274

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TABLE OF CONTENTS

INTRODUCTION

The experimental observation of the separation in the 2s and 2p levels of the Hydrogen atom[1] has stirred the interest of many physicists, and many proposals have been made to account for the discrepancy with the existing theory.

It is well known that the Dirac Hamiltonian for the electron does not include terms corresponding to the interaction between the electron and its transverse electromagnetic field. Many attempts have been and are being made to introduce this radiative reaction term[2] into the Dirac theory. These attempts have been partially successful in that they do give numerical results agreeing with experiment. These theories, however, require special assumptions because of the occurence of divergent integrals.

This paper will show that, on the basis of Podolsky's Generalized Quantum Electrodynamics[3], no new assumptions are necessary to account for the energy level shift.

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1. W. E. Lamb and R. C. Retherford, Phys. Rev., 72, 241 (1947)
2. H. A. Bethe, Phys. Rev., 72, 339 (1947); H. W. Lewis Phys. Rev., 73, 173 (1948); F. J. Dyson, Phys. Rev., 73, 617 (1948)
3. B. Podolsky and P. Schwed, Rev. Mod. Phys., 20, 40 (1948)

THEORY

Removal of Degeneracy

We shall show here how the degeneracy in the energy levels disappears in this theory. The wave equation for the Podolsky electron is given by Green[4] as:

$$(H + U)\psi = E'\psi \tag{1.1}$$

where

$$H = -\alpha_e \cdot cp_e - m_e c^2 \beta_e \tag{1.2}$$

$$E' = E + e^2/4a \tag{1.3}$$

$$U = -e^2 \Big\{ (1 - \tfrac{1}{2}\alpha_e \cdot \alpha_p)(1 - \exp(-r/a))/r - (\alpha_e \cdot \bar{r}/r)(\alpha_p \cdot \bar{r}/r)(1 - \exp(-r/a))/r - (\exp(-r/a))/a \Big\} \tag{1.4}$$

In the above equations U is the complete electrostatic and electromagnetic interaction. For the central field problem we may neglect those terms in U which involve the Dirac matrices, α_e and α_p. This is justified because α_p, which corresponds in the classical limit to v/c for the nucleus is negligible. This assumption is, of course, made in neglecting the vector potential term in Dirac's theory. Our wave equation now becomes:

$$(-\alpha_e \cdot cp_e - m_e c^2 \beta_e - e^2(1 - \exp(-r/a))/r)\,\psi = E'\psi \tag{1.5}$$

We see that our equation differs in two respects from Dirac's; first, the potential is Coulombic only for large radii, and, secondly, the energy contains an additional term, $-e^2/4a$, whose significance we shall discuss

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4. A. E. S. Green, Phys. Rev., 72, 339 (1947)

later. We may, in fact, make our equation identiaal to that of Dirac's[5] by placing:

$$V'(r) = -e^2(1 - \exp(-r/a))/r \qquad (1.6)$$

Eq. 1.5 then becomes:

$$(\alpha \cdot cp - mc^2\beta + V'(r))\psi = E'\psi \qquad (1.7)$$

which is Dirac's equation for the electron with our primed quantities replacing his unprimed quantities.

For the sake of clarity, we shall illustrate the essential steps necessary in solving eq. 1.7. Introducing explicit expressions for the 4 by 4 matrices α and β and the four-component ψ symbol, we get from eq. 1.7 the following simultaneous equations:

$$\begin{aligned}
(V' - mc^2)\psi_1 - cp_z\psi_3 - c(p_x - ip_y)\psi_4 &= E'\psi_1 \\
(V' - mc^2)\psi_2 + cp_z\psi_4 - c(p_x + ip_y)\psi_3 &= E'\psi_2 \\
(V' + mc^2)\psi_3 - cp_z\psi_1 - c(p_x - ip_y)\psi_2 &= E'\psi_3 \\
(V' + mc^2)\psi_4 + cp_z\psi_2 - c(p_x + ip_y)\psi_1 &= E'\psi_4
\end{aligned} \qquad (1.8)$$

