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Introduction

The following dissertation is an investigation of the developments of real, positive numbers, according to prescribed algorithms, into continued fractions having positive integral elements. The principal problems that present themselves are the ones relating to : (1) convergence and representation, if the development is infinite , (2) the uniqueness of the development , (3) the approximation properties of the convergents to the continued fractions , (4) the developments of rational numbers , (5) periodic developments, and (6) the irrationality of certain continued fractions. A total of five algorithms are studied systematically with the above problems foremost in mind . While some of the questions are not answered conclusively, it is hoped that this research will provide a spring-board for further investigations along the lines indicated.

This thesis was first undertaken at the suggestion of Dr. Otto Szász, Research Lecturer in Mathematics at the University of Cincinnati, and it was carried on to its present stage under his guidance. I am grateful to Dr. Szász for his help in the direction of the thesis, and for his encouragement in carrying out the work. I am indebted to Dr. Szász and to the members of the Department of Mathematics at the University of Cincinnati for the training in mathematical research and exposition that I received , and also for the many pleasant associations at the University of Cincinnati.

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Chapter 1

1.1 Appell Algorithm

Appell^{*} introduced the following algorithm :

Let κ be a given positive integer, $\kappa \geq 1$, and suppose that a real number $\xi_0 > 0$ is given. The number c_0 is defined to be the greatest integer contained in $\xi_0^{1/\kappa}$.

$$c_0 \leq \xi_0^{1/\kappa} < c_0 + 1 \quad ; \quad c_0 = [\xi_0^{1/\kappa}] \geq 0.$$

$$\text{Then :} \quad c_0^\kappa \leq \xi_0 < (c_0 + 1)^\kappa$$

$$\text{Write } \rho_0 = \xi_0 - c_0^\kappa, \text{ then } 0 \leq \rho_0 < (c_0 + 1)^\kappa - c_0^\kappa.$$

If $\rho_0 = 0$, then $\xi_0 = c_0^\kappa$, and we do not continue. Otherwise, $\rho_0 > 0$, and we proceed by defining the number ξ_1 by the relation :

$$\xi_1 = \frac{(c_0 + 1)^\kappa - c_0^\kappa}{\rho_0}.$$

$$\text{Thus, } \xi_1 > 1 \quad \text{and} \quad \xi_0 = c_0^\kappa + \frac{(c_0 + 1)^\kappa - c_0^\kappa}{\xi_1}.$$

The above procedure is repeated using ξ_1 in place of ξ_0 .

Thus, let c_1 be the greatest integer contained in $\xi_1^{1/\kappa}$.

$$c_1 \leq \xi_1^{1/\kappa} < c_1 + 1 \quad ; \quad c_1 = [\xi_1^{1/\kappa}] \geq 1. \quad \text{Then } c_1^\kappa \leq \xi_1 < (c_1 + 1)^\kappa.$$

$$\text{Write } \rho_1 = \xi_1 - c_1^\kappa \quad ; \quad 0 \leq \rho_1 < (c_1 + 1)^\kappa - c_1^\kappa$$

* References :

Appell, Paul : (1) " Sur une nouvelle mode de développement d'un nombre en fraction continue ". Bulletin des Sciences Mathématique, 1914, 2^{me} series, t.38, part 1, p118.

(2) L'Intermediare des Mathematiciens. vol 20, 1913. Paris. pp. 167 - 171

Szász, Otto / (1) Lectures on Continued Fractions , University of Cincinnati, 1940 - 41.

If $\rho_1 = 0$, then $\xi_1 = c_1^\kappa$, and the development terminates.

Then $\xi_0 = c_0^\kappa + \frac{(c_0+1)^\kappa - c_0^\kappa}{c_1^\kappa}$.

If $\rho_1 > 0$, then we continue as before, writing

$\xi_2 = \frac{(c_1+1)^\kappa - c_1^\kappa}{\rho_1}$; $\xi_2 > 1$, and so on.

If at any stage, $\rho_\nu = 0$, then $\xi_\nu = c_\nu^\kappa$, and the development terminates. One obtains in this case the finite

(terminating) continued fraction with integral elements

$$1.1.1 \quad \xi_0 = c_0^\kappa + \frac{(c_0+1)^\kappa - c_0^\kappa}{c_1^\kappa} + \dots + \frac{(c_{\nu-1}+1)^\kappa - c_{\nu-1}^\kappa}{c_\nu^\kappa}$$

with $c_0 \geq 0$, $c_n \geq 1$ for $1 \leq n \leq \nu-1$.

Moreover, as $\xi_\nu > 1$, it follows that $c_\nu \geq 2$. This leads to an ambiguity in the number of terms of a finite development

(1.1.1). For, as $c_\nu \geq 2$, is an integer, we may write:

$$c_\nu^\kappa = (c_\nu - 1)^\kappa + \frac{c_\nu^\kappa - (c_\nu - 1)^\kappa}{1}$$

Thus, there may be an even or an odd number of terms in a development.

If at every stage $\rho_\nu > 0$, the development does not terminate, and one obtains the infinite continued fraction

$$1.1.2 \quad \xi_0 \sim c_0^\kappa + \frac{(c_0+1)^\kappa - c_0^\kappa}{c_1^\kappa} + \dots$$

with positive integral elements $c_0 \geq 0$, $c_\nu \geq 1$ for $\nu \geq 1$.

If $\kappa = 1$, the above coincides with the regular continued fraction development of a real positive number.

Definition 1.1.1 The development of a real number $\xi_0 > 0$ by the "Appell Algorithm" into a continued fraction (1.1.1) or (1.1.2) is called a "development of type 1".

A continued fraction

$$1.1.3 \quad c_0^\kappa + \frac{(c_0+1)^\kappa - c_0^\kappa}{c_1^\kappa} + \dots + \frac{(c_{\nu-1}+1)^\kappa - c_{\nu-1}^\kappa}{c_\nu^\kappa} + \dots$$

with positive integral elements, and with $c_0 \geq 0$, $c_\nu \geq 1$ for $\nu \geq 1$,

and κ a positive integer, $\kappa \geq 1$, is called a continued fraction of type 1. If (1.1.3) is a terminating continued fraction, the final partial denominator c_n is at least 2; $c_n \geq 2$.

1.2 Convergence

We state first a lemma.

Lemma 1.2.1 *
 Let $K = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots$
 have positive elements, $a_\nu > 0$, $b_\nu > 0$, ($\nu \geq 1$), $b_0 \geq 0$.
 Then the convergents of even order $\frac{A_{2\nu}}{B_{2\nu}}$ increase monotonically to a limit g , the convergents of odd order $\frac{A_{2\nu+1}}{B_{2\nu+1}}$ decrease monotonically to a limit G , and, $0 < g \leq G$.
 That is, $\frac{A_{2\nu}}{B_{2\nu}} < \frac{A_{2\mu+1}}{B_{2\mu+1}}$ for $\nu, \mu \geq 0$.

Theorem 1.2.1

The infinite continued fraction of type 1
 1.2.1 $c_0^\kappa + \frac{(c_0+1)^\kappa - c_0^\kappa}{1 c_1^\kappa} + \frac{(c_1+1)^\kappa - c_1^\kappa}{1 c_2^\kappa} + \dots ; \left\{ \frac{A_\nu}{B_\nu} \right\}$

(with κ and c_ν ($\nu \geq 1$) positive integers)

converges, and if it is a development (1.1.2) of the number \sum_0^∞ then it represents \sum_0^∞ .

We give two proofs.

Proof: (a) As a short-hand notation, write $a_\nu = (c_{\nu+1})^\kappa - c_\nu^\kappa$ for $\nu \geq 1$, $b_\nu = c_\nu^\kappa$, $\nu \geq 0$ so that (1.1.2) becomes

$$1.2.3 \quad \sum_0^\infty \sim b_0 + \frac{a_1}{b_1} + \dots = b_0 + \left[\frac{a_\nu}{b_\nu} \right]_\infty$$

* For the proof of this Lemma, see

Perron, O. (1) "Die Lehre von den Kettenbrüchen", Satz 11 page 239. Leipzig, 1929.

Clearly $a_v = (c_{v-1} + 1)^\kappa - c_{v-1}^\kappa \geq 2^\kappa - 1 \geq 1$, and $b_v \geq 1$, $v \geq 1$ are positive integers, so that the B_v are all positive integers ($v \geq 1$). For $B_0 = 1$, $B_1 = b_1 = c_1^\kappa \geq 1$ and $B_v = b_v B_{v-1} + a_v B_{v-2}$, for $v \geq 2$. Then $B_v > b_v B_{v-1} \geq B_{v-1}$ and as the B_v are positive integers, it follows that $B_v \uparrow \infty$.

By Lemma 1.21, the convergents of even order $\frac{A_{2v}}{B_{2v}}$ have a limit g , also $\lim_{v \rightarrow \infty} \frac{A_{2v+1}}{B_{2v+1}}$ exists: call it G , and $0 < g \leq G$. The continued fraction (1.2.2) will converge if $g = G$, that is, if $\lim_{n \rightarrow \infty} \left(\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right) = 0$.

We proceed to demonstrate this. We have

$$1.2.4 \quad \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = (-1)^{n-1} \frac{a_1 a_2 \cdots a_n}{B_n B_{n-1}}; \quad n \geq 1$$

$$\text{Write } u_n = \frac{a_1 a_2 \cdots a_n}{B_n B_{n-1}} \quad \text{Then } \frac{u_n}{u_{n+1}} = \frac{B_{n+1}}{a_{n+1} B_{n-1}} = \frac{b_{n+1} B_n + a_{n+1} B_{n-1}}{a_{n+1} B_{n-1}}$$

$$\text{and } \frac{u_n}{u_{n+1}} = 1 + \frac{c_{n+1} B_n}{a_{n+1} B_{n-1}}; \quad n \geq 1$$

$$\text{Also, as } \frac{B_n}{B_{n-1}} > c_n^\kappa, \quad \frac{c_{n+1} B_n}{a_{n+1} B_{n-1}} > \frac{c_{n+1}^\kappa c_n^\kappa}{a_{n+1}} \geq \frac{c_{n+1}^\kappa}{(c_n + 1)^\kappa - c_n^\kappa},$$

$$\text{and } \frac{c_n^\kappa}{(c_n + 1)^\kappa - c_n^\kappa} \geq \frac{1}{2^\kappa - 1}, \quad \text{since } c_n \geq 1.$$

$$\text{Thus } \frac{u_n}{u_{n+1}} > 1 + \frac{1}{2^\kappa - 1} = \frac{2^\kappa}{2^\kappa - 1} \quad \text{and}$$

$$1.2.5 \quad \frac{u_{n+1}}{u_n} < \frac{2^\kappa - 1}{2^\kappa} = 1 - \frac{1}{2^\kappa} < 1, \quad n \geq 1.$$

$$\text{Write } \theta = 1 - \frac{1}{2^\kappa}. \quad \text{Then } u_{n+1} = u_n \cdot \theta = \theta^2 u_{n-1} = \cdots = \theta^n u_1.$$

$$\text{and } u_{n+1} < \theta^n \cdot \frac{(c_0 + 1)^\kappa - c_0^\kappa}{c_1^\kappa}, \quad (\theta < 1). \quad \text{Hence } \lim_{n \rightarrow \infty} u_n = 0$$

$$\text{that is } \lim_{n \rightarrow \infty} \frac{a_1 a_2 \cdots a_n}{B_n B_{n-1}} = 0, \quad \text{and so } \lim_{n \rightarrow \infty} \left(\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right) = 0$$

and $g = G$. Hence the continued fraction (1.2.2) converges for all integral values of $\kappa \geq 1$.

We show next that if (1.2.3) is the development (1.2.3) of a number ξ_0 , then it represents ξ_0 .

Let n be an arbitrary index, $n \geq 2$. From the algorithm we obtain:

$$\xi_0 = b_0 + \frac{a_1}{1b_1} + \frac{a_2}{1b_2} + \cdots + \frac{a_n}{1\xi_n},$$

where $\xi_n > 1$ is the n^{th} complete quotient, and $b_0 = c_0^k \geq 0$,

$$b_v = c_v^k \geq 1, \quad a_v = (c_{v+1}^k + 1) - c_{v+1}^k.$$

$$\text{Then } \xi_0 = \frac{\sum_n A_{n-1} + a_n A_{n-2}}{\sum_n B_{n-1} + a_n B_{n-2}}, \quad \text{and}$$

$$\xi_0 - \frac{A_{n-1}}{B_{n-1}} = \frac{-a_n (A_{n-1} B_{n-2} - A_{n-2} B_{n-1})}{B_{n-1} (\xi_n B_{n-1} + a_n B_{n-2})}$$

From $\xi_n = c_n^k + \frac{(c_{n+1}^k - c_n^k)}{\xi_{n+1}}$; $\xi_{n+1} > 1$, it follows that: $\xi_n > c_n^k$.

$$\text{Also, } |A_{n-1} B_{n-2} - A_{n-2} B_{n-1}| = a_{n-1} a_{n-2} \cdots a_1.$$

$$\text{Thus: } \left| \xi_0 - \frac{A_{n-1}}{B_{n-1}} \right| < \frac{a_n a_{n-1} \cdots a_1}{B_n B_{n-1}}$$

But from the preceding page 4, $\lim_{n \rightarrow \infty} \frac{a_1 a_2 \cdots a_n}{B_n B_{n-1}} = 0$.

Hence $\lim_{n \rightarrow \infty} \frac{A_{n-1}}{B_{n-1}} = \xi_0$, and the development (1.2.3)

represents ξ_0 . This proves Theorem 1.21.

Proof: (b) By theorem 10, page 279, Perron (1), the continued fraction $K = b_0 + \left[\frac{a_v}{b_v} \right]_{v=1}^{\infty}$, with positive elements a_v, b_v is surely convergent when the series $\sum_1^{\infty} \sqrt{\frac{b_v b_{v+1}}{a_{v+1}}}$ is divergent.

If we apply this criterion to the continued fraction (1.2.2), then we have that (1.2.2) converges

when $\sum_1^{\infty} \sqrt{\frac{c_v^k c_{v+1}^k}{(c_{v+1}^k - c_v^k)}}$ diverges.

Now $c_v \geq 1, (v \geq 1)$, and from the middle of page 4,

$$\frac{c_v^k}{(c_{v+1}^k - c_v^k)} \geq \frac{1}{2^k - 1} > \frac{1}{2^k}. \quad \text{Hence } \sum_{v=1}^{\infty} \sqrt{\frac{c_v^k c_{v+1}^k}{(c_{v+1}^k - c_v^k)}} > \frac{1}{2^{k/2}} \sum_1^{\infty} \sqrt{c_{v+1}^k}.$$

But $K \geq 1$, and $\sqrt{c_v^k} \geq 1$ for $v \geq 0$. Therefore, $\sum_1^{\infty} \sqrt{\frac{c_v^k c_{v+1}^k}{(c_{v+1}^k - c_v^k)}}$ diverges and (1.2.2) converges. It follows then that $\lim_{n \rightarrow \infty} \frac{A_n}{B_n}$ exists and is finite. Consequently,

$$\lim_{n \rightarrow \infty} \left(\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right) = \lim_{n \rightarrow \infty} (-1)^{n-1} \frac{a_1 a_2 \cdots a_n}{B_n B_{n-1}} = 0.$$

The remainder of the proof follows exactly as in (a).

1.3 The Functions u_n and $\gamma_n = \frac{u_{n+1}}{u_n}$

Let (1.3.1) $c_0^k + \frac{a_1}{c_1^k} + \frac{a_2}{c_2^k} + \dots$

be a continued fraction of type 1, with $\left\{ \frac{A_v}{B_v} \right\}$ as its sequence of convergents. Write

$$1.3.2 \quad u_n = \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{a_1 a_2 \dots a_n}{B_n B_{n-1}} \quad \text{and}$$

$$1.3.3 \quad \gamma_n = \frac{u_{n+1}}{u_n} = \frac{a_{n+1} B_{n-1}}{B_{n+1}}, \quad n \geq 1.$$

$$\text{Then : } \begin{cases} u_n = u_n(c_0, c_1, \dots, c_n) \\ \gamma_n = \gamma_n(c_1, c_2, \dots, c_{n+1}). \end{cases}$$

In the sequel, we shall consider the problems of determining the maximum of γ_n , the minimum of $\frac{B_n}{B_{n-1}}$, and also the maximum and the minimum of u_n , for $n \geq 1$.

We have, from (1.3.3), that :

$$\gamma_n = \frac{a_{n+1} B_{n-1}}{c_{n+1}^k B_n + a_{n+1} B_{n-1}} \leq \frac{a_{n+1} B_{n-1}}{B_n + a_{n+1} B_{n-1}}$$

that is : $\gamma_n(c_1, c_2, \dots, c_n, c_{n+1}) \leq \gamma_n(c_1, c_2, \dots, c_n, 1)$,

and equality holds only for $c_{n+1} = 1$. Therefore, we need

only consider the maximum of the function $\gamma_n(c_1, c_2, \dots, c_n, 1)$.

$$\text{Now } B_n + a_{n+1} B_{n-1} = c_n^k B_{n-1} + a_n B_{n-2} + [(c_{n+1})^k - c_n^k] B_{n-1} \\ = (c_{n+1})^k B_{n-1} + a_n B_{n-2}$$

so that $\frac{a_{n+1} B_{n-1}}{B_n + a_{n+1} B_{n-1}} = \frac{(c_{n+1})^k - c_n^k}{(c_{n+1})^k + \frac{a_n B_{n-2}}{B_{n-1}}}$, that is

$$1.3.4 \quad \gamma_n(c_1, c_2, \dots, c_n, 1) = \frac{(c_{n+1})^k - c_n^k}{(c_{n+1})^k + \frac{a_n B_{n-2}}{B_{n-1}}}$$

$$\text{Write (1.3.5) } \begin{cases} d_v = \frac{a_{v+1} B_{v-1}}{B_v}, (v \geq 1) ; d_0 = 0. \\ d_v = d_v(c_1, \dots, c_v). \end{cases}$$

Then (1.3.4) becomes : (1.3.4 a) $\gamma_n(c_1, \dots, c_n, 1) = \frac{(c_{n+1})^k - c_n^k}{(c_{n+1})^k + d_{n-1}}$

$$\text{From (1.3.5), } d_v = \frac{a_{v+1} B_{v-1}}{c_v^k B_v + a_v B_{v-2}} = \frac{(c_{v+1})^k - c_v^k}{c_v^k + \frac{a_v B_{v-2}}{B_{v-1}}}$$

$$\text{Thus : (1.3.5 a) } d_v = \frac{a_{v+1}}{c_v^k + d_{v-1}} = \frac{(c_{v+1})^k - c_v^k}{c_v^k + d_{v-1}}, (v \geq 1).$$

As $d_v > 0$ for $v \geq 1$, $d_0 = 0$, $d_v \leq \frac{(c_v+1)^\kappa - c_v^\kappa}{c_v^\kappa} \leq \left(1 + \frac{1}{c_v}\right)^\kappa - 1$.
Hence, $0 < d_v \leq 2^\kappa - 1$, ($v \geq 1$) with equality only if $v=1$, $c_v=1$.

Theorem 1.3.1

Let $n \geq 2$. The functions

$$\begin{aligned} \delta_n(c_1, \dots, c_{n-1}, c_n, 1) &= \frac{(c_n+1)^\kappa - c_n^\kappa}{(c_n+1)^\kappa + d_{n-1}}; \\ d_n(c_1, \dots, c_{n-1}, c_n) &= \frac{(c_n+1)^\kappa - c_n^\kappa}{c_n^\kappa + d_{n-1}} \end{aligned}$$

satisfy the relations

- (1) $\delta_n(c_1, \dots, c_{n-1}, c_n, 1) < \delta_n(c_1, c_2, \dots, c_{n-1}, 1, 1) < 1 - \frac{1}{2^\kappa}$
(2) $d_n(c_1, \dots, c_{n-1}, c_n) < d_n(c_1, \dots, c_{n-1}, 1, 1) < 2^\kappa - 1$

for $c_{n-1} \geq 1$ and $\kappa = 2$, and for $c_{n-1} \geq 2$ and $\kappa \geq 3$.

(3) $\lim_{c_{n-1} \rightarrow \infty} \delta_n(c_1, \dots, c_{n-1}, 1, 1) = 1 - \frac{1}{2^\kappa}$

(4) $\lim_{c_{n-1} \rightarrow \infty} d_n(c_1, \dots, c_{n-1}, 1) = 2^\kappa - 1$

In the set $c_v \geq 1$, $v=1, 2, \dots, n$, the least upper bounds of δ_n and d_n are respectively $1 - \frac{1}{2^\kappa}$ and $2^\kappa - 1$.

The two functions approach their upper bounds when $c_n = 1$, $c_{n-2} = 1$, $c_{n-4} = 1, \dots$ and when c_{n-1}, c_{n-3}, \dots are large.

Let $n=1$, ($\kappa \geq 2$) Then $\delta_1(c, 1) = \frac{(c+1)^\kappa - c^\kappa}{(c+1)^\kappa} \leq 1 - \frac{1}{2^\kappa}$
 $d_1(c) = \frac{(c+1)^\kappa - c^\kappa}{c^\kappa} \leq 2^\kappa - 1$.

with equality in both when $c=1$.

Proof:

We need first the following lemmas.

Lemma 1.3.1

Let $\phi(x) = x^\kappa + C$, where κ is a positive integer, $\kappa \geq 2$ and C is a constant, $0 < C \leq 2^\kappa - 1$. Then

(1.3.6) $\phi^2(x+1) > \phi(x)\phi(x+2)$ for $x=1, 2, \dots$, $\kappa=2$
for $x=2, 3, \dots$, $\kappa \geq 3$.

Moreover, if $C < \frac{4^\kappa - 3^\kappa}{3^\kappa - 2^{\kappa+1}}$, then the above inequality holds for $x=1$, $\kappa \geq 3$.

Proof of Lemma 1.3.1

The inequality $\varphi^2(x+1) > \varphi(x)\varphi(x+2)$ is equivalent to

$$\log \varphi(x+1) > \frac{\log \varphi(x) + \log \varphi(x+2)}{2}$$

Thus, we need only prove that $\log \varphi(x)$ is convex downwards for the indicated values of x and κ . We do this.

Let $y = \log \varphi(x)$. Then $y''(x) = \frac{\kappa x^{\kappa-2}}{(x^\kappa + C)^2} \{ C(\kappa-1) - x^\kappa \}$

As $C \leq 2^{\kappa-1}$, the factor $C(\kappa-1) - x^\kappa$ will be negative when :

$$(2^{\kappa-1})(\kappa-1) - x^\kappa < 0$$

. Consider the two cases.

1) $\kappa = 2$ We have $(2^{\kappa-1})(\kappa-1) - x^\kappa = 3 - x^2$.

This is negative for $x \geq 2$. Then $y''(x) < 0$, and accordingly, $\log \varphi(x)$ is convex downwards for $x \geq 2$.

That is, $\varphi^2(x+1) > \varphi(x)\varphi(x+2)$

When $x = 1$, this inequality becomes $(2^\kappa + C)^2 > (1+C)(3^\kappa + C)$.

This will be true if : $4^\kappa - 3^\kappa > C(3^\kappa - 2^{\kappa+1} + 1)$

and certainly if : $4^\kappa - 3^\kappa > (2^{\kappa-1})(3^\kappa - 2^{\kappa+1} + 1)$

[$3^\kappa - 2^{\kappa+1} + 1 > 0$ for $\kappa \geq 2$ as x^κ is convex upwards, i.e. $x^\kappa + (x+2)^\kappa > 2(x+1)^\kappa$]

Put $\kappa = 2$ in the last inequality above. We obtain :

$16 - 9 > 3(9 - 8 + 1)$, which is true, as $7 > 6$.

Hence, the inequality (1.3.6) is true for $x = 1, 2, \dots$ and $\kappa = 2$.

Case 2. $\kappa \geq 3$.

Let $x \geq 3$. Then $(2^{\kappa-1})(\kappa-1) - x^\kappa < 2^\kappa \cdot \kappa - 3^\kappa$

The function $g(t) = t^{\frac{1}{t}}$ has a maximum at $t = e$, and is decreasing for $t > e$. Hence $\kappa^{\frac{1}{\kappa}} \leq 3^{\frac{1}{3}}$. As $3^{\frac{1}{3}} < \frac{3}{2}$

we have $\kappa < \left(\frac{3}{2}\right)^\kappa$ and $2^\kappa \cdot \kappa < 3^\kappa$; i.e. $2^\kappa \cdot \kappa - 3^\kappa < 0$

Then $(2^{\kappa-1})(\kappa-1) - x^\kappa < 0$ for $x \geq 3$ and $\kappa \geq 3$, and $y''(x)$

is negative, $\kappa \geq 3$. Therefore, $y = \log \varphi(x)$ is convex downwards and $\varphi^2(x+1) > \varphi(x) \varphi(x+2)$ for $x \geq 3, \kappa \geq 3$.

We prove next that this inequality holds for $x = 3$, namely that (1) $(3^\kappa + C)^2 > (2^\kappa + C)(4^\kappa + C)$ for $\kappa \geq 3$

This inequality becomes (on multiplying out)

$$9^\kappa > 8^\kappa + C(4^\kappa - 2 \cdot 3^\kappa + 2^\kappa)$$

As $C \leq 2^\kappa - 1$, (1) will surely hold if: $9^\kappa > 8^\kappa + (2^\kappa - 1)(4^\kappa - 2 \cdot 3^\kappa + 2^\kappa)$

or $9^\kappa > 2(8^\kappa - 6^\kappa + 3^\kappa) - 2^\kappa$ and certainly, if

$$(2) \quad 9^\kappa > 2(8^\kappa - 6^\kappa + 3^\kappa)$$

The truth of (2) follows by induction. It is verified that (2) is true when $\kappa = 3$. Assume that (2) is true for an arbitrary index $\kappa \geq 3$. Consider the difference

$$(3) \quad 9 \cdot 2(8^\kappa - 6^\kappa + 3^\kappa) - 2(8^{\kappa+1} - 6^{\kappa+1} + 3^{\kappa+1})$$

This is equal to: $2[8^\kappa - 3 \cdot 6^\kappa + 6 \cdot 3^\kappa] = 2 \cdot 6^\kappa [(\frac{4}{3})^\kappa - 3 + 6(\frac{1}{2})^\kappa]$.

For $\kappa \geq 4$, $(\frac{4}{3})^\kappa \geq 3$, and for $\kappa = 3$,

$$(\frac{4}{3})^3 - 3 + 6(\frac{1}{2})^3 = \frac{64}{27} - 3 + \frac{6}{8} > 0$$

Therefore, the difference (3) is positive, $\kappa \geq 3$, i.e.

$$9 \cdot 2(8^\kappa - 6^\kappa + 3^\kappa) > 2(8^{\kappa+1} - 6^{\kappa+1} + 3^{\kappa+1})$$

By our assumption, $9^\kappa > 2(8^\kappa - 6^\kappa + 3^\kappa)$; then

$$9^{\kappa+1} > 9 \cdot 2(8^\kappa - 6^\kappa + 3^\kappa) \quad \text{and by the above, } 9^{\kappa+1} > 2(8^{\kappa+1} - 6^{\kappa+1} + 3^{\kappa+1}).$$

This is the inequality (2) with κ replaced by $\kappa+1$. Hence

(2) is true for all integral $\kappa, \kappa \geq 3$.

Therefore, we have $(3^\kappa + C)^2 > (2^\kappa + C)(4^\kappa + C)$ and

$$(1.3.6) \quad \varphi^2(x+1) > \varphi(x) \varphi(x+2) \quad \text{for } x = 2, 3, \dots, \kappa \geq 3.$$

In particular, for $x = 1$, this becomes

$$(2^\kappa + C)^2 > (1 + C)(3^\kappa + C), \quad \text{or } 4^\kappa - 3^\kappa > C(3^\kappa - 2^{\kappa+1} + 1)$$

Then, as $3^k - 2^{k+1} + 1 > 0$, (as already noted before), the inequality (1.3.6) will hold for $x = 1$, $k \geq 3$ when

$$C < \frac{4^k - 3^k}{3^k - 2^{k+1} + 1} \quad . \text{ This proves Lemma 1.3.1}$$

Lemma 1.3.2

The functions : (1) $\frac{(x+1)^k - x^k}{x^k + C} = \frac{\varphi(x+1) - \varphi(x)}{\varphi(x)}$
 (2) $\frac{(x+1)^k - x^k}{(x+1)^k + C} = \frac{\varphi(x+1) - \varphi(x)}{\varphi(x+1)}$

(where $\varphi(x) = x^k + C$, as defined in Lemma 1.3.1, $0 < C \leq 2^k - 1$, are both decreasing for increasing x , $x = 1, 2, \dots$

when $k = 2$, and $x = 2, 3, \dots$ when $k \geq 3$.

The value $x = 1$ will be included when $k \geq 3$

provided that $C < \frac{4^k - 3^k}{3^k - 2^{k+1} + 1}$

Proof of Lemma 1.3.2

We wish to prove the two inequalities

$$(3) \quad \frac{\varphi(x+1) - \varphi(x)}{\varphi(x)} > \frac{\varphi(x+2) - \varphi(x+1)}{\varphi(x+1)}, \quad \text{and}$$

$$(4) \quad \frac{\varphi(x+1) - \varphi(x)}{\varphi(x+1)} > \frac{\varphi(x+2) - \varphi(x+1)}{\varphi(x+2)}$$

Both (3) and (4) are equivalent to $\varphi^2(x+1) > \varphi(x)\varphi(x+2)$.

Therefore, by Lemma 1.3.1, (3) and (4) are both true for the indicated values of x and k . This proves the Lemma.

Theorem 1.3.1 now follows directly.

Let $n \geq 2$. From (1.3.4 a); $\delta_n(c_1, c_2, \dots, c_{n-1}, c_n, 1) = \frac{(c_n + 1)^k - c_n^k}{(c_n + 1)^k + c_{n-1}}$
 and from (1.3.5 a) $\delta_{n-1} = \frac{(c_{n-1} + 1)^k - c_{n-1}^k}{c_{n-1}^k + c_{n-2}} \leq 2^k - 1$.

Suppose $k = 2$. By Lemma 1.3.2,

$$\delta_n(c_1, c_2, \dots, c_{n-1}, 1) \leq \delta_n(c_1, \dots, c_{n-1}, 1, 1) = \frac{2^k - 1}{2^k + c_{n-1}}$$

Also, $d_{n-1} = d_{n-1}(c_{n-1}) > d_{n-1}(c_{n-1}+1)$ as d_{n-1} is a decreasing function of c_{n-1} , by the preceding lemma; and as $d_{n-1} > 0$,

$$\frac{2^k - 1}{2^k + d_{n-1}} < \frac{2^k - 1}{2^k + d_{n-1}(c_{n-1}+1)} < 1 - \frac{1}{2^k} \quad \text{Thus,}$$

$$(1) \quad \gamma_n(c_1, c_2, \dots, c_{n-1}, c_n, 1) < \gamma_n(c_1, c_2, \dots, c_{n-1}+1, 1, 1) < 1 - \frac{1}{2^k}$$

and $1 - \frac{1}{2^k}$ is an upper bound of γ_n . Similarly, by the

$$\text{Lemma 1.3.2, } d_n(c_1, \dots, c_{n-1}, c_n) \leq d_n(c_1, \dots, c_{n-1}, 1) = \frac{2^k - 1}{1 + d_{n-1}}$$

and, as above, we obtain

$$(2) \quad d_n(c_1, c_2, \dots, c_{n-1}, c_n) < d_n(c_1, c_2, \dots, c_{n-1}+1, 1) < 2^k - 1$$

and $2^k - 1$ is an upper bound of d_n . Indeed, $2^k - 1$ is

the least upper bound of d_n . For as $d_{n-1}(c_{n-1})$ is a monoton-

ic decreasing function of c_{n-1} , $\lim_{c_{n-1} \rightarrow \infty} d_{n-1}$ exists. From

$$d_{n-1} = \frac{(1 + \frac{1}{c_{n-1}})^k - 1}{1 + \frac{d_{n-2}}{c_{n-1}^k}}, \quad \text{it is apparent that } \lim_{c_{n-1} \rightarrow \infty} d_{n-1} = 0.$$

Then, from $d_n(c_1, c_2, \dots, c_{n-1}, 1) = \frac{2^k - 1}{1 + d_{n-1}}$, it follows that:

as $c_{n-1} \rightarrow \infty$, $d_n \uparrow$ to $2^k - 1$. Hence, $2^k - 1$ is the least

upper bound of d_n . Similarly, from $\gamma_n(c_1, \dots, c_{n-1}, 1, 1) = \frac{2^k - 1}{2^k + d_{n-1}}$

we have $\gamma_n(c_1, c_2, \dots, c_{n-1}, 1, 1) \uparrow$ to $1 - \frac{1}{2^k}$ as $c_{n-1} \rightarrow \infty$;

and hence, the least upper bound of γ_n is $1 - \frac{1}{2^k}$.

From the above discussion, and from the inequalities

(1) and (2), we note that γ_n and d_n will approach their upper bounds when $c_n = 1$ and when d_{n-1} is small. Also, d_{n-1} is small when c_{n-1} is large and when d_{n-2} is large. By Lemma 1.3.2 d_{n-2} is large when $c_{n-2} = 1$ and d_{n-3} is small, and so on.

Thus $\gamma_n(c_1, \dots, c_n, 1)$ and $d_n(c_1, \dots, c_{n-1}, c_n)$ approach their upper bounds when $c_n = 1$, $c_{n-2} = 1$, ... and when c_{n-1}, c_{n-3}, \dots

are large. ($\kappa=2$) The functions δ_n and δ_n attain their least upper bounds in the special case $n=1$ and $c_1=1$. For

$$\delta_1 = \frac{(c_1+1)^\kappa - c_1^\kappa}{(c_1+1)^\kappa} \leq 1 - \frac{1}{2^\kappa} ; \quad \delta_1 = \frac{(c_1+1)^\kappa - c_1^\kappa}{c_1^\kappa} \leq 2^\kappa - 1.$$

with equality in each when $c_1 = 1$.

The above results are also true for $\kappa \geq 3$. Let $\kappa \geq 3$.

Suppose first that $\delta_{n-1} < \frac{4^\kappa - 3^\kappa}{3^\kappa - 2^{\kappa+1} + 1}$. This will be true when $c_{n-1} \geq 3$, as $\delta_{n-1} < \frac{(c_{n-1}+1)^\kappa - c_{n-1}^\kappa}{c_{n-1}^\kappa} \leq \frac{4^\kappa - 3^\kappa}{3^\kappa}$,

By Lemma 1.3.2, $\delta_n(c_1, c_2, \dots, c_n, 1) \leq \delta_n(c_1, c_2, \dots, c_{n-1}, 1, 1)$, $(c_{n-1} \geq 3)$, and $\delta_n(c_1, c_2, \dots, c_{n-1}, 1, 1) = \frac{2^\kappa - 1}{2^\kappa + \delta_{n-1}}$. Also, $\delta_{n-1}(c_{n-1} + 1) < \delta_{n-1}(c_{n-1})$. Then $\delta_n(c_1, c_2, \dots, c_{n-1}, 1, 1) < \delta_n(c_1, \dots, c_{n-1} + 1, 1)$ and finally, $\delta_n(c_1, c_2, \dots, c_{n-1}, c_n, 1) < \delta_n(c_1, c_2, \dots, c_{n-1} + 1, 1, 1)$ for $c_{n-1} \geq 3$.

This inequality is also true for $c_{n-1} = 2$. Suppose that

$\delta_{n-1} \geq \frac{4^\kappa - 3^\kappa}{3^\kappa - 2^{\kappa+1} + 1}$ Then $c_{n-1} = 1$ or 2 . By Lemma 1.3.2,

$$\delta_n(c_1, c_2, \dots, c_{n-1}, c_n, 1) \leq \delta_n(c_1, c_2, \dots, c_{n-1}, 2, 1) = \frac{3^\kappa - 2^\kappa}{3^\kappa + \delta_{n-1}}$$

Let $c_{n-1} = 2$ As $c_{n-1} + 1 = 3$, $\delta_{n-1}(3) < \frac{4^\kappa - 3^\kappa}{3^\kappa - 2^{\kappa+1} + 1}$.

Also, $\delta_{n-1}(3) < \delta_{n-1}(2)$ by the Lemma 1.3.2. Then

$$\frac{3^\kappa - 2^\kappa}{3^\kappa + \delta_{n-1}(2)} < \frac{3^\kappa - 2^\kappa}{3^\kappa + \delta_{n-1}(3)} < \frac{2^\kappa - 1}{2^\kappa + \delta_{n-1}(3)}$$

the last inequality by the lemma. Thus the inequality

$$\delta_n(c_1, \dots, c_{n-1}, c_n, 1) < \delta_n(c_1, \dots, c_{n-1} + 1, 1)$$
 is true for $c_{n-1} \geq 2$, ($\kappa \geq 3$).

From this inequality, and from $\delta_n(c_1, c_2, \dots, c_{n-1}, 1, 1) = \frac{2^\kappa - 1}{2^\kappa + \delta_{n-1}} < 1 - \frac{1}{2^\kappa}$

it follows that $\delta_n(c_1, c_2, \dots, c_n, 1)$; ($\kappa \geq 3$) is large when $c_n = 1$ and

when δ_{n-1} is small. Therefore, by the argument on page 11,

for the case $\kappa = 2$, we have $\delta_n(c_1, \dots, c_n, 1)$ approaches

its upper bound when $c_n = 1$, $c_{n-2} = 1, \dots$ and when c_{n-1}, c_{n-3}, \dots

are large.

It is evident that the same conditions hold for δ_n , $n \geq 3$.

The remaining results regarding the least upper bounds of δ_n and δ_n follow exactly as in the case $n = 2$, page 11.

This proves all that was required.

Corollary 1.3.1 $\frac{B_n}{B_{n-1}}$ ($n \geq 2$) tends to its lower bound when $c_n = 1$, $c_{n-2} = 1$, ... and when c_{n-1}, c_{n-3}, \dots are large. The least upper bound of $\frac{B_n}{B_{n-1}}$ is 1, and it is attained when $n = 1$ and $c_1 = 1$; and also in the limiting case, when $n \geq 2$, $c_n = 1$, and $c_{n-1} \rightarrow \infty$.

Proof :

The corollary follows immediately from Theorem 1.3.1 when we note that $\frac{B_n}{B_{n-1}} = c_n^{\uparrow} + \frac{a_n B_{n-2}}{B_{n-1}} = c_n^{\uparrow} + \delta_{n-1}$, and for $n = 1$, $\frac{B_1}{B_0} = c_1^{\uparrow}$. ($n \geq 2$)

We consider next the problem of determining the maximum and the minimum of u_n

$$\text{By (1.3.2), page 6, } u_n = u_n(c_0, c_1, \dots, c_n) = \frac{a_1 a_2 \dots a_n}{B_n B_{n-1}} \quad (n \geq 1).$$

In the following, we assume that c_0 has a fixed value. This is done because each u_v , $v \geq 1$ is multiplied by the factor $a_v = (c_0 + 1)^v - c_0^v$, which is the only term containing c_0 . Clearly, this is no restriction.

$$\text{As } B_n = c_n^{\uparrow} B_{n-1} + a_n B_{n-2}; \quad B_n \leq B_{n-1} + a_n B_{n-2}.$$

$$\text{Then from (1.3.2), } u_n(c_0, \dots, c_n) \leq \frac{a_1 a_2 \dots a_n}{(B_{n-1} + a_n B_{n-2}) B_{n-1}} = u_n(c_0, \dots, c_{n-1}, 1).$$

for $n \geq 1$, and there is equality if $c_n = 1$. Therefore, we only consider the maximum of the function $u_n(c_0, c_1, \dots, c_{n-1}, 1)$.

Theorem 1.3.2 For $\kappa = 2, 3, 4, 5$, the function $u_n(c_0, c_1, \dots, c_{n-1}, 1)$ ($n \geq 2$) is a maximum when $c_v = 1$, $v = 1, 2, \dots, n-1$. In this case,

$$\max u_n = u_n(c_0, 1, 1, \dots, 1, 1) = \frac{a_1 (2^{\kappa-1})^n}{\beta_n \beta_{n-1}}$$
 where $\{\beta_v\}$ is the sequence of denominators of the convergents to the continued fraction $1 + \frac{2^{\kappa-1}}{1} + \frac{2^{\kappa-1}}{1} + \dots$

Let $\kappa \geq 6$. Then $\max u_n(c_0, \dots, c_{n-1}, 1)$ occurs when $c_1 = 1$, $c_{n-1} = c_{n-2} = 1$. The values of c_v , ($2 \leq v \leq n-3$) for which u_n is a maximum will be either 1 or 2, depending on n, v, κ .

u_n has a minimum when all the c_v are large. In the limiting case, $\lim_{c_v \rightarrow \infty} u_n(c_0, c_1, \dots, c_v, \dots, c_n) = 0$; $1 \leq v \leq n$.

Proof :

We need the following lemmas.

