INTEGRAL SOLUTIONS OF THE EQUATION

$$\xi^2 + \eta^2 + \dots = \zeta^2 + \tau^2 + \dots$$

IN THE QUADRATIC REALM OF RATIONALITY

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C O N T E N T S.

INTRODUCTION.

The following Thesis offers a generalized treatment of the extended forms of the so called Pythagorean numbers and offers proof of the existence of integral solutions of such numbers in the general quadratic realm of algebraic numbers. Investigation has been made by Dr.Harris Hancock* regarding the solutions of the Diophantine equation $\xi^2 + \eta^2 = \zeta^2$. It will be shown in the following that in the quadratic realm of rationality there exist integral solutions of all forms of quadratic Diophantine equations in any arbitrary number of unknown quantities.

An algebraic integer in the quadratic realm of rationality is defined as the root of an irreducible quadratic equation of the form

$$(1) \qquad t^2 + At + B = 0,$$

where A and B are rational integers and where the coefficient of t is unity. The following investigations should therefore be based upon an equation of form (1). It will be seen however by inspection that the following two forms of the quadratic, viz:

$$(2a) \quad x^2 - a_1x - a_2 = 0 \qquad \text{with the roots} \qquad \xi_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

and

$$(2b) \quad x^2 - 2a_1x - a_2 = 0 \qquad \text{with the roots} \qquad \xi_{1,2} = a_1 \pm \sqrt{a_1^2 + a_2}$$

*Integral Solutions of the Equation in Lionville's Journal page 327 ff.

lead to the same results, and these results will not be effected in any way whether we use the positive or the negative sign of the root. For brevity we shall therefore base our investigations upon quadratic equations of form (2b) and use the positive sign of their roots.

Before attempting to prove the existence of algebraic integral solutions of Diophantine equations in the quadratic realm of rationality of the kind defined above it will be of interest to first examine some of the integral solutions of Diophantine equations in the realm of rational numbers.

I.° To solve the equation $\underline{x^2+y^2=z^2}$, put with Kronecker *

$$x = 2pqt; \quad y = (p^2-q^2)t; \quad z = (p^2+q^2)t;$$

where p and q are any two integers relatively prime to one another, $p > q > 0$, and t any integer > 0; e.g:

take $p=5, q=3, t=1$; it follows $x=30; y=16; z=34$, and $30^2+16^2=34^2$

or $p=7, q=4, t=1$; it follows $x=56; y=33; z=65$, and $56^2+33^2=65^2$, a.s.o.

II.° Applying the previous method to the form

$\underline{x^2+y^2 = z^2+v^2 = w^2}$ by putting

$$x = 2pqt; \quad y = (p^2-q^2)t; \quad w = (p^2+q^2)t;$$
$$z = 2mnt; \quad v = (m^2-n^2)t; \quad w = (m^2+n^2)t;$$

where $p > q > 0$; $m > n > 0$; $t > 0$;

we find the equation $p^2+q^2 = m^2+n^2$; e.g:

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* See Dickson: History of the Theory of Numbers

take $p = 8$; $q = 1$; $m = 7$; $n = 4$; $t = 1$; it follows:

$x = 32$; $y = 126$; $z = 112$; $v = 66$ and $32^2 + 126^2 = 112^2 + 66^2 = 16900$.

Another method of determining numbers of the form $x^2 + y^2 = z^2 + v^2$ is given by Diophantos (book III,22) where he shows how to obtain four right triangles with equal hypothenuses; e.g:

Let (3, 4, 5,) and (8, 15, 17,) be the sides of two right triangles Multiplying the first group by 17 and the second group by 5 we find two triangles with equal hypothenuses, viz:

(51, 68, 85) and (40, 75, 85):

and since $85 = 5.17 = 9^2 + 2^2 = 7^2 + 6^2$,

we have $85^2 = (9^2 - 2^2) + 4.9^2.2^2 = 77^2 + 72^2$

and $85^2 = (7^2 - 6^2)\ 4.7^2.6^2 = 13^2 + 168^2$.

Hence $77^2 + 72^2 = 13^2 + 168^2 = 51^2 + 68^2 = 40^2 + 75^2 = 85$.

Numbers which satisfy the equation $p^2 + q^2 = m^2 + n^2$ are easily procured by trial (with a table of squares); for every prime number of the form $p = 4n+1$-or more exactly (according to Gauss' theorem *):

every prime number of the form

$$a = 2^{e}.Q.p_1^{e_1}.p_2^{e_2}.\dots.p_\nu^{e_\nu}$$

where $p_1, p_2, \dots, p_\nu$ are factors of the form $4n+1$ with any arbitrary exponents $e_1, e_2, \dots, e_\nu$ and where Q is a product of factors of the form $4n+3$ with even exponents, may be represented as the sum of two squares in one or more ways.

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* cf. Disquis. arith. V, 182, note.

Sommer: Vorlesungen über Zahlentheorie pag. 124.

If the product $k = (e_1+1)\cdot(e_2+1)\ldots\ldots(e_\nu+1)$ is an even number there are $k/2$ ways of solutions; if k is odd the number of solutions is $(k-1)/2$. - For example the integer $1105 = 5.13.17$ can be decomposed in $1/2(1+1)(1+1)(1+1) = 4$ different ways, viz: see Example (21) pag 69.

$$33^2+4^2 = 32^2+9^2 = 31^2+12^2 = 24^2+23^2.$$

But the integer $4225 = 5^2.13^2$ can be decomposed in $(k-1)/2$ ways, where $k = (2+1)(2+1) = 9$, i.e. in 4 ways, viz: $\begin{cases} 63^2+16^2 = 36^2+25^2 = \\ 56^2+33^2 = 52^2+39^2. \end{cases}$

III. The solutions of an equation of the form

$$\underline{x_1^2+x_2^2+x_3^2+\ldots\ldots+x_n^2 = v^2}$$

the sum of the left hand side containing any number of squares, can always be determined. One method is this :*

Let $x = -a_1^2+a_2^2+a_3^2+\ldots.+a_n^2$, where $(a_1^2+a_2^2+\ldots.+a_n^2) = v$,

and $x = 2a_1a_r\,(r = 2, 3, 4, \ldots.n)$;

then $\sum x_i^2\ (i = 1, 2, 3, \ldots.n)$ will satisfy v^2;

hence we have to assign any arbitrary values to the a's and calculate the $x_1, x_2, x_3, \ldots. x_n$. - e.g:

To solve $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = v^2$

let $a_1 = 2,\ a_2 = 5,\ a_3 = 1,\ a_4 = 4,\ a_5 = 6$; it follows

$x_1 = -4\quad 5\quad 1\quad 4\quad 6 = 74$

$x_2 = 2.2.5 = 20$; $x_3 = 2.2.1 = 4$; $x_4 = 2.2.4 = 16$; $x_5 = 2.2.6 = 24$;

and it is

$$74^2+20^2+4^2+16^2+24^2 = 82^2\ [= v^2 = (2^2+5^2+1^2+4^2+6^2)^2]$$

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*See: Dickson: History of the Theory of Numbers, II, 318.

Note that this method may also successfully be applied in order to find numbers whose squares may be decomposed into the sum of three (or more) squares in one (or more) ways. e.g:

$$1^2+3^2+8^2 = 3^2+4^2+7^2 = 74 = v.$$

Using the previous method we find for

$$a_1 = 1,\ a_2 = 3,\ a_3 = 8: \qquad x_1 = 72,\ x_2 = 6,\ x_3 = 16$$

and for

$$a_1 = 3,\ a_2 = 4,\ a_3 = 7: \qquad x_1 = 56, x_2 = 24,\ x_3 = 42,$$

and it follows

$$v^2 = 72^2+6^2+16^2 = 56^2+24^2+42^2 = 74^2.$$

Again, take

$$1^2+2^2+4^2+13^2 = 1^2+3^2+6^2+12^2 = 4^2+5^2+7^2+10^2 = 190 = v.$$

Here we obtain for

$$a_1 = 1,\ a_2 = 2,\ a_3 = 4, a_4 = 13: \quad x_1 = 188,\ x_2 = 4,\ x_3 = 8,\ x_4 = 26;$$

$$a_1 = 1,\ a_2 = 3,\ a_3 = 6, a_4 = 12: \quad x_1 = 188,\ x_2 = 6,\ x_3 = 12, x_4 = 24;$$

$$a_1 = 4,\ a_2 = 5,\ a_3 = 7, a_4 = 10: \quad x_1 = 158,\ x_2 = 40, x_3 = 56, x_4 = 80;$$

and it follows

$$v^2 = 188^2+4^2+8^2+26^2 =$$
$$= 188^2+6^2+12^2+24^2 =$$
$$= 158^2+40^2+56^2+80^2 = 190^2.$$

An other method of solving the previos form would be:

To solve: $x^2+y^2+z^2 = v^2$, write $x^2+y^2 = w^2$; then: $w^2+z^2 = v^2$;

Let $\qquad x = 2pqt$ and $\qquad w = 2mnt$;

$$y = (p^2-q^2)t; \qquad z = (m^2-n^2)t;$$

$$w = (p^2+q^2)t; \qquad v = (m^2+n^2)t;$$

it follows that $p^2 + q^2 = 2mn$, where $p > q$; $m > n$; $t > 0$ and p and q both even or both odd.- e.g:

Take $p = 6$; $q = 2$; $m = 5$; $n = 4$; $t = 2$;

it follows that $x = 48$; $y = 64$; $z = 18$; $v = 82$;

and it is seen that $48^2 + 64^2 + 18^2 = 82^2 = 6724$.

IV. Both the preceeding methods may be applied to the form $\underline{x^2 + y^2 + z^2 = v^2 + u^2}$.- By writing $v^2 + u^2 = w^2$ the form reduces to that of III and I.- Again by writing $x^2 + y^2 = w_1^2$ and $v^2 + u^2 = w_2^2$ we have the relations

$$\left.\begin{aligned} p^2 + q^2 &= 2kl \\ m^2 + n^2 &= k^2 + l^2 \end{aligned}\right\} \text{where} \quad \begin{aligned} &x = 2pqt; \quad z = (k^2 - l^2)t; \quad u = (m^2 - n^2)t; \\ &y = (p^2 - q^2)t; \quad v = 2mnt; \end{aligned}$$

e.g: take $m = 11$; $n = 8$; $k = 13$; $l = 4$; $p = 10$; $q = 2$; $t = 1$;

it follows that

$$x = 40; \quad y = 96; \quad z = 153; \quad v = 176; \quad u = 57;$$

and that is

$$40^2 + 96^2 + 153^2 = 176^2 + 57^2 = 185^2 = 34225.$$

V. We may finally consider the forms

$$\underline{x^2 + y^2 + z^2 + v^2 = r^2 + s^2} \qquad \text{and} \qquad \underline{x^2 + y^2 + z^2 = v^2 + r^2 + s^2},$$

which may be solved by putting

$$\begin{aligned} x^2 + y^2 &= r^2 \\ z^2 + v^2 &= s^2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x^2 + y^2 &= v^2 \\ r^2 + s^2 &= z^2. \end{aligned}$$

For example take $x = 5$; $y = 12$; $r = 13$; $z = 40$; $v = 9$; $s = 41$;

it follows that $5^2 + 12^2 + 40^2 + 9^2 = 13^2 + 41^2 = 1850$.

Again take $x = 33$; $y = 56$; $v = 65$; $r = 3$; $s = 4$; $z = 5$;

it follows that $33^2 + 56^2 + 5^2 = 65^2 + 3^2 + 4^2 = 4250$;

We shall now proceed to prove the existence of integral solutions of equations of the forms $\xi^2+\eta^2+\cdots = \zeta^2+\tau^2+\cdots$ in the quadratic realm of rationality. We shall first treat the following seven cases:

1. $\xi^2+\eta^2=\zeta^2+\tau^2$
2. $\xi^2+\eta^2+\zeta^2=\tau^2$
3. $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2$
4. $\xi^2+\eta^2+\zeta^2=\tau^2+\rho^2$
5. $\xi^2+\eta^2+\zeta^2+\tau^2+\rho^2=\sigma^2$
6. $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2+\sigma^2$
7. $\xi^2+\eta^2+\zeta^2=\tau^2+\rho^2+\sigma^2$

The general existence of such algebraic integers will first be demonstrated by three different methods and the results obtained thereby will then be applied to the general case in the universal quadratic realm of rationality.

I. – First Solution of the Equation $\xi^2+\eta^2+\dots = \zeta^2+\tau^2+\dots$

1° Case: $\xi^2+\eta^2 = \zeta^2+\tau^2$

Let
$$x^2-2a_1x-a_2=0 \qquad z^2-2c_1z-c_2=0$$
$$y^2-2b_1y-b_2=0 \qquad t^2-2d_1t-d_2=0$$
be four equations of the second degree and let

(i)
$$\xi = a_1+\sqrt{a_1^2+a_2} \qquad \zeta = c_1+\sqrt{c_1^2+c_2}$$
$$\eta = b_1+\sqrt{b_1^2+b_2} \qquad \tau = d_1+\sqrt{d_1^2+d_2}$$

be their respective roots, where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are rational integers such that consequently ξ, η, ζ, τ represent four algebraic integers.

If then $\xi^2+\eta^2=\zeta^2+\tau^2$, we have the two relations:

(1) $$2a_1\xi+a_2+2b_1\eta+b_2 = 2c_1\zeta+c_2+2d_1\tau+d_2$$

and hence

(1') $$2a_1^2+a_2+2a_1\sqrt{a_1^2+a_2}+2b_1^2+b_2+2b_1\sqrt{b_1^2+b_2} =$$
$$=2c_1^2+c_2+2c_1\sqrt{c_1^2+c_2}+2d_1^2+d_2+2d_1\sqrt{d_1^2+d_2}$$

From (1') we derive two equations by equating rational and irrational terms:

(2) $$2a_1^2+a_2+2b_1^2+b_2 = 2c_1^2+c_2+2d_1^2+d_2$$

and

(3) $$a_1\sqrt{a_1^2+a_2}+b_1\sqrt{b_1^2+b_2} = c_1\sqrt{c_1^2+c_2}+d_1\sqrt{d_1^2+d_2}$$

To solve these two equations, let

(4) $\sqrt{b_1^2+b_2} = m\sqrt{a_1^2+a_2}$ and $\sqrt{d_1^2+d_2} = n\sqrt{c_1^2+c_2}$,

where m and n are any (positive or negative) rational integers.

It follows from (3) that

(5) $(a_1+b_1m)\sqrt{a_1^2+a_2} = (c_1+d_1n)\sqrt{c_1^2+c_2}$,

from which it is seen that

(6') $a_1+b_1m = c_1+d_1n$, or $c_1 = a_1+b_1m-d_1n$

and

(6'') $\sqrt{a_1^2+a_2} = \sqrt{c_1^2+c_2}$

Substituting into (6'') the value for c_1 from (6'), we have

(6''') $\sqrt{a_1^2+a_2} = \sqrt{(a_1+b_1m-d_1n)^2+c_2}$

$$= \sqrt{a_1^2+b_1^2m^2+d_1^2n^2+2a_1b_1m-2a_1d_1n-2b_1d_1mn+c_2}$$

If this relation shall hold it must be

(7) $a_2 = b_1^2m^2+d_1^2n^2+2a_1b_1m-2a_1d_1n-2b_1d_1mn+c_2$.

In order to simplify this expression we obtain from (2) and (6'):

(8) $a_2+2b_1^2+b_2 = 2b_1^2m^2+2d_1^2n^2+4a_1b_1m-4a_1d_1n-4b_1d_1mn+c_2+2d_1^2+d_2$

and from (7) and (8) by ~~equating~~ forming (a_2-c_2) and substracting:

(9) $a_2-c_2 = 2b_1^2+b_2-2d_1^2-d_2$,

and by equating the expressions for a_2

(10) $b_1^2m^2+d_1^2n+2a_1b_1m-2a_1d_1n-2b_1d_1mn = 2b_1^2+b_2-2d_1^2-d_2$.

This equation in m and n may serve for solving the problem in question; there are an infinite number of solutions.

Make the left hand side a perfect square by putting

(i) $\quad 2a_1 = 3b_1 m - 3d_1 n$.

It follows

(10') $\quad (2b_1 m - 2d_1 n)^2 = 2b_1^2 + b_2 - 2d_1^2 - d_2$

Solutions will now be obtained by giving to each m and n the values $1, 2, 3, 4, \ldots$

Example (1):

Let $\quad b_1 = 2, \; d_1 = 3; \quad m = 1, \; n = 2$.

We have

from (i) $\quad a_1 = \frac{1}{2}(3b_1 m - 3d_1 n) = -6$

from (6') $\quad c_1 = a_1 + b_1 m - d_1 n = -10$

from (10') $\quad b_2 - d_2 = 74$

from (4), (6") $\quad a_2 = -59$

from (4) $\quad b_2 = -27; \; d_2 = -101; \; c_2 = -123$

Hence the roots of the four quadratics

$$x^2 + 12x + 59 = 0 \qquad z^2 + 20z + 123 = 0$$

$$y^2 - 4y + 27 = 0 \qquad t^2 - 6t + 101 = 0$$

are respectively

$$\xi = -6 + \sqrt{-23} \qquad \zeta = -10 + \sqrt{-23}$$

$$\eta = 2 + \sqrt{-23} \qquad \tau = 3 + 2\sqrt{-23}$$

and it is seen that

$$\xi^2 + \eta^2 = -6 - 8\sqrt{-23} = \zeta^2 + \tau^2$$

2° Case: $\xi^2+\eta^2+\zeta^2=\tau^2$

The original equations and their respect. roots being the same as in the preceding case we have corresponding to the equations (1), (2) and (3) the three relations:

$$(11)\quad 2a_1^2+a_2+2a_1\sqrt{a_1^2+a_2}+2b_1^2+b_2+2b_1\sqrt{b_1^2+b_2}+2c_1^2+c_2+2c_1\sqrt{c_1^2+c_2}=2d_1^2+d_2+2d_1\sqrt{d_1^2+d_2}$$

$$(12)\quad 2a_1^2+a_2+2b_1^2+b_2+2c_1^2+c_2=2d_1^2+d_2$$

$$(13)\quad a_1\sqrt{a_1^2+a_2}+b_1\sqrt{b_1^2+b_2}+c_1\sqrt{c_1^2+c_2}=d_1\sqrt{d_1^2+d_2}$$

Let

$$(14)\quad \sqrt{b_1^2+b_2}=m\sqrt{a_1^2+a_2} \quad \text{and} \quad \sqrt{c_1^2+c_2}=n\sqrt{d_1^2+d_2},$$

where m and n are two rational integers.

