Orbit of asteroid 1910 JR (Taunton 99).

Thesis submitted to the Faculty of the University of Cincinnati, in partial fulfillment of the requirements for the degree of Doctor of Philosophy,

by

Everett I. Yowell.

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Orbit of asteriod 1910 TR (Taunton 99).

This minor planet was discovered on a plate taken by Rev. Joel H. Metcalf at Taunton, Mass., on February 4, 1910. Five other plates were taken on February 5, March 3 and 11, April 1 and 3. He has furnished the following data for determining the observed positions:

Plate 999 taken Feb. $4^d\ 16^h\ 6^m\ 55^s$ G. M. T.

		x	y
B. D. 20° 2328	Star 1	12.398	32.850
	Asteroid	13.167	32.200
B. D. 19° 2226	Star 2	17.010	27.215

Plate 1002 taken Feb. 5 $15^h\ 7^m\ 30^s$ G.M. T.

B. D. 19° 2218	3	8.738	32.250
	Asteroid	11.738	32.208
B. D. 20, 2328	4	12.489	32.549

Plate 1024 taken March $3^d\ 15^h\ 1^m\ 50^s$ G. M. T.

B. D. 20 2274	Star 5	(12.769 (12.675	39.979 39.978	Take mean of two images.
	Asteroid	(13.632 (13.570	40.009 40.008	
B. D. 20 2275	Star 6	(13.749 (13.650	39.065 39.065	

Plate 1035 taken March $11^d\ 13^h\ 34^m\ 30^s$ G. M. T.

B. D. 20 2260	Star 7	12.282	23.978
	Asteroid	14.669	23.487
B. D. 19 2159	Star 8	17.218	19.069

57410

Plate 1052 taken April $1^d\ 15^h\ 15^m$ G. M. T.

B. D. 20 2250	Star 9	7.178	24.040
	Asteroid	9.582	21.270
	Star 10	11.915	18.188

Plate 1055 taken April $3^d\ 14^h\ 15^m\ 0^s$ G. M. T.

B. D. 20 2250	Star 11	19.350	35.951
	Asteroid	21.926	32.530
B. D. 19 2139	Star 12	24.097	30.110

To convert the rectangular coordinates into right ascension and declination, the following equations are used:

$$\tan\varphi = \frac{\Delta\delta}{15\,\Delta\alpha\cos\frac{\delta_1+\delta_2}{2}}, \quad \tan(\varphi-\vartheta) = \frac{\Delta y}{\Delta x}$$

$$\Delta\delta = \sigma(\Delta x\sin\vartheta + \Delta y\cos\vartheta)$$

$$\Delta\alpha = \sigma(\Delta x\cos\vartheta - y\sin\vartheta)$$

The scale value σ and angle ϑ are determined from the two given stars, and then with these values, the asteroid is determined with reference to each star. By means of these equations the following positions of the asteroid wwre obtained for 1910.0.

	Date G. M. T.	h	m	s				ms	
1910	Feb. 4.67147	9	26	27.00	19	46	28.5	1	2
	Feb. 5.63021	9	25	36.38	19	48	40.8	3	4
	Mar. 3.62627	9	04	26.09	20	23	27.4	5	6
	Apr.11.56564	8	59	55.80	20	21	28.0	7	8

		x						**
	h	m	s					
Apr. 1.63542	8	55	03.40	19	47	25.2	9	10
Apr. 3.59375	8	55	08.91	19	42	17.8	11	12

*		(1910.0)		(1910.0)			Authority
1	9	26	01.09	19	51	26.1	1/3(Rbg + A.G.Ber A + A.G.Ber B)
2	9	28	37.32	19	08	08.6	1/2(Rbg + A.G.Ber A + A.G.Ber.B)
3	9	23	57.45	19	40	48.3	1/3(A.G. Ber A + Par_3 + Rbg)
4	9	26	01.10	19	51	26.4	1/2(A.G.Ber A + A.G.Ber B)
5	9	03	56.98	20	06.3		A.G.Ber B.
6	9	04	29.90	20	16	08.5	A.G.Ber B
7	8	58	36.24	20	25	14.7	1/2(Bo VI + Ber B)
8	9	01	22.08	19	47	24.9	A.G.Ber A.
9	8	53	43.62	20	09	01.1	Kü
10	8	56	20.65	19	23	23.3	1/3(A.G.Ber A + Par_3 + Ba)
11	8	53	43.62	20	09	01.1	Kü
12	8	56	20.65	19	23	23.3	1/3(A.G.Ber A + Par_3 + Ba.)