Lemma 1.3.3

Let $\phi(x) = x^\kappa + C$ as defined in Lemma 1.3.1, Then the function $\frac{1}{\phi(x)}$ is convex upwards for $x \geq 1$ and $\kappa \geq 2$.

$$(1.3.7) \quad \frac{1}{x^\kappa + C} + \frac{1}{(x+2)^\kappa + C} > \frac{2}{(x+1)^\kappa + C}, \text{ for } x=1, 2, \dots \text{ and } \kappa \geq 2.$$

Proof of Lemma 1.3.3

Write $y = \frac{1}{\phi(x)} = \frac{1}{x^\kappa + C}$. Then

$$y' = \frac{-\kappa x^{\kappa-1}}{(x^\kappa + C)^2} \quad \text{and} \quad y'' = \frac{\kappa x^{\kappa-2}}{(x^\kappa + C)^3} [x^\kappa(\kappa+1) - C(\kappa-1)]$$

As $0 < C \leq 2^\kappa - 1$, the factor $x^\kappa(\kappa+1) - C(\kappa-1)$ will be positive when $x^\kappa(\kappa+1) - (2^\kappa - 1)(\kappa-1)$ is positive.

Consider the two cases.

Case 1. $\kappa = 2$.

Then $x^\kappa(\kappa+1) - (2^\kappa - 1)(\kappa-1) = 3x^2 - 3$. This is zero when $x = 1$, and is positive for $x > 1$. Then $y''(x) \geq 0$ for $x \geq 1$, with equality when $x = 1$. Therefore, $y = \frac{1}{\phi(x)}$ is convex upwards

Case 2. Let $\kappa \geq 3$.

The factor $x^{\kappa}(\kappa+1) - (2^{\kappa-1})(\kappa-1)$ is positive when $x \geq 2$, so that $y = \frac{1}{\varphi(x)}$ is convex upwards for $x \geq 2$.

When $x = 1$, the inequality (1.3.7) becomes

$$\frac{1}{1+C} + \frac{1}{3^{\kappa}+C} > \frac{2}{2^{\kappa}+C} \quad \text{or} \quad 6^{\kappa} + 2^{\kappa} - 2 \cdot 3^{\kappa} - C [3^{\kappa} - 2^{\kappa+1} + 1] > 0,$$

This will be positive if $6^{\kappa} + 2^{\kappa} - 2 \cdot 3^{\kappa} - (2^{\kappa-1}) [3^{\kappa} - 2^{\kappa+1} + 1] > 0$, or $2 \cdot 4^{\kappa} - 2 \cdot 2^{\kappa} + 1 - 3^{\kappa} > 0$. Now $2 \cdot 4^{\kappa} - 2 \cdot 2^{\kappa} + 1 - 3^{\kappa} = 4^{\kappa} - 3^{\kappa} + (2^{\kappa-1})^2 > 0$,

for $\kappa \geq 3$. Hence, the inequality (1.3.7) is true for $x = 1$, and $\kappa \geq 3$. This proves Lemma 1.3.3.

Remark : From the inequality (1.3.7), we obtain

$$(1.3.8) \quad \frac{1}{x^{\kappa}+C} - \frac{1}{(x+1)^{\kappa}+C} > \frac{1}{(x+1)^{\kappa}+C} - \frac{1}{(x+2)^{\kappa}+C};$$

Hence, $\max_{x \geq 1} \left[\frac{1}{x^{\kappa}+C} - \frac{1}{(x+1)^{\kappa}+C} \right] = \frac{1}{1+C} - \frac{1}{2^{\kappa}+C}$ for $x=1, 2, \dots, \kappa \geq 2$.

Denote by a superscript a raising of the indices in a

given expression by the amount of the superscript. Thus,

$$B_{-1}^{(v)} = 0, \quad B_0^{(v)} = 1, \quad B_1^{(v)} = (c_r^{\kappa})^v = c_{r+1}^{\kappa}, \quad \text{etc.}$$

From Perron, O (1), § 5, p.14, formula (24), we obtain

$$(1.3.9) \quad B_n = B_v B_{n-v}^{(v)} + a_{v+1} B_{v-1} B_{n-v-1}^{(v+1)}$$

This relation enables to express B_n in terms of c_v

where v is an arbitrary index $v \leq n$. The right member of (1.3.9)

$$\text{is : } B_{n-v}^{(v)} [c_v^{\kappa} B_{v-1} + a_v B_{v-2}] + [(c_v+1)^{\kappa} - c_v^{\kappa}] B_{v-1} B_{n-v-1}^{(v+1)}$$

$$= B_{v-1} \left\{ [c_v^{\kappa} + \frac{a_v B_{v-2}}{B_{v-1}}] B_{n-v}^{(v)} + [(c_v+1)^{\kappa} - c_v^{\kappa}] B_{n-v-1}^{(v+1)} \right\} \text{ and finally}$$

$$(1.3.10) \quad B_n = B_{v-1} \left\{ (c_v^{\kappa} + d_{v-1}) B_{n-v}^{(v)} + [(c_v+1)^{\kappa} - c_v^{\kappa}] B_{n-v-1}^{(v+1)} \right\}$$

for $1 \leq v \leq n$. (d_v defined previously by (1.3.5))

In (1.3.10) let $v=1$. Then $d_0 = 0$ and

$$B_n = c_1^{\kappa} B_{n-1}^{(1)} + [(c_1+1)^{\kappa} - c_1^{\kappa}] B_{n-2}^{(2)}$$

$$B_{n-1} = c_1^{\kappa} B_{n-2}^{(1)} + [(c_1+1)^{\kappa} - c_1^{\kappa}] B_{n-3}^{(2)}$$

We have : $U_n(c_0, c_1, \dots, c_n) = \frac{a_1 a_2 \dots a_n \{ (c_1+1)^{\kappa} - c_1^{\kappa} \}}{B_n \cdot B_{n-1}}$

$$= \frac{a_1 a_2 \dots a_n}{\left\{ \frac{c_1^{\kappa}}{(c_1+1)^{\kappa} - c_1^{\kappa}} \cdot B_{n-1}^{(1)} + B_{n-2}^{(2)} \right\} \left\{ c_1^{\kappa} B_{n-2}^{(1)} + [(c_1+1)^{\kappa} - c_1^{\kappa}] B_{n-3}^{(2)} \right\}}$$

As $\frac{c_1^{\kappa}}{(c_1+1)^{\kappa} - c_1^{\kappa}} \geq \frac{1}{2^{\kappa}-1}$, and $c_1 \geq 1$, the denominator is a minimum when $c_1 = 1$. Hence, $U_n(c_0, c_1, \dots, c_n) \leq U_n(c_0, 1, c_2, \dots, c_n)$ ($\kappa \geq 2$) with equality if $c_1 = 1$.

Lemma 1.3.4Let $n \geq 3$. Then

(a) $U_n(c_0, c_1, \dots, c_{n-1}, 1) \leq U_n(c_0, c_1, \dots, c_{n-2}, 1, 1)$ for all $\kappa \geq 2$, and there is equality only when $c_{n-1} = 1$.

Let $n \geq 4$. Then

(b) $U_n(c_0, c_1, \dots, c_{n-2}, 1, 1) \leq U_n(c_0, \dots, c_{n-3}, 1, 1, 1)$ for all $\kappa \geq 2$, and with equality if $c_{n-2} = 1$.

Proof :(a) In (1.3.10), let $r = n-1$. Then

$$B_n = B_{n-2} \left\{ (c_{n-1}^{\kappa} + d_{n-2}) B_{n-1}^{(n-1)} + [(c_{n-1}+1)^{\kappa} - c_{n-1}^{\kappa}] B_0^{(n)} \right\}.$$

As $c_n = 1$, $B_{n-1}^{(n-1)} = 1$, also $B_0^{(n)} = 1$. Then $B_n = B_{n-2} \{ (c_{n-1}+1)^{\kappa} + d_{n-2} \}$ Similarly, $B_{n-1} = B_{n-2} \{ c_{n-1}^{\kappa} + d_{n-2} \}$ Thus,

$$U_n(c_0, c_1, \dots, c_{n-1}, 1) = \frac{a_1 a_2 \dots a_{n-1}}{B_{n-2}^2} \left[\frac{(c_{n-1}+1)^{\kappa} - c_{n-1}^{\kappa}}{\{ (c_{n-1}+1)^{\kappa} + d_{n-2} \} \{ c_{n-1}^{\kappa} + d_{n-2} \}} \right]$$

The first factor $\frac{a_1 a_2 \dots a_{n-1}}{B_{n-2}^2}$ is independent of c_{n-1} . Thesecond factor is: $\frac{(c_{n-1}+1)^{\kappa} - c_{n-1}^{\kappa}}{\{ (c_{n-1}+1)^{\kappa} + d_{n-2} \} \{ c_{n-1}^{\kappa} + d_{n-2} \}} = \frac{1}{c_{n-1}^{\kappa} + d_{n-2}} - \frac{1}{(c_{n-1}+1)^{\kappa} + d_{n-2}}$ Now $0 < d_{n-2} \leq 2^{\kappa}-1$, ($n \geq 3$). Hence, by Lemma 1.3.3,

and the remark following (cp. page 15, (1.3.8))

$$\frac{1}{c_{n-1}^{\kappa} + d_{n-2}} - \frac{1}{(c_{n-1}+1)^{\kappa} + d_{n-2}} \leq \frac{1}{1 + d_{n-2}} - \frac{1}{2^{\kappa} + d_{n-2}}$$

with equality only if $c_{n-1} = 1$. Therefore,

$$U_n(c_0, c_1, \dots, c_{n-1}, 1) \leq U_n(c_0, \dots, c_{n-2}, 1, 1), \text{ for } n \geq 3 \text{ and } \kappa \geq 3.$$

$$\frac{\varphi(x+2)\varphi(x+1) - \varphi(x)\varphi(x+2)}{\varphi(x+2)\varphi(x+1) - \varphi^2(x+1)} = 1 + \frac{\varphi^2(x+1) - \varphi(x)\varphi(x+2)}{\varphi(x+2)(\varphi(x+1) - \varphi^2(x+1))}$$

and this will be greater than 1 if $\varphi^2(x+1) > \varphi(x)\varphi(x+2)$.

By Lemma 1.3.1, $\varphi^2(x+1) > \varphi(x)\varphi(x+2)$ } for $x=1, 2, \dots, \kappa=2$
 " $x=2, 3, \dots, \kappa \geq 3$.

Hence the product (2) is greater than 1 for $\kappa=2, x=1, 2, \dots$

and $\kappa \geq 3, x=2, 3, \dots$.

Let $\kappa \geq 3$. If $\varphi^2(x+1) \geq \varphi(x)\varphi(x+2)$ for $x=1$,

then we need not continue, and the Lemma is proved. Assume

that $\varphi^2(x+1) < \varphi(x)\varphi(x+2)$, when $x=1$. By (1.3.8)

$$\frac{1}{\varphi(x)} - \frac{1}{\varphi(x+1)} > \frac{1}{\varphi(x+1)} - \frac{1}{\varphi(x+2)}, \quad \text{and}$$

$$(3) \quad \frac{\varphi(x+1) - \varphi(x)}{\varphi(x+2) - \varphi(x+1)} \cdot \frac{\varphi(x+2)}{\varphi(x)} > 1, \quad \text{for } \kappa \geq 2, x=1, 2, \dots$$

The inequality (2) to be proved can be written in the form

$$\left\{ \frac{\varphi(x+1) - \varphi(x)}{\varphi(x+2) - \varphi(x+1)} \cdot \frac{\varphi(x+2)}{\varphi(x)} \right\} \frac{\varphi(x)}{\varphi(x+1)} \cdot \frac{\varphi(x+2) + (2^{\kappa}-1)\varphi(x+1)}{\varphi(x+1) + (2^{\kappa}-1)\varphi(x)} > 1.$$

By (3), the term in brackets is larger than 1. The remain-

ing factors, when multiplied out, become :

$$\frac{\varphi(x)\varphi(x+2) + (2^{\kappa}-1)\varphi(x+1)\varphi(x)}{\varphi^2(x+1) + (2^{\kappa}-1)\varphi(x+1)\varphi(x)} = 1 + \frac{\varphi(x)\varphi(x+2) - \varphi^2(x+1)}{\varphi^2(x+1) + (2^{\kappa}-1)\varphi(x+1)\varphi(x)}$$

From our assumption, $\varphi^2(x+1) < \varphi(x)\varphi(x+2)$. Then the above is surely larger than 1 when $x=1$. Therefore, the inequality (2) is true for $\kappa \geq 2$ and $x=1, 2, \dots$.

This proves Lemma 1.3.4 .

Theorem 1.3.2 follows by induction. Let $n \geq 5$, and let ν be an arbitrary index, $2 \leq \nu \leq n-3$.

We have seen that $u_n(c_0, \dots, c_{n-1}, 1)$ is a maximum when $c_0 = 1$ and $c_{n-1} = c_{n-2} = 1$, (Lemma 1.3.4) for $\kappa \geq 2$. That is:

$$u_n(c_0, c_1, \dots, c_{n-1}, 1) \leq u_n(c_0, 1, c_2, \dots, c_{n-3}, 1, 1, 1).$$

Assume that u_n is a maximum when $c_n = c_{n-1} = \dots = c_{v+1} = 1$. Then

$$u_n(c_0, c_1, \dots, c_v, 1, 1, \dots, 1) = \frac{a_1 a_2 \dots a_{v+1} (2^{n-v-1})^{n-v-1}}{\beta_n \beta_{n-1}}$$

Using (1.3.10), the factor containing c_v in $u_n(c_0, c_1, \dots, c_v, 1, 1, \dots, 1)$ is found to be

$$(1) \quad w(c_v) = \frac{(c_v + 1)^{\beta_{n-v}} - c_v^{\beta_{n-v}}}{\{(c_v^{\beta_{n-v}} + d_{v-1})\beta_{n-v} + [(c_v + 1)^{\beta_{n-v}} - c_v^{\beta_{n-v}}]\beta_{n-v-1}\} \{(c_v^{\beta_{n-v}} + d_{v-1})\beta_{n-v-1} + [(c_v + 1)^{\beta_{n-v}} - c_v^{\beta_{n-v}}]\beta_{n-v-2}\}}$$

We wish to prove that $w(c_v) > w(c_{v+1})$, so that $\max w(c_v) = w(1)$, and accordingly, $u_n(c_0, c_1, \dots, c_v, 1, 1, \dots, 1) \leq u_n(c_0, \dots, c_{v-1}, 1, 1, \dots, 1)$.

In $w(c_v)$, replace c_v by x , d_{v-1} by C , $0 < C \leq 2^{\beta_{n-v}}$, and write $x^{\beta_{n-v}} + C = \varphi(x)$. Then $w(c_v)$ takes the form

$$(2) \quad w(x) = \frac{\varphi(x+1) - \varphi(x)}{\{\beta_{n-v} \varphi(x) + [\varphi(x+1) - \varphi(x)]\beta_{n-v-1}\} \{\beta_{n-v-1} \varphi(x) + [\varphi(x+1) - \varphi(x)]\beta_{n-v-2}\}}$$

The inequality $w(x) > w(x+1)$ becomes

$$(3) \quad \frac{\varphi(x+2) - \varphi(x)}{\varphi(x+2) - \varphi(x+1)} \cdot \frac{\{\beta_{n-v} \varphi(x+1) + [\varphi(x+2) - \varphi(x+1)]\beta_{n-v-1}\}}{\{\beta_{n-v} \varphi(x) + [\varphi(x+1) - \varphi(x)]\beta_{n-v-1}\}} \cdot \frac{\{\beta_{n-v-1} \varphi(x+1) + [\varphi(x+2) - \varphi(x+1)]\beta_{n-v-2}\}}{\{\beta_{n-v-1} \varphi(x) + [\varphi(x+1) - \varphi(x)]\beta_{n-v-2}\}} > 1.$$

The factor III is greater than 1 since $\varphi(x)$ is an increasing function of x . The product of the first two factors I, II

$$\text{is : } \frac{\beta_{n-v} \varphi(x+1) (\varphi(x+1) - \varphi(x)) + \beta_{n-v-1} [(\varphi(x+1) - \varphi(x))(\varphi(x+2) - \varphi(x+1))]}{\beta_{n-v} \varphi(x) (\varphi(x+2) - \varphi(x+1)) + \beta_{n-v-1} [\varphi(x+1) - \varphi(x)](\varphi(x+2) - \varphi(x+1))}$$

This will be at least as large as 1 when $\varphi^2(x+1) \geq \varphi(x)\varphi(x+2)$

By Lemma 1.3.1, this is true when $\kappa=2$ and $x=1, 2, \dots$, and

when $\kappa \geq 3$, and $x=2, 3, \dots$. Hence, for $\kappa=2$

$$\max w(c_v) = w(1), \text{ and } u_n(c_0, c_1, \dots, c_v, 1, 1, \dots, 1) \leq u_n(c_0, \dots, c_{v-1}, 1, 1, \dots, 1).$$

with equality when $c_v = 1$. This completes the induction for

$\kappa=2$. Hence $\max u_n(c_0, c_1, \dots, c_n) = u_n(c_0, 1, 1, \dots, 1)$ for $\kappa=2$

There is equality only if $c_i = 1$, $i=1, 2, \dots, n$.

Consider next the remaining case, $\kappa \geq 3$. From the

above, $w(c_v) > w(c_{v+1})$ for $c_v = 2, 3, \dots$, and $\kappa \geq 3$

so that $\max w(c_v)$ is either $w(1)$ or $w(2)$. When $c_v = 1$, the inequality (3) to be proved becomes

$$(4) \frac{2^r - 1}{3^r - 2^r} \cdot \frac{\beta_{n-r}(2^r + C) + (3^r - 2^r)\beta_{n-r-1}}{\beta_{n-r}(1+C) + (2^r - 1)\beta_{n-r-1}} \cdot \frac{\beta_{n-r-1}(2^r + C) + (3^r - 2^r)\beta_{n-r-2}}{\beta_{n-r-1}(1+C) + (2^r - 1)\beta_{n-r-2}} > 1.$$

The two factors containing the parameter C , $0 < C \leq 2^r$, are both larger than 1. If C is increased to $2^r - 1$, the

fractions will be decreased. Then (4) will certainly be true if:

$$5) \frac{2^r - 1}{3^r - 2^r} \cdot \frac{\beta_{n-r}(2^r + 2^r - 1) + (3^r - 2^r)\beta_{n-r-1}}{\beta_{n-r}(1 + 2^r - 1) + (2^r - 1)\beta_{n-r-1}} \cdot \frac{\beta_{n-r-1}(2^r + 2^r - 1) + (3^r - 2^r)\beta_{n-r-2}}{\beta_{n-r-1}(1 + 2^r - 1) + (2^r - 1)\beta_{n-r-2}} > 1$$

Multiply this out and collect terms, obtaining

$$(6) \beta_{n-r-1} \beta_{n-r-2} (3^r - 2^r)(2^r - 1)(3^r - 2^{r+1} + 1) + \{ \beta_{n-r-1}^2 + \beta_{n-r} \beta_{n-r-2} \} (3^r - 2^r)(2^r - 1)^2 - \beta_{n-r} \beta_{n-r-1} \{ (3^r - 2^r)4^r - (2^r - 1)(2^{r+1} - 1)^2 \} > 0.$$

From the recursion formula for β_v ; $(2^r - 1)\beta_{n-r-2} = \beta_{n-r} - \beta_{n-r-1}$.

Using this in (6), we obtain:

$$\left(\beta_{n-r} \beta_{n-r-1} - \beta_{n-r-1}^2 \right) (3^r - 2^r)(3^r - 2^{r+1} + 1) + \beta_{n-r-1}^2 (3^r - 2^r)(2^r - 1)^2 + \{ \beta_{n-r}^2 - \beta_{n-r} \beta_{n-r-1} \} (3^r - 2^r)(2^r - 1) - \beta_{n-r} \beta_{n-r-1} \{ (3^r - 2^r)(4^r) - (2^r - 1)(2^{r+1} - 1)^2 \} > 0$$

and finally; $\beta_{n-r}^2 (3^r - 2^r)(2^r - 1) + \beta_{n-r-1}^2 (3^r - 2^r)(4^r - 3^r) - \beta_{n-r} \beta_{n-r-1} \{ (3^r - 2^r)(4^r - 3^r + 3 \cdot 2^r - 2) - (2^r - 1)(2^{r+1} - 1)^2 \} > 0.$

Since $2 \leq r \leq n-3$, we have $n-r-1 \geq 2$ and $\beta_{n-r-1} \geq \beta_2 > 0$.

The last inequality above can be written as:

$$(7) \left(\frac{\beta_{n-r}}{\beta_{n-r-1}} \right)^2 (3^r - 2^r) - \left(\frac{\beta_{n-r}}{\beta_{n-r-1}} \right) \left\{ (3^r - 2^r) \frac{(4^r - 3^r + 3 \cdot 2^r - 2)}{(2^r - 1)(2^{r+1} - 1)^2} - (3^r - 2^r)(4^r - 3^r) \right\} > 0.$$

Consider the quadratic in $\left(\frac{\beta_{n-r}}{\beta_{n-r-1}} \right)$.

$$(8) f\left(\frac{\beta_{n-r}}{\beta_{n-r-1}} \right) = \left(\frac{\beta_{n-r}}{\beta_{n-r-1}} \right)^2 (3^r - 2^r)(2^r - 1) - \left(\frac{\beta_{n-r}}{\beta_{n-r-1}} \right) \left\{ (3^r - 2^r) \frac{(4^r - 3^r + 3 \cdot 2^r - 2)}{(2^r - 1)(2^{r+1} - 1)^2} - (3^r - 2^r)(4^r - 3^r) \right\}$$

The coefficient of $\left(\frac{\beta_{n-r}}{\beta_{n-r-1}} \right)$, namely,

$$(3^r - 2^r) \frac{(4^r - 3^r + 3 \cdot 2^r - 2)}{(2^r - 1)(2^{r+1} - 1)^2} - (3^r - 2^r)(4^r - 3^r)$$

is negative for $\kappa = 3, 4$, and is positive for $\kappa \geq 5$. In addition, the discriminant of the quadratic (8) is negative for $\kappa = 5$, and is positive for $\kappa \geq 6$. As $f(0) \geq 0$, the quadratic (8) is positive definite for $\kappa = 5$. Therefore from the above, the inequality (6) page 20, is true for $\kappa = 3, 4, 5$. Accordingly, $\max W(c_r) = W(1)$. Then $u_n(c_0, c_1, \dots, c_{r-1}, c_r, 1, \dots, 1) \leq u_n(c_0, c_1, \dots, c_{r-1}, 1, \dots, 1)$ with equality if $c_r = 1$, ($\kappa = 3, 4, 5$).

This completes the induction for $\kappa = 3, 4, 5$. Hence, $\max u_n(c_0, c_1, \dots, c_n) = u_n(c_0, 1, \dots, 1)$, for $\kappa = 2$ (already proved) and for $\kappa = 3, 4, 5$. There is equality only if $c_i = 1$, $i = 1, 2, \dots, n$.

$$\text{Clearly } u_n(c_0, 1, 1, \dots, 1) = \frac{a_1 (2^{\kappa} - 1)^{n-1}}{\beta_n \beta_{n-1}}$$

Let $\kappa \geq 6$.

We have

$$1 \leq \frac{\beta_1}{\beta_0} < \frac{\beta_2}{\beta_1} < \dots < \frac{\beta_{2n+1}}{\beta_{2n}} < \frac{\beta_{2n}}{\beta_{2n-1}} < \dots < \frac{\beta_4}{\beta_3} < \frac{\beta_2}{\beta_1} = 2^{\kappa}$$

This follows immediately as $\frac{\beta_{r+1}}{\beta_r}$ are the convergents to the continued fraction $1 + \frac{2^{\kappa}-1}{1} + \frac{2^{\kappa}-1}{1} + \dots$.

$$\text{Also, } \frac{\beta_1}{\beta_0} = 1, \quad \frac{\beta_2}{\beta_1} = 2^{\kappa}, \quad \frac{\beta_3}{\beta_2} = 2 - \frac{1}{2^{\kappa}}; \quad \frac{\beta_4}{\beta_3} = *$$

For the quadratic (8), page 20, we note that $f(2) < 0$, and $f(2^{\kappa-1}-2) < 0$ for $\kappa = 6$. Then $f(2) < 0$, $f(2^{\kappa-1}-2) < 0$ for all $\kappa \geq 6$, since, when $1 < \frac{\beta_{2r}}{\beta_{2r-1}} < 2^{\kappa}$, the dominant term in $f\left(\frac{\beta_{2r}}{\beta_{2r-1}}\right)$ is the coefficient of $\frac{\beta_{2r}}{\beta_{2r-1}}$ namely $-\left\{ (3^{\kappa}-2^{\kappa})(4^{\kappa}-3^{\kappa}+3 \cdot 2^{\kappa-2}) - (2^{\kappa-1})(2^{\kappa+1}-1)^2 \right\}$.

Therefore, for $\kappa \geq 6$, the roots of $f\left(\frac{\beta_{2r}}{\beta_{2r-1}}\right)$ lie between 1 and 2^{κ} . Indeed, these roots tend to 1 and

$$* \frac{\beta_4}{\beta_3} = 1 + \frac{2^{\kappa}(2^{\kappa}-1)}{2^{\kappa+1}-1} < 1 + 2^{\kappa-1}$$

infinity as $\kappa \rightarrow \infty$. This can be seen when, in the equation $f(x) = 0$, f given by (8) page 20, we replace x by $\frac{1}{y}$ and clear of fractions, obtaining

$$(9) \quad y^2(3^\kappa - 2^\kappa)(4^\kappa - 3^\kappa) - y\{(3^\kappa - 2^\kappa)(4^\kappa - 3^\kappa + 3 \cdot 2^{\kappa-2}) - (2^{\kappa-1})(2^{\kappa+1}-1)^2\} + (3^\kappa - 2^\kappa)(2^\kappa - 1) = 0.$$

In (9), divide each term by $3^\kappa \cdot 4^\kappa$ and let $\kappa \rightarrow \infty$. The quadratic (9) then takes the form $y^2 - y = 0$

This equation has the roots $y=0, y=1$. Therefore, the roots of the quadratic (8) tend to 1 and infinity when $\kappa \rightarrow \infty$.

On the preceding pages, we have shown that

$$2 - \frac{1}{2^\kappa} = \frac{\beta_3}{\beta_2} < \dots < \frac{\beta_{2n+1}}{\beta_{2n}} < \frac{\beta_{2n}}{\beta_{2n-1}} < \dots < \frac{\beta_4}{\beta_3} = 1 + 2^{\kappa-1}$$

and also that $f(2) < 0, f(2^{\kappa-1}-1) < 0$, for $\kappa \geq 6$

Hence, $f(\frac{\beta_{n-v}}{\beta_{n-v-1}})$ will be negative for sufficiently large values of $\kappa \geq 6$, and for values of $v, 2 \leq v \leq n-3, (n \geq 5)$.

From this, it follows that the values of $c_v, (2 \leq v \leq n-3)$ for which $u_n(c_0, c_1, \dots, c_v, \dots, c_{n-1}, 1)$ is a maximum, can not all be 1. They will be either 1 or 3 depending on the particular values of n, v and κ .

By use of (1.3.10) p.15, we have :

$$u_n = \frac{a_1 a_2 \dots a_n}{B_n B_{n-1}} = \frac{a_1 a_2 \dots a_v \cdot a_{v+2} \dots a_n}{B_{v-1}^2} \cdot \omega(c_v) \quad \text{where } [\omega = \text{omega}]$$

$$\omega(c_v) = \frac{(c_v+1)^\kappa - c_v^\kappa}{\{(c_v^\kappa + d_{v-1}) B_{n-v}^{(\kappa)} + [(c_v+1)^\kappa - c_v^\kappa] B_{n-v-1}^{(\kappa+1)}\} \cdot \{(c_v^\kappa + d_{v-1}) B_{n-v-1}^{(\kappa)} + [(c_v+1)^\kappa - c_v^\kappa] B_{n-v-2}^{(\kappa+1)}\}}$$

Now $\omega(c_v) > \omega(c_v+1)$ for $c_v \geq 2$. The proof of this inequality follows exactly as that given for $W(c_v) > W(c_v+1)$ p.19, for we need only replace $\beta_{n-v}, \beta_{n-v-1}$ in $W(c_v)$ by $B_{n-v}^{(\kappa)}, B_{n-v-1}^{(\kappa)}$ to obtain $\omega(c_v)$. Then $u_n(c_0, c_1, \dots, c_v, \dots, c_{n-1}, c_n) < u_n(c_0, c_1, \dots, c_v+1, \dots, c_n)$ when $c_v \geq 2, (v=1, 2, \dots, n), (\kappa \geq 2)$.

~~Therefore, the minimum value of u_n occurs when all the~~

Therefore, the minimum value of u_n occurs when all the c_i , ($i=1, 2, \dots, n$) are large.

From the preceding expression for $w(c_v)$, upon dividing numerator and denominator by c_v^n , we obtain: $w(c_v) =$

$$\frac{(1 + \frac{1}{c_v})^n - 1}{\left\{ (1 + \frac{d_{v-1}}{c_v^n}) B_{n-v}^{(v)} + [(1 + \frac{1}{c_v})^n - 1] B_{n-v-1}^{(v+1)} \right\} \left\{ (c_v^n + d_{v-1}) B_{n-v-1}^{(v)} + [(c_v+1)^n - c_v^n \right\} B_{n-v-2}^{(v+1)} \right\}}$$

Then clearly, $\lim_{c_v \rightarrow \infty} w(c_v) = 0$

Therefore, $\lim_{c_v \rightarrow \infty} u_n(c_0, c_1, \dots, c_v, \dots, c_n) = 0$.

This concludes the proof of Theorem 1.3.2.

1.4 Convergence Properties

Definition 1.4.1 Unconditional Convergence * (Pringsheim)

A convergent continued fraction

$$1.4.1 \quad K_0 = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots = b_0 + \left[\frac{a_v}{b_v} \right]_1^\infty, \quad \text{with } a_v \neq 0, \quad (v \geq 1)$$

is said to be unconditionally convergent if all the continued fractions $K_n = b_n + \left[\frac{a_1}{b_1} \right]_{n+1}^\infty$ are convergent for $n = 0, 1, 2, \dots$.

Otherwise, the convergence of (1.4.1) is conditional.

Definition 1.4.2 Convergence by Bounded Variation †

The continued fraction (1.4.1) with convergents $\left\{ \frac{A_v}{B_v} \right\}$

is said to be convergent by bounded variation when the series $\sum_{n=1}^{\infty} \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right|$ converges, that is, when the sequence $\left\{ \frac{A_v}{B_v} \right\}$ is of bounded variation.

* Perron, O. (1) loc. cit.

† Szász, O. (1) loc. cit.

Criterion 1.4.1

The convergence of the series $\sum_{v=1}^{\infty} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right|$ insures the convergence of the corresponding continued fraction (1.4.1).

Proof :

If $\sum_{v=1}^{\infty} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right| < \infty$, then the remainders $\sum_{v=1}^{\infty} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right|$ tend to zero as $n \rightarrow \infty$. That is, given a positive ϵ , there exists an index n_0 such that $\sum_{v=1}^{\infty} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right| < \epsilon$ when $n \geq n_0$. Let p be a positive integer, $p \geq 1$. Then $\left| \frac{A_{n+p}}{B_{n+p}} - \frac{A_n}{B_n} \right| \leq \sum_{v=1}^{n+p} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right| < \epsilon$; Hence, by the Cauchy criterion, $\lim_{n \rightarrow \infty} \frac{A_n}{B_n}$ exists. ($n \geq n_0$).

An examination of the Proof (a) of Theorem 1.2.1, p.3 yields the following :

Theorem 1.4.1

The continued fraction of type 1,
 1.4.2 $c_0^{\kappa} + \frac{(c_0+1)^{\kappa} - c_0^{\kappa}}{|c_1^{\kappa}|} + \frac{(c_1+1)^{\kappa} - c_1^{\kappa}}{|c_2^{\kappa}|} + \dots$
 is unconditionally convergent.

Proof:

The continued fractions $c_n^{\kappa} + \frac{(c_n+1)^{\kappa} - c_n^{\kappa}}{|c_{n+1}^{\kappa}|} + \dots$ for $n = 0, 1, 2, \dots$, are all of type 1. Hence, by Theorem 1.2.1 they are all convergent. Therefore, by Definition 1.4.1, the continued fraction (1.4.2) is unconditionally convergent.

Theorem 1.4.2

The continued fraction of type 1, (1.4.2) is convergent by bounded variation.

Proof:

From (1.2.4) page 4, we have: $\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = (-1)^{n-1} \frac{a_1 a_2 \dots a_n}{B_n B_{n-1}}$
 and $\sum_{v=1}^{\infty} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right| = \sum_{v=1}^{\infty} \frac{a_1 a_2 \dots a_v}{B_v B_{v-1}} = \sum_{v=1}^{\infty} u_v$.
 Now from (1.2.5) page 4, $\frac{u_{n+1}}{u_n} < 1 - \frac{1}{2^{\kappa}}$ ($n \geq 1$).

Hence, $\sum_{v=1}^{\infty} u_v$ converges, for it is majorised by the

Convergent geometric series $\sum_{n=0}^{\infty} \theta^n$, $\theta = 1 - \frac{1}{2^\kappa} < 1$. Thus by Definition 1.4.2, (1.4.2) is convergent by bounded variation.

1.5 Uniqueness of Representation

We have seen that a real number $\xi_0 > 0$ can be developed uniquely by the Appell Algorithm into a continued fraction

$$1.1.2 \quad \xi_0 \sim c_0^\kappa + \frac{(c_0+1)^\kappa - c_0^\kappa}{1 \quad c_1^\kappa} + \frac{(c_1+1)^\kappa - c_1^\kappa}{1 \quad c_2^\kappa} + \dots$$

where $\kappa \geq 1$ is a prescribed integer, and the c_ν are integers, $c_0 \geq 0$, $c_\nu \geq 1$, ($\nu = 1, 2, \dots$), and also that the development converges and represents ξ_0 . We can write then:

$$1.5.1 \quad \xi_0 = c_0^\kappa + \frac{(c_0+1)^\kappa - c_0^\kappa}{1 \quad c_1^\kappa} + \dots$$

The question is now raised: Can two different continued fractions of type 1 represent the same number?

Theorem 1.5.1

A real number $\xi_0 > 0$ can be represented in one and only one way by a continued fraction of type 1. That is, ξ_0 is uniquely represented by its development of type 1.

Proof: Let the development of ξ_0 be (1.5.1) above.

Suppose that ξ_0 were represented by another continued fraction of type 1, namely: $\xi_0 = \gamma_0^\kappa + \frac{(\gamma_0+1)^\kappa - \gamma_0^\kappa}{1 \quad \gamma_1^\kappa} + \dots$

By Theorem 1.4.1, a continued fraction of type 1 is unconditionally convergent. The numbers ξ_ν, η_ν are defined by

$$1.5.2 \quad \begin{cases} \xi_\nu = c_\nu^\kappa + \frac{(c_\nu+1)^\kappa - c_\nu^\kappa}{1 \quad c_{\nu+1}^\kappa} + \dots \\ \eta_\nu = \gamma_\nu^\kappa + \frac{(\gamma_\nu+1)^\kappa - \gamma_\nu^\kappa}{1 \quad \gamma_{\nu+1}^\kappa} + \dots \end{cases} \quad \text{for } \nu = 1, 2, \dots$$

As the c_ν and γ_ν are positive integers, $c_\nu \geq 1$, $\gamma_\nu \geq 1$,

for $v \geq 1$, it follows that $\xi_v > 1$, $\eta_v > 1$; ($v = 1, 2, \dots$)

We can write, from (1.5.2), that

$$1.5.3 \quad \begin{cases} \xi_v = c_v + \frac{(c_v+1)^v - c_v^v}{\xi_{v+1}} \\ \eta_v = \delta_v + \frac{(\delta_v+1)^v - \delta_v^v}{\eta_{v+1}} \end{cases} \quad \text{for } v = 0, 1, 2, \dots, \\ \xi_0 = \delta_0$$

In particular, for $v = 0$, $\xi_0 = c_0 + \frac{(c_0+1)^0 - c_0^0}{\xi_1}$; $\delta_0 = \delta_0 + \frac{(\delta_0+1)^0 - \delta_0^0}{\eta_1}$
and as $\xi_1 > 1$, $\eta_1 > 1$, then $c_0^0 < \xi_0 < (c_0+1)^0$; $\delta_0^0 < \delta_0 < (\delta_0+1)^0$.

Thus, $c_0 = \delta_0 = [\xi_0]^{(k)}$. Therefore, $\xi_1 = \eta_1$.

Again, from (1.5.3), for $v = 1$ we have $\xi_1 = c_1 + \frac{(c_1+1)^1 - c_1^1}{\xi_2}$
 $\eta_1 = \delta_1 + \frac{(\delta_1+1)^1 - \delta_1^1}{\eta_2}$, and as $\xi_2 > 1$, $\eta_2 > 1$, it follows as

above that $c_1 = [\xi_1]^{(k)}$; $\delta_1 = [\eta_1]^{(k)}$. But $\eta_1 = \xi_1$. Hence

$\delta_1 = c_1$, and so $\eta_2 = \xi_2$. By repeating this procedure, we

obtain $\delta_v = c_v$ successively for $v = 0, 1, 2, \dots$. Hence

the representation of a real number ξ_0 by a continued

fraction of type 1 is unique. In particular, the number ξ_0

is uniquely represented by its continued fraction

development of type 1.

Corollary 1.5.1

A real number ξ_0 can not be represented simultaneously by a finite continued fraction of type 1 and also by an infinite one.

Proof:

Suppose, on the contrary, that ξ_0 were represented

by the two continued fractions of type 1:

$$\xi_0 = c_0 + \frac{(c_0+1)^0 - c_0^0}{c_1} + \dots + \frac{(c_{n-1}+1)^0 - c_{n-1}^0}{c_n}, \quad (c_n \geq 2)$$

$$\xi_0 = \delta_0 + \frac{(\delta_0+1)^0 - \delta_0^0}{\delta_1} + \dots$$

$$\text{Write: } \xi_v = c_v + \frac{(c_v+1)^v - c_v^v}{c_{v+1}} + \dots + \frac{(c_{n-1}+1)^v - c_{n-1}^v}{c_n}; \quad \text{for } v \geq 1, 2, \dots, n-1,$$

and $\xi_n = c_n^k \geq 2^k$. Also write: $\eta_v = \gamma_v^k + \frac{(\gamma_v+1)^k - \gamma_v^k}{1 - \gamma_v^k} + \dots$ ($v \geq 1$).

As in Theorem 1.5.1, we have that $\gamma_v = c_v$ for $v=0, 1, \dots, n-1$,

and $\eta_v = \xi_v$, $v=1, 2, \dots, n$. Then $\eta_n = \xi_n = c_n^k$.

But $\eta_n = \gamma_n^k + \frac{(\gamma_n+1)^k - \gamma_n^k}{\eta_{n+1}}$, and, as $\eta_{n+1} > 1$, $\gamma_n^k < \eta_n < (\gamma_n+1)^k$.

Thus, η_n can not be the k^{th} power of an integer. This

contradicts the above result that $\eta_n = c_n^k$. Hence Corollary

1.5.1 is true.

1.6 Rational ξ_0 and the Appell Algorithm

Let x_0, x_1 be two positive integers, $(x_0, x_1) = 1$, and write $\xi_0 = \frac{x_0}{x_1} > 0$. The integer $k \geq 2$ is given. We may assume that $\frac{x_0}{x_1} > 1$; for if $x_0 < x_1$, we would set $\xi_0 = \frac{1}{x_1/x_0} = \frac{1}{\xi_1}$ with $\xi_1 > 1$, and we would begin with ξ_1 instead of ξ_0 .

As before, write $c_0 = \left[\left(\frac{x_0}{x_1} \right)^{1/k} \right]$. Then: $c_0^k \leq \frac{x_0}{x_1} < (c_0+1)^k$. Write $x_0 = c_0^k x_1 + ((c_0+1)^k - c_0^k) x_2$.

This relation defines the rational number x_2 ; $x_2 \geq 0$. If $\frac{x_0}{x_1} = c_0^k$, then $x_2 = 0$ and the development terminates with the one equation: $x_0 = c_0^k x_1$.

Let $x_2 > 0$. As $\frac{x_0}{x_1} - c_0^k < (c_0+1)^k - c_0^k$, then $\frac{x_2}{x_1} < 1$. Clearly x_2 need not be an integer; for if $x_2 = 1 \neq c_0^k$, $0 < x_2 < x_1 = 1$.

Write $\xi_1 = \frac{x_1}{x_2} > 1$, and $c_1 = \left[\xi_1^{1/k} \right] \geq 1$. The equation $x_1 = c_1^k x_2 + [(c_1+1)^k - c_1^k] x_3$, defines the rational number $x_3 \geq 0$. If $\frac{x_1}{x_2} = c_1^k$, then $x_3 = 0$, and the development ends with the

two equations:
$$\begin{cases} x_0 = c_0^k x_1 + a_1 x_2 \\ x_1 = c_1^k x_2 \end{cases};$$

where: $a_1 = (c_0+1)^k - c_0^k$, and $c_1 \geq 2$.

From these, we obtain readily the corresponding continued fraction of type 1, namely: $\frac{x_0}{x_1} = c_0^{\wedge} + \frac{a_1}{1/c_1^{\wedge}}$.

Otherwise, $x_3 > 0$, and write $\xi_2 = \frac{x_2}{x_3} > 1$, as above. In general, $\xi_v = \frac{x_v}{x_{v+1}} > 1$, $c_v = \lceil \xi_v^{\wedge} \rceil \geq 1$, and x_{v+2} is defined by the equation: $x_v = c_v^{\wedge} x_{v+1} + a_{v+1} x_{v+2}$, ($v \geq 0$) where we have written $a_{v+1} = (c_{v+1})^{\wedge} - c_v^{\wedge}$; ($v = 0, 1, 2, \dots$).