The eqs. 1.8 have three constants of motion, and the corresponding simultaneous eigen-ψ's are

$$\psi(j, k=j+\tfrac{1}{2}, m) = \begin{pmatrix} (j+m)^{\frac{1}{2}}R^{I}G(j-\tfrac{1}{2}, m-\tfrac{1}{2}) \\ (j-m)^{\frac{1}{2}}R^{I}G(j-\tfrac{1}{2}, m+\tfrac{1}{2}) \\ (j+1-m)^{\frac{1}{2}}R^{II}G(j+\tfrac{1}{2}, m-\tfrac{1}{2}) \\ -(j+1+m)^{\frac{1}{2}}R^{II}G(j+\tfrac{1}{2}, m+\tfrac{1}{2}) \end{pmatrix} \qquad (1.9)$$

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5. See any book on Quantum Mechanics. With minor changes the notation used here is that of Rojansky's " Introductory Quantum Mechanics."

and

$$\psi(j,k=-j-\tfrac{1}{2},m) = \begin{pmatrix} (j+1-m)^{\frac{1}{2}}R^{I}G(j+\tfrac{1}{2},\ m-\tfrac{1}{2}) \\ -(j+1+m)^{\frac{1}{2}}R^{I}G(j+\tfrac{1}{2},\ m+\tfrac{1}{2}) \\ (j+m)^{\frac{1}{2}}R^{II}G(j-\tfrac{1}{2},\ m-\tfrac{1}{2}) \\ (j-m)^{\frac{1}{2}}R^{II}G(j-\tfrac{1}{2},\ m+\tfrac{1}{2}) \end{pmatrix} \qquad (1.10)$$

where the radial functions R are unrelated in (1.9) and (1.10). Either of the functions (1.9) and (1.10) may be substituted into eq.(1.8). In both cases, it is found that the substitution gives only two independent equations. After some manipulations of the equations so obtained, the tesseral functions, G, cancel out and yield the following radial equations:

$$(E'-V'(r)+mc^2)(R_a/c\hbar) - dR_b/dr - (k+1)(R_b/r) \qquad (1.11)$$

$$(E'-V'(r)-mc^2)(R_b/c\hbar) + dR_a/dr - (k-1)(R_a/r) \qquad (1.12)$$

where

$$R_a = R^{I}, \qquad R_b = i((k-\tfrac{1}{2})/(k-\tfrac{1}{2}))^{\frac{1}{2}}\ R^{II} \qquad (1.13)$$

and the quantum number k has the values:

$$k = \pm 1,\ \pm 2,\ \pm 3,\ \ldots\ -n. \qquad (1.14)$$

We may further simplify eqs. (1.11) and (1.12) by the substitutions:

$$X_a = rR_a, \qquad X_b = rR_b \qquad (1.15)$$

and

$$A = (mc^2 + E'),\quad B = (mc^2 - E'),\quad \alpha = e^2/c\hbar \qquad (1.16)$$

Substituting these quantities and reintroducing the explicit expression for $V'(r)$ we obtain the X equations:

$$AX_a + \alpha X_a/r - \alpha \exp(-r/a)X_a/r - X_b' - kX_b/r = 0 \qquad (1.17)$$

$$BX_b - \alpha X_b/r + \alpha \exp(-r/a)X_b/r - X_a' + kX_a/r = 0 \qquad (1.18)$$

where X' means dX/dr. The above equations are the same as Dirac's except for the exponential term. We note that, except for small radii, the exponential term may be neglected.[6] We are therefore justified in assuming the same series solutions for (1.17) and (1.18) as in the Dirac theory:

$$X_a = e^{-Cr} \sum_m a_{t+m} r^{t+m}, \qquad m = 0, 1, 2, \ldots \quad (1.19)$$

$$X_b = e^{-Cr} \sum_m b_{t+m} r^{t+m}, \qquad m = 0. 1. 2, \ldots \quad (1.20)$$

Substituting these into eqs. (1.17) and (1.18), we obtain two new equations which may be satified only if the coefficients of like powers of r are equal to zero. Hence, we obtain a sequence of equations from which we can solve for the a's, b's and t. Our sequence of equations is, of course, more complicated than the corresponding one's in Dirac's theory due to our non-Coulombic potential. Fortunately, however, we can draw many conclusions from the first pair of equations; this pair of equations give a relationship between t, the exponent of the first term of the series solution, and the quantum number k. These equations are:

Here	Dirac's Case	
$tb_t + kb_t = 0$	$\alpha a_t - (t + k)b_t = 0$	
$ta_t - ka_t = 0$	$\alpha b_t + (t - k)\, a_t = 0$	(1.21)

6. The magnitude of a is of the order of nuclear dimensions; thus, it is difficult to detect, through experiment, the deviation of our potential from Coulomb's law.