In the absence of any knowledge of the orbit, it is a matter of choice as to the simplest method of procedure. The method determined upon was to compute a provisional ellipse and correct this by varying the geocentric distances. Since the majority of the minor planets have nearly circular orbits, and the arc described in the period of observation is small, we may with advantage utilize the curvature of the orbit in our computation. For a short interval, the trajectory thru the apparent places will coincide with a small circle whose poles are α_0 δ_0 and whose polar distance is ν; for a longer interval we obtain a mean circle which best satisfies the observations; the less eccentric the orbit, the better will be the provisional result. Each (α, δ) gives us an equation of the form *

$$\cos\nu = \sin\delta \sin\delta_0 + \cos\delta \cos\delta_0 \sin(\alpha-\alpha_0)$$

$$\text{or} \quad \frac{\cos\nu}{\sin\delta_0} = \sin\delta + \cot\delta \quad \cos\alpha \cot\delta_0 \cos\alpha_0 + \cos\delta \sin\alpha \cot\delta_0 \sin\alpha_0$$

By subtracting each equation from the mean of all, $\frac{\cos\nu}{\sin\delta_0}$ is eliminated and we have a set of equations, linear in the two unknowns $\cot\delta_0 \cos\alpha_0$ and $\cot\delta_0 \sin\alpha_0$, which may be solved by the method of least squares or a similar method. The values of α_0, δ_0 & ν may be ~~solved~~ introduced into the condition that the three selected positions of the planet are in a plane passing thru the Sun:

$$c'\Delta' \cos\delta' \cos\alpha' - c\Delta \cos\delta \cos\alpha - c''\Delta'' \cos\delta'' \cos\alpha'' = (E-e)X,$$
$$c'\Delta' \cos\delta' \sin\alpha' - c\Delta \sin\alpha \cos\delta - c''\Delta'' \cos\delta'' \cos\alpha'' = (E-e)Y,$$
$$c'\Delta' \sin\delta' - c\Delta \sin\delta - c''\Delta'' \sin\delta'' = (E-e)Z,$$

The solution for Δ' may be written in determinant form.

$$\Delta' = \frac{E-e}{c'} \; \frac{\begin{vmatrix} X, & \cos\delta \cos\alpha & \cos\delta'' \cos\alpha'' \\ Y, & \cos\delta \sin\alpha & \cos\delta'' \sin\alpha'' \\ Z, & \sin\delta & \sin\delta'' \end{vmatrix}}{\begin{vmatrix} \cos\delta' \cos\alpha' & \cos\delta \cos\alpha & \cos\delta'' \cos\alpha'' \\ \cos\delta' \sin\alpha' & \cos\delta \sin\alpha & \cos\delta'' \sin\alpha'' \\ \sin\delta' & \sin\delta & \sin\delta'' \end{vmatrix}}$$

* Annals of Paris Observatory, Memoirs, Vol 23, p. G 53.

Let $\Omega = X_1 \cos\delta_0 \cos\alpha_0 + Y_1 \cos\delta_0 \sin\alpha_0 + Z_1 \sin\delta_0$; putting this value of Z_1 in the numerator, it may be reduced to the form

$$\frac{\Omega}{\sin\delta_0}(AB'' - BA'') \quad \text{where } A = \cos\delta\cos\alpha - \frac{X_1}{\Omega}\cos\nu$$
$$B = \cos\delta\sin\alpha - \frac{Y_1}{\Omega}\cos\nu$$

so that $\Delta' = \dfrac{E-e}{c'} \; \dfrac{\Omega(AB'')}{\sin\delta_0} \; \dfrac{1}{|\cos\delta'\cos\alpha',\ \cos\delta\sin\alpha,\ \sin\delta''|}$

As the c's are the ratios of the triangles included by pairs of radii vectores, we have, calling $\theta = K(t''-t)$ etc.,

$c = \dfrac{\theta}{\theta} + \gamma \quad c'' = \dfrac{\theta''}{\theta'} - \gamma \quad c' = 1 - e$, where for the first approximation $e = \dfrac{\theta\theta''}{2r'^3}$ and $\dfrac{\gamma}{e} = 3\theta'$

In reference to the earth a similar relation holds for C, C', C'', and E; assuming $\dfrac{C-c}{E-e} = \dfrac{\theta''-\theta}{3\theta'}$, $\quad X_1 = X' - \dfrac{C-c}{E-e}(X''-X)$

The determinant in the denominator can be replaced by $\dfrac{a\,b\,c}{2}\cot\nu$ where $b^2 = 4\left(\cos\delta\cos\delta''\sin^2\dfrac{\alpha''-\alpha}{2} + \sin^2\dfrac{\delta''-\delta}{2}\right)$ and our value becomes

$$\Delta' = \frac{(AB'')}{\theta(1-e)}\left(\frac{2E}{\theta\theta''} - \frac{2e}{\theta\theta'}\right)\frac{\Omega}{\sin\delta_0}\;\frac{\theta}{M'M''}\;\frac{\theta'}{MM''}\;\frac{\theta''}{MM'}\;\tan\nu$$

$$\text{or } \Delta' = \frac{h}{1-e}\left\{\frac{2E}{\theta\theta''} - \frac{2e}{\theta\theta'}\right\}$$

and $\quad \dfrac{\left\{1+\gamma\dfrac{\theta'}{\theta}\right\}\Delta}{\dfrac{(A'B'')}{\theta}} = \dfrac{\left\{1-\gamma\dfrac{\theta'}{\theta''}\right\}\Delta''}{\dfrac{(AB')}{\theta''}} = \dfrac{(1-e)\Delta'}{\dfrac{(AB'')}{\theta'}}$