If at a certain stage, $\xi_n = \frac{x_n}{x_{n+1}} = c_n^{\wedge}$, the development terminates, and we obtain the finite system of equations:

$$1.6.1 \quad \begin{cases} x_0 = c_0^{\wedge} x_1 + a_1 x_2 \\ x_1 = c_1^{\wedge} x_2 + a_2 x_3 \\ \dots \\ x_{n-1} = c_{n-1}^{\wedge} x_n + a_n x_{n+1} \\ x_n = c_n^{\wedge} x_{n+1} \end{cases}$$

with $c_v \geq 1$, ($v = 0, 1, \dots, n-1$); $c_n \geq 2$, and the x_v rational, ($v = 2, 3, \dots, n$); x_0, x_1 are integers, $(x_0, x_1) = 1$.

As $\xi_v = \frac{x_v}{x_{v+1}} > 1$, ($v \geq 0$), we have: $x_0 > x_1 > x_2 > \dots > x_{n+1}$.

If at every stage in the sequence of operations, $\xi_v \neq c_v^{\wedge}$, we obtain the infinite system of equations:

$$1.6.3 \quad \begin{cases} x_0 = c_0^{\wedge} x_1 + a_1 x_2 \\ x_1 = c_1^{\wedge} x_2 + a_2 x_3 \\ \dots \\ x_v = c_v^{\wedge} x_{v+1} + a_{v+1} x_{v+2} \\ \dots \end{cases}$$

and $x_0 > x_1 > x_2 > \dots > x_v > x_{v+1} > \dots$. The x_v are rational, $v \geq 2$; $(x_0, x_1) = 1$, and the c_v are integers, $c_v \geq 1$, ($v = 0, 1, \dots$).

From (1.6.1), we have: $\frac{x_0}{x_1} = c_0^{\wedge} + \frac{a_1}{x_1/x_2}$; $\frac{x_1}{x_2} = c_1^{\wedge} + \frac{a_2}{x_2/x_3}$; \dots ; $\frac{x_n}{x_{n+1}} = c_n^{\wedge}$; and so, $\frac{x_0}{x_1} = c_0^{\wedge} + \frac{a_1}{1/c_1^{\wedge}} + \dots + \frac{a_n}{1/c_n^{\wedge}}$, and $\frac{x_0}{x_1}$ has

a finite development of type 1 .

Similarly, we obtain from (.6.2)

$$\frac{x_0}{x_1} \sim c_0^n + \frac{a_1}{1c_1^n} + \frac{a_2}{1c_2^n} + \dots$$

Thus, with each system of equations we may associate a corresponding continued fraction of type 1 which is the development of the rational number $\frac{x_0}{x_1}$ by the Appell Algorithm.

Suppose now that we are given a system of equations

$$1.6.3 \quad x_v = c_v^n x_{v+1} + a_{v+1} x_{v+2}, \quad (v=0, 1, 2, \dots).$$

and the associated continued fraction: $\frac{x_0}{x_1} \sim c_0^n + \frac{a_1}{1c_1^n} + \dots$,

$$[a_{v+1} = (c_{v+1})^n - c_v^n], \quad \text{with convergents } \left\{ \frac{A_n}{B_n} \right\}.$$

In the recursion formulae

$$1.6.4 \quad \begin{cases} A_n = c_n^n A_{n-1} + a_n A_{n-2} \\ B_n = c_n^n B_{n-1} + a_n B_{n-2} \end{cases}, \quad n \geq 2,$$

define $A_{-1} = 1$, $B_{-1} = 0$ in order that (1.6.4) be true for $n=1$.

If we consider only a finite number, say the first v equations, of the system (1.6.3), we obtain :

$$1.6.5 \quad \frac{x_0}{x_1} = c_0^n + \frac{a_1}{1c_1^n} + \dots + \frac{a_v x_{v+1}}{1 x_v}, \quad (v \geq 1).$$

Then, from (1.6.4) and (1.6.5)

$$1.6.6 \quad \frac{x_0}{x_1} = \frac{A_v}{B_v} = \frac{x_v A_{v-1} + a_v x_{v+1} A_{v-2}}{x_v B_{v-1} + a_v x_{v+1} B_{v-2}}, \quad (v=1, 2, \dots).$$

From the system (1.6.3), we see that x_0 and x_1 each can be expressed as linear combinations of x_v and x_{v+1} . Indeed, in (1.6.6) there is equality of each corresponding numerator and denominator, that is :

$$1.6.7 \quad \begin{cases} x_0 = x_v A_{v-1} + x_{v+1} a_v A_{v-2} \\ x_1 = x_v B_{v-1} + x_{v+1} a_v B_{v-2} \end{cases}, \quad v=1, 2, \dots \quad *$$

We may solve the equations (1.6.7) for x_v, x_{v+1} in terms of x_0 and x_1 , as the determinant of the coefficients is: $a_v (A_{v-1} B_{v-2} - A_{v-2} B_{v-1}) = (-1)^v a_1 a_2 \dots a_v \neq 0$.

Thus:
$$x_{v+1} = (-1)^v \begin{vmatrix} A_{v-1} & x_0 \\ B_{v-1} & x_1 \end{vmatrix} \cdot \frac{1}{a_1 a_2 \dots a_v}$$
 and

$$1.6.8 \quad x_{v+1} = (-1)^{v+1} \frac{x_0 B_{v-1} - x_1 A_{v-1}}{a_1 a_2 \dots a_v}; \quad v \geq 1.$$

Theorem 1.6.1

Let the rational number $\xi_0 = \frac{x_0}{x_1} > 1$, with $(x_0, x_1) = 1$, have an infinite development of type 1, i.e.

$$\frac{x_0}{x_1} = c_0^k + \frac{a_1}{1 c_1^k} + \frac{a_2}{1 c_2^k} + \dots; \quad [a_v = (c_{v+1})^k - c_v^k]$$

Then in the equivalent system of equations

$$x_v = c_v^k x_{v+1} + a_{v+1} x_{v+2}; \quad (v=0, 1, 2, \dots), \quad \lim_{n \rightarrow \infty} x_n = 0.$$

Proof:

As $\xi_v = \frac{x_v}{x_{v+1}} > 1$ for $v \geq 0$, we have $x_0 > x_1 > x_2 > \dots > x_v > x_{v+1} > \dots$. Hence $\lim_{n \rightarrow \infty} x_n$ exists.

From (1.6.8) $x_{n+1} = \frac{x_1 B_{n-1} \left| \frac{x_0}{x_1} - \frac{A_{n-1}}{B_{n-1}} \right|}{a_1 a_2 \dots a_n}$ and as

$$\left| \frac{x_0}{x_1} - \frac{A_{n-1}}{B_{n-1}} \right| < \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{a_1 a_2 \dots a_n}{B_n B_{n-1}}; \quad 0 < x_{n+1} < \frac{x_1}{B_n}.$$

Now $B_0 = 1$; $B_1 = c_1^k \geq 1$, and $B_n = c_n^k B_{n-1} + a_n B_{n-2}$; ($n \geq 1$)

$B_n > B_{n-1}$, and as the B_n are integers, it follows that

$$\lim_{n \rightarrow \infty} B_n = +\infty \quad \text{Therefore,} \quad \lim_{n \rightarrow \infty} x_n = 0$$

* Perron, o. (1) § 5, p.13, formula 22, loc. cit.

Theorem 1.6.2

Let the rational number $\xi_0 = \frac{x_0}{x_1} > 1$

with $(x_0, x_1) = 1$, have a finite development of type 1, i.e.

$$\frac{x_0}{x_1} = c_0^{\wedge} + \frac{a_1}{c_1^{\wedge}} + \dots + \frac{a_n}{c_n^{\wedge}} \quad ; \quad c_n \geq 2.$$

Then in the equivalent system of equations

$$1.6.1 \quad \begin{cases} x_0 = c_0^{\wedge} x_1 + a_1 x_2 \\ x_1 = c_1^{\wedge} x_2 + a_2 x_3 \\ \dots \\ x_{n-1} = c_{n-1}^{\wedge} x_n + a_n x_{n+1} \\ x_n = c_n^{\wedge} x_{n+1} \end{cases}$$

the rational number x_{n+1} is the reciprocal of an integer;

$$x_{n+1} = \frac{1}{g} \quad \text{and } g \text{ is the greatest common divisor of } A_n, B_n ; \quad g = (A_n, B_n) \quad \text{where } \frac{A_n}{B_n} = c_0^{\wedge} + \left[\frac{a_n}{c_n^{\wedge}} \right].$$

Proof:

From (1.6.1), $\frac{x_n}{x_{n+1}}$ is an integer c_n^{\wedge} . Then $\frac{x_{n-1}}{x_{n+1}} = c_{n-1}^{\wedge} \frac{x_n}{x_{n+1}} + a_n$ is also an integer. By repeated

applications of this result, it follows that $\frac{x_0}{x_{n+1}}$ and $\frac{x_1}{x_{n+1}}$ are both integers.

Let $x_{n+1} = \frac{p}{g}$; $(p, g) = 1$. Then $\frac{x_0 g}{p}$ and $\frac{x_1 g}{p}$ are integers. Hence p divides $x_0 g$ and $x_1 g$. But $(x_0, x_1) = 1$, so that there exist integers s, t such that $s x_0 - t x_1 = 1$. Then p divides $g(s x_0 - t x_1) = g$. Hence $p = 1$, and $x_{n+1} = \frac{1}{g}$; $g \geq 1$.

From (1.6.7) page 30, we obtain for $v = n$,

$$x_0 = x_n A_{n-1} + x_{n+1} a_n A_{n-2} \quad \text{and as } x_n = c_n^{\wedge} x_{n+1},$$

$$\text{we have } x_0 = x_{n+1} (c_n^{\wedge} A_{n-1} + a_n A_{n-2}) = x_{n+1} A_n = \frac{A_n}{g}.$$

$$\text{Similarly, } x_1 = x_{n+1} B_n = \frac{B_n}{g}.$$

Thus q divides A_n and B_n . But $(x_0, x_1) = 1$.
Hence q is the greatest common divisor of A_n, B_n ;
$$g = (A_n, B_n).$$

Corollary 1.6.2

The g.c.d. of A_n, B_n namely q is odd. Also
$$g = \frac{A_n}{x_0} = \frac{B_n}{x_1}.$$

Proof:

As $A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1} a_1 a_2 \dots a_n$
and $a_v = (c_{v-1} + 1)^r - c_{v-1}^r$; ($v \geq 1$) is odd, it follows that every divisor of A_n and B_n must divide $a_1 a_2 \dots a_n$ and hence must be odd.

The second statement that $g = \frac{A_n}{x_0} = \frac{B_n}{x_1}$ follows immediately from the relations given above.

In the preceding, we showed that the development of type 1 of a rational number $\xi_0 = \frac{x_0}{x_1} > 1$, $(x_0, x_1) = 1$, can be associated with a system of linear equations, and that the development and the system are either both finite or both infinite. Thus, the system (1.6.2) $x_0 = c_0^r x_1 + a_1 x_2$,
 $x_1 = c_1^r x_2 + a_2 x_3, \dots$ where $\frac{x_v}{x_{v+1}} = \xi_v > 1$; ($v \geq 0$);
and the continued fraction development

$$1.6.2 \text{ a } \xi_0 = c_0^r + \frac{a_1}{1 c_1^r} + \frac{a_2}{1 c_2^r} + \dots \quad [a_v = (c_{v-1} + 1)^r - c_{v-1}^r].$$

are equivalent.

In the system (1.6.2), the rational numbers x_v ($v \geq 2$) need not be integers. In general, they are not. For, by Theorem 1.6.1, $\lim_{n \rightarrow \infty} x_n = 0$, and as the sequence $\{x_n\}$ is a strictly decreasing one, it follows that from some index, say μ , x_n is a fraction less than one, ($n \geq \mu$).

However, we can obtain a system of e linear equations equivalent to the continued fraction (1.6.2 a) above, and in which all the quantities are integers. This is indicated in the next paragraph.

Define $a_v = 1$. From (1.6.8),
 $a_0 a_1 \dots a_v x_{v+1} = (-1)^{v+1} (x_0 B_{v+1} - x_1 A_{v+1})$, so that $a_0 a_1 \dots a_v x_{v+1}$
 is an integer, $v \geq 0$.

Write $y_0 = x_0$; $y_1 = a_0 x_1 = x_1$, and $y_{v+1} = a_0 a_1 \dots a_v x_{v+1}$.
 The y_v are all integers, ($v \geq 0$).

Multiply the equations of the system (1.6.2) by $a_0, a_0 a_1,$
 $a_0 a_1 a_2, \dots$ in that order respectively, obtaining

$$1.6.9 \quad \begin{cases} y_0 = c_0^+ y_1 + y_2 \\ a_0 y_1 = c_1^+ y_2 + y_3 \\ \dots \\ a_v y_v = c_v^+ y_{v+1} + y_{v+2} \end{cases}$$

In this system, the y_v are positive integers, $y_v \geq 1$.
 Also, from $\xi_v = \frac{x_v}{x_{v+1}} = \frac{a_v y_v}{y_{v+1}} > 1$, it follows
 that $c_v = \left[\left(\frac{a_v y_v}{y_{v+1}} \right)^{\frac{1}{2}} \right] \geq 1$, and that $\frac{y_{v+2}}{y_{v+1}} < (c_v + 1)^+ - c_v^+ = a_{v+1}$
 i.e. $\xi_{v+1} = \frac{a_{v+1} y_{v+1}}{y_{v+2}} > 1$. This implies that the system
 (1.6.9) can be obtained independently by the Appell Algorithm.
 For, let $c_0 = \left[\xi_0^{\frac{1}{2}} \right] = \left[\left(\frac{a_0 y_0}{y_1} \right)^{\frac{1}{2}} \right]$. The integer y_2 is
 defined by the relation $a_0 y_0 = c_0^+ y_1 + y_2$
 In general, $\xi_v = \frac{a_v y_v}{y_{v+1}} > 1$, $c_v = \left[\xi_v^{\frac{1}{2}} \right] \geq 1$, and the integer
 y_{v+2} is defined by the equation: $a_v y_v = c_v^+ y_{v+1} + y_{v+2}$

From (1.6.8), we have

$$1.6.10 \quad y_{v+1} = (-1)^{v+1} (y_0 B_{v+1} - y_1 A_{v+1}), \quad (v \geq 0)$$

1.7 Examples of the Development of Rational Numbers by the Appell Algorithm.

Example 1. $\xi_0 = 5$, $K = 2$.

We have $2 = [\xi_0^{\frac{1}{2}}]$, so that $5 = 2^2 + \frac{5}{5}$.

Thus: $5 = 2^2 + \frac{5}{1 \cdot 2^2} + \frac{5}{1 \cdot 3^2} + \dots$

Let $\left\{ \frac{A_n}{B_n} \right\}$ be the sequence of convergents. Clearly, as $A_{-1} = 1$, $A_0 = 2^2$; $B_0 = 1$, $B_1 = 2^2$ and as A_n, B_n obey the same recursion formulas, we have $B_n = A_{n-1}$; $n \geq 0$.

Then $A_n = 4A_{n-1} + 5A_{n-2} = 4B_n + 5B_{n-1}$, and

$$A_n = 5B_n - [B_n - 5B_{n-1}], \quad n \geq 0.$$

$$\begin{aligned} \text{Also, } B_n - 5B_{n-1} &= 4B_{n-1} + 5B_{n-2} - 5B_{n-1} \\ &= -(B_{n-1} - 5B_{n-2}) = (-1)^n (B_0 - 5B_{-1}) = (-1)^n. \end{aligned}$$

Hence $A_n = 5B_n - (-1)^n$, and $\left| \frac{A_n}{B_n} - 5 \right| = \frac{1}{B_n}$.

As $B_n \uparrow \infty$, it follows that each convergent $\frac{A_n}{B_n}$ is closer to $\xi_0 = 5$ than the preceding one $\frac{A_{n-1}}{B_{n-1}}$. Also, as $A_n - 5B_n = \pm 1$, the numerators and the denominators of each convergent are relatively prime, $(A_n, B_n) = 1$.

Example 2. $\xi_0 = 2$, $K = 2$.

$$\begin{aligned} \xi_0 = 2 &= 1^2 + \frac{3}{3}; & \xi_1 = 3 &= 1^2 + \frac{3}{3/2} \\ \xi_2 = \frac{3}{2} &= 1^2 + \frac{3}{6}; & \xi_3 = 6 &= 2^2 + \frac{5}{5/2} \\ \xi_4 = \frac{5}{2} &= 1^2 + \frac{3}{2}; & \xi_5 = \xi_0 = 2 &= 1^2 + \frac{3}{3}; \end{aligned}$$

Thus we have:

$$(a) \quad 2 = 1^2 + \frac{3}{1 \cdot 1^2} + \frac{3}{1 \cdot 1^2} + \frac{3}{1 \cdot 2^2} + \frac{5}{1 \cdot 1^2} + \frac{3}{1 \cdot 1^2} + \frac{3}{1 \cdot 1^2} + \dots$$

(.) signifies period.

$$(b) \quad 3 = 1^2 + \frac{3}{1^2} + \frac{3}{1^2} + \frac{5}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \dots$$

$$(c) \quad 6 = 2^2 + \frac{5}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \frac{5}{1^2} + \dots$$

The rational numbers 3, 3, 6, $\frac{3}{2}$ and $\frac{5}{2}$ each have purely periodic developments with 5 term periods.

Corresponding to the continued fraction development for

$\xi_0 = 2$, we have the system of equations: $a_v y_v = c_v y_{v+1} + y_{v+2}$,
($v \geq 0$) with the initial conditions, $y_0 = 2$, $y_1 = 1$. Thus:

$$(1) \quad 2 = 1^2 \cdot 1 + 1$$

$$3 \cdot 1 = 1^2 \cdot 1 + 2$$

$$3 \cdot 1 = 1^2 \cdot 2 + 1$$

$$3 \cdot 2 = 2^2 \cdot 1 + 2$$

$$5 \cdot 1 = 1^2 \cdot 2 + 3$$

$$(2) \quad 3 \cdot 2 = 1^2 \cdot 3 + 3 \quad \longrightarrow \text{Equation (1) multiplied by 3}$$

$$3 \cdot 3 = 1^2 \cdot 3 + 6$$

$$3 \cdot 3 = 1^2 \cdot 6 + 3$$

$$3 \cdot 6 = 2^2 \cdot 3 + 6$$

$$5 \cdot 3 = 1^2 \cdot 6 + 9$$

$$(3) \quad 3 \cdot 6 = 1^2 \cdot 9 + 9 \quad \longrightarrow \text{Equation (2) multiplied by 3}$$

As the development for $\xi_0 = 2$ is periodic, the c_v are bounded. Indeed, $c_v \leq 2$. However, $y_{v+5} = 3 \cdot y_v$, $v \geq 1$ and therefore, the ~~system~~ sequence $\{y_v\}$ is unbounded.

For, $y_{v+5n} = 3^n y_{v+5(n-1)} = 3^n y_v$, ($v \geq 1$).

The same holds for the developments of 3, 6, $\frac{5}{2}$ and $\frac{3}{2}$

Consider the convergents $\frac{A_n}{B_n}$ to

$$\xi_0 = 6 = 2^2 + \frac{5}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \frac{3}{1^2} + \frac{5}{1^2} + \dots$$

Here $\frac{A_0}{B_0} = 4$, $\frac{A_1}{B_1} = 9$ and $|\xi_0 - \frac{A_0}{B_0}| = 2$; $|\xi_0 - \frac{A_1}{B_1}| = 3$,

so that $\frac{A_0}{B_0}$ is closer to ξ_0 than $\frac{A_1}{B_1}$.

$$\text{Also, } \frac{A_2}{B_2} = \frac{1 \cdot 9 + 3 \cdot 4}{1 \cdot 1 + 3 \cdot 1} = \frac{21}{4} = 6 - \frac{3}{4},$$

$$\frac{A_3}{B_3} = \frac{1 \cdot 21 + 3 \cdot 9}{1 \cdot 4 + 3 \cdot 1} = \frac{48}{7} = 6 + \frac{6}{7}, \text{ and } |\xi_0 - \frac{A_2}{B_2}| = \frac{3}{4}; |\xi_0 - \frac{A_3}{B_3}| = \frac{6}{7}.$$

Thus $\frac{A_2}{B_2}$ is closer to $\xi_0 = 6$ than $\frac{A_3}{B_3}$.

$$\text{Now } \frac{A_5}{B_5} = \frac{588}{97} = 6 + \frac{6}{97}; \quad \frac{A_4}{B_4} = \frac{111}{19} = 6 - \frac{3}{19},$$

and, as $\frac{6}{97} < \frac{3}{19}$, $\frac{A_5}{B_5}$ is closer to $\xi_0 = 6$ than $\frac{A_4}{B_4}$.

The behavior of the error-sequence $\{\theta_n\}$; $\theta_n = |\xi_0 - \frac{A_n}{B_n}|$, is determined by the following method. This method is applicable to all rational numbers having periodic developments of type 1.

As above, $\xi_0 = 6$ has a purely periodic development, with $c_{r+s} = c_r$. From the corresponding system of equations, $a_r y_r = c_r^2 y_{r+1} + y_{r+2}$; ($y_0 = 6, y_1 = 1$) we have: $y_{r+s+1} = 3 y_{r+1}$, $r \geq 0$. By a previous formula, (1.6.10) $y_{r+1} = (-1)^{r+1} (y_0 B_{r-1} - y_1 A_{r-1})$, ($r \geq 0$) where $\frac{A_n}{B_n}$ are the convergents to the development of $\frac{y_0}{y_1} = \xi_0$.

$$\text{Then: } |y_0 B_{r+4} - y_1 A_{r+4}| = 3 |y_0 B_{r+1} - y_1 A_{r+1}|$$

and, as $\xi_0 = \frac{y_0}{y_1} = \frac{6}{1}$, $y_0 = 6, y_1 = 1$ we have:

$$1.7.1 \quad |\xi_0 B_{r+4} - A_{r+4}| = 3 |\xi_0 B_{r+1} - A_{r+1}|, \quad (r \geq 0).$$

In general, $|\xi_0 B_{s(n+1)+\lambda} - A_{s(n+1)+\lambda}| = 3^s |\xi_0 B_{s(n)+\lambda} - A_{s(n)+\lambda}|$, so:

$$1.7.2 \quad |\xi_0 B_{s(n+1)+\lambda} - A_{s(n+1)+\lambda}| = 3^n |\xi_0 B_{s+\lambda} - A_{s+\lambda}|, \quad \text{for } \begin{cases} n \geq 0 \\ -1 \leq \lambda \leq 3 \end{cases}$$

Write $E_\lambda = |\xi_0 B_{s+\lambda} - A_{s+\lambda}|$; $-1 \leq \lambda \leq 3$. Then,

$$1.7.3 \quad \theta_{s(n+1)+\lambda} = \left| \xi_0 - \frac{A_{s(n+1)+\lambda}}{B_{s(n+1)+\lambda}} \right| = \frac{3^n E_\lambda}{B_{s(n+1)+\lambda}}, \quad n \geq 0.$$

We have:

$$E_{-1} = |\xi_0 B_{-1} - A_{-1}| = 1,$$

$$E_0 = |\sum_0 B_0 - A_0| = |6-4| = 2 ; E_1 = |\sum_0 B_1 - A_1| = |6-9| = 3$$

$$E_2 = |\sum_0 B_2 - A_2| = |6 \cdot 4 - 21| = 3$$

$$E_3 = |\sum_0 B_3 - A_3| = |6 \cdot 7 - 48| = 6$$

1° Compare θ_{5n-1} , θ_{5n} , ($n \geq 1$)

$$\theta_{5n-1} = \frac{3^n E_{-1}}{B_{5n-1}} = \frac{3^n}{B_{5n-1}}$$

$$\theta_{5n} = \frac{3^n E_0}{B_{5n}} = \frac{3^n \cdot 2}{B_{5n}}$$

(These are obtained from (1.7.3))

Table, $\sum_0 = 6$

$$c_{5v} = c_0 = 2$$

$$c_{5v+1} = c_1 = 1$$

$$c_{5v+2} = c_2 = 1$$

$$c_{5v+3} = c_3 = 1$$

$$c_{5v+4} = c_4 = 1$$

$$\text{Now, } B_{5n} = c_{5n}^2 B_{5n-1} + (2c_{5n-1} + 1) B_{5n-2}$$

$$> 4 B_{5n-1} > 2 B_{5n-1}, \quad (n \geq 1)$$

Hence, $\theta_{5n-1} > \theta_{5n}$, $n \geq 1$

2° Compare θ_{5n} with θ_{5n+1}

$$\theta_{5n} = \frac{3^n \cdot 2}{B_{5n}} ; \theta_{5n+1} = \frac{3^n E_1}{B_{5n+1}} = \frac{3^n \cdot 3}{B_{5n+1}}$$

$$\begin{aligned} \text{Now } (2 B_{5n+1} - 3 B_{5n}) &= 2 [B_{5n} + 5 B_{5n-1}] - 3 B_{5n} \\ &= 10 B_{5n-1} - B_{5n} = 10 B_{5n-1} - [4 B_{5n-1} + 3 B_{5n-2}] \\ &= 6 B_{5n-1} - 3 B_{5n-2} \end{aligned}$$

$$\text{Thus, } (2 B_{5n+1} - 3 B_{5n}) = 3 (2 B_{5n-1} - B_{5n-2}) > 0$$

as $B_{5n-1} - B_{5n-2} > 0$; $n \geq 1$ Hence $\theta_{5n} > \theta_{5n+1}$, ($n \geq 1$)

3° Compare θ_{5n+1} with θ_{5n+2}

$$\theta_{5n+1} = \frac{3^n \cdot 3}{B_{5n+1}} ; \theta_{5n+2} = \frac{3^n E_2}{B_{5n+2}} = \frac{3^n \cdot 3}{B_{5n+2}}$$

Clearly, as $B_{5n+2} > B_{5n+1}$ we have, $\theta_{5n+1} > \theta_{5n+2}$

4° Compare θ_{5n+2} with θ_{5n+3}

$$\theta_{5n+2} = \frac{3^n \cdot 3}{B_{5n+2}} ; \theta_{5n+3} = \frac{3^n E_3}{B_{5n+3}} = \frac{3^n \cdot 6}{B_{5n+3}}$$

$$\text{Now } B_{5n+3} = B_{5n+2} + 3 B_{5n+1},$$

$$\begin{aligned} \text{and } B_{5n+3} - 2B_{5n+2} &= 3B_{5n+1} - B_{5n+2} \\ &= 3B_{5n+1} - [B_{5n+1} + 3B_{5n}] \\ &= 2B_{5n+1} - 3B_{5n}. \end{aligned}$$

From 2^o, $2B_{5n+1} - 3B_{5n} > 0$. Then $B_{5n+3} > 2B_{5n+2}$,
and therefore, $\theta_{5n+2} > \theta_{5n+3}$, ($n \geq 1$)

$$5^o \text{ Compare } \theta_{5n+3} \text{ with } \theta_{5n+4} = \theta_{5(n+1)-1}$$

$$\theta_{5n+3} = \frac{3^n \cdot 6}{B_{5n+3}}; \quad \theta_{5n+4} = \frac{3^{n+1} E_{-1}}{B_{5n+4}} = \frac{3^{n+1}}{B_{5n+4}}.$$

Clearly, $2B_{5n+4} > B_{5n+3}$, so that $\theta_{5n+3} > \theta_{5n+4}$,

From our preceding results, $\theta_0 = 2$, $\theta_1 = 3$, $\theta_2 = \frac{3}{4}$, $\theta_3 = \frac{6}{7}$;

The above results then can be summarized as follows :

For $\xi_0 = 6$, the error-sequence $\{\theta_v\}$, $\theta_v = \left| \xi_0 - \frac{A_v}{B_v} \right|$
is strictly decreasing (to zero) for $v \geq 3$. When $v \leq 3$,
we have $\theta_0 < \theta_1$, $\theta_1 > \theta_2$, $\theta_2 < \theta_3$.

In addition, we have the following .

$$1. \quad \lim_{n \rightarrow \infty} \left| \xi_0 B_n - A_n \right| = +\infty \quad (\xi_0 = 6)$$

For, from the equation (1.7.2), p.36,

$$\left| \xi_0 B_{5n+\lambda} - A_{5n+\lambda} \right| = 3^n E_\lambda; \quad n \geq 1; \quad -1 \leq \lambda \leq 3,$$

and as $1 \leq E_\lambda \leq 6$, the above is evident.

$$2. \quad \lim_{n \rightarrow \infty} \frac{3^n}{B_{5n}} = 0$$

$$\text{For } \left| \xi_0 - \frac{A_{5n}}{B_{5n}} \right| = \frac{3^n E_0}{B_{5n}} = \frac{3^n \cdot 2}{B_{5n}}$$

and $\left| \xi_0 - \frac{A_{5n}}{B_{5n}} \right| \downarrow 0$ as $n \rightarrow \infty$.

Similar results may be obtained from the developments
for $\xi_0 = 2$ and $\xi_0 = 3$. We state them below.

(a) $\xi_0 = 2$

The error - sequence $\{\theta_v\}$ is strictly decreasing (to zero) for $v \geq 2$. That is, $\theta_v > \theta_{v+1}$, ($v \geq 2$)

(b) $\xi_0 = 3$. $\theta_v > \theta_{v+1}$ for $v \geq 0$.

Example 3. $\xi_0 = 7$. $\kappa = 2$

$$\xi_0 = 7 = 2^2 + \frac{5}{5/3} ; \quad \xi_1 = \frac{5}{3} = 1 + \frac{3}{9/2}$$

$$\xi_2 = \frac{9}{2} = 2^2 + \frac{5}{10} ; \quad \xi_3 = 10 = 3^2 + \frac{7}{7}$$

$$\xi_4 = \xi_0 = 7 :$$

Hence the development of $\xi_0 = 7$ is purely periodic, and :

$$7 = 2^2 + \frac{5/1}{1/1^2} + \frac{3/1}{1/2^2} + \frac{5/1}{1/3^2} + \frac{7/1}{1/2^2} + \frac{5/1}{1/1^2} + \dots$$

Similarly, $\xi_0 = 10$ has a periodic development with a period of 4 terms.

The corresponding system of equations for $\xi_0 = 7$,

$$a_v y_v = c_v^{\kappa} y_{v+1} + y_{v+2}, \quad (v \geq 0) \quad \text{is :}$$

$$(1) \quad 7 = 2^2 \cdot 1 + 3$$

$$5 \cdot 1 = 1^2 \cdot 3 + 2$$

$$3 \cdot 3 = 2^2 \cdot 2 + 1$$

$$5 \cdot 2 = 3^2 \cdot 1 + 1$$

$$(2) \quad 7 \cdot 1 = 2^2 \cdot 1 + 3 \quad \longrightarrow \quad \text{Same as equation (1).}$$

Here we note that $y_{v+4} = y_v$, $v \geq 0$. Also, as the c_v are bounded, $a_v y_v = (2c_{v+1})y_v$ is also bounded for all $v \geq 1$.

We now apply the method given in Example 3 to obtain further information on the behavior of the error - sequence

$\{\theta_n\}$ for $\xi_0 = 7$. From the above system, $\xi_0 = 7$; $y_0 = 7$, $y_1 = 1$,

and $y_{v+4} = y_v$, $v \geq 0$. By formula (1.6.10), p. 33,

$$y_{v+1} = |y_0 B_{v-1} - y_1 A_{v-1}| = |\xi_0 B_{v-1} - A_{v-1}|,$$

Then, $|\sum_0 B_{r+3} - A_{r+3}| = |\sum_0 B_{r+1} - A_{r+1}|$, and, in general

$$1.7.4 \quad |\sum_0 B_{4n+\lambda} - A_{4n+\lambda}| = |\sum_0 B_\lambda - A_\lambda|, \text{ for } \begin{cases} n \geq 0 \\ 0 \leq \lambda \leq 3 \end{cases}.$$

As before, write $E_\lambda = |\sum_0 B_\lambda - A_\lambda|$. Then

$$1.7.5 \quad \theta_{4n+\lambda} = \left| \sum_0 \frac{A_{4n+\lambda}}{B_{4n+\lambda}} \right| = \frac{E_\lambda}{B_{4n+\lambda}}, \quad \begin{matrix} n \geq 0 \\ 0 \leq \lambda \leq 3. \end{matrix}$$

	Table :	$\sum_0 = 7.$	
$C_{4r} = C_0 = 2$		$A_{-1} = 1$	$B_{-1} = 0$
$C_{4r+1} = C_1 = 1$		$A_0 = 4$	$B_0 = 1$
$C_{4r+2} = C_2 = 2$		$A_1 = 9$	$B_1 = 4$
$C_{4r+3} = C_3 = 3$		$A_2 = 48$	$B_2 = 7$
$E_0 = 3$, $E_1 = 2$,	$E_2 = 1$;	$E_3 = E_{-1} = 1.$

1° Compare θ_{4n} with θ_{4n+1} , $n \geq 0$.

$$\theta_{4n} = \frac{3}{B_{4n}}, \quad \theta_{4n+1} = \frac{2}{B_{4n+1}}$$

Clearly, $3 \cdot B_{4n+1} > 2 \cdot B_{4n}$; Hence $\theta_{4n} > \theta_{4n+1}$
($n \geq 0$)

2° Compare θ_{4n+1} with θ_{4n+2} ,

$$\theta_{4n+1} = \frac{2}{B_{4n+1}}, \quad \theta_{4n+2} = \frac{1}{B_{4n+2}}$$

and, evidently, $\theta_{4n+1} > \theta_{4n+2}$. ($n \geq 0$)

3° Compare θ_{4n+2} with θ_{4n+3}

$$\theta_{4n+2} = \frac{1}{B_{4n+2}}, \quad \theta_{4n+3} = \frac{1}{B_{4n+3}}$$

and again, as in 2°, $\theta_{4n+2} > \theta_{4n+3}$.

4° Compare θ_{4n+3} with θ_{4n+4}

$$\theta_{4n+3} = \frac{1}{B_{4n+3}}, \quad \theta_{4n+4} = \frac{3}{B_{4n+4}}$$

Now $B_{4n+4} > C_{4n+4}^2 B_{4n+3}$; $C_{4n+4} = 2$; and surely,

$B_{4n+4} > 3 B_{4n+3}$. Hence $\theta_{4n+3} > \theta_{4n+4}$, $n \geq 0$.

(a) We have then, for $\xi_0 = 7$ that the error - sequence $\{\theta_n\}$ decreases steadily (to zero) for $n \geq 0$; $\theta_n > \theta_{n+1}$.

(b) Also, the sequence $\{|\xi_0 B_n - A_n|\}$ is bounded. For, by (1.7.4), $|\xi_0 B_n - A_n| \leq \max E_\lambda$ and $E_\lambda \leq 3$.
 Thus : $|\xi_0 B_n - A_n| \leq 3$ ($n \geq 0$) ($0 \leq \lambda \leq 3$).

(c) As $|\xi_0 B_{4n+\lambda} - A_{4n+\lambda}| = 1$, for $\lambda = 2, 3$, it follows that $7 B_{4n+\lambda} - A_{4n+\lambda} = \pm 1$, and $(A_{4n+\lambda}, B_{4n+\lambda}) = 1$, for $n \geq 0$ and $\lambda = 2, 3$

Example 4.

The development of type 1 of $\xi_0 = 8$ is not purely periodic, it has a mixed period, with the period beginning after 17 terms.

$$\begin{aligned} \xi_0 = 8 = & 2^2 + \frac{51}{11^2} + \frac{31}{13^2} + \frac{71}{11^2} + \frac{31}{11^2} + \frac{31}{11^2} + \frac{31}{11^2} + \\ & + \frac{31}{11^2} + \frac{31}{11^2} + \frac{31}{11^2} + \frac{31}{11^2} + \frac{31}{14^2} + \frac{9}{13^2} + \frac{71}{11^2} \\ & + \frac{31}{11^2} + \frac{31}{12^2} + \frac{51}{11^2} + \frac{31}{12^2} + \frac{51}{12^2} + \frac{51}{12^2} + \dots \end{aligned}$$

Example 5.

This is an example to illustrate Theorem 1.6.2 p.31, on the greatest common divisor of A_n, B_n .

Consider $\xi_0 = \frac{32}{19}$; $\chi_0 = 32$, $\chi_1 = 19$.

We have :

$$\begin{aligned} \chi_0 = 32 &= 1^2 \cdot 19 + 3 \cdot \frac{13}{3} \\ \chi_1 = 19 &= 2^2 \cdot \frac{13}{3} + 5 \cdot \frac{1}{3} ; \quad \chi_2 = \frac{13}{3} = 3^2 \cdot \frac{1}{3} + 7 \cdot \frac{4}{21} \\ \chi_3 = \frac{1}{3} &= 1^2 \cdot \frac{4}{21} + 3 \cdot \frac{1}{21} ; \quad \chi_4 = \frac{4}{21} = 2^2 \cdot \frac{1}{21} \\ \chi_5 &= \frac{1}{21} ; \quad \chi_6 = 0. \end{aligned}$$

Here, $\chi_5 = \frac{1}{21}$ is the reciprocal of an integer, $\frac{1}{\chi_5} = 21$, and $\frac{\chi_0}{\chi_1} = \frac{32}{19} = 1^2 + \frac{31}{12^2} + \frac{51}{13^2} + \frac{71}{11^2} + \frac{31}{12^2}$;

$$A_0 = 1, B_0 = 1; A_1 = 7, B_1 = 4.$$

and a simple calculation yields, $A_2 = 672, B_2 = 399$.

$$\text{Now } 672 = 21 \cdot 32; \quad 399 = 21 \cdot 19$$

and as $(32, 19) = 1$, we have $21 = (672, 399)$;

$$\text{i.e. } \frac{1}{x_5} = (A_2, B_2)$$

1.8 Rational ξ_0 and Periodic Developments of Type 1 ($\kappa=2$).

In this section, we consider the problem: "What rational numbers have periodic developments of type 1?"

We restrict ourselves first to purely periodic developments.

Theorem 1.8.1

The only rational number having a purely periodic development with a period of one term is $\xi_0 = 5$.

Proof :

We can assume that ξ_0 is larger than 1. For if $0 < \xi_0 < 1$, we would have $c_0 = [\xi_0^{1/2}] = 0$; $\xi_0 = c_0^2 + \frac{2c_0+1}{\xi_1}$ where $\xi_1 = \frac{1}{\xi_0} > 1$. Then $c_1 = [\xi_1^{1/2}] \geq 1$, ~~and so~~ $\xi_1 = c_1^2 + \frac{2c_1+1}{\xi_2} > 1$, etc; and so $\xi_0 = \frac{1}{\frac{1}{c_1^2} + \frac{2c_1+1}{\xi_2} + \dots}$

We would then consider the continued fraction :

$$\xi_1 = \frac{1}{\xi_0} = c_1^2 + \frac{2c_1+1}{\xi_2} + \dots$$

The case $\xi_0 = 1$ can evidently be dismissed. Hence we can assume that $\xi_0 > 1$.

Let the rational number ξ_0 be greater than one, and let it have a purely periodic development with a one-term period, that is; let: $\xi_0 = c_0^2 + \frac{2c_0+1}{\xi_0} + \frac{2c_0+1}{\xi_0} + \dots$

ξ_0 satisfies the relation : $x = c_0^2 + \frac{2c_0 + 1}{x}$

That is, ξ_0 satisfies the quadratic equation

$$x^2 - c_0^2 x - (2c_0 + 1) = 0.$$

This quadratic then must have rational roots. Hence, its discriminant must be the square of an integer, namely ,

$$D_1 = (c_0^2)^2 + 4(2c_0 + 1) = d_1^2$$

where d_1 is an integer, $d_1 \geq 1$. Then

$$d_1^2 - (c_0^2)^2 = 4(2c_0 + 1).$$

In order that $d_1^2 - (c_0^2)^2$ be divisible by 4, c_0 and d_1 must be either both odd or both even. If c_0 and d_1 are

both odd, then $d_1^2 - (c_0^2)^2$ is divisible by 8. For then,

$$d_1 = 2k+1, \quad c_0^2 = 2m+1, \quad \text{and} \quad d_1^2 - (c_0^2)^2 = 4(k+m+1)(k-m).$$

As one of the two factors $(k+m+1)$, $(k-m)$ is even,

$d_1^2 - (c_0^2)^2$ is divisible by 8. But $4(2c_0 + 1)$ is at most divisible by 4, as $(2c_0 + 1)$ is odd. Hence c_0 and d_1 must be both even. Then $c_0 \geq 2$.

From $d_1^2 = (c_0^2)^2 + 4(2c_0 + 1)$, $d_1^2 > (c_0^2)^2$; $d_1 > c_0^2$

$$\text{Then } d_1 \geq c_0^2 + 2, \quad d_1^2 \geq (c_0^2 + 2)^2$$

Now $d_1^2 = (c_0^2)^2 + 4c_0^2 + 4 - 4(c_0^2 - 2c_0 + 1 - 1)$ and

$$1.8.1 \quad d_1^2 = (c_0^2 + 2)^2 - 4((c_0 - 1)^2 - 1)$$

As $c_0 \geq 2$, $(c_0 - 1)^2 - 1 \geq 0$; then $d_1^2 \leq (c_0^2 + 2)^2$, and so: $(c_0^2 + 2)^2 \leq d_1^2 \leq (c_0^2 + 2)^2$. Hence $d_1^2 = (c_0^2 + 2)^2$.

From the above relation (1.8.1), $d_1^2 = (c_0^2 + 2)^2$

when and only when $(c_0 - 1)^2 - 1 = 0$, that is, when $c_0 = 2$.

