279

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I hereby recommend that the thesis prepared under my supervision by Charles C. Goldman

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ON MULTIPLE FOURIER SERIES

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TABLE OF CONTENTS

INTRODUCTION

We consider essentially two problems: That of convergence almost everywhere of multiple Fourier series, and the strong summability of double Fourier series, power series and general orthogonal developments.

We generalize to several variables a paper by A. Plessner (25) on a sufficient condition on the Fourier coefficients for convergence almost everywhere of the Fourier series. We succeed in effecting this generalization without adding more restrictive conditions.

For the strong summability problem we found it necessary to add a cross condition in each of the two variables in generalizing the Hardy-Littlewood theorem (11). And then we prove only restricted strong summability. This is to be expected however, as this strong summability theorem is essentially a localization theorem and the localization property does not generalize unmodifiedly to two variables. We make similar modifications in generalizing analogous results of Fejèr (5) for power series and Zygmund (37)(p. 240)for general orthogonal developments.

I wish to express my sincere appreciation to Professor Otto Szasz for his invaluable aid. He gave generously of both his time and energy in indicating fruitful means of research, in pointing out the most significant features of a problem and in constantly and insistently emphasizing the importance of accuracy in writing, of clarity and completeness of discussion. I Hope this paper indicates I have profited somewhat by his direction.

CHAPTER I – PRELIMINARIES

A. NOTATIONS

We denote by capital letters vectors in the k dimensional space, so that $X = (x_1, x_2, \dots, x_k)$, $N = (n_1, n_2, \dots, n_k)$; $NX = n_1 x_1 + n_2 x_2 + \dots + n_k x_k$. The x_i $(i = 1, 2, \dots, k)$ are integers. $f(X) = f(x_1, x_2, \dots, x_k)$ is a real valued integrable function of period 2π in each variable.

We denote a k-triple summation

$$\sum_{n_1 = 1}^{m_1} \sum_{n_2 = 1}^{m_2} \cdots \sum_{n_k = 1}^{m_k} \quad \text{by} \quad \sum_{N = 1}^{M}$$

We denote a k-iterated integral

$$\int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} dx_1 \, dx_2 \, \dots \, dx_k \quad \text{by} \quad \int_Q dX$$

The formal multiple Fourier series of $f(X)$ is

$$f(X) \sim \sum_{N = -\infty}^{\infty} C_N \, e^{iNX} =$$

$$= \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} \cdots \sum_{n_k = -\infty}^{\infty} C_{n_1 \dots n_k} e^{i(n_1 x_1 \dots n_k x_k)}$$

where

$$2C_N = a_N - i\, b_N , \qquad C_{-N} = \bar{C}_N .$$

When we work in two variable no special notation will be employed.

Throughout this paper, unless otherwise specified, we take the multiple sum in the sense of Pringsheim (26). For example, in the case of two variables $\sum_{m,n=0}^{\infty} u_{mn}$ converges to the sum S, or $\lim_{m,n} S_{mn} = S$ in the Pringsheim sense, if there corresponds to every number $\epsilon > 0$ an integer N_0 such that for $m \geq N_0$, $n \geq N_0$ $| S_{mn} - S | \leq \epsilon$.

Integers in parenthesis (e.g.(15)) refer to the bibliography at the end of this paper. Decimals in parenthesis (e.g. (2.21)) refer to theorems, etc. within the paper itself.

B. Auxiliary Theorems

Lemma 10 <u>Fubini's Theorem</u> Hobson, (14) p. 630.

If f(x,y) be any function, bounded or unbounded, that is integrable L over the measurable and bounded set E , then

$$\iint_E f(x,y)\, dx\, dy = \int dx \int f(x,y)\, dy = \int dy \int f(x,y)\, dx$$

Lemma 11 . Hobson (15) p. 302.

Let $\{ S_n(x) \}$, a sequence of functions, summable in a measurable set E, of finite or infinite measure, and of any number of dimensions, be such that $S_n(X) \geq 0$ for all value of n and all x in E. If, for each value of X the sequence is monotome non-diminishing and if

$$\lim_{n \to \infty} \int_E S_n(X)\, dx$$

has a definite value, then $S_n(X) \to S$ p. p. in E.

Lemma 1.2 Young- Hausdorff inequality. Szasz (32)

If $1 < p \leq 2$ and

$$f(X) \sim \sum C_n e^{i N X}$$

then

$$\left\{ \sum | C_n |^{p'} \right\}^{\frac{1}{p'}} \leq M_p (f)$$

and

$$M_{p'}^{p}(f) \leq \sum | C_N |^{p}$$

Where

$$\frac{1}{p} + \frac{1}{p'} = 1, \text{ and}$$

$$M_{p}^{p} (f) = \frac{1}{(2\pi)^k} \int_Q | f (X) |^{p} d X$$

If $p = 2$, equality holds. (Parseval's Theorem).

Lemma 1.3 Holders Inequality (13) p. 24.

$$\int_Q f(X)\, g(X)\, dX < \left[\int |f(X)|^P\, dX\right]^{\frac{1}{P}} \left[\int |g(X)|^{P'}\, dX\right]^{\frac{1}{P'}}$$

where $\frac{1}{P} + \frac{1}{P'} = 1.$

Lemma 1.4 Minkowski's Inequality (13) p. 30.

$$\left[\sum (a_n + b_n)^r\right]^{\frac{1}{r}} < \left[\sum (a_n)^r\right]^{\frac{1}{r}} + \left[\sum (b_n)^r\right]^{\frac{1}{r}}, \quad r > 1$$

Lemma 1.5 Jensen's Inequality (13) p. 71.

If $\phi(x_{kl})$ is continuous in the open interval

$(0,0;\infty,\infty)$

$$\phi\left(\frac{\sum_1^{m,n} g_{kl}\, x_{kl}}{\sum_1^{m,n} g_{kl}}\right) < \frac{\sum_1^{m,n} g_{kl}\, \phi(x_{kl})}{\sum_1^{m,n} g_{kl}}$$

If we take $\phi(X) = X^r$ we get , $r > 1$

$$\left(\sum_1^{m,n} x_{kl}\right)^r < C \sum_1^{m,n} (x_{kl})^r$$

where $C = (mn)^{r-1}$ (constant).

CHAPTER II : CONVERGENCE ALMOST EVERYWHERE

A. History of the Problem.

The problem of convergence almost everywhere of single Fourier series has been developed from the extensive research into the problem for general orthogonal developments.

H. Weyl (33) first proved that if $\sum a_n^2 n^{1/2} < \infty$ then the orthogonal series $\sum a_n \psi_n(x)$ converges p. p (almost everywhere.) in the interval of orthogonality of the ψ_n (X).
Hobson generalized this result to $\sum n^p a_n^2 < \infty$, $p > o$ for the condition for convergence p. p. Then Plancherol (24) obtained this same result with the less restrictive condition $\sum A_n^2 (\log n)^3 < \infty$, and Rademacher (27) and Menchoff (22) imporved on this by establishing the weaker condition for convergence p. p. : $\sum A_n^2 (\log n)^2$

Menchoff further showed that this was the best result provided the system $\{\psi_n(x)\}$ was not specialized.

In the specialized case of Fourier series Plessner (25) showed that $(\log n)^2$ can be replaced by $(\log n)$, making the condition for convergence p. p. of Fourier series $\sum A_n^2 \log n < \infty$.
Whether or not this is the best result is an open question.

In extending Plessner's result to multiple Fourier series we have employed essentially the method Plessner used in his paper.

It is perhaps pertinent to call to the reader's attention the fact that the criteria for convergence p. p. are conditions on the Fourier coefficients and not on the function itself. Attempts at finding conditions on the function for convergence p. p. without proving more than that have been unsuccessful.

B. Convergence almost Everywhere : Coefficient Condition.

Theorem 2.0 if

$$(2.00)\quad \sum_{N=-\infty}^{\infty} | C_N |^2 \prod_{j=1}^{k} \log (|n_j| + 1)$$

then the multifle Fourier series

$$(2.01)\qquad f(X) \sim \sum_{-\infty}^{\infty} c_N e^{iNX}$$

converges almost everywhere.

PROOF:

Let us denote the partial sum of the sertie s (2.01) by

$$(2.02)\qquad S_M (X) = \sum_{N=-M}^{M} C_n e^{iNX}$$

It follows from the Young-Hausdorff theorem (Lemma 1.2) that there exists a function $q(X) \in L_2$ whose Fourier coefficients satisfy (2.00). The Fourier series of $q(x)$ in the interval $(o, 2\pi)$ is

$$(2.03)\qquad q(X) \sim \sum C_N \prod_{j=1}^{k} (\log(|n_j|+1)^{\frac{1}{2}} e^{iNX}, \quad q(X) \in L_2$$

The Fourier coefficientsof $q(X)$ are then

$$(2.04)\qquad \prod_{j=1}^{k} \left[\log (|n_j| + 1) \right] c_N = \frac{1}{\pi^k} \int q(X)\, e^{-iNX}\, dX$$

Substituting (2.04) into (2.02) we get

$$S_M(X) = \sum_{-M}^{M} \frac{1}{\pi^k} \int_{\mathcal{G}} q(R) \frac{e^{-iN(X-R)}}{\prod_{j=1}^{k} \left[\log(|n_j| + 1)\right]^{\frac{1}{2}}} dR$$

$$= \frac{1}{\pi^k} \int_{\mathcal{G}} q(R) \sum_{-M}^{M} \prod_{j=1}^{k} \frac{e^{-in_j(x_j - r_j)}}{\left[\log(|n_j| + 1)\right]^{\frac{1}{2}}} dR$$

$$(2.05) \qquad = \frac{1}{\pi^k} \int_{\mathcal{G}} q(R) \quad E_M(X - R) \; dR$$

where we have set

$$(2.06) \qquad E_M(X - R) = \sum_{-M}^{M} \prod_{j=1}^{k} \frac{e^{-in_j(x_j - r_j)}}{\left[\log(|n_j|+1)\right]^{\frac{1}{2}}} =$$

$$\sum_{N=1}^{M} \prod_{j=1}^{k} \frac{\cos n_j (x_j - r_j)}{\left[\log((n_j + 1)\right]^{\frac{1}{2}}}$$

We let $u_j(X_j)$ $(j = 1, 2, \ldots, k)$ be measurable functions which take only the values $1, 2, \ldots m_j$. We set $U(X) \equiv U$. In (2.05) we replace the subscript M by U (X) and then integrate (2.05). Then, after taking absolute values and using Fubini's theorem (Lemma 1.0), we get

$$\left| \int_Q S_U(X)\, dX \right| = \left| \frac{1}{\pi^k} \int_Q d(X) \int_Q q(R)\, E_U(X - R)\, dR \right|$$

$$= \left| \frac{1}{\pi^k} \int_Q \left[q(R) \int_Q E_U(X - R) \right] dR \right|$$

Applying Holder's Inequality (Lemma 1.3)

$$(2.07) \quad \left| \int_Q S_U(X)\, dX \right| \leq \frac{1}{\pi^k} \left[\int_Q |q(R)|^2\, dR \right]^{\frac{1}{2}} \left[\int_Q \left| \int_Q E_U(X - R)\, dX \right|^2 dR \right]^{\frac{1}{2}}$$