With the values of Δ and Δ'', we find the constants of the orbit and the positions computed for the given times of observation using the following equations:

$$2(r''^2 + r^2) = K(x + x'')^2 + K(x'' - x)^2 \qquad r''^2 - r^2 = K(x''-x)(x''+x)$$

$$K^2 = K(x'' - x)^2$$

$$\sin \omega' = \frac{r''^2 - r^2}{r''^2 + r^2}$$

$$\sin \omega' = \tan \omega$$

$$\sin^2 \frac{v'' - v}{2} = \frac{K^2 - (r'' - r)^2}{4rr''}$$

$$m = \frac{\theta'^2}{\left\{(r + r'')\cos \omega' \cos \frac{v'' - v}{2}\right\}^3}$$

$$\ell = \frac{K^2}{2(r + r'')^2 \cos \omega' \cos \frac{v'' - v}{2} \left(1 + \cos \omega' \cos \frac{v'' - v}{2}\right)}$$

$$h = \frac{m}{5/6 + \ell}$$

η^2 found from h by table.

$$\frac{\sqrt{rr''}}{p} \cos \omega' = \frac{2m \cos \omega' \cos \frac{v'' - v}{2}}{\eta^2 \sin^2 \frac{v'' - v}{2}}$$

$$\left(\frac{e\sqrt{rr''}}{p} \cos \omega'\right) \sin \frac{v + v''}{2} = \frac{\tan \omega}{\sin \frac{v'' - v}{2}}$$

$$\left(\frac{e\sqrt{rr''}}{p} \cos \omega'\right) \cos \frac{v + v''}{2} = \frac{1 - \frac{rr''}{p} \cos \omega'}{\cos \frac{v'' - v}{2}}$$

$$a = \frac{p}{\cos^2 \varphi} \qquad e = \sin \varphi$$

$$\sin a \cos \left(A' + \frac{v + v''}{2}\right) = \frac{(x'' - x) - (x'' + x) \tan \omega}{\sin \frac{v'' - v}{2} (r + r'') \cos^2 \omega'}$$

$$\sin a \sin \left(A' + \frac{v + v''}{2}\right) = \frac{(x'' + x) - (x'' - x) \tan \omega}{\cos \frac{v'' - v}{2} (r + r'') \cos^2 \omega'}$$

$$x = r \sin a \sin (A' + v)$$

Taking February 4, March 3 and April 3 as bases for the provisional orbit, we have in addition to the given uncorrected times and positions the following data:

	Feb. 4		Apr. 3
X	0.703236		0.972944
Y	-0.634153		- 0.213251
Z	0.092508		- 0.275093
θ	0.532705	0.996385	0.463679

For our circle we get $\alpha_0 = 137\ 47\ 04.7$, $\frac{\Omega}{\sin\delta_0} = (0.666905)$

$\delta_0 = 11\ 09\ 57.9$

$\nu = 9\ 21\ 50$

$\frac{(A'B'')}{\theta}$	(7.699373)	(7.714643)	(7.777234)
$\frac{(MM'')^2}{\theta}$	(7.749713)	(8.220843)	(8.583982)

Δ' is found by successive approximations and from it

Δ	1.948565	2.332233
r	2.93217	2.92907
$\sin i_0$	(7.121784)	
v	328 16 46.4	339 44 15.6
e	(8.186209)	
a	(0.472985)	

Time		(9.999765)			(9.939080)			(9.694988)		
	A	258	37	57.1	167	38	54.4	171	41	53.4
	α	9	26	27.43	9	03	55.75	8	55	08.31
	δ	19	46	28.0	20	25	15.2	19	42	09.4
O - C	$\Delta\alpha$	-.49			30.34			0.67		
	$\Delta\delta$	2.2			- 106.2			9.9		

The failure of the residuals of the first and last places to approach 0, indicates some error in the computation, but it is hardly worth seeking in the provisional results.

To improve this orbit we follow the method of varying the geocentric.

distances. The residuals from each observation are due to the fact that the used values of Δ and Δ'' are inaccurate: calling their corrections $\Delta\Delta$ and $\Delta\Delta''$, we have from each observation

$$\Delta\alpha \cos\delta = \cos \frac{d\alpha}{d\Delta}\Delta\Delta + \cos\delta \frac{d\alpha}{d\Delta''}\Delta\Delta''$$

$$\Delta\delta = \frac{d\delta}{d\Delta}\Delta\Delta + \frac{d\delta}{\Delta\Delta'}\Delta\Delta''$$

provided we can neglect the higher powers. The coefficients are found by varying Δ and Δ'' separately and computing the changes in α and δ for each date.