Then $\xi_0 = 2^2 + \frac{5!}{12^2} + \frac{5!}{12^2} + \dots$

is the only purely periodic continued fraction of type 1 having a period of one term, and which represents a rational number. Clearly $\xi_0 = 5$.

Let the real number $\xi_0 > 1$ have a finite development of type 1. (K. 2); $\xi_0 = c_0^2 + \frac{2c_0+1}{1/c_1^2} + \dots + \frac{2c_{n-1}+1}{1/\xi_n}$

where ξ_n is the square of an integer, $\xi_n = c_n^2$. As $\xi_n > 1$, $c_n \geq 2$.

We remarked previously that we can write

$$\xi_n = c_n^2 = (c_n - 1)^2 + \frac{c_n^2 - (c_n - 1)^2}{1}, \quad \text{and so there is an}$$

ambiguity in the number of terms in the finite development of ξ_0 . We avoid this by assuming, from this point on, that the last partial denominator in such a finite development of type 1 is at least 4, namely, $c_n^2 \geq 4$, $c_n \geq 2$.

Theorem 1.8.2

The only rational numbers having purely periodic developments of type 1 and with a period of two terms are (1) $\xi_0 = \frac{9}{8}$, (2) $\xi_0 = 24$; and we have:

$$\frac{9}{8} = 1^2 + \frac{31}{14^2} + \frac{91}{11^2} + \frac{31}{14^2} + \dots$$

$$24 = 4^2 + \frac{91}{11^2} + \frac{31}{14^2} + \frac{91}{11^2} + \dots$$

Proof:

Let the rational number $\xi_0 > 1$ have a purely periodic development with a two-term period,

$$1.8.2 \quad \xi_0 = c_0^2 + \frac{2c_0+1}{1/c_1^2} + \frac{2c_1+1}{1/c_0^2} + \frac{2c_0+1}{1/c_1^2} + \dots$$

Then $\xi_0 = c_0^2 + \frac{2c_0+1}{1/c_1^2} + \frac{2c_1+1}{1/\xi_0}$ so that

$$\xi_0 = \frac{\xi_0 A_1 + (2c_1+1)A_0}{\xi_0 B_1 + (2c_0+1)B_0}$$

Then ξ satisfies the quadratic equation

$$1.8.3 \quad \xi^2 B_1 - \xi (A_1 - (2c_1 + 1)B_0) - (2c_1 + 1)A_0 = 0.$$

This equation must have rational roots. Hence, its discriminant D_2 must be the square of an integer, namely,

$$D_2 = [A_1 - (2c_1 + 1)B_0]^2 - 4(2c_1 + 1)A_0 B_1 = d_2^2 > 0$$

where d_2 is an integer, $d_2 > 0$. Clearly,

$$d_2^2 = [A_1 + (2c_1 + 1)B_0]^2 - 4(2c_1 + 1)(A_1 B_0 - A_0 B_1),$$

and replacing the A'_2 and B'_2 by their respective values, we obtain,

$$1.8.4 \quad d_2^2 = [c_0^2 c_1^2 + (2c_0 + 1) + (2c_1 + 1)]^2 - 4(2c_0 + 1)(2c_1 + 1) \\ = u_2^2 - 4(2c_0 + 1)(2c_1 + 1).$$

where we have written $u_2 = c_0^2 c_1^2 + (2c_0 + 1) + (2c_1 + 1)$.

As $u_2^2 - d_2^2 = 4(2c_0 + 1)(2c_1 + 1)$, u_2, d_2 must both be even, by the argument in Theorem 1.8.1. As

$$u_2 = c_0^2 c_1^2 + 2(c_0 + c_1 + 1), \quad c_0 c_1 \text{ must be even, } c_0 c_1 \geq 2.$$

From (1.8.4), $d_2^2 < u_2^2$ and $d_2 < u_2$. Hence,

$$d_2 \leq u_2 - 2, \quad \text{and} \quad d_2^2 \leq (u_2 - 2)^2$$

We have: $d_2^2 = u_2^2 - 4(2c_0 + 1)(2c_1 + 1)$, and from this,

$$1.8.5 \quad d_2^2 = (u_2 - 2)^2 - 4[(2c_0 + 1)(2c_1 + 1) + 1 - u_2]$$

Also, $(u_2 - 2)^2 = (u_2 - 4)^2 + 4(u_2 - 3)$. Then

$$d_2^2 - (u_2 - 4)^2 = 4[u_2 - 3 - (2c_0 + 1)(2c_1 + 1) - 1 + u_2] \\ = 4[2u_2 - (2c_0 + 1)(2c_1 + 1) - 4] \\ = 4[2c_0^2 c_1^2 + (2c_0 + 1) + (2c_1 + 1) + 4c_0 c_1 - 3] \\ = 4[2c_0 c_1 (c_0 c_1 - 2) + (2c_0 + 1)(2c_1 + 1) - 3]$$

the last two lines obtained by using the relation

$$u_2 = c_0^2 c_1^2 + (2c_0 + 1) + (2c_1 + 1).$$

As $c_0 c_1 \geq 2$; $c_0 + c_1 \geq 3$, and $2(c_0 + c_1 + 1) \geq 8$.
 Thus, $d_2^2 - (u_2 - 4)^2 \geq 4[8 - 3] > 0$ and $d_2^2 > (u_2 - 4)^2$
 $d_2 > u_2 - 4$. Then $d_2 \geq u_2 - 2$, $d_2^2 \geq (u_2 - 2)^2$;
 But, from page 45, $d_2^2 \leq (u_2 - 2)^2$. Hence

$d_2^2 = (u_2 - 2)^2$, and from (1.8.5), we must have

$$1.8.6 \quad (2c_0 + 1)(2c_1 + 1) + 1 - u_2 = 0$$

The solutions c_0, c_1 of this diophantine equation are the only admissible values yielding a periodic development for ξ_0 . Equation (1.8.6) may be written as
 $(2c_0 + 1) + (2c_1 + 1) + 4c_0 c_1 - [c_0^2 c_1^2 + (2c_0 + 1) + (2c_1 + 1)] = 0$,
 or $c_0 c_1 (c_0 c_1 - 4) = 0$. Hence, the solutions are :
 (1) $c_0 = 1, c_1 = 4$, (2) $c_0 = 4, c_1 = 1$.

The other possibility, $c_0 = c_1 = 2$, gives the preceding case, Theorem 181.

Let ξ_0 be the continued fraction corresponding to $c_0 = 1, c_1 = 4$, and let η_0 be that one corresponding to $c_0 = 4, c_1 = 1$. Then,

$$\xi_0 = 1^2 + \frac{31}{14^2} + \frac{91}{11^2} + \frac{31}{14^2} + \dots$$

$$\eta_0 = 4^2 + \frac{91}{11^2} + \frac{31}{14^2} + \frac{91}{11^2} + \dots$$

$$\text{From (1.8.3) ; } x = \frac{A_1 - (2c_1 + 1)B_1 + \sqrt{D_2}}{2B_1}$$

(We choose the positive root of D_2 because the continued fraction (1.8.2) is the positive solution of the quadratic equation (1.8.3))

As $D_2 = d_2^2 = (u_2 - 2)^2$, and from (1.8.6), we have
 $x = \frac{c_0^2 c_1^2 + 2c_0}{c_1^2}$

Thus, $\xi_0 = \frac{16+2}{16} = \frac{9}{8}$, $\eta_0 = \frac{16+8}{1} = 24$,

and $\frac{9}{8}$ and 24 are the only rational numbers having purely periodic developments of type 1 with a two - term period. This proves Theorem 1.8.2 .

Theorem 1.8.3

There are no rational numbers $\xi_0 > 1$ having periodic developments of type 1 with a period of three terms.

Proof :

We restrict ourselves to purely periodic continued fractions, for, if no rational number has a purely periodic development with a three - term period, then surely no rational number can have a mixed periodic development with the same number of terms in the period. This is so because, if the period begins with the v^{th} partial denominator, then

$$\xi_0 = \frac{\xi_v A_{v-1} + (2c_{v-1} + 1) A_{v-2}}{\xi_v B_{v-1} + (2c_{v-1} + 1) B_{v-2}} \quad , \quad \text{and}$$

$$\xi_0 = \frac{A_{v-1}}{B_{v-1}} + (-1)^v \frac{(2c_{v-1} + 1)(2c_{v-2} + 1) \cdots (2c_0 + 1)}{B_{v-1} [\xi_v B_{v-1} + (2c_{v-1} + 1) B_{v-2}]}$$

Then, as ξ_v is not rational, ξ_0 can not be rational.

Suppose then that the rational number $\xi_0 > 1$ had a purely periodic development with a three - term period, namely

$$\xi_0 = c_0 + \frac{2c_0+1}{1c_1^2} + \frac{2c_1+1}{1c_2^2} + \frac{2c_2+1}{1c_0^2} + \frac{2c_0+1}{1c_1^2} + \dots$$

Then $\xi_0 = c_0 + \frac{2c_0+1}{1c_1^2} + \frac{2c_1+1}{1c_2^2} + \frac{2c_2+1}{1\xi_0}$, and so,

and so
$$\xi_0 = \frac{\xi_0 A_2 + (2c_2 + 1) A_1}{\xi_0 B_2 + (2c_2 + 1) B_1}$$

ξ_0 satisfies the quadratic equation

$$1.8.7 \quad \xi^2 B_2 - \xi [A_2 - (2c_2 + 1) B_1] - (2c_2 + 1) A_1 = 0$$

This equation has one positive and one negative root, and as $\xi_0 > c_0^2 \geq 1$, ξ_0 is the positive root.

If (1.8.7) is to have rational roots, the discriminant

D_3 must be the square of an integer, that is

$$D_3 = (A_2 - (2c_2 + 1) B_1)^2 + 4(2c_2 + 1) A_1 B_2 = d_3^2 > 0,$$

where d_3 is an integer, $d_3 > 0$. Clearly,

$$\begin{aligned} d_3^2 &= (A_2 + (2c_2 + 1) B_1)^2 - 4(2c_2 + 1) (A_2 B_1 - A_1 B_2) \\ &= (A_2 + (2c_2 + 1) B_1)^2 + 4(2c_2 + 1) (2c_2 + 1) (2c_0 + 1). \end{aligned}$$

Write $u_3 = A_2 + (2c_2 + 1) B_1$. Then

$$1.8.8 \quad d_3^2 = u_3^2 + 4(2c_0 + 1)(2c_2 + 1)(2c_2 + 1).$$

As in Theorem 1.8.1, u_3 and d_3 must be both even.

Now,
$$u_3 = A_2 + (2c_2 + 1) B_1 = c_2^2 (c_1^2 c_0^2 + 2c_0 + 1) + (2c_2 + 1) c_0^2 + (2c_2 + 1) c_1^2$$

and
$$u_3 = c_0^2 c_1^2 c_2^2 + c_0^2 (2c_2 + 1) + c_1^2 (2c_2 + 1) + c_2^2 (2c_0 + 1).$$

Evidently, u_3 is a cyclic function of c_0, c_1, c_2 ; that is, $u_3(c_0, c_1, c_2) = u_3(c_1, c_2, c_0) = u_3(c_2, c_0, c_1)$.

Hence, d_3^2 is also a cyclic function of c_0, c_1, c_2 .

$$d_3^2 = f_3(c_0, c_1, c_2).$$

From (1.8.8), $d_3^2 > u_3^2$, $d_3 > u_3$ and as d_3 and u_3 are both even, $d_3 \geq u_3 + 2$, $d_3^2 \geq (u_3 + 2)^2$.

We have
$$d_3^2 = u_3^2 + 4u_3 + 4 - 4[u_3 + 1 - (2c_0 + 1)(2c_2 + 1)(2c_2 + 1)].$$

and:

$$1.8.9 \quad \begin{cases} d_3^2 = (u_3 + 2)^2 - 4w_3 \\ w_3 = u_3 + 1 - (2c_0 + 1)(2c_1 + 1)(2c_2 + 1) \end{cases}$$

Now $(u_3 + 4)^2 - (u_3 + 2)^2 = 4(u_3 + 3)$, so that

$$(u_3 + 4)^2 - d_3^2 = 4[2u_3 + 4 - (2c_0 + 1)(2c_1 + 1)(2c_2 + 1)]$$

Also, $2u_3 = 2c_0^2 c_1^2 c_2^2 + 2c_0^2(2c_1 + 1) + 2c_1^2(2c_2 + 1) + 2c_2^2(2c_0 + 1)$

and $(2c_0 + 1)(2c_1 + 1)(2c_2 + 1) = 8c_0 c_1 c_2 + 2c_0(2c_1 + 1) + 2c_1(2c_2 + 1) + 2c_2(2c_0 + 1) + 1$.

Thus $(u_3 + 4)^2 - d_3^2 = 4[2c_0 c_1 c_2 (c_0 c_1 c_2 - 4) + 2c_0(c_0 - 1)(2c_1 + 1) + 2c_1(c_1 - 1)(2c_2 + 1) + 2c_2(c_2 - 1)(2c_0 + 1) + 3]$

If $c_0, c_1, c_2 \geq 4$, then as $c_v \geq 1, v = 0, 1, 2$.

$$(u_3 + 4)^2 - d_3^2 > 0$$

Consider ~~the~~ next the c_v such that $c_0, c_1, c_2 \leq 3$.

We rule out $c_0 = c_1 = c_2 = 1$ by Theorem 1.8.1. The only possible combinations are the two triples (a) $(1, 1, 2)$

and (b) $(1, 1, 3)$. As u_3 and d_3 are cyclic, we

can take (a) $c_0 = 1, c_1 = 1, c_2 = 2$. Then

$$(u_3 + 4)^2 - d_3^2 = 4[4(-2) + 6 \cdot 4 \cdot 3 + 3] > 0$$

(b) Let $c_0 = 1, c_1 = 1, c_2 = 3$. Then

$$(u_3 + 4)^2 - d_3^2 = 4[6(-1) + 6 \cdot 2 \cdot 3 + 3] > 0.$$

Hence $(u_3 + 4)^2 > d_3^2$ for all c_v such that

$c_0 \geq 1, c_1 \geq 1, c_2 \geq 2$. Then $u_3 + 4 > d_3$, and as u_3

and d_3 are both even, $u_3 + 2 \geq d_3$ and $(u_3 + 2)^2 \geq d_3^2$

But from the preceding, $d_3^2 \geq (u_3 + 2)^2$. Hence, $d_3^2 = (u_3 + 2)^2$

and from (1.8.9) we must have $w_3 = 0$ (w_3 defined by (1.8.9)).

We prove next that the diophantine equation $w_3 = 0$

has no integral solutions c_0, c_1, c_2 . This will prove the theorem.

We may write $w_3 = u_3 + 1 - (2c_0 + 1)(2c_1 + 1)(2c_2 + 1)$ in the following forms :

$$1.8.10 \text{ a} \quad w_3 = c_0 c_1 c_2 (c_0 c_1 c_2 - 8) + c_0 (c_0 - 2)(2c_1 + 1) \\ + c_1 (c_1 - 2)(2c_2 + 1) + c_2 (c_2 - 2)(2c_0 + 1)$$

$$1.8.10 \text{ b} \quad w_3 = c_0 c_1 c_2 (c_0 c_1 c_2 - 8) + (c_0 - 2)^2 + (c_1 - 2)^2 \\ + (c_2 - 2)^2 + 2(c_0 - 1)^2 c_1 + 2(c_1 - 1)^2 c_2 + 2(c_2 - 1)^2 c_0 - 12.$$

Evidently, w_3 is also a cyclic function of c_0, c_1, c_2 as u_3 and $(2c_0 + 1)(2c_1 + 1)(2c_2 + 1)$ have that property.

Now $u_3 = c_0^2 c_1^2 c_2^2 + c_0^2 (2c_1 + 1) + c_1^2 (2c_2 + 1) + c_2^2 (2c_0 + 1)$ must be even. Then $c_0^2 c_1^2 c_2^2 + c_0^2 + c_1^2 + c_2^2 + c_0^2$ must be even, and therefore $c_0 c_1 c_2 + c_0 + c_1 + c_2$ must be even. Then, (1) c_0, c_1, c_2 may all be odd or all even. (2) Any two of c_0, c_1, c_2 may be odd, the other one even. We can not have two even and the other one odd.

Consider the following cases.

(α)

Two of the c_i are equal to 1.

We can take $c_0 = 1, c_1 = 1$. Then from (1.8.10 a)

$$w_3 = c_2 (c_2 - 8) - 3 [3 - 2c_2 + 1 - 3c_2 (c_2 - 2)] \\ = 4c_2^2 - 16c_2 - 4 = 4[(c_2 - 2)^2 - 5]$$

Thus w_3 is never zero for integral c_2 and with $c_0 = c_1 = 1$.

(β) $c_0 = 1, c_1 = 3$

Using (1.8.10 a), we obtain ,

$$w_3 = 3c_2(3c_2 - 8) - 7 + 3c_2(c_2 - 2) = 12c_2^2 - 24c_2 - 4$$

$$w_3 = 12[(c_2 - 1)^2] - 16.$$

This quantity is never zero for integral c_2 . Thus, $w_3 \neq 0$ for $c_0 = 1$, $c_1 = 3$ and $c_2 \geq 1$

(γ) Let $c_0 = 3$, $c_1 = 1$ Then

$$w_3 = 3c_2(3c_2 - 8) + 9 - (2c_2 + 1) + 7c_2(c_2 - 3)$$

$$= 16c_2^2 - 40c_2 + 8$$

$$w_3 = 8(2c_2^2 - 5c_2 + 1).$$

Now $2c_2^2 - 5c_2 + 1 < 0$ for $c_2 = 2$, $c_2 = 1$
 $2c_2^2 - 5c_2 + 1 > 0$ for $c_2 = 3$

Hence w_3 can not vanish for $c_0 = 3$, $c_1 = 1$, $c_2 \geq 1$

(δ) If $c_0 \geq 2$, $c_1 \geq 2$, $c_2 \geq 2$, then evidently, from (1.8.10a)
 $w_3 > 0$ unless $c_0 = c_1 = c_2 = 2$.

The preceding discussion, (α), (β) (γ) and (δ) disposes of those cases where $c_0, c_1, c_2 \leq 8$, and also of those where $c_0, c_1, c_2 \geq 9$, with two of the c_i equal to one, and the third $c_i \geq 9$. Then the only remaining cases are those where $c_0, c_1, c_2 \geq 9$ and where at least two of the c_i are greater than or equal to two. But (1.8.10 b) shows that $w_3 \geq 9 + 1 + 4 - 12 > 0$
Hence $w_3 = 0$ has no integral solutions other than $c_0 = c_1 = c_2 = 2$, This establishes the theorem for purely periodic developments, and therefore, from the remarks at the beginning of the proof, for all periodic developments having a three = term period.

This proves Theorem 1.8.3 .

Definition 1.8.1

A function of n variables, $f(x_1, x_2, \dots, x_n)$ is said to be cyclic when it is unchanged by every cyclic permutation of the variables x_1, x_2, \dots, x_n .

Thus, if f is cyclic, then, $f(x_1, x_2, \dots, x_n) = f(x_2, x_3, \dots, x_n, x_1) = \dots = f(x_n, x_1, \dots, x_{n-1})$.

Lemma 1.8.1

A sufficient condition that a function $f(x_1, x_2, \dots, x_n)$ be cyclic is that $f(x_1, x_2, \dots, x_n) = f(x_2, x_3, \dots, x_n, x_1)$.

Proof :

By hypothesis, $f(x_1, x_2, \dots, x_n)$ remains unchanged when the variables x_1, x_2, \dots, x_n are permuted cyclicly to $x_2, x_3, \dots, x_n, x_1$. That is,

$$f(x_1, x_2, \dots, x_n) = f(x_2, x_3, \dots, x_n, x_1)$$

In this relation, replace x_1 by x_2 , x_2 by x_3 , \dots and x_n by x_1 . On the right side we obtain $f(x_3, x_4, \dots, x_n, x_1, x_2)$.

The left member $f(x_1, x_2, \dots, x_n)$ is unchanged, by this cyclic permutation of the variables. Hence, $f(x_1, x_2, \dots, x_n) = f(x_2, x_3, \dots, x_n, x_1) = f(x_3, x_4, \dots, x_n, x_1, x_2)$.

In this manner, it can be shown that $f(x_1, x_2, \dots, x_n) = f(x_\lambda, x_{\lambda+1}, \dots, x_n, x_1, \dots, x_{\lambda-2}, x_{\lambda-1})$ for $\lambda = 2, 3, \dots, n$.

Hence, by Definition 1.8.1, $f(x_1, \dots, x_n)$ is a cyclic function.

$$\text{Let } \xi_0 = c_0 + \frac{a_1}{1c_1} + \frac{a_2}{1c_2} + \dots$$

be a continued fraction of type 1, ($\kappa \geq 1$) with convergents $\left\{ \frac{A_n}{B_n} \right\}$. From Perron, O (1), "Kettenbrüche", § 5, formula 27, page 15, we have :

$$1.8.11 \quad \begin{cases} B_v = A_v^{(1)} \\ A_v = c_0^{\kappa} A_{v-1}^{(1)} + a_1 B_{v-1}^{(1)} \end{cases}, \quad v \geq 1.$$

By the first relation, $B_{v-1}^{(1)} = A_{v-2}^{(2)}$ and so, the second relation becomes

$$1.8.11 \text{ a} \quad A_v = c_0^{\kappa} A_{v-1}^{(1)} + a_1 A_{v-2}^{(2)}, \quad v \geq 1 \quad (A_{-1} = 1).$$

Theorem 1.8.4

Let the real number $\xi_0 > 1$ have a purely periodic development with a period of $(n+1)$ terms :

$$\xi_0 = c_0^{\kappa} + \frac{a_1}{1 c_1^{\kappa}} + \dots + \frac{a_n}{1 c_n^{\kappa}} + \frac{a_{n+1}}{1 c_0^{\kappa}} + \frac{a_1}{1 c_1^{\kappa}} + \dots$$

and with convergents $\frac{A_v}{B_v}$; $a_{v+1} = (c_v + 1)^{\kappa} - c_v^{\kappa}$.

Then the function : $u_{n+1}(c_0, c_1, \dots, c_n) = A_n + a_{n+1} B_{n-1}$ is cyclic, $n \geq 0$ ($B_{-1} = 0$).

Proof :

We have already seen that for $\kappa = 2$,

$u_{v+1} = A_v + a_{v+1} B_{v-1}$ is cyclic for $v = 0, 1, 2, \dots$

In the general case, $\kappa \geq 1$ it is evident that $u_0 = c_0^{\kappa}$, $u_1 = c_0^{\kappa} c_1^{\kappa} + a_1 + a_2$, or $u_1 = c_0^{\kappa} c_1^{\kappa} + (c_0 + 1)^{\kappa} - c_0^{\kappa} + (c_1 + 1)^{\kappa} - c_1^{\kappa}$ are cyclic functions of their variables.

Let n be an arbitrary index, $n \geq 1$. We have :

$$u_{n+1} = A_n + a_{n+1} B_{n-1} = c_n^{\kappa} A_{n-1} + a_n A_{n-2} + a_{n+1} B_{n-1}$$

In u_{n+1} , replace c_0 by c_1 , c_1 by c_2 , \dots and

c_n by c_0 , obtaining \bar{u}_{n+1} . Thus, using (1.8.11)

$$\begin{aligned} \bar{u}_{n+1} &= c_0^{\kappa} A_{n-1}^{(1)} + a_{n+1} A_{n-2}^{(1)} + a_1 B_{n-1}^{(1)} \\ &= c_0^{\kappa} A_{n-1}^{(1)} + a_1 A_{n-2}^{(2)} + a_{n+1} A_{n-2}^{(2)} = A_n + a_{n+1} B_{n-1} \end{aligned}$$

and $\bar{u}_{n+1} = u_{n+1}(c_1, c_2, \dots, c_n, c_0) = u_{n+1}(c_0, c_1, \dots, c_n)$.

Hence, by Lemma 1.8.1, $u_{n+1} = A_n + a_{n+1} B_{n-1}$ is a cyclic function.

Let the real number $\xi_0 > 1$ have a purely periodic development of type 1 and with a period of $n + 1$ terms. ($n \geq 2$)

$$1.8.12 \quad \xi_0 = c_0^2 + \frac{2c_0+1}{c_1^2} + \dots + \frac{2c_{n-1}+1}{c_n^2} + \frac{2c_n+1}{c_0^2} + \dots$$

Then $\xi_0 = \frac{\xi_0 A_n + (2c_n+1) A_{n-1}}{\xi_0 B_n + (2c_n+1) B_{n-1}}$, and

satisfies the quadratic equation :

$$x^2 B_n - x [A_n - (2c_n+1) B_{n-1}] - (2c_n+1) A_{n-1} = 0$$

Write $u_{n+1} = A_n + (2c_n+1) B_{n-1}$.

The discriminant of this quadratic is :

$$D_{n+1} = [A_n - (2c_n+1) B_{n-1}]^2 + 4(2c_n+1) A_{n-1} B_n = d_{n+1}^2 > 0$$

If d_{n+1} is rational, and hence an integer, then ξ_0 is rational. Otherwise, ξ_0 is a quadratic irrationality.

We may write d_{n+1}^2 as follows, :

$$1.8.13 \quad \begin{cases} d_{n+1}^2 = [A_n + (2c_n+1) B_{n-1}]^2 - 4(2c_n+1)(A_n B_{n-1} - A_{n-1} B_n) \\ d_{n+1}^2 = u_{n+1}^2 - (-1)^{n+1} \cdot 4 \prod_0^n (2c_v+1) \end{cases}$$

Let n be odd, $n = 2m + 1$, $m \geq 0$. Then

$$1.8.14 \text{ a} \quad d_{2m+2}^2 = u_{2m+2}^2 - 4 \prod_0^{2m+1} (2c_v+1), \text{ and } d_{2m+2}^2 < u_{2m+2}^2. \text{ Also, } \\ d_{2m+2}^2 - (u_{2m+2} - 2)^2 = 4 \left[u_{2m+2} + (-1)^{2m+1} - \prod_0^{2m+1} (2c_v+1) \right].$$

In 1.8.13 let n be even, $n = 2m$, $m \geq 0$

$$1.8.14 \text{ b} \quad \text{Then } d_{2m+1}^2 = u_{2m+1}^2 + 4 \prod_0^{2m} (2c_v+1), \text{ and } d_{2m+1}^2 > u_{2m+1}^2. \text{ Also, } \\ (u_{2m+1} + 2)^2 - d_{2m+1}^2 = 4 \left[u_{2m+1} + (-1)^{2m} - \prod_0^{2m} (2c_v+1) \right]$$

Lemma 1.8.2

Let the real number $\xi_0 > 1$ have a purely periodic development (1.8.12) with a $(n+1)$ term period. Then if $c_v \geq 2$, $v = 0, 1, 2, \dots$ we will have :

$$1.8.15 \quad u_{n+1} + (-1)^n \geq \prod_0^n (2c_v + 1), \quad n \geq 0$$

with equality only if $c_v = 2$, $v = 0, 1, 2, \dots, n$.

$$(\text{Here } u_{n+1} = A_n + (2c_n + 1) B_{n-1})$$

Proof :

$$\text{For } n=0 \quad (1.8.15) \text{ becomes } c_0^2 + 1 \geq 2c_0 + 1.$$

This is obviously true for $c_0 \geq 2$, and with equality only

if $c_0 = 2$. For $n = 1$, (1.8.15) becomes: $u_2 - 1 \geq (2c_0 + 1)(2c_1 + 1)$

$$\text{or } c_0^2 c_1^2 + (2c_0 + 1) + (2c_1 + 1) - 1 \geq (2c_0 + 1) + 4c_0 c_1 + 2c_1,$$

$$\text{or } c_0 c_1 (c_0 c_1 - 4) \geq 0 \quad \text{This proves the Lemma for } n=1.$$

The Lemma is true for $n=2$. For, in Theorem 1.8.3, we

$$\text{saw that } w_3 = u_3 + 1 - (2c_0 + 1)(2c_1 + 1)(2c_2 + 1)$$

could be written in the form (1.8.10 a),

$$w_3 = c_0 c_1 c_2 (c_0 c_1 c_2 - 8) + c_0 (c_0 - 2)(2c_1 + 1) \\ + c_1 (c_1 - 2)(2c_2 + 1) + c_2 (c_2 - 2)(2c_0 + 1).$$

Clearly, then $w_3 \geq 0$ if $c_v \geq 2$, and there is

equality only if $c_0 = c_1 = c_2 = 2$ That is,

$$u_3 + 1 > (2c_0 + 1)(2c_1 + 1)(2c_2 + 1), \text{ which is (1.8.15) for } n=2.$$

Assume that the Lemma is true for two arbitrary

successive indices, $k-1, k$, ($k \geq 2$) That is,

$$u_{n+1} + (-1)^n \geq \prod_0^n (2c_v + 1) \quad \text{holds for } n = k-1, n = k.$$

Now $u_{k+2} = A_{k+1} + (2c_{k+1} + 1) B_k$, and by Theorem 1.8.4

u_{k+2} is a cyclic function of c_0, c_1, \dots, c_{k+1} . As $\prod_0^{k+1} (2c_v + 1)$

is also cyclic, we can assume that c_{k+1} is the least of the

c'_v , i.e. $2 \leq c_{k+1} \leq c_v$, for $v = 0, 1, 2, \dots, k$ We do this.

$$\text{Now } u_{k+2} = (c_{k+1}^2 A_k + (2c_k + 1) A_{k-1}) \\ + (2c_{k+1} + 1) [c_k^2 B_{k-1} + (2c_{k-1} + 1) B_{k-2}]$$

$$u_{k+2} = c_{k+1}^2 (A_k + (2c_k + 1) B_{k-1}) + (2c_k + 1) [A_{k-1} + (2c_{k-1} + 1) B_{k-2}] \\ + B_{k-1} [c_k^2 (2c_{k+1} + 1) - c_{k+1}^2 (2c_k + 1)] \\ + (2c_{k-1} + 1) B_{k-2} [(2c_{k+1} + 1) - (2c_k + 1)]$$

$$\text{and : } U_{k+2} = C_{k+1}^2 U_{k+1} + (2C_{k+1}) U_k \\ + (C_k - C_{k+1}) B_{k-1} [C_{k+1} + C_k + 2C_{k+1} C_k] \\ + 2(C_{k+1} - C_k) (2C_{k-1} + 1) B_{k-2}.$$

$$\text{By assumption, } U_{k+1} \geq \prod_0^k (2C_v + 1) (-1)^k > 0$$

$$\text{so that : } U_k \geq \prod_0^{k-1} (2C_v + 1) (-1)^{k-1} > 0$$

$$U_{k+2} \geq C_{k+1}^2 \left[\prod_0^k (2C_v + 1) (-1)^k \right] + (2C_{k+1}) \left[\prod_0^{k-1} (2C_v + 1) (-1)^{k-1} \right] \\ + (C_k - C_{k+1}) B_{k-1} [C_{k+1} + C_k + 2C_{k+1} C_k] \\ + 2(C_{k+1} - C_k) (2C_{k-1} + 1) B_{k-2}, \quad \text{and:}$$

$$U_{k+2} \geq \prod_0^k (2C_v + 1) [C_{k+1}^2 + 1] (-1)^{k+1} (2C_k + 1) C_{k+1} \\ + (C_k - C_{k+1}) \delta_{k+2}$$

where we have written:

$$\delta_{k+2} = B_{k-1} (C_{k+1} + C_k + 2C_{k+1} C_k) - 2(2C_{k-1} + 1) B_{k-2}$$

If all the C_v equal 2, then evidently $C_{k+1}^2 + 1 = 2C_{k+1} + 1$,
and $2C_k = C_{k+1}^2$, so that $U_{k+2} = \prod_0^{k+1} (2C_v + 1) (-1)^{k+1}$.

Otherwise, at least one $C \geq 3$. Now either $C_k = C_{k+1} \geq 2$,
or $C_k > C_{k+1} \geq 2$. In the first case, $C_k = C_{k+1} \geq 2$,
we have $C_v \geq 3$ for at least one index, $v \leq k$, and so

$$U_k > \prod_0^{k-1} (2C_v + 1) (-1)^{k-1}. \quad \text{Also}$$

$$U_{k+2} > \prod_0^k (2C_v + 1) [C_{k+1}^2 + 1] (-1)^{k+1} [2C_k + 1 - C_{k+1}^2]$$

$$\text{If } C_k = C_{k+1} = 2 \quad \text{then } U_{k+2} > \prod_0^{k+1} (2C_v + 1) (-1)^{k+1}$$

$$\text{If } C_k = C_{k+1} \geq 3 \quad \text{then } (C_{k+1} - 1)^2 \geq 4,$$

$$\text{or } C_{k+1}^2 + 1 \geq (2C_{k+1} + 1) + 3, \quad \text{and}$$

$$U_{k+2} > \prod_0^{k+1} (2C_v + 1) (-1)^{k+1} + 3 \prod_0^k (2C_v + 1) (-1)^{k+1} (2C_k - C_{k+1}^2).$$

Then, as $C_v \geq C_{k+1} = C_k \geq 3$; ($v=0, 1, \dots, k-1$)

and clearly, $3 \prod_0^k (2C_v + 1) > C_k^2 - 2C_k$ & we have

$$\text{readily that } U_{k+2} > \prod_0^{k+1} (2C_v + 1) (-1)^{k+1}$$

The only remaining case is that where $C_k > C_{k+1} \geq 2$

Then $c_k - c_{k+1} \geq 1$, and $c_k \geq 3$. We have in this case :

$$u_{k+2} > \prod_0^k (2c_v + 1) [c_{k+1}^2 + 1] - (-1)^{k+1} - (-1)^{k+1} (2c_k - c_{k+1}) + \delta_{k+2},$$

$$\text{where } \delta_{k+2} = B_{k-1} (c_{k+1} + c_k + 2c_{k+1}c_k) - 2B_{k-2} (2c_{k-1} + 1).$$

$$\text{As } c_{k+1} \geq 2, c_{k-1} \geq 2, \text{ then } c_{k+1}B_{k-1} \geq 2B_{k-1} > 2c_{k-1}^2 B_{k-2} \geq 2B_{k-2} (2c_{k-1})$$

that is, $c_{k+1}B_{k-1} > 2B_{k-2} (2c_{k-1})$. Also,

$$(c_{k-1})B_{k-1} > 2B_{k-2} (2c_{k-1} + 1). \quad \text{Thus :}$$

$$B_{k-1} (c_{k+1} + c_k - 1) > 2B_{k-2} (2c_{k-1} + 1), \text{ and } \delta_{k+2} > 0. \text{ Indeed,}$$

$$\delta_{k+2} > B_{k-1} (1 + 2c_{k+1}c_k) > 0.$$

Now $B_{k-1} \geq c_{k-1}^2$, as $k \geq 2$, and clearly $\delta_{k+2} > 2c_k + c_{k-1}^2$.

Furthermore, as $c_{k+1} = \min c_v \geq 2$,

$$\delta_{k+2} > 2c_k + c_{k+1}^2 > |2c_k - c_{k+1}|^2$$

and $c_{k+1}^2 + 1 \geq 2c_{k+1} + 1$. Thus

$$u_{k+2} > \prod_0^{k+1} (2c_v + 1) - (-1)^{k+1}.$$

The Lemma 1.8.3 is then true for the next higher index.

Hence, it is true for all indices $n \geq 0$.

Theorem 1.8.5

If the real number $\xi_0 > 1$ has a purely periodic development with a period of $(n+1)$ terms, (1.8.12) and if $c_v \geq 2$, ($v = 0, 1, 2, \dots, n$) with at least one $c \geq 3$, then ξ_0 is a quadratioirrationality.

Proof :

Suppose on the contrary, that ξ_0 were rational.

Then from page 54, the discriminant D_{n+1} must be the square of an integer, $D_{n+1} = d_{n+1}^2$, where d_{n+1} is an integer.

$$\text{From (1.8.13), } d_{n+1}^2 = u_{n+1} - (-1)^{n+1} \cdot 4 \prod_0^n (2c_v + 1).$$

If n is odd, $d_{n+1} < u_{n+1}$, and as d_{n+1} and u_{n+1} are both

even, $d_{n+1} \leq u_{n+1} - 2$. But from (1.8.14 a),
 $d_{n+1}^2 - (u_{n+1} - 2)^2 = 4 \left[u_{n+1} + (-1)^n - \frac{1}{T} (2c_v + 1) \right]$,
 and the quantity in brackets is positive by Lemma 1.8.2.

Then $d_{n+1} > u_{n+1} - 2$. This contradicts the result $d_{n+1} \leq u_{n+1} - 2$.

If n is even, $d_{n+1}^2 > u_{n+1}^2$; $d_{n+1} > u_{n+1}$, and $d_{n+1} \geq u_{n+1} + 2$,
 as d_{n+1} and u_{n+1} are both even. But from (1.8.14 b)
 $(u_{n+1} + 2)^2 - d_{n+1}^2 = 4 \left[u_{n+1} + (-1)^n - \frac{1}{T} (2c_v + 1) \right]$

The quantity in brackets $u_{n+1} + (-1)^n - \frac{1}{T} (2c_v + 1)$

is positive by Lemma 1.8.2. Then $d_{n+1} < u_{n+1} + 2$.

This contradicts the previous result that $d_{n+1} \geq u_{n+1} + 2$.

Hence, ξ_0 is irrational.

Suppose that the real number $\xi_0 > 1$ has a mixed
 periodic development, the period beginning with the v^{th}
 partial denominator, $v \geq 1$. From page 47,

$$\xi_0 = \frac{A_{v-1}}{B_{v-1}} + \frac{(-1)^v (2c_{v-1} + 1)(2c_{v-2} + 1) \cdots (2c_v + 1)}{B_{v-1} \left[\xi_v B_{v-1} + (2c_{v-1} + 1) B_{v-2} \right]}$$

where ξ_v has a purely periodic development. Clearly,
 if ξ_v is irrational, ξ_0 is also irrational. This
 yields the Corollary.

Corollary 1.8.5

If the real number $\xi_0 > 1$ has a
 mixed periodic development, with a period of $n+1$ terms,
 beginning with the v^{th} partial denominator, $v \geq 1$, and
 if $c_i \geq 2$, ($i = v, v+1, v+2, \dots$) with at least one $c_i \geq 3$,
 then ξ_0 is a quadratic irrationality.

Remarks :

From O. Szasz , " Lectures on Continued Fractions and Allied Topics " , University of Cincinnati (1940-41), section on Irrationality Criteria , the continued fraction
$$K = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots$$

with positive integral elements a_v, b_v , converges and its value is irrational if there exist a sequence of positive increasing integers $n_1 < n_2 < \dots$ such that

$$\prod_{v=1+n_k}^{n_{k+1}} b_v \geq \prod_{v=1+n_k}^{n_{k+1}} a_v, \quad \text{for } k=1, 2, \dots$$

Apply this to the continued fraction (1.8.12) of the preceding Theorem 1.8.5 , page 57 . Take $n_1 = 0, n_2 = n+1$, and, in general, $n_k = (k-1)(n+1)$. Then

$$\prod_{v=1+n_k}^{n_{k+1}} b_v = c_0^2 c_1^2 \dots c_n^2 \quad \text{and}$$

$$\prod_{v=1+n_k}^{n_{k+1}} a_v = (2c_0+1)(2c_1+1) \dots (2c_n+1).$$

Clearly, if $c_v \geq 3$, ($v=0, 1, 2, \dots, n$) then $\prod c_v^2 > \prod (2c_v+1)$ and ξ_0 is irrational by the above criterion.

Theorem 1.8.5 shows that in the periodic case, only one of the c_v need be greater or equal to 3 , while all the other $c_v \geq 2$ in order that the particular continued fraction (1.8.12) represent an irrational number .

Irrationality Criterion 1.8.1

A real number $\xi_0 > 0$ is irrational if there exist two sequences of integers $\{A_n\}, \{B_n\}$ such that either
(a) $|\xi_0 B_n - A_n|$ is bounded from above and takes on infinitely many distinct values different from zero, or

$$(b) \quad |\xi_0 B_n - A_n| \neq 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} |\xi_0 B_n - A_n| = 0.$$

Proof :

(a) Suppose that ξ_0 were rational, $\xi_0 = \frac{p}{q}$, p and q integers. Then $|\xi_0 B_n - A_n| = \frac{|pB_n - qA_n|}{q}$. As $|\xi_0 B_n - A_n|$ is bounded, the numerator $|pB_n - qA_n|$ is also bounded. But this quantity is an integer for $n = 0, 1, 2, \dots$, and therefore, it can take on but a finite number of different values. This contradicts the hypothesis (a). Hence ξ_0 is irrational.

(b) Assume that ξ_0 is rational, $\xi_0 = \frac{p}{q}$. Then $|\xi_0 B_n - A_n| = \frac{|B_n p - q A_n|}{q} \neq 0$. As the numerator is an integer $\neq 0$, the fraction $\frac{|B_n p - q A_n|}{q}$ can not become less than $\frac{1}{q}$. This contradicts the hypothesis (b) that $\lim_{n \rightarrow \infty} |\xi_0 B_n - A_n| = 0$. Hence ξ_0 is irrational.

Theorem 1.8.6

If a real number $\xi_0 > 1$ has a continued fraction development of κ type 1, ($\kappa \geq 2$), $\xi_0 = c_0 + \left[\frac{a_\nu}{c_\nu} \right]_1^\infty$, such that $c_\nu \geq 3$ for $\nu \geq \lambda$, (λ a positive integer, $\lambda \geq 0$), then ξ_0 is irrational.

