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*I hereby recommend that the thesis prepared under my supervision by* Kenneth Charles Schraut  
*entitled* Convergence Factor Theorems for Double and Multiple Series,  
having Unbounded Partial Sums.

*be accepted as fulfilling this part of the requirements for the degree of* Doctor of Philosophy

*Approved by:*

Charles A. Moore



CONVERGENCE FACTOR THEOREMS FOR DOUBLE AND MULTIPLE SERIES,  
HAVING UNBOUNDED PARTIAL SUMS.

A dissertation submitted to the

Graduate School  
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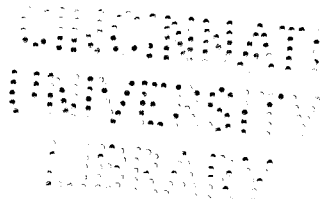
1940

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Convergence Factor Theorems for Double and Multiple  
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## Introduction.

In Volume 22 of the American Mathematical Society Colloquium Publications C. N. Moore [5]\* established for bounded, convergent double and multiple sequences necessary and sufficient conditions for the existence of non-factorable sequence to function transformations of a very general nature; and he also established for convergent, double and multiple series, having bounded partial sums, necessary and sufficient conditions for the existence of a general non-factorable convergence factor transformation of type I and of type II.

As early as 1922 T. Kojima [4] stated for unbounded double sequences which converge in Pringsheim's sense necessary and sufficient conditions for the existence of non-factorable triangular matrix transformations which preserve convergence, as well as transformations which preserve also the sum of the sequence. The conditions which he was forced to impose upon the elements of his matrix of transformation are of course very stringent.

In 1931 C. R. Adams [1] showed that if the elements of the non-factorable triangular matrix of transformation treated by Kojima satisfy the Robison-Kojima set of necessary

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\* Numbers in square brackets refer to bibliography.

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and sufficient conditions that the transformation be regular for bounded sequences, then that transformation is also regular for a large class of unbounded sequences which he introduces. In the same paper, he shows that if one assumes the transformation to be factorable, then it is regular if the transformed sequence is bounded.

In 1932 R. P. Agnew [3] stated that a sufficient condition for a factorable triangular matrix transformation which is regular for bounded convergent sequences to be regular for unbounded convergent sequences is that the transformed sequences be ultimately bounded. The following year, C. R. Adams [2] showed by means of an example that a non-factorable triangular matrix transformation which is regular for bounded sequences is not in general regular for convergent unbounded sequences when the transformed sequence is either bounded or only ultimately bounded.

The present work deals in part with non-factorable square matrix transformations which are regular for bounded, convergent sequences and which are of the type considered by C. N. Moore. There are stated conditions on the sequence to function transformations which preserve the sum of the sequence, as well as conditions which preserve only its convergence. The corresponding theorems for convergence factors of type I and of type II are also given; and the theorems on double sequences and on double series are generalized to theorems on multiple sequences and multiple

series. Finally, the work of C. R. Adams on double sequences is generalized to a more general type of non-factorable transformation, the corresponding theorem for double series is established, and both theorems are generalized to the multiple case.

In conclusion, the writer is very happy to express here his deep gratitude to Professor Charles N. Moore, Director of Graduate Studies in Mathematics at the University of Cincinnati, for proposing this problem and for offering much encouragement and many valuable suggestions during its investigation.

Kenneth C. Schraut.

University of Cincinnati,

April, 1940

P A R T I.

Double Sequences and Series.

1. Sequence to function transformations for double sequences of a certain class  $(\mathcal{A})_2$ .

We are given a double sequence,

1(1)  $\{s_{ij}\}$   $(i, j = 0, 1, \dots),$

converging to  $\mathcal{A}$  and a doubly infinite set of functions,

1(2)  $\{\varphi_{ij}(\alpha)\}$   $(i, j = 0, 1, \dots),$

defined in a set of points  $E(\alpha)$  in a space of any number of dimensions with coordinates real or complex.

Moore<sup>1</sup> has shown that a necessary and sufficient condition that the series

1(3)  $\sum_{i=0, j=0}^{\infty, \infty} s_{ij} \varphi_{ij}(\alpha)$

converge in  $E(\alpha)$  for all convergent sequences 1(1) which are such that

1(4)  $|s_{ij}| < M$   $(\text{all } i, j)$

is that the functions 1(2) satisfy

$(A^*)_2 \sum_{i=0, j=0}^{\infty, \infty} |\varphi_{ij}(\alpha)| < K(\alpha),$

-----  
1. Moore, C. N., Summable Series and Convergence Factors. Vol. 22, American Mathematical Society Colloquium Publications, 1938.

where  $K(\alpha)$  is a positive function of  $\alpha$  defined for each  $\alpha$  in  $E(\alpha)$ .

We wish to show that the condition  $(A^*)_2$  is sufficient for the convergence of 1(3) for sequences 1(1) which need not satisfy so narrow a restriction as 1(4) but which do satisfy two alternate conditions stated below.

We make the two following definitions:

DEFINITION 1.1. Set of functions of class  $(\pi)_2$ .

The set of functions 1(2) is of class  $(\pi)_2$  if the functions of the set satisfy  $(A^*)_2$ .

DEFINITION 1.2. Sequences of a certain class  $(\mathcal{O})_2$ .

The convergent sequence 1(1) is of class  $(\mathcal{O})_2$  relative to a given set of functions 1(2) of class  $(\pi)_2$  if the following two conditions are satisfied:

$(\alpha_1)_2$  The sequence 1(1) is such that the series on the right of

$$1(5) \quad \Phi_p^{(1)}(\alpha) \equiv \sum_{i=p}^{\infty} \left( \sum_{j=0}^{\infty} a_{ij} \varphi_{ij}(\alpha) \right)$$

converges for each  $\alpha$  in  $E(\alpha)$  and for all  $p > p_0$ .

$(\alpha_2)_2$  The sequence 1(1) is also such that the series on the right of

$$1(6) \quad \Phi_q^{(2)}(\alpha) \equiv \sum_{j=q}^{\infty} \left( \sum_{i=0}^{\infty} a_{ij} \varphi_{ij}(\alpha) \right)$$

converges for each  $\alpha$  in  $E(\alpha)$  and for all  $q > q_0$ .

Let us now show that the set of all sequences  $l(1)$  of class  $(\mathcal{A})_2$  relative to a given set of functions  $l(2)$  of class  $(\mathcal{M})_2$  includes all sequences satisfying  $l(4)$  and others as well. One must show that if  $l(1)$  satisfies  $l(4)$ , then conditions  $(\mathcal{A})_2$  and  $(\mathcal{A}^*)_2$  are also satisfied for functions  $l(2)$  of class  $(\mathcal{M})_2$ . To show that  $l(5)$  converges, we have but to show that for any  $\epsilon > 0$  there is an  $m > p_0$  such that

$$\left| \sum_{i=m}^{\mu} \left( \sum_{j=0}^{\infty} s_{ij} \phi_{ij}(\alpha) \right) \right| < \epsilon \quad (\text{for all } \mu > m),$$

Assuming  $l(4)$  and  $(\mathcal{A}^*)_2$ , we have for any  $\epsilon$  an  $m > p_0$  such that

$$\left| \sum_{i=m}^{\mu} \left( \sum_{j=0}^{\infty} s_{ij} \phi_{ij}(\alpha) \right) \right| \leq M \sum_{i=m}^{\mu} \left( \sum_{j=0}^{\infty} |\phi_{ij}(\alpha)| \right) < M \frac{\epsilon}{M} = \epsilon,$$

for all  $\mu > m$ . Similarly, we may show that  $l(6)$  converges by showing that for any  $\epsilon > 0$  there is an  $h > q_0$  such that

$$\left| \sum_{j=h}^{\nu} \left( \sum_{i=0}^{\infty} s_{ij} \phi_{ij}(\alpha) \right) \right| < \epsilon \quad (\text{for all } \nu > h).$$

We now exhibit a sequence  $l(1)$  which does not satisfy  $l(4)$  but which is of class  $(\mathcal{A})_2$  relative to a given set of functions  $l(2)$  of class  $(\mathcal{M})_2$ .

We define

$$1(7) \quad A_{i0} = i + 2, \quad (i = 0, 1, \dots),$$

$$A_{ij} = 0, \quad (\text{all other } i, j).$$

$$1(8) \quad \varphi_{i0}(\alpha) = \frac{\alpha}{i(i+1)(i+2)}, \quad (i = 1, 2, \dots),$$

$$\varphi_{0j}(\alpha) = \frac{\alpha}{j(j+1)(j+2)}, \quad (j = 1, 2, \dots),$$

$$\varphi_{00}(\alpha) = 0,$$

$$\varphi_{ij}(\alpha) = \frac{\alpha}{(i+j)(i+j+1)(i+j+2)}, \quad (\text{all other } i, j).$$

Condition  $(\alpha_1)_2$  yields

$$1(9) \quad \overline{\Phi}_p^{(1)}(\alpha) \equiv \sum_{i=p}^{\infty} \left( \sum_{j=0}^{\infty} A_{ij} \varphi_{ij}(\alpha) \right) = \sum_{i=p}^{\infty} \frac{(i+2)\alpha}{i(i+1)(i+2)},$$

which is certainly a convergent series for  $p > 0$ ; and

condition  $(\alpha_2)_2$  yields

$$1(10) \quad \overline{\Phi}_q^{(2)}(\alpha) \equiv \sum_{j=q}^{\infty} \left( \sum_{i=0}^{\infty} A_{ij} \varphi_{ij}(\alpha) \right) = 0, \quad (q > 0).$$

We now show that the series 1(3) converges for each  $\alpha$  in  $E(\alpha)$  for all sequences 1(1) of class  $(\alpha)_2$  relative

to a given set of functions  $l(2)$  of class  $(\mathcal{N})_2$ . If the sequence  $l(1)$ , converging to  $\mathcal{A}$ , is of class  $(\mathcal{O})_2$  relative to a given set of functions  $l(2)$  of class  $(\mathcal{N})_2$ , one can clearly see that the sequence

$$1(11) \quad \{s_{ij} - \mathcal{A}\} \equiv \{S_{ij}\}, \quad (i, j=0, 1, \dots),$$

is also of class  $(\mathcal{O})_2$  relative to the same given set of functions  $l(2)$  of class  $(\mathcal{N})_2$ , and conversely.

To establish the convergence of  $l(3)$  for sequences  $l(1)$  converging to  $\mathcal{A}$  and of class  $(\mathcal{O})_2$  relative to a given set of functions  $l(2)$  of class  $(\mathcal{N})_2$ , it suffices to show that for any  $\epsilon > 0$  there are numbers  $p$  and  $q$  such that for all  $m > p$  and all  $n > q$  the expression

$$1(12) \quad \left| \left[ \sum_{p+1,0}^{m,q} + \sum_{0,q+1}^{p,n} + \sum_{p+1,q+1}^{m,n} \right] S_{ij} \varphi_{ij}(\alpha) \right| < \epsilon$$

for all  $\alpha$  in  $E(\alpha)$ .

Given any  $\epsilon > 0$ , we choose numbers  $p_1$ , and  $q_1$ , such that for all  $i > p_1$ ,  $j > q_1$ ,

$$1(13) \quad |S_{ij}| < \frac{\epsilon}{5K(\alpha)},$$

which choice is possible because the sequence  $l(11)$  is assumed to converge to zero.

In view of the convergence of  $l(5)$  for any  $\epsilon > 0$ , we may choose a number  $p_2$  such that for all  $m > p \geq p_2$  and

for all  $\alpha$  in  $E(\alpha)$

$$1(14) \quad \left| \sum_{i=p+1}^m \left( \sum_{j=0}^{\infty} S_{ij} \phi_{ij}(\alpha) \right) \right| < \frac{\epsilon}{5}.$$

In view of the convergence of 1(6) for any  $\epsilon > 0$ , we may choose a number of  $q_2$  such that for all  $n > q \geq q_2$  and for all  $\alpha$  in  $E(\alpha)$

$$1(15) \quad \left| \sum_{j=q+1}^n \left( \sum_{i=0}^{\infty} S_{ij} \phi_{ij}(\alpha) \right) \right| < \frac{\epsilon}{5}.$$

We may select

$$1(16) \quad p > \max(p_0, p_1, p_2); \quad q > \max(q_0, q_1, q_2).$$

We are now in a position to consider the first term on the right of 1(12). We have

$$1(17) \quad \left| \sum_{i=p+1,0}^{m,q} S_{ij} \phi_{ij}(\alpha) \right| \leq \left| \sum_{i=p+1,0}^{m,\infty} S_{ij} \phi_{ij}(\alpha) \right| + \left| \sum_{i=p+1,q+1}^{m,\infty} S_{ij} \phi_{ij}(\alpha) \right|.$$

In view of 1(14), 1(16), 1(13), and condition  $(A^*)_2$  the term on the left of 1(17) is less than  $2\epsilon/5$ . Similarly, in view of 1(15), 1(16), and 1(13), and conditions  $(A^*)_2$  the second term on the left of 1(12) is in absolute value less than  $2\epsilon/5$ . Finally, the last term on the left of 1(12) can be shown to be less than  $\epsilon/5$  by an application of 1(13), 1(16), and condition  $(A^*)_2$ . We have thus established the validity of 1(12) for all  $p$  and  $q$  satisfying 1(16) and have thereby proved

**Theorem 1.**

The series  $l(3)$  will converge for each  $\alpha$  in  $E(\alpha)$  whenever the sequence  $l(1)$  converges and is of class  $(\alpha)_2$  relative to a given set of functions  $l(2)$  of class  $(\alpha)_2$ .

2. On the existence of convergence factors of type I for convergent double series of a certain class  $(\mathcal{O})_2$ .

We are given a convergent double series,

$$2(1) \quad \sum_{i=0, j=0}^{\infty, \infty} u_{ij} \quad ,$$

which converges to zero and a doubly infinite sequence of functions,

$$2(2) \quad \left\{ f_{ij}(\alpha) \right\} \quad (i, j = 0, 1, \dots),$$

defined in a set of points  $E(\alpha)$  of the type considered in section one.

We define

$$2(3) \quad s_{mn} \equiv \sum_{0,0}^{m,n} u_{ij} \quad ,$$

$$2(4) \quad \Delta_{10} f_{ij}(\alpha) \equiv f_{ij}(\alpha) - f_{i+1,j}(\alpha) \quad ,$$

$$\Delta_{01} f_{ij}(\alpha) \equiv f_{ij}(\alpha) - f_{i,j+1}(\alpha) \quad ,$$

$$\begin{aligned} \Delta_{11} f_{ij}(\alpha) &\equiv \Delta_{10} (\Delta_{01} f_{ij}(\alpha)) \equiv \Delta_{01} (\Delta_{10} f_{ij}(\alpha)) \\ &\equiv f_{ij}(\alpha) - f_{i+1,j}(\alpha) - f_{i,j+1}(\alpha) + f_{i+1,j+1}(\alpha) \end{aligned}$$

In view of  $(A^*)_2$  the series

$$\sum_{i=p, j=q}^{\infty, \infty} \varphi_{ij}(\alpha)$$

Converges for each  $\alpha$  in  $E(\alpha)$ , so that we may define the functions 2(2) in terms of the functions 1(2), thus:

$$2(5) \quad f_{pq}(\alpha) \equiv \sum_{i=p, j=q}^{\infty, \infty} \varphi_{ij}(\alpha),$$

$$2(6) \quad \Delta_{||} f_{ij}(\alpha) \equiv \varphi_{ij}(\alpha).$$

In view of  $(A^*)_2$ , the functions defined by 2(5) satisfy the condition

$$(A)_2 \quad \sum_{i=0, j=0}^{\infty, \infty} |\Delta_{||} f_{ij}(\alpha)| < K(\alpha) \quad (\text{in } E(\alpha)).$$

Since also in view of  $(A^*)_2$  the series

$$\sum_{i=p}^{\infty} \varphi_{ij}(\alpha)$$

Converges for each  $j$  and the series

$$\sum_{j=q}^{\infty} \varphi_{ij}(\alpha)$$

converges for each  $i$ , we have by virtue of 2(5)

$$2(7) \quad \Delta_{01} f_{pq}(\alpha) \equiv \sum_{i=p}^{\infty} \phi_{iq}(\alpha); \quad \Delta_{10} f_{pq}(\alpha) \equiv \sum_{j=q}^{\infty} \phi_{pj}(\alpha).$$

From the convergence of the two series in 2(7) it follows that the functions 2(2) satisfy the two further conditions

$$(D_1)_2 \quad \lim_{n \rightarrow \infty} \Delta_{10} f_{in}(\alpha) = 0 \quad (i = 0, 1, \dots, E(\alpha));$$

$$(D_2)_2 \quad \lim_{m \rightarrow \infty} \Delta_{01} f_{mj}(\alpha) = 0 \quad (j = 0, 1, \dots; E(\alpha)).$$

We now introduce the following two definitions:

DEFINITION 2.1.     Set of functions of class  $(\alpha')_2$ .

The set of functions 2(2) is of class  $(\alpha')_2$  if the functions of the set satisfy conditions  $(A)_2$ ,  $(D_1)_2$ , and  $(D_2)_2$ .

DEFINITION 2.2.     Double series of a certain class  $(\alpha)_2$ .

The convergent double series 2(1) is of class  $(\alpha)_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  if the sequence  $u_{ij}$  defined by 2(3) is of class  $(\alpha)_2$  relative to the given set of functions 2(6) of class  $(\alpha')_2$ , as this latter concept is defined in Definition 1.1.

We now raise the question of the existence of a convergence factor theorem corresponding to the theorem of section one; that is, we ask whether the series

$$2(8) \quad \sum_{i=0, j=0}^{\infty, \infty} u_{ij} f_{ij}(\alpha)$$

will converge for each  $\alpha$  in  $E(\alpha)$  whenever the series 2(1)

converges and is of class  $(\mathcal{O})_2$  relative to a given set of functions 2(2) of class  $(\mathcal{O}')_2$ .

We consider now the sequence of the partial sums of the series 2(8), namely:

$$2(9) \quad \sigma_{\mu\nu}(\alpha) \equiv \sum_{i=0, j=0}^{\mu, \nu} u_{ij} f_{ij}(\alpha),$$

which we break up in the following way according to the Abel Transformation for Double Series:

$$2(10) \quad \sigma_{\mu\nu}(\alpha) = \sum_{i=0, j=0}^{\mu-1, \nu-1} \Delta_{ij} f_{ij}(\alpha) + \sum_{i=0}^{\mu-1} \Delta_{i\nu} f_{i\nu}(\alpha) + \sum_{j=0}^{\nu-1} \Delta_{\mu j} f_{\mu j}(\alpha) + \Delta_{\mu\nu} f_{\mu\nu}(\alpha).$$

If there is a convergence factor theorem corresponding to the theorem of section one, the sequence defined by 2(9) should tend to a finite limit as  $\mu, \nu \rightarrow \infty$ . However, we shall show that one portion of the right hand side of 2(10) becomes infinite as  $\mu, \nu \rightarrow \infty$  for at least one series of class  $(\mathcal{O})_2$  relative to a given set of functions 2(2) of class  $(\mathcal{O}')_2$ , whereas the other portions of the right hand side of 2(10) for that same series tend to a finite limit as  $\mu, \nu \rightarrow \infty$ .

Let us first observe that for all series 2(1) of class  $(\mathcal{O})_2$  relative to a given set of functions 2(2) of class  $(\mathcal{O}')_2$  the first and last terms on the right of 2(10)

tend to a finite limit as  $\mu, \nu \rightarrow \infty$ . The first term on the right of 2(10) tends to a finite limit as  $\mu, \nu \rightarrow \infty$  by the theorem of section one, for we have assumed that the sequence  $\{f_{\mu\nu}(\alpha)\}$  defined by 2(3) is of class  $(\mathcal{O})_2$  relative to a given set of functions 2(6) of class  $(\mathcal{O})_2$ .

From the definitions 2(4) and condition (A)<sub>2</sub>, we have

$$\begin{aligned} |f_{00}(\alpha) - f_{\mu 0}(\alpha)| &= \left| \sum_{i=0}^{\mu-1} \Delta_{i0} f_{i0}(\alpha) \right| = \left| \sum_{0,0}^{\mu-1, \infty} \Delta_{ii} f_{ij}(\alpha) \right| < K(\alpha); \\ |f_{00}(\alpha) - f_{0\nu}(\alpha)| &= \left| \sum_{j=0}^{\nu-1} \Delta_{0j} f_{0j}(\alpha) \right| = \left| \sum_{0,0}^{\infty, \nu-1} \Delta_{ii} f_{ij}(\alpha) \right| < K(\alpha); \\ |f_{\mu 0}(\alpha) - f_{\mu\nu}(\alpha)| &= \left| \sum_{j=0}^{\nu-1} \Delta_{\mu j} f_{\mu j}(\alpha) \right| = \left| \sum_{\mu,0}^{\infty, \nu-1} \Delta_{ii} f_{ij}(\alpha) \right| < K(\alpha). \end{aligned}$$

It follows that

$$|f_{00}(\alpha) - f_{\mu\nu}(\alpha)| \leq |f_{00}(\alpha) - f_{\mu 0}(\alpha)| + |f_{\mu 0}(\alpha) - f_{\mu\nu}(\alpha)| < 2K(\alpha).$$

Since  $|f_{00}(\alpha)|$  is certainly finite for each  $\alpha$  in  $E(\alpha)$ , we have that  $|f_{\mu\nu}(\alpha)|$  for each  $\alpha$  is bounded for all  $\mu, \nu$ . Hence, if  $\Delta_{\mu\nu} \rightarrow 0$  as  $\mu, \nu \rightarrow \infty$ , we have for each  $\alpha$  in  $E(\alpha)$  that

$$\Delta_{\mu\nu} f_{\mu\nu}(\alpha) \rightarrow 0, \quad \text{as } \mu, \nu \rightarrow \infty.$$

We now construct an example which satisfies the hypotheses but which causes the second series on the right of 2(10) to diverge and the third series to converge as  $\mu, \nu \rightarrow \infty$ .

A particular set of functions of class  $(\mathcal{O})_2$  is the

following:

$$2(11) \quad f_{ij}(\alpha) = \frac{\alpha}{i+j}, \quad (i, j = 1, 2, \dots);$$

$$= 0, \quad (i = 0 \text{ or } j = 0).$$

We form  $\Delta_{10} f_{ij}(\alpha) = \frac{\alpha}{(i+j)(i+j+1)}, \Delta_{01} f_{ij}(\alpha) = \frac{\alpha}{(i+j)(i+j+1)}$

$$2(12) \quad \Delta_{11} f_{ij}(\alpha) = \frac{\alpha}{(i+j)(i+j+1)(i+j+2)}$$

We construct a particular sequence  $l(1)$ :

$$2(13) \quad \Delta_{ij} = (-1)^j j^{\frac{5}{2}}, \quad (i = 1, j = 1, 2, \dots);$$

$$= 0, \quad (i \neq 1, j = 1, 2, \dots);$$

The sequence  $l(1)$  defined in 2(13) is of class  $(\alpha)_2$  relative to the given set of functions 2(6) defined in 2(12), which is of class  $(\mu)_2$ . For, condition  $(\alpha_1)_2$  is satisfied:

$$\sum_{i=p}^{\infty} \left( \sum_{j=0}^{\infty} \Delta_{ij} f_{ij}(\alpha) \right) = 0, \quad (p > 1).$$

and condition  $(\alpha_2)_2$  is also satisfied:

$$\sum_{j=q}^{\infty} \left( \sum_{i=0}^{\infty} \Delta_{ij} f_{ij}(\alpha) \right) = \sum_{j=q}^{\infty} \frac{(-1)^j j^{\frac{5}{2}} \alpha}{(i+j)(2+j)(3+j)},$$

which certainly converges for all  $q > 0$ .