For our first hypothesis assume the values of Δ and Δ'' to be Δ + .05, Δ'' + .05 as the equations indicate that the values of Δ and Δ'' obtained above are too small. The times are corrected for aberration and the α's and δ's for parallax from the provisional data, so that our corrected data are:

Date	α			δ			X	Y	Z	
Jan 35.65994	9	26	26.94	19	46	30.2	.703196	-.634186	-.275109	1st
36.61866	9	25	36.27	19	48	42.5	.714948	-.623273	-.270376	
62.61426	9	04	26.09	20	23	29.0	.946688	-.271123	-.117615	middle
70.53904	8	59	55.75	20	21	39.5	.980511	-.148688	-.064504	
91.62180	8	55	03.51	19	47	26.7	.979697	.182837	.079310	
93.58000	8	55	08.98	19	42	19.3	.972969	.213120	.092446	last

With these values our computation follows:

Δ	1.998565	2.382233
$\sin\delta$	9.529338	9.527866
$\log\Delta$	0.300718	0.376984
$\cos\delta\cos\alpha$	9.867823	9.813888
$\cos\delta\sin\alpha$	9.766681	9.832276
X + x	-1.474148	- 1.551930
X	0.703196	0.972969
x	-2.177344	- 2.524899
Y + y	1.167882	1.619049
Y	-0.634186	0.213120
y	1.802068	1.405929
Z + z	0.676170	0.803249
Z	-0.275109	.092445
z	0.951279	.710804

	Nos	Logs	Logs	Nos	
$x'' \pm x$	-4.702243	0.672305	9.541204	-0.347555	
$y'' \pm y$	3.207997	0.506234	9.597848	-0.396139	
$Z'' \pm z$	1.662083	0.220653	9.381070	-0.240475	
$(x'' \pm x)^2$	22.11108	1.344610	9.082848	0.1207947	
$(y'' \pm x)^2$	10.29125	1.012468	9.195696	0.1569264	
$(Z'' \pm Z)^2$	2.76252	0.441306	8.762140	0.0578282	
$((x'' \pm x)^2$	35.16485		9.525756	0.3355493	
N^2	0.33555				
$2(r'' + r^2)$	35.50040	1.550233	m	7.677658	
$r''^2 + r^2$	17.75020	1.249203	$\sin^2 \frac{v''-v}{2}$	7.975476	
$(x''^2 - x^2)$	1.634289	0.213329	K	9.997938	
$(y''^2 - y^2)$	- 1.270814	0.104082	1 + K	0.300000	
$(Z'' - z^2)$	- 0.399690	9.601723	Z	0.301030	
$\Sigma = r''^2 - r^2$	- 0.036215	8.558888	$(r + r'')^2$	1.550233	
$\sin 2\omega = \frac{r''^2 - r^2}{r''^2\ r^2}$		7.309685y	col N^2	0.474244	
$1 + \cos 2\omega$		0.301030	colog 1	2.623445	
$\tan \omega = \sin \omega'$		7.008655	8 ℓ	7.376555	0.0023799
$\cos \omega'$		000000	5/6 + ℓ	9.922057	0.8357132
			h	7.755601	0.0056964
r''^2	8.856992	0.947286	N^2	0.0054427	
r''		0.473643	$\frac{m}{N^2}$	7.672215	0.0047013
r^2	8.893207	0.949058	x	7.365750	0.0023214
r		0.474529			

	Logs	Nos		
			K	9.997938
			m	-8.702182
$2\sqrt{r^2 + r''^2}$	0.775116		$\sin^2 (\frac{v''-v}{2})$	
$\cos \omega'$	.000000		2	0.301030
$r'' + r$	0.775116		col N^2	9.994557

$45 - \frac{\varphi}{2}$	44 34 58.9	
Check		
v	307 42 20.9	318 51 48.9
cos v	9.786473	9.876879
e	8.163105	8.163105
e cos v	7.949578	8.039984
1+e cos v	0.003850	0.004736
p	0.478379	0.478379
r	0.474529	0.473643 3
$\frac{v}{2}$	153 51 10.4	159 25 54.4
tan $\frac{v}{2}$	9.691006	9.574312
$\tan(45 - \frac{\varphi}{2})$	9.993679	9.993679
tan $\frac{E}{2}$	9.684685	9.567991
$\frac{E}{2}$	154 10 52.7	159 42 16.4
E	308 21 45.4	319 24 32.9
Sin E	9.894371	9.813350
e"	3.477530	3.477530
l(e sinE)"	3.371901	3.290880
(e sin E)"	- 0 39 14.5	-0 32 33.8
M	309 00 59.9	319 57 06.7
$a^{3/2}$	0.717707	
K	3.550007	
$\frac{K}{a^{3/2}}$	2.832300	
(t" - t)	1.762829	
l(M"-M)	4.595129	
M"-M	10 56 06.7	
direct	10 56 06.8	