The first member of the right hand side of the inequality is finite by hypothesis and by the Young - Hausdorff inequality (Lemma 1.2) applied to q (X). We need only consider the second member to get a proper bound for (2.07).

$$\left[\int_Q \left| \int_Q E_U(X - R)\, dX \right|^2 dR \right]^{\frac{1}{2}} =$$

$$= \left[\int_Q \left\{ \int_Q E_U(X - R)\, dX \int_Q E_{U'}(X' - R)\, dX' \right\} dR \right]^{\frac{1}{2}}$$

$$= \left[\int_Q \left\{ \int_Q \int_Q E_U(X - R) E_{U'}(X' - R)\, dX dX' \right\} dR \right]^{\frac{1}{2}}$$

and, upon interchaging orders of integration, we have

$$(2.08) \qquad = \left[\int_Q \int_Q \left\{ \int_Q E_U(X - R)E_{U'}(X' - R)dr \right\} dX\, dX' \right]^{\frac{1}{2}}$$

However

$$(2.09) \qquad \int_Q E_U(X - R)E_{U'}(X' - R)dR =$$

$$\int_Q \sum_{N=1}^{U} \left(\prod_{j=1}^{k} \frac{\cos n_j(x_j - r_j)}{\sqrt{\log (n_j + 1)}} \right) \left(\sum_{N'=1}^{U'} \prod_{j=1}^{k} \frac{\cos n'_j(x'_j - r_j)}{\sqrt{\log (n'_j + 1)}} \right) dR$$

Because of the orthogonality of the trigonometric functions the integral of all terms will vanish except those which consist of a product of sines or cosines (of the argument R) (2.09) becomes

$$\int_Q E_U\,(X - R)\,E_{U'}\,(X' - R)\,dR = \sum_{N=1}^{B} \prod_{j=1}^{k} \frac{\cos\ n_j\,(x_j - x'_j\,)}{\log\ (n_j\ + 1)}$$

where $b_j \equiv b_j\,(X_j, x'_j) \equiv \min\,(u_j, u'_j)\ (j = 1, 2, \ldots, k)$;

$$\equiv (b_1, b_2, \ldots, b_k)$$

Setting

$$(2.010) \qquad H_B\ (X - X') = \sum_{N=1}^{B} \prod_{j=1}^{k} \frac{\cos n_j\ (x_j - x_j')}{\log\ (n_j + 1)}$$

we get

$$(2.011) \qquad \left| \int_Q S_U(X)\, dX \right| \leq \left[\int_Q \int_Q H\ (X - X')\ dX\ dX' \right]^{\frac{1}{2}}$$

Since $\int_Q \int_Q H_B\ ((X - X')\ dX\ dX'$ is positive it would seem a relatively simple matter to integrate by simply reversing the orders of summation and integration. However, since B is a function of X , X', we would be able to do this only if each term of the sum were positive. (We could then replace the variable upper summation limit by its maximum). This not being the case we shall express $H_B\ (X - X')$ in terms of the Fejèr kernel.

Toward this end we set:

$$D_M \equiv \prod_{j=1}^{K} \frac{1}{2} + \sum_{n=1}^{m_j} \cos n\ (x_j - x'_j) =$$

$$\prod_{j=1}^{k} \left(\frac{\sin\ (m_j + \frac{1}{2})\ (x_j - x'_j\ )}{2\ \sin \frac{x_j - x'_j}{2}} \right)$$

$$D_o = o$$

$$T_M \equiv \sum_{N=1}^{M} D_N = \prod_{j=1}^{K} \frac{\sin^2\ (m_j + 1)\ \ (x_j - x'_j\ )}{2\ \sin^2\left(\frac{x_j - x'_j}{2}\right)}$$

$\Phi_0 \equiv 0$

D_M and Φ_M are the Dirichlet and Fejèr Kernels in K variables. We note that both are product functions, i.e.

$$D_M = \prod_{j=1}^{k} D_{m_j} \qquad \Phi_M = \prod_{j=1}^{k} \Phi_{m_j}$$

We now return to (2.010)

$$(2.0.12) \qquad H_B(x - x') = \sum_{N=1}^{B} \prod_{j=1}^{k} \frac{\cos n_j (x_j - x_j')}{\log (n_j + 1)}$$

$$= \sum_{N=1}^{B} \prod_{j=1}^{k} \frac{(D_j - D_{j-1})}{\log (n_j + 1)}$$

Setting

$$(2.0.13) \qquad \Delta \Phi_{m_j} = (\Phi_{m_j} - \Phi_{m_{j-1}}) = D_{m_j}$$

$$\Delta^2 \Phi_{m_j} = \Delta D_{m_j}$$

and, due to the mulitplicativity of the Fejèr and Dirichlet kernels

$$(2.0.14) \qquad \Delta^2 \Phi_M = \prod_{j=1}^{k} \Delta^2 \Phi_{m_j}$$

$$\Delta^2 D_M = \prod_{j=1}^{k} \Delta^2 D_{m_j}$$

Applying (2.0.13)and (2.0.14) to (2.0.12) we have

$$H_B\ (X - X^{1}) = \sum_{=1}^{B} \prod_{j=1}^{k} \frac{\Delta^{D} n_j}{\log\ (n_j + 1)} = \sum_{N=1}^{B} \prod_{j=1}^{k} \frac{\Delta^{2}\ T_{n_j}}{\log\ (n_j + 1)}$$

and we see that $H_B\ (X - X^{1})$ is also multiplicative. We now have

$$(2.0.15)\quad \int_Q \int_Q H_B\ (X - X^{1})\ dX\ dX^{1} =$$

$$\int_Q \int_Q \left(\sum_{N=1}^{B\ (X,\ X^{1})} \prod_{j=1}^{k} \frac{\Delta^{2}\ T_{n_j}}{\log\ (n_j + 1)} \right) dX\ dX^{1}$$

$$= \int_Q \int_Q \left(\sum_{N=1}^{B} \prod_{j=1}^{k} \frac{\Delta^{2}\ T_{n_j}}{l_{n+1}} \right) dX\ dX^{\prime 1}$$

where we have set $l_n = \log n$.

The problem of finding an upper bound for (2.0.10) has now been resolved into finding one for (2.0.15). To do this, we need only consider the case of a single variable for $k = 1$. Then because of its property of multiplicativity, an upper bound for (2.15) in the case of k variables will be $(\max A_j)^k$ where A_j denotes an upper bound for the single variable X_j.

Hence, setting $k = 1$ in (2.15) and dropping the subscript one, we have

$$\int_Q H(X - X')\, dX\, dX' = \int_Q \sum_{n=1}^{B(X,X')} \frac{\Delta 2\, T_n}{l_{n+1}}\, dX\, dX'$$

Now

$$H(X - X') = \sum_{n=1}^{b(X,X')} \frac{\Delta 2\, T_n}{l_{n+1}} = \sum_{1}^{b} \frac{T_n - 2T_{n-1} + T_{n-2}}{l_{n+1}}$$

$$= \sum_{1}^{b} \frac{T_n}{l_{n+1}} - 2 \sum_{o}^{b-1} \frac{T_n}{l_{n+2}} + \sum_{o}^{b-2} \frac{T_n}{l_{n+3}}$$

$$= \sum_{1}^{b-2} \left(\frac{1}{l_{n+3}} - \frac{2}{l_{n+2}} + \frac{1}{l_{n+1}} \right) T_n$$

$$+ \left(\frac{1}{l_b} - \frac{1}{l_{b+1}} \right) T_{b-1} + \frac{T_b - T_{b-1}}{l_{b+1}} - \left(\frac{2}{l_2} - \frac{1}{l_3} \right) T_o$$

$$= \sum_{1}^{b-2} \left(\Delta^2 l_{n+3} \quad T_n \right) - \Delta l_{b+1} \quad T_{b-1} + l_{b+1}^{-1} \, \Delta T_b$$

$$- (2l_2 - l_3)\ T_o$$

Where $\qquad l_n \equiv l_n - l_{n-1} \equiv \dfrac{1}{\log n} - \dfrac{1}{\log (n-1)}$

Because of the positiveness of $\int_Q H(X - X')\,(dXdX'$, we may consider only the positive terms, Thus

$$(2.0.16)\qquad \int_Q H(X - X')\; dXdX' \leq \int_Q \sum_{1}^{b-2} \Delta^2\, 1_{n+3}\; T_n \; dx\, dx'$$

$$+ \int_Q - \; 1_{b+1}\; T_{b-1}\; dX\, dX' + \int_Q \frac{\Delta T_b}{1_{b+1}}\; dXdX'$$

$$= \qquad I_1 \qquad + \qquad I_2 \qquad + \qquad I_3$$

In evaluating I_1 , I_2 , I_3 we need the following aids.

<u>Lemma (2.0)</u> If $F(X, X', t) = F(X', X, t)$, then

$$(2.0.17)\qquad \left| \int_Q F(X, X', b(X,X'))\; dXdX' \right| \leq 2 \int_Q \left| F(X,X' b(X) \right| dXdX'$$

<u>Proof:</u>

Let the regions Q_1 and Q_2 be defined as

$$Q_1 = Q\big(b(X) \leq b(X')\big) \quad 0 \leq x \leq 2\pi \;; \qquad Q_2 \equiv Q\big(b(X) > b(X')\big)$$

Then

in Q_1 : $b(X,X^1) \equiv \min \left(B(X),\ b(X^1)\right) = b(X)$

in Q_2 : $b(X,X^1) \equiv \min \left(b(x),\ b(X^1)\right) = b(X^1)$

and $Q_1 + Q_2 = Q$

$$\int_Q F(X,X^1,\ b(X,X^1))\ dXdX^1 = \int_Q F\ (X,X^1;\ b(X))\ dXdX^1 +$$

$$+ \int_{Q_2} F(X,X^1,\ b(X^1))\ dXdX^1$$

$$\left| \int_Q F(X,X^1,\ b(X,X^1))\ dXdX^1 \right. \leq \int_{Q_1} \left| F(X,X^1,\ b(X)) \right| dXdX^1 +$$

$$+ \int_{Q_2} \left| F(X,X^1,\ b(X^1)) \right| dXdX^1$$

$$= \int_{Q_1} \left| F(X,X^1,\ b(X)) \right| dXdX^1 + \int_{Q_2} \left| F(X^1,X,\ b(X)) \right| dX^1\ dX$$

$$\leq \int_Q \left[\left| F(X, X^1, b(X)) \right| + \left| F(X, X^1, b(X)) \right| \right] dX dX^1$$

$$= 2 \int_Q \left| F(X, X^1, b(X)) \right| dX dX^1$$

The following relations will also be used:

$$(2.0.18) \qquad -\Delta\, l_{n+2} = \frac{1}{\log(n+1)} - \frac{1}{\log(n+2)} < \frac{1}{(n+1)\log 2} \quad (n \geq 1)$$

$$(2.0.19) \qquad \Delta^2\, l_{n+3} = \frac{1}{\log(n+1)} - \frac{1}{\log(n+2)} + \frac{1}{\log(n+3)}$$

$$< \left(\frac{2}{\log 2} + 1 \right) \frac{1}{(n+1)^2 \log(n+1)}$$

To prove these set:

