.

Here, $m+1$ is any integer $1, 2, 3, \cdots$.

Proof of Theorem 2.

Necessity: Suppose the conjugate of the double Fourier series is summable (A); then $A(x,y;r,\rho)$ tends to a limit as $\begin{matrix}r\\ \rho\end{matrix}\Big\}\to 1$, and hence if we relate $\begin{cases} r \\ \rho\end{cases}$ to $\begin{cases}\epsilon\\ \eta\end{cases}$ by $\begin{cases}\epsilon = \arcsin(1-r) \\ \eta = \arcsin(1-\rho)\end{cases}$, then

$$A(x,y;r,\rho)-\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi}\underset{m}{\chi}(x,y;\underset{m}{\gamma},\delta)\,\frac{\cot\frac{\gamma}{2}}{2}\,\frac{\cot\frac{\delta}{2}}{2}\,d\gamma\, d\delta\to 0 \text{ a. } \left\{\begin{matrix}\epsilon = \arcsin(1-r)\\ \eta = \arcsin(1-\rho)\end{matrix}\right\}\to 0$$

by Theorem 1,

and hence the integral

$$\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \underset{m\ m}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma d\delta$$ exists and equals the limit of

$A(x,y;r,\rho)$ as $\left.\begin{matrix} r \\ \rho \end{matrix}\right\}\to 1$.

Sufficiency: Suppose the integral

$$(16.5)\qquad \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \underset{m\ m}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma d\delta \text{ converges,}$$

then $\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \underset{m\ m}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma d\delta$ tends to a limit as $\left.\begin{matrix} \epsilon \\ \eta \end{matrix}\right\}\to 0$,

and if we relate $\left\{\begin{matrix} \epsilon \\ \eta \end{matrix}\right.$ to $\left\{\begin{matrix} r \\ \rho \end{matrix}\right.$ by $\left\{\begin{matrix} \epsilon = \text{arc sin }(1-r) \\ \eta = \text{arc sin }(1-\rho) \end{matrix}\right.$, then

$$A(x,y;r,\rho) - \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \underset{m\ m}{\chi}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma d\delta \to 0$$

as $\left.\begin{matrix} \epsilon = \text{arc sin }(1-r) \\ \eta = \text{arc sin }(1-\rho) \end{matrix}\right\}\to 0$ by Theorem 1, and hence $A(x,y;r,\rho)$ tends to a limit as $\left.\begin{matrix} r \\ \rho \end{matrix}\right\}\to 1$, which is the integral (16.5) above. Therefore, the series is summable (A) and summable (A) to the value of the integral (16.5) above.

(17.) Theorem 3. Let $f(x,y)$ be integrable (L) in $Q(-\pi,-\pi;\pi,\pi)$ and defined outside Q by periodicity.

Then, at any point where

$$\underset{m+1\ m+1}{\chi}(x,y;s,t)=\begin{cases} o(1) \text{ as } \left.\begin{matrix}s\\ t\end{matrix}\right\}\to 0 \\ o\left(\frac{1}{\sin t}\right) \text{ as } s\to 0 \text{ uniformly on } 0<t<\pi, \\ o\left(\frac{1}{\sin s}\right) \text{ as } t\to 0 \text{ uniformly on } 0<s<\pi \end{cases}$$

m being a positive integer or 0, a necessary and sufficient condition for the summability (A) of the conjugate of the double Fourier series in the restricted sense is that the integral

$$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi\left[f(x+\gamma,y+\delta)-f(x-\gamma,y+\delta)-f(x+\gamma,y-\delta)+f(x-\gamma,y-\delta)\right]\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta$$

exist $(C;m,m)$ in the restricted sense, which integral will then represent the sum of the series.

Proof of Theorem 3.

Necessity: Suppose the conjugate of the double Fourier series is summable (A) in the restricted sense. Then $A(x,y;r,\rho)$ tends to a limit as $\left.\begin{matrix}r\\ \rho\end{matrix}\right\}\to 1$ in the restricted sense. If we relate $\left\{\begin{matrix}r\\ \rho\end{matrix}\right.$ to $\left\{\begin{matrix}\epsilon\\ \eta\end{matrix}\right.$ by

$$\begin{cases}\epsilon=\arcsin(1-r)\\ \eta=\arcsin(1-\rho)\end{cases},$$ then by Theorem 1

$$A(x,y;r,\rho)-\frac{1}{\pi^2}\int_\eta^\pi\int_\epsilon^\pi \underset{m\ m}{\chi}(x,y;\gamma,\delta)\cot\frac{\gamma}{2}\cot\frac{\delta}{2}\,d\gamma\,d\delta\to 0 \text{ as } \left.\begin{cases}\epsilon=\arcsin(1-r)\\ \eta=\arcsin(1-\rho)\end{cases}\right\}\to 0$$

in the restricted sense,

and hence

$$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi(x,y;\gamma,\delta)\,\frac{\cot\gamma}{2}\,\frac{\cot\delta}{2}\,d\gamma\,d\delta$$ exists in the restricted sense, or we

can say that it exists $(C;0,0)$ in the restricted sense and represents the sum of the series.

Therefore, by Lemma 10,

$$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi(x,y;\gamma,\delta)\,\frac{\cot\gamma}{2}\,\frac{\cot\delta}{2}\,d\gamma\,d\delta$$ exists $(C;m,m)$ in the restricted sense

and represents the sum of the series.

Sufficiency: Suppose

(17.5) $$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi(x,y;\gamma,\delta)\,\frac{\cot\gamma}{2}\,\frac{\cot\delta}{2}\,d\gamma\,d\delta$$ exists $(C;m,m)$ in the restricted

sense. Then, by Lemma 10,

$$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi(x,y;\gamma,\delta)\,\frac{\cot\gamma}{2}\,\frac{\cot\delta}{2}\,d\gamma\,d\delta$$ exists $(C;m,m)$ in the restricted sense

and equals (17.5) above.

But this means that

$$\frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \chi_{m\, m}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta \quad \text{tends to a limit as } \left.\begin{matrix}\epsilon\\ \eta\end{matrix}\right\}\to 0$$

in the restricted sense, and if we relate $\left\{\begin{matrix}\epsilon\\ \eta\end{matrix}\right.$ to $\left\{\begin{matrix}r\\ \rho\end{matrix}\right.$ by the

relation $\left\{\begin{matrix}\epsilon = \text{arc sin } (1-r)\\ \eta = \text{arc sin } (1-\rho)\end{matrix}\right.$, then, by Theorem 1

$$A(x,y;r,\rho) - \frac{1}{\pi^2}\int_{\eta}^{\pi}\int_{\epsilon}^{\pi} \chi_{m\, m}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta \to 0 \text{ as } \left\{\begin{matrix}\epsilon = \text{arc sin } (1-r)\\ \eta = \text{arc sin } (1-\rho)\end{matrix}\right\}\to 0$$

in the restricted sense, and hence $A(x,y;r,\rho)$ tends to a limit as

$\left.\begin{matrix}r\\ \rho\end{matrix}\right\}\to 1$ which is

$$\frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi} \chi_{m\, m}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta.$$

Therefore, the series is summable (A) in the restricted sense to

$$\frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi} \chi_{m\, m}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta \quad (C;0,0) \text{ and hence, by Lemma 10, to}$$

$$\frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi} \chi_{0\, 0}(x,y;\gamma,\delta)\ \frac{\cot\frac{\gamma}{2}}{2}\ \frac{\cot\frac{\delta}{2}}{2}\ d\gamma\, d\delta \quad (C;m,m).$$

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