	X A'	Nos	Y B'	Nos
$\sin \frac{v'' - v}{2}$	8.987738			
$(r+r'') \cos^2\omega'$	0.775116			
$\cos(\frac{v'' - v}{2})$	9.997938			
$(x + x'')(\tan \omega)$	7.680960	0.004797	7.514889	−0.003273
$x'' - x$	9.541024	−0.347555	9.597848	−0.396139
$\tan \omega$	7.008655		7.008655	
$x'' + x$	0.672305	−4.702243	0.506234	3.207997
$(x'' - x) \tan \omega$	6.549679	0.000355	6.606503	0.000404
$(x'' + x) - (x'' - x) \tan \omega$	0.672338	−4.702598	0.506179	3.207293
$\cos \frac{v'' - v}{2} (r + r'') \cos^2_\omega$	0.773054		0.773054	
$(x'' - x) - (x'' + x) \tan \omega$	9.546977	−0.352352	9.594244	−0.392866
$\sin \frac{v'' - v}{2} (r + r'') \cos^2_\omega{}'$	9.762854		9.762854	
$\sin (A' + \frac{v + v''}{2}) \sin a$	9.899284		9.733125	
sin / cos	9.899519		9.893106	
$\sin a \cos(A' + \frac{v'' + v}{2})$	9.784123		9.831390	
$\tan (A' + \frac{v'' + v}{2})$	0.115161		9.901735	
$A' + \frac{v'' + v}{2}$	232 30 32.5		141 25 38.5	
$\frac{v'' + v}{2}$	313 17 04.9		313 17 04.9	
A'	279 13 27.6		188 08 33.6	
$\sin a$	9.999765		9.938284	
$\sin 2 A'$	9.500312		9.447811	
$\sin^2 a$	9.999530		9.876568	
$\cos 2 A'$	9.977087		9.982216	

	Z C'	Nos
$\sin \frac{v'' - v}{2}$		
$(r + r'') \cos^2 \omega'$		
$\cos \left(\frac{v'' - v}{2}\right)$		
$(x + x''(\tan \omega')$	7.229308	$-$ 0.001696
$x'' - x$	9.381070	$-$ 0.240475
$\tan \omega'$	7.008655	
$x'' + x$	0.220653	1.662083
$(x'' - x) \tan \omega$	6.389725	0.000245
$(x'' + x) - (x'' - x) \tan \omega$	0.220589	1.661838
$\cos \frac{v'' - v}{2} (r + r'') \cos^2 \omega'$	0.773054	
$(x'' - x) - (x'' + x) \tan \omega$	9.377996	$-$ 0.238779
$\sin \frac{v'' - v}{2} (r + r'') \cos^2 \omega'$	9.762854	
$\sin \left(A' + \frac{v + v''}{2}\right) \sin a$	9.447535	
sin cos	9.917504	
$\sin a \cos \left(A' + \frac{v'' + v}{2}\right)$	9.615142	
$\tan \left(A' + \frac{v'' + v}{2}\right)$	9.832393	
$A' + \frac{v'' + v}{2}$	145 47 29.7	
$\frac{v'' + v}{2}$	313 17 04.9	
A'	192 30 24.8	
$\sin a$	9.697638	
$\sin 2 A'$	9.626172	
$\sin^2 a$	9.395276	
$\cos 2 A'$	9.957227	

$\Sigma \sin^2 a$	$\Sigma \sin^2 a \sin 2A'$	$\Sigma \sin^2 a \cos 2A'$
0.998918	-0.316113	-0.947582
0.752607	0.251047	0.722410
0.248471	0.105063	0.225166
1.999996	-0.000003	-0.000006

(r,v) for

	Feb. 5	Mar. 3
t" - t		1.490881
$\frac{K}{a^{3/2}}$	2.832300	2.832300
t' - t	9.981692	1.430628
M" - M'		5 50 46.5
M"		319 57 06.7
M	309 00 59.9	309 00 59.9
M' - M	0 10 51.6	5 05 20.1
M'	309 11 51.5	314 06 20.2 20.0
e"	3.477530	3.477530
sin M'	9.889285	9.856160
$\frac{(e^2)"}{2}$	1.339605	1.339605
sin 2M'	9.991030	9.999788
M'	309 11 51.5	314 06 20.1
e sin M'	-38 47.1	-35 56.2
$\frac{e^2}{2}$ sin 2M'	-21.4	-21.8
E'	308 32 43.0	313 30 02.1
sin E'	9.893271	9.860558