$$f(X) = \frac{1}{\log(X)} = (\log X)^{-1}$$

$$-\Delta\, l_{n+2} = f(n+1) - f(n+2) = -f'(n+1+r_n), \quad (0 < r_n < 1)$$

$$= \frac{1}{(X+1)\log^2(X+1)} \Bigg|_{X = n+1+r_n} = \frac{1}{(n+1+r_n)\log^2(n+1+r_n)}$$

$$\leq \frac{1}{(n+1)\log^2(n+1)} < \frac{1}{(n+1)\log 2} \quad (n \geq 1)$$

$$\Delta^2 l_{n+3} = \Delta l_{n+2} - \Delta l_{n+3}$$

$$= \frac{1}{(n+1+r_n)\log^2(n+1+r_n)} - \frac{1}{(n+2+q_n)\log^2(n+2+q_n)}$$

$$= f'(n+1+r_n) - f'(n+2+q_n) = -(1+q_n-r_n)\, f''(n+1+p_n)$$

where $\quad n+1+r_n < n+1+p_n < n+2+q_n$

i.e. $\quad r_n < p_n < 1+q_n$

$$f'(X) = - \frac{\left[\log^2 X + 2\log X\right]}{X^2\log^4 X} = -\left[\frac{1}{X^2\log^2 X} + \frac{2}{X^2\log^3 X}\right]$$

$$\therefore \Delta^2 l_{n+3} = \left[1+q_n-r_n\right]\Big[(n+1+p_n)^{-2}\log^{-2}(n+1+p_n) +$$

$$+ 2(n+1+p_n)^{-2}\log^{-3}(n+1+p_n)\Big]$$

$$= \left[1+q_n - r_n\right]\left[n+1+p_n\right]^{-2} \log^{-2}(n+1+p_n)\left[1 + \frac{2}{\log(n+1+p_n)}\right]$$

$$\left[1 + \frac{2}{\log 2}\right] \cdot \left[\frac{1}{(n+1)^2 \log^2 (n+1)}\right] \quad (n \geq 1)$$

We have now verified (2.0.18) and(2.0.19)

We proceed to evaluate (2.0.16).

$$0 \leq I_1 \equiv \int_Q \sum_1^{b-2} \Delta^2 1_{n+3} \; T_n \; dX dX^1 =$$

$$\int_0^{2\pi} \int_0^{2\pi} \sum_1^{b-2} \Delta^2 1_{n+3} \frac{\sin^2 (n+1) \frac{X - X^1}{2}}{2 \sin^2 \frac{X - X^1}{2}} \; dX dX^1$$

Since each term in this sum is positive, we can replace the variable upper limit by its maximum, m - 2. Then, reversing orders of integration and summation.

$$I_1 \leq \sum_1^{m-2} \Delta^2 1_{n\ 3} \int_0^{2\pi} \int_0^{2\pi} \frac{\sin^2 (n+1) \frac{X - X^1}{2}}{2 \sin^2 \frac{X - X^1}{2}} \; dX dX^1$$

$$= 2 \sum_1^{m-2} \Delta^2 1_{n\ 3} (n+1), \text{ as } \int_0^{2\pi} \frac{\sin^2 nX}{2 \sin^2 X} \, dx = \pi n$$

$$\leq 2\pi^2\left(1+\frac{2}{\log 2}\right)\sum_{1}^{m-2}\left(\frac{(n+1)}{(n+1)^2\log^2(n+1)}\right)$$

$$\leq 2\pi^2\left(1+\frac{2}{\log 2}\right)\sum_{1}^{\infty}\frac{1}{(n+1)\log^2(n+1)}$$

$$P \leq I_2 = \int_0^{2\pi}\int_0^{2\pi} -\Delta 1_{b+1}\, T_{b-1}\, dX dX^1$$

$$\leq \int_0^{2\pi}\int_0^{2\pi}\left(\frac{T_b - 1}{b(x,x^1)}\right) dX dX^1, \text{ by } (2.0.18)$$

$$\leq \frac{2\pi}{\log 2}\int_0^{2\pi}\frac{dx}{b(X)}\int_0\left(\frac{\sin^2 b(X)\,\frac{X-X^1}{2}}{2\sin^2\frac{X-X^1}{2}}\right) dX^1, \text{ by } (2.0.17)$$

$$= \frac{2\pi}{\log 2}\int_0^{2\pi}\frac{b(X)}{b(X)}\, dX = \frac{4\pi^2}{\log^2}$$

$$|I_3| \leq \int_0^{2\pi}\int_0^{2\pi}\frac{|\Delta T_b|}{1_{b+1}}\, dX dX^1 \leq \int_0^{2\pi}\left[\frac{1}{\log(b(X)+1)}\right.$$

$$\left.\int_0^{2\pi}\frac{|\sin(b(X)+\frac{1}{2})(X-X^1)|}{\left|\sin\frac{X-X^1}{2}\right|}\, dX^1\right] dX$$

But

$$\int_0^{2\pi} \frac{|\sin(b(X)+\frac{1}{2})(X-X^1)|}{2\left|\sin \frac{X-X^1}{2}\right|}\,dX = 2\int_0^{\pi} \frac{\sin(b(X)+\frac{1}{2})\,X^1}{2\sin\frac{X^1}{2}}\,dX^1$$

$$= \left[\int_0^{\frac{1}{b(X)}} + \int_{\frac{1}{b(X)}}^{\pi}\right] \frac{\sin\,(b(X)+\frac{1}{2})\,X^1}{\sin\frac{X^1}{2}}\,dX^1$$

$$\leq \int_0^{\frac{1}{b(X)}} \frac{(b(X)+\frac{1}{2})X^1}{\frac{X^1}{2}}\,dX^1 + \pi\int_{\frac{1}{b(X)}}^{\pi} \frac{1}{X^1}\,dX^1$$

$$= 2\,(b(X)+\tfrac{1}{2})\,X^1\Big|_0^{\frac{1}{b(X)}} + \log b(X)\Big|_{\frac{1}{b(X)}}^{\pi}$$

$$= \frac{2\,b(X)+1}{b\,(X)} + \pi\log\pi + \pi\log\,b(X)$$

$$\leq \frac{5}{2} + \log\pi + \pi\log b(X) < 3\,\pi\log\,b(X)$$

$$\therefore \left| I_3 \right| \leq 2 \int_0^{2\pi} \frac{3\pi \log b(X)}{\log (b(X)+1)} dX \quad 2 \int_0^{2\pi} \frac{3\pi \log b(X)}{\log(b(X)+1)} dX$$

$$< 12\pi$$

Applying the bounds for I, I_2, I_3 to (2.0.16)

$$\int_0^{2\pi} \int_0^{2\pi} H(X - X^1) \; dX \; dX^1 < A, \text{ (constant)}$$

Hence, for k variables we have

$$\int_Q \int_Q H_B(X-X^1) \; dX \; dX^1 < (\max A_j)^k = A^k \tag{2.0.20}$$

Substituting (3.4) into 2.21) we get

$$\int_Q S_v(X) \; dX \leq c \; A^k \; M_2^2 \left[q \right] \tag{2.0.21}$$

where $M_2^2 \left[(q) \right] = \int_Q \left| q(R) \right|^2 \, d\,R$

From this essential inequality we proceed to establish the convergence p. p. of the series (2.01)

Let

$$W_M(X) = \operatorname{Sup} S_N(X) \qquad (0 \leq N \leq M)$$

$$V_M(X) = \operatorname{Sup} (-S_N(X))$$

Since

$$S_0(X) = 0 \quad , W_M, V_M \geq 0$$

From (2.0.21) we have

$$\left| \int_Q S_U(X)\, dX \right| \leq C' M_2^2 \left[(q)\right], C^1 \equiv C A^k$$

We can choose $U(X)$ such that

(2.0.22)
$$\left| \int_Q W_M(X)\, dX \right| \leq C^1 M_2^2 \left[q \right]$$

$$\left| \int_Q V_M(X)\, dX \right| \leq C^1 M_2^2 \left[q \right]$$

Be definition $W_M(X)$, $V_M(X)$ are non-decreasing functions, and thus applying (2.0.22) their limits exist. Setting

$$W = \lim_{M \to \infty} W_M \quad , \qquad V = \lim_{M \to \infty} V_M$$

we have

$$\left| \int_Q W(X)\, dX \right| \leq C^1 M_2^2 \left[q \right]$$

$$\left| \int_Q V(X)\, dX \right| \leq C^1 \quad M_2^2 \, [\cdot q]$$

from which we conclude that V, W are finite p. p.; and since $W(X) = \sup S_M(X)$, $V(X) = \sup(-S_M(X))$, the sequence $(S_M(X))$ is bounded p. p.

Now let

$$\text{L}(X) = \operatorname{Sup} \left| S_M(X) + S_{M^1}(X) \right| \text{ for all } M, M^1$$

then $\quad \text{L}(X) \leq W + V$

for

$$\operatorname{Sup} \left| S_M + (-S_{M^1}) \right| \leq \left| \operatorname{Sup} S_M + \operatorname{Sup} (-S_{M^1}) \right|$$

$$= \left| W_M + V_M \right|$$

$$\leqq \left| W + V \right|$$

Hence

$$\left| \int_Q \text{L}(X)\, dX \right| \leq \left| \int_Q W(X)\, dX \right| + \left| \int_Q V(X)\, dX \right|$$

$$\leq 2\, C^1 \quad M_2^2 \, [q]$$

Set

$$\text{L}_M(X) = \operatorname{Sup} \left| S_N(X) - S_{N'}(X) \right| \text{ for all } N, N^1 > M$$

and let

$$q_M(X) \sim \sum_{J = |M|}^{\infty} C_J \prod_{n=1}^{k} (\log (J_n + 1))^{\frac{1}{2}} e^{iJX}$$

The function $L_M(X)$ is the corresponding to $q_M(X)$, so

$$\left| \int_Q L_M(X)\, dX \right| \leq 2\, c^1\, M_2^2 \left[q_M \right]$$

But $M_2^2 \left[q_M \right] \to o$ as $M \to o$ as it is less than the remainder of the convergent series (2.00) Hence

$$\lim_{M \to \infty} \left| \int_Q L_M(X) \quad dX \right| \to o$$

Since $\left\{ L_M(X) \right\}$ is a non-increasing sequence

$$\int_Q \lim_M L_M(X)\, dX = o$$

and so $\lim_{M \to \infty} L_M(X) = o$ for almost every (X) which means that $S_M(X)$ converges p. p.

C. Convergence Almost Everywhere: Consideration of Function Condition.

The problem of finding convergence p. p. conditions on the function without actually having more than convergence p. p. has not been solved, even for the case of the single variable. A rather trivial result - trivial in the sense that convergence p. p. is proved by showing that the convergence p. p. condition on the function is equivalent to the condition (2.00) on the coefficients - has been obtained by Plessner (25), and this result we generalize to several variables.