The eqs. (1.21) have the following solutions:

Here	Dirac's Case	
$t = -k, \quad a_t = 0$ $t = k, \quad b_t = 0$	$t = (k^2-\alpha^2)^{\frac{1}{2}}$	(1.21)

Recalling that k takes on positive and negative integral values, we see that our first pair in (1.21) holds for negative values of k and the second pair for positive values of k; otherwise, the solutions will not be well behaved at the origin. That is to say, the radial functions will diverge at the origin unless t is positive.

Starting with the values for t as given by (1.21), we may proceed to solve for the energy levels E_{nk} which render the functions R_a and R_b bounded at $r =$ infinity. In Dirac's theory E_{nk} is chosen in such a way that the power series in (1.19) and (1.20) terminate with a finite number of terms. There is one solution for E_{nk} corresponding to both positive and negative values of k. Notice, however, that in our case we have two separate energy levels corresponding to positive and negative values of k; thus, we have removed the degenerate energy levels.

Difficulty in solving for E_{nk} arises in our theory because we cannot simply terminate the power series as is done in Dirac's theory. This is due to the appearance of the transcendental exponential function in the modified potential function.

It may at first glance appear that we have assumed a wrong solution since Dirac's energy levels are found by choosing E_{nk} so that the power series (1.19) and (1.20) terminate, while in our case we cannot let the same series terminate. It is evident, however, that, if our series does not terminate, there must be a condition on the ratio of the coefficients of successive powers of r. Such a condition would be necessary, for example in (1.19), so that as r approaches infinity the factor exp(-Cr) converges to zero more "powerfully" than the divergent infinite series $\sum_{m} a_{t+m} r^{t+m}$. Such a condition would then permit us to solve for E_{nk}. The solution at best is very cumbersome to obtain. We, therefore, resort to the standard perturbation method to obtain an approximate value of the separation in the 2s and 2p levels.

Perturbation Calculation

We rewrite our wave equation (1.5) as follows:

$$(H + V_0)\psi = (E + e^2/4a)\psi \qquad (2.0)$$

where

$$V_0 = e^2 \exp(-r/a)/r \qquad (2.1)$$

In eq. (2.0) H is the Dirac Hamiltonian, V_0 the perturbing potential, and E the perturbed energy levels. The term $e^2/4a$ represents the electrodynamic self-energy of the electron. This term may be dropped from our wave equation (2.0) for our present calculation because, in taking the difference of the energies of the 2s and 2p states, the self-energy will cancel. Thus, (2.0) becomes

$$(H + V_0)\psi = E\psi \qquad (2.2)$$

In this form we can apply the standard perturbation method with Dirac's wave equation representing the unperturbed system. The linear energy correction, which is sufficiently accurate for our purposes, is given by the formula:

$$w = \int \psi_0^* V_0 \psi_0 \, dv \qquad (2.3)$$

where ψ_0 is the Dirac ψ function and ψ_0^* its complex conjugate.

Strictly speaking, the correction (2.3) holds only for non-degenerate states, but we may use it here because our perturbing potential V_0 is a function of radius only. Had V_0 contained angular variables it would be necessary to use methods pertaining to degenerate states.

In order to evaluate w_{2s} and w_{2p} we need explicit expressions for the ψ functions, ψ_{2s} and ψ_{2p}. In the Dirac theory the quantum numbers corresponding to different stationary states is given in the table below:

Quantum No.			States				
	$1^2S_{\frac{1}{2}}$	$2^2S_{\frac{1}{2}}$	$2^2P_{\frac{1}{2}}$	$3^2S_{\frac{1}{2}}$	$3^2P_{\frac{1}{2}}$	3^2P	3^2D
j	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	3/2	3/2
k	-1	-1	1	-1	1	-2	2
ℓ	0	0	1	0	1	1	2