For the sequence 2(13) relative to the functions 2(6) defined by 2(12) the second term on the right of 2(10) becomes infinite as  $\mu, \nu \rightarrow \infty$  :

$$\left| \sum_{i=0}^{\mu-1} s_{i\nu} \Delta_{i0} f_{i0}(\alpha) \right| = \left| \frac{(-1)^\nu \nu^{\frac{\nu}{2}} \alpha}{(1+\nu)(2+\nu)} \right| = \left| \frac{\nu^{\frac{\nu}{2}} \alpha}{(1+\nu)(2+\nu)} \right| \rightarrow \infty, \nu \rightarrow \infty.$$

On the other hand, for the sequence 2(13) relative to the functions 2(6) defined by 2(12) the third term on the right of 2(10) is zero for all  $\mu, \nu$  :

$$\left| \sum_{j=0}^{\nu-1} s_{\mu j} \Delta_{0j} f_{\mu j}(\alpha) \right| = \sum_{j=1}^{\nu-1} \frac{0 \cdot \alpha}{(\mu+j)(\mu+j+1)} = 0.$$

Thus, we have shown that there is a sequence 2(13), which may be considered as the sequence 2(3) of the partial sums of a double series 2(1) of class  $(\mathcal{A})_2$  relative to a given set of functions 2(11) of class  $(\mathcal{M}')_2$  for which the series 2(8) will diverge in  $E(\alpha)$ . Hence, in general, the series 2(8) will not converge for each  $\alpha$  in  $E(\alpha)$  whenever the series 2(1) converges and is of class  $(\mathcal{A})_2$  relative to a given set of functions 2(2) of class  $(\mathcal{M}')_2$  or, which is the same, whenever the sequence 2(3) of the partial sums of the series 2(1) is of class  $(\mathcal{A})_2$  relative to a given set of functions 2(6) of class  $(\mathcal{M})_2$ .

3. Convergence factors of type I for convergent double series of a certain class  $(\mathcal{L})_2$ .

Considering the double sequence of functions 2(2) to be of class  $(\mathcal{A}')_2$ , we make the following definition:

DEFINITION 3.1 Double series of a certain class  $(\mathcal{L})_2$ .

The convergent double series 2(1) is of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\mathcal{A}')_2$  if the sequence 2(3) of the partial sums of 2(1) is of class  $(\mathcal{A})_2$  relative to the given set of functions 2(6) of class  $(\mathcal{A})_2$  and if in addition the following two conditions are satisfied:

$$(\beta_1)_2 \quad \lim_{j \rightarrow \infty} \mathcal{A}_{ij} \Delta_{10} f_{ij}(\alpha) = \Psi_i^{(1)}(\alpha), \quad (i = 0, 1, \dots; E(\alpha));$$

$$\sum_{i=0}^{\infty} \Psi_i^{(1)}(\alpha) = \Psi^{(1)}(\alpha), \quad E(\alpha),$$

where  $\Psi_i^{(1)}(\alpha)$  is a bounded function of  $\alpha$ ;

$$(\beta_2)_2 \quad \lim_{i \rightarrow \infty} \mathcal{A}_{ij} \Delta_{01} f_{ij}(\alpha) = \Psi_j^{(2)}(\alpha), \quad (j = 0, 1, \dots; E(\alpha));$$

$$\sum_{j=0}^{\infty} \Psi_j^{(2)}(\alpha) = \Psi^{(2)}(\alpha), \quad E(\alpha),$$

where  $\Psi_j^{(2)}(\alpha)$  is a bounded function of  $\alpha$ .

We wish to show that the series 2(8) will converge for each  $\alpha$  in  $E(\alpha)$  whenever the series 2(1) converges and is of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$ .

Before attempting to prove the statement just made, it would be well to determine the relationship which exists between bounded sequences and sequences of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  and also the relationship which exists between sequences of class  $(\mathcal{A})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  and sequences of class  $(\mathcal{B})_2$  relative to the same given set of functions of class  $(\alpha')_2$ . In the first section we showed that the set of sequences of class  $(\mathcal{A})_2$  relative to a given set of functions of class  $(\alpha)_2$  contains all bounded sequences and others as well. In the present section we shall show that the set of sequences which are partial sums of series of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  contains all bounded sequences and others as well and that the set of series of class  $(\mathcal{A})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  contains the set of all series of class  $(\mathcal{L})_2$  relative to the same given set of functions 2(2) of class  $(\alpha')_2$  and other series as well.

We first show that all bounded sequences are contained in the set of sequences which are partial sums of

series of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$ . In section one we showed that 1(4) implies that  $(\alpha_1)_2$  and  $(\alpha_2)_2$  are both satisfied, and we now show that 1(4) also implies that  $(\beta_1)_2$  and  $(\beta_2)_2$  are satisfied: if we assume 1(4) and  $(D_1)_2$ , then

$$\lim_{j \rightarrow \infty} s_{ij} \Delta_{10} f_{ij}(\alpha) = 0, \quad (i = 0, 1, \dots; E(\alpha)),$$

which is equivalent to  $(\beta_1)_2$ ; and if we assume 1(4) and  $(D_2)_2$ , then

$$\lim_{i \rightarrow \infty} s_{ij} \Delta_{01} f_{ij}(\alpha) = 0, \quad (j = 0, 1, \dots; E(\alpha)),$$

which is equivalent to  $(\beta_2)_2$ . Therefore, if the sequence 2(3) satisfies 1(4) it may be considered as the sequence of the partial sums of a series of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$ .

We now show that there are series of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  which are such that the sequences 2(3) of their partial sums do not satisfy 1(4). Such a sequence is the following:

$$\begin{aligned} 3(1) \quad s_{ij} &= (i+2) && (i = 1, 2, \dots); \\ s_{ij} &= 0 && (\text{all other } i, j). \end{aligned}$$

A simple computation shows that the sequence 2(3) defined in 3(1) is of class  $(\mathcal{L})_2$  relative to the functions 2(2) of class  $(\alpha')_2$  defined in 2(11).

That the set of series of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  includes the set of series of class  $(\mathcal{L})_2$  relative to the same given set of functions 2(2) of class  $(\alpha')_2$  is obvious from the definitions of these two concepts. By recalling the example of section two, one observes that there are series of class  $(\mathcal{A})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  which are not series of class  $(\mathcal{L})_2$  relative to the same given set of functions 2(2) of class  $(\alpha')_2$ .

We are now ready to prove that the series 2(8) will converge for each  $\alpha$  in  $E(\alpha)$  whenever the series 2(1) converges and is of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$ .

It suffices to show that the sequence defined in 2(9), tends to a limit as  $\mu, \nu \rightarrow \infty$ . If, as in section two, the series on the right of 2(9) is broken up according to the Abel Transformation for Double Series, we need only show that the terms on the right of 2(10) approach a limit as  $\mu, \nu \rightarrow \infty$ . In section two we showed that for series of class  $(\mathcal{A})_2$  relative to a given set of functions of class  $(\alpha')_2$  the first and last terms on the right of 2(10) tend to a limit as  $\mu, \nu \rightarrow \infty$ . Since

series of class  $(L)_2$  relative to a given set of functions  $2(2)$  of class  $(\alpha')_2$  are also of class  $(L)_2$  relative to the same given set of functions  $2(2)$  of class  $(\alpha')_2$ , the first and last terms on the right of 2(10) will also tend to a limit as  $\mu, \nu \rightarrow \infty$  for series of class  $(L)_2$  relative to a given set of functions  $2(2)$  of class  $(\alpha')_2$ . There remains now only to show that the second and third terms on the right of 2(10) tend to a finite limit as  $\mu, \nu \rightarrow \infty$ .

To show that the second term,

$$\sum_{i=0}^{\mu-1} s_{i\nu} \Delta_{10} f_{i\nu}(\alpha),$$

tends to a finite limit as  $\mu, \nu \rightarrow \infty$  for each  $\alpha$  in  $E(\alpha)$ , it suffices to show that for any  $\epsilon > 0$ , there are numbers  $M$  and  $N$  such that for all  $\mu > M$  and for all  $\nu > N$  we have

$$3(2) \quad \left| \sum_{i=0}^{\mu-1} s_{i\nu} \Delta_{10} f_{i\nu}(\alpha) - \sum_{i=0}^{\infty} \Psi_i^{(1)}(\alpha) \right| < \epsilon.$$

Given any  $\epsilon > 0$ , there are numbers  $p, q$  such that

$$3(3) \quad |s_{ij}| < \frac{\epsilon}{3K(\alpha)} \quad (i > p, j > q),$$

in virtue of the assumption at the beginning of section two that the sequence of the partial sums 2(3) converges to zero. Corresponding to the  $\epsilon > 0$  just chosen, there is a number  $p$ , such that

$$3(4) \quad \left| \sum_{i=\bar{p}+1}^{\infty} \Psi_i^{(u)}(\alpha) \right| < \frac{\epsilon}{3}, \quad (\text{all } \bar{p} > p),$$

in virtue of the convergence of the series  $\sum \Psi_i^{(u)}(\alpha)$  in  $(\beta, \gamma)$ . We define

$$3(5) \quad M = \max(p, p_1).$$

Corresponding to the same  $\epsilon > 0$ , there is a  $q$ , such that

$$3(6) \quad \left| \mathcal{L}_{i\gamma} \Delta_{10} f_{i\gamma}(\alpha) - \Psi_i^{(u)}(\alpha) \right| < \frac{\epsilon}{3(M+1)} \quad (\text{all } \gamma > q),$$

We define

$$3(7) \quad N = \max(q, q_1),$$

and then we break up the left hand side of 3(2) to form

$$3(8) \quad \left| \sum_{i=0}^M \mathcal{L}_{i\gamma} \Delta_{10} f_{i\gamma}(\alpha) + \sum_{i=M+1}^{\mu-1} \mathcal{L}_{i\gamma} \Delta_{10} f_{i\gamma}(\alpha) - \sum_{i=0}^M \Psi_i^{(u)}(\alpha) \right. \\ \left. - \sum_{i=M+1}^{\infty} \Psi_i^{(u)}(\alpha) \right| \leq \left| \sum_{i=0}^M (\mathcal{L}_{i\gamma} \Delta_{10} f_{i\gamma}(\alpha) - \Psi_i^{(u)}(\alpha)) \right| \\ + \left| \sum_{i=M+1}^{\mu-1} \mathcal{L}_{i\gamma} \Delta_{10} f_{i\gamma}(\alpha) \right| + \left| \sum_{i=M+1}^{\infty} \Psi_i^{(u)}(\alpha) \right|.$$

In virtue of 3(6), 3(5), and 3(7) a consideration of the first term on the right of 3(8) yields

$$\left| \sum_{i=0}^M (\mathcal{L}_{i\gamma} \Delta_{10} f_{i\gamma}(\alpha) - \Psi_i^{(u)}(\alpha)) \right| < \frac{\epsilon}{3} \quad (\text{for } \gamma > N);$$

in virtue of 3(3) a consideration of the second term yields

$$\left| \sum_{i=M+1}^{\mu-1} \Delta_{i\mathcal{V}} f_{i\mathcal{V}}(\alpha) \right| < \frac{\epsilon}{3K(\alpha)} \sum_{i=M+1}^{\mu-1} \left( \sum_{j=\mathcal{V}}^{\infty} |\Delta_{ij} f_{ij}(\alpha)| \right) < \frac{\epsilon}{3}$$

for  $\mu > M$ ,  $\mathcal{V} > N$ ; and finally, a consideration of the last term yields by 3(4)

$$\left| \sum_{i=M+1}^{\infty} \Psi_i^{(1)}(\alpha) \right| < \frac{\epsilon}{3}.$$

We have thus established the inequality 3(2) for all

$\mu > M$  and all  $\mathcal{V} > N$ .

The third term on the right of 2(10) may be treated in a manner wholly analogous to the treatment of the second term. To show that the third term tends to a limit as  $\mu, \mathcal{V} \rightarrow \infty$ , we may thus show that for any  $\epsilon > 0$  there are numbers  $M$  and  $N$  such that

$$3(9) \quad \left| \sum_{j=0}^{\mathcal{V}-1} \Delta_{\mu j} f_{\mu j}(\alpha) - \sum_{i=0}^{\infty} \Phi_j^{(2)}(\alpha) \right| < \epsilon, \quad (\mu > M, \mathcal{V} > N).$$

In the above we have considered only series 2(1) converging to zero. Any convergent double series which does not converge to zero can be reduced to the case treated above by alteration of the term  $u_{00}$ , and since such an alteration does not affect the convergence

of the series 2(8), we have now completely established the following

Theorem 2.

The series 2(8) will converge for each  $\alpha$  in  $E(\alpha)$  for all series 2(1) converging to  $\mathcal{A}$  and of class  $(\mathcal{L})_2$  relative to a given set of functions 2(2) of class  $(\mathcal{A}')_2$ .

4. Sequence to function transformations for double sequences of a certain class  $(\overline{\mathcal{A}})_2$ .

We consider in this section the case where each of the functions in the set 1(2) of section one approaches a finite limit as  $\alpha \rightarrow \alpha_0$ , not in  $E(\alpha)$ . Let us first make the following two definitions:

DEFINITION 4.1. Set of functions of class  $(\overline{\mathcal{A}})_2$ .

A set of functions 1(2) is of class  $(\overline{\mathcal{A}})_2$  if it is of class  $(\mathcal{A})_2$  as this class was defined by Definition 1.1. and if in addition the functions of the set satisfy the four following conditions:

$$(C^*)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i=0, j=0}^{\infty, \infty} \varphi_{ij}(\alpha) = 1;$$

$$(E_1^*)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{j=0}^{\infty} |\varphi_{ij}(\alpha)| = 0, \quad \text{each } i;$$

$$(E_2^*)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i=0}^{\infty} |\varphi_{ij}(\alpha)| = 0, \quad \text{each } j;$$

$$(A_1^*)_2 \quad \sum_{i=0, j=0}^{\infty, \infty} |\varphi_{ij}(\alpha)| < K \quad \text{each } \alpha \text{ in } E'(\alpha),$$

where  $E'(\alpha)$  is a subset of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all  $\alpha$  of  $E(\alpha)$  in a certain

neighborhood of  $\alpha_0$ .

DEFINITION 4.2. Sequence of a certain class  $(\overline{\mathcal{A}})_2$ .

The convergent double sequence 1(1) is of class  $(\overline{\mathcal{A}})_2$  relative to a given set of functions 1(2) of class  $(\overline{\mathcal{A}})_2$  if the sequence 1(1) is of class  $(\overline{\mathcal{A}})_2$  relative to the given set of functions 1(2) of class  $(\overline{\mathcal{A}})_2$  as this concept was defined in Definition 1.2. and if in addition the functions of conditions  $(\alpha_1)_2$  and  $(\alpha_2)_2$  are such that they satisfy also the following two conditions:

$$(\overline{\alpha}_1)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \int_p^{(1)} (\alpha) = 0, \quad (p > p_0);$$

$$(\overline{\alpha}_2)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \int_q^{(2)} (\alpha) = 0, \quad (q > q_0).$$

It is the object of this section to show that the limit of the series 1(3) as  $\alpha \rightarrow \alpha_0$  shall be  $\mathcal{A}$  for every sequence 1(1) converging to  $\mathcal{A}$  and of class  $(\overline{\mathcal{A}})_2$  relative to a given set 1(2) of class  $(\overline{\mathcal{A}})_2$ .

It suffices to show that given an  $\epsilon > 0$ , there is a subset  $\mathcal{E}(\alpha)$  of  $E(\alpha)$ , having  $\alpha_0$  as a limit point and containing all  $\alpha$  of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$4(1) \quad \left| \sum_{i=0, j=0}^{\infty, \infty} s_{ij} \phi_{ij}(\alpha) - \mathcal{A} \right| < \epsilon.$$

Since we have assumed that the double sequence 1(1) converges to  $a$ , for any  $\epsilon > 0$  there are numbers  $p, q$  such that for all  $i > p, j > q$

$$4(2) \quad |s_{ij} - a| < \frac{\epsilon}{9K}$$

and also in view of  $(C^*)_{1,2}, (E^*)_{1,2}$ , and  $(E^*)_{2,2}$  for the numbers  $p, q$  there is a set  $\mathcal{E}(\alpha)$  which is a subset of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$4(3) \quad \left| \sum_{i=p+1, j=q+1}^{\infty, \infty} \phi_{ij}(\alpha) - 1 \right| < \frac{\epsilon}{9|a|}$$

Let us break up the left hand side of 4(1) in the following manner:

$$4(4) \quad \left| \sum_{i=0, j=0}^{\infty, \infty} s_{ij} \phi_{ij}(\alpha) - a \right| \leq \sum_{i=0, j=0}^{p, q} |s_{ij} \phi_{ij}(\alpha)| + 3 \sum_{i=p+1, j=q+1}^{\infty, \infty} |(s_{ij} - a)| |\phi_{ij}(\alpha)| + \left| \sum_{i=p+1, j=0}^{\infty, \infty} s_{ij} \phi_{ij}(\alpha) \right| + \left| \sum_{i=0, j=q+1}^{\infty, \infty} s_{ij} \phi_{ij}(\alpha) \right| + 3|a| \cdot \left| \sum_{i=p+1, j=q+1}^{\infty, \infty} \phi_{ij}(\alpha) - 1 \right|$$

For the consideration of the first term on the right of 4(4) it is convenient to define

$$M = \max |s_{ij}|, \quad (i = 0, 1, \dots, p; j = 0, 1, \dots, q).$$

In view of  $(E^*)_{1,2}, (E^*)_{2,2}$ , and the definition of  $M$  there

is a set  $\mathcal{E}_2(\alpha)$  which is a subset of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$4(5) \quad |\varphi_{ij}(\alpha)| < \frac{\epsilon}{9M(p+1)(q+1)} \quad (i = 0, 1, \dots, p; j = 0, 1, \dots, q).$$

For each in

$$4(6) \quad \sum_{0,0}^{p,q} |\rho_{ij} \varphi_{ij}(\alpha)| \leq M \sum_{0,0}^{p,q} |\varphi_{ij}(\alpha)| < \frac{\epsilon}{9}.$$

In view of  $(A^*)_2$  and 4(2) we have for all  $\alpha$  in  $E(\alpha)$

$$4(7) \quad 3 \sum_{p+1, q+1}^{\infty, \infty} |(\rho_{ij} - \rho) \varphi_{ij}(\alpha)| \leq \frac{3\epsilon}{9K} \sum_{p+1, q+1}^{\infty, \infty} |\varphi_{ij}(\alpha)| < \frac{\epsilon}{3}.$$

Considering the third term on the right of 4(4) we have in view of  $(\bar{\alpha}_1)_2$  a subset  $\mathcal{E}_3(\alpha)$  of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which  $p > p_0$

$$4(8) \quad |\Phi_p^{(1)}(\alpha)| < \frac{\epsilon}{9};$$

and thus for each  $\alpha$  in  $\mathcal{E}_3(\alpha)$  the third term on the right hand side of 4(4) is in absolute value less than  $\epsilon/9$ .

The fourth term may be treated in a manner wholly analogous to the treatment of the third term; so that in view of condition  $(\bar{\alpha}_2)_2$  there is a subset  $\mathcal{E}_4(\alpha)$  having  $\alpha_0$  as a

limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  for all  $\alpha$  in which for  $q > q_0$

$$4(9) \quad \left| \Phi_q^{(2)}(\alpha) \right| < \frac{\epsilon}{9}$$

and thus for each  $\alpha$  in  $E_4(\alpha)$  the fourth term on the right of 4(4) is in absolute value less than  $\epsilon/9$ .

Finally, in view of 4(3) we have for each  $\alpha$  in  $E_1(\alpha)$  that

$$4(10) \quad \left| a \right| \left| \sum_{p+1, q+1}^{\infty, \infty} a_{ij}(\alpha) - 1 \right| < \frac{\epsilon}{9}$$

In view of 4(6), 4(7), 4(8), 4(9), and 4(10) there is a set  $E(\alpha)$  containing only points common to the sets  $E_1(\alpha)$ ,  $E_2(\alpha)$ ,  $E_3(\alpha)$ ,  $E_4(\alpha)$ , and  $E_5(\alpha)$  and which is a subset of  $E(\alpha)$  having  $\alpha_0$  and containing only points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which 4(2) subsists.

Thus, we have proved the following:

### Theorem 3.

The series 1(3) as  $\alpha \rightarrow \alpha_0$  will tend to  $a$  for every sequence 1(1) converging to  $a$  and of class  $(\bar{\alpha})_1$  relative to a given set of functions 1(2) of class  $(\bar{\alpha})_2$ .

5. Convergence factors of type II for convergent double series of a certain class  $(\overline{\mathcal{L}})_2$ .

We consider in this section the case where each of the functions  $z(z)$  of section two approaches a limit as  $\alpha \rightarrow \alpha_0$ , a limit point of  $E(\alpha)$  not in  $E(\alpha)$ . In view of conditions  $(E_1^*)_2$  and  $(E_2^*)_2$  and equation 2(6) there immediately follow the two conditions

$$(E_1)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{j=0}^{\infty} |\Delta_{ij} f_{ij}(\alpha)| = 0, \quad (\text{all } i);$$

$$(E_2)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i=0}^{\infty} |\Delta_{ij} f_{ij}(\alpha)| = 0, \quad (\text{all } j).$$

Similarly, condition  $(A_1^*)_2$  and equation 2(6) imply condition

$$(A_1)_2 \quad \sum_{i=0, j=0}^{\infty, \infty} |\Delta_{ij} f_{ij}(\alpha)| < K \quad \text{in } E'(\alpha),$$

where  $E'(\alpha)$  contains all points of  $E(\alpha)$  lying in a certain neighborhood of  $\alpha_0$  and where  $K$  is a positive number.

Finally,  $(E_1)_2$ ,  $(E_2)_2$ ,  $(C^*)_2$ , and 2(6) also imply condition

$$(C)_2 \quad \lim_{\alpha \rightarrow \alpha_0} f_{ij}(\alpha) = 1 \quad (i, j = 0, 1, 2, \dots).$$

DEFINITION 5.1. Set of functions of class  $(\overline{\mathcal{A}})_2$ .

The set of functions  $z(z)$  is of class  $(\overline{\mathcal{A}})_2$  if it

is of class  $(\alpha')_2$  and if in addition the functions of the set satisfy conditions  $(E)_2$ ,  $(E_2)_2$ ,  $(A)_2$ , and  $(C)_2$ .