Theorem (2.1) If the function

$$\frac{\Phi(T)}{\prod_{j=1}^{k} t_j} = \prod_{j=1}^{k} \frac{1}{t_j} \cdot \int_Q \left| \varphi(X,T) \right|^2 dX$$

is integrable then

$$\sum_{N=-\infty}^{\infty} \left| C_N \right|^2 \prod_{j=1}^{k} \left[\log (|n_j| + 1) \right] < \infty, \; D_{-N} = \overline{C_n}$$

and conversely, where

$$\varphi(X, T) = \prod_{j=1}^{k} \left[f(X_j + t_j) + f(X_j + t_j) \right] - 2^k f(X)$$

and we define the symbolical product operation by

$$\prod_{j=1}^{k} f(X_j + t_j) \equiv f(X_1 + t_1, X_2 + t_2, \ldots, X_k + t_k).$$

Proof

$$f(X) \sim \sum_{N=-\infty}^{\infty} C_N \, e^{iNX}$$

It is well known that

$$Q(X, T) \sim \sum_{N=1}^{\infty} i^k \quad 2^k \left(\prod_{j=1}^{k} \sin n_j t_j \right) C_N \; e^{iNX}$$

Applying Parseval's theorem (Lemma 1.2) to the function $Q(X,T)$, we have

$$\Phi (T) \equiv \int_Q \left| Q(X, T) \right|^2 dX$$

$$= (4\pi)^k \sum_{N=1} C_N^2 \prod_{j=1}^{k} \sin^2 n_j \; t_j$$

We set

$$\Phi_M(T) = (4\pi)^k \sum_{N=1}^{M} C_N^2 \prod_{j=1}^{k} \sin^2 n_j \, t_j$$

We see that

$$0 < \frac{\Phi_{M'}(T)}{\prod_{j=1}^{k} t_j} \leq \frac{\Phi_M(T)}{\prod_{j=1}^{k} t_j} , \; m_j \geq m'_j (j = 1,2,\ldots k)$$

i.e. $\left\{ \dfrac{\Phi_M (T)}{\prod_{j=1}^{k} t_j} \right\}$ is a positive non-decreaseing sequence.

Since

$$\lim_{M \to \infty} \frac{\Phi_M(T)}{\prod_{j=1}^{k} t_j} = \frac{\Phi(T)}{\prod_{j=1}^{k} t_j}$$

we have, using the Lebesque theory of integration

$$\lim_{M \to \infty} \int_Q \frac{\Phi_M(T)}{\prod_{j=1}^{k} t_j} \, dT = \int_Q \frac{\Phi(T)}{\prod_{j=1}^{k} t_j} \, dT$$

provided either side exists. But, by hypothesis, the right side is finite; the left hand side is also finite.

Thus we can write

$$\int_Q \frac{\Phi_M(T)}{\prod_{j=1}^{k} t_j} \, dT = (4\pi)^k \int_Q \sum_{1}^{M} C_N^2 \prod_{j=1}^{k} \frac{\sin^2 n_j t_j}{t_j} \, dT$$

$$= (4\pi)^k \sum_{1}^{M} C_N^2 \int_Q \prod_{j=1}^{k} \frac{\sin^2 n_j t_j}{t_j} \, dT$$

Since

$$\int_0^{2\pi} \frac{\sin^2 n\,t}{t}\, d\,t = O\,(\log n)$$

$$\int_Q \prod_{j=1}^{k} \frac{\sin^2 n_j t_j}{t_j}\, dT = O\left(\prod_{j=1}^{k} \log n_j\right)$$

and thus

$$\lim_{M \to \infty} \int_Q \frac{\Phi_M(T)}{\prod_{j=1}^{k} t_j}\, dT < \infty$$

if $$\sum_{N=1}^{\infty} C_N^2 \prod_{j=1}^{k} \log\,(n_j + 1) < \infty$$

Retracing our proof, we have the converse.

Hardy and Littlewood (11) employeda method of "generating" a theorem concerning Fourier series from a known theorem by replacing sums by integrals, functions be ooefficients, the summation index by the variable, etc., thus obtaining a "new" theorem which they dalled the "transform" of the original one. The genesis of this principle is to be found in the theory of "Fourier Transforms."

Using this method we obtain a "transform" theorem - one which we have been unalbe to prove. Our conjecture is the transform of theorem (2.0):

Conjecture

If $\displaystyle\int_{\epsilon}^{\pi} |\varphi(t)|^2 \log \frac{1}{t}\, dT = \underset{\epsilon \to 0}{O}(1)$ then

the Fourier series

$$\sum_{0}^{\infty} (a_n \cos nX + b_n \sin nX)$$

converges p. p. where

$$\varphi(t) \equiv \varphi_X(t) = f(X+t) + f(x-t) - 2f(X).$$

CHAPTER III

A. Definition and Auxiliary Lemma.

A sequence (S_{mn}) is said to be strongly summable of order $B > 0$ and index $K > 0$ to a number S if a number S exists so that

$$\left[\frac{\sum_{kl=0}^{m\,n} \left| \sigma_{k\,l}^{(B-1)} - S\right|^{K}}{(m+1)(n+1)}\right]^{1/k} \to 0 \text{ as } m,\, n \to \infty$$

where $\sigma_{mn}^{(B-1)} = \dfrac{\sum_{0}^{m,n} \sigma_{kl}^{(B-2)}}{(m+1)\ (n+1)}$ and $\sigma_{mn}^{0} = S_{mn}$

We designate this by (S, B, K), B an integer.

We note that (S, 1, K) is

$$\left[\frac{\sum_{0}^{m,n} \left| S_{kl} - S\right|^{K}}{(m+1)\ (n+1)}\right]^{1/k} \to 0 \text{ as } m,\, n \to \infty$$

Henceforth, unless otherwise specified, when we speak of strong summability we will mean of order one.

A sequence (S_{mn}) is said to be restrictedly strongly summable (Moore (23) p. 79) if, under the above condition, m, n in such a manner that $C\,m < n < C^{-1}m$, C a constant; i. e. the m and n become infinite "together."

Lemma (3.0):

$$A_K\left[S_{kl}-S\right] \equiv \left[\frac{\sum\limits_{k,l=o}^{m,n} \left| S_{kl}-S\right|^{K}}{(m+1)\ (n+1)}\right]^{\frac{1}{K}}$$ is a monotonic

increasing function of $K > o$

Proof:

Let b be an arbitrary number $o < b < K$. Applying Holders Inequality (Lemma 1.3) we have

$$\frac{\sum\limits_{o}^{m,n} \left| S_{kl}-S\right|^{b}}{(m+1)\ (n+1)} \leq \left(\frac{1}{(m+1)\ (n+1)}\left[(m+1)(n+1)\right]^{\frac{1}{P}}\right).$$

$$\cdot\left[\sum_{o}^{m,n} \left| S_{kl}-S\right|^{bp'}\right]^{\frac{1}{P'}}$$

where $\frac{1}{P} + \frac{1}{P'} = 1$ and $bp' = K > b.$

$$\left[\frac{\sum\limits_{o}^{m,n} \left| S_{kl}-S\right|^{b}}{(m+1)\ (n+1)}\right]^{\frac{1}{b}} \leq \left[(m+1)(n+1)\right]^{\frac{1-p}{bp}}\left[\sum_{o}^{m,n} \left| S_{kl}-S\right|^{bp'}\right]^{\frac{1}{bp'}}$$

$$\leq \left[\frac{1}{(m+1)(n+1)} \sum_{o}^{m,n} \left| S_{kl}-S\right|^{K}\right]^{\frac{1}{K}}$$

as $bp^* = K$, $\frac{1-p}{b\,p} = \frac{1}{b\,p^*} = -\frac{1}{K}$. Since $K > b$ our lemma is proved.

B. History of the problem.

The notion of strong summability was first introduced by Hardy and Littlewood in 1913 (10). In this paper it was proved that if $f(X) \in L_2$ in a neighborhood of X_0 and if

$$\int_0^t \left\{ Q(u) \right\}^2 du = o\,(t)$$

then

$$\sum_0^n \left| S_m - S \right|^2 = o\,(n).$$

Hardy and Littlewood say of this theorem: "The classical theorem of Fejér and its generalization by Lebesgue show that in these circumstances

$$\sum_0^n (S_m - S) = o(n).$$

The interest of our theorem is that it shows (for example) that, when the Fourier series of a continuous function is divergent, its summability is not merely a consequence of the cancelling of the various deviations summed up in Fejér's mean, but rather of the comparative smallness of the devaitions."

With the same assumptions Fejér (5) has given a rather ingenious and simple proof of the Hardy - Littlewood theorem by proving Abel-

Poisson summability of power series and hence for the associated Fourier series.

The results of Hardy and Littlewood were extended by Carleman (1), Sutton (30) and Hardy and Littlewood themselves(11). The culmination of their combined efforts resulted in the proof of strong summability of index $K > 0$ under the assumptions

$$\int_0^t |Q(u)|^P \, du = O(t) \qquad P > 1$$

$$\int_0^t Q(u) \, du = o(t)$$

We make a generalization to two variables of this final result. With the addition of several single variable assumptions on the function we succeed in proving restricted strong summability. This "restriction" is to be expencted however as we are concerned with a localization property which does not generalize to several variables without such a restriction on the type of summability employed.

We also generalize Fejèr's result to two variables, proving restricted strong summability for power series. We note that in two variables the real and imaginary parts of the power series are not the general Fourier series and its conjugate as they are for the single variable case.

For general orthogonal developments Zygmund (37) p240 gave a nice proof of strong summability of index two under the assumptions of

(C,1) summability plus the convergence of the sum of the squares of the coefficients of the orthogonal development. We are unable to generalize this to two variables without replaceing the hypothesis of (C,1) summability by another triangular type double summability method which we call R-summability (ref. Ch. III, Sec. E.). For R summability of "product" double series (i.e $\sum a_m b_n$) we need more than (C,1) summability. However, there are double series whose sequences are unbounded and yet R. summable. Exactly how this R. summability method comparies with (C,1) summability we have been unable to ascertain.

C. Strong Summability of Power Series.

Theorem 3.0: if $f(z_1, z_2)$ is regular in $|z_1| < 1$, $|z_2| < 1$ and continuous at the point (1,1) and if

$$(3.0.1) \qquad \int_0^{\pi} |f(z_1, z_2)|^{p'} \, d\theta_1 = o\left(\frac{1}{(1-r_1)^{p'-1}}\right)$$

for all $|z_1| < 1$

$$(3.0.2) \qquad \int_0^{\pi} |f(z_1, z_2)|^{p'} \, d\theta_2 = o\left(\frac{1}{(1-r_2)^{p'-1}}\right)$$

for all $|z_2| < 1$

then

$$(3.0.3) \qquad S_{mn} = \sum_{0}^{m,n} C_{kl}$$

is restrictedly strongly summable of index p to S where

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and}$$

(3.0.4) $$f(z_1, z_2) = \sum_0^{\infty} C_{mn} z_1^m z_2^n, \quad f(1,1) = S.$$

Remark: We need only prove the theorem for an even index p = 2K (K > o) since $A_k (S_{mn} - S)$ is a monotonic increasing function of K (Lemma (3.0)).