It follows from (12)

$$(15)\quad (a_1+b_1m)\sqrt{a_1^2+a_2}=(d_1-c_1n)\sqrt{d_1^2+d_2},$$

whence

$$(16')\quad a_1+b_1m=d_1-c_1n \quad \text{or} \quad d_1=a_1+b_1m+c_1n$$

and

$$(16'')\quad \sqrt{a_1^2+a_2}=\sqrt{d_1^2+d_2}.$$

The same process as in the previous case leads to the formulas corresponding to (7) – (10'):

$$(16''')\quad \sqrt{a_1^2+a_2}=\sqrt{(a_1+b_1m+c_1n)^2+d_2};$$

$$(17)\quad a_2=b_1^2m^2+c_1^2n^2+2a_1b_1m+2a_1c_1n+2b_1c_1mn+d_2;$$

From (12) and (16') we have

$$(18)\quad a_2+2b_1^2+b_2+2c_1^2+c_2=2b_1^2m^2+2c_1^2n^2+4a_1b_1m+4a_1c_1n+4b_1c_1mn+d_2$$

Forming expressions for (a_2-d_2) from (17) and (18) and substracting we have

(19) $a_2 - d_2 = 2b_1^2 + b_2 + 2c_1^2 + c_2$

and equating the expressions for a_2 in (17) and (18):

(20) $b_1^2m^2 + c_1^2n^2 + 2a_1b_1m + 2a_1c_1n + 2b_1c_1mn = 2b_1^2 + b_2 + 2c_1^2 + c_2$.

Making the left hand side a perfect square, put

(i) $2a_1 = 3(b_1m + c_1n)$;

it follows

(20') $(2b_1m + 2c_1n)^2 = 2b_1^2 + b_2 + 2c_1^2 + c_2$.

<u>Example (2):</u>

Let: $b_1 = 4$; $c_1 = 5$; $m = 3$; $n = 2$.

We have

from (i) $a_1 = \frac{1}{2}(3b_1m + 3c_1n) = 33$

" (16') $d_1 = a_1 + b_1m + c_1n = 55$

" (20') $b_2 + c_2 = 1854$

" (14), (16'') $a_2 = -944$

" " (14) $d_2 = -2880$; $b_2 = 1289$; $c_2 = 555$

Hence the roots of the original quadratics are:

$\xi = 33 + \sqrt{145}$ $\zeta = 5 + 2\sqrt{145}$

$\eta = 4 + 3\sqrt{145}$ $\tau = 55 + \sqrt{145}$,

and it is seen that

$$\xi^2 + \eta^2 + \zeta^2 = 3160 + 110\sqrt{145} = \tau^2$$

3° Case: $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2$

Let ξ, η, ζ, τ be the respective roots of four quadratics as before and let $\rho = e_1+\sqrt{e_1^2+e_2}$ be the root of $r^2-2e_1r-e_2=0$. Corresponding to (1), (2) and (3) we derive the following equations:

$$(21)\quad 2a_1^2+a_2+2a_1\sqrt{a_1^2+a_2}+2b_1^2+b_2+2b_1\sqrt{b_1^2+b_2}+2c_1^2+c_2+2c_1\sqrt{c_1^2+c_2}+$$
$$+2d_1^2+d_2+2d_1\sqrt{d_1^2+d_2}=2e_1^2+e_2+2e_1\sqrt{e_1^2+e_1}.$$

Hence

$$(22)\quad 2a_1^2+a_2+2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2=2e_1^2+e_2$$

and

$$(23)\quad a_1\sqrt{a_1^2+a_2}+b_1\sqrt{b_1^2+b_2}+c_1\sqrt{c_1^2+c_2}=e_1\sqrt{e_1^2+e_2}-d_1\sqrt{d_1^2+d_2}.$$

Solving let

$$(24')\quad \sqrt{b_1^2+b_2}=m\sqrt{a_1^2+a_2}$$

$$(24'')\quad \sqrt{c_1^2+c_2}=n\sqrt{a_1^2+a_2}$$

$$(24''')\quad \sqrt{d_1^2+d_2}=p\sqrt{e_1^2+e_2},$$

where m, n, p are any three rational integers.

It follows from (23):

$$(25)\quad (a_1+b_1m+c_1n)\sqrt{a_1^2+a_2}=(e_1-d_1p)\sqrt{e_1^2+e_2}$$

whence

$$(26')\quad a_1+b_1m+c_1n=e_1-d_1p \qquad \text{or} \qquad e_1=a_1+b_1m+c_1n+d_1p$$

and

$$(26'')\quad \sqrt{a_1^2+a_2}=\sqrt{e_1^2+e_2}$$

As in the previous cases we find accordingly

$$(26''') \quad \sqrt{a_1^2+a_2} = \sqrt{(a_1+b_1m+c_1n+d_1p)^2+e_2} \;;$$

$$(27) \quad a_2 = b_1^2m^2+c_1^2n^2+d_1^2p^2+2a_1b_1m+2a_1c_1n+2a_1d_1p+2b_1c_1mn+$$
$$+2b_1d_1mp+2c_1d_1np+e_2 \;;$$

from (22) and (26')

$$(28) \quad a_2+2b_1^2+b_2+2c_1^2+c_2 = 2b_1^2m^2+2c_1^2n^2+2d_1^2p^2+4a_1b_1m+$$
$$+4a_1c_1n+4a_1d_1p+4b_1c_1mn+4b_1d_1mp+4c_1d_1pn+e_2-2d_1^2-d_2$$

Forming the expressions for (a_2-e_2) from (27) and (28) and substracting we have

$$(29) \quad a_2-e_2 = 2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2$$

and equating expressions for a_2 in (27) and (28):

$$(30) \quad b_1^2m^2+c_1^2n^2+d_1^2p^2+2a_1b_1m+2a_1c_1n+2a_1d_1p+2b_1c_1mn+$$
$$+2b_1d_1np+2c_1d_1pn =$$
$$= 2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2 .$$

Making the left hand side a perfect square, put

$$(i) \quad 2a_1 = 3(b_1m+c_1n+d_1p);$$

it follows

$$(30') \quad (2b_1m+2c_1n+2d_1p)^2 = 2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2$$

Example (3):

Let $\quad b_1 = 2, \quad c_1 = 3, \quad d_1 = 5.$

$\quad m = 4, \quad n = 3, \quad p = 1$

We have

from (i) $\quad a_1 = \frac{1}{2}(3b_1m + 3c_1n + 3d_1p) = 33$

" (26') $\quad e_1 = a_1 + b_1m + c_1n + d_1p = 55$

" (30') $\quad b_2 + c_2 + d_2 = 1860$

" (24), (26'') $\quad a_2 = -1016$

" (14) $\quad b_2 = 1164, \quad c_2 = 648, \quad d_2 = 48, \quad e_2 = -3063.$

Hence the roots of the original equations are:

$$\xi = 33 + \sqrt{73} \qquad \zeta = 3 + 3\sqrt{73}$$

$$\eta = 2 + 4\sqrt{73} \qquad \tau = 5 + \sqrt{73}$$

$$\rho = 55 + \sqrt{73}$$

and it is

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = 3098 + 110\sqrt{73} = \rho^2$$

4° Case: $\xi^2+\eta^2+\zeta^2=\tau^2+\rho^2$

Let $\xi, \eta, \zeta, \tau, \rho$ be the respective roots of five quadratics as before. We derive the following three equations:

(31) $2a_1^2+a_2+2a_1\sqrt{a_1^2+a_2}+2b_1^2+b_2+2b_1\sqrt{b_1^2+b_2}+2c_1^2+c_2+2c_1\sqrt{c_1^2+c_2}=$
$$=2d_1^2+d_2+2d_1\sqrt{d_1^2+d_2}+2e_1^2+e_2+2e_1\sqrt{e_1^2+e_2}$$

Hence

(32) $2a_1^2+a_2+2b_1^2+b_2+2c_1^2+c_2=2d_1^2+d_2+2e_1^2+e_2$

and

(33) $a_1\sqrt{a_1^2+a_2}+b_1\sqrt{b_1^2+b_2}+c_1\sqrt{c_1^2+c_2}=d_1\sqrt{d_1^2+d_2}+e_1\sqrt{e_1^2+e_1}$.

Put

(34′) $\sqrt{b_1^2+b_2}=m\sqrt{a_1^2+a_2}$

(34″) $\sqrt{c_1^2+c_2}=n\sqrt{a_1^2+a_2}$

(34‴) $\sqrt{e_1^2+e_2}=p\sqrt{d_1^2+d_2}$,

where m, n, p are three rational integers.

It follows from (33):

(35) $(a_1+b_1m+c_1n)\sqrt{a_1^2+a_2}=(d_1+e_1p)\sqrt{d_1^2+d_2}$

whence

(36′) $a_1+b_1m+c_1n=d_1+e_1p$ or $d_1=a_1+b_1m+c_1n-e_1p$

and

(36″) $\sqrt{a_1^2+a_2}=\sqrt{d_1^2+d_2}$

Hence

(36‴) $\sqrt{a_1^2+a_2}=\sqrt{(a_1+b_1m+c_1n-e_1p)^2+d_2}$.

or

$$(37)\quad a_2 = b_1^2m^2 + c_1^2n^2 + c_1^2n^2 + e_1^2p^2 + 2a_1b_1m + 2a_1c_1n - 2a_1e_1p + 2b_1c_1mn - 2b_1e_1mp - 2c_1e_1np + d_2$$

From (32) and (36'):

$$(38)\quad a_2 + 2b_1^2 + b_2 + 2c_1^2 + c_2 = 2b_1^2m^2 + 2c_1^2n^2 + 2e_1^2p^2 + 4a_1b_1m + 4a_1c_1n - 4a_1e_1p + 4b_1c_1mn - 4b_1e_1mp - 4c_1e_1np + d_2 + 2e_1^2 + e_2$$

Forming the difference $(a_2 - d_2)$ from equations (37) and (38) and substracting we have

$$(39)\quad a_2 - d_2 = 2b_1^2 + b_2 + 2c_1^2 + c_2 - 2e_1^2 - e_2,$$

and equating the expressions for a_2 in (37) and (38):

$$(40)\quad b_1^2m^2 + c_1^2n^2 + e_1^2p^2 + 2a_1b_1m + 2a_1c_1n - 2a_1e_1p + 2b_1c_1mn - 2b_1e_1mp - 2c_1e_1np = 2b_1^2 + b_2 + 2c_1^2 + c_2 - 2e_1^2 - e_2.$$

Making the left hand side a perfect square put

$$(i)\quad 2a_1 = 3(b_1m + c_1n - e_1p).$$

It follows

$$(40')\quad (2b_1m + 2c_1n - 2e_1p)^2 = 2b_1^2 + b_2 + 2c_1^2 + c_2 - 2e_1^2 - e_2.$$

<u>Example (4)</u>: – To show that b_1, c_1 or e_1 may also assume negative values, let

$$b_1 = 5, \quad c_1 = -3 \quad e_1 = 4$$
$$m = 6 \quad n = 2 \quad p = 3$$

We have

from (i) $a_1 = \frac{1}{2}(3b_1m + 3c_1n - 3e_1p) = 18$

" (36') $d_1 = a_1 + b_1m + c_1n - e_1p = 30$

" (40') $b_2 + c_2 - e_2 = 540$

" (34),(36') $a_2 = -306$

" (34) $b_2 = 623$; $c_2 = 63$; $d_2 = -882$; $e_2 = 146$

Hence we find as roots of the original quadratics;

$$\xi = 18 + 3\sqrt{2} \qquad \zeta = -3 + 6\sqrt{2}$$
$$\eta = 5 + 18\sqrt{2} \qquad \tau = 30 + 3\sqrt{2}$$

$$\rho = 4 + 9\sqrt{2}$$

and it is

$$\xi^2 + \eta^2 + \zeta^2 = 1096 + 252\sqrt{2} = \tau^2 + \rho^2$$

5° Case: $\xi^2+\eta^2+\zeta^2+\tau^2+\rho^2=\sigma^2$.

To the five quadratics of the previous case we add a sixth one, viz: $s^2-2f_1s-f_2=0$ the root of which is $\sigma=f_1+\sqrt{f_1^2+f_2}$.

We obtain the three equations:

$$(41)\quad 2a_1^2+a_2+2a_1\sqrt{a_1^2+a_2}+2b_1^2+b_2+2b_1\sqrt{b_1^2+b_2}+2c_1^2+c_2+2c_1\sqrt{c_1^2+c_2}+$$
$$+2d_1^2+d_2+2d_1\sqrt{d_1^2+d_2}+2e_1^2+e_2+2e_1\sqrt{e_1^2+e_2}=$$
$$=2f_1^2+f_2+2f_1\sqrt{f_1^2+f_2}.$$

Hence

$$(42)\quad 2a_1^2+a_2+2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2+2e_1^2+e_2=2f_1^2+f_2$$

and

$$(43)\quad a_1\sqrt{a_1^2+a_2}+b_1\sqrt{b_1^2+b_1}+c_1\sqrt{c_1^2+c_2}+d_1\sqrt{d_1^2+d_2}+e_1\sqrt{e_1^2+e_2}=f_1\sqrt{f_1^2+f_2}.$$

Put

$$(44')\quad \sqrt{b_1^2+b_2}=m\sqrt{a_1^2+a_2}$$

$$(44'')\quad \sqrt{c_1^2+c_2}=n\sqrt{a_1^2+a_2}$$

$$(44''')\quad \sqrt{d_1^2+d_2}=p\sqrt{f_1^2+f_2}$$

$$(44^{IV})\quad \sqrt{e_1^2+e_2}=q\sqrt{f_1^2+f_2},$$

where m, n, p, q are four rational integers.

It follows from (43)

$$(45)\quad (a_1+b_1m+c_1n)\sqrt{a_1^2+a_2}=(f_1-d_1p-e_1q)\sqrt{f_1^2+f_2}$$

whence

$$(46') \quad a_1+b_1m+c_1n = f_1-d_1p-e_1q \quad \text{or} \quad f_1=a_1+b_1m+c_1n+d_1p+e_1q$$

and

$$(46'') \quad \sqrt{a_1^2+a_2} = \sqrt{f_1^2+f_2}$$

Hence

$$(46''') \quad \sqrt{a_1^2+a_2} = \sqrt{(a_1+b_1m+c_1n+d_1p+e_1q)^2+f_2} \; ;$$

or

$$(47) \; a_2 = b_1^2m^2+c_1^2n^2+d_1^2p^2+e_1^2q^2+2a_1b_1m+2a_1c_1n+2a_1d_1p+2a_1e_1q+2b_1c_1mn+$$
$$+2b_1d_1mp+2b_1e_1mq+2c_1d_1np+2c_1e_1nq+2d_1e_1pq+f_2.$$

From (42) and (46')

$$(48) \quad a_2+2b_1^2+2c_1^2+c_2+2d_1^2+d_2+2e_1^2+e_2 =$$
$$=2b_1^2m^2+2c_1^2n^2+2d_1^2p^2+2e_1^2q^2+4a_1b_1m+4a_1c_1n+4a_1d_1p+$$
$$+4a_1e_1q+4b_1c_1mn+4b_1d_1mp+4b_1e_1mq+4c_1d_1np+$$
$$+4c_1e_1nq+4d_1e_1pq+f_2$$

Forming the difference (a_2-f_2) from equations (47) and (48) and substracting we have

$$(49) \quad a_2-f_2 = 2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2+2e_1^2+e_2 \, ,$$

and equating the expressions for a_2 in (47) and (48):

$$(50) \; b_1^2m^2+c_1^2n^2+d_1^2p^2+e_1^2q^2+2a_1b_1m+2a_1c_1n+2a_1d_1p+2a_1e_1q+2b_1c_1mn$$
$$+2b_1d_1mp+2b_1e_1mq+2c_1d_1np+2c_1e_1nq+2d_1e_1pq =$$
$$=2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2+2e_1^2+e_2$$

Making the left hand side a perfect square put

$$(i) \quad 2a_1 = 3(b_1m+c_1n+d_1p+e_1q).$$

It follows

$$(50') \quad (2b_1m+2c_1n+2d_1p+2e_1q)^2 = 2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2+2e_1^2+e_2$$

Example (5): –

Let $\quad b_1 = 2;\ c_1 = 4;\ d_1 = 5;\ e_1 = 1$

$m = 4;\ n = 3;\ p = 1;\ q = 7.$

We have

from (i): $\quad a_1 = \frac{1}{2}(3b_1m + 3c_1n + 3d_1p + 3e_1q) = 48$

" (46'): $\quad f_1 = a_1 + b_1m + c_1n + d_1p + e_1q = 80$

" (50'): $\quad b_2 + c_2 + d_2 + e_2 = 4004$

" (44), (46'') $\quad a_2 = -2250$

" (44) $\quad b_2 = 860;\ c_2 = 470;\ d_2 = 29;\ e_2 = 2645$

$f_2 = -6346$

Hence the roots of the original quadratics are

$$\xi = 48 + 3\sqrt{6} \qquad \zeta = 4 + 9\sqrt{6} \qquad \rho = 1 + 21\sqrt{6}$$

$$\eta = 2 + 12\sqrt{6} \qquad \tau = 5 + 3\sqrt{6} \qquad \sigma = 80 + 3\sqrt{6}$$

and it is:

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 + \rho^2 = 6454 + 480\sqrt{6} = \sigma^2$$

The following two cases

$$6^\circ \text{ Case: } \xi^2+\eta^2+\zeta^2+\tau^2=\rho^2+\sigma^2$$

and

$$7^\circ \text{ Case: } \xi^2+\eta^2+\zeta^2=\tau^2+\rho^2+\sigma^2$$

differ from the previous 5° case only in regard to the signs: and we have at once corresponding to formulas (45), (46'), (46"), (i) and (50) the expressions:

In Case 6°:

(55) $(a_1+b_1m+c_1n)\sqrt{a_1^2+a_2}=(f_1-d_1p+e_1q)\sqrt{f_1^2+f_2}$

whence

(56') $a_1+b_1m+c_1n=f_1-d_1p+e_1q$ or $f_1=a_1+b_1m+c_1n+d_1p-e_1q$

and

(56") $\sqrt{a_1^2+a_2}=\sqrt{f_1^2+f_2}$

Further follows, if

(i) $2a_1=3(b_1m+c_1n+d_1p-e_1q)$

the perfect square

(60') $(2b_1m+2c_1n+2d_1p-2e_1q)^2=2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2-2e_1^2-e_2$.

In Case 7°:

(65) $(a_1+b_1m+c_1n)\sqrt{a_1^2+a_2}=(d_1+e_1p+f_1q)\sqrt{d_1^2+d_2}$

whence

(66') $a_1+b_1m+c_1n=d_1+e_1p+f_1q$ or $d_1=a_1+b_1m+c_1n-e_1p-f_1q$

and

(66") $\sqrt{a_1^2+a_2}=\sqrt{d_1^2+d_2}$

Further follows if

(i) $2a_1=3(b_1m+c_1n-e_1p-f_1q)$

the perfect square

(70') $(2b_1m+2c_1n-2e_1p-2f_1q)^2=2b_1^2+b_2+2c_1^2+c_2-2e_1^2-e_2-2f_1^2-f_2$

Example (6): – Case 6°; Let: $b_1 = -5$; $c_1 = -2$; $d_1 = 4$; $e_1 = 1$;
$m = 2$; $n = 5$; $p = 9$; $q = 10$;

We have

from (i): $a_1 = \frac{1}{2}(3b_1m + 3c_1n + 3d_1p - 3e_1q) = 9$.