DEFINITION 5.2.     Double series of a certain class  $(\bar{L})_2$ .

The convergent double series 2(1) is of class  $(\bar{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  if the sequence of the partial sums  $\Psi$  defined by 2(3) is of class  $(\bar{\alpha})_2$  relative to the given set of functions 2(6) of class  $(\bar{\alpha})_2$  and if in addition the two following conditions are satisfied:

$$(\bar{\beta}_1)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \Psi^{(1)}(\alpha) = 0 ;$$

$$(\bar{\beta}_2)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \Psi^{(2)}(\alpha) = 0 .$$

We wish to show that the series 2(8) will converge in  $E(\alpha)$  and the function of  $\alpha$  defined by it will tend to the same value as the convergent series 2(1) of class  $(\bar{L})_2$  relative to a given set of functions 2(2) of class  $(\alpha')_2$  whenever  $\alpha \rightarrow \alpha_0$ .

If we assume that the series 2(1) converges to zero, we may use with advantage the results of section three, as well as those of section one. We again form the Abel Transformation for Double Series, which is as follows:

$$5(1) \quad \mathcal{T}_{\mu\nu}(\alpha) \equiv \sum_{i=0, j=0}^{\mu, \nu} a_{ij} f_{ij}(\alpha) = \sum_{i=0, j=0}^{\mu-1, \nu-1} s_{ij} \Delta_{ij} f_{ij}(\alpha) + \\ \sum_{i=0}^{\mu-1} s_{i\nu} \Delta_{i\nu} f_{i\nu}(\alpha) + \sum_{j=0}^{\nu-1} s_{\mu j} \Delta_{\mu j} f_{\mu j}(\alpha) + s_{\mu\nu} f_{\mu\nu}(\alpha).$$

In view of section three we know that under our present conditions the limit as  $\mu, \nu \rightarrow \infty$  of the sequence defined by 5(1) exists for each  $\alpha$  in  $E(\alpha)$ ; it thus suffices for our present purposes to show that there is a set  $\bar{E}(\alpha)$  which is a subset of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which for a given  $\epsilon > 0$ ,

$$5(2) \quad \left| \lim_{\mu, \nu \rightarrow \infty} \mathcal{T}_{\mu\nu}(\alpha) \right| < \epsilon.$$

In view of Theorem 3 just proved, there is a set  $\bar{E}_1(\alpha)$  which is a subset of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$5(3) \quad \left| \sum_{i=0, j=0}^{\infty, \infty} s_{ij} \Delta_{ij} f_{ij}(\alpha) \right| \equiv \left| \sum_{i=0, j=0}^{\infty, \infty} s_{ij} \phi_{ij}(\alpha) \right| < \frac{\epsilon}{4}.$$

Since we have shown in section three that the second term on the right of 5(1) converges to  $\bar{\Psi}^{(u)}(\alpha)$  as  $\mu, \nu \rightarrow \infty$ , we need only observe that in view of  $(\bar{\beta}, )_2$  there is a subset  $\bar{E}_2(\alpha)$  of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing

all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$5(4) \quad \left| \Psi^{(1)}(\alpha) \right| < \frac{\epsilon}{4}.$$

Similarly, since by section three the third term on the right of 5(1) converges to  $\bar{\Psi}^{(2)}(\alpha)$  as  $\mu, \nu \rightarrow \infty$ , we need only observe that in view of  $(\bar{\beta}_2)_2$  there is a subset  $\bar{E}_3(\alpha)$  of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$5(5) \quad \left| \Psi^{(2)}(\alpha) \right| < \frac{\epsilon}{4}.$$

In section two we pointed out that the limit of the fourth term on the right of 5(1) as  $\mu, \nu \rightarrow \infty$  is zero for all  $\alpha$  in  $E(\alpha)$ . Thus, in view of 5(3), 5(4), and 5(5), there is a subset  $\bar{E}(\alpha)$  of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points common to  $\bar{E}_1(\alpha)$ ,  $\bar{E}_2(\alpha)$ , and  $\bar{E}_3(\alpha)$  in which the inequality 5(2) subsists.

If the series 2(1) converges to  $A \neq 0$ , one may reduce its treatment to the case of a series converging to zero by alteration of the first term  $u_{00}$ , of series 2(1).

We have thus established the following:

**Theorem 4.**

The series 2(8) will converge in  $E(\alpha)$  and the function of  $\alpha$  defined by it will tend to the same value as the convergent series 2(1) of Class  $(\bar{L})_2$  relative to a given set of functions 2(2) of class  $(\bar{u})_2$  whenever  $\alpha \rightarrow \alpha_0$ .

6. Sequence to function transformations for double sequences of a certain class  $(\bar{\mathcal{L}})_2$ .

At the outset we make the following definition:

DEFINITION 6.1. Double sequence of a certain class  $(\bar{\mathcal{L}})_2$ .

The convergent double sequence  $l(1)$  is of class  $(\bar{\mathcal{L}})_2$  relative to the given set of functions  $l(2)$  of class  $(\bar{\mathcal{L}})_2$  if the following two conditions are satisfied:

$(\bar{\mathcal{J}}_1)_2$  For each fixed  $j$  the simple sequence  $l(1)$  is such that the series on the right of

$$6(1) \quad \int_j^{(1)}(\alpha) \equiv \sum_{i=0}^{\infty} s_{ij} \varphi_{ij}(\alpha)$$

converges for each  $j$ , and the function on the left of 6(1) is defined for each  $\alpha$  in  $E(\alpha)$  and is also such that

$$6(2) \quad \lim_{\alpha \rightarrow \alpha_0} \int_j^{(1)}(\alpha) = 0, \quad (\text{each } j);$$

$(\bar{\mathcal{J}}_2)_2$  For each fixed  $i$  the simple sequence  $l(1)$  is such that the series on the right of

$$6(3) \quad \int_i^{(2)}(\alpha) \equiv \sum_{j=0}^{\infty} s_{ij} \varphi_{ij}(\alpha)$$

converges for each  $i$ , and the function on the left of 6(3) is defined for each  $\alpha$  in  $E(\alpha)$  and is also such that

$$6(4) \quad \lim_{\alpha \rightarrow \alpha_0} \sqrt[i]{(\alpha)} = 0, \quad (\text{each } i).$$

We wish to show that given a double sequence 1(1) converging to  $s$  and of class  $(\mathcal{L}^{\bar{v}})_2$  relative to a given set of functions 1(2) of class  $(\bar{a})_2$  and corresponding to any  $\epsilon > 0$ , there is a number  $N$  and a set  $E(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , such that for all  $\alpha$  in  $E(\alpha)$  and all  $\mu, \nu > N$

$$6(5) \quad \left| \sum_{0,0}^{\mu,\nu} s_{ij} \phi_{ij}(\alpha) - s \right| < \epsilon.$$

We have

$$6(6) \quad \left| \sum_{0,0}^{\mu,\nu} s_{ij} \phi_{ij}(\alpha) - s \right| = \left| \sum_{0,0}^{\mu,\nu} s_{ij} \phi_{ij}(\alpha) - s \sum_{0,0}^{\mu,\nu} \phi_{ij}(\alpha) + s \sum_{0,0}^{\mu,\nu} \phi_{ij}(\alpha) - s \right|$$

$$\leq \left| \sum_{0,0}^{\mu,\nu} (s_{ij} - s) \phi_{ij}(\alpha) \right| + |s| \left| \sum_{0,0}^{\mu,\nu} \phi_{ij}(\alpha) - 1 \right|.$$

In view of the definition of  $f_{00}(\alpha)$  in section two,

$$6(7) \quad f_{00}(\alpha) = \sum_{0,0}^{\infty,\infty} \phi_{ij}(\alpha),$$

and hence for any  $\epsilon > 0$  there is a number  $N_1$ , such that for all  $\mu, \nu > N_1$  and each  $\alpha$  in  $E(\alpha)$

$$6(8) \quad \frac{\epsilon}{4|\mathcal{A}|} > \left| \sum_{0,0}^{\mu,\nu} \phi_{ij}(\alpha) - f_{00}(\alpha) \right| \geq \left| \sum_{0,0}^{\mu,\nu} \phi_{ij}(\alpha) - 1 \right| - \left| f_{00}(\alpha) - 1 \right|.$$

In view of condition  $(C^*)_2$  there is a set  $\mathcal{E}'(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which

$$\left| f_{00}(\alpha) - 1 \right| < \frac{\epsilon}{4|\mathcal{A}|};$$

and hence for all  $\mu, \nu > N$  and for each  $\alpha$  in  $\mathcal{E}'(\alpha)$

$$6(9) \quad \left| \sum_{0,0}^{\mu,\nu} \phi_{ij}(\alpha) - 1 \right| < \frac{\epsilon}{4|\mathcal{A}|} + \frac{\epsilon}{4|\mathcal{A}|} = \frac{\epsilon}{2|\mathcal{A}|}.$$

Thus, it suffices to show that given an  $\epsilon > 0$ , there is a number  $N \geq N_1$  and a set  $\mathcal{E}''(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which for all  $\mu, \nu > N$

$$6(10) \quad \left| \sum_{0,0}^{\mu,\nu} (\mathcal{A}_{ij} - \mathcal{A}) \phi_{ij}(\alpha) \right| < \frac{\epsilon}{2}.$$

Let us first observe that if the sequence 1(1) converges to  $\mathcal{A} \neq 0$  and if it is of class  $(\mathcal{L})_2$  relative to

a given set of functions  $l(2)$  of class  $(\overline{\sigma})_2$ , then the sequence,

$$6(11) \quad \left\{ s_{ij} \right\} \equiv \left\{ s_{ij} - s \right\} \quad (i, j = 0, 1, \dots),$$

is of class  $(\overline{\mathcal{L}})_2$  relative to the same given set of functions  $l(2)$  of class  $(\overline{\sigma})_2$ . For, we have

$$\begin{aligned} \left| \sum_{i=0}^{\infty} (s_{ij} - s) \phi_{ij}(\alpha) \right| &\leq \left| \sum_{i=0}^{\infty} s_{ij} \phi_{ij}(\alpha) \right| + \left| \sum_{i=0}^{\infty} s \phi_{ij}(\alpha) \right| \\ &\leq |s| \sum_{i=0}^{\infty} |\phi_{ij}(\alpha)| + \left| \sum_{i=0}^{\infty} s_{ij} \phi_{ij}(\alpha) \right|, \end{aligned}$$

of which the first term on the right tends to zero as  $\alpha \rightarrow \alpha_0$  in virtue of  $(E_2^*)$  and of which the second term tends to zero as  $\alpha \rightarrow \alpha_0$  in virtue of  $(\overline{\delta})_2$ . Similarly, we may show that

$$\left| \sum_{j=0}^{\infty} (s_{ij} - s) \phi_{ij}(\alpha) \right|$$

tends to zero as  $\alpha \rightarrow \alpha_0$  in virtue of  $(E_1^*)$  and  $(\overline{\delta}_2)_2$ .

Thus, to arrive at our desired result, it suffices to show that given an  $\epsilon > 0$ , there is a number  $N$  and a set  $\mathcal{E}''(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , such that for all  $\mu, \nu > N$  and all  $\alpha$

in  $\mathcal{E}''(\alpha)$

$$6(12) \quad \left| \sum_{0,0}^{\mu,\nu} s_{ij} \phi_{ij}(\alpha) \right| < \frac{\epsilon}{2}.$$

We break up the series 6(12) into four parts, as follows:

$$6(13) \quad \left| \sum_{0,0}^{\mu,\nu} s_{ij} \phi_{ij}(\alpha) \right| \leq \left[ \sum_{0,0}^{p,q} + \sum_{p+1,q+1}^{\mu,\nu} + \sum_{0,q+1}^{p,\nu} + \sum_{p+1,0}^{\mu,q} \right] s_{ij} \phi_{ij}(\alpha),$$

where  $p, q$  are such that for any  $\epsilon > 0$ , we have for all  $i > p, j > q$

$$6(14) \quad |s_{ij}| < \frac{\epsilon}{16K}.$$

We define

$$M = \max |s_{ij}| \quad (i = 0, 1, \dots, p; j = 0, 1, \dots, q).$$

In view of  $(E_1^*)_2, (E_2^*)_2$ , and the definition of  $M$ , there is a set  $\mathcal{E}'(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which

$$6(15) \quad |\phi_{ij}(\alpha)| < \frac{\epsilon}{16M(p+1)(q+1)} \quad (i = 0, 1, \dots, p; j = 0, 1, \dots, q).$$

Thus, for each  $\alpha$  in  $\mathcal{E}'(\alpha)$

$$6(16) \quad \left| \sum_{0,0}^{p,q} s_{ij} \phi_{ij}(\alpha) \right| \leq M \sum_{0,0}^{p,q} |\phi_{ij}(\alpha)| < \frac{\epsilon}{16}.$$

In view of  $(A_{1,2}^*)$  we have for all  $\alpha$  in  $E'(\alpha)$

$$6(17) \quad \left| \sum_{p+1, q+1}^{\mu, \nu} s_{ij} \phi_{ij}(\alpha) \right| \leq \frac{\epsilon}{16K} \sum_{p+1, q+1}^{\mu, \nu} |\phi_{ij}(\alpha)| < \frac{\epsilon}{16}.$$

Considering the third term on the right of 6(13), we may write

$$\left| \sum_{0, q+1}^{p, \nu} s_{ij} \phi_{ij}(\alpha) \right| \leq \sum_{i=0}^p \left[ \left| \sum_{j=0}^q s_{ij} \phi_{ij}(\alpha) \right| + M \sum_{j=0}^q |\phi_{ij}(\alpha)| \right].$$

Given  $\epsilon > 0$ , there is an  $N_1$ , such that in view of 6(3) for all  $\nu > N$

$$\left| \sum_{j=0}^q s_{ij} \phi_{ij}(\alpha) \right| - \left| \Gamma_i^{(2)}(\alpha) \right| \leq \left| \sum_{j=0}^q s_{ij} \phi_{ij}(\alpha) - \Gamma_i^{(2)}(\alpha) \right| < \frac{\epsilon}{16(p+1)},$$

$$\left| \sum_{j=0}^q s_{ij} \phi_{ij}(\alpha) \right| < \frac{\epsilon}{16(p+1)} + \left| \Gamma_i^{(2)}(\alpha) \right|.$$

Further, in view of 6(4) there is a set  $E_2(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which for  $i = 0, 1, \dots, p$ ;  $j = 0, 1, \dots, q$

$$\left| \Gamma_i^{(2)}(\alpha) \right| < \frac{\epsilon}{16(p+1)} \quad ; \quad |\phi_{ij}(\alpha)| < \frac{\epsilon}{16M(p+1)(q+1)}.$$

Thus, in  $E_2(\alpha)$  and for all  $\nu > N_1$

$$6(18) \quad \left| \sum_{0, q+1}^{p, \nu} s_{ij} \phi_{ij}(\alpha) \right| \leq \frac{\epsilon}{8} + \frac{\epsilon}{16} = \frac{3\epsilon}{16}.$$

The fourth term on the right of 6(13) may be treated in a manner analogous to the treatment of the third term, showing that for any given  $\epsilon > 0$  there is a number  $N_2$  and a set  $\mathcal{E}_3(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which for all  $\mu > N_2$

$$6(19) \quad \left| \sum_{p+1,0}^{\mu,9} s_{ij} \phi_{ij}(\alpha) \right| \leq \frac{\epsilon}{8} + \frac{\epsilon}{16} = \frac{3\epsilon}{16}.$$

Hence, in view of 6(16), 6(17), 6(18), and 6(19) for any given  $\epsilon > 0$  there is a set

$$\mathcal{E}''(\alpha) = \mathcal{E}'(\alpha) \cdot E'(\alpha) \cdot \mathcal{E}_2(\alpha) \cdot \mathcal{E}_3(\alpha),$$

containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which for all

$$\mu, \nu > N \equiv \max(N_1, N_2)$$

the inequality 6(12) subsists. Thus, we have now established

Theorem 5.

Corresponding to every positive number  $\epsilon$  there is a number  $N$  and a set  $\mathcal{E}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which the sequence

$$S_{\mu, \nu}(\alpha) \equiv \sum_{i=0, j=0}^{\mu, \nu} s_{ij} \phi_{ij}(\alpha)$$

differs from  $\alpha$  in absolute value by less than  $\epsilon$  for all  $\mu, \nu > N$  and for every sequence  $l(1)$  converging to  $\alpha$  and of class  $(\bar{L})_2$  relative to a given set of functions  $l(2)$  of class  $(\bar{\pi})_2$ .

It is of interest to note that Theorem 5 is a generalization of a theorem established by C. R. Adams<sup>1</sup>, for, to obtain Adams' result which he states as his theorem 2, we have merely to replace the point set  $E(\alpha)$  by the set  $(\mu, \nu)$  of all lattice points in the first quadrant, to replace  $\alpha_0$  by the point  $(\infty, \infty)$ , to replace the function  $\phi_{ij}(\alpha)$  by  $a_{\mu\nu ij}$  and  $S_{\mu\nu}(\alpha)$  by  $S_{\mu\nu}(\mu, \nu)$  or more simply by  $S_{\mu\nu}$ , and finally to sum the series in  $(A^*)_2$ ,  $(C^*)_2$ , and  $(A^*)_2$  from  $(0, 0)$  to  $(\mu, \nu)$  and to sum the series in  $(E^*)_2$  from zero to  $\nu$  and the series in  $(E^*)_2$  from zero to  $\mu$ .

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 1. Adams, C. R. Transformations of Double Sequences with Application to Cesàro Summability of Double Series. Bulletin of the American Mathematical Society. Vol. 37 (1931) pp. 741-748.

7. On convergent double series of a certain class  
 $(\overline{\mathcal{P}})_2$ .

We make first the following definition:

DEFINITION 7.1. Double series of a certain class  $(\overline{\mathcal{P}})_2$ .

The convergent double series 2(1) is of class  $(\overline{\mathcal{P}})_2$  relative to a given set of functions 2(2) of class  $(\overline{\mathcal{U}})_2$  if the sequence of the partial sums 2(3) of the partial sums of 2(1) is of class  $(\overline{\mathcal{L}})_2$  relative to the given set of functions 2(6) of class  $(\overline{\mathcal{U}})_2$  and if in addition the two following conditions are satisfied:

$$(\overline{\delta}_1)_2 \quad \lim_{j \rightarrow \infty} \mathcal{A}_{ij} \Delta_{i0} f_{ij}(\alpha) = \Lambda_i^{(1)}(\alpha), \quad (i=0,1,\dots; E(\alpha));$$

$$\lim_{\alpha \rightarrow \alpha_0} \Lambda_i^{(1)}(\alpha) = 0, \quad (i=0,1,\dots);$$

$$(\overline{\delta}_2)_2 \quad \lim_{i \rightarrow \infty} \mathcal{A}_{ij} \Delta_{01} f_{ij}(\alpha) = \Lambda_j^{(2)}(\alpha), \quad (j=0,1,\dots; E(\alpha));$$

$$\lim_{\alpha \rightarrow \alpha_0} \Lambda_j^{(2)}(\alpha) = 0, \quad (j=0,1,\dots).$$

We wish to show that given any positive number  $\epsilon$ , there will exist a number  $N$  and a set  $\mathcal{E}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  such that for all  $\alpha$  in  $\mathcal{E}(\alpha)$  and for all indices  $\mu, \nu > N$

$$7(1) \quad \left| \sigma_{\mu\nu}(\alpha) \right| \equiv \left| \sum_{i=0, j=0}^{\mu, \nu} u_{ij} f_{ij}(\alpha) \right| < \epsilon,$$

for all series 2(1) converging to zero and of class  $(\overline{\mathcal{F}})_2$  relative to a given set of functions 2(2) of class  $(\overline{\mathcal{A}})_2$ .

Let us choose an  $\epsilon > 0$ , then since the series 2(1) converges to zero, there are numbers  $m$  and  $n$  such that for all  $i > m$  and all  $j > n$

$$7(2) \quad |s_{ij}| < \frac{\epsilon}{6K}.$$

We form again the Abel Transformation for Double Series of 7(1), where  $\mu > m$ ,  $\nu > n$ , as follows:

$$7(3) \quad \begin{aligned} \sigma_{\mu\nu}(\alpha) &= \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} s_{ij} \Delta_{ij} f_{ij}(\alpha) + \sum_{i=0}^{\mu-1} s_{i\nu} \Delta_{i\nu} f_{i\nu}(\alpha) \\ &+ \sum_{j=0}^{\nu-1} s_{\mu j} \Delta_{\mu j} f_{\mu j}(\alpha) + s_{\mu\nu} f_{\mu\nu}(\alpha). \end{aligned}$$

Since the sequence 2(3) is of class  $(\overline{\mathcal{L}})_2$  relative to the given set of functions 2(6) of class  $(\overline{\mathcal{A}})_2$ , we may invoke theorem 5 to show that for any  $\epsilon > 0$  there is a set  $\mathcal{E}_1(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which for all  $\mu > m$  and all  $\nu > n$

$$7(4) \quad \left| \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} s_{ij} \Delta_{ij} f_{ij}(\alpha) \right| < \frac{\epsilon}{6}.$$

We now break up the second term on the right of 7(3), as follows:

$$\left| \sum_{i=0}^{\mu-1} s_{i\nu} \Delta_{10} f_{i\nu}(\alpha) \right| \leq \left| \sum_{i=0}^m s_{i\nu} \Delta_{10} f_{i\nu}(\alpha) \right| + \left| \sum_{i=m+1}^{\mu-1} s_{i\nu} \Delta_{10} f_{i\nu}(\alpha) \right|.$$

In view of condition  $(\bar{\delta}_1)_2$ , for any given  $\epsilon > 0$  there is a number  $N$ , such that for all  $\nu > N$ ,

$$\left| s_{i\nu} \Delta_{10} f_{i\nu}(\alpha) - \Lambda_i^{(1)}(\alpha) \right| < \frac{\epsilon}{12(m+1)}$$

and also there is a set  $\mathcal{E}_2^{(1)}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$\left| \Lambda_i^{(1)}(\alpha) \right| < \frac{\epsilon}{12(m+1)}$$

and also in view of  $(A_1)_2$  in which

$$\sum_{i=0}^{\infty} \left| \Delta_{11} f_{ij}(\alpha) \right| < K.$$

Thus, for all

$$\mu > m, \quad \nu > N_2 \equiv \max(N_1, n)$$

and for each  $\alpha$  in  $\mathcal{E}_2^{(1)}(\alpha)$  we have

$$\begin{aligned}
7(5) \quad & \left| \sum_{i=0}^{\mu-1} \omega_{i,\nu} \Delta_{10} f_{i,\nu}(\alpha) \right| \leq \left| \sum_{i=0}^m \omega_{i,\nu} \Delta_{10} f_{i,\nu}(\alpha) \right| + \left| \sum_{i=m+1}^{\mu-1} \omega_{i,\nu} \Delta_{10} f_{i,\nu}(\alpha) \right| \\
& \leq \sum_{i=0}^m \left| \omega_{i,\nu} \Delta_{10} f_{i,\nu}(\alpha) - \Lambda_i^{(1)}(\alpha) \right| + \sum_{i=0}^m \left| \Lambda_i^{(1)}(\alpha) \right| + \frac{\epsilon}{6K} \sum_{m+1, \nu}^{\mu-1, \infty} \left| \Delta_{11} f_{ij}(\alpha) \right| \\
& \leq \frac{\epsilon}{12} + \frac{\epsilon}{12} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.
\end{aligned}$$

The third term on the right of 7(3) may be treated in a manner analogous to the treatment of the second term, showing that for any given  $\epsilon > 0$  there is a number  $M_2$  and a set  $\mathcal{E}_2^{(2)}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which for all  $\mu > M_2$  and all  $\nu > n$

$$\begin{aligned}
7(6) \quad & \left| \sum_{j=0}^{\nu-1} \omega_{\mu j} \Delta_{01} f_{\mu j}(\alpha) \right| \leq \left| \sum_{j=0}^n \omega_{\mu j} \Delta_{01} f_{\mu j}(\alpha) \right| + \left| \sum_{j=n+1}^{\nu-1} \omega_{\mu j} \Delta_{01} f_{\mu j}(\alpha) \right| \\
& \leq \sum_{j=0}^n \left| \omega_{\mu j} \Delta_{01} f_{\mu j}(\alpha) - \Lambda_j^{(2)}(\alpha) \right| + \sum_{j=0}^n \left| \Lambda_j^{(2)}(\alpha) \right| + \frac{\epsilon}{6K} \sum_{\mu, n+1}^{\infty, \nu-1} \left| \Delta_{11} f_{ij}(\alpha) \right| \\
& \leq \frac{\epsilon}{12} + \frac{\epsilon}{12} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.
\end{aligned}$$

Thus, in a set  $\mathcal{E}_2(\alpha)$  containing all points common to both  $\mathcal{E}_2^{(1)}(\alpha)$  and  $\mathcal{E}_2^{(2)}(\alpha)$  we have the second and third terms on the right of 7(3) each less in absolute value than  $\frac{\epsilon}{3}$ .