Proof :

We can assume that $\lambda = 0$, for as #

$$\xi_0 = \frac{A_{\lambda-1}}{B_{\lambda-1}} + \frac{(-1)^\lambda a_1 a_2 \dots a_\lambda}{(\xi_\lambda B_{\lambda-1} + a_\lambda B_{\lambda-2}) B_{\lambda-1}},$$

if ξ_λ irrational. then so is ξ_0 . Then we have :

$$\xi_0 = \frac{\xi_{n+1} A_n + a_{n+1} A_{n-1}}{\xi_{n+1} B_n + a_{n+1} B_{n-1}}, \quad \text{and}$$

$$|\xi_0 B_n - A_n| = \frac{a_{n+1}}{\xi_{n+1}} |\xi_0 B_{n-1} - A_{n-1}|.$$

Now $\frac{a_{n+1}}{\xi_{n+1}} = \xi_n - c_n^2 > 0$, for otherwise, ξ_0 would have a finite development. Thus:

$$|\xi_0 B_n - A_n| \neq |\xi_0 B_{n-1} - A_{n-1}|.$$

Also, from the above recursion relation for $|\xi_0 B_n - A_n|$, we obtain,

$$|\xi_0 B_n - A_n| = \frac{a_{n+1} a_n \cdots a_2 a_1}{\xi_{n+1} \xi_n \cdots \xi_2 \xi_1} \neq 0.$$

As $\xi_v > c_v^2$, we have $\frac{a_{v+1}}{\xi_{v+1}} < \frac{2c_v + 1}{c_v^2} = \frac{2}{c_v} + \frac{1}{c_v^2} \leq \frac{2}{3} + \frac{1}{9}$

Thus $\frac{a_{v+1}}{\xi_{v+1}} < \frac{7}{9}$ for $v \geq 0$, and

$$|\xi_0 B_n - A_n| < \left(\frac{7}{9}\right)^n.$$

Then $\lim_{n \rightarrow \infty} |\xi_0 B_n - A_n| = 0$.

Hence, by the Criterion 1.8.1, ξ_0 is irrational.

Chapter 2

2.1 Semi - Regular Analogue of the Appell Algorithm

In the development of the preceding chapter, we took the greatest integer c_0 contained in the κ^{th} root of a given positive number ξ_0 , $c_0 = [\xi_0^{1/\kappa}]$. One obtains an analogous development in taking the least integer containing the κ^{th} root. (κ a positive integer, $\kappa \geq 1$).

Thus, let κ be a given positive integer, $\kappa \geq 1$, and let $\xi_0 > 0$ be a prescribed real number. The integer c_0 is defined to be the least integer containing $\xi_0^{1/\kappa}$. Then $c_0 \geq \xi_0^{1/\kappa} > c_0 - 1$; $c_0 \geq 1$, and $c_0^\kappa \geq \xi_0 > (c_0 - 1)^\kappa$.

If ξ_0 is the κ^{th} power of an integer, $\xi_0 = c_0^\kappa$, the development ends. Otherwise, $c_0 = [\xi_0^{1/\kappa}] + 1 \geq 1$.

Write $\xi_0 = c_0^\kappa - \rho_0$; then $0 \leq \rho_0 < c_0^\kappa - (c_0 - 1)^\kappa$. If $\rho_0 \neq 0$, the number ξ_1 is defined by: $\xi_1 = \frac{c_0^\kappa - (c_0 - 1)^\kappa}{\rho_0}$. Thus $\xi_1 > 1$, and $\xi_0 = c_0^\kappa - \frac{c_0^\kappa - (c_0 - 1)^\kappa}{\xi_1}$.

One can apply to ξ_1 the same sequence of operations as applied to ξ_0 obtaining: $\xi_1 = c_1^\kappa + \rho_1$ with $0 \leq \rho_1 < c_1^\kappa - (c_1 - 1)^\kappa$. If $\xi_1 = c_1^\kappa$, then $\rho_1 = 0$ and the development terminates. One obtains for ξ_0

$$\xi_0 = c_0^\kappa + \frac{c_0^\kappa - (c_0 - 1)^\kappa}{c_1^\kappa}$$

As $\xi_1 > 1$, $c_1 \geq 2$.

Otherwise, $\rho_1 > 0$, and $c_1 = [\xi_1^{1/\kappa}] + 1 \geq 2$. We proceed as above to define ξ_2 , $\xi_2 = \frac{c_1^\kappa - (c_1 - 1)^\kappa}{\rho_1}$; $\xi_2 > 1$, and so on.

Appell, P (1) and (2), loc. cit. page 1.

If at any stage in the sequence of operations ξ_v^{κ} is an integer c_v , then $\rho_v = 0$, and the development terminates.

One obtains in this case the finite development :

$$2.1.1 \quad \xi_0 = c_0^{\kappa} - \frac{c_0^{\kappa} - (c_0 - 1)^{\kappa}}{1 \ c_1^{\kappa}} - \frac{c_1^{\kappa} - (c_1 - 1)^{\kappa}}{1 \ c_2^{\kappa}} - \dots - \frac{c_{v-1}^{\kappa} - (c_{v-1} - 1)^{\kappa}}{1 \ c_v^{\kappa}}$$

If at every stage $\rho_v > 0$, the development does not terminate, and one obtains the infinite continued fraction

$$2.1.2 \quad \xi_0 \sim c_0^{\kappa} - \frac{c_0^{\kappa} - (c_0 - 1)^{\kappa}}{1 \ c_1^{\kappa}} - \frac{c_1^{\kappa} - (c_1 - 1)^{\kappa}}{1 \ c_2^{\kappa}} - \dots$$

In the developments (2.1.1) and (2.1.2), we have $c_0 \geq 1$, $c_n \geq 2$ for $n \geq 1$.

Definition 2.1.1

The development of a real number $\xi_0 > 0$ by the algorithm of section 2.1 into a continued fraction (2.1.1) or (2.1.2) is called the development of type 2 of ξ_0 .

A continued fraction $c_0^{\kappa} + \left[- \frac{c_{v-1}^{\kappa} - (c_{v-1} - 1)^{\kappa}}{c_v^{\kappa}} \right]_1^{\infty}$, with κ a positive integer, $\kappa \geq 1$ and where the c_v are integers, $c_0 \geq 1$, $c_v \geq 2$ for $v \geq 1$, is called a continued fraction of type 2.

For $\kappa = 1$, see P. Epstein, "Über Möbius Kettenbrüche und Elementar - Kettenbrüche", Mathematische Zeitschrift, 2, 1918.

As an illustration of the above algorithm, let $\xi_0 = 3$, $\kappa = 2$. Then one obtains, $c_0 = [\sqrt{3}] + 1 = 2$; $3 = 2^2 - \frac{3}{3}$. Clearly, $\xi_1 = 3 = \xi_0$. Hence $3 \sim 2^2 - \frac{3}{2^2} - \frac{3}{2^2} - \dots$

2.2 Convergence

We prove convergence for a special case.

Theorem 2.2.1

The development of type 2 of the number

 $\xi_0 = 2^k - 1$, ($k > 1$), is convergent and represents ξ_0 .Proof :We have: $C_0 = [(2^k - 1)^{\frac{1}{k}}] + 1 = 2$; $\xi_0 = 2^k - \frac{2^k - 1}{\xi_0}$.Evidently, $\xi_1 = 2^k - \xi_0$. It follows that

$$\begin{aligned} \xi_0 &\sim 2^k - \frac{2^k - 1}{1 \cdot 2^k} - \frac{2^k - 1}{1 \cdot 2^k} - \dots \left\{ \frac{A_v}{B_v} \right\} \\ &\sim 2^k + \left[-\frac{2^k - 1}{2^k} \right]_1 \sim \xi_0 + 1 + \left[-\frac{\xi_0}{\xi_0 + 1} \right]_1 \end{aligned}$$

Now: $A_0 = 2^k = \xi_0 + 1 = \frac{\xi_0^2 - 1}{\xi_0 - 1}$ and

$$A_1 = (\xi_0 + 1)A_0 - \xi_0 = \frac{\xi_0^3 + \xi_0^2 - \xi_0 - 1 - \xi_0^2 + \xi_0}{\xi_0 - 1} = \frac{\xi_0^3 - 1}{\xi_0 - 1}$$

In general, assume that $A_v = \frac{\xi_0^{v+2} - 1}{\xi_0 - 1}$ and $A_{v-1} = \frac{\xi_0^{v+1} - 1}{\xi_0 - 1}$ are true for an arbitrary index $v \geq 1$. Then

$$\begin{aligned} A_{v+1} &= 2^k A_v - (2^k - 1)A_{v-1} = (\xi_0 + 1) \left(\frac{\xi_0^{v+2} - 1}{\xi_0 - 1} \right) - \xi_0 \left(\frac{\xi_0^{v+1} - 1}{\xi_0 - 1} \right) \\ &= \frac{1}{\xi_0 - 1} \left(\xi_0^{v+3} + \xi_0^{v+2} - \xi_0 - 1 - \xi_0^{v+2} + \xi_0 \right) \end{aligned}$$

$$A_{v+1} = \frac{\xi_0^{v+3} - 1}{\xi_0 - 1}$$

Hence, $A_v = \frac{\xi_0^{v+2} - 1}{\xi_0 - 1}$ is true for $v \geq 0$.From (1.8.11) p. 53, $B_v = A_v^{(1)}$ and as the develop-

ment is purely periodic with a one term period, clearly

 $A_{v-1}^{(1)} = A_{v-1}$ Thus $B_v = A_{v-1}$, ($v \geq 0$) We have then

$$2.2.1 \quad A_{v-1} = B_v = \frac{\xi_0^{v+1} - 1}{\xi_0 - 1} = \frac{(2^k - 1)^{v+1} - 1}{2^k - 2}$$

$$\text{Thus } \frac{A_v}{B_v} = \frac{\xi_0^{v+2} - 1}{\xi_0 - 1} \times \frac{\xi_0 - 1}{\xi_0^{v+1} - 1} = \frac{\xi_0^{v+2} - 1}{\xi_0^{v+1} - 1}$$

$$\text{and } \frac{A_v}{B_v} - \xi_0 = \frac{\xi_0 - 1}{\xi_0^{v+1} - 1} > 0$$

Then $0 < \frac{A_v}{B_v} - \xi_0 < \frac{\xi_0}{\xi_0^{v+1} - \xi_0} = \frac{1}{\xi_0^v - 1}$. As $\xi_0 = 2^k - 1 > 3$,we have $\lim_{v \rightarrow \infty} \frac{A_v}{B_v} - \xi_0 = 0$. Hence the development of type2 of $\xi_0 = 2^k - 1$ converges and represents ξ_0 . ($k > 1$).

Suppose now that we wish to develop a positive number ξ_0 into a continued fraction of type 2 with $\kappa > 1$, and that the number ξ_0 were such that $1 < \xi_0 < 2^{\kappa}-1$. Then, $c_0 = 1 + [\xi_0^{\kappa}] = 2$, and $\xi_1 = 2^{\kappa} - \frac{2^{\kappa}-1}{\xi_0}$. Thus, $2^{\kappa} - \xi_0 = \frac{2^{\kappa}-1}{\xi_1}$; $\xi_1 = \frac{2^{\kappa}-1}{2^{\kappa}-\xi_0}$.

From our assumptions on ξ_0 , we have: $2^{\kappa}-1 > 2^{\kappa}-\xi_0 > 1$, and accordingly $\frac{1}{2^{\kappa}-1} < \frac{1}{2^{\kappa}-\xi_0} < 1$, or $1 < \frac{2^{\kappa}-1}{2^{\kappa}-\xi_0} < 2^{\kappa}-1$.

Then $1 < \xi_1 < 2^{\kappa}-1$, and ξ_1 has the same property as ξ_0 .

It follows by induction that all the ξ_v have the same property, $v \geq 0$, namely, $1 < \xi_v < 2^{\kappa}-1$. Hence, $c_v = 2, (v \geq 0)$ and $\xi_0 \sim 2^{\kappa} + \left[-\frac{2^{\kappa}-1}{2^{\kappa}} \right]_{\infty}$.

But $2^{\kappa} + \left[-\frac{2^{\kappa}-1}{2^{\kappa}} \right]_{\infty} = 2^{\kappa}-1$, by Theorem 2.2.1, and $\xi_0 \neq 2^{\kappa}-1$.

We have then the theorem.

Theorem 2.2.2

The development of type 2 of any number $\xi_0 > 1$ (with $\kappa > 1$), and such that $1 < \xi_0 < 2^{\kappa}-1$, is the same for all the ξ_0 , and it represents the number $2^{\kappa}-1$.

It follows readily that, if in the development of type 2 of a number $\xi_0 > 1$, we obtain a complete quotient ξ_v such that $1 < \xi_v < 2^{\kappa}-1$, then the development will be periodic from this point on; and also, the continued fraction development of ξ_0 converges but does not represent ξ_0 .

In connection with the result of Theorem 2.2.2, we point out that if $1 < \xi_0 < 2^{\kappa}-1$, then the sequence $\{\xi_v\}$ is

strictly decreasing, $\xi_v < \xi_{v-1} < \dots < \xi_0$. For, $1 < \xi_1 < 2^{\kappa}-1$, and $\xi_1 - \xi_0 = \xi_1 - 2^{\kappa} + \frac{2^{\kappa}-1}{\xi_1} = \frac{\xi_1^2 - 2^{\kappa}\xi_1 + 2^{\kappa}-1}{\xi_1}$,
 $\xi_1 - \xi_0 = \frac{[\xi_1 - (2^{\kappa}-1)](\xi_1 - 1)}{\xi_1} < 0$

Then $\xi_1 < \xi_0$. Similarly, $\xi_v < \xi_{v-1}$, for $v \geq 1$. Also, as $\xi_v > 1$ for $v \geq 0$, it follows that $\lim_{v \rightarrow \infty} \xi_v$ exists and that $2^{\kappa}-1 > \lim_{v \rightarrow \infty} \xi_v \geq 1$. From the relation

$$\xi_0 = 2^{\kappa} - \frac{2^{\kappa}-1}{1 \cdot 2^{\kappa}} - \dots - \frac{2^{\kappa}-1}{1 \cdot \xi_v}, \quad \left\{ \frac{A_v}{B_v} \right\}$$

where $1 < \xi_0 < 2^{\kappa}-1$; $1 < \xi_v < 2^{\kappa}-1$, ($v \geq 1$)

we have: $\xi_0 = \frac{\xi_v A_{v-1} - (2^{\kappa}-1)A_{v-2}}{\xi_v B_{v-1} - (2^{\kappa}-1)B_{v-2}}$, and $\xi_v = \frac{(\xi_0 - \frac{A_{v-2}}{B_{v-2}})(2^{\kappa}-1)B_{v-2}}{(\xi_0 - \frac{A_{v-1}}{B_{v-1}})B_{v-1}}$.

From equation (2.2.1), p. 64, $B_v = \frac{(2^{\kappa}-1)^{v+1} - 1}{2^{\kappa}-2}$, ($v \geq 0$). Then

$$(2^{\kappa}-1) \frac{B_{v-2}}{B_{v-1}} = (2^{\kappa}-1) \frac{(2^{\kappa}-1)^{v-1} - 1}{(2^{\kappa}-1)^v - 1} = \frac{(2^{\kappa}-1)^v - (2^{\kappa}-1)}{(2^{\kappa}-1)^v - 1} < 1.$$

Thus $1 < \xi_v < \frac{\xi_0 - \frac{A_{v-2}}{B_{v-2}}}{\xi_0 - \frac{A_{v-1}}{B_{v-1}}}$. As $\xi_0 < 2^{\kappa}-1$,
 $\lim_{v \rightarrow \infty} \frac{A_v}{B_v} = 2^{\kappa}-1$,

by Theorem 2.2.2. Then $\lim_{v \rightarrow \infty} (\xi_0 - \frac{A_{v-1}}{B_{v-1}}) = \xi_0 - (2^{\kappa}-1) \neq 0$,

so that $1 < \lim_{v \rightarrow \infty} \xi_v < 1$. Hence $\lim_{v \rightarrow \infty} \xi_v = 1$.

We consider next the general continued fraction of type 2:

$$2.2.2 \quad K = c_0^{\kappa} - \frac{c_0^{\kappa} - (c_0-1)^{\kappa}}{1 \cdot c_1^{\kappa}} - \frac{c_1^{\kappa} - (c_1-1)^{\kappa}}{1 \cdot c_2^{\kappa}} - \dots = c_0^{\kappa} + \left[- \frac{c_0^{\kappa} - (c_0-1)^{\kappa}}{c_1^{\kappa}} \right]_0^{\infty}$$

with κ and integer, $\kappa \geq 2$ and $c_0 \geq 1$, $c_v \geq 2$: $\left\{ \frac{A_n}{B_n} \right\}$.

Then $B_0 = 1$, $B_1 = c_1^{\kappa} \geq 2^{\kappa}$, and $B_n = c_n^{\kappa} B_{n-1} - [c_n^{\kappa} - (c_{n-1}+1)^{\kappa}] B_{n-2}$

Let n be an arbitrary index, $n \geq 2$, We have

$$B_n - [c_n^{\kappa} - (c_{n-1})^{\kappa}] B_{n-1} = (c_{n-1})^{\kappa} B_{n-1} - [c_{n-1}^{\kappa} - (c_{n-2}+1)^{\kappa}] B_{n-2}, \text{ and}$$

$$2.2.3 \quad B_n - [c_n^{\kappa} - (c_{n-1})^{\kappa}] B_{n-1} \geq B_{n-1} - [c_{n-1}^{\kappa} - (c_{n-2}+1)^{\kappa}] B_{n-2}$$

with equality if $c_n = 2$; By successive applications of (2.2.3),

$$B_n - [c_n^{\kappa} - (c_{n-1})^{\kappa}] B_{n-1} \geq B_1 - [c_1^{\kappa} - (c_0-1)^{\kappa}] B_0 \geq (c_0-1)^{\kappa} \geq 1,$$

$$2.2.4 \quad B_n \geq 1 + [c_n^{\kappa} - (c_{n-1})^{\kappa}] B_{n-1}, \quad n \geq 2$$

(with equality if $c_v = 2$ for $v = 1, 2, \dots, n$.)

Write $a_{r+1} = c_r^k - (c_r - 1)^k$; (2.2.4) becomes

$$B_n \geq 1 + a_{n+1} B_{n-1}$$

By repeated applications of this

inequality, $B_n \geq 1 + a_{n+1} + a_{n+1} a_n + a_{n+1} a_n a_{n-1} + \dots + a_{n+1} \dots c_1^k$;

with equality only if $c_r = 2$, $r=1, 2, \dots, n$. Hence, for $n \geq 2$

$$B_n \geq 1 + (2^k - 1) + (2^k - 1)^2 + \dots + (2^k - 1)^{n-1} \cdot 2^k$$

$$B_n \geq 1 + (2^k - 1) + (2^k - 1)^2 + \dots + (2^k - 1)^{n-1} + (2^k - 1)^n$$

As $B_1 = c_1^k \geq 2^k$, This formula is true $n=1$. Thus

$$2.2.5 \quad B_n \geq \frac{(2^k - 1)^{n+1} - 1}{2^k - 2}, \quad n \geq 1,$$

with equality if $c_r = 2$.

The expression (2.2.5) was obtained before, p. 64, (2.2.1) in the case $\xi_0 = 2^k - 1$.

We can state the above result as a minimum problem

Theorem 2.2.3

Let $\left\{ \frac{A_n}{B_n} \right\}$ be the sequence of convergents to the continued fraction of type 2

$$2.2.2 \quad K = c_0^k + \left[- \frac{c_{v-1}^k - (c_{v-1} - 1)^k}{c_v^k} \right]_{v=1}^{\infty}, \quad k \geq 2$$

Then in the domain $c_r \geq 2$, the function $B_n = B_n(c_1, c_2, \dots, c_n)$

satisfies the relation: $B_n \geq \frac{(2^k - 1)^{n+1} - 1}{2^k - 2}$, ($n \geq 1$)

and B_n attains its minimum value at the point $c_r = 2$ ($r=1, \dots, n$).

This minimum value is $\frac{(2^k - 1)^{n+1} - 1}{2^k - 2}$.

The inequality (2.2.4) $B_n \geq 1 + [c_n^k - (c_n - 1)^k] B_{n-1}$,

is also true for $n=1$, for evidently, $B_1 = c_1^k \geq 1 + c_1^k - (c_1 - 1)^k$.

Then, surely, $B_n > [c_n^k - (c_n - 1)^k] B_{n-1}$; ($n \geq 1$). Hence

$$2.2.6 \quad B_n > \prod_{i=1}^n (c_i^k - (c_i - 1)^k) \geq (2^k - 1)^n.$$

Theorem 2.2.4

The continued fraction of type 2

$$2.2.2 \quad K = c_0^n + \left[- \frac{c_{v-1}^n - (c_{v-1} - 1)^n}{c_v^n} \right]_1^\infty, \quad \left\{ \frac{A_n}{B_n} \right\}$$

is convergent for all integral values of $n \geq 2$, and

$$c_0^n > K > c_0^n - \frac{c_0^n - (c_0 - 1)^n}{2^n - 2} > 0.$$

Proof:

$$\text{Write } a_v = c_{v-1}^n - (c_{v-1} - 1)^n, \quad v \geq 1.$$

$$\text{We have: } \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} = (-1)^{v-1} \frac{a_1 a_2 \cdots a_v}{B_v B_{v-1}}, \quad \text{and}$$

$$2.2.7 \quad \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} = - \frac{\prod_{i=1}^v a_i}{B_v B_{v-1}}.$$

Thus, $\frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} < 0$, $\frac{A_v}{B_v} < \frac{A_{v-1}}{B_{v-1}}$, ($v \geq 1$). Hence

$$2.2.8. \quad \frac{A_0}{B_0} > \frac{A_1}{B_1} > \frac{A_2}{B_2} > \cdots > \frac{A_{v-1}}{B_{v-1}} > \frac{A_v}{B_v} > \cdots \geq 0$$

This property of the sequence $\left\{ \frac{A_n}{B_n} \right\}$ is sufficient to insure the existence of a unique limit. We use another proof to obtain additional properties of convergence.

$$\text{From (2.2.7) and (2.2.6), } \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right| = \frac{\prod_{i=1}^v a_i}{B_v B_{v-1}} < \frac{a_1}{(2^n - 1)^v}.$$

As $n \geq 2$, $(2^n - 1) \geq 3$. The series $\sum_{v=1}^{\infty} \frac{1}{(2^n - 1)^v}$ converges, and its value is $\frac{1}{2^n - 2}$. Then

$$2.2.9 \quad \sum_{v=1}^{\infty} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right| < \frac{c_0^n - (c_0 - 1)^n}{2^n - 2}.$$

The series $\sum_{v=1}^{\infty} \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right|$ converges, and by the Criterion 1.4.1, p. 24, $\lim_{n \rightarrow \infty} \frac{A_n}{B_n}$ exists. (The Criterion 1.4.1 applies to continued fractions with positive or negative elements.)

Also, from (2.2.7) and (2.2.8)

$$\frac{A_n}{B_n} = \frac{A_0}{B_0} - \sum_{v=1}^n \left| \frac{A_v}{B_v} - \frac{A_{v-1}}{B_{v-1}} \right|,$$

so that $c_0^n > \lim_{n \rightarrow \infty} \frac{A_n}{B_n} > \frac{c_0^n - (c_0 - 1)^n}{2^n - 1}$.

This establishes Theorem 2.2.4

Corollary 2.2.4 a. of type 2.

The continued fraction of type 2, (2.2.2) is convergent by bounded variation.

This follows by virtue of (2.2.9) above and Definition 1.4.2, p. 23

Corollary 2.2.4 b.

The continued fraction of type 2, (2.2.2) is unconditionally convergent.

This is evident as the continued fractions

$K_n = c_n^n + \left[- \frac{a_\lambda}{c_\lambda^n} \right]_{\lambda=n+1}^\infty$ are all of type 2, with $c_\nu \geq 2$, ($\nu \geq n$) for $n=1, 2, \dots$; and by Definition 1.4.1 p. 23.

We state a theorem on representation.

Theorem 2.2.5

Let the real number $\xi_0 > 0$ have the development of type 2, ($\kappa \geq 2$)

$$2.2.10 \quad \xi_0 \sim c_0^n + \left[- \frac{c_{\nu-1}^\nu - (c_{\nu-1} - 1)^\nu}{c_\nu^\nu} \right]_1^\infty$$

Then a necessary and sufficient condition that (2.2.10) represent ξ_0 is that $\xi_n \geq 2^{\kappa-1}$ for $n \geq 1$, where ξ_n is the n^{th} complete quotient in the development of type 2 of ξ_0 .

Proof: The condition is necessary.

For, if ξ_0 is represented by (2.2.10), then $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = \xi_0$.

We can then write $\xi_0 = c_0^n + \left[- \frac{c_{\nu-1}^\nu - (c_{\nu-1} - 1)^\nu}{c_\nu^\nu} \right]_1^\infty$

$$\text{Also: } \xi_0 = c_0^n - \frac{c_0^n - (c_0 - 1)^\kappa}{1 \cdot c_0^\kappa} - \dots - \frac{c_{\lambda-1}^\kappa - (c_{\lambda-1} - 1)^\kappa}{1 \cdot \xi_\lambda}$$

Thus : (2.2.11)
$$\xi_\lambda = c_\lambda^n - \frac{c_\lambda^n - (c_\lambda - 1)^n}{1 - c_{\lambda+1}^n} - \dots$$
 ($c_\lambda \geq 2$).

It follows that the minimal possible value of ξ_λ is 2^{n-1} .

For, if $1 < \xi_\lambda < 2^{n-1}$, then, the development of type 2 of ξ_λ would converge and represent the number $2^{n-1} \neq \xi_\lambda$.

This contradicts the relation (2.2.11) above. Hence $\xi_\lambda \geq 2^{n-1}$, ($\lambda \geq 1$).

The condition is also sufficient.

We have:
$$\xi_0 = \frac{\xi_n A_{n-1} - [c_{n-1}^n - (c_{n-1} - 1)^n] A_{n-2}}{\xi_n B_{n-1} - [c_{n-1}^n - (c_{n-1} - 1)^n] B_{n-2}}$$
,

and
$$\xi_n = \frac{[c_{n-1}^n - (c_{n-1} - 1)^n] B_{n-2}}{B_{n-1}} \cdot \left(\frac{\frac{A_{n-2} - \xi_0}{B_{n-2}}}{\frac{A_{n-1} - \xi_0}{B_{n-1}}} \right).$$

From the inequality (2.2.4), p.66,

$$B_{n-1} > [c_{n-1}^n - (c_{n-1} - 1)^n] B_{n-2}, \quad n \geq 2.$$

Thus, (2.2.12)
$$1 < \xi_n < \frac{\frac{A_{n-2} - \xi_0}{B_{n-2}}}{\frac{A_{n-1} - \xi_0}{B_{n-1}}}$$
.

By (2.2.8) $\frac{A_0}{B_0} > \frac{A_1}{B_1} > \dots > \frac{A_n}{B_n} > \dots > 0$ so that

$\lim_{n \rightarrow \infty} \frac{A_n}{B_n}$ exists and is unique. By hypothesis, $\xi_n \geq 2^{n-1} > 1$;

Now we prove that $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = \xi_0$.

Suppose, to the contrary, that $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} \neq \xi_0$.

It follows then, from (2.2.12) that

$$\lim_{n \rightarrow \infty} \left(\frac{\frac{A_{n-2} - \xi_0}{B_{n-2}}}{\frac{A_{n-1} - \xi_0}{B_{n-1}}} \right) = 1,$$

and therefore, that $\lim_{n \rightarrow \infty} \xi_n = 1$. This contradicts the

hypothesis that $\xi_n \geq 2^{n-1}$ for all n ; i.e. for $n = 1, 2, \dots$.

Therefore, $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = \xi_0$. This proves Theorem 2.3.5

Chapter 3

3.1 Approximation Theorems

Let the real number $\xi_0 > 0$ have the development of type 1, namely

$$3.1.1 \quad \xi_0 = c_0^\kappa + \frac{a_1}{1c_1^\kappa} + \frac{a_2}{1c_2^\kappa} + \dots \quad ; \quad \left\{ \frac{A_n}{B_n} \right\}$$

where $a_v = (c_{v+1})^\kappa - c_v^\kappa$, ($v \geq 0$) and κ is a positive integer, $\kappa \geq 2$. Note that a_v depends on κ when $\kappa \geq 2$. When $\kappa = 1$, we have $a_v = 1$, that is, we have the regular continued fraction development.

From the Appell Algorithm, we obtain the relation

$$3.1.2 \quad \xi_0 = c_0^\kappa + \frac{a_1}{1c_1^\kappa} + \dots + \frac{a_v}{1\xi_v} \quad , \quad (v \geq 1)$$

where ξ_v is the v^{th} complete quotient, $\xi_v > 1$, and with

$$\xi_v \approx c_v^\kappa + \frac{a_{v+1}}{1c_{v+1}^\kappa} + \dots$$

By Theorem 1.2.1, (On Convergence and representation), we

can write : (3.1.3) $\xi_v = c_v^\kappa + \frac{a_{v+1}}{1c_{v+1}^\kappa} + \dots$

Let n be an arbitrary index, $n \geq 1$. Then from (3.1.2)

$$\xi_0 = \frac{\xi_{n+1} A_n + a_{n+1} A_{n-1}}{\xi_{n+1} B_n + a_{n+1} B_{n-1}} \quad , \quad \text{or}$$

$$\xi_{n+1} (\xi_0 B_n - A_n) = -a_{n+1} (\xi_0 B_{n-1} - A_{n-1}) \quad , \quad \text{and}^*$$

Definition 3.1.1.

We shall write $\theta_n = \left| \xi_0 - \frac{A_n}{B_n} \right|$ for $n = 0, 1, \dots$

and call $\{\theta_n\}$ the error sequence corresponding to the development (3.1.1).

$$* \quad 3.1.4 \quad \left| \xi_0 - \frac{A_n}{B_n} \right| = \frac{a_{n+1} B_{n-1}}{\xi_{n+1} B_n} \left| \xi_0 - \frac{A_{n-1}}{B_{n-1}} \right| \quad ; \quad n \geq 1$$

Theorem 3.1.1

(a) The error-sequences $\{\theta_{2n}\}$, $\{\theta_{2n+1}\}$ both decrease steadily to zero.

(b) Of two successive convergents to the development of type 1 of $\xi_0 > 0$, at least one of them has the property that the error $\theta_n = \left| \xi_0 - \frac{A_n}{B_n} \right| < \frac{1}{2} \frac{a_1 a_2 \dots a_{n+1}}{B_n^2}$

Proof: (a) We have, from Lemma 1.2.1, (p. 3).

$$(1) \quad \frac{A_0}{B_0} < \frac{A_2}{B_2} < \dots < \frac{A_{2n}}{B_{2n}} < \dots < \xi_0 < \dots < \frac{A_{2n+1}}{B_{2n+1}} < \dots < \frac{A_1}{B_1}$$

$$\text{Hence, } \left| \xi_0 - \frac{A_{n+1}}{B_{n+1}} \right| < \left| \xi_0 - \frac{A_{n-1}}{B_{n-1}} \right|, \quad n \geq 1,$$

$$\text{Also, } \left| \xi_0 - \frac{A_{n+1}}{B_{n+1}} \right| < \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| = \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}},$$

$$\text{and } u_{n+1} = \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}} < \left(\frac{2^k - 1}{2^k} \right) \cdot u_n, \quad (n \geq 1),$$

by (1.2.5). Then $u_{n+1} < \left(\frac{2^k - 1}{2^k} \right)^n \cdot \frac{a_1}{C_1^k}$, and $\lim_{n \rightarrow \infty} u_{n+1} = 0$. Hence

$\lim_{n \rightarrow \infty} \theta_n = 0$, and from (1) above, $\theta_{2n} \downarrow 0$, $\theta_{2n+1} \downarrow 0$.

$$(b) \text{ We have } \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| = \left| \frac{A_{n+1}}{B_{n+1}} - \xi_0 + \xi_0 - \frac{A_n}{B_n} \right|$$

$$\text{Then } \left| \xi_0 - \frac{A_{n+1}}{B_{n+1}} \right| + \left| \xi_0 - \frac{A_n}{B_n} \right| = \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}}$$

$$\text{or } \theta_{n+1} + \theta_n = \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}}, \quad (n \geq 0).$$

We can not have $\theta_{n+1} > \frac{1}{2} \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}}$, and

$$\theta_n > \frac{1}{2} \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}}. \text{ Hence, one of them is } \leq \frac{1}{2} \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}}.$$

Now $B_{n+1} > B_n$, so that surely, of two successive errors

$$\theta_{n+1}, \theta_n \text{ one of them is less than } \frac{1}{2} \frac{a_1 a_2 \dots a_{n+1}}{B_n^2}.$$

~~Remark~~ ~~Theorem 3.1.1~~ is an extension of Theorem

Remark Theorem 3.1.1 (b) is an extension of Theorem 14, p.48, Perron, O, (1) "Kettenbrüche", which is the case $\kappa=1$. This is due to K. Vahlen,

"Über Näherungsewerte und Kettenbrüche", Journal für die reine und angewandte Math. v.115 (1895) p48.

Corollary 3.1.1

Of two successive errors, θ_n, θ_{n+1} at least one of them is less than $\frac{1}{2} \left(\frac{2^{\kappa}-1}{2^{\kappa}} \right)^n \cdot \frac{a_1}{C_1^{\kappa}}$.

(b) If for a certain index $n \geq 1$, $\theta_{n+1} > \theta_n$, then: $\theta_n < \theta_{n-1}$, $\theta_{n+2} < \theta_{n+1}$.

Proof: (a) This is immediate from Theorem 3.1.1 (b),

and the previous result (1.2.5), (repeated in the Proof

of Theorem 3.1.1) that $u_{n+1} = \frac{a_1 a_2 \dots a_{n+1}}{B_n B_{n+1}} < \frac{(2^{\kappa}-1)^n}{(2^{\kappa})^n} \cdot \frac{a_1}{C_1^{\kappa}}$.

(b) From Theorem 3.1.1, p.72; if $\theta_{n+1} > \theta_n$, we obtain readily that $\theta_{n-1} > \theta_{n+1} > \theta_n > \theta_{n+2}$. Hence $\theta_n < \theta_{n-1}$, and $\theta_{n+2} < \theta_{n+1}$.

This corollary 3.1.1 b shows that there can not be two consecutive errors θ_n, θ_{n+1} , each greater than the preceding one. That is, we can not have both $\left\{ \begin{array}{l} \theta_n > \theta_{n-1} \\ \theta_{n+1} > \theta_n \end{array} \right.$.

Theorem 3.1.2.

Of three successive convergents to the continued fraction development of a number ξ_0 at least

one has the property that the error $\theta_n = \left| \xi_0 - \frac{A_n}{B_n} \right| < \frac{a_1 a_2 \dots a_{n+1}}{\sqrt{2^{\kappa+2}-3} B_n^2}$.

In particular, for those ξ_0 having infinite developments of type 1, there are infinitely many convergents having the the above property.

Proof : From the formula

$$\xi_v = \frac{\xi_v A_{v-1} + a_v A_{v-2}}{\xi_v B_{v-1} + a_v B_{v-2}}, \quad v \geq 1 \quad \left(\begin{array}{l} A_{-1} = 1 \\ B_{-1} = 0 \end{array} \right)$$

we obtain $\theta_{v-1} = \frac{a_1 a_2 \dots a_v}{B_{v-1} (\xi_v B_{v-1} + a_v B_{v-2})}$; or $\theta_{v-1} = \frac{a_1 a_2 \dots a_v}{B_{v-1}^2 (\xi_v + \frac{a_v B_{v-2}}{B_{v-1}})}$

The theorem will be proved if we can show that of three successive indices, there is at least one such that

$$\xi_v + \frac{a_v B_{v-2}}{B_{v-1}} > \sqrt{2^{v+2} - 3}.$$

Write $\phi_v = \frac{B_{v-2}}{B_{v-1}}$, ($v = 1, 2, \dots$) ; $B_{-1} = 0$.

It remains to be shown that for every positive integer λ , at least one of the numbers

$$(I) \quad \xi_\lambda + a_\lambda \phi_\lambda, \quad \xi_{\lambda+1} + a_{\lambda+1} \phi_{\lambda+1}, \quad \xi_{\lambda+2} + a_{\lambda+2} \phi_{\lambda+2}$$

is greater than $\sqrt{2^{\lambda+2} - 3}$.

Now for $v \geq 0$, $\xi_v = c_v + \frac{a_{v+1}}{\xi_{v+1}}$, and

$$\frac{1}{\phi_{v+1}} = \frac{B_v}{B_{v-1}} = c_v + \frac{a_v B_{v-2}}{B_{v-1}} = c_v + a_v \phi_v.$$

That is :

$$(II) \quad \frac{1}{\phi_{v+1}} = c_v + a_v \phi_v, \quad (v \geq 1).$$

Thus,

$$\xi_v = \frac{1}{\phi_{v+1}} - a_v \phi_v + \frac{a_{v+1}}{\xi_{v+1}}, \quad \text{or}$$

$$(III) \quad \xi_v + a_v \phi_v = \frac{a_{v+1}}{\xi_{v+1}} + \frac{1}{\phi_{v+1}}, \quad (v \geq 1).$$

Assume, on the contrary, that the three numbers (I) are all not greater than $\sqrt{2^{\lambda+2} - 3}$. Then we can write

$$(IV) \quad \left\{ \begin{array}{l} \xi_\lambda + a_\lambda \phi_\lambda = \frac{a_{\lambda+1}}{\xi_{\lambda+1}} + \frac{1}{\phi_{\lambda+1}} \leq \sqrt{2^{\lambda+2} - 3} \\ \xi_{\lambda+1} + a_{\lambda+1} \phi_{\lambda+1} = \frac{a_{\lambda+2}}{\xi_{\lambda+2}} + \frac{1}{\phi_{\lambda+2}} \leq \sqrt{2^{\lambda+2} - 3} \\ \xi_{\lambda+2} + a_{\lambda+2} \phi_{\lambda+2} = \frac{a_{\lambda+3}}{\xi_{\lambda+3}} + \frac{1}{\phi_{\lambda+3}} \leq \sqrt{2^{\lambda+2} - 3} \end{array} \right.$$

By addition, we obtain from IV ,

$$V \quad \begin{cases} \bar{S}_{\lambda+1} + \frac{a_{\lambda+1}}{\bar{S}_{\lambda+1}} + a_{\lambda+1} \phi_{\lambda+1} + \frac{1}{\phi_{\lambda+1}} \leq 2\sqrt{2^{k+2}-3} \\ \bar{S}_{\lambda+2} + \frac{a_{\lambda+2}}{\bar{S}_{\lambda+2}} + a_{\lambda+2} \phi_{\lambda+2} + \frac{1}{\phi_{\lambda+2}} \leq 2\sqrt{2^{k+2}-3} \end{cases}$$

$$\text{As } a_{v+1} = (c_v + 1)^k - c_v^k \geq 2^k - 1, \quad v \geq 0,$$

we have :

$$VI \quad \begin{cases} \bar{S}_{\lambda+1} + \frac{2^k - 1}{\bar{S}_{\lambda+1}} + (2^k - 1) \phi_{\lambda+1} + \frac{1}{\phi_{\lambda+1}} \leq 2\sqrt{2^{k+2}-3} \\ \bar{S}_{\lambda+2} + \frac{2^k - 1}{\bar{S}_{\lambda+2}} + (2^k - 1) \phi_{\lambda+2} + \frac{1}{\phi_{\lambda+2}} \leq 2\sqrt{2^{k+2}-3} \end{cases}$$

Now it can be shown that, of two successive indices, there is at least one for which $\phi_v < \frac{1}{2^k - 1} \cdot \frac{\sqrt{2^{k+2}-3} - 1}{2}$.

For, if the contrary were true, then

$$\phi_v \geq \frac{1}{2^k - 1} \cdot \frac{\sqrt{2^{k+2}-3} - 1}{2}; \quad \phi_{v+1} \geq \frac{1}{2^k - 1} \cdot \frac{\sqrt{2^{k+2}-3} - 1}{2}.$$

$$\text{Thus: } \frac{1}{\phi_{v+1}} \leq (2^k - 1) \cdot \frac{2}{\sqrt{2^{k+2}-3} - 1} = \frac{\sqrt{2^{k+2}-3} + 1}{2}$$

$$\text{or } \frac{1}{\phi_{v+1}} \leq \frac{\sqrt{2^{k+2}-3} + 1}{2}.$$

Now $(2^{k+2} - 3)$ is an odd number, and $(2^{k+2} - 3) = 4(2^k - 1) + 1$.

As every odd square must be of the form $8N + 1$, and as $2^k - 1$, is odd, it is apparent that $2^{k+2} - 3$ can never be the square of an integer. Also, as $\phi_{v+1} = \frac{B_{v+1}}{B_v}$ is rational, ($v \geq 0$)

we must have strict inequality in the last expression

~~written above~~ p. 74: that is: $\frac{1}{\phi_{v+1}} < \frac{\sqrt{2^{k+2}-3} + 1}{2}$

From (II) p. 74, $\frac{1}{\phi_{v+1}} = c_v^k + a_v \phi_v$;

$$\text{Also: } a_v \phi_v \geq \frac{a_v}{2^k - 1} \cdot \frac{\sqrt{2^{k+2}-3} - 1}{2} \geq \frac{\sqrt{2^{k+2}-3} - 1}{2}$$

$$\text{Thus; } \frac{\sqrt{2^{n+2}-3}+1}{2} > \frac{1}{\phi_{n+1}} \geq c_v^n + \frac{\sqrt{2^{n+2}-3}-1}{2}$$

$$\text{or } \frac{\sqrt{2^{n+2}-3}+1}{2} > c_v^n + \frac{\sqrt{2^{n+2}-3}-1}{2}$$

Hence, $c_v^n < 1$. This is impossible. Therefore our assumption is false, and of two successive indices, at least one is such that $\phi_v < \frac{1}{2^{n-1}} \frac{\sqrt{2^{n+2}-3}-1}{2}$.