We note that conditions (3.0.1) and 3.0.2) imply

(3.0.5) $$\int_0^{\pi} \int_0^{\pi} |f(z_1, z_2)|^{p'} d\theta_1 \, d\theta_2 =$$

$$O\left(\frac{1}{[(1 - r_1)(1 - r_2)]^{p'-1}}\right)$$

Proof:

Consider the uniformly convergent series

(3.0.6) $$\frac{1}{(1 - z_1)(1 - z_2)} = \sum_0^{\infty} z_1^m z_2^n \quad \text{for} \quad \begin{matrix} |z_1| < 1 \\ |z_2| < 1 \end{matrix}$$

Since $f(z_1, z_2)$ is regular in $|z_1| < 1$, $|z_2| < 1$

the series

(3.0.7) $$\sum_0^{\infty} C_{mn} z_1^m z_2^n \quad , |z_1| < 1, |z_2| < 1$$

is also uniformly convergent. Hence the product series is uniformly convergent and representsthe product function for $|z_1| < 1$, $|z_2| < 1$. After formal multiplication of (3.0.6) and (3.0.7) we have

$$(3.0.8) \qquad \frac{f(z_1, z_2)}{(1 - z_1)(1 - z_2)} = \sum_{o}^{\infty} S_{mn} \, z_1^{m} \, z_2^{n}$$

where S_{mn} is defined by (3.0.3)

Subtracting

$$\frac{S}{(1 - z_1)(1 - z_2)} = \sum_{o}^{\infty} S \, z_1^{m} \, z_2^{n} \qquad (\text{where } S = f(1,1))$$

from (3.0.8) we obtain

$$F(z_1, z_2) = \frac{f(z_1, z_2) - S}{(1 - z_1)(1 - z_2)} = \sum_{o}^{\infty} (S_{mn} - S) \, z_1^{m} \, z_2^{n}$$

Applying the Young - Hausdorff inequality (Lemma (1.2)) to $F(z_1, z_2)$ we have

$$I \equiv \sum_{o} \left| S_{mn} - S \right|^{2K} r_1^{2mK} \, r_2^{2nK} \leq$$

$$\leq \left[\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \frac{f(z_1, z_2) - S}{(1 - z_1)(1 - z_2)} \right|^{\frac{2K}{2K - 1}} d\theta_1 \, d\theta_2 \right]^{(2K - 1)}$$

which, after rewriting as an integral from o to π is

$$I^{2b - 1} \leq \left[\frac{1}{4\pi^2} \int_{o}^{\pi} \int_{o}^{\pi} \frac{\left| f(z_1, z_2) - S \right|^{2b} + \left| f(z_1, \bar{z}_2) - S \right|^{2b}}{\left[(1 - z_1)(1 - z_2) \right]^{2b}} + \right.$$

$$\left. + \frac{\left| f(\bar{z}_1, z_2) - S \right|^{2b} + \left| f(\bar{z}_1, \bar{z}_2) - S \right|^{2b}}{\left[(1 - z_1)(1 - z_2) \right]^{2b}} \, d\theta_1 \, d\theta_2 \right]$$

where we have written $2b = \frac{2K}{2K-1}$, $2b - 1 = \frac{2K}{2K - 1} - 1 = \frac{1}{2K-1}$

$$I^{2b-1} = \frac{1}{4\pi^2} \int_0^{\pi} \int_0^{\pi} \frac{L}{G} \, d\theta_1 \, d\theta_2$$

$$= \left\{ \int_0^{\delta_1} \int_0^{\delta_2} + \int_0^{\delta_1} \int_{\delta_2}^{\pi} + \int_{\delta_1}^{\pi} \int_0^{\delta_2} + \int_{\delta_1}^{\pi} \int_{\delta_2}^{\pi} \right\} \frac{L}{G} \, d\theta_1 \, d\theta_2$$

$$I^{2b-1} = I_1 + I_2 + I_3 + I_4$$

We estimate I_4

$$I_4 = \int_{\delta_1}^{\pi} \int_{\delta_2}^{\pi} \frac{L}{G} \, d\theta_1 \, d\theta_2$$

For the denominator G of the integrand we have

$$G = \left| 1 - r_1 e^{i\theta_1} \right|^{2b} \cdot \left| 1 - r_2 e^{i\theta_2} \right|^{2b} >$$

$$> \left[16 \, r_1^2 \, r_2^2 \, \frac{\delta_1^2 \, \delta_2^2}{\pi^4} \right]^b$$

for

$$\left| 1 - r_1 e^{i\theta_1} \right|^2 > \left| 1 - r_1 e^{i\delta_1} \right|^2 = (1 - r_1 \cos \delta_1)^2 \; r^2 \sin^2 \delta_1 =$$

$$= 1 + r_1^2 - 2r_1 \cos \delta_1 = (1 - r_1)^2 + 2r_1(1 - \cos \delta_1)$$

(3.0.9)

$$2 r_1 2 \sin^2 \frac{\delta_1}{2} > 4r_1 \frac{\delta_1^2}{\pi^2}$$

Similarly

$$\left| 1 - r_2 e^{i\theta_2} \right|^2 > 4r_2 \frac{\delta_2^2}{\pi^2}$$

Using (3.0.5) we have ($p' = \frac{p}{p-1} = \frac{2K}{2K-1} = 2b$)

(3.0.10) $$\int_0^{\pi} \int_0^{\pi} L \, d\theta_1 \, d\theta_2 = o\left(\frac{1}{[(1-r_1)(1-r_2)]^{2b-1}} \right)$$

Combining the estimates for G and (3.0.10) we get

$$I_4 = o\left(\frac{1}{[(1-r_1)(1-r_2)]^{2b-1}} \right) \text{ as } r_1, r_2 \to 1$$

We now consider I_1.

Because of the continuity of $f(\ ,\ )$ at the point(1,1)we have

$$I_1 \equiv \int_0^{\delta_1} \int_0^{\delta_2} \frac{L}{G} \, d\theta_1 \, d\theta_2 <$$

$$< \epsilon \int_0^{\infty}\int_0^{\infty} \frac{d\theta_1 \quad d\theta_2}{\left|1 - r_1 e^{i\theta_1}\right|^2 \cdot \left|1 - r_2 e^{i\theta_2}\right|^2}$$

$$(3.0.11) \qquad < \epsilon \int_0^{\pi}\int_0^{\pi} \frac{d\theta_1 \, d\theta_2}{\left[(1 - r_1)^2 \;\; 4r_1 \; \frac{\theta_1^2}{\pi^2}\right]^b \left[(1 - r_2)^2 \; 4r_2 \; \frac{\theta_2^2}{\pi^2}\right]^b}$$

as $\sin \frac{X}{2} < \frac{X}{\pi}$ for $0 \le x \le \pi$

Now let

$$\theta_1 = \frac{1 - r_1}{\sqrt{r_1}} \, \varphi_1 \quad , \quad \theta_2 = \frac{1 - r_2}{\sqrt{r_2}} \, \varphi_2$$

We have

$$I_1 < \epsilon \int_0^{\infty}\int_0^{\infty} \frac{(1 - r_1)\,(1 - r_2)\; d\varphi_1 \; d\varphi_2}{\sqrt{r_1 r_2}\,\left[(1 - r_1)(1 - r_2)\right]^{2b}\left[(1 + \frac{4\varphi_1^2}{\pi^2})(1 + \frac{4\varphi_2^2}{\pi^2})\right]^b}$$

$$\le \frac{\epsilon}{\left[(1 - r_1)(1 - r_2)\right]^{2b - 1}} \int_0^{\infty}\int_0^{\infty} \frac{d\varphi_1 \; d\varphi_2}{\sqrt{r_1 r_2}\left[(1 + \frac{4\varphi_1^2}{2})\,(1 + \frac{4\varphi_2^2}{2})\right]^b}$$

$$\le \frac{B\epsilon}{\left[(1 - r_1)(1 - r_2)\right]^{2b - 1}}$$

where B is a bound (the better the closer r_1, r_2 are to 1) for the above integral. Or we can write

$$(3.0.12) \qquad I_1 = 0\left(\frac{1}{\left[(1 - r_1)(1 - r_2)\right]^{2b - 1}}\right) \quad \text{as } r_1 , r_2 \to 1$$

We estimate I_2.

$$I_2 = \int_0^{\delta_1} \int_{\delta_2}^{\pi} \frac{L(\theta_1, \theta_2)\, d\theta_1\, d\theta_2}{\left[\left|1 - r_1 e^{i\theta_1}\right| \cdot \left|1 - r_2 e^{i\theta_2}\right|\right]^{2b}}$$

$$\leq \left(\frac{4r_2 \delta_2^2}{\pi^2}\right)^{-b} \int_0^{\delta_1} \frac{1}{\left|1 - r_1 e^{i\theta_1}\right|^{2b}} \left\{ \int_0^{\pi} L(\theta_1, \theta_2)\, d\theta_2 \right\} d\theta_1$$

as in (3.09).

Since $\displaystyle\int_0^{\pi} L\left(\theta_1, \theta_2\right) d\theta_2 = o\left(\frac{1}{(1 - r_2)^{2b-1}}\right)$ for all θ_1

and $\displaystyle\int_0^{\delta_1} \frac{d\theta_1}{\left|1 - r_1 e^{i\theta_1}\right|^{2b}} = O\left(\frac{1}{(1 - r_1)^{2b-1}}\right)$ as in (3.0.11)

we obtain

$$I_2 = O\left(\frac{1}{(1 - r_1)^{2b-1}}\right) \quad o\left(\frac{1}{(1 - r_2)^{2b-1}}\right)$$

Similarly, for I_3 we have

$$I_3 = o\left(\frac{1}{(1-r_1)^{2b-1}}\right) O\left(\frac{1}{(1-r_2)^{2b-1}}\right)$$

Combining estimates for I_1, I_2, I_3, I_4

$$I^{2b-1} = o\left(\frac{1}{\left[(1-r_1)(1-r_2)\right]^{2b-1}}\right) + O\left(\frac{1}{(1-r_1)^{2b-1}}\right) \cdot o\left(\frac{1}{(1-r_2)^{2b-1}}\right) + o\left(\frac{1}{(1-r_1)^{2b-1}}\right) O\left(\frac{1}{(1-r_2)^{2b-1}}\right)$$

Now, if we restrict r_1, r_2 in such a manner that $r_1, r_2 \to 1$ while $C < \dfrac{1-r_2}{1-r_1} < C^{-1}$ (C constant) i.e. $r_1, r_2 \to 1$ "together" we get

$$\left[(1-r_1)(1-r_2)\right]^{2b-1} I^{2b-1} = o(1,1)$$

or

$$(3.0.13) \qquad (1-r_1)(1-r_2) \sum_{0}^{\infty} \left|S_{mn} - S\right|^{2K} r_1^{2mK} r_2^{2nk} = o(1,1) \quad r_1, r_2 \to 1$$

But this is restricted strong summability of $(S_{mn} - S)$, for, as we proceed to show (3.0.13) is equivalent to

$$(3.0.14) \quad \sum_{k,l=0}^{m,n} \left| S_{kl} - S \right|^{2K} = o(m,n)$$

$$m, n \to \infty \quad \text{where} \quad (nC < m < n \cdot C^{-1})$$

Setting

$$r_1 = 1 - \frac{1}{m}$$

$$r_2 = 1 - \frac{1}{n}$$

where $\quad C < \frac{m}{n} < C^{-1}$ for $C < \dfrac{1 - r_2}{1 - r_1} < C^{-1}$

Substituting into (3.0.13) we obtain

$$\sum_{0}^{m,n} \left| S_{kl} - S \right|^{2K} (1 - \tfrac{1}{m})^{2k\ K} (1 - \tfrac{1}{n})^{2\ lK} = o\ (m\ n)$$

Since $(1 - \frac{1}{m})^m \sim e$ we have

$$\sum_{0}^{m,n} \left| S_{kl} - S \right|^{2K} = o\ (m\ n)$$

$$m, n \to \infty$$

$$C < \frac{m}{n} < C^{-1}$$

i.e. restricted strong summability.