There remains now for consideration only the last term on the right hand side of 7(3). In view of  $(C)_2$  there is a subset  $\mathcal{E}_3(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which

$$\left| f_{\mu\nu}(\alpha) - 1 \right| < K - 1 \quad (\text{all } \mu, \nu).$$

Thus, for  $\mu > m$  and  $\nu > n$

$$\begin{aligned} \left| a_{\mu\nu} f_{\mu\nu}(\alpha) \right| &= \left| a_{\mu\nu} f_{\mu\nu}(\alpha) - a_{\mu\nu} + a_{\mu\nu} \right| \leq \\ \left| a_{\mu\nu} [f_{\mu\nu}(\alpha) - 1] \right| + |a_{\mu\nu}| &\leq \frac{\epsilon}{6K} (K-1) + \frac{\epsilon}{6K} = \frac{\epsilon}{6}. \end{aligned}$$

Therefore, in a set

$$\mathcal{E}(\alpha) \equiv \mathcal{E}_1(\alpha) \cdot \mathcal{E}_2(\alpha) \cdot \mathcal{E}_3(\alpha)$$

containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , we have for  $\mu > M_2$  and  $\nu > N_2$  and  $N = \max(M_2, N_2)$  the inequality 7(1) subsisting.

If the series 2(1) under consideration converges to  $a \neq 0$ , we may reduce it to the case just considered by considering the series  $\sum u_{ij}'$  which converges to zero and which is defined as follows

$$u'_{00} = u_{00} - \mathcal{A},$$

$$u'_{ij} = u_{ij},$$

(all other  $i, j$ ).

We have now completely established

Theorem 6.

Corresponding to any positive number  $\epsilon$ , there will be a number  $N$  and a set  $\mathcal{E}(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , such that for all  $\alpha$  in  $\mathcal{E}(\alpha)$  and all indices  $\mu, \nu > N$  the inequality

$$|\overline{\sigma}_{\mu\nu}(\alpha) - \mathcal{A}| < \epsilon$$

subsists for all series 2(1) converging to  $\mathcal{A}$  and of class  $(\overline{\mathcal{J}})_2$  relative to a given set of functions 2(2) of class  $(\overline{\mathcal{A}'})_2$ .

P A R T II.

Multiple Sequences and Series.

8. Sequence to function transformations for multiple sequences of a certain class  $(\mathcal{A})_n$ .

We consider the  $n$ -tuple sequence,

$$8(1) \quad \{s_{[i]}\},$$

which converges to  $s$  and where  $[i]$  denotes the subscripts  $(i_1, \dots, i_n)$ , and  $i_k = 0, 1, \dots$ , for  $k = 1, 2, \dots, n$ . We also consider a set of  $n$ -tuply infinite functions,

$$8(2) \quad \{ \varphi_{[i]}(\alpha) \} \quad (i_k = 0, 1, \dots; k = 1, \dots, n),$$

defined for each  $\alpha$  in a set  $E(\alpha)$  of the type specified in section one.

As with double series so also with multiple series, Moore<sup>1</sup> has shown that a necessary and sufficient condition that the series

$$8(3) \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} s_{[i]} \varphi_{[i]}(\alpha)$$

converge in  $E(\alpha)$  for all convergent sequences 8(1) which are such that

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1. Moore, C.N. loc. cit.

$$8(4) \quad |A_{I_k J_k}| < M, \quad (i_k = 0, 1, \dots; k = 1, \dots, n)$$

is that the functions 8(2) satisfy

$$(A^*)_n \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} |\varphi_{I_k J_k}(\alpha)| < K(\alpha),$$

for each  $\alpha$  in  $E(\alpha)$  and where  $K(\alpha)$ , as above, is a positive function of  $\alpha$ . We wish here to establish for multiple sequences the analog of Theorem 1.

Let us now introduce the following two definitions:

DEFINITION 8.1. Set of functions of class  $(\mathcal{M})_n$ .

The set of functions 8(2) is of class  $(\mathcal{M})_n$  if the functions of the set satisfy condition  $(A^*)_n$ .

DEFINITION 8.2. Multiple sequences of certain class  $(\mathcal{A})_n$

The convergent sequence 8(1) is of class  $(\mathcal{A})_n$  relative to a given set of functions 8(2) of class  $(\mathcal{M})_n$  if the following  $(2^n - 2)$  conditions are satisfied:

$n$  conditions  $(\alpha_s^{(1)})_n$  of the type

$(\alpha_i^{(1)})_n$  The sequence 8(1) is such that there is a set of integers  $(p_1, \dots, p_{n-1})$  such that the series on the right of

$$8(5) \quad \Phi_{m_1, \dots, m_{n-1}}^{(1,1)}(\alpha) \equiv \sum_{i_1=m_1}^{\infty} \dots \sum_{i_{n-1}=m_{n-1}}^{\infty} \left( \sum_{i_n=0}^{\infty} A_{I_k J_k} \varphi_{I_k J_k}(\alpha) \right)$$

converges for all  $m_i > p_i$  ( $i = 1, \dots, n-1$ ) and for all  $\alpha$  in  $E(\alpha)$ .

$(2^n - n - 2)$  conditions  $(\alpha_s^{(k)})_n$  of the type  $(\alpha_1^{(k)})_n$

The sequence 8(1) is also such that there is a set of integers  $(p_1, \dots, p_{n-k})$  such that the series on the right of

$$8(6) \quad \Phi_{m_1, \dots, m_{n-k}}^{(k,1)}(\alpha) \equiv \sum_{i_1=m_1}^{\infty} \dots \sum_{i_{n-k}=m_{n-k}}^{\infty} \left( \sum_{i_{n-k+1}=0}^{\infty} \dots \sum_{i_n=0}^{\infty} \Delta_{I_i J} \phi_{I_i J}^{(\alpha)} \right)$$

converges for all  $m_i > p_i$  ( $i = 1, \dots, n-k$ ) and for all  $\alpha$  in  $E(\alpha)$ , where  $k$  ranges from 2 to  $(n-1)$  inclusive and  $s$  from 1 to  $\binom{n}{k}$  for each  $k$ .

We wish to show in this section that the series 8(3) converges for each  $\alpha$  in  $E(\alpha)$  whenever the convergent sequence 8(1) is of class  $(\alpha)_n$  relative to a given set of functions 8(2) of class  $(\kappa)_n$ .

As in the first section, here also it can easily be shown that if the sequence 8(1), converging to  $\mathcal{L}$ , is of class  $(\alpha)_n$  relative to a given set of functions 8(2) of class  $(\kappa)_n$ , then is also the sequence

$$8(7) \quad \{ \Delta_{I_i J} - \mathcal{L} \} \equiv \{ S_{I_i J} \}, \quad (i_k = 0, 1, \dots; k=1, \dots, n),$$

of class  $(\alpha)_n$  relative to a given set of functions 8(2) of class  $(\kappa)_n$ , and conversely. Thus, to establish our desired results, it suffices to show that for any  $\epsilon > 0$  there is

determined a set of integers  $(\bar{p}_1, \dots, \bar{p}_n)$ , such that for all

$$r_i > p_i \quad (i = 1, \dots, n)$$

$$8(8) \quad \left| \sum_{l_1=0}^{r_1} \dots \sum_{l_n=0}^{r_n} S_{I_{ij}} \phi_{I_{ij}}(\alpha) - \sum_{l_1=0}^{\bar{p}_1} \dots \sum_{l_n=0}^{\bar{p}_n} S_{I_{ij}} \phi_{I_{ij}}(\alpha) \right| < \epsilon.$$

It is convenient now to define the following notation

$$8(9) \quad P_0 \equiv 0; \quad P_1 \equiv 1; \quad P_h \equiv \sum_{i=0}^{h-1} \binom{h}{i} (P_i + 1), \quad (h = 2, 3, \dots).$$

If we perform the subtraction indicated by the left hand side of 8(8) and transform the resulting terms in a manner analogous to a way in which the remainder 1(12) was transformed and approximated, there will result  $P_n$  terms, where  $P_n$  is defined in 8(9).

Corresponding to any  $\epsilon > 0$  we can find numbers

$p_k^{(1)}$  ( $k = 1, \dots, n$ ), such that for all  $i_k > p_k^{(1)}$  ( $k = 1, \dots, n$ ) and for each  $\alpha$  in  $E(\alpha)$

$$8(10) \quad |S_{I_{ij}}| < \frac{\epsilon}{P_n K(\alpha)}.$$

Similarly, corresponding to any  $\epsilon > 0$ , we can find numbers  $p_k^{(2)}$  ( $k = 1, \dots, n$ ), such that for all

$$\bar{p}_i > p_i^{(2)} > m_i \quad (i = 1, \dots, n)$$

each of the  $2^n - 2$  series defined in Definition 8(2) have

remainders which for each  $\alpha$  in  $E(\alpha)$  can be made less than  $\epsilon/p_n$  in absolute value.

We select for  $i = 1, \dots, h$

$$8(11) \quad \bar{p}_i > \max(p_i, p_i^{(1)}, p_i^{(2)}).$$

By use of the convergence of the  $2^n - 1$  series in Definition 8.2., condition  $(A^*)_n$ , 8(10), and 8(11) one can easily show that each of the  $P_n$  terms resulting from the subtraction of the two terms on the left of 8(8) is in absolute value less than  $\epsilon/p_n$ . We have thus established the validity of inequality 8(8) and have thereby proved

**Theorem 7.**

The series 8(3) converges for each  $\alpha$  in  $E(\alpha)$  whenever the sequence 8(1) converges to  $\mathcal{A}$  and is of class  $(\mathcal{A})_n$  relative to a given set of functions 8(2) of class  $(\mathcal{A})_n$ .

9. Convergence factors of type I for convergent multiple series of a certain class  $(\mathcal{L})_n$ .

Let us consider the multiple series

$$\vartheta(1) \quad \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} u_{[i]},$$

which converges to zero and the multiple sequence of functions

$$\vartheta(2) \quad \left\{ f_{[i]}(\alpha) \right\} \quad (i_k = 0, 1, \dots; k = 1, \dots, n),$$

defined in a set of points  $E(\alpha)$  of the type previously described. We introduce the following notations

$$\vartheta(3) \quad \Delta_{\parallel \dots \parallel} f_{[i]}(\alpha) = \sum_{k_1=0}^1 \sum_{k_2=0}^1 \cdots \sum_{k_n=0}^1 (-1)^{k_1} (-1)^{k_2} \cdots (-1)^{k_n} f_{[i+k]}(\alpha);$$

$$\Delta_{\parallel \dots \parallel 0} f_{[i]}(\alpha) = \sum_{k_1=0}^1 \sum_{k_2=0}^1 \cdots \sum_{k_{n-1}=0}^1 (-1)^{k_1} (-1)^{k_2} \cdots (-1)^{k_{n-1}} f_{i+k_1, \dots, i_{n-1}+k_{n-1}, i_n}(\alpha);$$

---


$$\Delta_{1, 0 \dots 0} f_{[i]}(\alpha) = f_{i, i_2, \dots, i_n}(\alpha) - f_{i+1, i_2, \dots, i_n}(\alpha);$$

there are analogous conventions for the symbols  $\Delta$  with zero subscripts in different positions. We shall represent

by

$$\Delta_{1,0}^{n-k,k}$$

a  $\Delta$  symbol having as subscripts  $(n-k)$  unities and  $k$  zeros.

We now introduce the following definition:

DEFINITION 9.1. Set of functions of class  $(\alpha')_n$ .

A set of functions  $\mathcal{F}$  to be of class  $(\alpha')_n$  if the functions of the set satisfy the following  $(2^n - 1)$  conditions:

$n$  conditions  $(D_s^{(1)})$  of the type

$$(D_s^{(1)}) \quad \lim_{i_1 \rightarrow \infty} \Delta_{11 \dots 10} f_{[i]}^{(\alpha)} = 0 \quad (E(\alpha); \text{ all } i_1, \dots, i_{n-1});$$

$(2^n - n - 2)$  conditions  $(D_s^{(k)})$  of the type

$$(D_s^{(k)}) \quad \lim_{i_{n-k+1} \rightarrow \infty, \dots, i_n \rightarrow \infty} \Delta_{1,0}^{n-k,k} f_{[i]}^{(\alpha)} = 0$$

$$(E(\alpha); \text{ all } i_1, \dots, i_{n-k}),$$

$k$  ranging from 2 to  $(n - 1)$  inclusive, and  $s$  from 1 to

$\binom{n}{k}$  for each  $k$ ; and finally the condition

$$(A)_n \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} |\Delta_{11 \dots 1} f_{[i]}^{(\alpha)}| < K(\alpha),$$

where  $K(\alpha)$  is a positive function of  $\alpha$  defined for each  $\alpha$  in  $E(\alpha)$ .

As Moore<sup>1</sup> has shown, the  $(2^n - 1)$  conditions just stated imply the validity of  $\binom{n}{k}$  equations of the form

$$9(4) \quad \sum_{i_1=m_1}^{\infty} \dots \sum_{i_k=m_k}^{\infty} \Delta_{i_1, \dots, i_k} f_{[i, n-k, m]} = \Delta_{0, 1}^{k, n-k} f_{[m]},$$

where the symbol  $[i, n-k, m]$ , for  $k = 1, 2, \dots, n$  stands for the set of indices  $i_1, \dots, i_k, m_{k+1}, \dots, m_n$ .

It is now desirable to introduce another definition, namely:

DEFINITION 9.2. Multiple series of a certain class  $(\mathcal{L})_n$ .

A convergent series 9(1) is of class  $(\mathcal{L})_n$  relative to a given set of functions 9(2) of class  $(\mathcal{A})_n$  if the sequence of the partial sums of 9(1), defined by

$$9(5) \quad S_{[m]} \equiv \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} u_{[i]},$$

is of class  $(\mathcal{A})_n$  relative to a given set of functions

$$9(6) \quad \varphi_{[i]}(\alpha) = \Delta_{i_1, \dots, i_n} f_{[i]}(\alpha), \quad (E(\alpha)),$$

of class  $(\mathcal{A})_n$  as these latter classes were defined by definitions 8.2 and 8.1., respectively; and if in addition the sequence defined in 9(5) satisfies the following

1. Moore, C.N. loc. cit.

$(2^n - 2)$  conditions

$n$  conditions  $(\beta_s^{(1)})$  of the type

$$(\beta_s^{(1)}) \quad \lim_{i_n \rightarrow \infty} \omega_{[i]} \Delta_{11 \dots 10} f_{[i]}(\alpha) = \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha),$$

which is such that

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) = \Psi^{(1,1)}(\alpha),$$

and also such that there are  $(2^{n-1} - 2)$  pairs of uniformly convergent series, namely:

$(n-1)$  pairs of series  $(\beta_s^{(1)})_t^{(1)}$  of the type

$$(\beta_s^{(1)})_t^{(1)} \quad \sum_{i_1=0}^{\infty} \left( \sum_{i_2=0}^{m_2} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right) \text{ and}$$

$$\sum_{i_1=0}^{\infty} \left( \sum_{i_2=0}^{m_2} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \omega_{[i]} \Delta_{11 \dots 10} f_{[i]}(\alpha) \right),$$

of which the first series is uniformly convergent in the variables  $m_2, \dots, m_{n-1}$ , each on the range  $0, 1, \dots$ , and the second series is uniformly convergent in the variables  $m_2, \dots, m_{n-1}, i_n$ , each on the range  $0, 1, \dots$ , and

$(2^{n-1} - [n-1] - 2)$  pairs of series  $(\beta_s^{(1)})_t^{(2)}$  of the type

$$\left( \beta_s^{(l)} \right), \sum_{i_1=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} \left( \sum_{i_{l+1}=0}^{m_{l+1}} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(l,1)}(\alpha) \right) \text{ and}$$

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} \left( \sum_{i_{l+1}=0}^{m_{l+1}} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \omega_{[i]} \Delta_{11 \dots 10} f_{[i]}^{(l)}(\alpha) \right),$$

of which the first series is uniformly convergent in the variables  $m_{l+1}, \dots, m_{n-1}$ , each on the range  $0, 1, \dots$ , and the second series is uniformly convergent in the variables  $m_{l+1}, \dots, m_{n-1}, i_n$ , each on the range  $0, 1, \dots$ , where  $l$  ranges from 2 to  $(n-2)$  inclusive and  $t$  from 1 to  $\binom{n-1}{l}$  for each  $l$ .

$(2^{n-n-2})$  conditions  $(\beta_s^{(k)})$  of the type

$$\left( \beta_s^{(k)} \right) \lim_{m_{n-k+1} \rightarrow \infty, \dots, m_n \rightarrow \infty} \omega_{[i, k, m]} \Delta_{11 \dots 10 \dots 0} f_{[i, k, m]}^{(k)}(\alpha) =$$

$$\Psi_{i_1, \dots, i_{n-k}}^{(k,1)}(\alpha) = \Psi_{i_1, \dots, i_{n-k}}^{(k,1)}(\alpha),$$

which is such that

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_{n-k}=0}^{\infty} \Psi_{i_1, \dots, i_{n-k}}^{(k,1)}(\alpha) = \Psi_{i_1, \dots, i_{n-k}}^{(k,1)}(\alpha),$$

and also such that there are  $(2^{n-k} - 2)$  pairs of uniformly

convergent series, namely:

(n - k) pairs of series  $(\beta_i^{(k)})_t^{(1)}$  of the type

$$(\beta_i^{(1)})_t^{(1)} = \sum_{i_1=0}^{\infty} \left( \sum_{i_2=0}^{m_2} \cdots \sum_{i_{n-k}=0}^{m_{n-k}} \Psi_{i_1, \dots, i_{n-k}}^{(k, 1)}(\alpha) \right) \text{ and}$$

$$\sum_{i_1=0}^{\infty} \left( \sum_{i_2=0}^{m_2} \cdots \sum_{i_{n-k}=0}^{m_{n-k}} \mathcal{S}_{[i, k, m]} \Delta_{11 \dots 10 \dots 0} f_{[i, k, m]}^{(\alpha)} \right),$$

of which the first series is uniformly convergent in the variables  $m_2, \dots, m_{n-k}$ , each on the range  $0, 1, \dots$ , and the second series is uniformly convergent in the variables  $m_2, \dots, m_{n-k}$ ,  $i_{n-k+1}, \dots, i_n$ , each on the range  $0, 1, \dots$ , and

$(2^{n-k} - [n - k] - 2)$  pairs of series  $(\beta_i^{(k)})_t^{(l)}$  of the type

$$(\beta_i^{(k)})_t^{(l)} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} \left( \sum_{i_{l+1}=0}^{m_{l+1}} \cdots \sum_{i_{n-k}=0}^{m_{n-k}} \Psi_{i_1, \dots, i_{n-k}}^{(k, l)}(\alpha) \right) \text{ and}$$

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} \left( \sum_{i_{l+1}=0}^{m_{l+1}} \cdots \sum_{i_{n-k}=0}^{m_{n-k}} \mathcal{S}_{[i, k, m]} \Delta_{11 \dots 10 \dots 0} f_{[i, k, m]}^{(\alpha)} \right),$$

of which the first series is uniformly convergent in the variables  $m_{l+1}, \dots, m_{n-k}$ , each on the

range  $0, 1, \dots$ , and the second series is uniformly convergent in the variables  $m_{\ell+1}, \dots, m_{n-k}$ ,  $i_{n-k+1}, \dots, i_n$ , each on the range  $0, 1, \dots$ , where  $\ell$  ranges from 2 to  $(k-1)$  inclusive and  $t$  from 1 to  $\binom{n-k}{\ell}$  for each  $\ell$ .

The number  $k$  ranges from 2 to  $(n-1)$  inclusive and  $s$  from 1 to  $\binom{n}{k}$  for each  $k$ .

We wish to show that the series

$$9(7) \quad \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} u_{[i]} f_{[i]}(\alpha)$$

converges for each  $\alpha$  in  $E(\alpha)$  whenever the series 9(1) converges and is of class  $(\mathcal{L})_n$  relative to a given set of functions 9(2) of class  $(\mathcal{A}')_n$ .

We consider the sequence of the partial sums of the series 9(7) namely:

$$9(8) \quad \mathcal{T}_{[m]}(\alpha) \equiv \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} u_{[i]} f_{[i]}(\alpha),$$

which we break up according to the Generalization of the Abel Transformation to multiple series in the following way:

$$\begin{aligned}
9(9) \quad \sigma_{[m]}(\alpha) &= \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} u_{[i]} f_{[i]}(\alpha) \\
&= \sum_{i_1=0}^{m_1-1} \cdots \sum_{i_n=0}^{m_n-1} u_{[i]} \Delta_{11 \dots 1} f_{[i]}(\alpha) \\
&+ \left[ \sum_{i_1=0}^{m_1-1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}-1} u_{[i, 1, m]} \Delta_{11 \dots 10} f_{[i, 1, m]}(\alpha) \right] + \dots \\
&+ \left[ \sum_{i_1=0}^{m_1-1} \cdots \sum_{i_{n-k}=0}^{m_{n-k}-1} u_{[i, k, m]} \Delta_{11 \dots 10 \dots 0} f_{[i, k, m]}(\alpha) \right] + \dots \\
&+ u_{[m]} f_{[m]}(\alpha),
\end{aligned}$$

where the large brackets enclosing an expression indicate the sum of the whole group of terms of similar form, obtained by permutation of indices.