The angular factors for the 2s state are given by (1.10) with the appropriate quantum numbers taken from the preceeding table. It remains for us to find the radial functions R_a and R_b. We substitute $k = -1$ into (1.17) and (1.18) and drop the exponential terms. This gives

$$AX_a + \alpha X_a/r - X_b' + X_b/r = 0 \qquad (2.4)$$

$$BX_b - \alpha X_b/r - X_a' - X_a/r = 0 \qquad (2.5)$$

and the series solution (1.19) and (1.20) here become

$$X_a = \exp(-Cr)\ (a_t r^t + a_{t+1} r^{t+1}) \qquad (2.6)$$

$$X_b = \exp(-Cr)\ (b_t r^t + b_{t+1} r^{t+1}) \qquad (2.7)$$

where the constants are related as follows:

$$E_{2s} = mc^2 g, \quad g = (\tfrac{1}{2} + \tfrac{1}{2}(1 - \alpha^2))^{\frac{1}{2}}, \quad t = 2g^2 - 1$$

$$C = (mc/\hbar)(1 - g^2)^{\frac{1}{2}}, \quad t = (1 - \alpha^2)^{\frac{1}{2}}; \quad \alpha = 2g(1 - g^2)^{\frac{1}{2}}$$

Substituting (2.6) and (2.7) into (2.4) and (2.5), we get two equations in powers of r. These two equations can be satisfied only if the coefficients of each power of r are equal to zero. Hence we get for the coefficients of r^{t-1}, r^t, r^{t+1}, respectively, the following equations:

$$\left.\begin{aligned} &\alpha a_t - (t - 1)b_t = 0 \\ &\alpha b_t + (t + 1)a_t = 0 \end{aligned}\right\} \qquad (2.8)$$

$$\left.\begin{aligned} &(mc/\hbar)\left[(1 + g)a_t - (1 - g^2)^{\frac{1}{2}} b_t\right] + \alpha a_{t+1} - t b_{t+1} = 0 \\ &(mc/\hbar)\left[(g-1)b_t - (1-g^2)^{\frac{1}{2}} a_t\right] + \alpha b_{t+1} - (t+2)a_{t+1} = 0 \end{aligned}\right\} \qquad (2.9)$$

and

$$\left.\begin{aligned} &(1 + g)a_{t+1} - (1 - g^2)^{\frac{1}{2}} b_{t+1} = 0 \\ &(g - 1)b_{t+1} - (1 - g^2)^{\frac{1}{2}} a_{t+1} = 0 \end{aligned}\right\} \qquad (2.10)$$

It can be readily shown that the determinants of eqs. (2.8) and (2.10) vanish. We can therefore solve for the a's in terms of the b's and eliminate two of the four unknowns in (2.9). The resulting determinant of (2.9) is also found to vanish; hence we can solve for all the constants in terms of one of them, say, b_t. We have,

$$
\begin{aligned}
b_{t-1} &= -(mc/\hbar)\left((1-g^2)^{\frac{1}{2}}/(2g-1)\right) b_t \\
a_{t-1} &= (mc/\hbar)\left((1-g)/(2g-1)\right) b_t \qquad (2.11) \\
a_t &= \left((g^2-1)/g(1-g^2)^{\frac{1}{2}}\right) b_t
\end{aligned}
$$

Using the above constants, we obtain at once the X's, and, recalling that $X_a = rR_a$ and $X_b = rR_b$, we finally obtain the radial functions:

$$
R_a = e^{-Cr}\left\{ -\frac{(1-g^2)^{\frac{1}{2}}}{g}\, r^{(2g^2-2)} + \frac{mc(1-g)}{\hbar(2g-1)}\, r^{(2g^2-1)} \right\}
$$

$$
R_b = e^{-Cr}\left\{ r^{(2g^2-2)} + \frac{mc(1-g^2)^{\frac{1}{2}}}{\hbar(2g-1)}\, r^{(2g^2-1)} \right\} \qquad (2.12)
$$

where we have, for convenience, retained only the constants C and g. The normalizing constants have been omitted in (2.12).