Write $\phi_{\lambda+i} < \frac{1}{2^{n-1}} \frac{\sqrt{2^{n+2}-3}-1}{2}$, where i is one of the numbers 1, 2. Now $\frac{1}{2^{n-1}} \frac{\sqrt{2^{n+2}-3}-1}{2} = \frac{2}{\sqrt{2^{n+2}-3}+1} < 1$

so that $\phi_{\lambda+i}$ satisfies the relation

$$(VII) \quad \phi_{\lambda+i} < \frac{2}{\sqrt{2^{n+2}-3}+1}$$

The function $f(x) = (2^{n-1})x + \frac{1}{x}$ is decreasing in the interval $0 < x < \frac{1}{\sqrt{2^{n-1}}}$, as its derivative is negative then.

$$\text{As } \frac{2}{\sqrt{2^{n+2}-3}+1} < \frac{2}{\sqrt{2^{n+2}-4}} = \frac{1}{\sqrt{2^{n-1}}}$$

we have from VII that $\phi_{\lambda+i} < \frac{2}{\sqrt{2^{n+2}-3}+1} < \frac{1}{\sqrt{2^{n-1}}}$

Hence, from the remark above concerning the function

$(2^{n-1})x + \frac{1}{x}$, we have:

$$\text{or } (2^{n-1}) \phi_{\lambda+i} + \frac{1}{\phi_{\lambda+i}} > \frac{(2^{n-1}) (\sqrt{2^{n+2}-3}-1)}{(2^{n-1}) \cdot 2} + \frac{\sqrt{2^{n+2}-3}+1}{2}$$

$$VIII \quad (2^{n-1}) \phi_{\lambda+i} + \frac{1}{\phi_{\lambda+i}} > \sqrt{2^{n+2}-3}$$

$$\text{From IV, as } a_{\lambda+i} \geq 2^{n-1}; \quad \frac{2^{n-1}}{\sum_{\lambda+i}} \leq \frac{a_{\lambda+i}}{\sum_{\lambda+i}}$$

and, using VII, $\leq \sqrt{2^{n+2}-3} - \frac{1}{\phi_{\lambda+i}}$

$$\frac{2^{n-1}}{\sum_{\lambda+i}} < \sqrt{2^{n+2}-3} - \frac{\sqrt{2^{n+2}-3}+1}{2} = \frac{\sqrt{2^{n+2}-3}-1}{2}$$

$$\text{or } \frac{1}{S_{\lambda+i}} < \left(\frac{1}{2^k-1}\right) \cdot \frac{\sqrt{2^{k+2}-3}-1}{2} = \frac{2}{\sqrt{2^{k+2}-3}+1}$$

That is, $S_{\lambda+i}$ satisfies the relations

$$\text{IX} \quad S_{\lambda+i} > \frac{\sqrt{2^{k+2}-3}+1}{2} = (2^k-1) \frac{2}{\sqrt{2^{k+2}-3}-1}$$

The function $g(x) = x + \frac{2^k-1}{x}$ is increasing for $x > \sqrt{2^k-1}$, as its derivative is then positive.

$$\text{From IX, } S_{\lambda+i} > \frac{\sqrt{2^{k+2}-3}+1}{2} > \frac{\sqrt{2^{k+2}-4}}{2}$$

$$\text{and } S_{\lambda+i} > \sqrt{2^k-1}.$$

$$\text{Hence, } S_{\lambda+i} + \frac{2^k-1}{S_{\lambda+i}} > \frac{\sqrt{2^{k+2}-3}+1}{2} + \frac{(2^k-1)\sqrt{2^{k+2}-3}-1}{(2^k-1) \cdot 2}$$

or

$$\text{X} \quad S_{\lambda+i} + \frac{2^k-1}{S_{\lambda+i}} > \sqrt{2^{k+2}-3}.$$

Adding VIII and X we obtain

$$S_{\lambda+i} + \frac{2^k-1}{S_{\lambda+i}} + (2^k-1)\varphi_{\lambda+i} + \frac{1}{\varphi_{\lambda+i}} > 2\sqrt{2^{k+2}-3}$$

As $i=1$ or 2 , the relation just obtained contradicts VI

p. 75.

Hence, our original assumption is false. Therefore,

of the three numbers: $S_{\lambda} + a_{\lambda} \varphi_{\lambda}$, $S_{\lambda+1} + a_{\lambda+1} \varphi_{\lambda+1}$, $S_{\lambda+2} + a_{\lambda+2} \varphi_{\lambda+2}$, at least one of them is greater than $\sqrt{2^{k+2}-3}$.

This proves Theorem 3.1.2

Remark : Theorem 3.1.2 is an extension of Theorem 15, p. 49,

- (i) Perron, O: "Kettenbrüche", which is the case $k=1$. For $k=1$, the theorem was first remarked by Borel, E, "Contributions à l'analyse arithmétique du continu"; Journal de Mathématiques pures et appliquées (5) 9, 1903, p 49

See also ; Hurwitz, A, "Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche", Math-Annalen, v. 39 (1891) p. 49.

Lemma 3.1.3.

If, in the development $\xi_0 = c_0^r + \left[\frac{a_r}{c_r^r} \right]_1^\infty$ we have either $\xi_n \leq 2c_n^r$, or $\xi_{n+1} \geq \frac{a_{n+1}}{c_n^r} = \left(1 + \frac{1}{c_n}\right)^r - 1$, for a certain index $n \geq 1$, then $\theta_n < \theta_{n-1}$.

Proof:

First we show that the two conditions are equivalent. For, as $\xi_n = c_n^r + \frac{a_{n+1}}{\xi_{n+1}}$, if $\xi_n \leq 2c_n^r$, then $\frac{a_{n+1}}{\xi_{n+1}} \leq c_n^r$, and $\xi_{n+1} \geq \frac{a_{n+1}}{c_n^r}$.

Conversely, if $\xi_{n+1} \geq \frac{a_{n+1}}{c_n^r}$, then $\xi_n \leq 2c_n^r$.

From (3.1.4) page 74, we have (i) $\theta_n = \frac{a_{n+1}}{\xi_{n+1}} \cdot \frac{B_{n-1}}{B_n} \cdot \theta_{n-1}$

As $B_n > c_n^r B_{n-1}$, that is, $\frac{B_{n-1}}{B_n} < \frac{1}{c_n^r}$, and by hypothesis $\frac{a_{n+1}}{\xi_{n+1}} \leq c_n^r$, we obtain $\frac{a_{n+1}}{\xi_{n+1}} \cdot \frac{B_{n-1}}{B_n} < 1$. Hence $\theta_n < \theta_{n-1}$

for that particular index n.

Theorem 3.1.3 a

If, in the development $\xi_0 = c_0^r + \left[\frac{a_r}{c_r^r} \right]_1^\infty$, we have $c_n \geq \frac{1}{2^{1/r} - 1}$ for a certain index n, then $\theta_n < \theta_{n-1}$.

Proof: If $c_n \geq \frac{1}{2^{1/r} - 1}$, then $\frac{1}{c_n} \leq 2^{1/r} - 1$, and $\left(\frac{c_n+1}{c_n}\right)^r \leq 2$.

That is, $(c_n+1)^r < 2c_n^r$. Now, $c_n^r \leq \xi_n < (c_n+1)^r$, so that $\xi_n < 2c_n^r$. Hence, by Lemma 3.1.3, $\theta_n < \theta_{n-1}$.

Theorem 3.1.3 (b)

If, in the development $\xi_0 = c_0^r + \left[\frac{a_r}{c_r^r} \right]_1^\infty$, we have $c_{n+1} \geq 2$

for a certain index n , then $\theta_n < \theta_{n-1}$ for that index.

Proof:

From Lemma 3.1.3, $\theta_n < \theta_{n-1}$ if $\xi_{n+1} \geq (1 + \frac{1}{c_n})^r - 1$.

Now, $\max (1 + \frac{1}{c_n})^r - 1 = 2^r - 1$. If $c_{n+1} \geq 2$, then $c_{n+1}^r \geq 2^r$,

As $\xi_{n+1} \geq c_{n+1}^r$, surely, $\xi_{n+1} > (1 + \frac{1}{c_n})^r - 1$. Hence $\theta_n < \theta_{n-1}$.

Remark

The above Theorems 3.1.3 a and (b)

enables one to compare the errors θ_n, θ_{n-1} , without a knowledge of the complete quotients ξ_n, ξ_{n+1} ; instead, one need know only the elements of the continued fraction.

Remarks and Examples 3.1.1

1. We point out that the condition $\xi_n \leq 2c_n^r$ in Lemma 3.1.3 is a sufficient one but not a necessary one that $\theta_n < \theta_{n-1}$. For example, in section 1.7, Example 2, pages 34, 35, in the development of $\xi_0 = 6$, we obtained $\xi_{5r} = 6$, $\xi_{5r+1} = \frac{5}{2}$, $\xi_{5r+2} = 2$, $\xi_{5r+3} = 3$, $\xi_{5r+4} = \frac{3}{2}$, for $r \geq 0$. Now $\xi_{5r+1} = \frac{5}{2} > 2c_{5r+1}^2 = 2$; yet we saw there (p.38) that $\theta_{5r+1} < \theta_{5r}$ for $r \geq 1$, but not for $r=0$. Similarly for $\xi_{5r+3} = 3 > 2c_{5r+3}^2 = 2$.

2. As an illustration of the Lemma 3.1.3, we give the following: ($r=2$). (a) $\xi_0 = 24 = 4^2 + \frac{9}{\xi_1}$

$$\xi_1 = \frac{9}{8} = 1 + \frac{3}{\xi_2}; \quad \xi_2 = 24 = \xi_0$$

Thus, $\xi_{2r} = 24$, $\xi_{2r+1} = \frac{9}{8}$ for $r = 0, 1, 2, \dots$

and $\xi_0 = 4^2 + \frac{9}{11^2} + \frac{31}{14^2} + \frac{91}{11^2} + \dots$

As $\xi_{2v} = 24$, $c_{2v} = 4$, we have $\xi_{2v} < 2c_{2v}^2$

Similarly, $\xi_{2v+1} = \frac{9}{8} < 2c_{2v+1}^2 = 2$ for $v \geq 0$

Therefore, by the Lemma 3.1.3, we have $\theta_n < \theta_{n-1}$,

for $n \geq 1$

for the above continued fraction.

(b) From section 1.7, Example 3, page 39, we obtained the development :

$$\xi_0 = 7 = 2^2 + \frac{5/1}{1/1^2} + \frac{3/1}{1/2^2} + \frac{5/1}{1/3^2} + \frac{7/1}{1/2^2} + \frac{5/1}{1/1^2} + \dots$$

with $\xi_1 = \frac{5}{3}$, $\xi_2 = \frac{9}{2}$, $\xi_3 = 10$, $\xi_4 = 10$. We have then:

$$\xi_{4v} = 7 < 2c_{4v}^2 = 8 \quad ; \quad \xi_{4v+2} = \frac{9}{2} < 2c_{4v+2}^2 = 8$$

$$\xi_{4v+1} = \frac{5}{3} < 2c_{4v+1}^2 = 2 \quad ; \quad \xi_{4v+3} = 10 < 2c_{4v+3}^2 = 18.$$

Hence, by Lemma 3.1.3, $\theta_n < \theta_{n-1}$ ($n \geq 1$) for the above continued fraction development.

$$3. \quad \text{We have } \frac{a_{n+1}}{c_n^k} = \frac{(c_n+1)^k - c_n^k}{c_n^k} = \left(1 + \frac{1}{c_n}\right)^k - 1$$

and $\left(1 + \frac{1}{c_n}\right)^k - 1 \leq 2^k - 1$; if $\xi_{n+1} \geq 2^k - 1$,

then surely $\theta_n < \theta_{n-1}$, by Lemma 3.1.3.

For the particular value $k=2$, this condition becomes:

If for an index n , $\xi_{n+1} \geq 3$, then $\theta_n < \theta_{n-1}$.

We note that for $\xi_0 = 5$, $\xi_v = \xi_0 = 5$,

and $\xi_0 = 5 = 2^2 + \frac{5/1}{1/2^2} + \frac{5/1}{1/2^2} + \dots$. Then, as $\xi_v > 3$,

$\theta_v < \theta_{v-1}$, ($v=1, 2, \dots$) by the above result. This

verifies the results in Example 1, (1.7) page 34.

4. Let $k=2$. We have $2c_n^2 - (c_n+1)^2 = (c_n-1)^2 - 2$,

and $(c_n-1)^2 - 2 > 0$ if $c_n \geq 3$. Thus, if $c_n \geq 3$,

then $2c_n^2 > (c_n+1)^2$. Now, as $c_n^2 \leq \xi_n < (c_n+1)^2$,

We shall always have $\bar{\xi}_n < 2c_n^2$ if $c_n \geq 3$.

Hence, if $c_n \geq 3$, $\theta_n < \theta_{n-1}$ by Lemma 3.1.3.

Therefore, we need ~~only~~ investigate only those (θ_n, θ_{n-1}) where $c_n \leq 2$.

If $c_n = 1$ and $1 \leq \bar{\xi}_n \leq 2$, or if $c_n = 2$ and $4 < \bar{\xi}_n \leq 8$ then $\theta_n < \theta_{n-1}$ by Lemma 3.1.3. Thus, there are only two instances where we can not tell readily the behavior of two successive errors θ_n, θ_{n-1} , namely, where $4 > \bar{\xi}_n > 2$, and $9 > \bar{\xi}_n > 8$.

As an example of this, let us examine the continued fraction development of type 1 of $\bar{\xi}_0 = 6$, Example 2, p.35

$$\bar{\xi}_0 = 6 = 2^2 + \frac{5/1}{1/1^2} + \frac{3/1}{1/1^2} + \frac{3/1}{1/1^2} + \frac{3/1}{1/1^2} + \frac{3/1}{1/2^2} + \frac{5/1}{1/1^2} + \dots$$

Here, $\bar{\xi}_{5v} = 6$, $c_{5v} = 2$; $\bar{\xi}_{5v+1} = 5/2$, $c_{5v+1} = 1$
 $\bar{\xi}_{5v+2} = 2$, $c_{5v+2} = 1$; $\bar{\xi}_{5v+3} = 3$, $c_{5v+3} = 1$
 $\bar{\xi}_{5v+4} = 3/2$; $c_{5v+4} = 1$, for $v \geq 0$.

Clearly, $\bar{\xi}_{5v} < 2c_{5v}^2$; $\bar{\xi}_{5v+2} \leq 2c_{5v+2}^2$, and $\bar{\xi}_{5v+4} < 2c_{5v+4}^2$, so that, by Lemma 3.1.3, $\theta_{5v} < \theta_{5v-1}$, $\theta_{5v+2} < \theta_{5v+1}$, and $\theta_{5v+4} < \theta_{5v+3}$ for $v \geq 0$. However, as $4 > \bar{\xi}_{5v+1} > 2$ and $4 > \bar{\xi}_{5v+3} > 2$, Lemma 3.1.3 gives no information about the behavior of the sequences $\{\theta_{5v+1}, \theta_{5v}\}$ and $\{\theta_{5v+3}, \theta_{5v+2}\}$

A criterion, Theorem 3.1.4, will be given later on p. 82 which will enable us to get the additional information about the behavior of these error - sequences.

5. Consider the continued fraction development of type 1 of $\bar{\xi}_0 = \frac{1 + \sqrt{13}}{2}$; $\bar{\xi}_0 = 1^2 + \frac{3/1}{1/1^2} + \frac{3/1}{1/1^2} + \dots$

Here, $\xi_0 = \xi_v = \frac{1+\sqrt{13}}{2}$, and $c_v = 1$, ($v \geq 1$)

Then, as $2 < \frac{1+\sqrt{13}}{2} < 3$, $\xi_v > 2c_v^2$.

Lemma 3.1.3 gives no information about $\{\theta_n\}$

In this example, we have that $B_n = A_{n-1}$, ($n \geq 0$)
for the above continued fraction. Also from (3.1.4), p.71,

namely, $\theta_n = \frac{a_{n+1}}{\xi_{n+1}} \cdot \frac{B_{n-1}}{B_n} \theta_{n-1}$, we obtain:

$$\theta_n = \frac{3}{\xi_0} \cdot \frac{B_{n-1}}{A_{n-1}} \cdot \theta_{n-1}. \quad \text{When } n = 1, \text{ this relation}$$

becomes: $\theta_1 = \frac{3}{\xi_0} \theta_0$. But $\xi_0 < 3$. Hence $\theta_1 > \theta_0$.

In general, from $\theta_n = \frac{3}{\xi_0} \cdot \frac{B_{n-1}}{A_{n-1}} \theta_{n-1}$, we will have

$$\theta_n < \theta_{n-1}, \quad \text{when } \frac{3}{\xi_0} < \frac{A_{n-1}}{B_{n-1}}. \quad \text{As } \xi_0 = \frac{1+\sqrt{13}}{2} > 2$$

$$\frac{3}{\xi_0} < \frac{3}{2}. \quad \text{Also } \frac{A_2}{B_2} = 1 + \frac{3}{1+3}, \quad \text{and } \frac{A_1}{B_1} = 1 + \frac{3}{4} > \frac{3}{2}$$

Finally, $\frac{A_{n-1}}{B_{n-1}} \geq \frac{A_2}{B_2} > \frac{3}{\xi_0}$, $n \geq 2$.

Therefore, $\theta_n < \theta_{n-1}$, $n \geq 2$

Theorem 3.1.4

if, for the continued fraction development
of type 1 of ξ_0 , $\xi_0 = c_0^k + \left[\frac{a_v}{c_1^k} \right]_1^\infty$,

we have $\theta_n < \theta_{n-1}$ for a certain index n , and in
addition

$$(3.1.5) \quad \frac{\xi_{n+2}}{c_{n+2}^k} - \frac{\xi_{n+1}}{c_{n+1}^k} \leq 1,$$

then $\theta_{n+2} < \theta_{n+1}$

Proof:

We have: $\xi_{n+2} = c_{n+2}^k + \frac{a_{n+3}}{\xi_{n+3}}$, or

$$3.1.6 \quad \xi_{n+3} = \frac{a_{n+3}}{\xi_{n+2} - c_{n+2}^k} = \frac{a_{n+3}}{c_{n+2}^k + \left(\frac{a_{n+2}}{\xi_{n+1} - c_{n+1}^k} - 2c_{n+2}^k \right)}$$

In (3.1.5), put $\xi_{n+2} = \frac{a_{n+2}}{\xi_{n+1} - c_{n+1}^k}$ obtaining:

$$\frac{1}{c_{n+2}^k} \left(\frac{a_{n+2}}{\xi_{n+1} - c_{n+1}^k} \right) \leq 1 + \frac{\xi_{n+1}}{c_{n+1}^k} = \frac{\xi_{n+1} + c_{n+1}^k}{c_{n+1}^k}$$

$$\text{or, } \frac{a_{n+2} \cdot 2c_{n+1}^k}{(\xi_{n+1} - c_{n+1}^k)(\xi_{n+1} + c_{n+1}^k)} \leq 2c_{n+2}^k \quad \text{or,}$$

$$\frac{a_{n+2}}{\xi_{n+1} - c_{n+1}^k} - \frac{a_{n+2}}{\xi_{n+1} + c_{n+1}^k} \leq 2c_{n+2}^k \quad \text{That is, } \curvearrowright$$

Thus, (3.1.6) becomes:

$$\frac{a_{n+2}}{\xi_{n+1} - c_{n+1}^k} - 2c_{n+2}^k \leq \frac{a_{n+2}}{\xi_{n+1} + c_{n+1}^k}$$

$$3.1.7 \quad \xi_{n+3} \geq \frac{a_{n+3}}{c_{n+2}^k + \frac{a_{n+2}}{c_{n+1}^k + \xi_{n+1}}}$$

$$\text{We have: } \theta_n = \frac{a_{n+1} B_{n-1}}{\xi_{n+1} B_n} \cdot \theta_{n-1} \quad [3.1.4, p. 73]$$

and, as $\theta_n < \theta_{n-1}$, by hypothesis, we obtain $\frac{a_{n+1} B_{n-1}}{\xi_{n+1} B_n} < 1$,

or $\xi_{n+1} > \frac{a_{n+1} B_{n-1}}{B_n}$. Put this in (3.1.7), obtaining:

$$3.1.8 \quad \xi_{n+2} > \frac{a_{n+3}}{c_{n+2}^k + \frac{a_{n+2}}{c_{n+1}^k + \frac{a_{n+1} B_{n-1}}{B_n}}} = \frac{a_{n+3} B_{n+1}}{B_{n+2}}$$

$$\text{But } \theta_{n+2} = \frac{a_{n+3} B_{n+1}}{\xi_{n+3} B_{n+2}} \cdot \theta_{n+1} \quad \text{Hence, by (3.1.8)}$$

$\theta_{n+2} < \theta_{n+1}$. This proves Theorem 3.1.4 .

Applications of Theorem 3.1.4

1. In the Example 4, (pages 80, 81) we considered the continued fraction development of type 1 of $\xi_0 = 6$, $k=2$,

$$\xi_0 = 6 = 2^2 + \frac{51}{112} + \frac{31}{112} + \frac{31}{112} + \frac{31}{112} + \frac{31}{12^2} + \frac{51}{112} + \dots$$

$$\text{with } \begin{cases} \xi_{5v} = 6 & , & \xi_{5v+1} = 5/2 & , & \xi_{5v+2} = 2 & , & \xi_{5v+3} = 3 \\ c_{5v} = 2 & & c_{5v+1} = 1 & & c_{5v+2} = 1 & , & c_{5v+3} = 1 \\ \xi_{5v+4} = 3/2 & , & c_{5v+4} = 1 & . \end{cases}$$

and we could get no information from Lemma 3.1.3 regarding the behavior of the error - sequences $\{\theta_{5v+1}, \theta_{5v}\}$ and $\{\theta_{5v+3}, \theta_{5v+2}\}$ for $v \geq 1$. The above theorem 3.1.4 allows

us to compare these errors. We know that $\theta_{s_{n+4}} < \theta_{s_{n+3}}$ for $n \geq 0$
 (by Lemma 3.1.3), i.e. $\theta_{s_{n-1}} < \theta_{s_{n-2}}$, ($n \geq 1$)

$$\text{Now } \frac{\xi_{s_{n+1}}}{c_{s_{n+1}}^2} - \frac{\xi_{s_n}}{c_{s_n}^2} = \frac{5}{2} - \frac{6}{4} = 1.$$

Hence, by Theorem 3.1.4, $\theta_{s_{n+1}} < \theta_{s_n}$ for $n \geq 1$

We can now apply this result to compare $\theta_{s_{n+3}}$ and $\theta_{s_{n+2}}$

$$\text{We have : } \frac{\xi_{s_{n+3}}}{c_{s_{n+3}}^2} - \frac{\xi_{s_{n+2}}}{c_{s_{n+2}}^2} = 3 - 2 = 1.$$

Hence, $\theta_{s_{n+3}} < \theta_{s_{n+2}}$ for $n \geq 1$.

The behavior of the error - sequence $\{\theta_n\}$ for $\xi_0 = 6$
 is summarized here : $\theta_1 > \theta_0$, $\theta_2 < \theta_1$, $\theta_3 > \theta_2$
 and for $n \geq 4$, $\theta_n < \theta_{n-1}$

2. In the Example 5, p. 81, 82, we considered the
 continued fraction development of type 1 of $\xi_0 = \frac{1 + \sqrt{13}}{2}$; ($H=2$)

$$\xi_0 = 1 + \frac{3}{1} + \frac{3}{1} + \frac{3}{1} + \dots$$

$$\text{Here, } \xi_0 = \xi_n = \frac{1 + \sqrt{13}}{2}, \quad c_0 = c_n = 1.$$

The criterion, Lemma 3.1.3, did not give any results
 concerning the error - sequence $\{\theta_n\}$. The Theorem 3.1.4
 gives some information. We saw in Example 5 that $\theta_1 > \theta_0$
 and consequently, $\theta_2 < \theta_1$, (by Corollary 3.1.1 b, p.73)
 and also, $\theta_3 < \theta_2$.

$$\text{Now } \frac{\xi_{n+1}}{c_{n+1}^2} - \frac{\xi_n}{c_n^2} = 0, \quad n \geq 0.$$

Hence $\theta_n < \theta_{n-1}$ ($n \geq 4$), by Theorem 3.1.4.

Lemma 3.1.5

Corresponding to a given index $\lambda \geq 2$ however large, there exists a number $\xi_0 > 1$ whose continued fraction development of type 1, namely, $\xi_0 = c_0^{\kappa} + \left[\frac{a_v}{c_v^{\kappa}} \right]_1^{\infty}$ is such that $\theta_{\lambda+1} > \theta_{\lambda}$.

Proof :

The proof consists in exhibiting a number ξ_0 with the above property.

Let the number ξ_0 be defined by the continued fraction of type 1,

$$\xi_0 = c_0^{\kappa} + \frac{a_1}{c_1^{\kappa}} + \frac{a_2}{c_2^{\kappa}} + \dots + \frac{a_{\lambda}}{c_{\lambda}^{\kappa}} + \frac{a_{\lambda+1}}{\xi_{\lambda+1}}$$

where the c_v and κ are positive integers, $\kappa \geq 2$, $c_v \geq 1$, $a_{v+1} = (c_v + 1)^{\kappa} - c_v^{\kappa}$, and $\xi_{\lambda+1} > 1$.

Let the c_v be fixed, $0 \leq v \leq \lambda-1$. Then $\xi_0 = \xi_0(c_{\lambda}, \xi_{\lambda+1})$.

Write $\xi_{\lambda+1} = c_{\lambda+1}^{\kappa} + \frac{a_{\lambda+2}}{\xi_{\lambda+2}}$; $\xi_{\lambda+2} > 1$.

From (3.1.4), page 74, $\theta_{\lambda+1} = \frac{a_{\lambda+2}}{\xi_{\lambda+2}} \cdot \frac{B_{\lambda}}{B_{\lambda+1}} \theta_{\lambda}$, or

$$(1) \quad \frac{\theta_{\lambda+1}}{\theta_{\lambda}} = \left(\xi_{\lambda+1} - c_{\lambda+1}^{\kappa} \right) \cdot \frac{B_{\lambda}}{B_{\lambda+1}}$$

The problem is to determine c_{λ} and $\xi_{\lambda+1}$, and therefore ξ_0 , so that $\theta_{\lambda+1} > \theta_{\lambda}$.

$$\text{We have: } \frac{B_{\lambda+1}}{B_{\lambda}} = c_{\lambda+1}^{\kappa} + \frac{a_{\lambda+1}}{c_{\lambda}^{\kappa} + \frac{a_{\lambda} B_{\lambda-2}}{B_{\lambda-1}}}; \quad (\lambda \geq 2)$$

$$\text{and } \frac{B_{\lambda}}{B_{\lambda+1}} \left(\xi_{\lambda+1} - c_{\lambda+1}^{\kappa} \right) > \frac{\xi_{\lambda+1} - c_{\lambda+1}^{\kappa}}{\left(c_{\lambda+1}^{\kappa} + \frac{a_{\lambda+1}}{c_{\lambda}^{\kappa}} \right)}$$

Comparing this with the equation (1) above, we see that $\frac{\theta_{\lambda+1}}{\theta_{\lambda}} > 1$

$$\text{when } (2) \quad \xi_{\lambda+1} > 2 c_{\lambda+1}^{\kappa} + \frac{a_{\lambda+1}}{c_{\lambda}^{\kappa}}$$

This inequality will be satisfied when $c_\lambda \geq 2$ and $2^\lambda > \bar{\xi}_{\lambda+1} > \left(\frac{3}{2}\right)^{\lambda+1}$; $c_{\lambda+1} = 1$. For then, we will have:

$$2c_{\lambda+1} + \frac{a_{\lambda+1}}{c_\lambda^\lambda} = 2 + \frac{(c_{\lambda+1})^\lambda - c_\lambda^\lambda}{c_\lambda^\lambda} = 2 + \left(1 + \frac{1}{c_\lambda}\right)^\lambda - 1 \geq 1 + \left(\frac{3}{2}\right)^\lambda.$$

Thus, choose $c_\lambda = 2$ and $\bar{\xi}_{\lambda+1}$ such that $2^\lambda > \bar{\xi}_{\lambda+1} > \left(\frac{3}{2}\right)^{\lambda+1}$. Then by the above remark, $\theta_{\lambda+1} > \theta_\lambda$.

Therefore, the continued fraction of type 1,

$$3.1.9 \quad \bar{\xi}_0 = c_0^\lambda + \frac{a_1}{c_0^\lambda} + \dots + \frac{a_{\lambda+1}}{1 \cdot 2^\lambda} + \frac{3^\lambda - 2^{\lambda+1}}{1 \cdot \bar{\xi}_{\lambda+1}}$$

where $2^\lambda > \bar{\xi}_{\lambda+1} > \left(\frac{3}{2}\right)^{\lambda+1}$; $\bar{\xi}_0$ has the property that $\theta_{\lambda+1} > \theta_\lambda$, i.e. $\left| \bar{\xi}_0 - \frac{A_{\lambda+1}}{B_{\lambda+1}} \right| > \left| \bar{\xi}_0 - \frac{A_\lambda}{B_\lambda} \right|$.

By Theorem 1.5.1, page 25, the number $\bar{\xi}_0$ defined by the continued fraction of type 1, (3.1.9) is uniquely represented by its continued fraction development of type 1. Hence, the continued fraction (3.1.9) is identical with the continued fraction development of type 1 of $\bar{\xi}_0$. This establishes the existence of the number $\bar{\xi}_0 > 0$, and the Lemma 3.1.5 is proved.

Remark.

The preceding Lemma 3.1.5 shows that the behavior of the error - sequence $\{\theta_n\}$ for $n \leq \lambda$ has no effect on the ultimate behavior of $\{\theta_n\}$, for $n \geq \lambda$. For, as the c_ν , $0 \leq \nu \leq \lambda-1$, were arbitrarily chosen in the Lemma, we could very well have chosen them so that $c_\nu \geq \frac{1}{2^{\nu-1}}$, $0 \leq \nu \leq \lambda-1$. Then, by Theorem 3.1.3 a, page 78, $\theta_\nu < \theta_{\nu-1}$ for $0 \leq \nu \leq \lambda-1$.

Corollary 3.1.5

There exists a number $\xi_0 > 0$ whose continued fraction development of type 1 is such that $\theta_{n+1} > \theta_n$ for infinitely many indices n .

Proof :

By Lemma 3.1.5, there exists a number $\xi_0 = \xi_0(\xi_{\lambda+1})$ such that $\theta_{\lambda+1} > \theta_\lambda$, and where the variable $\xi_{\lambda+1}$ ranges over the interval $2^k > \xi_{\lambda+1} > (\frac{3}{2})^k + 1$. Then $c_{\lambda+1} = 1$, and we have : $\xi_{\lambda+1} = 1 + \frac{2^k - 1}{\xi_{\lambda+2}} > (\frac{3}{2})^k + 1$

Thus : $1 < \xi_{\lambda+2} < (2^k - 1) (\frac{2}{3})^k < 2^{k-1}$, and $c_{\lambda+2} = 1$

Write $\xi_{\lambda+2} = 1 + \frac{2^k - 1}{\xi_{\lambda+3}}$. From the preceding inequality, $\xi_{\lambda+3} > \frac{2^k - 1}{(2^k - 1) (\frac{2}{3})^{k-1}}$, and $\xi_{\lambda+3} > (\frac{3}{2})^k > 1$.

As $\xi_{\lambda+3}$ is at our disposition, subject to the inequalities above, we choose $\xi_{\lambda+3}$ so that $c_{\lambda+3} \geq 2$ and

$$2^k > \frac{(c_{\lambda+3} + 1)^k - c_{\lambda+3}^k}{\xi_{\lambda+3} - c_{\lambda+3}^k} > (\frac{3}{2})^k + 1$$

Choose $\begin{cases} c_{\lambda+3} = 2 \\ 3^k > \xi_{\lambda+3} > 2^k \end{cases}$. Then $2^k > \frac{3^k - 2^k}{\xi_{\lambda+3} - 2^k} > (\frac{3}{2})^k + 1$.

Also, as $\xi_{\lambda+4} = \frac{a_{\lambda+4}}{\xi_{\lambda+3} - c_{\lambda+3}^k}$, we have from the preceding inequality that $2^k > \xi_{\lambda+4} > (\frac{3}{2})^k + 1$. Then $c_{\lambda+4} = 1$, and by Lemma 3.1.5, $\theta_{\lambda+4} > \theta_{\lambda+3}$. Therefore, the

number $\xi_0 = \xi_0(\xi_{\lambda+4})$, defined by the continued fraction

$$3.1.10 \quad \xi_0 = c_0^k + \frac{a_1}{1/c_0^k} + \dots + \frac{a_{\lambda+1}}{1/2^k} + \frac{3^k - 2^k}{1} + \frac{2^k - 1}{1} + \frac{2^k - 1}{1/2^k} + \frac{3^k - 2^k}{1/\xi_{\lambda+4}}$$

where $2^k > \xi_{\lambda+4} > (\frac{3}{2})^k + 1$, and $2^k > \xi_{\lambda+1} > (\frac{3}{2})^k + 1$, is such that $\theta_{\lambda+1} > \theta_\lambda$ and $\theta_{\lambda+4} > \theta_{\lambda+3}$.

It follows by repeated applications of the above procedure that there exists a number $\xi_0 > 0$ whose continued fraction development of type 1 is such that $\theta_{n+1} > \theta_n$ for infinitely many indices n .

Remark.

As $\xi_{\lambda+k_r}$, ($k_r = 1, 4, \dots$) can take on infinitely many values in $(2^r, (\frac{3}{2})^{r+1})$, it follows that there are infinitely many numbers ξ_0 having the property described in Corollary 3.1.5.

Example 1. (To illustrate Corollary 3.1.5)

Let $\xi_0 = \frac{4 + \sqrt{37}}{3}$, ($r=2$). It is easily verified that the continued fraction development of type 1 of $\xi_0 = \frac{4 + \sqrt{37}}{3}$ is purely periodic with a three-term period, namely:

$$3.1.1 \quad \xi_0 = \frac{4 + \sqrt{37}}{3} = 1 + \frac{3/1}{1/1^2} + \frac{3/1}{1/3^2} + \frac{7/1}{1/1^2} + \frac{3/1}{1/1^2} + \frac{3/1}{1/3^2} + \dots$$

Also, we have: $\xi_0 = \xi_{3n} = \frac{4 + \sqrt{37}}{3}$, $c_{3n} = 1$,

$$\xi_{3n+1} = \xi_1 = \frac{-1 + \sqrt{37}}{4}, \quad c_{3n+1} = 1, \quad \text{and}$$

$$\xi_{3n+2} = 5 + \sqrt{37}, \quad c_{3n+2} = 3, \quad (\text{for } n = 0, 1, 2, \dots).$$

As $\xi_{3n+1} = \xi_1 = \frac{-1 + \sqrt{37}}{4} < \frac{3}{2}$, and as $c_{3n+1} = 1$, $\xi_{3n+1} < 2c_{3n+1}$.

Then $\theta_{3n+1} < \theta_{3n}$ by Theorem 3.1.3 b, p.78. Similarly, it can be shown that $\theta_{3n+2} < \theta_{3n+1}$ for $n \geq 0$.

$$\text{Now } \theta_{3n} = \frac{a_{3n+1}}{\xi_{3n+1}} \cdot \frac{B_{3n-1}}{B_{3n}} \theta_{3n-1}, \quad (n \geq 1)$$

$$\text{and } \frac{a_{3n+1}}{\xi_{3n+1}} = \xi_{3n} - c_{3n}^2 = \frac{1 + \sqrt{37}}{3}. \quad \text{Also}$$

$$\frac{B_{3n}}{B_{3n-1}} < c_{3n}^2 + \frac{a_{3n}}{c_{3n-1}^2} = 1 + \frac{7}{9} = \frac{16}{9}. \quad \text{That is, } \frac{B_{3n-1}}{B_{3n}} > \frac{9}{16}.$$

Thus, $\frac{\theta_{3n}}{\theta_{3n+1}} > \left(\frac{1+\sqrt{37}}{3}\right) \times \frac{9}{16} = \frac{3}{16}(\sqrt{37}+1) > \frac{21}{16} > 1.$

and $\theta_{3n} > \theta_{3n-1}, (n \geq 1).$ Hence, there are infinitely many indices λ such that $\theta_\lambda > \theta_{\lambda+1}$, namely, $\lambda = 3n.$

This illustrates the validity of Corollary 3.1.5 .

Example 3.

The above continued fraction (3.1.11) offers an illustration where the sequence $\left\{ \frac{\theta_n}{\theta_{n-1}} \right\}$ does not have a unique limit, but, instead, has three distinct limit-points. It is of interest because one limit-point is greater than 1.

Theorem 3.1.5

Let the real number $\xi_0 > 1$ have the infinite development of type 1, $\xi_0 = c_0 + \left[\frac{a_n}{c_n} \right]_\infty$ with convergents $\left\{ \frac{A_n}{B_n} \right\}$, and let $\epsilon > 0$ be a given number, however small. Then, corresponding to the number ϵ there exists a number ξ_0' and an index $\lambda \geq 2$ such that $|\xi_0 - \xi_0'| < \epsilon$ and $\theta_\lambda' = \left| \xi_0' - \frac{A_{\lambda+1}'}{B_{\lambda+1}'} \right| < \theta_{\lambda+1}' < \epsilon$
(Here $\left\{ \frac{A_n'}{B_n'} \right\}$ is the sequence of convergents to the continued fraction development of type 1 of ξ_0')

Proof :

The sequence $\{u_n\}$ where $u_n = \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{a_1 a_2 \dots a_n}{B_n B_{n-1}}$, decreases steadily to zero as n increases, by (1.3.5) page 4 ; for $\frac{u_{n+1}}{u_n} < 1 - \frac{1}{2^n} < 1$, and $u_{n+1} < \left(1 - \frac{1}{2^n}\right) u_n$. Hence, there exists an index m such that $\left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| < \epsilon$ for $n \geq m$. Let λ be any integer satisfying: $\lambda \geq m+1$. By Lemma 3.1.5, page 85, we know that corresponding to the

integer λ , there exists a number ξ_0' whose continued fraction development k of type 1 is such that $\theta_{\lambda+1}' > \theta_\lambda'$, and that ξ_0' has the development :

$$\xi_0' = c_0'^r + \frac{a_1'}{1c_1'^r} + \dots + \frac{a_\lambda'}{12^r} + \frac{3^r - 2^r}{1\xi_{\lambda+1}'}$$

where $2^r > \xi_{\lambda+1}' > \left(\frac{3}{2}\right)^r + 1$. As the elements c_v' are arbitrary for $0 \leq v \leq \lambda-1$, (see the proof of the Lemma 3.1.5), we can choose $c_v' = c_v$, $0 \leq v \leq \lambda-1$.

Then we shall have $\frac{A_v'}{B_v'} = \frac{A_v}{B_v}$, $0 \leq v \leq \lambda-1$.

As $\lambda \geq m+1$, that is, $\lambda-1 \geq m$, it follows that

$$\left| \frac{A_{\lambda+1}'}{B_{\lambda+1}'} - \frac{A_\lambda'}{B_\lambda'} \right| < \left| \frac{A_{\lambda-1}'}{B_{\lambda-1}'} - \frac{A_{\lambda-2}'}{B_{\lambda-2}'} \right| = \left| \frac{A_{\lambda-1}}{B_{\lambda-1}} - \frac{A_{\lambda-2}}{B_{\lambda-2}} \right| < \epsilon$$

Also, since ξ_0' lies between $\frac{A_{\lambda-1}'}{B_{\lambda-1}'}$ and $\frac{A_{\lambda-2}'}{B_{\lambda-2}'}$, i.e., between $\frac{A_{\lambda-1}}{B_{\lambda-1}}$ and $\frac{A_{\lambda-2}}{B_{\lambda-2}}$, and the same is true of ξ_0 , we have $|\xi_0 - \xi_0'| < \epsilon$.

In addition, ξ_0' lies between $\frac{A_{\lambda+1}'}{B_{\lambda+1}'}$ and $\frac{A_\lambda'}{B_\lambda'}$.

Hence $\theta_{\lambda+1}' = \left| \xi_0' - \frac{A_{\lambda+1}'}{B_{\lambda+1}'} \right| < \epsilon$. Also, as

$$\theta_{\lambda+1}' > \theta_\lambda', \quad \text{we have} \quad \theta_\lambda' < \theta_{\lambda+1}' < \epsilon.$$

This proves all that is required.

It can be readily shown, as in Corollary 3.1.5, that in any interval $|\xi_0 - \xi| < \epsilon$ there exist infinitely many numbers ξ_0' whose continued fraction developments of k type 1 are such that $\theta_n' < \theta_{n+1}' < \epsilon$ for infinitely many indices n .