To arrive at our desired result, it suffices to show that each term on the right 9(9) tends to a limit as  $m_1, \dots, m_n$  each becomes infinite.

In virtue of Theorem 7 just proved, the first term on the right of 9(9) tends to a finite limit as  $m_1, \dots, m_n$  each becomes infinite, for the sequence of the partial sums defined by 9(5) is of class  $(\mathcal{A})_n$  relative to a given set of functions 9(6) of class  $(\mathcal{A})_n$ .

To show that a representative term selected from terms of the form of the expression in the first large bracket on the right of 9(9) tends to a finite limit as each of  $m_1, \dots, m_n$  becomes infinite, it suffices to show that for any  $\epsilon > 0$  there are numbers  $M_1, \dots, M_n$ , such that for all  $(m_1 > M_1, \dots, m_n > M_n)$  we have

$$9(10) \quad \left| \sum_{i_1=0}^{m_1-1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}-1} \mathcal{A}_{[i_1, m]} \Delta_{11 \dots 10} f_{[i_1, m]}^{(\alpha)} - \Psi^{(1,1)}(\alpha) \right| < \epsilon.$$

Since the sequence defined by 9(5) converges to zero, for any  $\epsilon > 0$  there are numbers  $M_{1,1}, \dots, M_{n,1}$ , such that

$$9(11) \quad \left| \mathcal{A}_{[i]} \right| < \frac{\epsilon}{(2^n - 5)K(\alpha)}, \quad (i_1 > M_{1,1}, \dots, i_n > M_{n,1}).$$

Since the first series in  $(\beta_1^{(1)})$  converges for any  $\epsilon > 0$  there are numbers  $M_{1,2}, \dots, M_{n-1,2}$ , such that

$$9(12) \quad \left| \sum_{i_1=m_1+1}^{\infty} \cdots \sum_{i_{n-1}=m_{n-1}+1}^{\infty} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right| < \frac{\epsilon}{2^n - 5},$$

$$(m_1 > M_{1,2}, \dots, m_{n-1} > M_{n-1,2}; E(\alpha)).$$

Since the  $(n-1)$  pairs of series  $(\beta_1^{(1)})_k$  are uniformly convergent, there are  $(n-1)$  numbers  $M_{1,3}, \dots, M_{n-1,3}$ , such that we have  $(n-1)$  pairs of inequalities of the form

$$9(13) \quad \left| \sum_{i_1=M_{1,3}+1}^{m_1-1} \left( \sum_{i_2=0}^{m_2} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \mathcal{A}_{[i_1, m]} \Delta_{11 \cdots 10} f_{[i_1, m]}^{(\alpha)} \right) \right| < \frac{\epsilon}{2^n - 5},$$

$E(\alpha),$

for all  $m_1 > M_{1,3} + 2$ , and all  $(m_2, \dots, m_n)$  each on the range  $0, 1, \dots$ ; and

$$9(14) \quad \left| \sum_{i_1=M_{1,3}+1}^{m_1-1} \left( \sum_{i_2=0}^{m_2} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right) \right| < \frac{\epsilon}{2^n - 5}, \quad E(\alpha),$$

for all  $m_1 > M_{1,3} + 2$ , all  $m_2, \dots, m_{n-1}$  each on the range  $0, 1, \dots$ ,

Since the  $[2^n - (n-1) - 2]$  pairs of series  $(\beta_{i_1}^{(1,1)})$  are uniformly convergent, there are  $(n-1)$  numbers  $M_{1,4}, \dots, M_{n-1,4}$ , such that we have  $(n-1)$  pairs of inequalities of the form

$$9(15) \quad \left| \sum_{i_1=M_{1,4}+1}^{m_1-1} \cdots \sum_{i_\ell=M_{\ell,4}+1}^{m_\ell-1} \left( \sum_{i_{\ell+1}=0}^{m_{\ell+1}-1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}-1} \mathcal{A}_{[i_1, m]} \Delta_{11 \cdots 10} f_{[i_1, m]}^{(\alpha)} \right) \right|$$

$$< \frac{\epsilon}{2^n - 5}, \quad E(\alpha), \quad (m_1 > M_{1,4} + 2, \dots, m_\ell > M_{\ell,4} + 2)$$

and all  $m_{\ell+1}, \dots, m_n$  each on the range  $0, 1, \dots$ ; and

$$9(16) \quad \left| \sum_{i_1=M_{1,4}+1}^{m_1-1} \cdots \sum_{i_\ell=M_{\ell,4}+1}^{m_\ell-1} \left( \sum_{i_{\ell+1}=0}^{m_{\ell+1}-1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}-1} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right) \right| < \frac{\epsilon}{2^n - 5},$$

$E(\alpha)$ ,  $(m_1 > M_{1,4} + 2, \dots, m_\ell > M_{\ell,4} + 2)$  and all

$m_{\ell+1}, \dots, m_n$  each on the range  $0, 1, \dots$ .

We define

$$9(17) \quad M_i = \max(M_{i,1}, M_{i,2}, M_{i,3}, M_{i,4})$$

$$(i = 1, \dots, n-1),$$

In view of the limit  $(\beta_i^{(1)})$ , we have for any  $\epsilon > 0$  an  $M_{n,2}$ , such that

$$9(18) \quad \left| \mathcal{L}_{[i]} \Delta_{11 \dots 10} f_{[i]}(\alpha) - \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right| < \frac{\epsilon}{(2^n - 5) \prod_{k=1}^{n-1} (M_k + 1)},$$

$E(\alpha)$  and all  $i_n > M_{n,2}$ .

We define

$$9(19) \quad M_n = \max(M_{n,1}, M_{n,2}).$$

(1.1) After substituting in 9(10) the series for which  $\underline{\Psi}(\alpha)$  is the sum and after computing the difference indicated on the left of 9(10), the following expression results:

9(20)

$$\left| \sum_{i_1=0}^{M_1} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \omega_{[i, l, m]} \Delta_{11 \cdots 10} f_{[i, l, m]}(\alpha) + \right.$$

$$\left. \left[ \sum_{i_1=M_1+1}^{m_1-1} \left( \sum_{i_2=0}^{M_2-1} \cdots \sum_{i_{n-1}=0}^{M_{n-1}-1} \omega_{[i, l, m]} \Delta_{11 \cdots 10} f_{[i, l, m]}(\alpha) \right) \right] + \cdots + \right.$$

$$\left. \left[ \sum_{i_1=M_1+1}^{m_1-1} \cdots \sum_{i_2=M_2+1}^{m_2-1} \left( \sum_{i_2+1=0}^{M_2+1} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \omega_{[i, l, m]} \Delta_{11 \cdots 10} f_{[i, l, m]}(\alpha) \right) \right] + \right.$$

$$\left. \left( \sum_{i_1=M_1+1}^{m_1-1} \cdots \sum_{i_{n-1}=M_{n-1}+1}^{m_{n-1}-1} \omega_{[i, l, m]} \Delta_{11 \cdots 10} f_{[i, l, m]}(\alpha) \right) - \sum_{i_1=0}^{M_1} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(l, l)}(\alpha) \right.$$

$$\left. - \left[ \sum_{i_1=M_1+1}^{\infty} \left( \sum_{i_2=0}^{M_2} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(l, l)}(\alpha) \right) \right] - \cdots - \right.$$

$$\left. \left[ \sum_{i_1=M_1+1}^{\infty} \cdots \sum_{i_2=M_2+1}^{\infty} \left( \sum_{i_2+1=0}^{M_2+1} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(l, l)}(\alpha) \right) \right] - \sum_{i_1=M_1+1}^{\infty} \cdots \sum_{i_{n-1}=M_{n-1}+1}^{\infty} \Psi_{i_1, \dots, i_{n-1}}^{(l, l)}(\alpha) \right|$$

$$\leq \left| \sum_{i_1=0}^{M_1} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \left( \omega_{[i, l, m]} \Delta_{11 \cdots 10} f_{[i, l, m]}(\alpha) - \Psi_{i_1, \dots, i_{n-1}}^{(l, l)}(\alpha) \right) \right| +$$

$$\left| \left[ \sum_{i_1=M_1+1}^{m_1-1} \left( \sum_{i_2=0}^{M_2-1} \cdots \sum_{i_{n-1}=0}^{M_{n-1}-1} \omega_{[i, l, m]} \Delta_{11 \cdots 10} f_{[i, l, m]}(\alpha) \right) \right] + \cdots + \right.$$

$$\left| \left[ \sum_{i_1=M_1+1}^{m_1-1} \cdots \sum_{i_2=M_2+1}^{m_2-1} \left( \sum_{i_{2+1}=0}^{M_{2+1}} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \omega_{[i_1, m]} \Delta_{11 \cdots 10} f_{[i_1, m]}^{(\alpha)} \right) \right] \right| +$$

$$\left| \sum_{i_1=M_1+1}^{m_1-1} \cdots \sum_{i_{n-1}=M_{n-1}+1}^{m_{n-1}-1} \omega_{[i_1, m]} \Delta_{11 \cdots 10} f_{[i_1, m]}^{(\alpha)} \right| +$$

$$\left| \left[ \sum_{i_1=M_1+1}^{\infty} \left( \sum_{i_2=0}^{M_2} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right) \right] \right| + \cdots +$$

$$\left| \left[ \sum_{i_1=M_1+1}^{\infty} \cdots \sum_{i_2=M_2+1}^{\infty} \left( \sum_{i_{2+1}=0}^{M_{2+1}} \cdots \sum_{i_{n-1}=0}^{M_{n-1}} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right) \right] \right| +$$

$$\left| \sum_{i_1=M_1+1}^{\infty} \cdots \sum_{i_{n-1}=M_{n-1}+1}^{\infty} \Psi_{i_1, \dots, i_{n-1}}^{(1,1)}(\alpha) \right|,$$

where the large brackets enclosing an expression indicate the sum of the whole group of terms of similar form, obtained by permutation of indices.

By virtue of 9(18) the first term on the right of 9(20) is less than  $\frac{\epsilon}{2^n - 5}$ . By virtue of 9(13), 9(17), and 9(19) the  $(n - 1)$  expressions indicated within the square brackets around the second term are each less than  $\frac{\epsilon}{2^n - 5}$ , and by

virtue of 9(15), 9(17), and 9(19) the  $\left[ 2^{n-1} - (n-1) - 2 \right]$  expressions indicated within the square brackets enclosing the next term written are each less than  $\frac{\epsilon}{2^n - 5}$ . The fourth term written is less than  $\frac{\epsilon}{2^n - 5}$  by virtue of 9(11), 9(17), 9(19), and 9(4), and condition (A)<sub>n</sub>. The fifth term written indicates (n - 1) terms each of which are less than  $\frac{\epsilon}{2^n - 5}$  by virtue of 9(16), 9(17), and 9(19); and the following term indicates  $\left[ 2^{n-1} - (n-1) - 2 \right]$  expressions each of which are less than  $\frac{\epsilon}{2^n - 5}$  by virtue of 9(16), 9(17), and 9(19). Finally, the last term is less than  $\frac{\epsilon}{2^n - 5}$  by virtue of 9(12), 9(17), and 9(19). Thus, we have established 9(10) and ipso facto have shown that a representative term selected from expressions of the form of the term enclosed in the first large bracket on the right of 9(9) tends to a limit as  $m_1, \dots, m_n$  become infinite.

To show that a representative term selected from expressions of the form of the term in the second large bracket tends to a finite limit as  $m_1, \dots, m_n$  become infinite, it suffices to show that for any  $\epsilon > 0$  there are numbers  $N_1, \dots, N_n$  such that for all ( $m_1 > N_1, \dots, m_n > N_n$ ) we have

$$9(21) \quad \left| \sum_{l_1=0}^{m_1-1} \cdots \sum_{l_{n-k}=0}^{m_{n-k}-1} \omega_{[l, k, m]} \Delta_{11 \cdots 10 \cdots 0} f_{[l, k, m]}^{(\alpha)} - \Psi_{(\alpha)}^{(k, 1)} \right| < \epsilon$$

The establishing of 9(21) is wholly analogous to the procedure used for 9(10): here we show that each term of the computed difference of the left hand side of 9(21) is less than  $\frac{\epsilon}{2^{n-k+1}5}$  by using the condition  $(\beta_i^{(k)})$  and the  $(2^{n-k} - 2)$  pairs of uniformly convergent series listed under  $(\beta_i^{(k)})$ .

Finally, we consider the last term. Recalling that we are considering numbers  $m_1, \dots, m_n$  each sufficiently large, we have for an  $\epsilon > 0$  numbers  $p_1, \dots, p_n$  such that

$$9(22) \quad \left| \omega_{[m]} \right| < \epsilon \quad (m_1 > p_1, \dots, m_n > p_n).$$

Moore<sup>1</sup> has shown that for each  $\alpha$  in  $E(\alpha)$ ,  $f_{[m]}(\alpha)$  remains bounded for all  $[m]$ . Hence, the last term on the right of 9(9) tends to zero as the  $m$ 's become infinite, provided the sequence defined by 9(5) tends to zero under the same conditions.

Thus, for any  $\epsilon > 0$  there is determined a number  $N$  such that for all  $(m_1 > N, \dots, m_n > N)$

$$\left| \sigma_{[m]}^{(\alpha)} \right| < \epsilon$$

Since any convergent series may be reduced to the case just considered, that is, to a series converging to zero by alteration of the term  $u_{00\dots 0}$ , we have now established completely the following:

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1. Moore, C.N. loc. cit.

**Theorem 8.**

The series 9(7) converges for each  $\alpha$  in  $E(\alpha)$  whenever the series 9(1) converges and is of class  $(L)_n$  relative to a given set of functions 9(2) of class  $(\alpha')_n$ .

10. Sequence to function transformations for multiple sequences of a certain class  $(\bar{\alpha})_n$ .

We consider in this section the case where each of the functions  $\mathcal{S}(2)$  of section eight approaches a limit as  $\alpha \rightarrow \alpha_0$ , not in  $E(\alpha)$ , and we make the two following definitions:

DEFINITION 10.1. Set of functions of class  $(\bar{\alpha})_n$ .

The set of functions  $\mathcal{S}(2)$  is of class  $(\bar{\alpha})_n$  if it is of class  $(\alpha)_n$  as the concept was defined by definition 8.1. and in addition the functions of the set satisfy the following  $(n + 2)$  conditions:

$$(A^*)_n \quad \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} |\varphi_{[i]}(\alpha)| < K \quad (\text{each } \alpha \text{ in } E'(\alpha)),$$

where  $E'(\alpha)$  contains all points of  $E(\alpha)$  lying in a certain neighborhood of  $\alpha_0$ , and  $K$  is a positive constant;

$$(C^*)_n \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \varphi_{[i]}(\alpha) = 1 ;$$

$n$  conditions  $(E_s^*)_n$  of the type

$$(E_s^*)_n \quad \lim_{\alpha \rightarrow \alpha_0} \left\{ \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} |\varphi_{[i]}(\alpha)| \right\} = 0 \quad (i_1 = 0, 1, \dots),$$

where  $S$  ranges from 1 to  $n$  inclusive.

DEFINITION 10.2. Sequence of a certain class  $(\bar{\alpha})_n$ .

The convergent sequence 8(1) is of class  $(\bar{\alpha})_n$  relative to a given set of functions 8(2) of class  $(\bar{\alpha})_n$  if the sequence 8(1) is of class  $(\alpha)_n$  relative to the given set of functions 8(2) of class  $(\bar{\alpha})_n$ , as this concept was defined by definition 8.2. and if in addition the functions of conditions  $(\alpha_s^{(k)})_n$  satisfy the following  $(2^n - 2)$  additional conditions:

$n$  conditions  $(\bar{\alpha}_s^{(n-1)})_n$  of the type

$$(\bar{\alpha}_s^{(n-1)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Phi_{m_i}^{(n-1,1)}(\alpha) = 0, \quad (m_i > p_i);$$

$(2^n - n - 2)$  conditions  $(\bar{\alpha}_s^{(n-k)})_n$  of the type

$$(\bar{\alpha}_s^{(n-k)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Phi_{m_1, \dots, m_k}^{(n-k,1)}(\alpha) = 0, \quad (m_i > p_i; i = 1, \dots, k),$$

where  $k$  ranges from 2 to  $(n-1)$  inclusive and  $S$  from 1 to  $\binom{n}{k}$  for each  $k$ .

It is the object of this section to show that the limit of the series

$$10(1) \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} A_{I_{ij}} \phi_{I_{ij}}(\alpha)$$

shall be  $\mathcal{A}$  as  $\alpha \rightarrow \alpha_0$  for every sequence 8(1) converging to  $\mathcal{A}$  and of class  $(\bar{\alpha})_n$  relative to a given set of functions 8(2) of class  $(\bar{\alpha})_n$ .

It suffices to show that given an  $\epsilon > 0$  there is a set  $\mathcal{E}(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$10(2) \quad \left| \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} \mathcal{A}_{I_{ij}} \phi_{I_{ij}}(\alpha) - \mathcal{A} \right| < \epsilon.$$

Since we have assumed that the sequence 8(1) converges to  $\mathcal{A}$ , for any  $\epsilon > 0$  there are numbers  $p_1, \dots, p_n$ , such that for all  $(i_k > p_k, k = 1, \dots, n)$

$$10(3) \quad \left| \mathcal{A}_{I_{ij}} - \mathcal{A} \right| < \frac{\epsilon}{(p_n + 2) K},$$

and also in view of  $(C^*)_2$ , the conditions  $(E^*)_n$ , and the choice of numbers  $p_1, \dots, p_n$ , there is a set  $\mathcal{E}_0(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$10(4) \quad \left| \sum_{i_1=p_1+1}^{\infty} \dots \sum_{i_n=p_n+1}^{\infty} \phi_{I_{ij}}(\alpha) - 1 \right| < \frac{\epsilon}{(p_n + 2) |\mathcal{A}|}.$$

We break up the left hand side of 10(2) into the following  $(2^n + 1)$  parts:

10(5)

$$\begin{aligned}
& \left| \sum_{i_1=0}^{p_1} \cdots \sum_{i_n=0}^{p_n} \Delta_{Ii} \phi_{Ii}(\alpha) + \sum_{i_1=p_1+1}^{\infty} \cdots \sum_{i_n=p_n+1}^{\infty} (\Delta_{Ii} - \Delta) \phi_{Ii}(\alpha) + \right. \\
& \left[ \sum_{i_1=p_1+1}^{\infty} \sum_{i_2=0}^{p_2} \cdots \sum_{i_n=0}^{p_n} \Delta_{Ii} \phi_{Ii}(\alpha) \right] + \cdots + \\
& \left[ \sum_{i_1=p_1+1}^{\infty} \cdots \sum_{i_k=p_k+1}^{\infty} \sum_{i_{k+1}=0}^{p_{k+1}} \cdots \sum_{i_n=0}^{p_n} \Delta_{Ii} \phi_{Ii}(\alpha) \right] + \cdots + \\
& \left. \sum_{i_1=p_1+1}^{\infty} \cdots \sum_{i_n=p_n+1}^{\infty} \Delta \phi_{Ii}(\alpha) - \Delta \right|,
\end{aligned}$$

where the large square brackets enclosing an expression indicate the sum of the whole group of terms of similar form obtained by permutation of the indices, where  $k$  ranges from 2 to  $(n-1)$  inclusive, and where  $p_1, \dots, p_n$  are the numbers defined in 10(3).

We define

$$M = \max |\Delta_{Ii}|, \quad (i_k = 0, 1, \dots, p_k; k = 1, \dots, n).$$

In view of condition  $(E^*)_n$  and the definition of  $M$ , there is a set  $\mathcal{E}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha$ , in which

$$10(6) \quad |\varphi_{I_{ij}}(\alpha)| < \frac{\epsilon}{(P_n+2)M \prod_{j=1}^n (P_j+1)}, \quad (i = 0, 1, \dots, p; k=1, \dots, n).$$

For each  $\alpha$  in  $E_1(\alpha)$

$$10(7) \quad \left| \sum_{i_1=0}^{P_1} \dots \sum_{i_n=0}^{P_n} \Delta_{I_{ij}} \varphi_{I_{ij}}(\alpha) \right| \leq M \sum_{i_1=0}^{P_1} \dots \sum_{i_n=0}^{P_n} |\varphi_{I_{ij}}(\alpha)| < \frac{\epsilon}{P_n+2}.$$

In view of  $(A_1^*)_n$  and 10(3) we have for all  $\alpha$  in  $E'(\alpha)$

$$10(8) \quad \sum_{P_1+1}^{\infty} \dots \sum_{P_n+1}^{\infty} |\Delta_{I_{ij}}^{-2}| |\varphi_{I_{ij}}(\alpha)| \leq \frac{\epsilon}{(P_n+2)K} \sum_{P_1+1}^{\infty} \dots \sum_{P_n+1}^{\infty} |\varphi_{I_{ij}}(\alpha)| < \frac{\epsilon}{P_n+2}.$$

Thus, the first two terms of 10(5) are each by 10(7) and 10(8) in absolute value less than  $\epsilon/P_n+2$  in a sufficiently small neighborhood of  $\alpha_0$ . The  $n$  terms indicated by the first set of large brackets in 10(5) may be treated in a manner wholly analogous to the following treatment of a representative term:

In view of conditions  $(\bar{\alpha}_5^{(k)})_n$  and condition  $(A_1^*)_n$  there is a set  $E_3^{(h)}(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which each of the  $n$  terms indicated within the first set of large square brackets is in absolute value less than  $\epsilon/P_n+2$ . Under the same conditions there is also a set  $E_3^{(k)}(\alpha)$

$\mathcal{E}_3^{(K)}(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which each of the  $\binom{h}{K}$  terms indicated within the second set of large square brackets is less than  $\epsilon/\rho_{n+2}$ .

We may define

$$\mathcal{E}_3(\alpha) = \prod_{K=1}^{h-1} \mathcal{E}_3^{(K)}(\alpha),$$

which is a set for all points  $\alpha$  of which each of the  $(2^h + 1)$  terms of 10(5), except the first two and the last two, is in absolute value less than  $\epsilon/\rho_{n+2}$ .

There remains now only to show that the difference of the last two terms of 10(5) is in absolute value less than  $\epsilon/\rho_{n+1}$  in a set  $\mathcal{E}_0(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ . An application of 10(4) yields the desired result immediately.

Thus, in a set

$$\mathcal{E}(\alpha) \equiv \mathcal{E}_0(\alpha) \cdot \mathcal{E}_1(\alpha) \cdot F'(\alpha) \cdot \mathcal{E}_3(\alpha),$$

which contains all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , we have shown that 10(2) subsists and have thereby at last completely established

**Theorem 9.**

The series 10(1) as  $\alpha \rightarrow \alpha_0$  will tend to  $\mathcal{L}$  for every sequence 8(1) converging to  $\mathcal{L}$  and of class  $(\bar{\alpha})_n$  relative to a given set of functions 8(2) of class  $(\bar{\mu})_n$ .