From eq. (1.13) we get for the 2s state,

$$
R^{I} = R_a, \qquad R^{II} = -i(3)^{\frac{1}{2}} R_b \qquad (2.13)
$$

Substituting these into (1.10) we get

$$
\psi(j=\tfrac{1}{2}, k=-1, m=-\tfrac{1}{2}) = \begin{bmatrix} 2^{\frac{1}{2}}\, R_a\, G^{-1}_{1} \\ -R_a\, G^{-0}_{1} \\ 0 \\ -i3^{\frac{1}{2}}\, R_b\, G^{0}_{0} \end{bmatrix} \qquad (2.14)
$$

We are now ready to use the perturbation formula (2.3) and calculate the shift of the 2s level; thus:

$$w_{2s} = \int \psi^* \, V_o \psi \, dv \tag{2.15}$$

where the ψ function is that given by (2.14). The ortho-normality in the angular function G allows us to simplify (2.15) into the following:

$$w_{2s} = N^2 \int_0^\infty V_o (R_a^2 + R_b^2) r^2 dr \tag{2.16}$$

where

$$1/N^2 = \int_0^\infty (R_a^2 + R_b^2) r^2 dr \tag{2.17}$$

The integrals in (216) and (2.17) may be evaluated in a straight forward manner (see Appendix), and we get

$$w_{2s} = (ea/\sqrt{2})^2 (mc\alpha/\hbar)^3 \tag{2.18}$$

where we have neglected terms of higher order than α^2 by the use of the following approximations,

$$\begin{aligned} g^2 &\doteq 1 - \alpha^2/4, \quad (1 - g^2)^{\frac{1}{2}} \doteq \alpha/2 \\ g &\doteq 1 - \alpha^2/8, \quad 2C \doteq (2mc/\hbar)(1 - g^2)^{\frac{1}{2}} \doteq (mc\alpha/h) \end{aligned} \tag{2.19}$$

The calculation for w_{2p} amounts to a repetition of the above work, and we need only outline it briefly. The radial functions for this case are

$$\begin{aligned} R_a &\doteq e^{-Cr} \left\{ \frac{g}{(1-g^2)^{\frac{1}{2}}} r^{2g^2-2} - \frac{mc}{\hbar(2g-1)} r^{2g^2-1} \right\} \\ R_b &\doteq e^{-Cr} \left\{ r^{2g^2-2} + \frac{mc(1+g)^{\frac{1}{2}}}{\hbar(2g-1)(1-g)^{\frac{1}{2}}} r^{2g^2-1} \right\} \end{aligned} \tag{2.20}$$

It is noted that these functions differ from those of the 2s state only in the coefficients of r.

with these radial functions, the integrations in (2.16) and (2.17) may be repeated to give

$$w_{2p} = (ea/\sqrt{2})^2(mc\alpha/\hbar)^3(\alpha^2/16), \qquad (2.21)$$

that is,

$$w_{2p} = w_{2s}(\alpha^2/16) \qquad (2.22)$$

Thus we see that the shift in the 2p state is negligible, and we get to a good approximation

$$w' = w_{2s} - w_{2p} = (ea/\sqrt{2})^2(mc\alpha/\hbar)^3 = e^2a^2/2a_o^3 \qquad (2.23)$$

where a_o is the Bohr radius.

Expressed in wave numbers, the separation of the 2s and 2p levels has been determined experimentally by Lamb and Retherford[7] as

$$v^{-1} = .0355 \text{ cm.}^{-1} \qquad (2.24)$$

Therefore,

$$w'/hc = e^2a^2/2hca_o^3 = .0355 \qquad (2.25)$$

Solving for a we get

$$a = 10.67\ r_o \qquad (2.26)$$

where r_o is the classical electronic radius ($r_o = (e^2/m_oc^2)$).

The above value for a agrees with that found by Kikuchi[8] (his calculation was made in non-relativistic approximation and was based on Pasternack's[9] hypothetical shift of .03 cm.$^{-1}$)

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7. Latest value quoted by Lamb in a paper presented before Am. Phys. Soc. (April 30, 1948) See ref. 1
8. C. Kikuchi, Phys. Rev., 69, 125 (1946)
9. S. Pasternack, Phys. Rev., 54, 1113 (1938)