" (56'): $f_1 = a_1 + b_1m + c_1n + d_1p - e_1q = 15$

" (60'): $b_2 + c_2 + d_2 - e_2 = 56$

" (54), (56"): $a_2 = -71$

" (54): $b_2 = 15$; $c_2 = 46$; $d_2 = 794$; $e_2 = 999$; $f_2 = -215$

Hence the following roots result:

$$\xi = 9 + \sqrt{10} \qquad \zeta = -2 + 5\sqrt{10} \qquad \rho = 1 + 10\sqrt{10}$$
$$\eta = -5 + 2\sqrt{10} \qquad \tau = 4 + 9\sqrt{10} \qquad \sigma = 15 + \sqrt{10}$$

and it is

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = 1236 + 50\sqrt{10} = \rho^2 + \sigma^2$$

Example (7): – Case 7°; Let: $b_1 = 2$; $c_1 = 4$; $e_1 = 3$; $f_1 = 1$
$m = 2$; $n = 4$; $p = 3$; $q = 1$.

We have

from (i): $a_1 = \frac{1}{2}(3b_1m + 3c_1n - 3e_1p - 3f_1q) = 15$

" (66'): $d_1 = a_1 + b_1m + c_1n - e_1p - f_1q = 25$

" (70'): $b_2 + c_2 - e_2 - f_2 = 380$

" (64), (66"): $a_2 = -186$

" (64): $b_2 = 152$; $c_2 = 608$; $d_2 = -586$; $e_2 = 342$; $f_2 = 38$.

Hence the following roots result:

$$\xi = 15 + \sqrt{39} \qquad \zeta = 4 + 4\sqrt{39} \qquad \rho = 3 + 3\sqrt{39}$$
$$\eta = 2 + 2\sqrt{39} \qquad \tau = 25 + \sqrt{39} \qquad \sigma = 1 + \sqrt{39}$$

and it is

$$\xi^2 + \eta^2 + \zeta^2 = 1064 + 70\sqrt{39} = \tau^2 + \rho^2 + \sigma^2$$

II. – Second Solution of the Equation $\xi^2+\eta^2+\ldots=\zeta^2+\tau^2+\cdots$

This solution is a different form of the first one and consists in expressing the different algebraic numbers of an equation through one another. The final results which are based upon the same fundamental equations are identical with those of the first solution.

1° Case: $\xi^2+\eta^2=\zeta^2+\tau^2$

Let

(1) $\eta = m\xi+p$ and $\tau = n\zeta+q$, $\left\{\begin{array}{l} m,p,n,q \text{ being} \\ \text{rational integers.}\end{array}\right.$

Squaring these expressions and substituting the values for η and τ as given in (i) pag. 8, we have

$$m^2\xi^2+2mp\,\xi+p^2 = 2b_1\eta+b_2$$
$$n^2\zeta^2+2nq\,\zeta+q^2 = 2d_1\tau+d_2.$$

Substituting the values for ξ^2, η and ζ^2, τ resp. we obtain two equations in ξ and ζ, viz:

$$2a_1m^2\xi+a_2m^2+2mp\,\xi+p^2 = 2b_1m\,\xi+2b_1p+b_2$$
$$2c_1n^2\zeta+c_2n^2+2nq\,\zeta+q^2 = 2d_1n\zeta+2d_1q+d_2$$

or

$$2a_1m^2(a_1+\sqrt{a_1^2+a_2})+a_2m^2+2mp(a_1+\sqrt{a_1^2+a_2})+p^2 = 2b_1m(a_1+\sqrt{a_1^2+a_2})+2b_1p+b_2$$
$$2c_1n^2(c_1+\sqrt{c_1^2+c_2})+c_2n^2+2nq(c_1+\sqrt{c_1^2+c_2})+q^2 = 2d_1n(c_1+\sqrt{c_1^2+c_2})+2d_1q+d_2$$

Separating rational and irrational terms it is seen that

(2) $a_1m+p = b_1$

$c_1n+q = d_1$

and

(3) $a_2m^2+p^2 = 2b_1p+b_2$

$c_2n^2+q^2 = 2d_1q+d_2$

and eliminating p and q from equations (2) and (3) respectively we have

(4) $m^2(a_1^2 + a_2) = b_1^2 + b_2$

$n^2(c_1^2 + c_2) = d_1^2 + d_2$.

Substituting into equation (1) pag. 8 the values for η and $\bar{\iota}$ as assumed in (1) pag. 24 we derive

$$2a_1\xi + a_2 + 2b_1 m\xi + 2b_1 p + b_2 = 2c_1\zeta + c_2 + 2d_1 n\zeta + 2d_1 q + d_2$$

or

$$2a_1^2 + 2a_1\sqrt{a_1^2+a_2} + a_2 + 2a_1 b_1 m + 2b_1 m\sqrt{a_1^2+a_2} + 2b_1 p + b_2 =$$
$$= 2c_1^2 + 2c_1\sqrt{c_1^2+c_2} + c_2 + 2c_1 d_1 n + 2d_1 n\sqrt{c_1^2+c_2} + 2d_1 q + d_2.$$

Equating rational and irrational parts it follows:

(5) $a_1 + b_1 m = c_1 + d_1 n$

and

$$2a_1^2 + a_2 + 2a_1 b_1 m + 2b_1 p + b_2 = 2c_1^2 + c_2 + 2c_1 d_1 n + 2d_1 q + d_2,$$

and eliminating p and q from (6) by the use of (2) and (5) it is seen that

(7) $a_2 + 2b_1^2 + b_2 - c_2 - 2d_1^2 - d_2 = 2b_1^2 m^2 + 2d_1^2 n^2 + 4a_1 b_1 m - 4a_1 d_1 n -$
$- 4b_1 d_1 mn.$

This expression is identical with equation (8) pag. 9 and the method heretofore applied has been reduced to that of the first solution. –

Since the treatment of the remaining cases is analogous to that of the first case it will be sufficient to state the substitutions and the resulting formulas.

2° Case: $\xi^2+\eta^2+\zeta^2=\tau^2$

Let:

(8) $\eta = m\xi + p$ and $\zeta = n\eta + q$.

It follows

(9) $a_1 m + p = b_1$; $b_1 n + q = c_1$

(10) $a_2 m^2 + p^2 = 2b_1 p + b_2$; $b_2 n^2 + q^2 = 2c_1 q + c_2$

and through elimination of p and q

(11) $m^2(a_1^2+a_2) = b_1^2 + b_2$; $n^2(b_1^2+b_2) = c_1^2+c_2$.

By use of equation (1) pag. 8 we have

(12) $a_1 + b_1 m + c_1 n = d_1$

and

(13) $2a_1^2 + a_2 + 2a_1 b_1 m + 2b_1 p + b_2 + 2b_1 c_1 n + 2c_1 q + c_2 = 2d_1^2 + d_2$

whence by eliminating p and q we obtain

(14) $a_2 + 2b_1^2 + b_2 + 2c_1^2 + c_2 = 2b_1^2 m^2 + 2c_1^2 n^2 + 4a_1 b_1 m + 4a_1 c_1 n + 4b_1 c_1 mn + d_2$

which expression proves to be identical with equation (18) pag. 11.

3° Case: $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2$

Let

(15) $\eta = m\xi + q;$ $\zeta = n\eta + r;$ $\tau = p\zeta + s;$

It follows

(16) $a_1 m + q = b_1$

$b_1 n + r = c_1$

$c_1 p + s = d_1$

(17) $a_2 m^2 + q^2 = 2b_1 q + b_2$

$b_2 n^2 + r^2 = 2c_1 r + c_2$

$c_2 p^2 + s^2 = 2d_1 s + d_2$

Eliminating q, r and s respectively we have:

(18) $m^2(a_1^2 + a_2) = b_1^2 + b_2$

$n^2(b_1^2 + b_2) = c_1^2 + c_2$

$p^2(c_1^2 + c_2) = d_1^2 + d_2;$

Again, using equation (1) pag. 8 we obtain

(19) $a_1 + b_1 m + c_1 n + d_1 p = e_1$

and

(20) $2a_1^2 + a_2 + 2a_1 b_1 m + 2b_1 q + b_2 + 2b_1 c_1 n + 2c_1 r + c_2 + 2c_1 d_1 p + 2d_1 s + d_2 =$

$= 2e_1^2 + e_2,$

whence by eliminating q, r and s we obtain

(21) $a_2 + 2b_1^2 + b_2 + 2c_1^2 + c_2 + 2d_1^2 + d_2 =$

$= 2b_1^2 m^2 + 2c_1^2 n^2 + 2d_1^2 p^2 + 4a_1 b_1 m + 4a_1 c_1 n + 4a_1 d_1 p +$

$+ 4b_1 c_1 mn + 4b_1 d_1 mp + 4c_1 d_1 np + e_2;$

which expression proves to be identical with equation (28) pag. 14.

4° Case: $\xi^2+\eta^2+\zeta^2=\tau^2+\rho^2$.

Let.

(22) $\eta = m\xi + q;\qquad \zeta = n\eta + r;\qquad \rho = p\zeta + s$

It follows

(23) $a_1 m + q = b_1$ (24) $a_2 m^2 + q^2 = 2b_1 q + b_2$

$b_1 n + r = c_1$. $b_2 n^2 + r^2 = 2c_1 r + c_2$

$c_1 p + s = e_1$ $c_2 p^2 + s^2 = 2e_1 s + d_2$

Eliminating q, r, s respectively

(25) $m^2(a_1^2 + a_2) = b_1^2 + b_2$

$n^2(b_1^2 + b_2) = c_1^2 + c_2$

$p^2(c_1^2 + c_2) = e_1^2 + e_2$.

From equation (1) pag 8. we obtain

(26) $a_1 + b_1 m + c_1 n - e_1 p = d_1$

and

(27) $2a_1^2 + a_2 + 2a_1 b_1 m + 2b_1 q + b_2 + 2b_1 c_1 n + 2c_1 r + c_2 - 2c_1 e_1 p -$

$- 2e_1 s - e_2 = 2d_1^2 + d_2$,

whence by eliminating q, r and s we obtain

(28) $a_2 + 2b_1^2 + b_2 + 2c_1^2 + c_2 - 2e_1^2 - e_2 =$

$= 2b_1^2 m^2 + 2c_1^2 n^2 + 2e_1^2 p^2 + 4a_1 b_1 m + 4a_1 c_1 n - 4a_1 e_1 p +$

$+ 4b_1 c_1 mn - 4b_1 e_1 mp - 4c_1 e_1 np + d_2$,

which expression proves to be identical with equation (38) pag. 17.

5° Case: $\xi^2+\eta^2+\zeta^2+\tau^2+\rho^2=\sigma^2$

Let

(29). $\eta = m\xi + r;\quad \zeta = n\eta + s;\quad \tau = p\zeta + v;\quad \rho = q\tau + w.$

It follows

(30) $a_1 m + r = b_1$
$b_1 n + s = c_1$
$c_1 p + v = d_1$
$d_1 q + w = e_1$

(31) $a_2 m^2 + r^2 = 2b_1 r + b_2$
$b_2 n^2 + s^2 = 2c_1 s + c_2$
$c_2 p^2 + v^2 = 2d_1 v + d_2$
$e_2 q^2 + w^2 = 2e_1 w + e_2$

Eliminating r, s, v and w respectively, we have

(32) $m^2(a_1^2+a_2) = b_1^2+b_2$ $\qquad p^2(c_1^2+c_2) = d_1^2+d_2$
$n^2(b_1^2+b_2) = c_1^2+c_2$ $\qquad q^2(d_1^2+d_2) = e_1^2+e_2$

From equation (1) pag 8. it follows that

(33) $a_1 + b_1 m + c_1 n + d_1 p + e_1 q = f_1$

and.

(34) $$2a_1^2 + a_2 + 2a_1 b_1 m + 2b_1 r + b_2 + 2b_1 c_1 n + 2c_1 s + c_2 + \\ + 2c_1 d_1 p + 2d_1 v + d_2 + 2d_1 e_1 q + 2e_1 w + e_2 = 2f_1^2 + f_2,$$

whence eliminating r, s, v and w we obtain

(35) $$a_2 + 2b_1^2 + b_2 + 2c_1^2 + c_2 + 2d_1^2 + d_2 + 2e_1^2 + e_2 = \\ = 2b_1^2 m^2 + 2c_1^2 n^2 + 2d_1^2 p^2 + 2e_1^2 q^2 + 4a_1 b_1 m + 4a_1 c_1 n + 4a_1 d_1 p + \\ + 4a_1 e_1 q + 4b_1 c_1 mn + 4b_1 d_1 mp + 4b_1 e_1 mq + 4c_1 d_1 np + \\ + 4c_1 e_1 nq + 4d_1 e_1 pq + f_2,$$

which expression proves to be identical with equation (48) pag. 20.

6° Case: $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2+\sigma^2$

the same substitutions (29) as used in the preceding case lead to the formulas (30) – (32).

Then again using equation (1) pag. 8 we obtain

(36) $a_1+b_1m+c_1n+d_1p-e_1q=f_1$

and

(37) $2a_1^2+a_2+2a_1b_1m+2b_1r+b_2+2b_1c_1n+2c_1s+c_2+$

$$+2c_1d_1p+2d_1v+d_2-2d_1e_1q-2e_1w-e_2=2f_1^2+f_2,$$

whence by elimination of r, s, v, w we obtain

(38) $a_2+2b_1^2+b_2+2c_1^2+c_2+2d_1^2+d_2-2e_1^2-e_2=$

$$=2b_1^2m^2+2c_1^2n^2+2d_1^2p^2+2e_1^2q^2+4a_1b_1m+4a_1c_1n+4a_1d_1p-$$

$$-4a_1e_1q+4b_1c_1mn+4b_1d_1mp-4b_1e_1mq+4c_1d_1np-$$

$$-4c_1e_1nq-4d_1e_1pq+f_2,$$

which expression differs from the analogous equation (35) of the previous case only as to the signs.

7° Case: $\xi^2+\eta^2+\zeta^2 = \tau^2+\rho^2+\sigma^2$.

Let

(39) $\eta = m\xi + r;\quad \zeta = n\eta + s;\quad \rho = p\tau + v;\quad \sigma = q\rho + w;$

It follows that

(40) $a_1 m + r = b_1$ (41) $a_2 m^2 + r^2 = 2b_1 r + b_2$

$b_1 n + s = c_1$ $b_2 n^2 + s^2 = 2c_1 s + c_2$

$d_1 p + v = e_1$ $d_2 p^2 + v^2 = 2e_1 v + e_2$

$e_1 q + w = f_1$ $e_2 q^2 + w^2 = 2f_1 w + f_2$

Eliminating r, s, v, w respectively we have

(42) $m^2(a_1^2+a_2) = b_1^2 + b_2$ $p^2(d_1^2+d_2) = e_1^2 + e_2$

$n^2(b_1^2+b_2) = c_1^2 + c_2$ $q^2(e_1^2+e_2) = f_1^2 + f_2$

It follows from equation (1) pag. 8:

(43) $a_1 + b_1 m + c_1 n = d_1 + e_1 p + f_1 q$

and

(44) $2a_1^2 + a_2 + 2a_1 b_1 m + 2b_1 r + b_2 + 2b_1 c_1 n + 2c_1 s + c_2 =$

$= 2d_1^2 + d_2 + 2d_1 e_1 p + 2e_1 v + e_2 + 2e_1 f_1 q + 2f_1 w + f_2,$

whence by elimination of r, s, v, w we obtain

(45). $a_2 + 2b_1^2 + b_2 + 2c_1^2 + c_2 = 2b_1^2 m^2 + 2c_1^2 n^2 + 2e_1^2 p^2 + 2f_1^2 q^2 +$

$+ 4a_1 b_1 m + 4a_1 c_1 n - 4a_1 e_1 p - 4a_1 f_1 q + 4b_1 c_1 mn -$

$- 4b_1 e_1 mp - 4b_1 f_1 mq - 4c_1 e_1 np - 4c_1 f_1 nq + 4e_1 f_1 pq +$

$+ d_2 + 2e_1^2 + e_2 + f_2,$

which expression also corresponds to the analogous equation (35)

III. – Third Solution of the Equation $\xi^2+\eta^2+\dots = \zeta^2+\tau^2+\dots$

The following method offers a more general solution of the problem.

1° Case: $\xi^2+\eta^2 = \zeta^2+\tau^2$

Again let $x^2-2a_1x-a_2=0$ $\qquad z^2-2c_1z-c_2=0$

$y^2-2b_1y-b_2=0$ $\qquad t^2-2d_1t-d_2=0$

be four quadratics and let

(i) $\qquad \xi = a_1+\sqrt{a_1^2+a_2} \qquad \zeta = c_1+\sqrt{c_1^2+c_2}$

$\eta = b_1+\sqrt{b_1^2+b_2} \qquad \tau = d_1+\sqrt{d_1^2+d_2}$

their respective roots representing four algebraic integers.
In order to express all these algebraic integers by one of them, say, ζ, let

(1′) $\qquad \eta = m\zeta+n$

(1″) $\qquad \tau = p\zeta+q$

(1‴) $\qquad \xi = r\zeta+s$,

where m, n, p, q, r, s are any (positive or negative) rational integers.

It follows from (1′) and (1″):

(i′) $\qquad \eta^2 = m^2\zeta^2+2mn\zeta+n^2$

$\tau^2 = p^2\zeta^2+2pq\zeta+q^2$

and since η and τ are roots of the quadratics in y and t we can substitute into (i') the values:

$$(i'') \quad \eta^2 = 2b_1\eta + b_2$$
$$\tau^2 = 2d_1\tau + d_2.$$

It follows:

$$(i''') \quad m^2\zeta^2 + 2mn\zeta + n^2 = 2b_1\eta + b_2$$
$$p^2\zeta^2 + 2pq\zeta + q^2 = 2d_1\tau + d_2,$$

and substituting into (i''') the values for $\zeta^2 = 2c_1\zeta + c_2$, for $\zeta = c_1 + \sqrt{c_1^2+c_2}$ and finally the values for η and τ from (1') and (1''), we find

$$2c_1m^2(c_1+\sqrt{c_1^2+c_2}) + c_2m^2 + 2mn(c_1+\sqrt{c_1^2+c_2}) + n^2 =$$
$$= 2b_1m(c_1+\sqrt{c_1^2+c_2}) + 2b_1n + b_2$$
$$2c_1p^2(c_1+\sqrt{c_1^2+c_2}) + c_2p^2 + 2pq(c_1+\sqrt{c_1^2+c_2}) + q^2 =$$
$$= 2d_1p(c_1+\sqrt{c_1^2+c_2}) + 2d_1q + d_2,$$

whence, by equating rational and irrational values respect. we have

$$(2) \quad c_1m + n = b_1 \quad \text{and} \quad 2c_1m(c_1m+n-b_1) + c_2m^2 + n^2 = 2b_1n + b_2$$
$$c_1p + q = d_1 \qquad 2c_1p(c_1p+q-d_1) + c_2p^2 + q^2 = 2d_1q + d_2,$$

or, since the factors of $2c_1m$ and $2c_1p$ vanish,

$$(3) \quad c_2m^2 + n^2 = 2b_1n + b_2$$
$$c_2p^2 + q^2 = 2d_1q + d_2.$$

Eliminating n and q respectively from (2) and (3) we find

$$(4) \quad m^2(c_1^2+c_2) = b_1^2 + b_2$$
$$p^2(c_1^2+c_2) = d_1^2 + d_2.$$

Introducing now the relation ($1'''$):

$$\xi = r\zeta + s$$

i.e.: $a_1 + \sqrt{a_1^2 + a_2} = c_1 r + \sqrt{c_1^2 r^2 + c_2 r^2} + s$,

we have

(5) $a_1 = c_1 r + s$

and

(6) $a_1^2 + a_2 = r^2(c_1^2 + c_2)$

Since further from (5)

$$c_1^2 r^2 = a_1^2 + s^2 - 2a_1 s$$

we have, substituting this value into (6):

(7) $2a_1 s + a_2 = c_2 r^2 + s^2$.