11. Convergence factors of type II for convergent multiple series of a certain class  $(\overline{L})_n$ .

We consider in this section the case where each of the functions  $g(2)$  approaches a limit as  $\alpha \rightarrow \alpha_0$ , not in  $E(\alpha)$ . It is convenient now to make the following two definitions:

DEFINITION 11.1. Set of functions of class  $(\overline{\alpha}')_n$ .

The set of functions  $g(2)$  is of class  $(\overline{\alpha}')_n$  if it is of class  $(\alpha')_n$  and if in addition the functions of the set satisfy the following  $(n + 2)$  conditions:

$$(C)_n \quad \lim_{\alpha \rightarrow \alpha_0} f_{[i]}(\alpha) = 1, \quad (\text{all } [i]);$$

$$(A)_n \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} |\Delta_{i_1 \dots i_n} f_{[i]}(\alpha)| < K \quad (E'(\alpha)),$$

where  $E'(\alpha)$ , as above, contains all points of  $E(\alpha)$  lying in a certain neighborhood of  $\alpha_0$ , and where  $K$  is a positive number;

$n$  conditions  $(E_s)_n$  of the following type:

$$(E_s)_n \quad \lim_{\alpha \rightarrow \alpha_0} \left\{ \sum_{i_2=0}^{\infty} \dots \sum_{i_n=0}^{\infty} |\Delta_{i_1 \dots i_n} f_{[i]}(\alpha)| \right\} = 0, \quad (i_1 = 0, 1, \dots).$$

DEFINITION 11.2. Series of a certain class  $(\overline{\mathcal{L}})_n$ .

The convergent multiple series 9(1) is of class  $(\overline{\mathcal{L}})_n$  relative to a given set of functions 9(2) of class  $(\overline{\sigma}')_n$  if the sequence of the partial sums of 9(1), defined by 9(5) is of class  $(\overline{\sigma})_n$  relative to the given set of functions 9(6) of class  $(\overline{\sigma})_n$  and if in addition the  $(2^n - 2)$  conditions are satisfied:

$n$  conditions  $(\overline{\beta}_s^{(1)})_n$  of the type

$$(\overline{\beta}_s^{(1)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Psi^{(1,1)}(\alpha) = 0;$$

$(2^n - n - 2)$  conditions  $(\overline{\beta}_s^{(k)})_n$  of the type

$$(\overline{\beta}_s^{(k)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Psi^{(k,1)}(\alpha) = 0.$$

We wish to show that the series 9(7) will converge in  $E(\alpha)$  and have the same value as the convergent series 9(1) of class  $(\overline{\mathcal{L}})_n$  relative to a given set of functions 9(2) of class  $(\overline{\sigma}')_n$  whenever  $\alpha \rightarrow \alpha_0$ .

If we assume that the series 9(1) converges to zero, we may use the results of section nine, as well as those of section eight.

We form again the Abel Transformation for multiple series, which is the expression 9(9). In view of section nine we know that under the conditions which we have imposed, the limit as  $m_k \rightarrow \infty$ ,  $k = 1, \dots, n$  of the sequence defined

by 9(9) exists; it thus suffices for our purposes to show that there is a set  $\bar{E}(\alpha)$  which contains all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  and in which for any given  $\epsilon > 0$

$$11(1) \quad \left| \lim_{m_1 \rightarrow \infty, \dots, m_n \rightarrow \infty} \sigma_{[m]}(\alpha) \right| < \epsilon.$$

In view of theorem nine just proved there is a set  $\bar{E}_1(\alpha)$ , containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$11(2) \quad \left| \sum_{l_1=0}^{\infty} \dots \sum_{l_n=0}^{\infty} \omega_{[l_1, \dots, l_n]} f_{[l_1, \dots, l_n]}(\alpha) \right| \equiv \left| \sum_{l_1=0}^{\infty} \dots \sum_{l_n=0}^{\infty} \omega_{[l_1, \dots, l_n]} \phi_{[l_1, \dots, l_n]}(\alpha) \right| < \frac{\epsilon}{2^n}.$$

Since the  $n$  terms indicated by the first of the large square brackets of 9(9) each by section nine converge in  $E(\alpha)$  to functions of the form  $\Psi^{(1,s)}(\alpha)$  as the  $m$ 's each become infinite, we need only observe that there is a subset  $\bar{E}_2(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which there are  $n$  inequalities involving  $\Psi^{(1,s)}(\alpha)$  of the following form

$$11(3) \quad \left| \Psi^{(1,1)}(\alpha) \right| < \frac{\epsilon}{2^n},$$

which is true in view of condition  $(\bar{\beta}_r^{(1)})_n$ .

Similarly in view of section nine and conditions  $(\bar{\beta}_s^{(k)})_n$  there is a set  $\bar{E}_3(\alpha)$  containing all points of

$E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , in which the  $(2^n - n - 2)$  terms indicated by the second of the large square brackets of 9(9) are each less than  $\frac{\epsilon}{2^n}$ , for there are  $(2^n - n - 2)$  inequalities involving  $\Psi^{(k,s)}(\alpha)$  and subsisting for each  $\alpha$  in  $\bar{E}_3(\alpha)$ . They are of the following form

$$11(4) \quad \left| \Psi^{(k,1)}(\alpha) \right| < \frac{\epsilon}{2^n}.$$

In section nine it was pointed out that the limit of the last term on the right of 9(9) is zero for all  $\alpha$  in  $E(\alpha)$  as  $m, \dots, m_n$  each become infinite. Thus, in view of 11(2), 11(3), and 11(4) there is a set  $\bar{E}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , which may be defined

$$\bar{E}(\alpha) \equiv \bar{E}_1(\alpha) \cdot \bar{E}_2(\alpha) \cdot \bar{E}_3(\alpha),$$

in which the inequality 11(1) subsists.

If the series 9(1) converges to  $u \neq 0$ , one may reduce the treatment of that case to the above treatment of the case where the series 9(1) converges to zero by alteration of the term  $u_{00} \dots 0$ .

We have now established

**Theorem 10.**

The series 9(7) will converge in  $E(\alpha)$  and have the same value as the convergent series 9(1) of class  $(\bar{L})_n$  relative to a given set of functions of class  $(\bar{a}')_n$  whenever  $\alpha \rightarrow \alpha_0$ .

12. Sequence to function transformations for multiple sequences of a certain class  $(\mathcal{L}^{\bar{v}})_n$ .

Given a set of functions  $\mathcal{S}(2)$  of class  $(\bar{\alpha})_n$ , we make the following definition:

DEFINITION 12.1. Sequence of a certain class  $(\mathcal{L}^{\bar{v}})_n$ .

The convergent sequence  $\mathcal{S}(1)$  is of class  $(\mathcal{L}^{\bar{v}})_n$  relative to a given set of functions  $\mathcal{S}(2)$  of class  $(\bar{\alpha})_n$  if the following  $(2^n - 2)$  conditions are satisfied:  $n$  conditions  $(\bar{\gamma}_s^{(1)})_n$  of the type  $(\bar{\gamma}_s^{(1)})_n$ . The sequence  $\mathcal{S}(1)$  is such that the series on the right of

$$12(1) \quad \prod_{i_2, \dots, i_n}^{(1,1)} (\alpha) \equiv \sum_{i_1=0}^{\infty} \omega_{[i_1]} \varphi_{[i_1]}(\alpha)$$

converges for each  $i_2, \dots, i_n$ ; and the function on the left of 12(1) is defined for each  $\alpha$  of  $E(\alpha)$  and is also such that

$$12(2) \quad \lim_{\alpha \rightarrow \alpha_0} \prod_{i_2, \dots, i_n}^{(1,1)} (\alpha) = 0 \quad (\text{each } i_2, \dots, i_n);$$

$(2^n - n - 2)$  conditions  $(\bar{\gamma}_s^{(k)})_n$  of the type  $(\bar{\gamma}_s^{(k)})_n$ . The sequence  $\mathcal{S}(1)$  is such that the series on the right of

$$12(3) \quad \left[ \begin{matrix} (k, 1) \\ i_{k+1}, \dots, i_n \end{matrix} \right] (\alpha) \equiv \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \mathcal{S}_{[i]} \varphi_{[i]}(\alpha)$$

converges for each  $i_{k+1}, \dots, i_n$ ; and the function on the left of 12(3) is defined for each  $\alpha$  in  $E(\alpha)$  and is also such that

$$12(4) \quad \lim_{\alpha \rightarrow \alpha_0} \left[ \begin{matrix} (k, 1) \\ i_{k+1}, \dots, i_n \end{matrix} \right] (\alpha) = 0, \quad (\text{each } i_{k+1}, \dots, i_n);$$

$k$  ranging from 2 to  $(n - 1)$  inclusive and  $S$  from 1 to  $\binom{n}{k}$  for each  $k$ .

We wish to show that given a multiple sequence 8(1) converging to  $\mathcal{U}$  and of class  $(\mathcal{U})_n$  relative to a given set of functions 8(2) of class  $(\bar{\pi})_n$  and corresponding to any  $\epsilon > 0$  there is a number  $N$  and a set  $\mathcal{E}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , such that for all  $\alpha$  in  $\mathcal{E}(\alpha)$  and all  $m_k > N$ ,  $k = 1, \dots, n$

$$12(5) \quad \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \mathcal{S}_{[i]} \varphi_{[i]}(\alpha) - \mathcal{U} \right| < \epsilon.$$

We have

$$12(6) \quad \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \mathcal{S}_{[i]} \varphi_{[i]}(\alpha) - \mathcal{U} \right| = \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \mathcal{S}_{[i]} \varphi_{[i]}(\alpha) - \right.$$

$$\left| \omega \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \varphi_{[i]}(\alpha) + \omega \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \varphi_{[i]}(\alpha) - \omega \right| \leq$$

$$\left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} (\omega - \omega) \varphi_{[i]}(\alpha) \right| + |\omega| \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \varphi_{[i]}(\alpha) - 1 \right|.$$

In view of the definition of  $f_{00\dots 0}(\alpha)$  in section nine, we have

$$12(7) \quad f_{00\dots 0}(\alpha) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \varphi_{[i]}(\alpha),$$

and hence for any  $\epsilon > 0$  there is a number  $N_1$ , such that for all  $m_k > N_1$ ,  $k = 1, \dots, n$  and each  $\alpha$  in  $E(\alpha)$

$$12(8) \quad \frac{\epsilon}{4|\omega|} > \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \varphi_{[i]}(\alpha) - f_{00\dots 0}(\alpha) \right| \geq$$

$$\left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \varphi_{[i]}(\alpha) - 1 \right| - \left| f_{00\dots 0}(\alpha) - 1 \right|.$$

In view of  $(C^*)_n$  there is a subset  $\mathcal{E}'(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$\left| f_{00\dots 0}(\alpha) - 1 \right| < \frac{\epsilon}{4|\omega|},$$

and hence for all  $m_k > N_1$ ,  $k = 1, \dots, n$  and each  $\alpha$  in  $\mathcal{E}'(\alpha)$

$$12(9) \quad \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \phi_{I_{i_j}}(\alpha) - 1 \right| \leq \frac{\epsilon}{4|A|} + \frac{\epsilon}{4|A|} = \frac{\epsilon}{2|A|}.$$

Thus, it suffices to show that given an  $\epsilon > 0$  there is a number  $N \geq N_1$  and a set  $E''(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which for all  $m_k \geq N$ ,  $k = 1, \dots, n$

$$12(10) \quad \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} (s_{I_{i_j}} - a) \phi_{I_{i_j}}(\alpha) \right| < \frac{\epsilon}{2}.$$

Let us first observe that if the sequence 8(1) converges to  $a \neq 0$  and if it is of class  $(\bar{L})_n$  relative to a given set of functions of class  $(\bar{\alpha})_n$ , then the sequence

$$12(11) \quad \{s_{I_{i_j}}\} \equiv \{s_{I_{i_j}} - a\}$$

is also of class  $(\bar{L})_n$  relative to the same given set of functions 8(2) of class  $(\bar{\alpha})_n$ . For, we have  $n$  relations of the type

$$\left| \sum_{i_1=0}^{\infty} (s_{I_{i_j}} - a) \phi_{I_{i_j}}(\alpha) \right| \leq \left| \sum_{i_1=0}^{\infty} s_{I_{i_j}} \phi_{I_{i_j}}(\alpha) \right| + |a| \left| \sum_{i_1=0}^{\infty} \phi_{I_{i_j}}(\alpha) \right|,$$

of which the first term tends to zero as  $\alpha \rightarrow \alpha_0$  in virtue

of  $(\bar{\gamma}^{(1)})_n$  and of which the second term tends to zero as  $\alpha \rightarrow \alpha_0$  in virtue of the conditions  $(E_S^*)_n$ .

We also have  $(2^n - n - 2)$  relations of the type

$$\left| \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} (s_{[i]} - s) \varphi_{[i]}(\alpha) \right| \leq$$

$$\left| \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} s_{[i]} \varphi_{[i]}(\alpha) \right| + |s| \cdot \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} |\varphi_{[i]}(\alpha)|,$$

of which the first term tends to zero as  $\alpha \rightarrow \alpha_0$  in virtue of  $(\bar{\gamma}^{(K)})_n$ , and of which the second term tends to zero as  $\alpha \rightarrow \alpha_0$  in virtue of  $(E_S^*)_n$ .

Thus, to arrive at our desired result, it suffices to show that given  $\epsilon > 0$ , there is a number  $N$  and a set  $E''(\alpha)$  containing all points in a certain neighborhood of  $\alpha_0$  such that for all  $m_k > N$ ,  $k = 1, \dots, n$  and all  $\alpha$  in  $E''(\alpha)$

$$12(12) \quad \left| \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} s_{[i]} \varphi_{[i]}(\alpha) \right| < \frac{\epsilon}{2}.$$

We break up the series 12(12) into  $2^n$  parts, as follows:

$$12(13) \quad \left| \sum_{i_1=0}^{p_1} \cdots \sum_{i_n=0}^{p_n} s_{[i]} \varphi_{[i]}(\alpha) + \sum_{i_1=p_1+1}^{m_1} \cdots \sum_{i_n=p_n+1}^{m_n} s_{[i]} \varphi_{[i]}(\alpha) + \right.$$

$$\left[ \sum_{i_1=p_1+1}^{m_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_n=0}^{p_n} s_{[i]} \varphi_{[i]}(\alpha) \right] + \cdots +$$

$$\left[ \sum_{i_1=p_1+1}^{m_1} \cdots \sum_{i_k=p_k+1}^{m_k} \sum_{i_{k+1}=0}^{p_{k+1}} \cdots \sum_{i_n=0}^{p_n} s_{[i]} \varphi_{[i]}(\alpha) \right] + \cdots \quad | ,$$

where the large square brackets enclosing an expression indicate the sum of the whole group of terms of similar form obtained by permutation of the indices, where  $k$  ranges from 2 to  $(n - 1)$  inclusive and where  $p_1, \dots, p_k$  are such that for any  $\epsilon > 0$  we have for all  $i_1 > p_1, \dots, i_n > p_n$

$$12(14) \quad |s_{[i]}| < \frac{\epsilon}{2^{n+1} K} .$$

We define

$$M = \max |s_{[i]}| \quad (i_k = 0, 1, \dots, p_k ; k = 1, \dots, n).$$

In view of condition  $(E_1^*)_n$  and the definition of  $M$ , there is a set  $\mathcal{E}_1(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$12(15) \quad \left| \varphi_{[i]}(\alpha) \right| < \frac{\epsilon}{2^{n+1} M \prod_{j=1}^n (p_j+1)} \quad (i_k = 1, \dots, p_k ; k = 1, \dots, n).$$

For each  $\alpha$  in  $E_1(\alpha)$

$$12(16) \left| \sum_{i_1=0}^{p_1} \dots \sum_{i_n=0}^{p_n} S_{I_{i_j}} \phi_{I_{i_j}}(\alpha) \right| \leq M \sum_{i_1=0}^{p_1} \dots \sum_{i_n=0}^{p_n} |\phi_{I_{i_j}}(\alpha)| < \frac{\epsilon}{2^{n+1}}.$$

In view of  $(A_n^*)$  and 12(14) we have for all  $\alpha$  in  $E'(\alpha)$

$$12(17) \left| \sum_{i_1=p_1+1}^{m_1} \dots \sum_{i_n=p_n+1}^{m_n} S_{I_{i_j}} \phi_{I_{i_j}}(\alpha) \right| \leq \frac{\epsilon}{2^{n+1}K} \sum_{i_1=p_1+1}^{m_1} \dots \sum_{i_n=p_n+1}^{m_n} |\phi_{I_{i_j}}(\alpha)| < \frac{\epsilon}{2^{n+1}}.$$

Thus, the first two terms of 12(13) are each by 12(16) and 12(17) in absolute value less than  $\frac{\epsilon}{2^{n+1}}$ . The  $n$  terms indicated by the first set of large brackets in 12(13) may be treated in a manner wholly analogous to the following treatment of a representative term:

In view of the  $n$  conditions  $(E_n^*)$  and the limit in condition  $(\bar{\delta}_n^{(1)})$ , there is a subset  $E_3^{(1)}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which for  $i = 0, 1, \dots, p$ ;  $k = 1, \dots, n$

$$|\phi_{I_{i_j}}(\alpha)| < \frac{\epsilon}{3 \cdot 2^{n+1} M \prod_{j=1}^n (p_j + 1)}$$

and also in which for  $i_1 = 0, 1, \dots, p_1$

$$\left| \prod_{i_2, \dots, i_n}^{(1)}(\alpha) \right| < \frac{\epsilon}{3 \cdot 2^{n+1} M \prod_{j=2}^n (p_j + 1)}.$$

Given  $\epsilon > 0$ , there is an  $N_1^{(1)}$  such that for all  $m_i > N_1^{(1)}$  in view of  $(\bar{\delta}_i^{(1)})_n$  for each  $\alpha$  in  $E(\alpha)$

$$\left| \sum_{i=1}^{m_1} s_{[i]} \phi_{[i]}(\alpha) \right| - \left| \Gamma_{i_2, \dots, i_n}^{(1,1)}(\alpha) \right| <$$

$$\left| \sum_{i=1}^{m_1} s_{[i]} \phi_{[i]}(\alpha) - \Gamma_{i_2, \dots, i_n}^{(1,1)}(\alpha) \right| < \frac{\epsilon}{3 \cdot 2^{n+1} \prod_{j=2}^n (p_j + 1)}.$$

$$\left| \sum_{i=1}^{m_1} s_{[i]} \phi_{[i]}(\alpha) \right| < \frac{\epsilon}{3 \cdot 2^{n+1} \prod_{j=2}^n (p_j + 1)} + \left| \Gamma_{i_2, \dots, i_n}^{(1,1)}(\alpha) \right|.$$

Thus, in  $E_3^{(1,1)}(\alpha)$  and for all  $m_i > N_1^{(1)}$

$$\begin{aligned} 12(18) \quad & \left| \sum_{i_1=p_1+1}^{m_1} \sum_{i_2=0}^{p_2} \dots \sum_{i_n=0}^{p_n} s_{[i]} \phi_{[i]}(\alpha) \right| \leq \\ & \left| \sum_{i_1=0}^{p_1} \dots \sum_{i_n=0}^{p_n} s_{[i]} \phi_{[i]}(\alpha) \right| + \left| \sum_{i_2=0}^{p_2} \dots \sum_{i_n=0}^{p_n} \left( \sum_{i_1=0}^{m_1} s_{[i]} \phi_{[i]}(\alpha) \right) \right| \\ & \leq \frac{\epsilon}{3 \cdot 2^{n+1}} + \frac{\epsilon}{3 \cdot 2^{n+1}} + \frac{\epsilon}{3 \cdot 2^{n+1}} = \frac{\epsilon}{2^{n+1}}. \end{aligned}$$

The  $(2^n - n - 2)$  terms indicated by the second set of large brackets in 12(13) may be treated in a manner

wholly analogous to the following treatment of a representative term which holds for  $k = 2, \dots, (n - 1)$  :

In view of the  $n$  conditions  $(E_s^*)_n$  and the limit in conditions  $(\bar{\gamma}^{(k)})_n$ , there is a subset  $\mathcal{E}_3^{(k,1)}(\alpha)$  of  $E(\alpha)$  having  $\alpha_0$  as a limit point and containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which for  $k = 1, \dots, n$ ;  $i_k = 0, 1, \dots, p_k$  :

$$12(19) \quad \left| \varphi_{[i]}(\alpha) \right| < \frac{\epsilon}{3 R_k^0 2^{n+1} M \prod_{j=1}^n (p_j + 1)}$$

and also in which for  $i_j = 0, 1, \dots, p_j$ ;  $j = 1, \dots, k$ , the following  $(2^k - 1)$  inequalities subsist:

$k$  inequalities  $(\eta_s^{(1)})_k$  of the type

$$(\eta_s^{(1)})_k \quad \left| \Gamma_{i_2, \dots, i_n}^{(1,1)}(\alpha) \right| < \frac{\epsilon}{3 R_k^0 2^{n+1} \prod_{j=2}^n (p_j + 1)},$$

$(2^k - k - 1)$  inequalities  $(\eta_s^{(m)})_k$  of the type

$$(\eta_s^{(m)})_k \quad \left| \Gamma_{i_{m+1}, \dots, i_n}^{(m,1)}(\alpha) \right| < \frac{\epsilon}{3 R_k^0 2^{n+1} \prod_{j=m+1}^n (p_j + 1)},$$

where  $m$  ranges from 2 to  $k$  inclusive and  $s$  from 1 to  $\binom{k}{m}$  for each  $m$ , where each  $\Gamma$  symbol has at least the indices  $i_{k+1}, \dots, i_n$  and where  $R_k^p$  is defined by 10(11).

We break up the term in the second of the large brackets of 12(13), as follows:

12(20)

$$\left| \sum_{i_1=p_1+1}^{m_1} \cdots \sum_{i_k=p_k+1}^{m_k} \left( \sum_{i_{k+1}=0}^{p_{k+1}} \cdots \sum_{i_n=0}^{p_n} s_{[i]} \varphi_{[i]}(\alpha) \right) \right| \leq$$

$$\left| \sum_{i_{k+1}=0}^{p_{k+1}} \cdots \sum_{i_n=0}^{p_n} \left( \sum_{i_1=0}^{m_1} \cdots \sum_{i_k=0}^{m_k} s_{[i]} \varphi_{[i]}(\alpha) \right) \right| +$$

$$\frac{R_k^{k-1}}{\binom{k}{k-1}} \left[ \left| \sum_{i_{k+1}=0}^{p_{k+1}} \cdots \sum_{i_n=0}^{p_n} \left( \sum_{i_k=0}^{p_k} \left\{ \sum_{i_1=0}^{m_1} \cdots \sum_{i_{k-1}=0}^{m_{k-1}} s_{[i]} \varphi_{[i]}(\alpha) \right\} \right) \right] \right] +$$

---


$$\frac{R_k^1}{\binom{k}{1}} \left[ \left| \sum_{i_{k+1}=0}^{p_{k+1}} \cdots \sum_{i_n=0}^{p_n} \left( \sum_{i_2=0}^{p_2} \cdots \sum_{i_k=0}^{p_k} \left\{ \sum_{i_1=0}^{m_1} s_{[i]} \varphi_{[i]}(\alpha) \right\} \right) \right] \right] +$$

$$R_k^0 \left| \sum_{i_1=0}^{p_1} \cdots \sum_{i_n=0}^{p_n} s_{[i]} \varphi_{[i]}(\alpha) \right|,$$

where the large brackets enclosing an expression indicate the sum of the whole group of terms of similar form, obtained by permutation of the indices of sums within the

parentheses.