Chapter 4

4.1 Algorithm:

O. Szász introduced the following algorithm :
(see references below)

A sequence of arbitrary positive integers $\{b_v\}$, $b_v \geq 1$, $v=1, 2, \dots$ and a real number $\xi_0 > 0$ are given. If $b_1 \xi_0$ is an integer, $b_1 \xi_0 \geq 1$, the integer a_1 is defined by $a_1 = [b_1 \xi_0] = b_1 \xi_0$. If $b_1 \xi_0$ is not an integer, a_1 is defined as: $a_1 = [b_1 \xi_0] + 1$.

Evidently then, in either case,

$$b_1 \xi_0 \leq a_1 < b_1 \xi_0 + 1 \quad ; \quad \text{or} \quad \xi_0 \leq \frac{a_1}{b_1}$$

Write $\xi_0 = \frac{a_1}{b_1 + \xi_1}$, defining the number ξ_1 .

Clearly, $\xi_1 \geq 0$, with equality if $b_1 \xi_0$ is an integer.

In this case, $\xi_1 = 0$, and the development terminates with the

equation: $\xi_0 = \frac{a_1}{b_1}$. Otherwise $\xi_1 > 0$, and

we continue with ξ_1 as with ξ_0 by writing: $a_2 = b_2 \xi_1$ if $b_2 \xi_1$ is an integer, and, if not, by writing $a_2 = [b_2 \xi_1] + 1$.

Then, as before, $b_2 \xi_1 \leq a_2 < b_2 \xi_1 + 1$; $\xi_1 = \frac{a_2}{b_2 + \xi_2}$; $\xi_2 \geq 0$.

There is equality in these relations only if $b_2 \xi_1$ is an integer.

If $\xi_2 = 0$, then $\xi_1 = \frac{a_2}{b_2}$ and we obtain for ξ_0 the finite

continued fraction development, $\xi_0 = \frac{a_1}{b_1 + \frac{a_2}{b_2}}$.

Otherwise, $\xi_2 > 0$, and we repeat the above operations with

ξ_2 in place of ξ_1 . In this manner, we define successively

the real, non-negative numbers ξ_0, ξ_1, \dots and the positive

O. Szász: (1) Lectures on Continued Fractions University of Cincinnati, 1940-41, Lecture 12, Algorithm 12.2.2.

integers a_1, a_2, \dots , by the relations

$$4.1.1 \quad \left\{ \begin{array}{l} a_{v+1} = b_{v+1} \xi_v \quad \text{if } b_{v+1} \xi_v \text{ is an integer.} \\ a_{v+1} = [b_{v+1} \xi_v] + 1 \quad \text{otherwise} \\ \xi_v = \frac{a_{v+1}}{b_{v+1} + \xi_{v+1}} \quad ; \quad \xi_{v+1} = 0 \text{ if } b_{v+1} \xi_v \text{ is an} \\ \text{integer; } \xi_{v+1} > 0 \text{ otherwise} \end{array} \right.$$

for $v = 0, 1, 2, \dots$

Then $b_{v+1} \xi_v \leq a_{v+1} < b_{v+1} \xi_v + 1$, with equality if $b_{v+1} \xi_v$ is an integer.

If, for some particular index λ , $b_{\lambda+1} \xi_\lambda$ is an integer, and consequently $\xi_{\lambda+1} = 0$, we shall say that the development terminates. Then, from (4.1.1), $\xi_\lambda = \frac{a_{\lambda+1}}{b_{\lambda+1}}$, and, we obtain, in this case, the finite continued fraction development of ξ_0 .

$$4.1.2 \quad \xi_0 = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_{\lambda+1}}{b_{\lambda+1}}$$

If at every stage $b_{v+1} \xi_v$ is not an integer, the development does not terminate. One obtains then the infinite continued fraction development of ξ_0 .

$$4.1.3 \quad \xi_0 \sim \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots$$

Definition 4.1.1

The continued fraction development of a real number $\xi_0 > 0$ according to the algorithm 4.1 (and corresponding to the given sequence $\{b_v\}$ of positive integers) will be called the development of type 3 of the number ξ_0 . ξ_0 is said to generate the development.

Theorem 4.1.1

A finite development of type 3 is the rational number ξ_0 which generated it.

Proof: The theorem follows immediately from equation (4.1.2). For, clearly, there is equality, and as a finite continued fraction with integral elements certainly represents a rational number, ξ_0 is rational.

The converse of Theorem 4.1.1 is also true; namely, that the development of type 3 of a rational number ξ_0 is finite.

Theorem 4.1.2

The development of type 3 of a rational number $\xi_0 > 0$ corresponding to the given sequence of positive integers $\{b_v\}$ is terminating.

Proof: Let $\xi_0 = \frac{p_0}{q_0}$, where p_0, q_0 are positive integers, $p_0 \geq 1, q_0 \geq 1$. By the algorithm 4.1, we have : $a_1 = b_1, \xi_1 = \frac{b_1 p_0}{q_0}$ if $b_1 \xi_0$ is an integer; or $a_1 = [b_1 \xi_0] + 1, a_1 = [\frac{b_1 p_0}{q_0}] + 1$, otherwise. Then, in either case, $\frac{b_1 p_0}{q_0} \leq a_1 < \frac{b_1 p_0}{q_0} + 1$ and so :

$$4.1.4 \quad 0 \leq a_1 q_0 - b_1 p_0 < q_0$$

From $\xi_0 = \frac{p_0}{q_0} = \frac{a_1}{b_1 + \xi_1}$, $\xi_1 \geq 0$, we have \rightarrow

$$\rightarrow \xi_1 = \frac{a_1 q_0 - b_1 p_0}{p_0} \geq 0$$

Write: $\begin{cases} q_1 = p_0 \geq 1 \\ p_1 = a_1 q_0 - b_1 p_0 \geq 0 \end{cases}$

$$\text{Then } \xi_1 = \frac{p_1}{q_1} \geq 0$$

If $b_1 \xi_0$ is an integer, there is equality in the preceding relations, and in (4.1.4)

That is, $a_1 g_0 - b_1 p_0 = p_1 = 0$, $\xi_1 = 0$,

and $\xi_0 = \frac{p_0}{g_0} = \frac{a_1}{b_1}$. Thus, the development terminates.

If $b_1 \xi_0$ is not an integer, then $\xi_1 > 0$, and $p_1 = a_1 g_0 - b_1 p_0 > 0$.
From (4.1.4), $0 < p_1 < g_0$.

As p_1 is an integer, $p_1 \geq 1$. Also, $g_1 = p_0 \geq 1$.

We continue with $\xi_1 = \frac{p_1}{g_1}$ as with ξ_0 , writing:
 $a_2 = b_2 \xi_1 = \frac{b_2 p_1}{g_1}$ ($b_2 \xi_1$, an integer)
or $a_2 = [b_2 \xi_1] + 1 = [b_2 \frac{p_1}{g_1}] + 1$, otherwise.

In either case, $\frac{b_2 p_1}{g_1} \leq a_2 < \frac{b_2 p_1}{g_1} + 1$, and

$$4.1.5 \quad 0 \leq a_2 g_1 - b_2 p_1 < g_1$$

From $\xi_1 = \frac{p_1}{g_1} = \frac{a_2}{b_2 + \xi_2}$; $\xi_2 \geq 0$, we have
 $\xi_2 = \frac{a_2 g_1 - b_2 p_1}{p_1} = \frac{p_2}{g_2} \geq 0$
where we write, as before: $\begin{cases} p_2 = a_2 g_1 - b_2 p_1 \geq 0 \\ g_2 = p_1 \geq 1. \end{cases}$

If $b_2 \xi_1$ is an integer, there is equality in (4.1.5) and in the subsequent relations. That is, $a_2 g_1 - b_2 p_1 = p_2 = 0$, and $\xi_2 = \frac{a_2}{b_2}$. The development then terminates yielding a finite continued fraction development of type 3 for ξ_0 namely,

$$\xi_0 = \frac{a_1}{b_1 + \frac{a_2}{b_2}}.$$

If $b_2 \xi_1$ is not an integer, then $\xi_2 > 0$ and $p_2 = a_2 g_1 - b_2 p_1 > 0$.
From (4.1.5) we have: $0 < p_2 < g_2$. As p_2 is an integer, $p_2 \geq 1$. Also, $g_2 = p_1 \geq 1$. Now $g_1 = p_0$ from the preceding. Thus, $0 < p_2 < p_0$.

In general, it follows as above, that $\xi_v = \frac{p_v}{g_v} \geq 0$
 4.1.6 { where $p_v = a_v g_{v-1} - b_v p_{v-1} \geq 0$; $g_v = p_{v-1} \geq 1$
 $0 \leq p_v < g_{v-1}$

Write $p_1 = g_0$. As $p_v < g_{v-1} = p_{v-2}$,
 We have : $(v \geq 1)$
 $p_0 > p_2 > \dots > p_{2v} > \dots$
 $p_{-1} > p_1 > p_3 > \dots > p_{2v-1} > \dots$

The integers p_{2v}, p_{2v-1} are strictly decreasing for $v \geq 0$
 and they are non-negative. Hence there exists an index $n \geq 1$

such that $p_n = 0$, $p_n = a_n g_{n-1} - b_n p_{n-1} = 0$

Then $\xi_n = \frac{p_n}{g_n} = 0$, and $\xi_{n-1} = \frac{p_{n-1}}{g_{n-1}} = \frac{a_n}{b_n}$.

Therefore, the development of ξ_0 terminates, and we obtain
 the finite continued fraction development of type 3 :

$$\xi_0 = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n}.$$

This proves Theorem 4.1.2.

Let $\xi_0 > 0$ be rational; $\xi_0 = \frac{p_0}{g_0}$. Write $p_1 = g_0$. By
 Theorem 4.1.2, the development of type 3 of ξ_0 is finite ;
 there exists then an index $n \geq 1$ such that $\xi_n = 0$. From the
 algorithm 4.1 and (4.1.6), we obtain the relations

$$\xi_v = \frac{p_v}{g_v} > 0 ; \quad p_v = a_v g_{v-1} - b_v p_{v-1} > 0 ,$$

$$1 \leq p_v < g_{v-1} ; \quad g_v = p_{v-1} \geq 1 \quad (\text{for } v=1, 2, \dots, n-1)$$

$$p_n = a_n g_{n-1} - b_n p_{n-1} = 0 , \quad g_n = p_{n-1} .$$

As $g_v = p_{v-1}$, we have the recursion formula for the p_v :

$$p_v = a_v p_{v-2} - b_v p_{v-1}, \quad \text{for } v = 1, 2, \dots, n-1$$

$$0 = a_n p_{n-2} - b_n p_{n-1}.$$

This can be written as the system of equations :

$$4.1.7 \quad \left\{ \begin{array}{l} a_1 p_1 = b_1 p_0 + p_1 \\ a_2 p_0 = b_2 p_1 + p_2 \\ \dots \\ a_{n-1} p_{n-3} = b_{n-1} p_{n-2} + p_{n-1} \\ a_n p_{n-2} = b_n p_{n-1} \end{array} \right. \quad \text{with : } \left\{ \begin{array}{l} p_{-1} > p_1 > p_3 > \dots \\ p_0 > p_2 > p_4 > \dots \\ p_n = 0, \text{ and} \\ \frac{p_{n-1}}{p_{n-2}} = \frac{a_n}{b_n} \end{array} \right.$$

We can associate the system 4.1.7 with a continued fraction, (i.e. with the continued fraction development of type 3 of $\xi_0 = \frac{p_1}{p_0}$. For, if v is any index, $v \geq 1$ then,

$$a_v p_{v-2} = b_v p_{v-1} + p_v ;$$

$$\frac{1}{p_{v-2}} = \frac{a_v}{b_v p_{v-1} + p_v}, \quad \text{and}$$

$$\frac{p_{v-1}}{p_{v-2}} = \frac{a_v}{b_v + \frac{p_v}{p_{v-1}}}, \quad \text{for } v = 1, 2, \dots, n-1$$

while for $v = n$,

$$\frac{p_{n-1}}{p_{n-2}} = \frac{a_n}{b_n}$$

The last two formulae yield : $\frac{p_0}{p_{-1}} = \frac{a_1}{b_1} + \dots + \frac{a_n}{b_n}$

which is the development of type 3 of the rational number $\xi_0 = \frac{p_0}{p_{-1}}$.

From the system (4.1.7), we see that every common divisor of p_0 and $p_1 = q_0$ is a divisor of p_1 , and consequently of every p_v , ($v = 1, 2, \dots, n-1$). Then every common divisor of p_0 and p_1 divides p_{n-2} and p_{n-1} . However, it does not follow that p_{n-1} divides p_{n-2} or any of the preceding p_v , or even that $p_{n-1} = (p_{-1}, p_0)$, and similarly for p_{n-2} ; (i.e.), that p_{n-2} need

$$p_{n-2} = (p_1, p_0)$$

not divide p_n , or that n . All that can be said is that the g.c.d. of p_1 and p_0 divides p_{n-2} and p_{n-1} .

The following examples will illustrate these remarks.

Example 4.1.1. Let $\xi_0 = \frac{37}{13}$ $\left\{ \begin{array}{l} p_0 = 37 \\ q_0 = 13 \end{array} \right.$

and let $b_v = 2v$, $v \geq 1$. Then $b_1 = 2$,

$$a_1 = \left[\frac{74}{13} \right] + 1 = 6, \quad q_1 = p_0 = 37,$$

$$p_1 = a_1 q_0 - b_1 p_0 = 78 - 74 = 4.$$

$$(1) \quad \xi_1 = \frac{p_1}{q_1} = \frac{4}{37}.$$

Similarly, we obtain:

$$(2) \quad \xi_2 = \frac{p_2}{q_2} = \frac{p_2}{p_1} = \frac{21}{4} \quad ; \quad \begin{cases} a_2 = 1 \\ b_2 = 4 \end{cases}$$

$$(3) \quad \xi_3 = \frac{p_3}{q_3} = \frac{p_3}{p_2} = \frac{2}{21} \quad ; \quad \begin{cases} a_3 = 32 \\ b_3 = 6 \end{cases}$$

$$(4) \quad \xi_4 = \frac{p_4}{q_4} = \frac{p_4}{p_3} = \frac{5}{2} \quad ; \quad \begin{cases} a_4 = 1 \\ b_4 = 8 \end{cases}$$

$b_5 = 10$. Then

$$b_5 \xi_4 = 10 \cdot \frac{5}{2} = 25, \text{ an integer.}$$

$$\xi_5 = 0,$$

and the development terminates. Here, $n=5$ $\left\{ \begin{array}{l} p_4 = p_{n-1} = 5 \\ p_3 = p_{n-2} = 2 \end{array} \right.$

and neither p_4, p_3 divide each other, nor is either the g.c.d. of 13 and 37.

Example 4.1.2 Let $\xi_0 = \frac{35}{14}$; $p_0 = 35$, $q_0 = 14$.

and let $b_v = 3v$ ($v \geq 1$).

$$\begin{aligned} \text{Then } b_1 &= 3 ; \quad a_1 = \left[3 \cdot \frac{35}{14} \right] + 1 = 8 \\ g_1 &= p_0 = 35 ; \quad p_1 = a_1 g_0 - b_1 p_0 = 7. \\ \xi_1 &= \frac{p_1}{g_1} = \frac{7}{35} ; \quad b_2 = 6, \quad a_2 = \left[6 \cdot \frac{7}{35} \right] + 1 = 2, \\ g_2 &= p_1 = 7, \quad p_2 = 2 \cdot 35 - 6 \cdot 7 = 28 \\ \text{and } \xi_2 &= \frac{p_2}{g_2} = \frac{28}{7} = 4 \end{aligned}$$

Thus ξ_2 is an integer, and surely $b_3 \xi_2$ is an integer. Hence, the development terminates, $\xi_3 = 0$. In this case, $n=3$ and

$p_{n-2} = p_1 = 7$ divides $28 = p_{n-1} = p_2$. Also,

$p_{n-2} = 7$ is the g.c.d. of $p_0 = 35, g_0 = 14$.

Example 4.1.3.

Let $\xi_0 = \frac{72}{42}$; $p_0 = 72$, $g_0 = 42$,
and let $b_v = 2v-1$, $v \geq 1$.

$$\begin{aligned} \text{Then } a_1 &= \left[\frac{72}{42} \right] + 1 = 2, \\ g_1 &= p_0 = 72, \quad p_1 = 2 \cdot 42 - 72 = 12 \\ \xi_1 &= \frac{p_1}{g_1} = \frac{12}{72}. \quad \text{Similarly:} \end{aligned}$$

$$\xi_2 = \frac{p_2}{g_2} = \frac{p_2}{p_1} = \frac{36}{12} = 3, \text{ an integer.}$$

Then $b_3 \xi_2$ is also an integer, $\xi_3 = 0$, and the development terminates.

Here, $n=3$, $p_{n-2} = p_1 = 12$ divides $36 = p_{n-1} = p_2$, but p_{n-2} is not the g.c.d. of 72 and 42, (namely p_0 and g_0).

Example 4.1.4.

This is an example to show that if $(p_0, g_0) = 1$, then the successive p_v need not be relatively prime.

$$\text{Let } \xi_0 = \frac{18}{5}, \quad p_0 = 18, \quad g_0 = 5$$

$$(18, 5) = 1$$

and let $b_v = 1$, $v \geq 1$. Then $b_1 = 1$,
 $a_1 = \left[\frac{18}{5} \right] + 1 = 4$, $g_1 = p_0 = 18$; and
 $p_1 = 4 \cdot 5 - 18 = 2$.
 And we see that $p_1 = 2$, $g_1 = p_0 = 18$
 are not relatively prime, $(p_1, p_0) = 2$.

4.2 Non-Terminating Continued Fraction Developments of Type 3

If $\rho_0 > 0$ is a rational number, its continued fraction development of type 3, corresponding to a given sequence of positive integers $\{b_v\}$ is obtained from the system of equations

$$4.2.1 \quad \rho_0 = \frac{a_1}{b_1 + \rho_1}, \quad \rho_1 = \frac{a_2}{b_2 + \rho_2}, \quad \dots$$

where the b_v are given positive integers, and the a_v and ρ_v are defined successively by $\begin{cases} a_v = b_v \rho_{v-1} \\ \rho_v = 0 \end{cases}$ if $b_v \rho_{v-1}$ is an integer, and otherwise, by

$$(4.2.2) \quad \begin{aligned} a_v &= \left[b_v \rho_{v-1} \right] + 1 \\ \rho_{v-1} &= \frac{a_v}{b_v + \rho_v} \end{aligned} \quad \text{for } v = 1, 2, \dots$$

As we have already seen (Theorem 4.1.2) the system (4.2.1) terminates when $b_v \rho_{v-1}$ is an integer, i.e. when ρ_0 is rational.

If the number ρ_0 is irrational, we can evidently apply to it the same sequence of operations as above, that is, develop ρ_0 into a continued fraction according to the Algorithm 4.1, and obtain a system of equations similar to (4.2.1) above. However, in this case, the sequence of operations, and therefore, the system of equations do not terminate, but continue on

indefinitely. For, as ρ_0 is irrational, it follows that $\rho_1 = \frac{a_1}{\rho_0} - b_1$ is also irrational, and similarly, all the ρ_v are irrational, $v \geq 0$, and $b_{v+1} \rho_v$ can not be an integer. Thus, the operations continue indefinitely, and the corresponding system of equations

$$\rho_{v-1} = \frac{a_v}{b_v + \rho_v}, \quad v = 1, 2, \dots$$

is non-terminating.

Theorem 4.2.1.

The development of type 3 of a real number $\rho_0 > 0$ converges and represents ρ_0 .

Proof: We assume that ρ_0 is irrational.

The case ρ_0 a rational number is trivial by Theorems 4.1.1 and 4.1.2, as the development is a terminating one.

Let ρ_0 be irrational. Then the continued fraction development of type 3, according to algorithm 4.1 is non-terminating. From (4.2.2) we have

$$a_v = [b_v \rho_{v-1}] + 1 \quad \text{for } v = 1, 2, \dots$$

$$\rho_{v-1} = \frac{a_v}{b_v + \rho_v}$$

As $b_v \rho_{v-1} < a_v < b_v \rho_{v-1} + 1$, with strict inequality, then (1) $0 < a_v - b_v \rho_{v-1} < 1$. Also $\rho_{v-1} \rho_v = a_v - b_v \rho_{v-1}$, and

thus (4.2.3) : $0 < \rho_v \rho_{v-1} < 1$

From (4.2.2) we obtain

$$4.2.4 \quad \rho_0 = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_v}{b_v + \rho_v}, \quad (v \geq 1).$$

Denote by $\left\{ \frac{A_n}{B_n} \right\}$ the sequence of convergents to the continued fraction $\left[\frac{a_v}{b_v} \right]_{v=1}^{\infty}$

Now: $B_0 = 1$, $B_1 = b_1 \geq 1$, and
 $B_v = b_v B_{v-1} + a_v B_{v-2} > B_{v-1}$, $v \geq 2$.

As the B_v are positive integers, we have $B_v \uparrow \infty$ as $v \rightarrow \infty$.

From (4.2.4), we obtain the well-known relation:

$$\rho_0 = \frac{(b_v + \rho_v) A_{v-1} + a_v A_{v-2}}{b_v + \rho_v B_{v-1} + a_v B_{v-2}}$$

$$\rho_0 = \frac{\rho_v A_{v-1} + A_v}{\rho_v B_{v-1} + B_v}, \quad (v \geq 0)$$

Then $\rho_0 B_v - A_v = -\rho_v (\rho_0 B_{v-1} - A_{v-1})$, or finally

$$4.2.5 \quad |\rho_0 B_v - A_v| = \rho_v |\rho_0 B_{v-1} - A_{v-1}| \quad \text{for } v=0, 1, 2, \dots$$

Thus $|\rho_0 B_v - A_v| = \rho_v \rho_{v-1} \dots \rho_1 \rho_0$

By (4.2.3) $\rho_v \rho_{v-1} < 1$; $v=1, 2, \dots$. Certainly then, $\begin{cases} |\rho_0 B_v - A_v| < 1 \\ |\rho_0 B_v - A_v| < \rho_0 \end{cases}$

and, in either case, $|\rho_0 B_v - A_v| < 1 + \rho_0$; $|\rho_0 - \frac{A_v}{B_v}| < \frac{1 + \rho_0}{B_v}$.

As $B_v \uparrow \infty$ as $v \rightarrow \infty$, it follows that $\lim_{v \rightarrow \infty} \frac{A_v}{B_v} = \rho_0$.

Thus, the continued fraction development of ρ_0 converges and its value is ρ_0 . This proves Theorem 4.2.1.

Remarks From the relations $|\rho_0 B_v - A_v| = \rho_v \rho_{v-1} \dots \rho_1 \rho_0 \neq 0$,
 for $v \geq 0$, and $\rho_\lambda \rho_{\lambda-1} < 1$, $\lambda = 1, 2, \dots$
 it follows that $\lim_{v \rightarrow \infty} |\rho_0 B_v - A_v| = 0$.

For, let $\epsilon > 0$ be a given number arbitrarily small. As $\rho_\lambda \rho_{\lambda-1} < 1$
 there exists an index $n_0 > 0$; $n_0 = n_0(\epsilon)$, such that

$(\rho_n \rho_{n-1}) (\rho_{n-2} \rho_{n-3}) \dots \rho_1 \rho_0 < \epsilon$ for $n \geq n_0$. Then:

$|\rho_0 B_v - A_v| < \epsilon$ for $v \geq n_0$, and so

$$\lim_{v \rightarrow \infty} |\rho_0 B_v - A_v| = 0.$$

This yields another proof of Theorem 4.2.1, for we have $|\rho_0 B_v - A_v| < \epsilon$ for $v \geq n$, and surely then, $|\rho_0 - \frac{A_v}{B_v}| < \epsilon$. Hence, $\lim_{v \rightarrow \infty} \frac{A_v}{B_v} = \rho_0$.

The above result, namely $\lim_{v \rightarrow \infty} |\rho_0 B_v - A_v| = 0$ shows that ρ_0 is irrational. For, as the sequences $\{B_v\}, \{A_v\}$ consist of integers, we have by the Irrationality Criterion 1.8.1 (b), p. 60, that ρ_0 is irrational.

4.3 Approximation Properties

Let $\rho_0 > 0$ be irrational or rational, and let the development of type 3 of ρ_0 be: (corresponding to the given sequence of positive integers b_v)

$$4.3.1 \quad \rho_0 = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_v}{b_v + \rho_v}$$

$$\rho_v = \frac{a_{v+1}}{b_{v+1}} + \frac{a_{v+2}}{b_{v+2}} + \dots$$

If ρ_0 is rational, the development (4.3.1) is a finite one. There exists an index n such that $\rho_{n+1} = 0$. Also $\rho_v > 0$ for $0 \leq v \leq n$.

Otherwise, ρ_0 is irrational, and (4.3.1) does not terminate. Then $\rho_v > 0$ for $v \geq 0$.

From (4.3.1)

$$\rho_0 = \frac{(b_v + \rho_v)A_{v-1} + a_v A_{v-2}}{b_v + \rho_v B_{v-1} + a_v B_{v-2}} = \frac{\rho_v A_{v-1} + A_v}{\rho_v B_{v-1} + B_v}$$

$$\text{and } |\rho_0 B_v - A_v| = \rho_v |\rho_0 B_{v-1} - A_{v-1}|.$$

Then

$$(4.3.2) \quad \left| \rho_0 - \frac{A_v}{B_v} \right| = \frac{\rho_v B_{v-1}}{B_v} \left| \rho_0 - \frac{A_{v-1}}{B_{v-1}} \right|, \quad (v \geq 1).$$

As $\rho_v \rho_{v-1} < 1$, ($v \geq 1$), of every two successive indices $i, i+1$, there is at least one, say v ($v=i$ or $i+1$) such that $\rho_v \leq 1$. If $\rho_v \leq 1$, then surely

$$\left| \rho_0 - \frac{A_v}{B_v} \right| < \left| \rho_0 - \frac{A_{v-1}}{B_{v-1}} \right|, \quad \text{as } B_v > B_{v-1}.$$

This occurs at least once for every two successive indices.

If $\rho_v > 1$, we can not say that $\theta_v = \left| \rho_0 - \frac{A_v}{B_v} \right| < \theta_{v-1}$.

From (4.3.2), namely, $\theta_v = \frac{\rho_v B_{v-1}}{B_v} \theta_{v-1}$, we obtain a sufficient condition that $\theta_v < \theta_{v-1}$.

Theorem 4.3.1

Let $\rho_0 = \left[\frac{a_v}{b_v} \right]_{v=1}^{\infty}$ with convergents $\left\{ \frac{A_n}{B_n} \right\}$, and denote by $\theta_n = \left| \rho_0 - \frac{A_n}{B_n} \right|$, the error - sequence corresponding to the development of type 3 of ρ_0 .

Then, a sufficient condition that $\theta_v < \theta_{v-1}$

for a particular index v is that

$$4.3.3 \quad b_v b_{v+1} \geq a_{v+1}.$$

Proof :

From (4.2.2) page 99, , as

$$\rho_v = \frac{a_{v+1}}{b_{v+1} + \rho_{v+1}} \leq \frac{a_{v+1}}{b_{v+1}}, \quad (\text{as } \rho_{v+1} \geq 0),$$

$$\text{Then } \frac{\rho_v B_{v-1}}{B_v} \leq \frac{a_{v+1} B_{v-1}}{b_{v+1} B_v} < \frac{a_{v+1} B_{v-1}}{b_{v+1} b_v B_{v-1}}$$

$$\text{or } \frac{\rho_v B_{v-1}}{B_v} < \frac{a_{v+1}}{b_v b_{v+1}} \leq 1, \quad \text{by hypothesis.}$$

But, from (4.3.2), $\theta_v = \frac{\rho_v B_{v-1}}{B_v} \theta_{v-1}$. Hence $\theta_v < \theta_{v-1}$.

Theorem 4.3.2

Let $\rho_0 > 0$ have the development of type 3 $\rho_0 = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots$ $\left\{ \frac{A_n}{B_n} \right\}$.

A sufficient condition that $\theta_v < \theta_{v-1}$ for a certain index v

is that $a_v \geq 2$.

Proof :

We have $\rho_v < \frac{1}{\rho_{v-1}}$ and

$$\frac{\rho_v B_{v-1}}{B_v} < \frac{B_{v-1}}{\rho_{v-1} B_v} = \frac{B_{v-1}}{b_v \rho_{v-1} B_{v-1} + \rho_{v-1} a_v B_{v-2}}$$

Now $b_v \rho_{v-1} < a_v < b_v \rho_{v-1} + 1$ (from (1), p. 100)

so that :

$$\frac{\rho_v B_{v-1}}{B_v} < \frac{B_{v-1}}{(a_v - 1) B_{v-1} + \rho_{v-1} a_v B_{v-2}}$$

Then, as $a_v \geq 2$; $a_v - 1 \geq 1$, and so $\frac{\rho_v B_{v-1}}{B_v} < 1$.

Hence, from the relation $\theta_v = \frac{\rho_v B_{v-1}}{B_v} \theta_{v-1}$, we have $\theta_v < \theta_{v-1}$.

Example 4.3.1

The following is an example of a development of type 3 of a number $\rho_0 > 0$ for which there are infinitely many indices v such that $\theta_v > \theta_{v-1}$.

Let $\rho_0 = 1 + \sqrt{5}$, and let $b_v = 1$, $v = 1, 2, \dots$

It is easily verified that the development of type 3 of ρ_0

is purely periodic with a two-term period. We have

$$\rho_0 = \frac{41}{11} + \frac{11}{11} + \frac{41}{11} + \dots + \frac{a_{2v} 1}{1 b_{2v} + \rho_{2v}}$$

with $a_{2v-1} = 4$, $a_{2v} = 1$; $b_v = 1$, ($v \geq 1$)

$$\rho_{2v} = \rho_0 = 1 + \sqrt{5} > 3$$

$$\rho_{2v+1} = \sqrt{5} - 2 < 1$$

We have : $\left| \rho_0 - \frac{A_{2v}}{B_{2v}} \right| = \frac{\rho_{2v} B_{2v-1}}{B_{2v}} \left| \rho_0 - \frac{A_{2v-1}}{B_{2v-1}} \right|$

and $B_{2v} = B_{2v-1} + B_{2v-2}$. Then $B_{2v} \leq 2 B_{2v-1}$

(equality if $v=1$). Also, $\frac{\rho_{2v} B_{2v-1}}{B_{2v}} > \frac{3 B_{2v-1}}{2 B_{2v-1}} = \frac{3}{2} > 1$.

Therefore, $\left| \rho_0 - \frac{A_{2v}}{B_{2v}} \right| > \left| \rho_0 - \frac{A_{2v-1}}{B_{2v-1}} \right|$ for $v = 1, 2, \dots$

Indeed, $\liminf_{v \rightarrow \infty} \frac{\theta_{2v}}{\theta_{2v-1}} \geq \frac{3}{2}$.

Chapter 5

5.1 Algorithm

Let $\xi_0 > 0$ be a given number, and let $\{c_r\}$ be a given sequence of positive integers. Write $b_0 = [\xi_0] \geq 0$. If ξ_0 is an integer, $\xi_0 \geq 1$, then $b_0 = \xi_0$, and we say that the development terminates. Otherwise, we define ρ_0 by the relation : $\rho_0 = \xi_0 - b_0$; Then $0 < \rho_0 < 1$.

The integer b_1 is defined to be the largest integer x such that $[x\rho_0] = x = [c_1, \rho_0]$. Hence $[(b_1+1)\rho_0] > [b_1, \rho_0]$ and $b_1 \geq c_1$. The integer b_1 is unique as $0 < \rho_0 < 1$.

It follows then that $[(b_1+1)\rho_0] = 1 + [b_1, \rho_0]$.

Write $a_1 = 1 + [b_1, \rho_0] = [(b_1+1)\rho_0] \geq 1$;

and define ξ_1 by the relation : $\xi_1 = \frac{a_1}{\rho_0}$.

Then $\xi_1 > a_1$, as $0 < \rho_0 < 1$. If $(b_1+1)\rho_0$

is an integer, then from the above, $a_1 = (b_1+1)\rho_0$, and

$\frac{a_1}{\rho_0} = b_1 + 1$. But $\xi_1 = \frac{a_1}{\rho_0}$. Hence $\xi_1 = b_1 + 1$.

and ξ_1 is an integer. The development then terminates, and

we have : $\xi_0 = b_0 + \frac{a_1}{b_1+1}$.

Otherwise, $(b_1+1)\rho_0$ is not an integer, and

consequently, $b_1 = [\xi_1]$. This can be seen as follows :

From $a_1 = [b_1, \rho_0] + 1 = [(b_1+1)\rho_0]$,

$b_1, \rho_0 < a_1 \leq (b_1+1)\rho_0$, or : $b_1 < \frac{a_1}{\rho_0} \leq b_1 + 1$

Hence $b_1 = [\frac{a_1}{\rho_0}] = [\xi_1]$, as $\xi_1 = \frac{a_1}{\rho_0}$.

Also, from the relation $b_1 + 1 > \frac{a_1}{\rho_0}$, as $0 < \rho_0 < 1$

we obtain $b_1 \geq a_1$.

We continue as before with ξ_1 , writing $\xi_1 = b_1 + \rho_1$ with $0 < \rho_1 < 1$. Define b_2 to be the largest integer x such that $[x\rho_1] = [c_2\rho_1]$. Hence $[(b_2+1)\rho_1] > [b_2\rho_1]$, and $b_2 \geq c_2$. It follows, as $0 < \rho_1 < 1$, that $[(b_2+1)\rho_1] = 1 + [b_2\rho_1]$. Write $a_2 = 1 + [b_2\rho_1] = [(b_2+1)\rho_1] \geq 1$, and define ξ_2 by $\xi_2 = \frac{a_2}{\rho_1}$; $\xi_2 > a_2$. If $(b_2+1)\rho_1$ is an integer, then $a_2 = (b_2+1)\rho_1$, $\frac{a_2}{\rho_1} = b_2+1$, and $\xi_2 = b_2+1$ is an integer. Accordingly the development terminates. Also, as $\xi_2 = b_2+1 > a_2$, $b_2 \geq a_2$.

We obtain, in the above case, the continued fraction

$$\xi_0 = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2+1}, \quad \text{with } b_1 \geq a_1, \quad b_2 \geq a_2.$$

Otherwise, $(b_2+1)\rho_1$ is not an integer, and it follows that $b_2 = [\xi_2]$. For, from $a_2 = [b_2\rho_1] + 1 = [(b_2+1)\rho_1]$, $b_2\rho_1 < a_2 < (b_2+1)\rho_1$, and $b_2 < \frac{a_2}{\rho_1} < b_2+1$ or $b_2 < \xi_2 < b_2+1$. Hence $b_2 = [\xi_2]$.

Also, $b_2+1 > \xi_2 > a_2$, and $b_2 \geq a_2$.

Write $\xi_2 = b_2 + \rho_2$. Then from the preceding, $0 < \rho_2 < 1$.

We continue the sequence of operations defined above with ρ_2 replacing ρ_1 , and c_3 replacing c_2 .

In general, $0 < \rho_n < 1$, and b_{n+1} is the largest integer x such that $[x\rho_n] = [c_{n+1}\rho_n]$; $b_{n+1} \geq c_{n+1}$. Then $[(b_{n+1}+1)\rho_n] = 1 + [b_{n+1}\rho_n]$. Write $a_{n+1} = 1 + [b_{n+1}\rho_n]$, and define ξ_{n+1} by: $\xi_{n+1} = \frac{a_{n+1}}{\rho_n}$. If $(b_{n+1}+1)\rho_n$ is an integer, then $a_{n+1} = (b_{n+1}+1)\rho_n$, and it follows that $\xi_{n+1} = (b_{n+1}+1)$, an integer. The

development terminates. We note that as $0 < \rho_n < 1$,
 $\xi_{n+1} > a_{n+1}$ and from $\xi_{n+1} = b_{n+1} + 1$, we obtain $b_{n+1} \geq a_{n+1}$.

The number ξ_0 then has the development

$$5.1.1 \quad \xi_0 = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_{n+1}}{b_{n+1} + 1}$$

with $b_r \geq a_r$, $r = 1, 2, \dots, n+1$.

Otherwise, $(b_{n+1} + 1)\rho_n$ is not an integer, and it follows that $b_{n+1} < \xi_{n+1} < b_{n+1} + 1$, and $b_{n+1} = [\xi_{n+1}]$; $b_{n+1} \geq a_{n+1}$.

Then write $\xi_{n+1} = b_{n+1} + \rho_{n+1}$, and $0 < \rho_{n+1} < 1$; and continue as before.

We note once more that, as $\xi_r = \frac{a_r}{\rho_{r-1}}$ and $0 < \rho_{r-1} < 1$, we have $\xi_r > a_r$. Also, as $b_r = [\xi_r]$, $b_r \geq a_r$, ($r = 1, 2, \dots$). (This last inequality holds when ξ_r is an integer, for in this case, $\xi_r = b_r + 1$, and as $\xi_r > a_r$, $b_r \geq a_r$.)

If at every stage, $(b_r + 1)\rho_r$ is not an integer, the development does not terminate, and one obtains an infinite continued fraction development for ξ_0 :

$$5.1.2 \quad \xi_0 \sim b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots$$

with $b_r \geq a_r$, ($r \geq 1$), and with ^{positive} integral elements a_r, b_r .

As a special case of the above algorithm, we note that if $c_r = 1$, ($r = 1, 2, \dots$) then $a_r = [c_r \rho_{r-1}] + 1 = 1$, as $0 < \rho_{r-1} < 1$, and we obtain the regular continued fraction development of ξ_0 .

Definition 5.1.1

1. The development of a real number $\xi_0 > 0$ according to the algorithm 5.1, and corresponding to the given sequence of positive integers $\{c_v\}$ is called the development of type 4 of the number ξ_0 . ξ_0 is said to generate the development.

2. A continued fraction $b_0 + \frac{a_1}{b_1} + \dots$, terminating or not, with positive integral elements a_v, b_v such that $b_v \geq a_v$, ($v=1, 2, \dots$) will be called a continued fraction of type 4. In the terminating case, the final $b_n > a_n$.

Theorem 5.1.1

A finite development of type 4 is the rational number ξ_0 which generated it.

Proof :

The proof is immediate from the algorithm 5.1 and the relation (5.1.1). For there is equality, and as the finite continued fraction (5.1.1) has integral elements, it represents a rational number.

The converse of this theorem is also true.

Theorem 5.1.2

The development of type 4 of a rational number ξ_0 , (corresponding to the given sequence $\{c_v\}$ of positive integers) is terminating.

Proof :

Let ξ_0 be rational. By the algorithm 5.1, we have : $b_0 = [\xi_0]$, $\xi_0 = b_0 + \rho_0$. If ξ_0 is an integer, $\rho_0 = 0$, and the development terminates with the equation

$$\xi_0 = b_0.$$

Otherwise, $0 < \rho_0 < 1$, $\rho_0 = \xi_0 - b_0$ is rational, and we write $\frac{\rho_0}{g_0} = \rho_0$, where p_0, g_0 are positive integers, $p_0 \geq 1, g_0 \geq 1$. As $\rho_0 < 1$, $1 \leq p_0 < g_0$.

Write $p_1 = g_0$. Next, write :

(1) $a_1 = 1 + [c, \rho_0] = 1 + [b, \rho_0]$, where b , is the largest integer x such that $[x\rho_0] = [c, \rho_0]$. Then

(2) $a_1 = [(b+1)\rho_0] = 1 + [b, \rho_0]$;

and we write $\xi_1 = \frac{a_1}{\rho_0}$. Thus $\xi_0 = b_0 + \frac{a_1}{\xi_1}$,

and $\xi_1 = \frac{a_1}{\rho_0} > a_1$. From (1) and (2) above

$$b, \rho_0 < a_1 \leq (b+1)\rho_0.$$

There is equality in this relation if $(b+1)\rho_0$ is an integer. Then $a_1 = (b+1)\rho_0$ and $\xi_1 = \frac{a_1}{\rho_0} = b+1$.

Thus, ξ_1 is an integer and the development ends. Otherwise

$(b+1)\rho_0$ is not an integer, and it follows that

$$b, < \frac{a_1}{\rho_0} < b+1, \quad \text{and so } b, = [\xi_1]$$

Write $\rho_1 = \xi_1 - b,$; $0 < \rho_1 < 1$, so that $\xi_1 = b, + \rho_1$.

$$\text{Now } \rho_1 = \frac{a_1}{\rho_0} - b, = \frac{a_1 g_0 - b, p_0}{p_0} = \frac{p_1}{g_1}$$

where we write: $p_1 = a_1 g_0 - b, p_0$; $g_1 = p_0$.

As $0 < \rho_1 < 1$, $1 \leq p_1 < g_1$, or $1 \leq p_1 < p_0$.

Also, from before, $1 \leq p_0 < p_{-1}$. Thus,

$$1 \leq p_1 < p_0 < p_{-1}$$

We have then :

$$\xi_0 = b_0 + \frac{a_1}{\xi_1}$$

$$b_0 = [\xi_0]$$

$$b_1 = [\xi_1]; \quad \xi_1 = b_1 + \rho_1; \quad 0 < \rho_1 < 1,$$

$$\text{and } \rho_1 = \frac{p_1}{g_1}, \quad p_1 = a_1 g_0 - b, p_0; \quad g_1 = p_0.$$

We continue with ρ_1 as with ρ_0 , to define a_2 by means of the sequence $\{c_v\}$, and so on.