The following procedure will now take place in the further solution of the problem:

1°: c_1 and c_2 will be expressed through the rational integers m, n, p, q, r, s.

2°: By eliminating n, q, s from the three substitutions (1) we shall obtain values for b_1, d_1, a_1 and b_2, d_2, a_2 and

3°: express the sums under the radical signs of the algebraic integers, i.e: $a_1^2 + a_2$, $b_1^2 + b_2$ and $d_1^2 + d_2$ by $c_1^2 + c_2$.

Equation (1) pag 8 was:

$$2a_1\xi + a_2 + 2b_1\eta + b_2 = 2c_1\zeta + c_2 + 2d_1\tau + d_2$$

Substituting the values for ξ, η, τ from equations (1) pag 32,

$$2a_1 r(c_1 + \sqrt{c_1^2 + c_2}) + 2a_1 s + a_2 + 2b_1 m(c_1 + \sqrt{c_1^2 + c_2}) + 2b_1 n + b_2 =$$

$$= 2c_1(c_1 + \sqrt{c_1^2 + c_2}) + c_2 + 2d_1 p(c_1 + \sqrt{c_1^2 + c_2}) + 2d_1 q + d_2,$$

and equating rational and irrational parts respect. we obtain

(8) $a_1 r + b_1 m = c_1 + d_1 p$, and

and from (11):

$$(14) \quad c_2 = \frac{R^2D^2 + M^2D^2 - P^2D^2}{D} = (R^2 + M^2 - P^2)D.$$

Similarly we express b_1, d_1, a_1 and b_2, d_2, a_2 by M, P and R.

From equations (2) and (13) we obtain:

$$(15) \quad b_1 = M + r(mR - rM) - p(mP - pM)$$

and

$$(16) \quad d_1 = P + r(pR - rP) - m(mP - pM)$$

Again, combining (5) and (13)

$$(17) \quad a_1 = R + m(rM - mR) - p(rP - pR).$$

From (3), (14), (15) we obtain

$$(18) \quad b_2 = [(mR - rM)^2 - (mP - pM)^2 - M^2]\cdot D;$$

From (3), (14), (16)

$$(19) \quad d_2 = [(pR - rP)^2 + (pM - mP)^2 + P^2]\cdot D;$$

and finally from (6), (14), (17):

$$(20) \quad a_2 = [(rM - mR)^2 - (rP - pR)^2 - R^2]\cdot D.$$

We are now in position to determine four algebraic integers ξ, η, ζ, τ such that $\xi^2 + \eta^2 = \zeta^2 + \tau^2$. This procedure may however be simplified considerably. For note that from equations (4) and (6) we have:

$$a_1^2 + a_2 = r^2(c_1^2 + c_2)$$
$$b_1^2 + b_2 = m^2(c_1^2 + c_2)$$
$$d_1^2 + d_2 = p^2(c_1^2 + c_2),$$

and we may express readily $c_1^2+c_2$ through M, P and R; it follows:

(21) $$c_1^2+c_2 = R^2+M^2-P^2-(mR-rM)^2+(rP-pR)^2+(mP-pM)^2 = \mathcal{C}, \text{ say.}$$

Example (8):

Let: $$M=4; \quad P=6; \quad R=7;$$
$$m=3; \quad p=5; \quad r=2;$$

we derive
$$a_1 = R+m(rM-mR)-p(rP-pR) = 83$$
$$b_1 = M+r(mR-rM)-p(mP-pM) = 40$$
$$c_1 = rR+mM-pP = -4$$
$$d_1 = P+r(pR-rP)-m(mP-pM) = 58.$$

It is further seen that
$$c_1^2+c_2 = 393;$$
and hence
$$a_1^2+a_2 = 2^2\cdot 393; \quad b_1^2+b_2 = 3^2\cdot 393; \quad d_1^2+d_2 = 5^2\cdot 393.$$
and the following roots result:
$$\xi = 83+2\sqrt{393} \qquad \zeta = -4+\sqrt{393}$$
$$\eta = 40+3\sqrt{393} \qquad \tau = 58+5\sqrt{393}$$
and it is
$$\xi^2+\eta^2 = 13598 + 572\sqrt{393} = \zeta^2+\tau^2$$

2° Case: $\xi^2+\eta^2+\zeta^2=\tau^2$

Since we have to deal with the same number of algebraic integers as in the first case we have to make the same substitutions which express ξ, η and τ by ζ, and the final results differ from those of the previous case only in the signs. We obtain accordingly the following group of formulas:

Let

(22) $\eta = m\zeta + n$;

$\tau = p\zeta + q$;

$\xi = r\zeta + s$;

It follows

(23) $c_1 m + n = b_1$ and $c_1 p + q = d_1$

(24) $c_2 m^2 + n^2 = 2b_1 n + b_2$ " $c_2 p^2 + q^2 = 2d_1 q + d_2$

(25) $m^2(c_1^2 + c_2) = b_1^2 + b_2$ " $p^2(c_1^2+c_2) = d_1^2 + d_2$

(26) $a_1 = c_1 r + s$

(27) $2a_1 s + a_2 = c_2 r^2 + s^2$

(28) $a_1^2 + a_2 = r^2(c_1^2 + c_2)$

(29) $a_1 r + b_1 m + c_1 = d_1 p$

(30) $2a_1 s + a_2 + 2b_1 n + b_2 + c_2 = 2d_1 q + d_2$

(31) $$c_1 = \frac{pq - rs - mn}{1 - p^2 + r^2 + m^2}$$

(32) $$c_2 = \frac{q^2 - s^2 - n^2}{1 - p^2 + r^2 + m^2}$$

Again write

$$1-p^2+r^2+m^2 = \mathcal{D}$$

and let

(33) $$n = \mathcal{M}\mathcal{D}; \quad q = \mathcal{P}\mathcal{D}; \quad s = \mathcal{R}\mathcal{D}$$

where $\mathcal{M}, \mathcal{P}, \mathcal{R}$ are rational integers.

It follows

(34) $$c_1 = p\mathcal{P} - r\mathcal{R} - m\mathcal{M}$$

(35) $$c_2 = (\mathcal{P}^2 - \mathcal{R}^2 - \mathcal{M}^2)\mathcal{D}$$

(36) $$b_1 = \mathcal{M} - r(m\mathcal{R} - r\mathcal{M}) + p(m\mathcal{P} - p\mathcal{M})$$

(37) $$d_1 = \mathcal{P} - r(p\mathcal{R} - r\mathcal{P}) + m(m\mathcal{P} - p\mathcal{M})$$

(38) $$a_1 = \mathcal{R} - m(r\mathcal{M} - m\mathcal{R}) + p(r\mathcal{P} - p\mathcal{R})$$

Further

(39) $$b_2 = [(m\mathcal{P} - p\mathcal{M})^2 - (m\mathcal{R} - r\mathcal{M})^2 - \mathcal{M}^2].\mathcal{D}$$

(40) $$d_2 = [-(p\mathcal{R} - r\mathcal{P})^2 - (p\mathcal{M} - m\mathcal{P})^2 - \mathcal{P}^2].\mathcal{D}$$

(41) $$a_2 = [(r\mathcal{P} - p\mathcal{R})^2 - (r\mathcal{M} - m\mathcal{R})^2 - \mathcal{R}^2].\mathcal{D}$$

and due to equations (25) and (28) we have

$$a_1^2 + a_2 = r^2(c_1^2 + c_2),$$

$$b_1^2 + b_2 = m^2(c_1^2 + c_2),$$

$$d_1^2 + d_2 = p^2(c_1^2 + c_2);$$

and finally

(42) $$c_1^2 + c_2 = \mathcal{P}^2 - \mathcal{R}^2 - \mathcal{M}^2 + (p\mathcal{R} - r\mathcal{P})^2 + (p\mathcal{M} - m\mathcal{P})^2 - (r\mathcal{M} - m\mathcal{R})^2 = \mathcal{C}.$$

Example (9):

Let $M = 2, \quad P = 5, \quad R = 7$

$m = 4, \quad p = 3, \quad r = 1$.

We have

$a_1 = 63; \quad b_1 = 18; \quad d_1 = 45; \quad c_1 = 0$

$c_1^2 + c_2 = C = -252 = -4 \cdot 9 \cdot 7$

Hence

$a_1^2 + a_2 = -1^2 \cdot 2^2 \cdot 3^2 \cdot 7; \quad b_1^2 + b_2 = -4^2 \cdot 2^2 \cdot 3^2 \cdot 7; \quad d_1^2 + d_2 = -3^2 \cdot 2^2 \cdot 3^2 \cdot 7$

and the following result:

$\xi = 63 + 6\sqrt{-7} \qquad \zeta = 0 + 6\sqrt{-7}$

$\eta = 18 + 24\sqrt{-7} \qquad \tau = 45 + 18\sqrt{-7}$

and it is

$$\xi^2 + \eta^2 + \zeta^2 = -243 + 1620\sqrt{-7} = \tau^2$$

Example (9a):

Again let: $M = 2; \quad P = 5; \quad R = 7$.

$m = 3; \quad p = 1; \quad r = 4$

We have

$a_1 = 59; \quad b_1 = -37; \quad d_1 = 96; \quad c_1 = -29;$

$c_1^2 + c_2 = C = 141$

Hence

$a_1^2 + a_2 = 4^2 \cdot 141; \quad b_1^2 + b_2 = 3^2 \cdot 141; \quad d_1^2 + d_2 = 1^2 \cdot 141;$

and the following roots result:

$\xi = 59 + 4\sqrt{141} \qquad \zeta = -29 + \sqrt{141}$

$\eta = -37 + 3\sqrt{141} \qquad \tau = 96 + \sqrt{141}$

and it is

$$\xi^2 + \eta^2 + \zeta^2 = 9357 + 192\sqrt{141} = \tau^2$$

3° Case: $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2$.

In this case we have to add to the four equations of the previous cases a fifth equation, namely

(i) $r^2-2e_1r-e_2=0$ with the root $\rho=e_1+\sqrt{e_1^2+e_2}$.

Expressing as before each algebraic integer ξ,η,ζ,τ,ρ by one of them, say, τ, let

(43') $\zeta=m\tau+n$ (43''') $\xi=r\tau+s$

(43'') $\rho=p\tau+q$ (43'''') $\eta=k\tau+l$,

where m,n,p,q,r,s,k,l are rational integers.

It follows from (43), since ζ and ρ are roots of the equations in z — (see (i) pag. 32) and in r (see (i) above):

(i') $m^2\tau^2+2mn\tau+n^2=\zeta^2=2c_1\zeta+c_2$

$p^2\tau^2+2pq\tau+q^2=\rho^2=2e_1\rho+e_2$

Substituting into (i') the values for $\tau^2=2d_1\tau+d_2$, for $\tau=d_1+\sqrt{d_1^2+d_2}$, and finally for ζ and ρ the values from (43) we obtain:

$$2d_1m^2(d_1+\sqrt{d_1^2+d_2})+d_2m^2+2mn(d_1+\sqrt{d_1^2+d_2})+n^2=2c_1m(d_1+\sqrt{d_1^2+d_2})+2c_1n+c_2$$

$$2d_1p^2(d_1+\sqrt{d_1^2+d_2})+d_2p^2+2pq(d_1+\sqrt{d_1^2+d_2})+q^2=2e_1p(d_1+\sqrt{d_1^2+d_2})+2e_1q+e_2,$$

whence, by equating rational and irrational values respectively we have:

(44) $d_1m+n=c_1$ and $2d_1m(d_1m+n-c_1)+d_2m^2+n^2=2c_1n+c_2$

$d_1p+q=e_1$ " $2d_1p(d_1p+q-e_1)+d_2p^2+q^2=2e_1q+e_2$,

or, since the factors of $2d_1m$ and $2d_1p$ vanish,

(45) $d_2m^2+n^2=2c_1n+c_2$

$d_2p^2+q^2=2e_1q+e_2$.

Eliminating n and q respectively from (44) and (45), we find

(46) $m^2(d_1^2+d_2)=c_1^2+c_2$

$p^2(d_1^2+d_2)=e_1^2+e_2$.

Introducing relations ($43'''$) and (43^{IV}) we have

$$a_1 + \sqrt{a_1^2 + a_2} = rd_1 + \sqrt{r^2 d_1^2 + r^2 d_2} + s$$

and

$$b_1 + \sqrt{b_1^2 + b_2} = kd_1 + \sqrt{k^2 d_1^2 + k^2 d_2} + l,$$

whence

(47) $$a_1 = rd_1 + s$$
$$b_1 = kd_1 + l$$

and

(48) $$a_1^2 + a_2 = r^2(d_1^2 + d_2)$$
$$b_1^2 + b_2 = k^2(d_1^2 + d_2),$$

and substituting the values for $r^2 d_1^2$ and $k^2 d_1^2$ resp. from (47):

(49) $$2a_1 s + a_2 = r^2 d_2 + s^2$$
$$2b_1 l + b_2 = k^2 d_2 + l^2$$

Making use of equation (1) pag 8 and equating rational and irrational parts respectively we obtain:

(50) $$d_1 = e_1 p - a_1 r - b_1 k - c_1 m$$

and

(51) $$2a_1 s + a_2 + 2b_1 l + b_2 + 2c_1 n + c_2 + d_2 = 2e_1 q + e_2 .$$

From equations (44), (47) and (50) we derive

(52) $$d_1 = \frac{pq - mn - rs - lk}{1 + m^2 - p^2 + r^2 + k^2},$$

and from (45), (49) and (51) we derive

(53) $$d_2 = \frac{q^2 - n^2 - s^2 - l^2}{1 + m^2 - p^2 + r^2 + k^2}$$

Further call the denominator of (52) and (53), namely $1 + m^2 - p^2 + r^2 + k^2 = D$

and let

(54) $n = MD$; $q = PD$; $s = RD$; $l = KD$;

where M, P, R, K are rational integers.

We have at once

(55) $d_1 = pP - mM - rR - kK$

and

(56) $d_2 = (P^2 - M^2 - R^2 - K^2)D$

Combining as before, previous results we derive:

from equations (44) and (52):

(57) $c_1 = r(rM - mR) + k(kM - mK) + mpP + (1-p^2)M$

(58) $e_1 = r(rP - pR) + k(kP - pK) - mpM + (1-m^2)P$

from equations (47) and (52):

(59) $a_1 = k(kR - rK) + r(pP - mM) + (1 + m^2 - p^2)R$

(60) $b_1 = r(rK - kK) + k(pP - mM) + (1 + m^2 - p^2)K$

from equations (45), (53), (57):

(61) $c_2 = [(mP - pM)^2 - (mK - kM)^2 - (mR - rM)^2 - M^2]D$

from equations (45), (53), (58):

(62) $e_2 = [-(pM - mP)^2 - (pR - rP)^2 - (pK - kP)^2 - P^2]D$

from equations (48) (53) (59) (60)

(63) $a_2 = [(rP - pR)^2 - (rM - mR)^2 - (rK - kR)^2 - R^2]D$

and

(64) $b_2 = [(kP - pK)^2 - (kM - mK)^2 - (kR - rK)^2 - K^2]D$;

Equations (46) and (48) are

$$a_1^2 + a_2 = r^2(d_1^2 + d_2)$$

$$b_1^2 + b_2 = k^2(d_1^2 + d_2)$$

$$c_1^2 + c_2 = m^2(d_1^2 + d_2)$$

$$e_1^2 + e_2 = p^2(d_1^2 + d_2),$$

and it follows from (55) and (56) that

(65) $$d_1^2 + d_1 = P^2 - M^2 - R^2 - K^2 + (mP - pM)^2 + (pR - rP)^2 + (pK - kP)^2 - \\ -(mR - rM)^2 - (mK - kM)^2 - (rK - kR)^2 = b.$$

Example (10):

Let: $M = 2;\quad P = 6;\quad R = 0;\quad K = 3;$

$m = 4;\quad p = 10;\quad r = 5;\quad k = 10$

We obtain

$$a_1 = 110;\quad b_1 = 346;\quad c_1 = 172;\quad d_1 = 22;\quad e_1 = 472$$

$$d_1^2 + d_2 = 1450 = 5^2 \cdot 58,$$

and the following roots result:

$$\xi = 110 + 25\sqrt{58} \qquad \zeta = 172 + 20\sqrt{58} \qquad \rho = 472 + 50\sqrt{58}$$

$$\eta = 346 + 50\sqrt{58} \qquad \tau = 22 + 5\sqrt{58}$$

and it is

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = 367784 + 47200\sqrt{58} = \rho^2$$

4° Case: $\xi^2+\eta^2+\zeta^2=\tau^2+\rho^2$

This form requires the same substitutions as the previous one.

Let:

(66) $\zeta = m\tau + n$

$\rho = p\tau + q$

$\xi = r\tau + s$

$\eta = k\tau + l$, where m, n, p, q, r, s, k, l are rational integers.

We derive the same formulas which differ from those of the previous case only in the signs:

(67) $d_1 m + n = c_1$ and $d_1 p + q = e_1$

(68) $d_2 m^2 + n^2 = 2c_1 n + c_2$ " $d_2 p^2 + q^2 = 2e_1 q + e_2$

(69) $m^2(d_1^2 + d_2) = c_1^2 + c_2$ " $p^2(d_1^2 + d_2) = e_1^2 + e_2$

(70) $a_1 = r d_1 + s$ " $b_1 = k d_1 + l$

(71) $a_1^2 + a_2 = r^2(d_1^2 + d_2)$ " $b_1^2 + b_2 = k^2(d_1^2 + d_2)$

(72) $2a_1 s + a_2 = r^2 d_2 + s^2$ " $2b_1 l + b_2 = k^2 d_2 + l^2$

(73) $d_1 = a_1 r + b_1 k + c_1 m - e_1 p$

(74) $2a_1 s + a_2 + 2b_1 l + b_2 + 2c_1 n + c_2 = d_2 + 2e_1 q + e_2$

(75) $$d_1 = \frac{mn - pq + rs + kl}{1 - m^2 + p^2 - r^2 - k^2}$$

(76) $$d_2 = \frac{n^2 - q^2 + s^2 + l^2}{1 - m^2 + p^2 - r^2 - k^2}$$

Again calling the denominator of (75) and (76)

viz: $$1-m^2+p^2-r^2-k^2 = D,$$

let $$n = \mathcal{M}D;\quad q = PD;\quad s = RD;\quad l = KD,$$

where $\mathcal{M}, P, R, K$ are rational integers.