We wish to show that there is a number  $N_{i,0}^{(k)}$  sufficiently large, such that for all  $m_i > N_{i,0}^{(k)}$  ( $i = 1, \dots, k$ ) and all  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$  the term on the left hand side of 12(20) is less than  $\frac{\epsilon}{2^{n+1}}$ . In view of condition  $(\bar{\gamma}_i^{(k)})_n$  and inequality  $(\eta_i^{(k)})_k$  there is a number  $N_{i,0}^{(k)}$ , such that for all  $m_i > N_{i,0}^{(k)}$  ( $i = 1, \dots, k$ ) and all  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$  the first term on the right of 12(20),

$$\begin{aligned}
 12(21) \quad & \left| \sum_{i_{k+1}=0}^{p_{k+1}} \dots \sum_{i_n=0}^{p_n} \left( \sum_{i_1=0}^{m_1} \dots \sum_{i_k=0}^{m_k} S_{[i]} \varphi_{[i]}(\alpha) \right) \right| \leq \\
 & \sum_{i_{k+1}=0}^{p_{k+1}} \dots \sum_{i_n=0}^{p_n} \left( \left| \frac{\epsilon}{3R_k^0 2^{n+1} \prod_{j=k+1}^n (p_j+1)} + \left[ \begin{matrix} (k,1) \\ i_{k+1}, \dots, i_n \end{matrix} \right] (\alpha) \right| \right) \\
 & \leq \frac{\epsilon}{3R_k^0 2^{n+1}} + \frac{\epsilon}{3R_k^0 2^{n+1}} = \frac{2\epsilon}{3R_k^0 2^{n+1}}.
 \end{aligned}$$

In view of conditions  $(\bar{\gamma}_s^{(k-1)})_n$  and inequalities  $(\eta_s^{(k-1)})_k$  there is a number  $N_{i,1}^{(k)}$ , such that for all  $m_i > N_{i,1}^{(k)}$  ( $i = 1, \dots, k$ ) and all  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$  each of the  $\binom{k}{k-1}$  terms indicated by the first of the large square brackets on the right of 12(20) is less than  $[2\epsilon/3R_k^0 2^{n+1}]$ . Hence, the second term on the right of 12(20) for all  $m_i > N_{i,1}^{(k)}$  and for all  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$  is less than  $[2\epsilon R_k^{k-1}/3R_k^0 2^{n+1}]$ . Similarly in view of conditions  $(\bar{\gamma}_s^{(1)})_n$  and inequalities  $(\eta_s^{(1)})_k$  there is a number  $N_{i,t}^{(k)}$ , such that for all  $m_i > N_{i,t}^{(k)}$  ( $i = 1, \dots, k$ ) and all  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$  each of the

$\binom{k}{1}$  terms indicated by the second of the large square brackets on the right of 12(20) is less than  $[2\epsilon/3R_k^0 2^{n+1}]$ .

Hence, the next to the last term on the right of 12(20) for all  $m_i > N_{i,t}^{(k)}$  and for all  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$  is less than  $[2\epsilon R'_k/3R_k^0 2^{n+1}]$ . In view of 12(19) and the definition M, the last term on the right of 12(20),

$$12(22) \quad R_k^0 \left| \sum_{i=0}^{p_1} \cdots \sum_{i_n=0}^{p_n} s_{[l]} \varphi_{[l]}(\alpha) \right| \leq R_k^0 M \sum_{i=0}^{p_1} \cdots \sum_{i_n=0}^{p_n} |\varphi_{[l]}(\alpha)| < \frac{\epsilon}{3 \cdot 2^{n+1}}$$

for each  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$ .

If we define

$$12(23) \quad N_i^{(k)} \equiv \max N_{i,t}^{(k)}, \quad (t = 0, 1, \dots, k-1),$$

we have shown that for each  $\alpha$  in  $\mathcal{E}_3^{(k,1)}(\alpha)$  and for all  $m_i > N_i^{(k)}$  ( $i = 1, \dots, k$ ) the expression on the left of 12(20) is less than

$$\frac{\epsilon}{3 \cdot 2^{n+1}} + \frac{2\epsilon}{3R_k^0 2^{n+1}} \left[ R_k^1 + \cdots + R_k^{k-1} + 1 \right] = \frac{\epsilon}{2^{n+1}}$$

Thus, there is a set  $\mathcal{E}''(\alpha)$ , containing all points common to the sets  $\mathcal{E}_i(\alpha)$ ,  $\mathcal{E}'(\alpha)$ , and  $\prod \mathcal{E}_3^{(k,s)}(\alpha)$  [ $k$  ranging from 2 to  $(n-1)$  inclusive and  $s$  from 1 to  $\binom{n}{k}$  for each  $k$ ], in which each of the  $2^n$  terms of 12(13) is in absolute value less than  $\frac{\epsilon}{2^{n+1}}$  for all  $m_i > N$  ( $i = 1, \dots, n$ ), where

$$N = \max N_s^{(k)}, \quad \left[ k = 1, \dots, (n-1); s = 1, \dots, \binom{n}{k} \right].$$

Hence, for each  $\alpha$  in  $\mathcal{E}(\alpha)$  (13) is less than  $\frac{\epsilon}{2}$  for all  $m_i > N$ , ( $i = 1, \dots, n$ ).

We have completely established:

Theorem 11.

Corresponding to every positive number  $\epsilon$  there is a number  $N$  and a set  $\mathcal{E}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which the sequence

$$S_{m_1, \dots, m_n} \equiv \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} \mathcal{S}_{[i]} \varphi_{[i]}(\alpha)$$

differs from  $\mathcal{S}$  in absolute value by less than  $\epsilon$  for all  $m_k > N$ ,  $k = 1, \dots, n$  and for every sequence (1) converging to  $\mathcal{A}$  and of class  $(\overline{\alpha})_n$  relative to a given set of functions (2) of class  $(\overline{\pi})_n$ .

If in the above, one sets  $n = 2$ , theorem 11 reduces to theorem 5. Since the theorem of Adams<sup>1</sup> referred to above is a special case of theorem 5, it is all the more a special case of theorem 11, just proved.

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1. Adams, C. R. Loc. cit.

13. On convergent multiple series of a certain class  $(\overline{\mathcal{P}})_n$ .

Given a multiply infinite set of functions  $\mathcal{G}(2)$  of class  $(\overline{\mathcal{A}})_n$ , as this concept was defined by Definition 11.1, we make the following definition:

DEFINITION 13.1. Multiple series of a certain class  $(\overline{\mathcal{P}})_n$ .

The convergent multiple series  $\mathcal{G}(1)$  is of class  $(\overline{\mathcal{P}})_n$  relative to a given set of functions  $\mathcal{G}(2)$  of class  $(\overline{\mathcal{A}})_n$  if the sequence of the partial sums of  $\mathcal{G}(2)$  as defined by  $\mathcal{G}(5)$  is of class  $(\overline{\mathcal{L}})_n$  relative to the given set of functions  $\mathcal{G}(6)$  of class  $(\overline{\mathcal{A}})_n$  and if in addition the following  $(2^n - 2)$  conditions are satisfied:

$n$  sets of conditions  $(\overline{\delta}_s^{(1)})_n$  of the type

$$(\overline{\delta}_1^{(1)})_n \quad \lim_{m_2 \rightarrow \infty, \dots, m_n \rightarrow \infty} \Delta_{[i, n-1, m]} \Delta_{10 \dots 0} f_{[i, n-1, m]}^{(\alpha)} = \Lambda_{i_1}^{(1)}(\alpha)$$

$$\lim_{\alpha \rightarrow \alpha_0} \Lambda_{i_1}^{(1)}(\alpha) = 0$$

(all  $i_1$ );

$\binom{n}{2}$  sets of conditions  $(\overline{\delta}_s^{(2)})_n$  of the type

$$(\overline{\delta}_1^{(2)})_n \quad \lim_{m_3 \rightarrow \infty, \dots, m_n \rightarrow \infty} \Delta_{[i, n-2, m]} \Delta_{110 \dots 0} f_{[i, n-2, m]}^{(\alpha)} = \Lambda_{i_1, i_2}^{(1, 2)}(\alpha)$$

$$\lim_{\alpha \rightarrow \alpha_0} \bigwedge_{i_1, i_2}^{(1,2)} (\alpha) = 0 \quad (\text{each } i_1, i_2);$$

$\binom{2}{1}$  pairs of conditions  $(\bar{\delta}_1^{(2)})_n^{(1, \bar{1})}$  :

$$\begin{aligned} (\bar{\delta}_1^{(2)})_n^{(1,1)} & \left| \sum_{i_1=0}^{\infty} \Delta_{[i_1, n-2, m]} \Delta_{110 \dots 0} f_{[i_1, n-2, m]}^{(\alpha)} \right| \\ & = \bigwedge_{i_2, m_3, \dots, m_n}^{(\bar{1}, 2)} (\alpha) \end{aligned}$$

$$\lim_{\alpha \rightarrow \alpha_0} \bigwedge_{i_2, m_3, \dots, m_n}^{(\bar{1}, 2)} (\alpha) = 0$$

(each  $i_2, m_3, \dots, m_n$ );

$$\begin{aligned} (\bar{\delta}_i^{(2)})_n^{(1,2)} & \left| \sum_{i_2=0}^{\infty} \Delta_{[i, n-2, m]} \Delta_{110 \dots 0} f_{[i, n-2, m]}^{(\alpha)} \right| \\ & = \bigwedge_{i_2, m_3, \dots, m_n}^{(1, \bar{2})} (\alpha), \end{aligned}$$

$$\lim_{\alpha \rightarrow \alpha_0} \bigwedge_{i_2, m_3, \dots, m_n}^{(1, \bar{2})} (\alpha) = 0 \quad (\text{each } i_2, m_3, \dots, m_n);$$

$\binom{n}{k}$  sets of conditions  $(\bar{\delta}_s^{(k)})_n$  of the type

$$\begin{aligned} (\bar{\delta}_1^{(k)})_n & \lim_{m_{k+1} \rightarrow \infty, \dots, m_n \rightarrow \infty} \Delta_{[i, n-k, m]} \Delta_{11 \dots 10 \dots 0} f_{[i, n-k, m]}^{(\alpha)} \\ & = \bigwedge_{i_1, \dots, i_k}^{(1, 2, \dots, k)} (\alpha), \end{aligned}$$

$$\lim_{\alpha \rightarrow \alpha_0} \bigwedge_{i_1, \dots, i_k}^{(1, 2, \dots, k)} (\alpha) = 0$$

(each  $i_1, \dots, i_k$ );

$\binom{k}{1}$  sets of conditions  $(\bar{\delta}_1^{(k)})_n^{(1,r)}$  of the type

$$\left( \bar{\delta}_1^{(k)} \right)_n^{(1,r)} \quad \left| \sum_{i_1=0}^{\infty} \mathcal{A}_{[i_1, n-k, m]} \Delta_{11 \dots 10 \dots 0} f_{[i_1, n-k, m]}^{(\alpha)} \right|$$

$$= \bigwedge_{i_2, \dots, i_k, m_{k+1}, \dots, m_n}^{(\bar{1}, 2, \dots, k)} (\alpha),$$

$$\lim_{\alpha \rightarrow \alpha_0} \bigwedge_{i_2, \dots, i_k, m_{k+1}, \dots, m_n}^{(\bar{1}, 2, \dots, k)} (\alpha) = 0,$$

(each  $i_2, \dots, i_k, m_{k+1}, \dots, m_n$ ),

$\binom{k}{\ell}$  sets of conditions  $(\bar{\delta}_1^{(k)})_n^{(\ell,r)}$  of the type

$$\left( \bar{\delta}_1^{(k)} \right)_n^{(\ell,r)} \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_{\ell}=0}^{\infty} \mathcal{A}_{[i_1, n-k, m]} \Delta_{11 \dots 10 \dots 0} f_{[i_1, n-k, m]}^{(\alpha)}$$

$$= \bigwedge_{i_{\ell+1}, \dots, i_k, m_{k+1}, \dots, m_n}^{(\bar{1}, \bar{2}, \dots, \bar{\ell}, \ell+1, \dots, k)} (\alpha),$$

$$\lim_{\alpha \rightarrow \alpha_0} \bigwedge_{i_{\ell+1}, \dots, i_k, m_{k+1}, \dots, m_n}^{(\bar{1}, \bar{2}, \dots, \bar{\ell}, \ell+1, \dots, k)} (\alpha) = 0,$$

(each  $i_{\ell+1}, \dots, i_k, m_{k+1}, \dots, m_n$ ),

where  $\ell$  ranges from 2 to  $(k-1)$  inclusive for each  $k$  and  $k$  from 3 to  $(n-1)$  inclusive,  $r$  ranging from 1 to  $\binom{k}{\ell}$  for each  $\ell$  and  $s$  from 1 to  $\binom{n}{k}$  for each  $k$ .

We wish to show that given a positive number  $\epsilon$ , there will exist a number  $N_0$  and a set  $\mathcal{C}(\epsilon)$  containing all

points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  such that for all indices  $M_k > N_0$ ,  $k = 1, \dots, n$  and for all  $\alpha$  in  $E(\alpha)$

$$13(1) \quad \left| \sigma_{[M]}(\alpha) \right| \equiv \left| \sum_{i_1=0}^{M_1} \cdots \sum_{i_n=0}^{M_n} u_{[i]} f_{[i]}(\alpha) \right| < \epsilon,$$

for all series 9(1) converging to zero and of class  $(\mathcal{P})_n$  relative to a given set of functions 9(2) of class  $(\mathcal{P}')_n$ .

Given any  $\epsilon > 0$ , then since 9(1) converges to zero, there are numbers  $m_1, \dots, m_n$  such that for all  $i_k > m_k$ ,  $k = 1, \dots, n$

$$13(2) \quad |u_{[i]}| < \frac{\epsilon}{T_n K},$$

where the sequence  $\{u_{[m]}\}$  is defined by 9(5) and the number  $T_n$  is defined

$$13(3) \quad T_n \equiv \sum_{i=0}^n \binom{n}{i} T_{n-i} + n, \quad T_0 \equiv 1.$$

On the sequence 13(1) we perform the generalization to multiple series of the Abel Transformation, as follows:

$$13(4) \quad \sigma_{[M]}(\alpha) = \sum_{i_1=0}^{M_1-1} \cdots \sum_{i_n=0}^{M_n-1} u_{[i]} \Delta_{|| \dots ||} f_{[i]}(\alpha).$$

$$\begin{aligned}
& + \left[ \sum_{i_1=0}^{M_1-1} \omega_{[i_1, n-1, M]} \Delta_{10 \dots 0} f_{[i_1, n-1, M]}^{(\alpha)} \right] + \dots \\
& + \left[ \sum_{i_1=0}^{M_1-1} \dots \sum_{i_k=0}^{M_k-1} \omega_{[i_1, n-k, M]} \Delta_{11 \dots 10 \dots 0} f_{[i_1, n-k, M]}^{(\alpha)} \right] + \dots \\
& + \omega_{[M]} f_{[M]}^{(\alpha)}.
\end{aligned}$$

where the large brackets enclosing an expression indicate the sum of the whole group of terms of similar form, obtained by permutation of the indices, and where  $k$  ranges from 2 to  $(n-1)$  inclusive, and where  $M_s > m_s$ ,  $s = 1, \dots, n$ .

Since the sequence 9(5) is of class  $(\bar{\omega})_n$  relative to the given set of functions 9(6) of class  $(\bar{\omega})_n$ , we may invoke theorem 8 to show that for any  $\epsilon > 0$  there is a set  $\mathcal{E}_\epsilon(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha$ , in which for all  $M_s^{(1)} > m_s$ ,  $s = 1, \dots, n$ .

$$13(5) \quad \left| \sum_{i_1=0}^{M_1^{(1)}-1} \dots \sum_{i_n=0}^{M_n^{(1)}-1} \omega_{[i]} \Delta_{11 \dots 1} f_{[i]}^{(\alpha)} \right| < \frac{\epsilon}{T_n}.$$

Each of the  $n$  terms indicated by the first set of large square brackets on the right of 13(4) may be treated in a manner wholly analogous to the following treatment of a representative term:

In view of condition  $(\bar{\delta}^{(1)})_n$  for a given  $\epsilon > 0$  there are numbers  $M_k^{(2)}$ ,  $k = 2, \dots, n$ , such that for all

$$M_k > M_k^{(2)}, k = 2, \dots, n$$

$$\left| \omega_{[i, n-1, M]} \Delta_{10 \dots 0} f_{[i, n-1, M]}^{(1)}(\alpha) - \Lambda_{i_1}^{(1)}(\alpha) \right| < \frac{\epsilon}{2 T_n(m_1+1)},$$

and also there is a set  $\mathcal{E}_2^{(1)}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$\left| \Lambda_{i_1}^{(1)}(\alpha) \right| < \frac{\epsilon}{2 T_n(m_1+1)}$$

and also in view of  $(A, )_n$  in which

$$\sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} \left| \Delta_{11 \dots 1} f_{[i]}^{(1)}(\alpha) \right| < K.$$

Thus, for all

$$M_1 > M_1^{(1)}, M_k > M_k^0 \equiv \max(M_k^{(2)}, M_k^{(1)}) \quad (k = 2, \dots, n),$$

and for each  $\alpha$  in  $\mathcal{E}_2^{(1,1)}(\alpha)$  we have

$$\begin{aligned} 13(6) \quad & \left| \sum_{i_1=0}^{M_1-1} \omega_{[i, n-1, M]} \Delta_{10 \dots 0} f_{[i, n-1, M]}^{(1)}(\alpha) \right| \leq \\ & \left| \sum_{i_1=0}^{m_1} \omega_{[i, n-1, M]} \Delta_{10 \dots 0} f_{[i, n-1, M]}^{(1)}(\alpha) \right| + \\ & \left| \sum_{i_1=m_1+1}^{M_1-1} \omega_{[i, n-1, M]} \Delta_{10 \dots 0} f_{[i, n-1, M]}^{(1)}(\alpha) \right| \leq \end{aligned}$$

$$\sum_{i=0}^{m_1} \left| \mathcal{A}_{[i, n-1, M]} \Delta_{10 \dots 0} f_{[i, n-1, M]}^{(\alpha)} - \Lambda_i^{(1)}(\alpha) \right| +$$

$$\sum_{i=0}^{m_1} \left| \Lambda_i^{(1)}(\alpha) \right| + \frac{\epsilon}{T_n K} \sum_{i_1=m_1+1}^{M_1-1} \sum_{i_2=M_2}^{\infty} \dots \sum_{i_n=M_n}^{\infty} \left| \Delta_{11 \dots 1} f_{[i]}^{(\alpha)} \right|$$

$$\leq \frac{\epsilon}{2T_n} + \frac{\epsilon}{2T_n} + \frac{\epsilon}{T_n} = \frac{2\epsilon}{T_n} .$$

For each of the n expressions indicated by the first of the sets of the large square brackets there is a set  $\mathcal{E}_2^{(i)}(\alpha)$  in which the expression is less than  $\frac{2\epsilon}{T_n}$ . Thus, there is a subset  $\mathcal{E}_2(\alpha)$  in which all of the n expressions indicated by the first of the sets of large square brackets are less than  $\frac{2\epsilon}{T_n}$ .

Each of the  $\binom{n}{k}$  terms indicated by the second set of large square brackets may be treated in a manner wholly analogous to the treatment of the following term which we break up into  $\frac{T_k}{2}$  parts which are each less than  $\frac{2\epsilon}{T_n}$  in a sufficiently small neighborhood of  $\alpha_0$  containing points of  $E(\alpha)$ ;

$$13(7) \sum_{i_1=0}^{M_1-1} \dots \sum_{i_k=0}^{M_k-1} \mathcal{A}_{[i, n-k, M]} \Delta_{1 \dots 10 \dots 0} f_{[i, n-k, M]}^{(\alpha)} .$$

The expression 13(7) is broken up in the same manner as 12(12) was broken up; and by appealing to Condition  $(\mathcal{S}_{,n}^{(k)})$  and its subconditions, one can see by use of the method employed in dealing with 13(6) that there is a set  $\mathcal{E}_{k+1}^{(1)}(\alpha)$ , containing all points of  $E(\alpha)$  in a certain

neighborhood of  $\alpha_0$  in which 13(7) is in absolute value less than  $\frac{T_k \epsilon}{T_n}$ . The set  $\mathcal{E}_{k+1}(\alpha)$ , containing all points common to the  $\binom{n}{k}$  sets  $\mathcal{E}_{k+1}^{(s)}(\alpha)$ , is a set in which all the  $\binom{n}{k}$  expressions indicated by the second set of large square brackets are each in absolute value less than  $\frac{T_k \epsilon}{T_n}$ .

There remains now for consideration only the last term on the right of 13(4). In view of  $(C)_n$  there is a subset  $\mathcal{E}_{n+1}(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  in which

$$13(8) \quad \left| f_{[M]}(\alpha) - 1 \right| < K - 1 \quad (\text{all } M_k, k = 1, \dots, n).$$

Thus, for all  $\alpha$  in  $\mathcal{E}_{n+1}(\alpha)$  and all  $M_k, k = 1, \dots, n$

$$\begin{aligned} \left| \omega_{[M]} f_{[M]}(\alpha) \right| &= \left| \omega_{[M]} f_{[M]}(\alpha) - \omega_{[M]} + \omega_{[M]} \right| \\ &\leq \left| \omega_{[M]} \left[ f_{[M]}(\alpha) - 1 \right] \right| + \left| \omega_{[M]} \right| \\ &< \frac{\epsilon}{T_n K} (K - 1) + \frac{\epsilon}{T_n K} = \frac{\epsilon}{T_n}. \end{aligned}$$

Hence, in a set

$$\mathcal{E}(\alpha) = \prod_{i=1}^{n+1} \mathcal{E}_i(\alpha)$$

containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$ , we have that

$$\left| \sigma_{[M]}(\alpha) \right| < \epsilon,$$

for all  $M_1, \dots, M_n > N_0$ .

If the series 9(1) under consideration converges to  $s \neq 0$ , we may reduce it to the case just considered by introducing the series  $\sum u'_{[i]}$  converging to zero and defined as follows:

$$u'_{i_0 \dots i_n} = u_{i_0 \dots i_n} - s$$

$$u'_{[i]} = u_{[i]} \quad (\text{all } i_k \neq 0, k=1, \dots, n).$$

Thus, we have completely established

**Theorem 12:**

Corresponding to a positive number  $\epsilon$ , there will exist a number  $N_0$  and a set  $E(\alpha)$  containing all points of  $E(\alpha)$  in a certain neighborhood of  $\alpha_0$  such that for all indices  $M_k > N_0, k = 1, \dots, n$  and for all  $\alpha$  in  $E(\alpha)$

$$\left| \sum_{i_1=0}^{M_1} \dots \sum_{i_n=0}^{M_n} u_{[i]} f_{[i]}(\alpha) - s \right| < \epsilon$$

for all series 9(1) converging to  $s$  and of class  $(\mathcal{P})_n$  relative to a given set of functions 9(2) of class  $(\bar{a}')_n$ .