As a partial check on our theory we must ascertain that the normal state is shifted a negligible amount by the perturbing potential. This requirement is of great importance since the separation between the normal state and the next higher state (2p state) has been determined to great accuracy by spectroscopists and found to check with existing theory. The function for the normal state is given in many texts and we need only state the result of the perturbation calculation. We find

$$w'_{1s} = (16/3)\, w'_{2s} = .190 \text{ cm.}^{-1} \qquad (2.27)$$

which is a negligible shift since the 1s - 2p separation is of the order of 82,000 cm.$^{-1}$

In order to find the level shift for other hydrogen-like atoms (ionized He, etc.) we need to go back in our calculations and replace e^2 by Ze^2 where Z is the atomic number. Thus, for example, for ionized helium we find

$$w'_{2s} = Z^4(.0355) = .567 \text{ cm.}^{-1} \qquad (2.28)$$

This value differs greatly from that found by Bethe[2] in his classical approach to the problem; he finds

$$w_{2s} = .46 \text{ cm.}^{-1} \qquad (2.29)$$

We, therefore, have a basis of comparison of the merits of the two theories pending further experiment. Such an experiment does seem possible in the immediate future because the frequency required of the microwave-oscillator is not too high (around 15,000 m.c.).

Conclusions

We have shown that on the basis of the Generalized Quantum Electrodynamics the degeneracy in Dirac's theory disappears. It was shown that in the new theory the central field problem reduces to a study of Dirac's wave equation for a modified Coulomb potential. The important thing to bear in mind is that the new potential function arose naturally from a simple basic assumption; that the Lagrangian contains the second as well as the usual first derivative of the generalized field coordinates.

If one does not accept the method of approach in the Generalized field theory, it is necessary to assume a cut-off frequency for high energies in order to avoid infinite integrals. Such an assumption was necessary, for example, in the calculation made by Bethe. Podolsky's theory does actually make the same assumption in a more general way; the constant of length a appearing in the theory is intimately connected with the cut-off energy for radiation and is to be determined experimentally as we have done in this paper. Once this constant has been determined, there will be no necessity for special assumptions in dealing with "high-energy" problems. Thus, the arbitrariness in the theory has been removed, and one may check the merits of the theory by solving other problems which involve high energies (such as the electron scattering problem.)

In the usual quantum electrodynamics[10] one finds it necessary to assume a cut-off energy limit for high frequency radiation. In the generalized theory we have seen that such an assumption is not necessary, but it is not difficult to visualize that the universal constant of length a is intimately connected with the cut-off limit.

We assume that the constant a is equivalent to the critical wavelength or cut-off wavelength:

$$a = \lambda_c \qquad (3.0)$$

On this assumption we find for the critical energy:

$$E_c = hc/\lambda_c = 80m_o c^2 \qquad (3.1)$$

Heitler[10] quotes $137mc^2$ as the order of magnitude of the critical energy limit, and our value appears to be compatible.

On the basis of the generalized theory, particles may interact in the ordinary manner with the real or virtual photons or in an extraordinary manner with the real or virtual low mass mesons. The mass of the low mass meson associated with the extraordinary field is:

$$m = \hbar/ac = 12.82m_o \qquad (3.2)$$

Finally we may evaluate the electromagnetic self-energy of the electron as given by Green[3]: $-\frac{e^2}{4a} = -\frac{m_o c^2}{42.6}$ (3.3)

10. W. Heitler, "Quantum Theory of Radiation," Oxford, 1936.

The relativistic energy momentum relation gives us

$$(E + e^2/4a)^2 = c^2p^2 + m_o^2c^4 \qquad (3.4)$$

or for $p = 0$,

$$E = m_oc^2 - e^2/4a = Mc^2 \qquad (3.5)$$

$$M = m_o - e^2/4ac^2 \qquad (3.6)$$

where M is the experimental mass, m_o the mechanical mass, and the last term, the electromagnetic mass.