It follows that

$$\left.\begin{aligned} d_1 &= m\mathcal{M} - pP + rR + kK \\ d_2 &= (\mathcal{M}^2 - P^2 + R^2 + K^2)\cdot D \end{aligned}\right\}$$

$$\left.\begin{aligned} c_1 &= r(mR - r\mathcal{M}) + k(mK - k\mathcal{M}) - mpP + (1+p^2)\mathcal{M} \\ e_1 &= r(pR - rP) + k(pK - kP) + mp\mathcal{M} + (1-m^2)\mathcal{M} \end{aligned}\right\}$$

$$\left.\begin{aligned} a_1 &= k(rK - kR) + r(m\mathcal{M} - pP) + (1-m^2+p^2)R \\ b_1 &= r(kR - rK) + k(m\mathcal{M} - pP) + (1-m^2+p^2)K \end{aligned}\right\}$$

Further

$$c_2 = [(mK - k\mathcal{M})^2 - (mP - p\mathcal{M})^2 + (mR - r\mathcal{M})^2 - \mathcal{M}^2]\cdot D$$

$$e_2 = [(p\mathcal{M} - mP)^2 + (pR - rP)^2 + (pK - kP)^2 - P^2]\cdot D$$

$$a_2 = [(r\mathcal{M} - mR)^2 - (rP - pR)^2 + (rK - kR)^2 - R^2]\cdot D$$

$$b_2 = [(k\mathcal{M} - mK)^2 - (kP - pK)^2 + (kR - rK)^2 - K^2]\cdot D.$$

Equations (71) and (69) are

$$a_1^2 + a_2 = r^2(d_1^2 + d_2)$$

$$b_1^2 + b_2 = k^2(d_1^2 + d_2)$$

$$c_1^2 + c_2 = m^2(d_1^2 + d_2)$$

$$e_1^2 + e_2 = p^2(d_1^2 + d_2)$$

and it follows from (78) and (79) that

$$d_1^2 + d_2 = \mathcal{M}^2 - P^2 + R^2 + K^2 + (mP - p\mathcal{M})^2 + (pR - rP)^2 + (pK - kP)^2 -$$
$$-(mR - r\mathcal{M})^2 - (mK - k\mathcal{M})^2 - (rK - kR)^2 = \mathcal{C}.$$

Example (11):

Let: $\mathcal{M} = 4$; $\mathcal{P} = 7$; $\mathcal{R} = 2$; $\mathcal{K} = 3$;

$m = 3$; $p = 6$; $r = 10$; $k = 2$;

we have

$a_1 = -192$; $b_1 = -236$; $c_1 = -316$; $d_1 = -4$; $e_1 = -556$;

$d_1^2 + d_2 = b = 1536 = 16^2 \cdot 6$

the following roots result:

$\xi = -192 + 160\sqrt{6}$ $\quad \zeta = -316 + 48\sqrt{6}$ $\quad \rho = -556 + 96\sqrt{6}$

$\eta = -236 + 32\sqrt{6}$ $\quad \tau = -4 + 16\sqrt{6}$

and it follows

$$\xi^2 + \eta^2 + \zeta^2 = 365984 - 106880\sqrt{6} = \tau^2 + \rho^2$$

The next three cases may be examined together since the same number of algebraic integers is involved in these cases. Adding one more equation to those applied in the previous cases, viz:

(i) $s^2 - 2f_1 s - f_2 = 0$ with the root $\sigma = f_1 + \sqrt{f_1^2 + f_2}$,

we derive according to previous methods the following formulas which hold good for all three cases:

Expressing all algebraic integers $\xi, \eta, \zeta, \tau, \rho, \sigma$ by one of them, say ρ, let:

(89) $\zeta = m\rho + n$; $\sigma = r\rho + s$; $\xi = k\rho + l$;

$\tau = p\rho + q$; $\eta = h\rho + j$;

where $m, n, p, q, r, s, k, l, h, j$ are rational integers.

Squaring the first three of these substitutions and introducing for $\zeta^2, \tau^2, \sigma^2$ respectively the values $2c_1\zeta + c_2$, $2d_1\tau + d_2$, $2f_1\sigma + f_2$, it follows

(i') $m^2\rho^2 + 2mn\rho + n^2 = \zeta^2 = 2c_1\zeta + c_2$

$p^2\rho^2 + 2pq\rho + q^2 = \tau^2 = 2d_1\tau + d_2$

$r^2\rho^2 + 2rs\rho + s^2 = \sigma^2 = 2f_1\sigma + f_2$

Substituting into (i') the values $\rho^2 = 2e_1\rho + e_2$, $\rho = e_1 + \sqrt{e_1^2 + e_2}$, and the values for ζ, τ and σ from (89) we obtain:

$$2m^2e_1(e_1 + \sqrt{e_1^2+e_2}) + m^2e_2 + 2mn(e_1 + \sqrt{e_1^2+e_2}) + n^2 = 2c_1m(e_1 + \sqrt{e_1^2+e_2}) + 2c_1n + c_2$$

$$2p^2e_1(e_1 + \sqrt{e_1^2+e_2}) + p^2e_2 + 2pq(e_1 + \sqrt{e_1^2+e_2}) + q^2 = 2d_1p(e_1 + \sqrt{e_1^2+e_2}) + 2d_1q + d_2$$

$$2r^2e_1(e_1 + \sqrt{e_1^2+e_2}) + r^2e_2 + 2rs(e_1 + \sqrt{e_1^2+e_2}) + s^2 = 2f_1r(e_1 + \sqrt{e_1^2+e_2}) + 2f_1s + f_2$$

and separating rational and irrational terms we derive

(90) $e_1m + n = c_1$,

$e_1p + q = d_1$,

$e_1r + s = f_1$.

and

(91) $$e_2 m^2 + n^2 = 2c_1 n + c_2$$
$$e_2 p^2 + q^2 = 2d_1 q + d_2$$
$$e_2 r^2 + s^2 = 2f_1 s + f_2 .$$

Eliminating n, q, s from (90) and (91) we find

(92) $$m^2(e_1^2 + e_2) = c_1^2 + c_2$$
$$p^2(e_1^2 + e_2) = d_1^2 + d_2$$
$$r^2(e_1^2 + e_2) = f_1^2 + f_2$$

Introducing now the substitutions for ξ and η from (89) it follows:

$$a_1 + \sqrt{a_1^2 + a_2} = k(e_1 + \sqrt{e_1^2 + e_2}) + l$$

and $$b_1 + \sqrt{b_1^2 + b_2} = h(e_1 + \sqrt{e_1^2 + e_2}) + j$$

whence

(93) $$a_1 = ke_1 + l; \qquad b_1 = he_1 + j;$$

and

(94) $$a_1^2 + a_2 = k^2(e_1^2 + e_2); \qquad b_1^2 + b_2 = h^2(e_1^2 + e_2);$$

and since from (93)

$$k^2 e_1^2 = a_1^2 + l^2 - 2a_1 l$$
$$h^2 e_1^2 = b_1^2 + j^2 - 2b_1 j ,$$

it follows

(95) $$a_2 + 2a_1 l = l^2 + k^2 e_2 ; \qquad b_2 + 2b_1 j = j^2 + h^2 e_2 . -$$

At the hand of these formulas (89-95) we shall now investigate the last three cases the final results of which will differ from one another in the signs only.

5° Case: $\xi^2+\eta^2+\zeta^2+\tau^2+\rho^2=\sigma^2$.

We derive from equation (1) pag 8:

$$2a_1(k\rho+l)+a_2+2b_1(h\rho+j)+b_2+2c_1(m\rho+n)+2d_1(p\rho+q)+$$
$$+d_2+2e_1\rho+e_2=2f_1(r\rho+s)+f_2.$$

Substituting $\rho=e_1+\sqrt{e_1^2+e_2}$ it follows:

(96) $e_1=f_1r-a_1k-b_1h-c_1m-d_1p$

and

(97) $2a_1l+a_2+2b_1j+b_2+2c_1n+c_2+2d_1q+d_2+e_2=2f_1s+f_2$

and consequently we obtain from (90), (93), (96):

(98) $$e_1=\frac{sr-lk-jh-mn-pq}{1-r^2+k^2+h^2+m^2+p^2},$$

and from (91), (95), (97)

(99) $$e_2=\frac{s^2-l^2-j^2-n^2-q^2}{1-r^2+k^2+h^2+m^2+p^2}.$$

Denoting the denominator of (98) and (99)

$$1-r^2+k^2+h^2+m^2+p^2=D,$$

let

(100) $n=\mathcal{N}D;\quad q=PD;\quad s=RD;\quad l=KD;\quad j=HD;$

and we derive according to previous methods the following formulas:

(101) $e_1=rR-kK-hH-m\mathcal{N}-pP$

(102) $e_2=(R^2-K^2-H^2-\mathcal{N}^2-P^2)D$

(103) $c_1=p(p\mathcal{N}-mP)-r(r\mathcal{N}-mR)-m(kK+hH)+(1+k^2+h^2)\mathcal{N}$

(104) $d_1=r(pR-rP)-m(p\mathcal{N}-mP)-p(kK+hH)+(1+k^2+h^2)P.$

(105) $f_1=m(mR-r\mathcal{N})-p(rP-pR)-r(kK+hH)+(1+k^2+h^2)R$

(106) $a_1=r(kR-rK)-h(kH-hK)-k(m\mathcal{N}+pP)+(1+m^2+p^2)K$

(107) $b_1=r(hR-rH)-k(hK-kH)-h(m\mathcal{N}+pP)+(1+m^2+p^2)H.$

Further

(108) $$c_2 = [(mR - r\mathcal{M})^2 - (mK - k\mathcal{M})^2 - (mH - h\mathcal{M})^2 - (mP - p\mathcal{M})^2 - \mathcal{M}^2] \cdot \mathcal{D}$$

(109) $$d_2 = [(pR - rP)^2 - (pK - kP)^2 - (pH - hP)^2 - (p\mathcal{M} - mP)^2 - P^2] \cdot \mathcal{D}$$

(110) $$f_2 = [-(rK - kR)^2 - (rH - hR)^2 - (r\mathcal{M} - mR)^2 - (rP - pR)^2 - R^2] \cdot \mathcal{D}$$

(111) $$a_2 = [(kR - rK)^2 - (kH - hK)^2 - (k\mathcal{M} - mK)^2 - (kP - pK)^2 - K^2] \cdot \mathcal{D}$$

(112) $$b_2 = [(hR - rH)^2 - (hK - kH)^2 - (h\mathcal{M} - mH)^2 - (hP - pH)^2 - H^2] \cdot \mathcal{D}.$$

Equations (92) and (94) are

$$a_1^2 + a_2 = k^2(e_1^2 + e_2) \qquad d_1^2 + d_2 = p^2(e_1^2 + e_2)$$
$$b_1^2 + b_2 = h^2(e_1^2 + e_2) \qquad f_1^2 + f_2 = r^2(e_1^2 + e_2),$$
$$c_1^2 + c_2 = m^2(e_1^2 + e_2)$$

and we finally derive

(113) $$e_1^2 + e_2 = R^2 - K^2 - H^2 - \mathcal{M}^2 - P^2 + (rK - kR)^2 + (hR - rH)^2 + (r\mathcal{M} - mR)^2 + (rP - pR)^2 - (hK - kH)^2 - (mK - k\mathcal{M})^2 - (kP - pK)^2 - (h\mathcal{M} - mH)^2 - (hP - pH)^2 - (mP - p\mathcal{M})^2 = \mathcal{C}.$$

Example (12):

Let $\mathcal{M} = 3;\ P = 5;\ R = 6;\ K = 4;\ H = 2;$

$m = 1;\ p = 2;\ r = 10;\ k = 3;\ h = 4;$

we obtain

$a_1 = -195;\ b_1 = -30;\ c_1 = -180;\ d_1 = -291;\ e_1 = 27;\ f_1 = -144;$

$e_1^2 + e_2 = 1971 = 3^2 \cdot 219.$

The following roots result:

$$\xi = -195 + 9\sqrt{219} \qquad \zeta = -180 + 3\sqrt{219} \qquad \rho = 27 + 3\sqrt{219}$$
$$\eta = -30 + 12\sqrt{219} \qquad \tau = -291 + 6\sqrt{219} \qquad \sigma = -144 + 30\sqrt{219}$$

and it is:

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 + \rho^2 = 217836 - 8640\sqrt{219} = \sigma^2$$

σ° Case: $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2+\sigma^2$

Here we have at once

(114) $$e_1=\frac{lk+jh+mn+pq-rs}{1+r^2-k^2-h^2-m^2-p^2},$$

(115) $$e_2=\frac{l^2+j^2+n^2+q^2-s^2}{1+r^2-k^2-h^2-m^2-p^2}.$$

Denoting again the denominator

$$1+r^2-k^2-h^2-m^2-p^2=D$$

and using the substitutions of equation (100) we derive:

(116) $e_1=kK+hH+mM+pP-rR$

(117) $e_2=(K^2+H^2+M^2+P^2-R^2):D$

(118) $c_1=p(mP-pM)-r(mR-rM)+m(kK+hH)+(1-k^2-h^2)M$

(119) $d_1=r(rP-pR)-m(mP-pM)+p(kK+hH)+(1-k^2-h^2)P$

(120) $f_1=m(rM-mR)-p(pR-rP)+r(kK+hH)+(1-k^2-h^2)R$

(121) $a_1=r(rK-kR)-h(hK-kH)+k(mM+pP)+(1-m^2-p^2)K$

(122) $b_1=r(rH-hR)-k(kH-hK)+h(mM+pP)+(1-m^2-p^2)H$

Further

(123) $c_2=[(mK-kM)^2+(mH-hM)^2+(mP-pM)^2-(mR-rM)^2-M^2].D$

(124) $d_2=[(pK-kP)^2+(pH-hP)^2+(pM-mP)^2-(pR-rP)^2-P^2].D$

(125) $f_2=[(rK-kR)^2+(rH-hR)^2+(rM-mR)^2+(rP-pR)^2-R^2].D$

(126) $a_2=[(kH-hK)^2+(kM-mK)^2+(kP-pK)^2-(kR-rK)^2-K^2].D$

(127) $b_2=[(hK-kH)^2+(hM-mH)^2+(hP-pH)^2-(hR-rH)^2-H^2].D$

Finally

$$(128)\quad e_1^2+e_2 = K^2+H^2+M^2+P^2-R^2-(hK-kH)^2-(mK-kM)^2-$$
$$-(kP-pK)^2+(kR-rK)^2-(hM-mH)^2-(hP-pH)^2+$$
$$+(hR-rH)^2-(mP-pM)^2+(mR-rM)^2+(pR-rP)^2 = \mathcal{C}.$$

Example (13):

Let
$$M=1;\quad P=2;\quad R=5;\quad K=3;\quad H=4;$$
$$m=1;\quad p=2;\quad r=3;\quad k=4;\quad h=5;$$

We obtain
$$a_1=-20;\ b_1=-34;\ c_1=-14;\ d_1=-28;\ e_1=22;\ f_1=-114;$$
$$e_1^2+e_2 = 304 = 4^2\cdot 19$$

The following roots result:

$$\xi = -20+16\sqrt{9} \qquad \zeta = -14+4\sqrt{19} \qquad \rho = 22+4\sqrt{19}$$
$$\eta = -34+20\sqrt{9} \qquad \tau = -28+8\sqrt{19} \qquad \sigma = -114+12\sqrt{19}$$

and we have

$$\xi^2+\eta^2+\zeta^2+\tau^2 = 16520-2560\sqrt{19} = \rho^2+\sigma^2$$

7° Case: $\xi^2+\eta^2+\zeta^2 = \tau^2+\rho^2+\sigma^2$

We derive

(129) $$e_1 = \frac{lk+jh+mn-pq-rs}{1+r^2-k^2-h^2-m^2+p^2}$$

and

(130) $$e_2 = \frac{l^2+j^2+n^2-q^2-s^2}{1+r^2-k^2-h^2-m^2+p^2}$$

Denoting again

$$1+r^2-k^2-h^2-m^2+p^2 = D$$

and making use of the substitutions of equations (100) we obtain

(131) $$e_1 = kK+hH+m\mathcal{M}-pP-rR$$

(132) $$e_2 = (K^2+H^2+\mathcal{M}^2-P^2-R^2)\cdot D$$

(133) $$c_1 = p(p\mathcal{M}-mP)-r(mR-r\mathcal{M})+m(kK+hH)+(1-k^2-h^2)\mathcal{M}$$

(134) $$d_1 = r(rP-pR)-m(mP-p\mathcal{M})+p(kK+hH)+(1-k^2-h^2)P$$

(135) $$f_1 = m(r\mathcal{M}-mR)-p(rP-pR)+r(kK+hH)+(1-k^2-h^2)R$$

(136) $$a_1 = r(rK-kR)-h(hK-kH)+k(m\mathcal{M}-pP)+(1-m^2+p^2)K$$

(137) $$b_1 = r(rH-hR)-k(kH-hK)+h(m\mathcal{M}-pP)+(1-m^2+p^2)H$$

Further

(138) $$c_2 = [(mK-k\mathcal{M})^2+(mH-h\mathcal{M})^2-(mP-p\mathcal{M})^2-(mR-r\mathcal{M})^2-\mathcal{M}^2]\cdot D$$

(139) $$d_2 = [(pK-kP)^2+(pH-hP)^2+(p\mathcal{M}-mP)^2-(pR-rP)^2-P^2]\cdot D$$

(140) $$f_2 = [(rK-kR)^2+(rH-hR)^2+(r\mathcal{M}-mR)^2-(rP-pR)^2-R^2]\cdot D$$

(141) $$a_2 = [(kH-hK)^2+(k\mathcal{M}-mK)^2-(kP-pK)^2-(kR-rK)^2-K^2]\cdot D$$

(142) $$b_2 = [(hK-kH)^2+(h\mathcal{M}-mH)^2-(hP-pH)^2-(hR-rH)^2-H^2]\cdot D$$

Finally

$$e_1^2+e_2=K^2+H^2+M^2-P^2-R^2-(hK-kH)^2-(mK-kM)^2+ \tag{140}$$
$$+(kP-pK)^2+(kR-rK)^2-(hM-mH)^2+(hP-pH)^2+$$
$$+(hR-rH)^2+(mP-pM)^2+(mR-rM)^2-(pR-rP)^2=\mathfrak{C}.$$

Example (14):

Let

$$M=1;\quad P=3;\quad R=4;\quad K=5;\quad H=6;$$
$$m=3;\quad p=1;\quad r=2;\quad k=4;\quad h=5;$$

We obtain

$$a_1=-52;\ b_1=-54;\ c_1=82;\ d_1=-90;\ e_1=42;\ f_1=-92$$
$$e_1^2+e_2=136=2^2\cdot 34.$$

The following roots result:

$$\xi=-52+8\sqrt{34}\qquad \zeta=82+6\sqrt{34}\qquad \rho=42+2\sqrt{34}$$
$$\eta=-54+10\sqrt{34}\qquad \tau=-90+2\sqrt{34}\qquad \sigma=-92+4\sqrt{34}$$

and it is:

$$\xi^2+\eta^2+\zeta^2=19144-928\sqrt{34}=\tau^2+\rho^2+\sigma^2$$

Note that if in the third solution the expression $\mathscr{C}$ becomes a perfect square we obtain results in rational integers.