D. Strong Summability of Fourier Series.

Theorem (3.1) If $f(x,y) \in L^r$, for some $r > 1$ and K is any positive number, then at every point (x,y) where

$$(3.1.0) \quad \Phi_v(s) \equiv \int_0^s |\varphi(u,v)|^r \, du = o(s) \text{ uniformly in } v.$$

$$(3.1.1) \quad \Phi_r(t) \equiv \int_0^t |\varphi(u,v)|^r \, dv = o(t) \text{ uniformly in } u.$$

$$(3.1.2) \quad \int_0^s \varphi(u,v) \frac{\sin m u}{2 \tan \frac{1}{2} u} \, du = O(s) \text{ uniformly in } v \text{ and}$$

for $m = 1, 2, 3, \ldots$

$$(3.1.3) \quad \int_0^t \varphi(u,v) \frac{\sin n v}{2 \tan \frac{1}{2} v} \, dv = O(t) \text{ uniformly in } u \text{ and}$$

for $n = 1, 2, 3, \ldots$

the Fourier series of $f(x,y)$ is restrictedly strongly summable to $f(x,y)$, i.e.

$$(3.1.4) \quad \frac{1}{m+1} \frac{1}{n+1} \sum_0^{m,n} \left| S_{kl}(x,y) - f(x,y) \right|^K \to 0$$

as $m,n \to \infty$ in such a manner that

$C n < m < C^{-1} n$ $\qquad$ C constant

Remarks: We note that conditions (3.1.0) and 3.1.1) imply

$$(3.1.5) \qquad \int_0^s \int_0^t \left| Q\,(u,v) \right|^r \; du \; dv \; = \; o\,(s,t)$$

Proof:

We first observe that if (3.14) is true for a certain K, it is a fortiori true for any smaller K; for $A_k\,(S_{kl} - f)$ is a non-decreasing function of K. (Lemma 3.0).

It is sufficient to prove (3.1.4) for $K = r' = \dfrac{r}{r-1}$ for $\left\{ \dfrac{\Phi_r\,(s)}{st} \right\}^{\frac{1}{r}}$ is a non decreasing function of r, and so if $\Phi r(s)$ $= o(s)$ for a specific value of r it remains true for any smaller r. Taking r sufficiently close to one we obtain K as large as we please.

We shall prove (3.1.4) using the modified partion sums

$$\overset{*}{S}_{kl}\,(x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f\,(x+s,\, y+t) \quad \frac{\sin ks \quad \sin lt}{2\,\tan\frac{1}{2}\quad 2\,\tan\frac{1}{2}\,t} \quad ds\; dt,$$

for

$$\left| S_{kl}(x,y) - f(x,y) \right|^k \leq \left[\left|\overset{*}{S}_{kl} - f\right| + \left|S_{kl} - \overset{*}{S}_{kl}\right| \right]^k , \qquad k > b$$

Applying Jensen's inequality (Lemma (1.5)).

$$\left| S_{kl} - f \right|^K \leq B \left[\left|\overset{*}{S}_{kl} - f\right|^K + \left|S_{kl} - \overset{*}{S}_{kl}\right| \right]^K , \qquad \text{Bconstant}$$

and $\left| S_{kl} - S^{*}_{kl} \right| \to 0$ uniformly to o for $\begin{cases} 0 \le k \le m \\ 0 \le l \le n \end{cases}$

(Hobson (15)) p. 700.

We write

$$S^{*}_{kl}(x,y) - f(x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+s, y+t) \frac{\sin ks}{2\tan\frac{1}{2}s} \frac{\sin lt}{2\tan\frac{1}{2}t} \, ds\, dt$$

$$- \frac{f(x,y)}{\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} \varphi(s,t;\, x,y)\, D_{kl}(s,t)\, ds\, dt$$

where $D_{kl}(s,t) = \dfrac{\sin ks \sin lt}{2\tan\frac{1}{2}s \;\; 2\tan\frac{1}{2}t}$

and $\varphi(st) = \varphi(s,t;\, s,y) =$

$= f(x+s, y+t) + f(x-s, y+t) + f(x+s, y-t) + f(x-s, y-t) - 4f(x,y)$

Thus

$$S^{*}_{kl}(x,y) - f(x,y) =$$

$$\frac{1}{\pi^2} \left[\int_{0}^{\frac{1}{m}} \int_{0}^{\frac{1}{n}} + \int_{0}^{\frac{1}{m}} \int_{\frac{1}{n}} + \int_{\frac{1}{m}}^{\pi} \int_{0}^{\frac{1}{n}} + \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \right] \varphi(s,t)\, D_{kl}(s,t)\, ds\, dt$$

$$= a_{kl} + b_{kl} + c_{kl} + d_{kl}$$

Applying Minkowski's inequality (Lemma 1.4) we get

$$\left[\frac{1}{m+1} \frac{1}{n+1} \sum_{0}^{m,n} \left| S^{*}_{kl} - f \right|^{K}\right]^{\frac{1}{K}} \leq \left[\frac{1}{m+1} \frac{1}{n+1} \sum_{0}^{m,n} \left| a_{kl}\right|^{K}\right]^{\frac{1}{K}}$$

$$+ \left[\frac{1}{m+1} \frac{1}{n+1} \sum_{0}^{m,n} \left| b_{kl} \right|^{K}\right]^{\frac{1}{K}} + \left[\frac{1}{m+1} \frac{1}{n+1} \sum_{0}^{m,n} \left| c_{kl} \right|^{K}\right]^{\frac{1}{K}} +$$

$$+ \left[\frac{1}{m+1} \frac{1}{n+1} \sum_{0}^{m,n} \left| d_{kl} \right|^{K}\right]^{\frac{1}{K}}$$

To establish the proof we must show each of these four terms approach zero as m and n become infinite.

We set:

$$D_k(s) = \frac{\sin k^s}{2 \tan \frac{1}{2}s} \quad , \qquad D_l(t) = \frac{\sin lt}{2 \tan \frac{1}{2} t}$$

so that

$$D_{kl}(s,t) = D_k(s) \; D_l(t)$$

We note that

$$\begin{aligned} & D_k(s) < k \text{ for } 0 < s \leq \pi \\ (3.1.6) \quad & D_l(t) < l \text{ for } 0 < t \leq \pi \\ & D_{kl}(s,t) < kl \quad \text{for } 0 < s \leq \pi, \; 0 < t \leq \pi \end{aligned}$$

We first consider a_{uv} :

Using (3.1.6) and (3.1.5) we obtain

$$\left| a_{kl} \right| < \frac{1}{\pi^2}\ kl \quad \Phi_1 \left(\frac{1}{m}, \frac{1}{n}\right) \leq \frac{1}{\pi^2}\ kl \quad \Phi_1 \left(\frac{1}{k}\ \frac{1}{l}\right)$$

$$= \quad kl \quad o\left(\frac{1}{k} \quad \frac{1}{l}\right) \quad = \quad N_{kl} \rightarrow 0$$

Hence

$$(3.17) \quad \left[\frac{1}{m+1}\ \frac{1}{n+1} \sum_{0}^{m,n} \left| a_{kl} \right|^{K} \right]^{\frac{1}{K}} \leq \left[\frac{1}{m+1} \quad \frac{1}{n+1} \quad \sum_{0}^{m,n} \quad {}_{kl}^{K} \right]^{\frac{1}{K}} \underset{m,n \to \infty}{\rightarrow 0}$$

For b_{kl}:

$$b_{kl} = \int_{\frac{1}{n}}^{\pi} \int_{0}^{\frac{1}{m}} \Phi (s,t) \quad D_{kl} (s,t) \, ds \, dt$$

$$= \int_{\frac{1}{n}}^{\pi} D_l (t) \left[\int_{0}^{\frac{1}{m}} (s,t) \quad D_k (S) \, dS \right] dt$$

Taking absolute values, and applying (3.1.2) we get

$$\left| b_{kl} \right| \leq o \left(\frac{1}{m}\right) \int_{\frac{1}{n}}^{\pi} \left| D_l (t) \right| dt \leq l \; o\left(\frac{1}{m}\right)$$

$$\sum_{0}^{m,n} \left| b_{kl} \right|^{K} \leq o \left(\frac{1}{m^K}\right) \sum_{0}^{n} \left| l \right|^{K} \leq n^{K+1} \quad o\left(\frac{1}{m^K}\right)$$

Hence

$$(3.1.8)\quad \left[\frac{1}{m+1}\,\frac{1}{n+1}\sum_{0}^{m,n}\left|b_{kl}\right|^{K}\right]^{\frac{1}{K}} \leq \frac{O(\frac{1}{m})\,n^{1+\frac{1}{K}}}{(n+1)^{\frac{1}{K}}\,(m+1)^{\frac{1}{K}}} \leq \frac{n}{m}\,\frac{O(1)}{m^{\frac{1}{K}}}$$

$$\leq C\,\frac{O(1)}{m^{\frac{1}{K}}} \longrightarrow 0$$

For C_{kl}:

Similarly we find that

$$(3.1.9)\left[\frac{1}{m+1}\,\frac{1}{n+1}\sum_{0}^{m,n}\left|C_{kl}\right|^{K}\right]^{\frac{1}{K}} \leq \frac{m}{n}\,\frac{O(1)}{n^{\frac{1}{K}}} \leq C\,\frac{O(1)}{n^{\frac{1}{K}}} \longrightarrow 0$$

For d_{kl}:

$$d_{kl} = \int_{\frac{1}{m}}^{\pi}\int_{\frac{1}{n}}^{\pi}\varphi(s,t)\,\frac{\sin ks \quad \sin lt}{2\tan\frac{1}{2}s\;2\tan\frac{1}{2}t}\,ds\;dt$$

We observe that the d_{uv} are the Fourier Coefficients of the function $F(s,t)$ defined in the following manner

$$F(s,t) = \varphi(s,t)\,\tfrac{1}{2}\cot\tfrac{1}{2}s\;\;\tfrac{1}{2}\cot\tfrac{1}{2}t \qquad \left\{\begin{array}{l}\frac{1}{m}\leq s\leq\pi\\ \frac{1}{n}\leq t\leq\pi\end{array}\right\}$$

$$F(s,t) \equiv 0 \qquad \left\{\begin{array}{l}-\pi\leq s<\frac{1}{m}\\ -\pi\leq t<\frac{1}{n}\end{array}\right\}$$