APPENDIX

The integrals in the text are all of the same type and may be evaluated readily in terms of the gamma function. We shall show here the details of the perturbation calculation for the 2s state. We have for the energy due to the perturbation,

$$w'_{2s} = N^2 \int_0^\infty (\exp(-r/a)/r)(R_a^2 r^2 + R_b^2 r^2)dr \qquad \text{(A1)}$$

where

$$N^{-2} = \int_0^\infty (R_a^2 r^2 + R_b^2 r^2)dr \qquad \text{(A2)}$$

Consider the first term:

$$R_a^2 r^2 = \exp(-2Cr) \left\{ \frac{(1-g^2)}{g^2} r^{4g^2-2} - \frac{2C(1-g)}{g(2g-1)} r^{4g^2-1} - \frac{C^2(1-g^2)}{(1+g)(2g-1)^2} r^{4g^2} \right\} \qquad \text{(A3)}$$

The second term leads to expressions differing only in the coefficients; thus, the integrals are of the type

$$I_1 = \int_0^\infty \exp(-2Cr)((1-g^2)/g^2)r^{4g^2-2}dr \qquad \text{(A4)}$$

Now, let

$$2Cr = s, \text{ or } r = s/2C, \text{ and } dr = ds/2C \qquad \text{(A5)}$$

then, (A4) becomes:

$$I_1 = \frac{1-g^2}{g^2}\left(\frac{1}{2C}\right)^{4g^2-1} \int_0^\infty e^{-s} s^{(4g^2-1)-1} ds \qquad \text{(A6)}$$

Thus, we see that the definite integral is simply the definition of the gamma function; hence,

$$I_1 = \frac{1-g^2}{g^2} \frac{1}{2C}^{4g^2-1} \Gamma(4g^2-1) \qquad \text{(A7)}$$

Similarly, all other terms may be transformed into gamma function. They have different arguments in the gamma function, but they may be expressed in terms of a common argument by the use of the relations,

$$\Gamma(p-1) = p\,\Gamma(p), \quad \Gamma(p) = (p-1)\,\Gamma(p-1) \qquad \text{(A8)}$$

One gets for the perturbation energy, finally,

$$W'_{2s} = e^2 \left\{ \frac{(3g^2-2)(2C+1/a)}{g^2(4g^2-2)} - \frac{2C}{g(2g-1)} - \frac{2m^2c^2(2g-1)(g-1)}{h^2(2g-1)(2C-1/a)} \right\}$$

$$\times \left\{ \frac{1-g-2g^2}{g^2} + \frac{2g^2(2g+1)}{(1+g)(2g-1)} \right\}^{-1} \times \left(\frac{2C}{2C+1/a} \right)^{4g^2-1} \qquad \text{(A9)}$$

<u>Approximation</u>

The exact linear energy correction (A9) may be simplified by neglecting all but the largest terms, assuming that <u>a</u> is of the order of magnitude 10^{-12}cm. Then, we have

$$W'_{2s} = \frac{e^2a^2}{2}\left(\frac{mc\alpha}{\hbar}\right)^3 = \frac{e^2a^2}{2}\,a_o^{-3} \qquad \text{(A10)}$$

where α is the fine structure constant and a_o is the Bohr radius. The terms neglected are roughly 10^{-4} times the size of the above term and are certainly negligible until experimental precision is greatly improved.

ADDITIONAL THEORY

On the completion of this thesis, it was called to the attention of this writer that there was experimental datum* for the 3s-3p separation of ionized helium. Here, we give the added calculation. It can be readily shown that, to the approximation we desire, the Schroedinger functions are adequate for the perturbation calculation.

Using the Schroedinger function, we get for the 3s energy level shift,

$$w'_{3s} = (8/27)(Z^4e^2a^2/cha_o^3) \quad \text{(in wave numbers)} \qquad \text{(B1)}$$

$$= (8/27)\, Z^4(.0355)\ \text{cm.}^{-1} \qquad \text{(B2)}$$

Therefore, for helium, we get

$$w'_{3s} = (8/27)(16)(.0355) = .168\ \text{cm.}^{-1} \qquad \text{(B3)}$$

The experimental value is

$$w'_{3s} = .16\ (\pm .02)\ \text{cm.}^{-1} \qquad \text{(B4)}$$

The value given by Bethe (here corrected for latest value of the hydrogen 2s shift) is

$$w'_{3s} = .140\ \text{cm.}^{-1} \qquad \text{(B5)}$$

The experimental value given in (B4) was determined spectroscopically by Mack and Austern*. In making the optimistic statement concerning the possibility of measuring the 2s separation of ionized helium, it did not occur to the writer that, in the microwave method, there would be difficulties in detecting excited states of ionized helium.

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* J. E. Mack and N. Austern, Phys. Rev., 72, 972 (1947)