For example (15) take the 1° Case.

Let $\quad M=3;\ P=2;\ R=4;$

$\quad m=2;\ p=1;\ r=3,$

It follows from equation (21) pag.

$$c_1^2+c_2=\mathscr{C}=25=5^2$$

and from (17) (15) (13) and (16)

$$a_1=4;\quad b_1=-1;\quad c_1=16;\quad d_1=-6.$$

The following roots result:

$$\xi=4+3.5=19 \qquad \zeta=16+5=21$$
$$\eta=-1+2.5=9 \qquad \tau=-6+1.5=-1$$

and it is

$$\xi^2+\eta^2=\zeta^2+\tau^2=19^2+9^2=21^2+1^2=442$$

Example (16): take the 6° Case.

Let $\left.\begin{matrix} M=1;\ P=3;\ R=2;\ K=5;\ H=15 \\ m=1;\ p=3;\ r=2;\ k=5;\ h=15 \end{matrix}\right\}$ or $\left\{\begin{matrix} M=1;\ P=3;\ R=2;\ K=9;\ H=13; \\ m=1;\ p=3;\ r=2;\ k=9;\ h=13; \end{matrix}\right.$

It follows from (121), (122), (118), (119), (116), (120):

$$a_1=5;\ b_1=15;\ c_1=1;\ d_1=3;\ e_1=256;\ f_1=2$$

or

$$a_1=9;\ b_1=13;\ c_1=1;\ d_1=3;\ e_1=256;\ f_1=2$$

and it is in either case from (128)

$$e_1^2 + e_2 = 6 = 256 = 16^2$$

The following roots result

$\xi = 5 + 5 \cdot 16 = 85$ $\qquad \zeta = 1 + 1 \cdot 16 = 17$ $\qquad \rho = 256 + 16 = 272$

$\eta = 15 + 15 \cdot 16 = 255$ $\qquad \tau = 3 + 3 \cdot 16 = 51$ $\qquad \sigma = 2 + 2 \cdot 16 = 34$

or

$\xi = 9 + 9 \cdot 16 = 153$ $\qquad \zeta = 1 + 1 \cdot 16 = 17$ $\qquad \rho = 256 + 16 = 272$

$\eta = 13 + 13 \cdot 16 = 221$ $\qquad \tau = 3 + 3 \cdot 16 = 51$ $\qquad \sigma = 2 + 2 \cdot 16 = 34$

and either case leads to the same result:

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = \rho^2 + \sigma^2$$

$$\left.\begin{array}{l} 85^2 + 255^2 + 17^2 + 51^2 = 272^2 + 34^2 \\ 153^2 + 221^2 + 17^2 + 51^2 = 272^2 + 34^2 \end{array}\right\} = 75140$$

IV.- General Solution of the Equation $\xi^2+\eta^2+\dots = \zeta^2+\tau^2+\dots$ in the Quadratic Realm of Rationality

An algebraic integer of a quadratic realm as previously defined in the root of an irreducible equation of the form

(i) $$t^2+At+B=0,$$

the coefficient of t^2 being 1 and the coefficient A and the absolute term B being rational integers. The special results obtained hereto fore regarding the properties of such algebraic integers allow us to extend our investigations to the general solution of the problem by proving the

Theorem: In every quadratic realm of rationality there exists an infinite number of integral solutions of the equation: $\xi^2+\eta^2+\dots = \zeta^2+\tau^2+\dots$

Any algebraic integer α of a quadratic realm $R(\sqrt{t})$ - t being any (positive or negative) rational integer not containing a squared factor - may be represented in two forms

(ii) $\alpha = a+b\sqrt{t}$ and $\alpha = a+b\frac{1+\sqrt{t}}{2}$

where a and b are rational integers and where

$$t \equiv 2 (\text{mod.} 4) \text{ or } \equiv 3 (\text{mod.} 4)$$

and

$$t \equiv 1 (\text{mod.} 4) \text{ respectively.}$$

All algebraic integers may further be written in the linear form

$$\alpha = x.1 + y.\omega$$

where $1, \omega$ form a basis of the realm and x and y are rational integers. In the first of the two cases (ii) $\omega = \sqrt{t}$, in the second $\omega = \frac{1+\sqrt{t}}{2}$

It is evident that the second of the above two cases, viz: $\left(1, \omega = \frac{1+\sqrt{t}}{2}\right)$ is involved in the first one $\left(1, \omega = \sqrt{t}\right)$; for

$$\alpha = g + g_1 \frac{1+\sqrt{t}}{2} = \frac{2g + g_1 + g_1\sqrt{t}}{2} = \bar{g} + \bar{g}_1\sqrt{t}.$$

It is therefore sufficient to confine the following investigations to the first case only, viz:

$$\xi = g + g_1\sqrt{t},$$

where g, g_1, t are rational integers, t not containing any squared factor and where ξ is a root of the equation $x^2 - 2a_1x - a_2 = 0$.

1° Case: $\xi^2 + \eta^2 = \zeta^2 + \tau^2$.

Let:

$$x^2 - 2a_1 x - a_2 = 0 \; ; \qquad z^2 - 2c_1 z - c_2 = 0 \; ;$$

$$y^2 - 2b_1 y - b_2 = 0 \; ; \qquad t^2 - 2d_1 t - d_2 = 0 \; ;$$

be four quadratics and let

(i) $$\xi = m_1 + m\sqrt{t} \; ; \qquad \zeta = p_1 + p\sqrt{t} \; ;$$

$$\eta = n_1 + n\sqrt{t} \; ; \qquad \tau = q_1 + q\sqrt{t} \; ;$$

be their resp. roots where $m_1, m, n_1, n, p_1, p, q_1, q$ and t are any rational integers, t not containing a quadratic factor. It follows, since

$$\xi^2 - 2a_1 \xi - a_2 = 0 : \text{ that}$$

(1) $$m_1^2 + 2m_1 m\sqrt{t} + m^2 t = 2a_1 m_1 + 2a_1 m\sqrt{t} + a_2$$

whence

$$m_1 = a_1 \qquad \text{and} \qquad m^2 t = a_1^2 + a_2$$

Consequently it is

(2) $$\xi = a_1 + m\sqrt{t} \; ;$$

and similarly we obtain

(3) $\eta = b_1 + n\sqrt{t}$; (4) $\zeta = c_1 + p\sqrt{t}$; (5) $\tau = d_1 + q\sqrt{t}$;

where m, n, p, q are four arbitrary rational integers such that

(6) $m^2 t = a_1^2 + a_2$; (7) $p^2 t = c_1^2 + c_2$;

(8) $n^2 t = b_1^2 + b_2$; (9) $q^2 t = d_1^2 + d_2$.

From equation (1) pag. 8 we derive

(i) $$2a_1^2 + 2a_1 m\sqrt{t} + a_2 + 2b_1^2 + 2b_1 n\sqrt{t} + b_2 = $$
$$= 2c_1^2 + 2c_1 p\sqrt{t} + c_2 + 2d_1^2 + 2d_1 q\sqrt{t} + d_2$$

and by equating rational and irrational terms we obtain

(10). $$2a_1^2 + a_2 + 2b_1^2 + b_2 = 2c_1^2 + c_2 + 2d_1^2 + d_2$$

and

(11) $$a_1 m + b_1 n = c_1 p + d_1 q.$$

Considering now equations (6), (7), (8), (9), (10), we write

$$2a_1^2 + a_2 = a_1^2 + m^2 t$$
$$2b_1^2 + b_2 = b_1^2 + n^2 t$$
$$2c_1^2 + c_2 = c_1^2 + p^2 t$$
$$2d_1^2 + d_2 = d_1^2 + q^2 t,$$

and it follows by addition

(12) $$a_1^2 + b_1^2 - c_1^2 - d_1^2 = t(p^2 + q^2 - m^2 - n^2),$$

or

(12') $$a_1^2 + b_1^2 + (m^2 + n^2)t = c_1^2 + d_1^2 + (p^2 + q^2)t.$$

Before we proceed with our investigations we may insert a solution of the problem in rational integers.

Eliminating successively a, b, c, d between the equations (12') and (11) we derive four new expressions:

(13) $$(c_1^2 + d_1^2 - b_1^2)m^2 - (m^2 + n^2 - p^2 - q^2)tm^2 = (c_1 p + d_1 q - b_1 n)^2$$

(14) $$(c_1^2 + d_1^2 - a_1^2)n^2 - (m^2 + n^2 - p^2 - q^2)tn^2 = (c_1 p + d_1 q - a_1 m)^2$$

(15) $$(a_1^2 + b_1^2 - d_1^2)p^2 + (m^2 + n^2 - p^2 - q^2)tp^2 = (a_1 m + b_1 n - d_1 q)^2$$

(16) $$(a_1^2 + b_1^2 - c_1^2)q^2 + (m^2 + n^2 - p^2 - q^2)tq^2 = (a_1 m + b_1 n - c_1 p)^2$$

Next putting

(ii) $m^2+n^2=p^2+q^2$

it is seen that

(13') $(c_1q-d_1p)^2-(c_1n-b_1p)^2-(d_1n-b_1q)^2=0$:

(14') $(c_1q-d_1p)^2-(c_1m-a_1p)^2-(d_1m-a_1q)^2=0$

(15)' $(a_1n-b_1m)^2-(a_1q-d_1m)^2-(b_1q-d_1n)^2=0$

(16') $(a_1n-b_1m)^2-(a_1p-c_1m)^2-(b_1p-c_1n)^2=0$;

comparing these four expressions we find through an easy transformation that

(17) $\frac{a_1}{m}=\frac{b_1}{n}=\frac{c_1}{p}=\frac{d_1}{q}$,

and since $m^2+n^2=p^2+q^2$, it follows that also $a_1^2+b_1^2=c_1^2+d_1^2$.

Example (17):

Let $m=8$; $n=1$; $p=4$; $q=7$;

then $\frac{a_1}{8}=\frac{b_1}{1}=\frac{c_1}{4}=\frac{d_1}{7}$. - Let $b_1=4$; it follows

$a_1=32$; $c_1=16$; $d_1=28$;

and $a_1^2+b_1^2=c_1^2+d_1^2=32^2+4^2=16^2+28^2=1040$.

The solution of the Problem proper depends evidently upon that of the two equations (12) and (11). The number t being a fixed integer let

(18) $p^2+q^2-m^2-n^2=k$,

where k is constant. Hence we have reduced the general solution of the problem to that of the two expressions

$$(19) \quad \left| \begin{array}{l} a_1^2+b_1^2-c_1^2-d_1^2 = tk, \\ a_1 m + b_1 n = c_1 p + d_1 q \end{array} \right| ; \qquad \begin{array}{l} k = p^2+q^2-m^2-n^2 \\ d_1 = \dfrac{a_1 m - b_1 n + c_1 p}{q}, \end{array}$$

from which by elimination of d_1 results the quadratic equation in the three unknown quantities a_1 b_1 and c_1 to be determined:

$$(20) \quad a_1^2(q^2-m^2) + b_1^2(q^2-n^2) - c_1^2(q^2+p^2) - 2a_1b_1mn + 2a_1c_1mp + 2b_1c_1np = tkq^2.$$

That this equation admits solution may be seen by assuming

$c_1 = 0.$

It follows a quadratic in two unknown quantities:

$$(21) \quad a_1^2(q^2-m^2) + b_1^2(q^2-n^2) - 2a_1b_1mn = tkq^2$$

where $k = q^2-m^2-n^2$.

which readily may be solved +) by multiplying both sides by q^2-m^2. It results:

$$[\underbrace{a_1(q^2-m^2)-b_1mn}_{S}]^2 + b_1^2q^2(q^2-m^2-n^2) = tkq^2(q^2-m^2)$$

or

$$(22) \quad S^2 = kq^2[(q^2-m^2)t - b_1^2].$$

Hence equation (20) admits solution if values can be found for b_1 which transform the right hand side into a perfect square.

<u>Example (18):</u>

Let $\quad q = 23; \quad m = 14; \quad n = 3; \quad t = 5.$

Of all values for b_1 that satisfy condition (21) only two will lead to integral values for a_1, viz:

$b_1 = 12; \quad a_1 = 50; \quad d_1 = 32;$

$b_1 = 33; \quad a_1 = 34; \quad d_1 = 25;$

+) cf. Dr. Harris Hancock: Integral Solutions of the Equation $\xi^2+\eta^2=\zeta^2$ pag. 337, (in Liouville's Journal) the resulting roots are

$$\xi = 50 + 14\sqrt{5} \qquad \text{and} \qquad \xi = 34 + 14\sqrt{5}$$
$$\eta = 12 + 3\sqrt{5} \qquad \qquad \eta = 33 + 3\sqrt{5}$$
$$\tau = 32 + 23\sqrt{5} \qquad \qquad \tau = 25 + 23\sqrt{5}$$

and it is

$$\xi^2 + \eta^2 = \tau^2, \text{ viz:}$$

$$3669 + 1472\sqrt{5} \qquad \text{and} \qquad 3270 + 1150\sqrt{5}$$

In order to solve equation (20) multiply by (q^2-m^2) and add the identity

(i) $$b_1^2m^2n^2 - b_1^2m^2n^2 + c_1^2m^2p^2 - c_1^2m^2p^2 + 2b_1c_1m^2np - 2b_1c_1m^2np = 0$$

It follows

$$[a_1(q^2-m^2) - b_1mn + c_1mp]^2 +$$
$$+ q^2(b_1^2q^2 - b_1^2m^2 - b_1^2n^2 - c_1^2q^2 - c_1^2p^2 + c_1^2m^2 + 2b_1c_1np) = tkq^2(q^2-m^2)$$

or denoting the squared term by S:

(23) $$S'^2 = q^2[(q^2-m^2)(tk - b_1^2 + c_1^2) + (b_1n - c_1p)^2].$$

In order to obtain solutions we have to transform the right-hand side into a perfect square which evidently will be reached by putting

(24) $$tk = b_1^2 - c_1^2 = \text{const};$$

it follows.

(25 $$S^2 = q^2(b_1n - c_1p)^2.$$

Example (19):

Let
$$q = 4; \quad p = 3; \quad m = 2; \quad n = 1; \quad t = 7;$$
then
$$k = 4^2 + 3^2 - 2^2 - 1^2 = 20; \quad [k \gtrless 0]$$
and we have to solve the equation
$$tk = b_1^2 - c_1^2 = 7 \cdot 20 = 140$$
Two integral solutions are found for b_1 and c_1: [+)]
$$tk = 12^2 - 2^2 = 140$$
$$= 36^2 - 34^2 = 140$$
It follows
$$S'^2 = 4^2(12 \cdot 1 - 2 \cdot 3)^2 = 4^2 \cdot 6^2$$
$$= 4^2(36 \cdot 1 - 34 \cdot 3)^2 = 4^2 \cdot (-66)^2$$
and hence
$$a_1 = \frac{4 \cdot 6 + 12 \cdot 2 \cdot 1 - 2 \cdot 3 \cdot 2}{12} = 3 \quad \text{and} \quad d_1 = 3$$
$$a_1 = \frac{-4 \cdot 66 + 36 \cdot 2 \cdot 1 - 34 \cdot 3 \cdot 2}{12} = -33 \quad \quad d_1 = -33 \quad \text{from (19)}$$
the following roots result making use of equations (2) – (9):

$$\xi = \pm 3 + 2\sqrt{7} \qquad\qquad \xi = \mp 33 + 2\sqrt{7}$$
$$\eta = \pm 12 + \sqrt{7} \qquad \text{and} \qquad \eta = \pm 36 + \sqrt{7}$$
$$\zeta = \pm 2 + 3\sqrt{7} \qquad\qquad \zeta = \pm 34 + 3\sqrt{7}$$
$$\tau = \pm 3 + 4\sqrt{7}, \qquad\qquad \tau = \mp 33 + 4\sqrt{7}.$$
and it is
$$\xi^2 + \eta^2 = \zeta^2 + \tau^2$$
$$= 188 \pm 36\sqrt{7} \qquad\qquad = 2420 \mp 60\sqrt{7}$$

+) To solve $x^2 - y^2 = \text{const.}$ let $(y+a)^2 - y^2 = c$ and let 'a' assume all values such that $a^2 < c$; this determines y. – (cf. Dickson, History of the Theory of Numbers, Vol. II. pag. 402).

Assigning negative values to b_1 and c_1 such that

$$tk = (\mp 12)^2 - (\pm 2)^2 = (\mp 36)^2 - (\pm 34)^2$$

we find

$$S^2 = 4^2(-18)^2; \qquad a_1 = d_1 = \mp 9$$

$$S^2 = 4^2(-138)^2; \qquad a_1 = d_1 = \mp 69$$

Hence

$$\xi = \mp 9 + 2\sqrt{7} \qquad \xi = \mp 69 + 2\sqrt{7}$$

$$\eta = \mp 12 + \sqrt{7} \quad \text{and} \quad \eta = \mp 36 + \sqrt{7}$$

$$\zeta = \pm 2 + 3\sqrt{7} \qquad \zeta = \pm 34 + 3\sqrt{7}$$

$$\tau = \mp 9 + 4\sqrt{7} \qquad \tau = \mp 69 + 4\sqrt{7}$$

and it is

$$\xi^2 + \eta^2 = \zeta^2 + \tau^2$$

$$= 260 \mp 60\sqrt{7} \qquad = 6092 \mp 348\sqrt{7}$$

Another form of Equation (25). –

Note that by introducing the identity

(ii) $b_1^2 q^2 p^2 - b_1^2 q^2 p^2 + c_1^2 q^2 n^2 - c_1^2 q^2 n^2 = 0$

besides (i) into the development of equation (20) pag. 7 formula (23) may be written in the following form:

(23') $S^2 = q^2 \left\{ k\left[(q^2 - m^2)t - b_1^2 + c_1^2\right] + (b_1 p - c_1 n)^2 \right\}$

and putting

(24') $(q^2 - m^2)t = b_1^2 - c_1^2$

it follows

(25') $S^2 = q^2 (b_1 p - c_1 n)^2$

Example (20):

Let $q=7$; $p=5$; $m=4$; $n=3$; $t=3$;

then $k=49$

and $(q^2-m^2)t = b_1^2 - c_1^2 = 99$

According to the method pag 65[+)] three integral values satisfy the equation of which only one leads to integral solutions; viz:

$$b_1^2-c_1^2 = 18^2-15^2 = 99.$$

It follows

$$S^2 = 7^2 \cdot 45^2; \quad a_1 = 7; \quad d_1 = 1;$$

and it results

$$\xi = a_1 + m\sqrt{t} = \pm 7 + 4\sqrt{3}$$

$$\eta = b_1 + n\sqrt{t} = \pm 18 + 3\sqrt{3}$$

$$\zeta = c_1 + p\sqrt{t} = \pm 15 + 5\sqrt{3}$$

$$\tau = d_1 + q\sqrt{t} = \pm 1 + 7\sqrt{3}$$

and

$$\xi^2 + \eta^2 = 448 \pm 164\sqrt{3} = \zeta^2 + \tau^2.$$

No integral solutions will result if b_1 and c_1 take negative values.