In general, we have :

$$5.1.3 \quad \begin{aligned} \xi_n &= b_n + \rho_n, & 0 < \rho_n < 1; & \quad b_n = [\xi_n], \\ \rho_n &= \frac{p_n}{g_n}, & p_n &= a_n g_{n-1} - b_n p_{n-1} \geq 1 \\ & & g_n &= p_{n-1} \geq 1, \\ 1 \leq p_n &< g_n; & \text{or} & \quad 1 \leq p_n < p_{n-1}, \quad (n=0, 1, 2, \dots). \end{aligned}$$

From (5.1.3), it follows that $p_{-1} > p_0 > p_1 > \dots > p_\nu > \dots \geq 1$.

As the p_ν are positive integers and are strictly decreasing, it follows that either,

- (a) there exists an index λ such that $p_{\lambda-1} \geq 2$ and $(b_\lambda + 1)\rho_{\lambda-1} = (b_\lambda + 1)\frac{p_{\lambda-1}}{g_{\lambda-1}}$ is an integer, or
 (b) there exists an index μ such that $p_{\mu-1} = 1$.

In the case (a), $(b_\lambda + 1)\rho_{\lambda-1}$ an integer, then, as $a_\lambda = (b_\lambda + 1)\rho_{\lambda-1}$, we have $\xi_\lambda = \frac{a_\lambda}{p_{\lambda-1}} = b_\lambda + 1$, an integer. Hence the development terminates, and the theorem is proved.

Otherwise, (b) $p_{\mu-1} = 1$, and $\rho_{\mu-1} = \frac{p_{\mu-1}}{g_{\mu-1}} = \frac{1}{g_{\mu-1}}$. Then, from $\xi_\mu = \frac{a_\mu}{p_{\mu-1}}$, and the above, $\xi_\mu = a_\mu g_{\mu-1}$. Thus, ξ_μ is an integer, and the development terminates.

This proves Theorem 5.1.2 .

5.2 Non - Terminating Continued Fractions of Type 4

If ξ_0 is rational, its continued fraction development of type 4 is terminating, by Theorem 5.1.2 . We saw there that the development ended when ξ_n is an integer. If now the number ξ_0 is irrational, it follows that all the complete quotients ξ_ν are also irrational. For, from

$$\xi_0 = b_0 + \frac{a_1}{b_1} + \dots + \frac{a_{\nu-1}}{b_{\nu-1}} + \frac{a_\nu}{\xi_\nu}$$

we have, as before, that if ξ_0 is irrational, so is ξ_v , and conversely. Therefore, the development of an irrational number $\xi_0 > 0$ is non-terminating.

From the algorithm 5.1 applied to the irrational number ξ_0 , we obtain

$$5.2.1 \quad \xi_0 \sim b_0 + \frac{a_1}{1b_1} + \dots$$

with the additional relations

$$5.2.2 \quad \begin{cases} b_v \geq a_v, & v=1, 2, \dots; & \xi_{v-1} = b_{v-1} + \rho_{v-1} \\ \rho_{v-1} = \frac{a_v}{\xi_v}, & 0 < \rho_v < 1, \\ \xi_v > a_v \geq 1, & & b_v = [\xi_v] \end{cases}$$

Theorem 5.2.1

(a) The development of type 4 of an irrational number $\xi_0 > 0$ converges and represents ξ_0 .

(b) Conversely, an infinite continued fraction of type 4 converges and represents an irrational number.

Proof :

$$\text{Let } (5.2.1) ; \xi_0 \sim b_0 + \frac{a_1}{1b_1} + \dots$$

$$\text{From the algorithm 5.1, } \xi_0 = b_0 + \frac{a_1}{1b_1} + \dots + \frac{a_n}{1\xi_n}$$

$$\text{and } \xi_0 = \frac{\xi_n A_{n-1} + a_n A_{n-2}}{\xi_n B_{n-1} + a_n B_{n-2}} \quad \text{Then}$$

$$\left| \xi_0 - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{a_1 a_2 \dots a_n}{(\xi_n B_{n-1} + a_n B_{n-2}) B_{n-1}} ; n \geq 1.$$

As $\xi_n > b_n$, we have

$$5.2.2 \quad \left| \xi_0 - \frac{A_{n-1}}{B_{n-1}} \right| < \frac{a_1 a_2 \dots a_n}{B_n B_{n-1}}$$

Also, as $b_v \geq a_v$ for $v=1, 2, \dots$ we have :

$B_v = b_v B_{v-1} + a_v B_{v-2} \geq a_v B_{v-1}$ (equality only if $v=1$),
and thus, $B_v > a_v a_{v-1} \cdots a_1$.

As the elements a_v, b_v are positive integers, we
have : $B_n > B_{n-1} > \cdots > B_2 > B_1$. Hence $B_n \uparrow \infty$ as $n \rightarrow \infty$.
Therefore, $\left| \xi_0 - \frac{A_{n-1}}{B_{n-1}} \right| < \frac{1}{B_{n-1}}$ and $\lim_{n \rightarrow \infty} \frac{A_{n-1}}{B_{n-1}} = \xi_0$.

Hence, the development (5.2.1) converges and represents ξ_0 .

We can then write equality in (5.2.1), namely,

$$\xi_0 = b_0 + \frac{a_1}{1b_1} + \frac{a_2}{1b_2} + \cdots$$

(b) Let (5.2.3) $b_0 + \frac{a_1}{1b_1} + \frac{a_2}{1b_2} + \cdots$

be an infinite continued fraction of type 4. As (5.2.3)

has positive elements, we need only prove that

$$\lim_{n \rightarrow \infty} \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = 0 \quad \text{in order that (5.2.3) converges.}$$

Now $\left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{a_1 a_2 \cdots a_n}{B_n B_{n-1}}$ and as $b_v \geq a_v$,
it follows as above that $B_v > a_1 a_2 \cdots a_v$, and

$B_v \uparrow \infty$ as $v \rightarrow \infty$. Thus $\left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| < \frac{1}{B_{n-1}}$,
and clearly, $\lim_{n \rightarrow \infty} \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = 0$. Therefore,

(5.2.3) converges. Call its value ξ_0 . That is, write :

$$\xi_0 = b_0 + \left[\frac{a_v}{b_v} \right]_{\lambda=1}^{\infty}. \quad \text{We prove next that } \xi_0 \text{ is irrational.}$$

The continued fractions $b_\lambda + \frac{a_{\lambda+1}}{1b_{\lambda+1}} + \cdots$, are
all of type 4, ($\lambda = 0, 1, 2, \dots$), and by the preceding, they
are all convergent. We write then : $\xi_\lambda = b_\lambda + \left[\frac{a_{v+1}}{b_v} \right]_{\lambda+1}^{\infty}$.
($\lambda = 1, 2, \dots$).

$$\text{Then, } \xi_0 = b_0 + \frac{a_1}{1b_1} + \cdots + \frac{a_\lambda}{1\xi_\lambda}$$

$$\text{and } \xi_0 = \frac{\xi_\lambda A_{\lambda-1} + a_\lambda A_{\lambda-2}}{\xi_\lambda B_{\lambda-1} + a_\lambda B_{\lambda-2}}, \quad (\lambda \geq 1).$$

It follows then that (a) $|\xi_0 B_{\lambda-1} - A_{\lambda-1}| = \frac{a_\lambda}{\xi_\lambda} |\xi_0 B_{\lambda-2} - A_{\lambda-2}|$,
 and (b) $|\xi_0 B_{\lambda-1} - A_{\lambda-1}| = \frac{a_1 a_2 \dots a_\lambda}{\xi_1 \xi_2 \dots \xi_\lambda}$, for $\lambda = 1, 2, \dots$

Also, as $\xi_\lambda > a_\lambda$, we have $|\xi_0 B_{\lambda-1} - A_{\lambda-1}| < 1$,
 and (c) $|\xi_0 B_{\lambda-1} - A_{\lambda-1}| < |\xi_0 B_{\lambda-2} - A_{\lambda-2}|$, for $\lambda = 2, 3, \dots$

Hence, by the Irrationality Criterion 1.8.1 a, page 59, 60,
 we have that ξ_0 is irrational.

This completes the proof of Theorem 5.2.1

Theorem 5.2.2

Each convergent $\frac{A_n}{B_n}$ to the continued fraction development of type 4 of $\xi_0 > 0$

5.2.3 $\xi_0 = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots$ (terminating or not)
 is a closer approximation to ξ_0 than the preceding convergent.

Proof : Write $\xi_v = b_v + \frac{a_{v+1}}{b_{v+1}} + \dots$; $v = 1, 2, \dots$

Then $\xi_0 = b_0 + \frac{a_1}{b_1} + \dots + \frac{a_{n+1}}{b_{n+1}}$, and, as on page 113,

we have : $\xi_0 = \frac{\xi_{n+1} A_n + a_{n+1} A_{n-1}}{\xi_{n+1} B_n + a_{n+1} B_{n-1}}$, and

$|B_n \xi_0 - A_n| = \frac{a_{n+1}}{\xi_{n+1}} |B_{n-1} \xi_0 - A_{n-1}|$, $n = 1, 2, \dots$

As $\xi_{n+1} > b_{n+1} \geq a_{n+1}$, we have: $|B_n \xi_0 - A_n| < |B_{n-1} \xi_0 - A_{n-1}|$,

or $|\xi_0 - \frac{A_n}{B_n}| < \frac{B_{n-1}}{B_n} |\xi_0 - \frac{A_{n-1}}{B_{n-1}}|$, and certainly, as

$B_n = b_n B_{n-1} + a_n B_{n-2} > B_{n-1}$, we have finally,

$|\xi_0 - \frac{A_n}{B_n}| < |\xi_0 - \frac{A_{n-1}}{B_{n-1}}|$ for $n = 1, 2, \dots$

This proves Theorem 5.2.2 .

Chapter 6

6.1 Algorithm

The following was suggested by Dr. W.C. Taylor.*

Let $\xi_0 > 0$ be a given real number, and suppose that $\{f_n(x)\}$ is a given sequence of strictly monotonic increasing functions, for $n = 0, 1, 2, \dots$. The integer c_0 is defined uniquely by : $f_0(c_0) \leq \xi_0 < f_0(c_0 + 1)$

If $f_0(x)$ has an inverse $f_0^{-1}(x)$, c_0 is defined as : $c_0 = [f_0^{-1}(\xi_0)]$. Clearly, this is equivalent to the preceding definition. For then : $c_0 \leq f_0^{-1}(\xi_0) < c_0 + 1$ and thus $f_0(c_0) \leq \xi_0 < f_0(c_0 + 1)$.

If $\xi_0 = f_0(c_0)$, the development terminates. Otherwise $f_0(c_0) < \xi_0 < f_0(c_0 + 1)$, and we continue by writing :

$$\xi_0 = f_0(c_0) + \frac{f_0(c_0 + 1) - f_0(c_0)}{\xi_1}$$

This defines the number ξ_1 . Evidently, $\xi_1 > 1$. We continue with ξ_1 as with ξ_0 before, replacing $f_0(x)$ by $f_1(x)$. Thus, the integer c_1 is defined by the relations : $f_1(c_1) \leq \xi_1 < f_1(c_1 + 1)$.

If $\xi_1 = f_1(c_1)$, the development terminates, and we have :

$$\xi_0 = f_0(c_0) + \frac{f_0(c_0 + 1) - f_0(c_0)}{f_1(c_1)}$$

Otherwise, $f_1(c_1) < \xi_1 < f_1(c_1 + 1)$, and we continue by writing, as before, $\xi_1 = f_1(c_1) + \frac{f_1(c_1 + 1) - f_1(c_1)}{\xi_2}$.

* See O. Szász (1) "Lectures on Continued Fractions", University of Cincinnati, 1940, - 41, Lecture 13.

Clearly, $\xi_2 > 1$, and so on, with ξ_2 and $f_2(x)$ replacing ξ_1 and $f_1(x)$ respectively. If, for a certain index $n \geq 0$, $\xi_n = f_n(c_n)$; c_n an integer, the development ends. We obtain in this case the finite continued

$$6.1.1 \quad \xi_0 \sim f_0(c_0) + \frac{f_0(c_0+1) - f_0(c_0)}{1} + \dots + \frac{f_{n-1}(c_{n-1}+1) - f_{n-1}(c_{n-1})}{1} + \frac{f_n(c_n)}{1}$$

If for every index v , $f_v(c_v) \neq \xi_v$, where c_v is an integer, the development does not terminate, and one obtains an infinite development for ξ_0 , namely,

$$6.1.2 \quad \xi_0 \sim f_0(c_0) + \frac{f_0(c_0+1) - f_0(c_0)}{1} + \dots$$

We note that for $f_n(x) = x$, $n = 0, 1, 2, \dots$ the above yields the regular continued fraction development of ξ_0 , while for $f_n(x) = x^n$, $n = 0, 1, 2, \dots$ κ an integer, we obtain the Appell development (type 1) of ξ_0 .

6.2 Developments Corresponding to Linear Functions

In this section, we consider the developments obtained for a real number $\xi_0 > 0$ when the linear function $f(x) = ax + b$, $a > 0$, is taken as the sequence of functions $\{f_n(x)\}$ in the Algorithm 6.1. That is,

$$f_v(x) = ax + b \quad \text{for } v = 0, 1, 2, \dots$$

The assumption $a > 0$ is made in order that $f(x) = ax + b$ be strictly increasing. In the following, we shall show that in studying these developments, we need only consider the class of functions $f(x) = ax + b$ satisfying

$$6.2.1 \quad 0 < a + b \leq 1.$$

First, we prove two lemmas .

Lemma 6.2.1

Two linear functions

$$f_1(x) = ax + b \quad ; \quad f_2(x) = ax + b' \quad (a > 0)$$

yield the same development of a number $\xi_0 > 0$ (corresponding to the algorithm 6.1) if and only if $\frac{b-b'}{a}$ is an integer κ ; $b - b' = a \cdot \kappa$.

Proof :

From the algorithm 6.1, the development of ξ_0 corresponding to $f_1(x)$ is: $\xi_0 \sim f_1(c_0) + \frac{f_1(c_0+1) - f_1(c_0)}{f_1(c_0)} + \dots$

or, as $f_1(x) = ax + b$

$$6.2.2 \quad \xi_0 \sim ac_0 + b + \frac{a}{1ac_0 + b} + \frac{a}{1ac_1 + b} + \dots$$

where the c_v are integers, $v = 0, 1, 2, \dots$

Similarly, the development of ξ_0 corresponding to

$$f_2(x) = ax + b' \quad \text{is :}$$

$$6.2.3 \quad \xi_0 \sim ax_0 + b' + \frac{a}{1ax_0 + b'} + \frac{a}{1ax_1 + b'} + \dots$$

where the x_v are integers, $v = 0, 1, \dots$

If the developments (6.2.2) and (6.2.3) are the same, then we must have : $ax_v + b' = ac_v + b$ for $v = 0, 1, \dots$

$$\text{Then } a(x_v - c_v) = b - b' \quad \text{and } x_v - c_v = \frac{b - b'}{a}.$$

As x_v, c_v are integers, $\frac{b - b'}{a}$ must be an integer.

Conversely, assume that $\frac{b - b'}{a}$ is an integer. From the algorithm 6.1, we have :

$$(1) \quad \begin{cases} \xi_0 = (ac_0 + b) + \frac{a}{\xi_1} ; \\ \xi_0 = (ax_0 + b') + \frac{a}{\xi_1} ; \end{cases}$$

with $\xi_1 > 1, \xi_1 > 1$. The integers c_0, x_0 are defined by:

$$c_0 = [f_1^{-1}(\xi_0)] \quad ; \quad x_0 = [f_2^{-1}(\xi_0)].$$

Now $f_1^{-1}(x) = \frac{1}{a}(x-b)$; $f_2^{-1}(x) = \frac{1}{a}(x-b')$

so that : $c_0 = \left[\frac{\xi_0 - b}{a} \right]$; $\gamma_0 = \left[\frac{\xi_0 - b'}{a} \right]$

Write : $\frac{\xi_0 - b}{a} = c_0 + \rho_0$; $0 \leq \rho_0 < 1$. Then

$$\frac{\xi_0 - b'}{a} = \frac{\xi_0 - b}{a} + \frac{b - b'}{a} = c_0 + \frac{b - b'}{a} + \rho_0 .$$

Hence $\gamma_0 = c_0 + \frac{b - b'}{a}$, and so $a\gamma_0 + b' = ac_0 + b$.

From (1), p.116, we have $\xi_1 = \xi_0$. It follows by

complete induction that $a\gamma_r + b' = ac_r + b$, ($r = 0, 1, 2, \dots$).

Hence the developments (6.2.2) and (6.2.3) are identical.

This proves the Lemma 6.2.1 .

We call $f_1(x)$ and $f_2(x)$ equivalent, $f_1(x) \sim f_2(x)$ if $f_1'(x) = f_2'(x) = a$, and $\frac{f_1 - f_2}{f_1'} = \kappa$, an integer.

Lemma 6.2.2

Let $f(x) = ax + b$, where $a \geq b > 1$.

Then all the numbers ξ , $1 < \xi < b$, have the same development, namely : $\xi \sim (-a + b) + \frac{a1}{1-a+b} + \frac{a1}{1-a+b} + \dots$

Proof :

Let $\xi_0 > 0$ be such that $1 < \xi_0 < b$

We have $c_0 = [f^{-1}(\xi_0)] = \left[\frac{\xi_0 - b}{a} \right] = -1$;

as $0 > \frac{\xi_0 - b}{a} > -\frac{b}{a} \geq -1$. Then $f(c_0) = -a + b \leq 0$,

and $\xi_1 = (-a + b) + \frac{a}{\xi_1}$, $\xi_1 > 1$.

Now $\xi_1 = \frac{a}{\xi_0 + a - b}$, and as $\xi_0 > 1$, $\xi_1 < \frac{a}{1 + a - b}$;

$$\xi_1 - b < \frac{a}{1 + a - b} - b = \frac{a - b - ab + b^2}{1 + a - b}$$

$$\xi_1 - b < \frac{(a-b)(1-b)}{1+a-b} \leq 0 \quad \text{as } a-b \geq 0 , 1-b < 0 .$$

Thus, $1 < \xi_1 < b$ and ξ_1 has the same properties as ξ_0

It follows by induction that all ξ_r , $r \geq 1$ have the

same properties, namely : $1 < \xi_v < b$ for $v = 1, 2, \dots$

Hence, $\xi_0 \sim (-a + b) + \frac{a}{1-a+b} + \dots$

Clearly, by the above, all ξ , $1 < \xi < b$, have the same development corresponding to the function $f(x) = ax + b$, $a \geq b > 1$. This proves Lemma 6.2.2.

In view of this Lemma, we shall exclude the development corresponding to a linear function $f(x) = ax + b$ with $a \geq b > 1$. Hence, we assume that either $a < b$ or $b \leq 1$ or both.

Theorem 6.2.1

Corresponding to the linear function

$f(x) = ax + b$, ($a > 0$) and where $a < b$ if $b > 1$,

used in the development of a real number $\xi_0 > 0$ according to the algorithm 6.1, there exists a linear function

$f_1(x) = a'x + b'$ with $0 < a + b' \leq 1$

and such that $f(x) \sim f_1(x)$: i.e. the two functions yield the same development of ξ_0 .

Proof :

The proof is divided into three parts.

1° . Consider the case where $f(x) = ax + b$, and $a = b$.

Then by the assumption above the theorem, $0 < a \leq 1$

The function $f_1(x) = ax$, $b' = 0$ is the desired one.

For, as $b' = 0$, $\frac{b-b'}{a} = \frac{b}{a} = 1$, and by Lemma 6.2.1,

$f(x) \sim f_1(x)$. Also, as $a + b' = a$, we have

$$0 < a + b' < 1.$$

2° Consider next the case where, in the function $f(x) = ax + b$, we have $b > a > 0$.

Let m be the largest integer γ such that $b \leq (\gamma+1)a$.
Then $m \geq 1$, and $m \cdot a < b \leq (m+1)a$.

Write $b' = b - (m+1)a$. Then $\frac{b-b'}{a} = m+1$, is an integer.

We show next that $0 < a + b' < 1$.

From the definition of m : $0 < b - ma \leq a$.

(a) Suppose now that $a \leq 1$. Then $0 < b - ma \leq 1$

But $a + b' = a + b - (m+1)a = b - ma$. Hence $0 < a + b' \leq 1$.

(b) Suppose that $a > 1$. Then either $b - ma > 1$ or $b - ma \leq 1$. Assume that $b - ma > 1$, and write $b_1 = b - ma$. Then $a \geq b_1 > 1$. Also $\frac{b-b_1}{a} = m$ is an integer, so that the developments of ξ corresponding to the functions $f_1(x) = ax + b$ and $f_2(x) = ax + b_1$ are the same. But from Lemma 6.2.2, and the remark on p.118, this case, namely, $a \geq b_1 > 1$ is excluded. Therefore, we must have $0 < b - ma \leq 1$. Accordingly, $0 < a + b' \leq 1$ and the function $f_1(x) = ax + b'$ is the desired one.

3° Consider the case where $a > b$, ($a > 0$). The case $b > 1$ is ruled out. There remains then the cases to consider where (a) $0 < b \leq 1$ and (b) $b \leq 0$.

(a) Write $b' = b - a$. Clearly, $\frac{b-b'}{a}$ is an integer, so that Lemma 6.2.1 is satisfied. Also, $a + b' = b$ and clearly, $0 < a + b' \leq 1$. The function $f_1(x) = ax + b'$ with $b' = b - a$ is then the required one.

Consider next the possibility (b) $b \leq 0$, $a > 0$.
 First, let $a > 1$. We need consider only those b
 for which $0 \geq b > 1-a$, and $b \leq -a$. For, other-
 wise, $-a \leq b \leq 1-a$ and so $0 < a+b \leq 1$,
 and we would choose $b' = b$.

Let then, $a > 1$, $0 \geq b > 1-a$.

This is the exceptional case. For, by writing $b_1 = a+b$,
 we have $\frac{b_1 - b}{a} = 1$. Also, as $0 \geq b > 1-a$,
 we have $a \geq a+b > 1$, so that $a \geq b_1 > 1$,
 which is the exceptional case.

Next consider $a > 1$, $b \leq -a$. We can not have
 equality, for if $b = -a < -1$, then by writing
 $b_1 = b + 2a = a$, we would have $a = b_1 > 1$,
 $\frac{b_1 - b}{a} = 2$, and this is the exceptional case. Thus, we
 must have $b < -a$ ($a > 1$). Let m be the least integer δ
 such that $b + \delta a > a$. Then $b + ma > a \geq b + (m+1)a$.
 and clearly, as $b < -a$, $m \geq 3$. Write $b_1 = b + ma$
 Then as $\frac{b_1 - b}{a} = m$ is an integer, the development of
 corresponding to $f(x) = ax + b$ and $g(x) = ax + b_1$,
 are the same. Now $b_1 > a > 1$, and by 2° (b), there
 exists a number b' such that $\frac{b_1 - b'}{a}$ is an integer
 and $0 < a + b' \leq 1$. The function $f_1(x) = ax + b'$,
 is the desired one.

The only remaining case to consider is that where $b \leq 0$
 and $0 < a \leq 1$. If $-a < b \leq 0$, then $0 < a+b \leq a \leq 1$,
 and the choice of $b' = b$ would suffice. Assume then that
 $b \leq -a < 0$

Let m be the least integer γ such that $b + (\gamma+1)a > 0$. Then $b + ma > a \geq b + (m-1)a$ and we have $0 < b + (m+1)a \leq a$. Write $b' = b + ma$. Then $0 < a + b' \leq a < 1$, and $\frac{b'-b}{a} = m$ is an integer. The required function is then $f(x) = ax + b'$.

This completes the proof of Theorem 6.2.1

Remarks :

In studying the developments of a real number ξ , ($\xi > 0$), by the algorithm 6.1, and corresponding to the linear function $f(x) = ax + b$, we can restrict ourselves to the class of linear functions $f(x) = ax + b$ where $a > 0$ and $0 < a + b \leq 1$, by Theorem 6.2.1.

Definition 6.2.1

The development of a number $\xi_0 > 0$ into a continued fraction by the algorithm 6.1, and corresponding to the linear function $f(x) = ax + b$ where $a > 0$, $0 < a + b \leq 1$, will be called the development of type 5 of ξ_0 .

Let $\xi_0 > 0$ and let $f(x) = ax + b$.

If we develop ξ_0 into a continued fraction by the algorithm 6.1, we have: $c_0 = [f^{-1}(\xi_0)] = \left[\frac{\xi_0 - b}{a} \right]$.

Then c_0 may be positive, zero, or negative. Also,

$\xi_0 = (ac_0 + b) + \frac{a_1}{\xi_1}$, $\xi_1 > 1$. As $\xi_1 > 1$, $\xi_1 > a + b$,

and $\frac{\xi_1 - b}{a} > 1$. Hence, we have: $c_1 = [f^{-1}(\xi_1)] = \left[\frac{\xi_1 - b}{a} \right] \geq 1$,

and by the same argument, we have $c_r \geq 1$ for $r = 1, 2, \dots$

Consequently, $f(c_r) = ac_r + b \geq a + b > 0$, so that the continued fraction development of type 5 of ξ_0 , namely :

$$6.2.4 \quad \xi_0 \sim ac_0 + b + \frac{a}{1ac_1 + b} + \frac{a}{1ac_2 + b} + \dots$$

has positive elements. ($ac_0 + b$ may be ≤ 0)

Definition 6.2.2

A continued fraction

$$6.2.5 \quad ac_0 + b + \frac{a}{1ac_1 + b} + \frac{a}{1ac_2 + b} + \dots$$

terminating or not, such that the c_r are positive integers, $c_r \geq 1$ for $r=1, 2, \dots$ while c_0 may be a negative integer or zero, and, in addition, $a > 0$, $0 < a + b \leq 1$ will be called a continued fraction \mathfrak{f} of type 5. If (6.2.5) is terminating, and n is the final index, then $c_n \geq 2$.

Theorem 6.2.2

A continued fraction of type 5 (6.2.5) converges, and if it is a development (6.2.4) of the number ξ_0 , then it represents ξ_0 .

Proof :

Denote by $\left\{ \frac{A_n}{B_n} \right\}$ the sequence of convergents to (6.2.5). Evidently, we need only consider the case where (6.2.5) is non-terminating. As (6.2.5) has positive elements, it is sufficient to show that $\lim_{n \rightarrow \infty} \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = 0$ in order to prove convergence.

$$\text{We have } \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{a^n}{B_n B_{n-1}} \quad (\text{Let } n \geq 2)$$

$$\text{Write } u_n = \frac{a^n}{B_n B_{n-1}}. \text{ Then } \frac{u_{n+1}}{u_n} = \frac{a^{n+1} B_{n-1} B_n}{a^n B_{n+1} B_n} = \frac{a B_{n-1}}{B_{n+1}}.$$

From $B_{n+1} = (ac_{n+1} + b)B_n + a B_{n-1}$ we obtain :

$$\frac{B_{n+1}}{a B_{n-1}} = 1 + \frac{(ac_{n+1} + b) B_n}{a B_{n-1}} > 1 + \frac{(a+b)(a+b)}{a}$$

since $B_n = (ac_n + b)B_{n-1} + aB_{n-2} > (a+b)B_{n-1}$

Thus : $\frac{B_{n+1}}{aB_{n-1}} > 1 + \frac{(a+b)^2}{a}$ and

$$\frac{aB_{n-1}}{B_{n+1}} < 1 - \frac{(a+b)^2}{a + (a+b)^2}$$

Write $\mu = 1 - \frac{(a+b)^2}{a + (a+b)^2}$. Then we have

$$\frac{u_{n+1}}{u_n} = \frac{aB_{n+1}}{B_{n+1}} < \mu < 1 \quad \text{for } n \geq 2.$$

and so, $u_{n+1} < \mu^{n-1} u_2$; $u_{n+1} < \mu^{n-1} \cdot \frac{a^2}{B_1 B_2}$.

Clearly, as $0 < \mu < 1$, $\lim_{n \rightarrow \infty} u_{n+1} = 0$. That is,

$$\lim_{n \rightarrow \infty} \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{a^n}{B_n B_{n-1}} = 0.$$

Hence the continued fraction (6.2.5) converges.

If (6.2.5) is the development of the number ξ_0 then, from the algorithm, we have : $\xi_0 = (ac_0 + b) + \frac{a1}{1ac_1 + b} + \dots + \frac{a1}{1\xi_n}$

$\xi_n > 1$. Indeed, $\xi_n > f(c_n) = ac_n + b$.

We have : $\xi_0 = \frac{\xi_n A_{n+1} + a A_{n+2}}{\xi_n B_{n+1} + a B_{n+2}}$ and

$$\left| \xi_0 - \frac{A_{n+1}}{B_{n+1}} \right| = \frac{a^n}{B_{n+1} (\xi_n B_{n+1} + a B_{n+2})} < \frac{a^n}{B_n B_{n+1}}$$

In the preceding paragraph, we showed that $\lim_{n \rightarrow \infty} \frac{a^n}{B_n B_{n+1}} = 0$.

Hence $\lim_{n \rightarrow \infty} \frac{A_{n+1}}{B_{n+1}} = \xi_0$, and the continued fraction development (6.2.4) represents ξ_0 , g. e. d.

In the following paragraphs, we specialize the class of linear functions used in the algorithm 6.1.

Consider those linear functions $f(x) = ax + b$ with $a > 0$, $0 < a + b \leq 1$, and such that $ax + b$ takes on each of the integral values 1, 2, 3, for integral values of x . This is the class of functions

$f(x) = \frac{x}{m}$ where $m \geq 1$ is a positive integer.

This can be verified as follows :

The slope of a line belonging to the class described above must be $\frac{1}{m}$, m a positive integer; $m \geq 1$.

Write $f(x) = \frac{1}{m}x + b$. Then, by assumption,
 $0 < \frac{1}{m} + b \leq 1$ That is, $f(1) = \frac{1}{m} + b \leq 1$.

Then, of the m values $1, 2, \dots, m$, there is exactly one say p , for which $f(p) = 1$; $1 \leq p \leq m$. That is,

$$\frac{p}{m} + b = 1 \quad \text{and so} \quad b = 1 - \frac{p}{m} = \frac{m-p}{m}.$$

Thus, $f(x) = \frac{1}{m}x + \frac{m-p}{m}$.

Now $\frac{b}{a} = \left(\frac{m-p}{m}\right)m = m-p$, is an integer, $(a = \frac{1}{m})$

Hence, by Lemma 6.2.2, the functions $f(x) = \frac{1}{m}x + b$,
 $f_1(x) = \frac{1}{m}x$ are equivalent.

Theorem 6.2.2

The development of type 5 of a rational number ξ_0 corresponding to the sequence of functions $\{f_n(x)\}$, $f_n(x) = \frac{x}{a_n}$ where the a_n are positive integers for $n=0, 1, 2, \dots$, $a_n \geq 1$, is terminating.

Proof : Let $\xi_0 = \frac{p_0}{p_1}$; (p_0, p_1 are positive integers

$p_0 \geq 1, p_1 \geq 1$). We note first that $f_n^{-1}(x) = a_n x$.

By the algorithm 6.1, we have : $c_0 = [f_0^{-1}(\xi_0)] = [a_0 p_0 / p_1]$

If $c_0 = \frac{a_0 p_0}{p_1}$; ($a_0 p_0 - c_0 p_1 = 0$), the development ends, and we have $\xi_0 = \frac{p_0}{p_1} = \frac{c_0}{a_0}$.

Otherwise, $c_0 < \frac{a_0 p_0}{p_1} < c_0 + 1$, and

$$(1) \quad 0 < a_0 p_0 - c_0 p_1 < p_1$$

and we continue by writing : $\xi_0 = \frac{p_0}{p_1} = c_0 \cdot \frac{1}{a_0} + \frac{1}{\xi_1}$
 ($\xi_1 > 1$)

Then $\frac{f}{\xi_1} = \frac{a_0 p_0 - c_0 p_1}{a_0 p_1 - c_0 p_1}$ and $\xi_1 = \frac{p_1}{a_0 p_0 - c_0 p_1} = \frac{p_1}{p_2}$

where we have written $p_2 = a_0 p_0 - c_0 p_1$.

Clearly from (1) (bottom of p. 124) $p_1 > p_2 > 0$,

and again we have $\xi_1 > 1$.

As before, write $c_1 = [f_1''(\xi_1)] = [a_1 \frac{p_1}{p_2}]$. Clearly $c_1 \geq a_1$.
 If $c_1 = \frac{a_1 p_1}{p_2}$, or $a_1 p_1 - c_1 p_2 = 0$, the development terminates with the equation, $\xi_1 = \frac{p_1}{p_2} = \frac{c_1}{a_1}$.
 Also, as $\xi_1 > 1$, $c_1 > a_1$.

Otherwise, we have : $c_1 < a_1 \frac{p_1}{p_2} < c_1 + 1$, and so
 (2) $0 < a_1 p_1 - c_1 p_2 < p_2$.

We continue by writing : $\xi_1 = \frac{p_1}{p_2} = c_1 \cdot \frac{1}{a_1} + \frac{1}{\xi_2}$
 $\xi_2 > 1$. Then $\xi_2 = \frac{p_2}{a_1 p_1 - c_1 p_2} = \frac{p_2}{p_3}$, with

$$p_3 = a_1 p_1 - c_1 p_2$$

From (2), it follows that

$$p_2 > p_3 > 0, \text{ and so } \xi_2 = \frac{p_2}{p_3} > 1.$$

We continue then with ξ_2 in place of ξ_1 , and

$$f_2(x) = \frac{1}{a_2} x \text{ in place of } f_1(x).$$

In general, for an arbitrary index $v \geq 1$, $\xi_{v-1} = \frac{p_{v-1}}{p_v} > 1$,
 (3) $p_{v+1} = a_{v+1} p_v - c_{v+1} p_{v+1} \geq 0$.

and if $p_{v+1} > 0$, $c_v = [a_v \frac{p_v}{p_{v+1}}] \geq a_v$, and $\xi_v = \frac{p_v}{p_{v+1}} > 1$.

We have then that : $p_0 > p_1 > p_2 > \dots > p_{v-1} > p_v > \dots > 0$.

The integers p_v are non - negative and are constantly

decreasing. Hence, there exists an index $n \geq 0$ such that

$$p_{n+2} = a_n p_n - c_n p_{n+1} = 0$$

Then $c_n = \frac{a_n p_n}{p_{n+1}} = [f_n^{-1}(\xi_n)] = f_n^{-1}(\xi_n)$,

and the development terminates. We obtain for ξ_0 the finite development

$$6.2.7 \quad \xi_0 = \frac{p_0}{p_1} = \frac{1}{a_0} c_0 + \frac{\frac{1}{a_0} |}{\frac{1}{a_1} \cdot c_1} + \frac{\frac{1}{a_1} |}{\frac{1}{a_2} \cdot c_2} + \dots + \frac{\frac{1}{a_{n-1}} |}{\frac{1}{a_n} \cdot c_n}$$

with $c_v \geq a_v$ for $1 \leq v \leq n-1$ and $c_n > a_n$.

This proves Theorem 6.2.3 .

With the continued fraction development (6.2.7), we can associate the system of equations (from (3) page 125)

$$6.2.8. \quad \begin{cases} a_0 p_0 = c_0 p_1 + p_2 \\ a_1 p_1 = c_1 p_2 + p_3 \\ \dots \\ a_{n-1} p_{n-1} = c_{n-1} p_n + p_{n+1} \\ a_n p_n = c_n p_{n+1} \end{cases}, \quad p_{n+2} = 0.$$

where the a_v, p_v, c_v , are positive integers, $p_v > p_{v+1} \geq 1$, and $c_v = [\frac{a_v p_v}{p_{v+1}}] \geq a_v$, $1 \leq v \leq n-1$ and $c_n > a_n$.

From (6.2.8), we obtain immediately that :

$$6.2.9 \quad \frac{a_0 p_0}{p_1} = c_0 + \frac{a_1 |}{1 c_1} + \frac{a_2 |}{1 c_2} + \dots + \frac{a_n |}{1 c_n}.$$

One can obtain this directly from (6.2.7) by an equivalence transformation, namely :

$$\gamma_0 \xi_0 = \frac{\gamma_0 p_0}{p_1} = \gamma_0 \frac{c_0}{a_0} + \frac{\gamma_0 \gamma_1 \frac{1}{a_1} |}{1 \gamma_1 \frac{c_1}{a_1}} + \dots + \frac{\gamma_{v-1} \gamma_v \frac{1}{a_{v-1}} |}{1 \gamma_v \frac{c_v}{a_v}} + \dots$$

where we choose $\gamma_v = a_v$ for $v=0, 1, 2, \dots, n$.

We have already noted that the c_v, a_v are positive integers, and also that $c_v \geq a_v$, ($v=0, 1, 2, \dots, n-1$); $c_n > a_n$. It follows that the continued fraction (6.2.9) is of type 4.

Indeed, (6.2.9) is exactly Leighton's^{*} " Proper Continued Fraction Development " of the number $\xi = \frac{a_0 p_0}{p_1}$ corresponding to the given sequence of positive integers $\{a_n\}$. Thus we see that the development of type 5 of a number $\xi_0 > 0$ corresponding to the given sequence of functions $\{f_n(x)\}$ $f_n(x) = \frac{1}{a_n} x$ where $a_n \geq 1$ are positive integers, is (except for an equivalence transformation) the proper continued fraction development of the number ξ_0 corresponding to the given sequence of positive integers $\{a_n\}$.

Let the given sequence of linear functions be :
 $f_n(x) = \frac{1}{g} \cdot x$, $(n = 0, 1, \dots)$ with g a positive integer, $g \geq 1$. Then, by Theorem 6.2.2 , the development of a rational number ξ_0 corresponding to this particular sequence of functions is terminating. However, if we take $f_n(x) = \frac{1}{g} x + \frac{1}{p}$, $(n \geq 0)$, where $g > 0$ and p are integers such that $0 < \frac{1}{g} + \frac{1}{p} \leq 1$, it does not follow that a rational number ξ_0 will necessarily have a terminating development corresponding to $f(x) = \frac{1}{g} x + \frac{1}{p}$.

In this connection, we prove the theorem :

* Leighton, W . American Mathematical Monthly, May 1940, vol.47, no.5, page 274.

Theorem 6.2.3 If $f(x) = \frac{1}{g}x + \frac{1}{p}$, $0 < \frac{1}{g} + \frac{1}{p} \leq 1$, where $g > 0$, and p are integers, and if, in addition, $\frac{g+1}{p}$ is an integer, then the development of type 5 of $\xi_0 = |p|$ corresponding to $f(x) = \frac{1}{g}x + \frac{1}{p}$ is periodic with a one - term period. It is purely periodic if p is positive, and the period begins after the first term if p is negative.

Proof :

Assume first that : (a) p is positive, $p > 0$. Then as $g \geq 1$, and $0 < \frac{1}{g} + \frac{1}{p} \leq 1$, we must have $p \geq 2$. We have $f'(x) = \frac{1}{g}(x - \frac{1}{p})$ and so, $c_0 = [f'(\xi_0)]$
 $c_0 = [g(p - \frac{1}{p})] = gp + [-\frac{g}{p}]$.

Now as $\frac{g+1}{p}$ is an integer, it follows that $[-\frac{g}{p}] = -\frac{g-1}{p}$.

Thus, $c_0 = gp - \frac{g-1}{p}$, and so, by the algorithm,

$$\xi_0 = p = \frac{1}{g} \left\{ gp - \frac{g-1}{p} \right\} + \frac{1}{p} + \frac{1}{\xi_1}, \quad \text{or}$$

$$pg = pg - \frac{g-1}{p} + \frac{g}{p} + \frac{1}{\xi_1}$$

$$\text{and } \frac{1}{\xi_1} = \frac{1}{p}; \quad \xi_1 = p = \xi_0.$$

Hence, the development of $\xi_0 = p > 0$ is purely periodic with a one - term period.

(b) Assume next that p is negative. $\xi_0 = |p|$.

Then as : $0 < \frac{1}{g} + \frac{1}{p} \leq 1$, we must have

$|p| > g$ Thus, $\frac{g+1}{p}$ can be an integer only if

$g+1 = -p$. Assume then that $\frac{g+1}{p} = -1$. As $\xi_0 = |p|$,

we have $c_0 = [f'(\xi_0)] = [g(|p| - \frac{1}{p})] = g \cdot |p| - [-\frac{g}{p}]$.

and as $\frac{g+1}{p} = -1$, $-\frac{g}{p} = 1 + \frac{1}{p}$,

and finally $0 < -\frac{g}{p} < 1$, ($p < 0$)

Thus $c_0 = g|p|$, and so $\xi_0 = |p| =$

$$|p| = \left(g|p| \cdot \frac{1}{g} + \frac{1}{p} \right) + \frac{1}{g/\xi_1}$$

$$= |p| + \frac{1}{p} + \frac{1}{g\xi_1},$$

and

$$\xi_1 = -\frac{p}{g} = \frac{|p|}{g} > 1.$$

Then $c_1 = \left[g \left(\frac{|p|}{g} - \frac{1}{p} \right) \right] = \left[|p| - \frac{g}{p} \right]$, and as $0 < -\frac{g}{p} < 1$,

$$c_1 = |p|.$$

It follows that: $\xi_1 = \frac{|p|}{g} = \left(\frac{1}{g}|p| + \frac{1}{p} \right) + \frac{1}{\xi_2}$,

$$\xi_2 = -\frac{p}{g} = \frac{|p|}{g}.$$

Thus $\xi_2 = \xi_1 = \frac{|p|}{g}$, and the development of type 5 of $\xi_0 = |p|$ is periodic from this point on, and with a one-term period.

This proves the Theorem 6.2.3 .