Applying the Young-Hausdorff inequality (Lemma 1.2) to the function $F(s,t)$ we have

$$(3.1.10)\quad \left[\sum_{0}^{m,n} |d_{kl}|^{K}\right]^{\frac{1}{K}} \leq \left[\frac{1}{\pi^2}\int_{\frac{1}{m}}^{\pi}\int_{\frac{1}{n}}^{\pi} \frac{|\varphi(s,t)|^{r}\, ds\, dt}{|2\tan\frac{1}{2}s \quad 2\tan\frac{1}{2}t|^{r}}\right]^{\frac{1}{r}}$$

$$\leq \left[\frac{1}{\pi^2}\int_{\frac{1}{m}}^{\pi}\int_{\frac{1}{n}}^{\pi} \frac{|\varphi(s,t)|^{r}}{s^{r}\quad t^{r}}\, ds\, dt\right]^{\frac{1}{r}}$$

where $K = r^{*} = \frac{r}{r-1}$

We now integrate the right hand member by parts (6). Setting

$$V = s^{-r}\,t^{-r} \qquad \frac{\partial^2 U}{\partial s\,\partial t} = |\varphi(s,t)|^{r}$$

$$\frac{\partial V}{\partial t} = -rt^{-r-1}s^{-r} \qquad \frac{\partial U}{\partial s} = \int_{0}^{t} |\varphi(s,t)|^{r}\, dt \equiv \Phi_{r}(t)$$

$$\frac{\partial V}{\partial s} = -rs^{-r-1}t^{-r} \qquad \frac{\partial U}{\partial t} = \int_{0}^{s} |\varphi(s,t)|^{r}\, ds \equiv \Phi_{r}(s)$$

$$\frac{\partial^2 V}{\partial s\,\partial t} = r^2 s^{-r-1}\,t^{-r-1} \qquad U = \int_{0}^{s}\int_{0}^{t} |\varphi(s,t)|^{r}\, ds\, dt$$

$$\equiv \Phi_{r}(s,t)$$

We have

$$(3.1.11)\quad \int_{\frac{1}{m}}^{\pi}\int_{\frac{1}{n}}^{\pi} \frac{|\varphi(s,t)|^{r}}{(st)^{r}}\, ds\, dt =$$

$$= \frac{1}{(st)^r} \Phi_r(s,t) \Bigg|_{(\frac{1}{m}, \frac{1}{n})}^{(\pi,\pi)} + r^2 \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_r(s,t)}{(st)^{r\ 1}} \, ds \, dt +$$

$$+ r \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_r(t)}{s^{r+1} t^r} \, ds \, dt + r \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_r(s)}{s^r \, t^{r+1}} \, ds \, dt$$

Considering each part separately, and applying (3.1.5), (3.1.0) and (3.1.1)

$$\frac{1}{(st)^r} \Phi_r(s,t) \Bigg|_{(\frac{1}{m},\frac{1}{n})}^{(\pi,\pi)} = m^r n^r \; o\left(\frac{1}{m} \frac{1}{n}\right) = o(m^{r-1} n^{r-1})$$

$$r^2 \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_r(s,t)}{(st)^{r\ 1}} \, ds \, dt = r^2 \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} o(s^{-r} t^{-r}) \, ds \, dt = o(m^{r-1} n^{r-1})$$

$$r^3 \int_{\frac{1}{n}}^{\pi} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_r(t)}{s^{r+1} t^r} \, ds \, dt =$$

$$= r \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \frac{o(t) \, ds \, dt}{s^{r+1} t^r} = r \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} o(s^{-1-r} t^{1-r}) \, ds \, dt$$

$$= o(m^r n^{r-2}) = \frac{m}{n} \, o(m^{r-1} n^{r-1}) = o(m^{r-1} n^{r-1}) \text{ as } \frac{m}{n} < C$$

And similarly

$$r \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_r(s)}{s^r \, t^{r+1}} \, ds \, dt = o(m^{r-1} n^{r-1})$$

Collecting these estimates for (3.1.11) and substituting into (3.1.10) we get

$$(3.1.12) \quad \left[\frac{1}{m+1}\,\frac{1}{n+1} \sum_{o}^{m,n} |d_{kl}|^K\right]^{\frac{1}{K}} \leq \left(\frac{1}{m+1}\,\frac{1}{n+1}\right)^{\frac{1}{K}} \cdot$$

$$\cdot\, o\left(m^{\frac{r-1}{r}}\, n^{\frac{r-1}{r}}\right) = o(1)$$

$$\text{as } K = r' = \frac{r}{r-1}$$

We find the estimates (3.1.7)(3.1.8)(3.1.9) and (3.1.12) give us our desired result:

$$\left[\frac{1}{m+1}\;\frac{1}{n+1} \sum_{o}^{m,n} \left| S^{*}_{kl} - f \right|^K\right]^{\frac{1}{K}}_{m,n \to \infty} \longrightarrow 0.$$

E. Strong Summability of Orthogonal Developments.

Definition: Corresponding to the definitions of summability for a single series we have the following definitions (Robison, (29)) for giving a value to a divergent double series. Let the given series be represented as follows:

$$\sum_{p,q=1} u_{pq}$$

then the double sequence S_{mn} for this series is given by

$$S_{mn} = \sum_{p,q=1}^{m,n} u_{pq}$$

We define a new sequence by the relation

$$T_{mn} = \sum_{p,q=1}^{m,n} a_{mnpq} \; S_{pq}$$

We shall call this transformation and its matrix A: (a_{mnpq}). a triangular type transformation. Here $p \leq m$, $q \leq n$.

It is a well known fact that if a simple series converges the corresponding sequence is bounded. This is not true for a double series. For example, consider the double series defined by

$$a_{1n} = 1$$
$$a_{2n} = -1$$
$$a_{mn} = o, m > 2$$

This series converges to zero and yet the partial sums S_{mu} are not bounded. Thus, convergent double series can be divided into two classes according to whether the corresponding partial sequences are bounded or not. We will now define regularity of a transformation with regard to a convergent bounded double sequence. Hence, even if a transformation is regular it need not give to an unbounded convergent double sequence the value to which it converges.

REGULARITY

If, whenever S_{mn} is a bounded convergent double sequence, T_{mn} converges to the same value, then the transformation is daid to be boundedly regular.

Robison (29) has proved the following theorem:

Lemma 3.2. A necessary and sufficient condition that any triangular type transformation T be boundedly regular is:

(a) $\lim_{m,n\to\infty} a_{mnpq} = 0$ for each p and q.

(b) $\lim_{m,n\to\infty} \sum_{p,q=1}^{m,n} a_{mnpq} = 1$

(c) $\lim_{m,n\to\infty} \sum_{p=1}^{m} \left| a_{mnpq} \right| = 0$ for each q.

(d) $\lim_{m,n\to\infty} \sum_{q=1}^{n} \left| a_{mnpq} \right| = 0$ for each p.

(e) $\sum_{p,q=1}^{m,n} \left| a_{mnpq} \right| \leq A$, A constant.

DEFINITION: R Summability.

The series $\sum_{p,q=1}^{\infty} u_{pq}$ is R Summable if

$$R_{mn} = \sum_{p,q=1}^{m,n} \left(1 - \frac{pq}{mn} \right) u_{pq} \xrightarrow[m,n\to\infty]{} s$$

Lemma 3.3 R summability is a regular triangular type double summability method.

PROOF :

Applying the Abel transformation for double series to R_{mn} we get

$$R_{mn} = \sum_{p,q=1}^{m,n} (1 - \frac{pq}{mn}) u_{pq}$$

$$= \sum_{p,q=1}^{m,n} (1 - \frac{pq}{mn})(s_{pq} - s_{p-1,q} - s_{p,q-1} + s_{p-1,q-1})$$

$$= \sum_{p,q=1}^{m,n} (1 - \frac{pq}{mn}) s_{pq} - \sum_{p,q=1}^{m-1,n} \left(1 - \frac{(p+1)(q)}{m\,n}\right) s_{pq}$$

$$- \sum_{p,q=1}^{m,n-1} \left(1 - \frac{p(q+1)}{m\,n}\right) s_{pq} + \sum_{p,q=1}^{m-1,n-1} \left(1 - \frac{(p+1)(q+1)}{m\,n}\right) s_{pq}$$

Thus

$$R_{mn} = \frac{1}{m} \sum_{p=1}^{m-1} s_{mn} + \frac{1}{n} \sum_{q=1}^{n-1} s_{mq} - \frac{1}{mn} \sum_{p,q=1}^{m-1,n-1} s_{pq} \tag{3.2.0}$$

$$= \sum_{p,q=1}^{m,n} a_{mnpq} \quad S_{pq}$$

where the a_{mnpq} are defined by

$$a_{mnpq} = \begin{cases} -\frac{1}{mn} & p < m, \quad q < n \\ \frac{1}{m} & p < m, \quad q = n \\ \frac{1}{n} & p = m, \quad q < n \\ o & \text{otherwise} \end{cases}$$

But these are, by definition, the coefficients of a double triangular type transformation. The transformation is also regular for, as one can readily verify, the coefficients a_{mnpq} satisfy the regularity conditions (Lemma 3.3).

Lemma 3.4 If the double series $\sum_{p,q=1}^{\infty} u_{pq}$ is boundedly convergent then

$$\sum_{p,q=1}^{m,n} pq u_{pq} = o\ (m,n)$$

PROOF:

Since R_{mn} is regular we have

$$\left| S_{mn} - R_{mn} \right| = \left| \sum_{p,q=1}^{m,n} u_{pq} - \sum_{p,q=1}^{m,n} \left(1 - \frac{pq}{mn} \right) u_{pq} \right| \to o$$

$$= \sum_{p,q=1}^{m,n} \frac{pq}{mn} u_{pq} \xrightarrow[m,n\to\infty]{} o$$

$$\text{i.e} \quad \sum_{p,q=1}^{mn} p\, q\, u_{pq} = o\,(mn)$$

Having defined a double summability method, R, it is natural to attempt to compare it with other summability methods. For example, let us consider the double Cesaro summability method of order one and attempt a comparison, i.e. is R summability more or less general than (C,1) summability, or do the methods overlap? We gain some insight into this problem but do not succeed in providing a complete description of their comparison.