2° Case: $\xi^2+\eta^2+\zeta^2=\tau^2$

According to the methods of the previous case the solution of this form depends upon that of the two equations:

$$(26)\qquad \left|\begin{array}{l} a_1^2+b_1^2+c_1^2-d_1^2=tk \\ a_1m+b_1n+c_1p=d_1q \end{array}\right| ;\qquad \begin{array}{l} k=q^2-m^2-n^2-p^2 \\ d_1=\dfrac{a_1m+b_1n+c_1p}{q}, \end{array}$$

from which by elimination of d_1 results the quadratic

$$(27)\qquad a_1^2(q^2-m^2)+b_1^2(q^2-n^2)+c_1^2(q^2-p^2)-2a_1b_1mn-2a_1c_1mp-2b_1c_1np=tkq^2$$

in three unknown quantities a_1, b_1 and c_1.

Solving multiply by q^2-m^2 and add the identity

$$(i)\qquad b_1^2n^2m^2-b_1^2n^2m^2+c_1^2p^2m^2-c_1^2p^2m^2+2b_1c_1m^2np-2b_1c_1m^2np=0.$$

It follows

$$[a_1(q^2-m^2)-b_1nm-c_1pm]^2+$$
$$+q^2(b_1^2q^2-b_1^2n^2-b_1^2m^2+c_1^2q^2-c_1^2p^2-c_1^2m^2-2b_1c_1np)=$$
$$=tkq^2(q^2-m^2)$$

or

$$(28)\qquad S_0^2=q^2[(q^2-m^2)(tk-b_1^2-c_1^2)+(b_1n+c_1p)^2]$$

then if

$$(29)\qquad tk=b_1^2+c_1^2=\text{const.},$$

the right hand side becomes a perfect square, viz:

$$(30)\qquad S'^2=q^2(b_1n+c_1p)^2$$

Example (21):

Let $q = 20; \quad m = 7; \quad n = 5; \quad p = 1; \quad t = 17;$

then $k = q^2 - m^2 - n^2 - p^2 = 325; \quad [q^2 > (m^2+n^2+p^2)]$

and $tk = b_1^2 + c_1^2 = 17 \cdot 325 = 5525.$

$5525 = 5^2 \cdot 13 \cdot 17$; 5, 13 and 17 being numbers of the form $(4n+1)$. The number of ways to decompose 5525 into the sum of two squares is +) $\frac{1}{2} k = \frac{1}{2}(a+1)(b+1)(c+1)$, where a, b, c are the exponents of numbers of the form $(4n+1)$. Hence we obtain $\frac{1}{2} k = \frac{1}{2}(2+1)(1+1)(1+1) = 6$ different ways, viz:

$$5525 = 74^2 + 7^2 \qquad = 70^2 + 25^2$$
$$= 73^2 + 14^2 \qquad = 62^2 + 41^2$$
$$= 71^2 + 22^2 \qquad = 55^2 + 50^2.$$

Applying formulas (30) and (28) we find for positive and negative values of b_1 and c_1 resp. the following table of integers:

a_1	b_1	c_1	d_1	a_1	b_1	c_1	d_1
± 29	± 74	± 7	± 29	—	± 74	∓ 7	—
—	± 73	± 14	—	± 27	± 73	∓ 14	± 27
± 29	± 71	± 22	± 29	—	± 71	∓ 22	—
—	± 70	± 25	—	± 25	± 70	∓ 25	± 25
± 27	± 62	± 41	± 27	—	± 62	∓ 41	—
± 25	± 55	± 50	± 25	—	± 55	∓ 50	—

+) See pag. 4.

the following 6 groups of roots result:

$\xi = \pm 29 + 7\sqrt{17}$	$\pm 29 + 7\sqrt{17}$	$\pm 27 + 7\sqrt{17}$
$\eta = \pm 74 + 5\sqrt{17}$	$\pm 21 + 5\sqrt{17}$	$\pm 62 + 5\sqrt{17}$
$\zeta = \pm 7 + \sqrt{17}$	$\pm 22 + \sqrt{17}$	$\pm 41 + \sqrt{17}$
$\tau = \pm 29 + 20\sqrt{17}$	$\pm 29 + 20\sqrt{17}$	$\pm 27 + 20\sqrt{17}$
$\xi = \pm 25 + 7\sqrt{17}$	$\pm 27 + 7\sqrt{17}$	$\pm 25 + 7\sqrt{17}$
$\eta = \pm 55 + 5\sqrt{17}$	$\pm 73 + 5\sqrt{17}$	$\pm 70 + 5\sqrt{17}$
$\zeta = \pm 50 + \sqrt{17}$	$\mp 14 + \sqrt{17}$	$\mp 25 + \sqrt{17}$
$\tau = \pm 25 + 20\sqrt{17}$	$\pm 27 + 20\sqrt{17}$	$\pm 25 + 20\sqrt{17}$

and it is respectively:

$$\xi^2 + \eta^2 + \zeta^2 = \tau^2$$

$= 7641 \pm 1160\sqrt{17}$	$= 7641 \pm 1160\sqrt{17}$	$= 7529 \pm 1080\sqrt{17}$
$= 7425 \pm 1000\sqrt{17}$	$= 7529 \pm 1080\sqrt{17}$	$= 7425 \pm 1000\sqrt{17}$

3° Case: $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2$.

In this case we have to deal with the following equations:

$$(31)\qquad \xi = a_1+m\sqrt{t}\ ;\qquad \zeta = c_1+p\sqrt{t}\ ;\qquad \rho = e_1+r\sqrt{t};$$
$$\eta = b_1+n\sqrt{t}\ ;\qquad \tau = d_1+q\sqrt{t}\ ;$$

further is

$$(31')\qquad m^2t=a_1^2+a_2\ ;\qquad p^2t=c_1^2+c_2\ ;\qquad r^2t=e_1^2+e_2\ ;$$
$$n^2t=b_1^2+b_2\ ;\qquad q^2t=d_1^2+d_2\ ;$$

and we consequently derive the two equations

$$(32)\qquad \left|\begin{array}{l} a_1^2+b_1^2+c_1^2+d_1^2-e_1^2=tk \\ a_1m+b_1n+c_1p+d_1q=e_1r \end{array}\right|\ ;\quad \begin{array}{l} k=r^2-m^2-n^2-p^2-q^2, \\ e_1=\dfrac{a_1m+b_1n+c_1p+d_1q}{r}, \end{array}$$

from which by eliminating e_1 there results a quadratic in the four unknown quantities a_1, b_1, c_1 and d_1; viz:

$$(33)\qquad a_1^2(r^2-m^2)+b_1^2(r^2-n^2)+c_1^2(r^2-p^2)+d_1^2(r^2-q^2)-$$
$$-2a_1b_1mn-2a_1c_1mp-2a_1d_1mq-2b_1c_1np-2b_1d_1nq-2c_1d_1pq=tkr^2.$$

Solving, multiply by r^2-m^2 and add the identity:

$$(i)\qquad b_1^2n^2m^2-b_1^2n^2m^2+c_1^2p^2m^2-c_1^2p^2m^2+d_1^2q^2m^2-d_1^2q^2m^2+$$
$$+2b_1c_1npm^2-2b_1c_1npm^2+2b_1d_1nqm^2-2b_1d_1nqm^2+$$
$$+2c_1d_1pqm^2-2c_1d_1pqm^2=0.$$

the following equation results:

$$[a_1(r^2-m^2)-b_1nm-c_1pm-d_1qm]^2+$$
$$+r^2(b_1^2r^2-b_1^2n^2-b_1^2m^2+c_1^2r^2-c_1^2p^2-c_1^2m^2+d_1^2r^2-d_1^2q^2-d_1^2m^2-$$
$$-2b_1c_1np-2b_1d_1nq-2c_1d_1pq)=tkr^2(r^2-m^2)$$

or

$$(34)\qquad S^2=r^2[(r^2-m^2)(tk-b_1^2-c_1^2-d_1^2)+(b_1n+c_1p+d_1q)^2].$$

If then

(3.5) $tk = b_1^2 + c_1^2 + d_1^2 = \text{const}$,

it follows

(36) $S^2 = r^2 (b_1 n + c_1 p + d_1 q)^2$

Example (22):

Let $r = 12$; $m = 11$; $n = 4$; $p = 2$; $q = 1$; $t = 7$;

then $k = r^2 - p^2 - q^2 - m^2 - n^2 = 2$; $[r^2 > (p^2 + q^2 + m^2 + n^2)]$

and $tk = b_1^2 + c_1^2 + d_1^2$.

Let $7 \cdot 2 = 3^2 + 2^2 + 1^2 = 14$;

it follows

$S^2 = 12^2 \cdot 17^2$; $a_1 = 17$; $e_1 = 17$;

and assigning to b_1, c_1, and d_1 all possible positive and negative values we have the following table:

a_1	b_1	c_1	d_1	e_1
± 17	± 3	± 2	± 1	± 17
± 15	± 3	± 2	∓ 1	± 15
± 9	± 3	∓ 2	± 1	± 9
± 7	± 3	∓ 2	∓ 1	± 7

Consequently the four groups of roots result:

$$\xi = \pm 17 + 11\sqrt{7}; \quad \pm 15 + 11\sqrt{7}; \quad \pm 9 + 11\sqrt{7}; \quad \pm 7 + 11\sqrt{7}$$
$$\eta = \pm 3 + 4\sqrt{7}; \quad \pm 3 + 4\sqrt{7}; \quad \pm 3 + 4\sqrt{7}; \quad \pm 3 + 4\sqrt{7}$$
$$\zeta = \pm 2 + 2\sqrt{7}; \quad \pm 2 + 2\sqrt{7}; \quad \mp 2 + 2\sqrt{7}; \quad \mp 2 + 2\sqrt{7}$$
$$\tau = \pm 1 + \sqrt{7}; \quad \mp 1 + \sqrt{7}; \quad \pm 1 + \sqrt{7}; \quad \mp 1 + \sqrt{7}$$
$$\rho = \pm 17 + 12\sqrt{7}; \quad \mp 15 + 12\sqrt{7}; \quad \pm 9 + 12\sqrt{7}; \quad \pm 7 + 12\sqrt{7}$$

and it is $\xi^2 + \eta^2 + \zeta^2 + \tau^2 = \rho^2$, viz:

$$= 1297 \pm 408\sqrt{7}; \quad = 1233 \pm 360\sqrt{7}; \quad = 1089 \pm 216\sqrt{7}; \quad = 1057 \pm 168\sqrt{7}.$$

4° Case: $\xi^2 + \eta^2 + \zeta^2 = \tau^2 + \rho^2$.

We have at once:

$$(37) \quad \left| \begin{array}{l} a_1^2 + b_1^2 + c_1^2 - d_1^2 - e_1^2 = tk \\ a_1 m + b_1 n + c_1 p = d_1 q + e_1 r \end{array} \right| ; \quad k = q^2 + r^2 - m^2 - n^2 - p^2 ; \quad e_1 = \frac{a_1 m + b_1 n + c_1 p - d_1 q}{r}.$$

Eliminating e_1 it follows the quadratic:

$$(38) \quad a_1^2(r^2-m^2) + b_1^2(r^2-n^2) + c_1^2(r^2-p^2) - d_1^2(r^2+q^2) -$$
$$- 2a_1b_1mn - 2a_1c_1mp + 2a_1d_1mq - 2b_1c_1np + 2b_1d_1nq +$$
$$+ 2c_1d_1pq = tkr^2$$

Solving multiply by r^2-m^2 and add on either side the expression

$$(i) \quad b_1^2n^2m^2 + c_1^2m^2p^2 + d_1^2m^2q^2 + 2b_1c_1npm^2 + 2b_1d_1nqm^2 + 2c_1d_1pqm^2.$$

It follows

$$[a_1(r^2-m^2) - b_1mn - c_1mp + d_1mq]^2 +$$
$$+ r^2(b_1^2r^2 - b_1^2n^2 - b_1^2m^2 + c_1^2r^2 - c_1^2p^2 - c_1^2m^2 - d_1^2r^2 - d_1^2q^2 + d_1^2m^2 -$$
$$- 2b_1c_1np + 2b_1d_1nq + 2c_1d_1pq) = tkr^2(r^2-m^2)$$

or

(39) $$S^2 = r^2\left[(r^2-m^2)(tk - b_1^2 - c_1^2 + d_1^2) + (b_1 n + c_1 p - d_1 q)^2\right]$$

If then

(40) $$tk = b_1^2 + c_1^2 - d_1^2 = \text{const.}$$

it follows

(41) $$S^2 = r^2(b_1 n + c_1 p - d_1 q)^2$$

Example (23):

Let $$q = 12; \quad r = 11; \quad m = 10; \quad n = 9; \quad p = 4; \quad t = 3;$$

then $$k = q^2 + r^2 - m^2 - n^2 - p^2 = 68; \quad [k \gtrless 0; \text{ but } r > m].$$

and $$tk = b_1^2 + c_1^2 - d_1^2$$

$$3.68 = \left\{\begin{matrix} 14^2 + 3^2 - 1^2 \\ 6^2 + 13^2 - 1^2 \end{matrix}\right\} = 204.$$

The possible combinations of positive and negative values of b, c, and d, result in the following two tables:

a_1	b_1	c_1	d_1	e_1	a_1	b_1	c_1	d_1	e_1
± 126	± 14	± 3	± 1	± 126	± 94	± 6	± 13	± 1	± 94
∓ 126	∓ 14	± 3	± 1	∓ 126	∓ 14	∓ 6	± 13	± 1	∓ 14
± 102	± 14	∓ 3	± 1	± 102	∓ 10	± 6	∓ 13	± 1	∓ 10
± 150	± 14	± 3	∓ 1	± 150	± 118	± 6	± 13	∓ 1	± 118

and it is

$$\xi^2 + \eta^2 + \zeta^2 = \tau^2 + \rho^2, \text{ viz.}$$

$16672 \pm 2796\sqrt{3}$; $16672 \mp 2748\sqrt{3}$; $11200 \pm 2268\sqrt{3}$; $23296 \pm 3276\sqrt{3}$,

$9632 \pm 2092\sqrt{3}$; $992 \mp 284\sqrt{3}$; $896 \mp 196\sqrt{3}$; $14720 \pm 2572\sqrt{3}$.

In the next three cases the following fundamental equations take place:

$$(42)\qquad \begin{aligned} &\xi = a_1 + m\sqrt{t}\ ; && \zeta = c_1 + p\sqrt{t}\ ; && \rho = e_1 + r\sqrt{t}\ ; \\ &\eta = b_1 + n\sqrt{t}\ ; && \tau = d_1 + q\sqrt{t}\ ; && \sigma = f_1 + s\sqrt{t}\ ; \end{aligned}$$

and it is further:

$$(42')\qquad \begin{aligned} &m^2 t = a_1^2 + a_2\ ; && p^2 t = c_1^2 + c_2\ ; && r^2 t = e_1^2 + e_2\ ; \\ &n^2 t = b_1^2 + b_2\ ; && q^2 t = d_1^2 + d_2\ ; && s^2 t = f_1^2 + f_2\ . \end{aligned}$$

5° Case: $\xi^2 + \eta^2 + \zeta^2 + \tau^2 + \rho^2 = \sigma^2$.

We have at once the two equations:

$$(43)\qquad \left|\begin{aligned} &a_1^2 + b_1^2 + c_1^2 + d_1^2 + e_1^2 - f_1^2 = tk \\ &a_1 m + b_1 n + c_1 p + d_1 q + e_1 r = f_1 s \end{aligned}\right| \begin{aligned} &;\ k = s^2 - m^2 - n^2 - p^2 - q^2 - r^2 \\ &;\ f_1 = \frac{a_1 m + b_1 n + c_1 p + d_1 q + e_1 r}{s} \end{aligned}$$

from which we derive by elimination of f_1 a quadratic in five unknown quantities: a_1, b_1, c_1, d_1 and e_1, viz:

$$(44)\qquad \begin{aligned} &a_1^2(s^2-m^2) + b_1^2(s^2-n^2) + c_1^2(s^2-p^2) + d_1^2(s^2-q^2) + e_1^2(s^2-r^2) - \\ &- 2a_1b_1mn - 2a_1c_1mp - 2a_1d_1mq - 2a_1e_1mr - 2b_1c_1np - \\ &- 2b_1d_1nq - 2b_1e_1nr - 2c_1d_1pq - 2c_1e_1pr - 2d_1e_1qr = \\ &= tks^2 \end{aligned}$$

Multiplying by $s^2 - m^2$ and adding on both sides the expression

$$(i)\qquad \begin{aligned} &b_1^2n^2m^2 + c_1^2p^2m^2 + d_1^2q^2m^2 + e_1^2r^2m^2 + 2b_1c_1npm^2 + 2b_1d_1nqm^2 + \\ &+ 2b_1e_1rnm^2 + 2c_1d_1pqm^2 + 2c_1e_1prm^2 + 2d_1e_1qrm^2. \end{aligned}$$

We obtain

$$\begin{aligned} &[a_1(s^2-m^2) - b_1nm - c_1pm - d_1qm - e_1rm]^2 + \\ &+ s^2(b_1^2s^2 - b_1^2n^2 - b_1^2m^2 + c_1^2s^2 - c_1^2p^2 - c_1^2m^2 + d_1^2s^2 - d_1^2q^2 - d_1^2m^2 + \\ &+ e_1^2s^2 - e_1^2r^2 - e_1^2m^2 - 2b_1c_1np - 2b_1d_1nq - 2b_1e_1nr - 2c_1d_1pq - \\ &- 2c_1e_1pr - 2d_1e_1qr) = tks^2(s^2-m^2) \end{aligned}$$

or

(45) $$S'^2 = s^2\left[(s^2-m^2)(tk-b_1^2-c_1^2-d_1^2-e_1^2)+(b_1n+c_1p+d_1q+e_1r)^2\right].$$

If then

(46) $$tk = b_1^2+c_1^2+d_1^2+e_1^2 = \text{const.}$$

it follows

(47) $$S'^2 = s^2(b_1n+c_1p+d_1q+e_1r)^2.$$

Example (24):

Let $s=16;\ m=10;\ n=9;\ p=7;\ q=4;\ r=2;\ t=11;$

then $k= s^2-m^2-n^2-p^2-q^2-r^2 = 6;\ [s>(m^2+n^2+p^2+q^2+r^2)]$

and $tk = b_1^2+c_1^2+d_1^2+e_1^2$

$11\cdot 6 = 1^2+5^2+6^2+2^2 = 66.$

Considering all possible combinations of positive and negative values of b, c, d, and e, we have the following table of integral numbers:

a_1	b_1	c_1	d_1	e_1	f_1
± 12	± 1	± 5	± 6	± 2	± 12
± 9	∓ 1	± 5	± 6	± 2	± 9
± 4	± 1	± 5	∓ 6	± 2	± 4
± 1	∓ 1	± 5	∓ 6	± 2	± 1

and it is:

$$\xi^2+\eta^2+\zeta^2+\tau^2+\rho^2 = \sigma^2, \text{ viz:}$$

$2960 \pm 384\sqrt{11};\quad 2897 \pm 288\sqrt{11};\quad 2832 \pm 128\sqrt{11};\quad 2817 \pm 32\sqrt{11};$

Note that of the 24 possible permutations of 1, 5, 6, 2 only two more will lead to integral values of a_1 and f_1, viz: 6, 2, 5, 1. and 5, 1, 2, 6, and the roots will partially be identical with those obtained above.