**GLOSSARY AND BIBLIOGRAPHY.**

## GLOSSARY

Since in the body of the text many sequences, series, conditions, and definitions are referred to frequently by number or by letter, the writer has thought it wise to collect many of them below for the convenience of the reader. To those definitions of classes and to those conditions which are complicated and which require a relatively large amount of space to state, references are given to that page of the text where they are introduced.

## I. Double Series and Sequences

1(1)  $\{s_{ij}\}$ ,  $(i, j = 0, 1, \dots)$ , converging to  $s$ .

1(2)  $\{\phi_{ij}(\alpha)\}$ ,  $(i, j = 0, 1, \dots)$ , defined in  $E(\alpha)$ .

1(3)  $\sum_{i=0, j=0}^{\infty, \infty} s_{ij} \phi_{ij}(\alpha)$ .

(A\*)  $\sum_{i=0, j=0}^{\infty, \infty} |\phi_{ij}(\alpha)| < K(\alpha)$ , positive function of  $\alpha$ .

DEFINITION 1.1

The set 1(2) as of class  $(\alpha)_2$  if  $(A^*)_2$  is satisfied.

DEFINITION 1.2.

The sequence 1(1) is of class  $(\alpha)_2$  relative to a given set

1(2) of class  $(\alpha)_2$  if

$(\alpha)_2$  for all  $p > p_0$  the series

$$1(5) \quad \Phi_p^{(1)}(\alpha) \equiv \sum_{i=p}^{\infty} \left( \sum_{j=0}^{\infty} \rho_{ij} \phi_{ij}(\alpha) \right)$$

converges for each  $\alpha$  in  $E(\alpha)$ ;

$(\alpha)_2$  and if for all  $q > q_0$  the series

$$1(6) \quad \Phi_q^{(2)}(\alpha) \equiv \sum_{j=q}^{\infty} \left( \sum_{i=0}^{\infty} \rho_{ij} \phi_{ij}(\alpha) \right)$$

converges for each  $\alpha$  in  $E(\alpha)$ .

$$2(1) \quad \sum_{i=0, j=0}^{\infty, \infty} u_{ij} \quad \text{converges to zero.}$$

$$2(2) \quad \{f_{ij}(\alpha)\}, \quad (i, j = 0, 1, \dots), \text{ defined in } E(\alpha).$$

$$2(3) \quad \rho_{mn} \equiv \sum_{i=0, j=0}^{m, n} u_{ij}.$$

$$2(5) \quad f_{pq}(\alpha) \equiv \sum_{i=p, j=q}^{\infty, \infty} \phi_{ij}(\alpha).$$

$$2(6) \quad \Delta_{11} f_{ij}(\alpha) = \phi_{ij}(\alpha).$$

$$(A)_2 \quad \sum_{i=0, j=0}^{\infty, \infty} |\Delta_{ij} f_{ij}(\alpha)| < K(\alpha), \quad (\text{in } E(\alpha)).$$

$$(D_1)_2 \quad \lim_{n \rightarrow \infty} \Delta_{i0} f_{in}(\alpha) = 0, \quad (i = 0, 1, \dots, E(\alpha)).$$

$$(D_2)_2 \quad \lim_{m \rightarrow \infty} \Delta_{0j} f_{mj}(\alpha) = 0, \quad (j = 0, 1, \dots, E(\alpha)).$$

DEFINITION 2.1.

The set 2(2) is of class  $(\mathcal{A}')_2$  if  $(A)_2$ ,  $(D_1)_2$ , and  $(D_2)_2$  are satisfied.

DEFINITION 2.2.

The series 2(1) is of class  $(\mathcal{A})_2$  relative to a given set 2(2) of class  $(\mathcal{A}')_2$  if 2(3) is of class  $(\mathcal{A})_2$  relative to the given set 2(6) of class  $(\mathcal{A})_2$ .

$$2(8) \quad \sum_{i=0, j=0}^{\infty, \infty} u_{ij} f_{ij}(\alpha).$$

$$2(9) \quad \sigma_{\mu, \nu}(\alpha) \equiv \sum_{i=0, j=0}^{\mu, \nu} u_{ij} f_{ij}(\alpha).$$

DEFINITION 3.1

Series 2(1) is of class  $(\mathcal{L})_2$  relative to a given set 2(2) of class  $(\mathcal{A}')_2$  if 2(3) is of class  $(\mathcal{A})_2$  relative to the given set 2(6) of class  $(\mathcal{A})_2$  and if the following two conditions are satisfied

$$(\beta_1)_2 \quad \lim_{j \rightarrow \infty} \Delta_{ij} \Delta_{10} f_{ij}(\alpha) = \Phi_i^{(1)}(\alpha), \quad (i = 0, 1, \dots; E(\alpha));$$

$$\sum_{i=0}^{\infty} \Phi_i^{(1)}(\alpha) = \Psi^{(1)}(\alpha), \quad E(\alpha),$$

$\Psi^{(1)}(\alpha)$  is bounded in absolute value;

$$(\beta_2)_2 \quad \lim_{i \rightarrow \infty} \Delta_{ij} \Delta_{01} f_{ij}(\alpha) = \Phi_j^{(2)}(\alpha), \quad (j = 0, 1, \dots; E(\alpha));$$

$$\sum_{j=0}^{\infty} \Phi_j^{(2)}(\alpha) = \Psi^{(2)}(\alpha), \quad E(\alpha),$$

$\Psi^{(2)}(\alpha)$  is bounded in absolute value.

#### DEFINITION 4.1

The set  $1(2)$  is of class  $(\overline{\alpha})_2$  if it is of class  $(\alpha)_2$  and if the four following conditions are satisfied:

$$(C^*)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i=0, j=0}^{\infty, \infty} \phi_{ij}(\alpha) = 1;$$

$$(E^*_1)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{j=0}^{\infty} |\phi_{ij}(\alpha)| = 0, \quad (\text{each } i);$$

$$(E^*_2)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i=0}^{\infty} |\phi_{ij}(\alpha)| = 0, \quad (\text{each } j);$$

$(A^*_1)_2 \quad \sum_{i=0, j=0}^{\infty, \infty} |\phi_{ij}(\alpha)| < K$ , in a sufficiently small neighborhood  $E'(\alpha)$  of  $\alpha_0$ .

DEFINITION 4.2.

The sequence  $l(1)$  is of class  $(\overline{\mathcal{O}})_2$  relative to a given set of functions  $l(2)$  of class  $(\overline{\mathcal{R}})_2$ , if it is of class  $(\mathcal{O})_2$  relative to the same set  $l(2)$  and if

$$(\overline{\alpha}_1)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \Phi_q^{(1)}(\alpha) = 0, \quad (\text{each } q);$$

$$(\overline{\alpha}_2)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \Phi_p^{(2)}(\alpha) = 0, \quad (\text{each } p).$$

DEFINITION 5.1.

A set  $2(2)$  is of class  $(\overline{\mathcal{R}}')_2$  if it is of class  $(\mathcal{R}')_2$  and if

$$(E_1)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{j=0}^{\infty} |\Delta_{ij} f_{ij}(\alpha)| = 0, \quad (\text{all } i);$$

$$(E_2)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i=0}^{\infty} |\Delta_{ij} f_{ij}(\alpha)| = 0, \quad (\text{all } j);$$

$$(A_1)_2 \quad \sum_{i=0, j=0}^{\infty, \infty} |\Delta_{ij} f_{ij}(\alpha)| < K, \quad (\text{in } E'(\alpha));$$

$$(C)_2 \quad \lim_{\alpha \rightarrow \alpha_0} f_{ij}(\alpha) = 1, \quad (i, j = 0, 1, \dots).$$

DEFINITION 5.2.

The series 2(1) is of class  $(\overline{\mathcal{L}})_2$  relative to a given set of functions 2(2) of class  $(\overline{\pi})_2$  if 2(3) is of class  $(\overline{\alpha})_2$  relative to the given set 2(6) of class  $(\overline{\alpha})_2$  and if

$$(\overline{\beta}_1)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \Phi^{(1)}(\alpha) = 0 ;$$

$$(\overline{\beta}_2)_2 \quad \lim_{\alpha \rightarrow \alpha_0} \Phi^{(2)}(\alpha) = 0.$$

DEFINITION 6.1.

The sequence 1(1) is of class  $(\overline{\mathcal{L}})_2$  relative to a given set 1(2) of class  $(\overline{\pi})_2$  if

$(\overline{\gamma}_1)_2$  The series on the right of

$$6(1) \quad \Gamma_j^{(1)}(\alpha) \equiv \sum_{i=0}^{\infty} \omega_{ij} \phi_{ij}(\alpha) \text{ converges for each } j,$$

$$6(2) \quad \lim_{\alpha \rightarrow \alpha_0} \Gamma_j^{(1)}(\alpha) = 0, \text{ (each } j\text{).}$$

$(\overline{\gamma}_2)_2$  The series on the right of

$$6(3) \quad \Gamma_i^{(2)}(\alpha) \equiv \sum_{j=0}^{\infty} \omega_{ij} \phi_{ij}(\alpha) \text{ converges for each } i,$$

$$6(4) \quad \lim_{\alpha \rightarrow \alpha_0} \Gamma_i^{(2)}(\alpha) = 0, \text{ (each } i\text{).}$$

.....

$$6(5) \quad \left| \sum_{0,0}^{\mu,\nu} s_{ij} f_{ij}(\alpha) - s \right| < \epsilon, \quad \mu, \nu > N, \text{ in } \mathcal{E}(\alpha).$$

$$6(11) \quad \{s_{ij}\} = \{s_{ij} - s\} \quad (i, j = 0, 1, \dots).$$

DEFINITION 7.1.

The series 2(1) is of class  $(\overline{\mathcal{J}})_2$  if relative to a given set 2(2) of class  $(\overline{\mathcal{A}})_2$  if 2(3) is of class  $(\overline{\mathcal{L}})_2$  relative to the given set 2(6) of class  $(\overline{\mathcal{A}})_2$  and if

$$(\overline{\mathcal{S}}_1)_2 \quad \lim_{j \rightarrow \infty} s_{ij} \Delta_{10} f_{ij}(\alpha) = \Lambda_i^{(1)}(\alpha), \quad (i = 0, 1, \dots; \mathcal{E}(\alpha));$$

$$\lim_{\alpha \rightarrow \alpha_0} \Lambda_i^{(1)}(\alpha) = 0, \quad (i = 0, 1, \dots);$$

$$(\overline{\mathcal{S}}_2)_2 \quad \lim_{i \rightarrow \infty} s_{ij} \Delta_{01} f_{ij}(\alpha) = \Lambda_j^{(2)}(\alpha), \quad (j = 0, 1, \dots; \mathcal{E}(\alpha));$$

$$\lim_{\alpha \rightarrow \alpha_0} \Lambda_j^{(2)}(\alpha) = 0 \quad (j = 0, 1, \dots).$$

## II. Multiple Series and Sequences

8(1)  $\{ \Delta_{I_k J_k} \}$ , ( $i_k = 0, 1, \dots$ ;  $k = 1, \dots, n$ ) converging to  $\alpha$ .

8(2)  $\{ \phi_{I_k J_k}(\alpha) \}$ , ( $i_k = 0, 1, \dots$ ;  $k = 1, \dots, n$ ).

8(3)  $\sum_{l_1=0}^{\infty} \dots \sum_{l_n=0}^{\infty} \Delta_{I_1 J_1} \phi_{I_1 J_1}(\alpha)$ .

(A\*)<sub>n</sub>  $\sum_{l_1=0}^{\infty} \dots \sum_{l_n=0}^{\infty} | \phi_{I_1 J_1}(\alpha) | < K(\alpha)$ , a positive function of  $\alpha$ .

DEFINITION 8.1.

The set of 8(2) is of class  $(\alpha)_n$  if (A\*)<sub>n</sub> is satisfied.

DEFINITION 8.2.

The sequence 8(1) is of class  $(\alpha)_n$  relative to a given set 8(2) of class  $(\alpha)_n$ , if  $(2^n - 2)$  conditions of the type  $(\alpha_s^{(k)})_n$  are satisfied:

$(\alpha_s^{(1)})_n$  If for all  $m_i > p_i$  ( $i = 1, \dots, n-1$ ) the series

8(5)  $\sum_{m_1, \dots, m_{n-1}}^{(1,1)} (\alpha) \equiv \sum_{l_1=m_1}^{\infty} \dots \sum_{l_{n-1}=m_{n-1}}^{\infty} \left( \sum_{l_n=0}^{\infty} \Delta_{I_1 J_1} \phi_{I_1 J_1}(\alpha) \right)$

converges for each  $\alpha$  in  $E(\alpha)$ .

$(\alpha_s^{(k)})_n$  and if for all  $m_i > p_i$  ( $i = 1, \dots, n-k$ ) the series

$$8(6) \quad \Phi_{m_1, \dots, m_{n-k}}^{(k, 1)}(\alpha) \equiv \sum_{l_1=m_1}^{\infty} \dots \sum_{l_{n-k}=m_{n-k}}^{\infty} \left( \sum_{i_{n-k+1}=0}^{\infty} \dots \sum_{i_n=0}^{\infty} \Delta_{i_1, \dots, i_n} \Phi_{i_1, \dots, i_n}(\alpha) \right)$$

converges for each  $\alpha$  in  $E(\alpha)$ .

$$8(9) \quad P_0 \equiv 1; \quad P_1 \equiv 1; \quad P_h \equiv \sum_{i=0}^{h-1} \binom{h}{i} (P_i + 1), \quad (h=2, 3, \dots).$$

$$9(1) \quad \sum_{l=0}^{\infty} \dots \sum_{l_n=0}^{\infty} U_{i_1, \dots, i_n}, \quad \text{converging to zero.}$$

$$9(2) \quad \left\{ f_{i_1, \dots, i_n}(\alpha) \right\}, \quad (i_k = 0, 1, \dots; \quad k = 1, \dots, n).$$

DEFINITION 9.1.

The set 9(2) is of class  $(\mathcal{A})_n$  if the following

$(2)^n - 1$  conditions are satisfied:

$$(A)_n \quad \sum_{l_1=0}^{\infty} \dots \sum_{l_n=0}^{\infty} |\Delta_{i_1, \dots, i_n} f_{i_1, \dots, i_n}(\alpha)| < K(\alpha), \quad \text{a positive function of } \alpha;$$

$n$  conditions  $(D)_s^{(1)}$  of the type

$$(D)_1^{(1)} \quad \lim_{i_n \rightarrow \infty} \Delta_{i_1, \dots, i_{n-1}, 0} f_{i_1, \dots, i_n}(\alpha) = 0, \quad (E(\alpha); \text{ all } i_1, \dots, i_{n-1});$$

$(2)^n - n - 2$  conditions  $(D)_s^{(k)}$  of the type

$$(D)_1^{(k)} \quad \lim_{i_{n-k+1} \rightarrow \infty, \dots, i_n \rightarrow \infty} \Delta_{i_1, \dots, i_{n-k}, 0}^{n-k, k} f_{i_1, \dots, i_n}(\alpha) = 0,$$

$(E(\alpha); \text{ all } (i_1, \dots, i_{n-k})),$

$k$  ranging from 2 to  $(n-1)$  inclusive, and  $s$  from 1 to  $\binom{n}{k}$

for each  $k$ .

DEFINITION 9.2.

The series 9(1) is of class  $(\mathcal{L})_h$  relative to a given set 9(2) of class  $(\mathcal{A}')_h$  if the sequence of the partial sums of 9(1), defined by

$$9(5) \quad \mathcal{A}_{[m]} \equiv \sum_{l_1=0}^{m_1} \cdots \sum_{l_n=0}^{m_n} u_{l_1 l_2 \dots l_n}$$

is of class  $(\mathcal{A})_h$  relative to a given set

$$9(6) \quad \varphi_{l_1 l_2 \dots l_n}(\alpha) \equiv \Delta_{l_1 \dots l_n} f_{l_1 l_2 \dots l_n}(\alpha), \quad (E(\alpha)),$$

of class  $(\mathcal{A})_h$  and if the  $(2^n - 2)$  conditions of pages 55-59 are satisfied.

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$$9(7) \quad \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} u_{l_1 l_2 \dots l_n} f_{l_1 l_2 \dots l_n}(\alpha).$$

DEFINITION 10.1.

The set 8(2) is of class  $(\mathcal{A})_h$  if it is of class  $(\mathcal{A})_h$  and if the following  $(n+2)$  conditions are also satisfied:

$$(A^*)_{l_1 \dots l_n} \quad \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} |\varphi_{l_1 l_2 \dots l_n}(\alpha)| < \kappa, \quad (\text{each } \alpha \text{ in } E'(\alpha)).$$

$E'(\alpha)$  contains all  $\alpha$  in a certain neighborhood of  $\alpha_0$ , and

K is a positive constant;

$$(C^*)_n \quad \lim_{\alpha \rightarrow \alpha_0} \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} \phi_{I_i}(\alpha) = 1;$$

n conditions  $(E_s^*)_n$  of the type

$$(E_s^*)_n \quad \lim_{\alpha \rightarrow \alpha_0} \left\{ \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} |\phi_{I_i}(\alpha)| \right\} = 0, \quad (i_1 = 0, 1, \dots);$$

where s ranges from 1 to n inclusive.

DEFINITION 10.2.

The sequence  $\mathcal{S}(1)$  is of class  $(\bar{\alpha})_n$  relative to a given set  $\mathcal{S}(2)$  of class  $(\underline{\alpha})_n$  if  $\mathcal{S}(1)$  is of class  $(\alpha)_n$  relative to a given set  $\mathcal{S}(2)$  of class  $(\bar{\alpha})_n$  and if the following  $(2^n - 2)$  conditions are satisfied:

n conditions  $(\bar{\alpha}_s^{(n-1)})_n$  of the type

$$(\bar{\alpha}_s^{(n-1)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Phi_{m_1}^{(n-1,1)}(\alpha) = 0, \quad (m_1 > p_1);$$

$(2^n - n - 2)$  conditions  $(\bar{\alpha}_s^{(n-k)})_n$  of the type

$$(\bar{\alpha}_s^{(n-k)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Phi_{m_1, \dots, m_k}^{(n-k,1)}(\alpha) = 0, \quad (m_i > p_i; i = 1, \dots, k),$$

where k ranges from 2 to (n - 1) inclusive and s from 1 to  $\binom{n}{k}$  for each k.



$$10(1) \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} A_{I_{ij}} f_{I_{ij}}(\alpha).$$

DEFINITION 11.1.

The set 9(2) is of class  $(\bar{\alpha}')_n$  if it is of class  $(\alpha')_n$  and if the following  $(n+2)$  conditions are satisfied:

$$(C)_n \quad \lim_{\alpha \rightarrow \alpha_0} f_{I_{ij}}(\alpha) = 1 \quad (\text{all } [i] );$$

$$(A)_n \quad \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} |A_{i_1 \dots i_n}| f_{I_{ij}}(\alpha) < K, \quad (E'(\alpha) ),$$

where  $E'(\alpha)$  contains all points of  $E(\alpha)$  a certain neighborhood of  $\alpha_0$  and where  $K$  is a positive number;  $n$  conditions  $(E_s)_n$  of the following type:

$$(E_s)_n \quad \lim_{\alpha \rightarrow \alpha_0} \left\{ \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} |A_{i_1 \dots i_n}| f_{I_{ij}}(\alpha) \right\} = 0,$$

( $i_s = 0, 1, \dots$ ).

DEFINITION 11.2.

The series 9(1) is of class  $(\bar{L})_n$  relative to a

given set 9(2) of class  $(\overline{\alpha}')_n$  if the sequence 9(5) is of class  $(\overline{\alpha})_n$  relative to the given set 9(6) of class  $(\overline{\alpha})_n$  and if the following  $(2^n - 2)$  conditions are satisfied:

$n$  conditions  $(\overline{\beta}_s^{(1)})_n$  of the type

$$(\overline{\beta}_s^{(1)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Psi^{(1,1)}(\alpha) = 0;$$

$(2^n - n - 2)$  conditions  $(\overline{\beta}_s^{(k)})_n$  of the type

$$(\overline{\beta}_s^{(k)})_n \quad \lim_{\alpha \rightarrow \alpha_0} \Psi^{(k,1)}(\alpha) = 0,$$

where  $k$  ranges from 2 to  $(n-1)$  inclusive and  $s$  from 1 to  $\binom{n}{k}$  for each  $k$ .

DEFINITION 12.1.

The sequence 8(1) is of class  $(\overline{\mathcal{L}})_n$  relative to a given set 8(2) of class  $(\overline{\alpha})_n$  if the following  $(2^n - 2)$  conditions are satisfied:

$n$  conditions  $(\overline{\delta}_s^{(1)})_n$  of the type

$(\overline{\delta}_s^{(1)})_n$  The series

$$12(1) \quad \Gamma_{i_2, \dots, i_n}^{(1,1)}(\alpha) \equiv \sum_{i_1=0}^{\infty} \omega_{[i_1]} \varphi_{[i_1]}(\alpha)$$

converges for each  $i_2, \dots, i_n$ ; and the function on the left is such that

$$12(2) \quad \lim_{\alpha \rightarrow \alpha_0} \Gamma_{i_2, \dots, i_n}^{(1,1)}(\alpha) = 0 \quad (\text{each } i_2, \dots, i_n);$$

$(2^n - n - 2)$  conditions  $(\bar{y}_s^{(k)})_n$  of the type

$(\bar{y}_1^{(k)})_n$  The series

$$12(3) \quad \Gamma_{i_{k+1}, \dots, i_n}^{(k,1)}(\alpha) \equiv \sum_{i_1=0}^{\infty} \dots \sum_{i_k=0}^{\infty} s_{[i]} \phi_{[i]}(\alpha)$$

converges for each  $i_{k+1}, \dots, i_n$ ; and the function on the left is such that

$$12(4) \quad \lim_{\alpha \rightarrow \alpha_0} \Gamma_{i_{k+1}, \dots, i_n}^{(k,1)}(\alpha) = 0, \quad (\text{each } i_{k+1}, \dots, i_n);$$

$k$  ranging from 2 to  $(n-1)$  inclusive and  $s$  from 1 to  $\binom{n}{k}$  for each  $k$ .

$$12(11) \quad \{s_{[i]}\} \equiv \{s_{[i]} - s\}$$

DEFINITION 13.1.

The series 9(1) is of class  $(\bar{y})_n$  relative to a given set 9(2) of class  $(\bar{y}')_n$  if the sequence 9(5) is of class  $(\bar{y})_n$  relative to the given set 9(6) of class  $(\bar{y})_n$  and if the  $(2^n - 2)$  conditions of pages 97-99 are satisfied.

$$13(3) \quad T_n \equiv \sum_{i=1}^n \binom{n}{i} T_{n-i} + n, \quad T_0 \equiv 1.$$

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