	Feb. 5	Mar. 3	(Cont'd.)
e"	3.477530	3.477530	
(e sin E)	-39 08.6	-36 18.2	
M'	309 11 51.6	314 06 20.3	
M'	- 0.1	- 0.2	
e			
cos E			
e cos E			
$E = \frac{M}{1 - e \cos E}$			
E	308 32 42.9	313 30 01.9	
$\frac{E}{2}$	154 16 21.4	156 45 00.9	
tan $\frac{E}{2}$	9.682918	9.633093	
tan $(45 - \frac{\varphi}{2})$	9.993679	9.993679	
tan $\frac{v}{2}$	9.689239	9.639414	
$\frac{v}{2}$	153 56 42.0	156 26 46.7	
v	307 53 24.0	315 53 33.4	
cos v	9.788273	9.832909	
e	8.163105	8.163105	
e cos v	7.951378	7.996014	
1 + e cos v	0.003866	0.004282	
p	0.478379	0.478379	
r	0.474513	0.474097	

	Mar. 11	Apr. 1
$t'' - t$	1.362501	0.291857
$\frac{K}{a^{3/2}}$	2.832300	2.832300
$t' - t$	1.542562	
$M'' - M'$	4 21 00.3	0 22 10.9
M''	319 57 06.7	319 57 06.7
M	309 00 59.9	
$M' - M$	6 35 06.4	
M'	315 36 06.4 3	319 34 55.8
e''	3.477530	3.477530
$\sin M'$	9.844876	9.811814
$\frac{(e^2)''}{2}$	1.339605	1.339605
$\sin 2M'$	9.999904	9.994421
M'	315 36 06.3	319 34 55.8
$e \sin M'$	-35 00.1	- 32 26.9
$\frac{e^2}{2} \sin 2M'$	-21.9	-21.6
E'	315 00 44.3	319 02 07.3
$\sin E'$	9.849392	9.816634
e''	3.477530	3.477530
$(e \sin E')''$	-35 22.9	- 32 48.6
M'	315 36 07.2	319 34 55.9
M'	- 0.9	- 0.1
e	8.163	
$\cos E$	9.849	
$e \cos E$	.010	

$$E = \frac{M}{1 - e\cos E} \qquad \frac{-0.9}{.99} = -0.9$$

	Mar. 11	Apr. 1 (Cont'd.)
E	315 00 43.4	319 02 07.2
$\frac{E}{2}$	157 30 21.7	159 31 03.6
$\tan \frac{E}{2}$	9.617095	9.572329
$\tan (45 - \frac{\varphi}{2})$	9.993679	9.993679
$\tan \frac{v}{2}$	9.623416	9.578650
$\frac{v}{2}$	157 12 35.0	159 14 34.0
v	314 25 10.0	318 29 08.0
cos v	9.845040	9.874359
e	8.163105	8.163105
e cos v	8.008145	8.037464
1 + e cos v	0.004403	0.004709
p	0.478379	0.478379
r	0.473976	0.473670

Check

$a^{1/2}$	0.239236	0.239236	0.239236	0.239236
col $r^{1/2}$	9.762744	9.762952	9.763012	9.763165
$\sin \frac{\varphi}{2}$	7.861986	7.861986	7.861986	7.861986
sin E	9.893271	9.860558	8.849394	9.816634
sin 1/2(v - E)	7.757237	7.724732	7.713628	7.681021
1/2(v - E)	-0 19 39.41	-0 18 14.36	-0 17 46.73	-0 16 29.58
v - E	-0 39 18.8	-0 36 28.7	-0 35 33.5	-0 32 59.2
direct	-0 39 18.9	-0 36 28.5	-0 35 33.4	-0 32 29.2

$A' + v$	2265548.5			
$B' + v$	135 50 54.5			
$c' + v$	140 12 45.7			
$r \sin \alpha$	0.474294	0.474278	0.473862	0.473741
$\sin (A' + v)$	9.863633	9.864934	9.897223	9.905998
$r \sin b$	0.412813	0.412797	0.412381	0.412260
$\sin (B' + v)$	9.842813	9.841515	9.798541	9.783806
$r \sin c$	0.172167	0.172151	0.171735	0.171614
$\sin (C' + v)$	9.806139	9.804457	9.754234	9.736936
x	− 2.177344	− 2.183795	− 2.350093	− 2.397392
X	0.703196	0.714948	0.946688	0.980511
y	1.802071	1.796024	1.625257	1.570602
Y	− 0.634186	− 0.623273	− 0.271123	− 0.148688
Z	0.951275	0.947563	0.843274	0.810121
Z	− 0.275109	− 0.270376	− 0.117615	− 0.064504
$Y + y$	0.067400	0.069206	0.131662	0.152873
sin cos	9.894219	9.892914	9.857107	9.850254
$X + x$	0.168541	0.166977	0.147183	0.151333
$\tan \alpha$	9.898859	9.902229	9.984479	0.001540
(α)	141 36 43.9	141 23 44.1	136 01 25.0	134 53 54.3
time	9 26 26.93	9 25 34.94	9 04 05.67	8 59 35.62
$Z + z$	9.830053	9.830709	9.860733	9.872516
sin cos	9.973603	9.973498	9.971842	9.971939
$\Delta \cos \delta$	0.274322	0.274063	0.290076	0.302619