6° Case: $\xi^2+\eta^2+\zeta^2+\tau^2=\rho^2+\sigma^2$.

We derive the two equations:

$$(48)\quad \left|\begin{array}{l} a_1^2+b_1^2+c_1^2+d_1^2-e_1^2-f_1^2=tk \\ a_1m+b_1n+c_1p+d_1q-e_1r=f_1s \end{array}\right| \begin{array}{l} ;\ k=r^2+s^2-m^2-n^2-p^2-q^2 \\ ;\ f_1=\dfrac{a_1m+b_1n+c_1p+d_1q-e_1r}{s}. \end{array}$$

from which follows a quadratic in five unknown quantities a_1,b_1,c_1,d_1,e_1:

$$(49)\quad a_1^2(s^2-m^2)+b_1^2(s^2-n^2)+c_1^2(s^2-p^2)+d_1^2(s^2-q^2)-e_1^2(s^2+r^2)-$$
$$-2a_1b_1mn-2a_1c_1mp-2a_1d_1mq+2a_1e_1mr-2b_1c_1np-$$
$$-2b_1d_1nq+2b_1e_1nr-2c_1d_1pq+2c_1e_1pr+2d_1e_1qr=$$
$$=tks^2.$$

Solving multiply by s^2-m^2 and add on both side the expression

$$(i)\quad b_1^2n^2m^2+c_1^2p^2m^2+d_1^2q^2m^2+e_1^2r^2m^2+2b_1c_1npm^2+2b_1d_1nqm^2+$$
$$+2b_1e_1rnm^2+2c_1d_1pqm^2+2c_1e_1prm^2+2d_1e_1qrm^2.$$

It follows

$$[a_1(s^2-m^2)-b_1mn-c_1pm-d_1qm+e_1rm]^2+$$
$$+s^2(b_1^2s^2-b_1^2n^2-b_1^2m^2+c_1^2s^2-c_1^2p^2-c_1^2m^2+d_1^2s^2-d_1^2q^2-$$
$$-d_1^2m^2-e_1^2s^2-e_1^2r^2+e_1^2m^2-2b_1c_1np-2b_1d_1nq+$$
$$+2b_1e_1nr-2c_1d_1pq+2c_1e_1pr+2d_1e_1qr)=tks^2(s^2-n^2).$$

or

$$(50)\quad S^2=s^2[(s^2-m^2)(tk-b_1^2-c_1^2-d_1^2+e_1^2)+(b_1n+c_1p+d_1q-e_1r)^2].$$

If then

$$(51)\quad tk=b_1^2+c_1^2+d_1^2-e_1^2$$

it follows.

$$(52)\quad S^2=s^2(b_1n+c_1p+d_1q-e_1r)^2$$

Example (25):

Let: $r=8;\quad s=11;\quad m=10;\quad n=5;\quad p=3;\quad q=1;\quad t=5;$

then $k=r^2+s^2-m^2-n^2-p^2-q^2=50;\ [k \gtrless 0;$ but $s>m];$

and $tk=b_1^2+c_1^2+d_1^2-e_1^2$

$$\left.\begin{aligned} 5\cdot 50 &= 1^2+3^2+16^2-4^2\\ &= 1^2+11^2+12^2-4^2\\ &= 4^2+5^2+15^2-4^2\\ &= 4^2+9^2+13^2-4^2 \end{aligned}\right\} = 250.$$

In each of these four rows we may permute the first three ciphers without changing the value of tk; hence we have $6\cdot 4=24$ different arrangements. Moreover we find that each of these 24 permutations results into 16 integral values of a_1 and f_1 considering all possible positive and negative values of b_1, c_1, d_1 and e_1. The example leads consequently to $24\cdot 16=384$ algebraic numbers with integral values for $a, b, c, d, e,$ and f.

e.g.: take the permutation

$$b_1^2+c_1^2+d_1^2-e_1^2=3^2+16^2+1^2-4^2=250.$$

the following table results:

a_1	b_1	c_1	d_1	e_1	f_1
± 32	± 3	± 16	± 1	± 4	± 32
± 2	∓ 3	± 16	± 1	± 4	± 32
∓ 64	± 3	∓ 16	± 1	± 4	∓ 64
± 30	± 3	± 16	∓ 1	± 4	± 30
± 96	± 3	± 16	± 1	∓ 4	± 96
∓ 94	∓ 3	∓ 16	± 1	± 4	∓ 94
0	∓ 3	± 16	∓ 1	± 4	0
± 66	∓ 3	± 16	± 1	∓ 4	± 66

and it is
$\xi^2+\eta^2+\zeta^2=\tau^2+\rho^2+\sigma^2$, viz.

$1965 \pm 768\sqrt{5}$
$945 \pm 108\sqrt{5}$
$5037 \mp 1344\sqrt{5}$
$1841 \pm 224\sqrt{5}$
$10157 \pm 2048\sqrt{5}$
$9777 \mp 2004\sqrt{5}$
$941 \pm 64\sqrt{5}$
$5297 \pm 1388\sqrt{5}$

7° Case: $\xi^2+\eta^2+\zeta^2=\tau^2+\rho^2+\sigma^2$.

the two fundamental equations are

$$(53)\quad \left|\begin{array}{l} a_1^2+b_1^2+c_1^2-d_1^2-e_1^2-f_1^2=tk \\ a_1m+b_1n+c_1p=d_1q+e_1r+f_1s \end{array}\right| \begin{array}{l} ;\ k=q^2+r^2+s^2-m^2-n^2-p^2 \\ ;\ f_1=\dfrac{a_1m+b_1n+c_1p-d_1q-e_1r}{s}, \end{array}$$

and it follows a quadratic in five unknown quantities a_1, b_1, c_1, d_1, e_1:

$$(54)\quad a_1^2(s^2-m^2)+b_1^2(s^2-n^2)+c_1^2(s^2-p^2)-d_1^2(s^2+q^2)-e_1^2(s^2+r^2)-$$
$$-2a_1b_1mn-2a_1c_1mp+2a_1d_1mq+2a_1e_1mr-2b_1c_1np+$$
$$+2b_1d_1nq+2b_1e_1nr+2c_1d_1pq+2c_1e_1pr-2d_1e_1qr=$$
$$=tks^2$$

Solving multiply by s^2-m^2 and add on either side the expression:

$$(c)\quad b_1^2n^2m^2+c_1^2p^2m^2+d_1^2m^2q^2+e_1^2m^2r^2+2b_1c_1npm^2+2b_1d_1nqm^2+$$
$$+2b_1e_1nrm^2+2c_1d_1pqm^2+2c_1e_1prm^2+2d_1e_1qrm^2.$$

It follows

$$[a_1(s^2-m^2)-b_1nm-c_1pm+d_1qm+e_1rm]^2+$$
$$+s^2(b_1^2s^2-b_1^2n^2-b_1^2m^2+c_1^2s^2-c_1^2p^2-c_1^2m^2-d_1^2s^2-$$
$$-d_1^2q^2+d_1^2m^2-e_1^2s^2-e_1^2r^2+e_1^2m^2-2b_1c_1np+2b_1d_1nq+$$
$$+2b_1e_1nr+2c_1d_1pq+2c_1e_1pr-2d_1e_1qr)=tks^2(s^2-m^2)$$

or

$$(55)\quad \mathfrak{J}^2=s^2[(s^2-m^2)(tk-b_1^2-c_1^2+d_1^2+e_1^2)+(b_1n+c_1p-d_1q-e_1r)^2].$$

If then

$$(56)\quad tk=b_1^2+c_1^2-d_1^2-e_1^2=\text{const.},$$

it follows

$$(57)\quad \mathfrak{J}^2=s^2(b_1n+c_1p-d_1q-e_1r)^2$$

Example (26):

Let $k = q^2+r^2+s^2-(m^2+n^2+p^2)$, $[k \gtrless 0;$ but $s > m]$

$= \quad 314 \quad - \quad 294 \quad = 20; \qquad t = 7$

$$= \begin{Bmatrix} 3^2+4^2+17^2 \\ 3^2+7^2+16^2 \\ 5^2+8^2+15^2 \end{Bmatrix} - \begin{Bmatrix} (13^2+11^2+2^2) \\ (13^2+10^2+5^2) \end{Bmatrix} = 20$$

and $tk = b_1^2+c_1^2-(d_1^2+e_1^2)$

$7.20 = 205 - 65 = 140$

$$\begin{Bmatrix} 14^2+3^2 \\ 13^2+6^2 \end{Bmatrix} - \begin{Bmatrix} (8^2+1^2) \\ (7^2+4^2) \end{Bmatrix} = 140$$

From the many possible combinations (the example will result into $6\cdot(4+4)\cdot16 = 768$ integers) take e.g.:

$q = 3;\ r = 4;\ s = 17;\ m = 13;\ n = 11;\ p = 2;$

$b_1^2+c_1^2-(d_1^2+e_1^2) = 14^2+3^2-(8^2+1^2) = 140.$

Considering all values of $b_1\ c_1\ d_1$ and e_1 we have the following table:

a_1	b_1	c_1	d_1	e_1	f_1
±33	±14	±3	±8	±1	±33
∓44	∓14	±3	±8	±1	∓44
±30	±14	∓3	±8	±1	±30
±45	±14	±3	∓8	±1	±45
±35	±14	±3	±8	∓1	±35
∓47	∓14	∓3	±8	±1	∓47
∓32	∓14	±3	∓8	±1	∓32
∓42	∓14	±3	±8	∓1	∓42

and it is:

$\xi^2+\eta^2+\zeta^2 = \tau^2+\rho^2+\sigma^2$, viz:

$3352 \pm 1128\sqrt{7}$

$4199 \mp 1440\sqrt{7}$

$3163 \pm 1076\sqrt{7}$

$4288 \pm 1490\sqrt{7}$

$3488 \pm 1230\sqrt{7}$

$4472 \pm 1542\sqrt{7}$

$3287 \mp 1128\sqrt{7}$

$4027 \mp 1388\sqrt{7}.$

Example (27).

In the following example it will be shown that the theorem pag 58. holds good for any amount of algebraic numbers in the quadratic realm of rationality.

We assume ten algebraic numbers such that

$$(58) \quad \xi^2+\eta^2+\zeta^2+\tau^2+\rho^2 = \sigma^2+\pi^2+\varphi^2+\psi^2+\chi^2$$

where

$$\xi = a_1+m^2t;\quad \zeta = c_1+p^2t;\quad \rho = e_1+r^2t;\quad \pi = g_1+o^2t;\quad \psi = j_1+v^2t;$$
$$\eta = b_1+n^2t;\quad \tau = d_1+q^2t;\quad \sigma = f_1+s^2t;\quad \varphi = h_1+u^2t;\quad \chi = l_1+w^2t;$$

We derive two equations as heretofore, viz:

$$(59) \quad \left|\begin{array}{l} a_1^2+b_1^2+c_1^2+d_1^2+e_1^2-f_1^2-g_1^2-h_1^2-j_1^2-l_1^2 = tk \\ a_1m+b_1n+c_1p+d_1q+e_1r = f_1s+g_1o+h_1u+j_1v+l_1w \end{array}\right|$$

where

$$k = s^2+o^2+u^2+v^2+w^2-m^2-n^2-p^2-q^2-r^2,$$

and

$$l_1 = \frac{a_1m+b_1n+c_1p+d_1q+e_1r-f_1s-g_1o-h_1u-j_1v}{w}.$$

Through elimination of l_1 we obtain a quadratic in 9 unknown quantities, viz:

$$(60) \quad a_1^2(w^2-m^2)+b_1^2(w^2-n^2)+c_1^2(w^2-p^2)+d_1^2(w^2-q^2)+e_1^2(w^2-r^2)-f_1^2(w^2+s^2)-$$
$$-g_1^2(w^2+o^2)-h_1^2(w^2+u^2)-j_1^2(w^2+v^2)-$$
$$-2a_1b_1mn-2a_1c_1mp-2a_1d_1mq-2a_1e_1mr+2a_1f_1ms+2a_1g_1mo+$$
$$+2a_1h_1mu+2a_1j_1mv-2b_1c_1np-2b_1d_1nq-2b_1e_1nr+2b_1f_1ns+$$
$$+2b_1g_1no+2b_1h_1nu+2b_1j_1nv-2c_1d_1pq-2c_1e_1pr+2c_1f_1ps+$$
$$+2c_1g_1po+2c_1h_1pu+2c_1j_1pv-2d_1e_1qr+2d_1f_1qs+2d_1g_1qo+$$
$$+2d_1h_1qu+2d_1j_1qo+2e_1f_1rs+2e_1g_1ro+2e_1h_1ru+2e_1j_1rv-$$
$$-2f_1g_1so-2f_1h_1su-2f_1j_1sv-2g_1h_1ou+2g_1j_1ov-2h_1j_1uv =$$
$$= tkw^2$$

Solving multiply by w^2-m^2 and add on both sides of the equation the expression

$$
\begin{array}{lll}
(i)\;\; b_1^2n^2m^2+ & 2b_1g_1nom^2+ & 2d_1h_1qum^2+\\
c_1^2p^2m^2+ & 2b_1h_1num^2+ & 2d_1j_1qvm^2+\\
d_1^2q^2m^2+ & 2b_1j_1nvm^2+ & 2e_1f_1rsm^2+\\
e_1^2r^2m^2+ & 2c_1d_1pqm^2+ & 2e_1g_1rom^2+\\
f_1^2s^2m^2+ & 2c_1e_1prm^2+ & 2e_1h_1rum^2+\\
g_1^2o^2m^2+ & 2c_1f_1psm^2+ & 2e_1j_1rvm^2+\\
h_1^2u^2m^2+ & 2c_1g_1pom^2+ & 2f_1g_1som^2+\\
j_1^2v^2m^2+ & 2c_1h_1pum^2+ & 2f_1h_1sum^2+\\
2b_1c_1npm^2+ & 2c_1j_1pvm^2+ & 2f_1j_1svm^2+\\
2b_1d_1nqm^2+ & 2d_1e_1qrm^2+ & 2g_1h_1oum^2+\\
2b_1e_1nrm^2+ & 2d_1f_1qsm^2+ & 2g_1j_1ovm^2+\\
2b_1f_1nsm^2+ & 2d_1g_1qom^2+ & 2h_1j_1uvm^2.
\end{array}
$$

It follows:

$$
\begin{aligned}
&[a_1(w^2-m^2)-b_1nm-c_1pm-d_1qm-e_1rm+f_1sm+g_1om+h_1um+j_1vm]^2+\\
&+w^2(b_1^2w^2-b_1^2n^2-b_1^2m^2+c_1^2w^2-c_1^2p^2-c_1^2m^2+d_1^2w^2-d_1^2q^2-d_1^2m^2+e_1^2w^2-\\
&-e_1^2r^2-e_1^2m^2-f_1^2w^2-f_1^2s^2+f_1^2m^2-g_1^2w^2-g_1^2o^2-g_1^2m^2-h_1^2w^2-h_1^2u^2+h_1^2m^2-\\
&-j_1^2w^2-j_1^2o^2+j_1^2m^2-2b_1c_1np-2b_1d_1nq-2b_1e_1nr+2b_1f_1ns+\\
&+2b_1g_1no+2b_1h_1nu+2b_1j_1nv-2c_1d_1pq-2c_1e_1pr+2c_1f_1ps+\\
&+2c_1g_1po+2c_1h_1pu+2c_1j_1pv-2d_1e_1qr+2d_1f_1qs+2d_1g_1qo+\\
&+2d_1h_1qu+2d_1j_1qv+2e_1f_1rs+2e_1g_1ro+2e_1h_1ru+2e_1j_1rv-\\
&-2f_1g_1so-2f_1h_1su-2f_1j_1sv-2g_1h_1ou-2g_1j_1ov-2h_1j_1uv=\\
&=tkw^2(w^2-m^2)
\end{aligned}
$$

or

$$
(61)\quad S^2=w^2\left[(w^2-m^2)(tk-b_1^2-c_1^2-d_1^2-e_1^2+f_1^2+g_1^2+h_1^2+j_1^2)+\right.
$$
$$
\left.+(b_1n+c_1p+d_1q+e_1r-f_1s-g_1o-h_1u-j_1v)^2\right]
$$

If then

(62) $tk = b_1^2 + c_1^2 + d_1^2 + e_1^2 - f_1^2 - g_1^2 - h_1^2 - j_1^2$

it follows

(63) $S^2 = r^2(b_1 n + c_1 p + d_1 q + e_1 r - f_1 s - g_1 o - h_1 u - j_1 v)^2$

Example (27):

Let $k = (s^2+o^2+u^2+v^2+w^2) - (m^2+n^2+p^2+q^2+r^2)$ [$k \gtrless 0$; but $w > m$]

$= (1^2+2^2+3^2+4^2+16^2) - (12^2+7^2+6^2+5^2+2^2)$

$= \quad 286 \quad - \quad 258 \quad = 28;$

$t = 7;$

and $tk = (b_1^2+c_1^2+d_1^2+e_1^2) - (f_1^2+g_1^2+h_1^2+j_1^2)$

$= (16^2+7^2+5^2+4^2) - (9^2+8^2+2^2+1^2)$

$7 \cdot 28 = \quad 346 \quad - \quad 150 \quad = 196;$

it follows from (61) and (63):

$S^2 = 16^2 \cdot 152^2; \quad a_1 = 38; \; f_1 = 38;$

Of all possible combinations of positive and negative values for $b_1, c_1, d_1, e_1, f_1, g_1, h_1, j_1$ (there are $2 \cdot 73 = 146$) only $2 \cdot 38 = 76$ result in integral solutions of a_1 and f_1. We consider the following two:

a_1	b_1	c_1	d_1	e_1	f_1	g_1	h_1	j_1	l_1
± 38	± 16	± 7	± 5	± 4	± 9	± 8	± 2	± 1	± 38
∓ 5	∓ 16	± 7	± 5	± 4	∓ 9	± 8	± 2	± 1	∓ 5

and it is

$\xi^2 + \eta^2 + \zeta^2 + \tau^2 + \rho^2 = \sigma^2 + \pi^2 + \varphi^2 + \psi^2 + \chi^2$, viz.

$3596 \pm 1286\sqrt{7}$ and $2177 \mp 194\sqrt{7}$.