We define a <u>product double series</u> to be a double series which is a product of two simple series. We denote its partial sum by

$$S_{mn} = S_m\, S'_n = \sum_{1}^{m,n} a_p\, b_q = \sum_{1}^{m} a_p \cdot \sum_{1}^{n} b_q$$

By definition, the Cesaro means of the first order of the product double series $\sum a_p\, b_q$ are:

$$\sigma_{mn} = \frac{1}{mn} \sum_{p,q=1}^{m,n} S_p \quad S'_q$$

$$= \left(\frac{1}{m} \sum_{p=1}^{m} S_p\right) \cdot \left(\frac{1}{n} \sum_{q=1}^{m} S'_q\right) = \sigma_m\, \sigma'_n$$

$$= \frac{1}{mn}\left(\sum_{1}^{m-1} s_p + s_m\right)\left(\sum_{1}^{n-1} s'_q + s'_m\right)$$

Hence,

$$(3.2.1)\quad \sigma_{mn} = \frac{1}{mn}\left[\sum_{1}^{m-1,n-1} s_p s'_q + s_m \sum_{1}^{m-1} s'_q + s'_n \sum_{1}^{m-1} s_p + s_m \quad s'_n\right]$$

From 3.2.0) we have for the R means:

$$(3.2.2)\quad R_{mn} = \frac{1}{mn}\left[-\sum_{1}^{m-1,n-1} s_p s'_q + m s_m \sum_{1}^{n-1} s'_q \quad n s'_n \sum_{1}^{m-1} s_p\right]$$

Subtracting (3.2.2) from 3.2.1) we get

$$(3.2.3)\quad \sigma_{mn} - R_{mn} = \frac{(m-1)(n-1)}{mn}\left[2\sigma_{m-1}\sigma'_{n-1} - s_m \sigma'_{n-1} - s'_n \sigma_{m-1}\right] + \frac{s_m s'_n}{mn}$$

Which becomes, after substituting

$$\Delta\sigma_m = \sigma_m - \sigma_{m-1} \qquad , \quad \Delta\sigma'_n = \sigma'_n - \sigma'_{n-1}$$

$$s_m = m\Delta\sigma_m + \sigma_{m-1} \quad , \quad s'_n = n\Delta\sigma'_n + \sigma'_{n-1}$$

$$(3.2.4)\quad \sigma_{mn} - R_{mn} = -\frac{(m-1)(n-1)}{mn}\left[\sigma_{m-1}\Delta\sigma'_n + \sigma'_{m-1}\Delta\sigma_m\right] + \frac{s_m s'_n}{mn}$$

It is well known that a necessary and sufficient condition for (C,1) summability of a single series is that $S_m = o\,(m)$. Hence, if the double product series is (C,1) summable then $S_m = o(m)$, $S'_n = o(n)$. From (3.2.3) we now observe that if, in addition to the assumption of (C,1) summability to zero, the partial sums are bounded then all of the terms on the right hand side of (3.2.3) become zero as $m,n \to \infty$ and we have proved Theorem (3.2.0).

If a product double series with bounded partial sums is (C,1) summable to zero, then the series is also R summable to zero.

From (3.2.4) we observe that the assumptions of (C,1) summability plus the additional conditions that

$$\Delta \sigma_m = o\left(\frac{1}{m}\right), \quad \Delta \sigma'_n = o\left(\frac{1}{n}\right)$$

imply restricted R summability. We can only conclude restricted R summability under these assumptions for, applying our assumptions to (3.2.4) we have

$$\sigma_{mn} - R_{mn} = -\frac{(m-1)(n-1)}{mn}\sigma_{m-1}\, o\left(\frac{1}{n}\right) - \frac{(m-1)(n-1)}{m\,n}\sigma'_{n-1}\, o\left(\frac{1}{m}\right) + o(1)$$

$$= -\left(\frac{m-1}{n}\right) o(1)\, \sigma_{m-1} - \left(\frac{n-1}{m}\right) o(1)\, \sigma'_{n-1} + o(1)$$

$\longrightarrow$ o provided $m,n \to \infty$ "together", i.e. in such a manner that

$$c\,n < m < c^{-1}\,n.$$

We have thus proved the theorem:

Theorem (3.2.1). If the double product series is (C,1) summable and if $\Delta \sigma_m = o\left(\frac{1}{m}\right)$, $\Delta \sigma'_n = o\left(\frac{1}{n}\right)$ then the series is restrictedly summable R.

It may appear from this discussion that we need more than (C,1) summability to get R. summability, i.e. that (C,1) summability is the more general method. That this is not the case, at least with respect to general double series is illustrated by the following example of a divergent double series which is R summable but not (C,1) summable. Consider the series whose unbounded partial sums are defined as:

$$S_{mn} = 0 \quad , \quad m \neq n$$

$$S_{nn} = \log \log n \ , \quad n > 1$$

We have

$$R_{mn} = \frac{1}{m} \sum_{p=1}^{m-1} S_{pn} + \frac{1}{n} \sum_{q=1}^{n-1} S_{mq} - \frac{1}{mn} \sum_{p,q=1}^{m-1,n-1} S_{pq} \ ,$$

hence, for $m > n$,

$$R_{mn} = \frac{\log \log n}{m} - \frac{1}{mn} \sum_{q=2}^{n-1} \log \log q$$

$$(3.2.0) \qquad = \frac{1}{mn} \left(n \log \log n - \sum_{q=2}^{n-1} \log \log q\right)$$

We now estimate $\sum_{q=2}^{n-1} \log \log q$

We have

$$0 < - \int_{q-1}^{q} \log \log x \, dx + \log \log q$$

$$= \int_{q-1}^{q} \log \frac{\log q}{\log x} \, dx < \log \frac{\log q}{\log(q-1)}$$

$$= \frac{\log\left(1+\frac{1}{q-1}\right)}{\log(q-1)} < \frac{1}{(q-1)\log(q-1)} = O\left(\frac{1}{q\log q}\right)$$

Now, summing over q from 3 to n we get

$$0 < -\int_2^n \log\log x\,dx + \sum_3^n \log\log p = O\left(\sum_3^n \frac{1}{p\log p}\right) = O(\log n)$$

and, integrating the first term by parts

$$\int_2^n \log\log x\,dx = x\log\log x\Big|_2^n - \int_2^n \frac{dx}{\log x}$$

$$= n\log\log n - 2\log\log 2 - \int_2^n \frac{dx}{\log x}$$

Estimating the last term, we get

$$\int_2^n \frac{dx}{\log x} = \left[\int_2^{\sqrt{n}} + \int_{\sqrt{n}}^n\right]\frac{dx}{\log x} = I_1 + I_2$$

$$I_1 < \frac{\sqrt{n}}{\log 2} = o(n)$$

$$I_2 < \frac{n}{\log\sqrt{n}} = o(n)$$

Hence

$$\sum_3^n \log\log p - n\log\log n = o(n)$$

(3.2.1) $$\sum_3^n \log\log p = n\log\log n + o(n)$$

Substituting this into (3.2.0) we have

$$R_{mn} = \frac{1}{mn}\left(n \log\log n - n \log\log n + o(n)\right)$$

$$= \frac{o(n)}{mn} \to o \quad m,n \to \infty$$

We thus have R summability.

The series is not (C,1) summable for

$$\sigma_{mn} = \frac{1}{mn} \sum_{p,q=2}^{m,n} \log\log q$$

$$= \frac{1}{m} \sum_{q=2}^{n} \log\log q \underset{m,n\to\infty}{\longrightarrow} \infty$$

Hence, R. and (C,1) summabilities are overlapping methods. To what extent they overlap is an extremely interesting but open question. Are there double series whose partial sums are bounded that are R. summable but not (C,1) summable? We intend to consider this method much more fully at some future date. Our interest in R. summability is centered in this paper in the following theorem:

<u>Theorem (3.22)</u>

If $\sum_{p,q=1}^{\infty} c_{pq}^2 < \infty$ and if the series $\sum_{p,q=1}^{\infty} c_{pq} \; \psi_{pq}(x,y)$ ( ) with partial sums $S_{mn}(x,y)$ is summable R in a set E, $|E| > o$, to a function $S(x,y)$, where $\{\psi_{pq}(x,y)\}$ is an orthonormal system, then

$$(3.2.2) \quad \frac{1}{mn} \sum_{p,q=1}^{m,n} \left| S_{pq}(x,y) - S(x,y) \right| \underset{m,n\to\infty}{\longrightarrow} 0$$

for almost every $(x,y) \in E$.

PROOF:

We shall need the following lemma.

Lemma 3.6

If $\sum_{p,q=1} C^2_{pq} < \infty$, the series

$$(3.2.3) \quad \sum_{p,q=1} \frac{\left[S_{pq}(x,y) - R_{pq}(x,y) \right]^2}{pq}$$

converges for almost every (x,y) in the interval $(0,1)$.

PROOF:

Since all the terms in the series (3.2.2)are positive it is sufficient to show convergence of the sum of the integrals of the terms of the series (Lemma 1.1), i.e. we must show

$$(3.2.4) \quad \sum_{r,s=1} \iint_E \frac{\left[S_{rs}(x,y) - R_{rs}(x,y) \right]^2}{rs} \, dx \, dy < \infty$$

We have

$$S_{mn} - R_{mn} = \sum_{r,s=1}^{m,n} \frac{rs}{mn} A_{rs}(x,y)$$

where $\quad A_{rs}(x,y) = C_{rs} \psi_{rs}(x,y)$

Substituting into (3.2.1) we get

$$\sum_{r,s=1} \iint_E \frac{\left[S_{rs}(x,y) - R_{rs}(x,y) \right]^2}{rs} \, dx \, dy$$

$$= \sum_{r,s=1}^{\infty} \frac{1}{rs} \iint \left[\sum_{p,q=1}^{rs} \frac{pq}{rs} \quad A_{pq} \right]^2 \quad dx\,dy$$

$$= \sum_{r,s=1}^{\infty} \quad \frac{1}{rs} \quad \sum_{p,q=1}^{r,s} \left(\frac{p2q2}{r^2s^2} \qquad C^2_{pq} \right)$$

because of the orthogonality property of the $_{rs}(x,y)$.

$$= \sum_{p,q=1}^{\infty} p^2q^2 \, C^2_{pq} \quad \sum_{r,s=p,q}^{\infty} \quad \frac{1}{r^3s^3}$$

$$\leq \sum_{p,q=1}^{\infty} \quad p^2 \, q^2 \quad C^2_{pq} \quad \frac{1}{p^2 \, q^2} \quad = \sum_{p,q=1}^{\infty} \quad C^2_{pq} < \infty \quad \text{by hyp.}$$

Hence

$$(3.2.5) \qquad \sum_{r,s=1}^{\infty} \frac{\left[S_{rs}(x,y) - R_{rs}(x,y)\right]^2}{rs} < \infty \quad \text{p.p. in E.}$$

Applying (lemma 3.4) to (3.2.5) we get

$$(3.2.6) \qquad \sum_{r,s=1}^{m,n} \left[S_{rs}(x,y) - R_{rs}(x,y)\right]^2 \quad = o\ (mn)$$

for almost every $(x,y) \in E$.

We now can prove our theorem easily. We set

$$S_{rs}(x,y) - S(x,y) \quad = \quad (S_{rs} - R_{rs}) + (R_{rs} - S)$$

$$\frac{1}{mn}\sum_{r,s=1}^{m,n}\left|S_{rs}(x,y)-S(x,y)\right|^2 \leq \frac{1}{mn}\sum_{r,s=1}^{m,n}\left|(S_{rs}-R_{rs})+(R_{rs}-S)\right|^2$$

and, applying <u>Minkowski's inequality</u> (Lemma 1.4)

$$\left[\frac{1}{mn}\sum_{r,s=1}^{m,n}\left|S_{rs}(x,y)-S(x,y)\right|^2\right]^{\frac{1}{2}} \leq \left[\frac{1}{mn}\sum_{r,s=1}^{m,n}(S_{rs}-R_{rs})^2\right]^{\frac{1}{2}}+\left[\frac{1}{mn}\sum_{r,s=1}^{m,n}\left|R_{rs}-S\right|^2\right]^{\frac{1}{2}}$$

The first term is o(1) by (3.2.6) and the second term is o(1) by hypothesis.

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