tan δ	9.555731	9.556646	9.570657	9.569897
δ	19 46 29.5	19 48 48.0	20 24 36.4	20 22 38.5
Δ	0.300719	0.300565	0.318234	0.330680
O - C $\Delta\alpha$	0.01	1.33	20.49	20.13
O - C $\Delta\delta$	- 0.7	- 5.5	- 67.4	- 59.0
A' + v			238 05 16.5	
B' + v			147 00 22.5	
C' + v			151 22 13.7	
r sin d		0.473435	0.473408	
sin (A' + v)		9.927039	9.928836	
r sin b		0.411954	0.411927	
sin (B' + v)		9.740417	9.736036	
r sin c		0.171308	0.171281	
sin (C' + v)		9.685675	9.680466	
x		- 2.514630	- 2.524899	
X		0.979697	0.972969	
y		1.420270	1.405928	
Y		0.182837	0.213120	
Z		0.719421	0.710799	
z		0.079310	0.092445	
Y + y		0.204962	0.209359	
sin cos		9.858737	9.858484	

$X + x$	0.186089	0.190872
$\tan \alpha$	0.018873	0.018387
α (arc)	133 45 19.6	133 47 14.9
time	8 55 01.31	8 55 08.99
$Z + z$	9.902401	9.904847
sin cos	9.973552	9.973792
$\Delta \cos \delta$	0.346225	0.350775
$\tan \delta$	9.556176	9.554072
δ	19 47 36.8	19 42 19.0
Δ	0.372673	0.376983
O - C $\Delta\alpha$	2.09	- 0.01
O - C $\Delta\delta$	- 11.6	0.3

For the second hypothesis, Δ is unchanged but Δ'' is decreased by .05 and for the third hypothesis Δ is increased by .10 and Δ'' constant. The computation is repeated with each of these two sets of values and the following values of certain quantities are noted.

	Hypoth II.		Hypoth.III.	Final
x, x"	-2.177344	-2.492327	-2.251103	-2.524899
Y, y"	1.802068	1.371949	1.860501	1.405929
Z, z"	0.951279	0.693945	0.985112	0.710804
$2(r''^2 + r^2)$	34.93737		36.71266	
M^2	0.350437		0.356845	
log r	0.474529		0.488847	
r"	0.466630		0.473643	

	Hypoth. II	Hypoth. III
tan ω	7.958780	8.243110
v	286 02 47.6	281 11 32.0
π''	297 29 44.8	292 19 52.6
e	9.008046	9.299092
p	0.486589	0.505317
a	0.491119	0.522885
A'	300 52 51.1	305 45 50.5
B'	209 50 58.2	214 35 37.5
C'	214 01 56.1	219 15 12.6
sin a	9.999778	9.999728
sin b	9.938650	9.937336
sin c	9.696418	9.700628

The resulting positions for the six dates are:

Hypothesis II

α 9 26 26.95, 9 25 36.22, 9 04 12.15, 8 59 39.88,
8 55 03.57, 8 55 08.77

δ 19 46 30.3, 19 48 45.2, 20 25 02.7, 20 23 11.0,
19 47 29.1, 19 42 18.9

Hypothesis III

α 9 26 26.94, 9 25 37.41, 9 04 38.83, 9 00 04.70,
8 55 03.95, 8 55 08.95

δ 19 46 30.4, 19 48 41.9, 20 23 51.1, 20 22 09.5,
19 47 34.8, 19 42 19.5

And the resulting residuals in the sense O − C are:

Hyp. II	−.01"	.05"	14.01"	15.87"	−.06"	.22'
	−0.1	−2.7	−93.7	−91.5	−2.4	0.4
Hyp. III	00	−1.14	−12.67	−8.95	−0.44	.03
	−0.2	0.6	−22.1	−30.0	−8.1	−0.2

The $\Delta\alpha$'s are reduced to the equator ($\Delta\alpha\cos\delta$) and the results tabulated:

	Feb. 4		Feb. 5		Mar. 3		Mar. 11	
	($\Delta\alpha\cos\delta$)"	$\Delta\delta$	$\Delta\alpha$ cos δ	$\Delta\delta$	$\Delta\alpha\cos\delta$	$\Delta\delta$	cos $\Delta\alpha\cos\delta$	$\Delta\delta$
Hyp. I	0.1	−0.7	18.8	−5.5	288.0	−67.4	283.0	−59.0
" II	−0.1	−0.1	0.7	−2.7	196.9	−93.7	223.1	−91.5
" III	0.0	−0.2	−16.1	0.6	−178.1	−22.1	−125.8	−30.0

	Apr. 1		Apr. 3	
	$\Delta\alpha$ cos δ	$\Delta\delta$	$\Delta\alpha\cos\delta$	$\Delta\delta$
Hyp. I	31.1	−10.1	−0.1	0.3
" II	−0.8	−2.4	3.1	0.4
" III	−6.2	−8.1	0.4	−0.2

Calling the fractional corrections to dΔ" (−.05) x and to d Δ' (+.10)y, each date gives us two equations of the form $ax + by + n = 0$:

Feb. 5
$$-18.1x \quad -34.9y \quad + 18.8 = 0$$
$$2.8x \quad + 6.1y \quad - 5.5 = 0$$

Mar. 3
$$-81.1x \quad -466.1y \quad + 288.0 = 0$$
$$-26.3x \quad + 45.3y \quad -67.4 = 0$$

$$\text{Mar. 11} \quad \begin{array}{rrrcl} -59.9x & -408.8y & +283.0 & = & 0 \\ -32.5x & +29.0y & -59.0 & = & 0 \end{array}$$

$$\text{Apr. 1} \quad \begin{array}{rrrcl} -31.9x & -37.3y & +31.1 & = & 0 \\ 7.7x & +2.0y & -10.1 & = & 0 \end{array}$$

By the method of least squares, these reduce to the two normal equations:

$$\begin{array}{rrrcl} 66669x & +389910y & -256561 & = & 0 \\ 15047x & +66669y & -40924 & = & 0 \end{array}$$

whose solution is

$$x = -0.807063$$

$$y = 0.795996$$

So that the corrections to Δ'' and Δ become respectively

$$(-.05) \times (-.807063) = .040353$$

$$10 \times .795996 = .079600$$

Hence $= 1.998565 + .079600 = 2.078165$

$$= 2.382233 + .040353 = 2.422586$$

With these values the final values are computed, giving

$$\log a = 0.496406$$

$$\log \mu = 2.805398$$

$$\varphi = 4^\circ\ 59'\ 30''.3$$

and the heliocentric coordinates

$$x = (9.999717)\ r \sin (v + 305^\circ\ 58'\ 19''.9)$$

$$y = (9.937187)\ r \sin (v + 214^\circ\ 46'\ 24''.0)$$

$$z = (9.701118)\ r \sin (v + 219^\circ\ 32'\ 09''.4)$$

The other constants are obtained from A', B', C' by means of the following equations:

$$\tan s = \frac{\sin c}{\sin b}$$

$$\tan\left(\frac{B' + C'}{2} - (\pi - \Omega)\right) = \frac{\sin(s + \varepsilon)}{\sin(s - \varepsilon)} \tan\left(\frac{B' - C'}{2}\right)$$

$$-\tan i = \frac{\sin b \sin c \sin(B' - C')}{\sin a \cos A} \qquad A = A' - (\pi - \Omega)$$

$$\tan \Omega = \frac{\sin b \sin B \sec \varepsilon}{\sin a \sin A}$$

Check

$$\sin i \sin \Omega = \sin b \sin c \sin(B' - C')$$

$\sin \varepsilon$	9.701118	$\sin b \sin c$	9.658308
$\sin b$	9.937187	$\sin(B' - C')$	8.919222
$\tan s$	9.763931	cosec a	0.000283
s	30 08 33.2	$\sec A$	0.528860
$s + \varepsilon$	53 35 36.8	$-\tan i$	9.086670
$s - \varepsilon$	6 41 29.6	i	6° 57' 33".3
$\frac{B' - C'}{2}$	− 2 22 52.7	$\sin a$	9.999717
$\sin(S + \varepsilon)$	9.905796	$\sin A$	9.980103
$\tan\left(\frac{B' - C'}{2}\right)$	8.618943	$\sin b$	9.937187
cosec $(S - \varepsilon)$	0.933583	$\sin B$	9.499437
tan ( )	9.458322	$\sec \varepsilon$	0.037441
( )	343° 58' 16".0	$\cos \Omega$	9.979820
$\frac{B' + C'}{2}$	2.7 09 16.7	$\sin \Omega$	9.474065
$\pi - \Omega$	233 11 00.7	$\tan \Omega$	9.494245
A	72 47 19.2	Ω −	342° 40' 07".0
		$\sin \Omega$	9.474065
		$\sin i$	9.083458
			8.557523
		check	8.557527

Hence the elements are:

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M	=	290	36	34.0
	=	215	51	08.0
i	=	6	57	38.3
	=	4	59	30.3
	=	$638''.849$		
log a	=	0.496406		

Comparing the elements wuth those given by Mascart*, it is seen that the mean motion corresponds to that of one of the largest groups ($630''$ to $650''$): the same is true of the eccentricity, inclination and longitude of ascending node. Only in the longitude of perihelion does it fall in with one of the smaller groups.

* Paris Memoirs, Vol.XXIII.