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ON SOME TRIANGULAR SUMMABILITY METHODS

1. Given a sequence-to-function transformation

$$(1.1) \quad \psi(x) = \sum_{\nu=0}^{\infty} s_{\nu} \psi_{\nu}(x)$$

where $\{\psi_{\nu}(x)\}$ is a sequence of functions defined on a point set E having limit point ξ not belonging to the set, then the generalized limit of the sequence $\{s_{\nu}\}$ is defined by

$$\lim_{x \rightarrow \xi} \psi(x) = s = \text{gen. lim. } s_{\nu}$$

if the left hand limit exists. In a recent paper^{1*}, Professor Szász considered the question of selecting a sequence of points $\{x_m\}$ belonging to the point set E , $x_m \rightarrow \xi$ as $n \rightarrow \infty$, and associating with it the sequence-to-sequence triangular transformation

$$(1.2) \quad B_m(x_m) = \sum_{\nu=0}^m s_{\nu} \psi_{\nu}(x_m), \quad m = 0, 1, 2, \dots,$$

defining now the generalized limit of the sequence $\{s_{\nu}\}$

by

$$\lim_{n \rightarrow \infty} B_m(x_m) = s = \text{gen. lim. } s_{\nu}$$

(It is noted that $B_m(x)$ is defined by taking the first $n+1$ terms of the series defining the function $\psi(x)$,

* The superscripts refer to the notes and references at the end of the paper.

and then constructing the summability method with the sequence $\{\chi_m\}$.)

As set forth in the aforementioned paper of Professor Szász, the regularity of either method (1.1) or (1.2) does not imply that of the other. The conditions of regularity for the sequence-to-sequence transformation (1.2) are

$$(1.3) \quad \lim_{m \rightarrow \infty} \psi_\nu(\chi_m) = 0, \quad \nu = 0, 1, 2, \dots,$$

$$(1.4) \quad \sum_{\nu=0}^m |\psi_\nu(\chi_m)| < K, \quad , K \text{ independent of } m,$$

$$(1.5) \quad \lim_{m \rightarrow \infty} \sum_{\nu=0}^m \psi_\nu(\chi_m) = 1.$$

The purpose of this paper is to consider the triangular matrix transformation which results when $\psi_\nu(x)$ is taken equal to $e^{-x} \frac{x^\nu}{\nu!}$; that is when the generating process is applied to Borel's exponential method. The generalized limit of the sequence $\{s_\nu\}$ is now defined by

$$(1.6) \quad \text{gen. lim } s_\nu = s = \lim_{m \rightarrow \infty} e^{-\chi_m} \sum_{\nu=0}^m \frac{s_\nu \chi_m^\nu}{\nu!}, \quad \chi_m \rightarrow \infty.$$

2. The first problem considered for this summability method is the determination of necessary and sufficient conditions on the sequence of points $\{\chi_m\}$, $\chi_m \rightarrow \infty$, so that the regularity conditions (1.3),

(1.4), and (1.5) are satisfied. It is easily seen that the conditions (1.3) and (1.4) are here fulfilled with every sequence $\{x_n\}$, so long as $x_n \rightarrow \infty$. Therefore, in order to insure regularity, it remains to choose the sequence $\{x_n\}$ so that

$$(2.1) \quad g_n(x_n) \equiv g_n = e^{-x_n} \sum_{\nu=0}^n \frac{x_n^\nu}{\nu!} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

The fundamental theorem in this section, solving the problem, is based on the Gaussian limit:

$$(A) \quad \lim_{x \rightarrow \infty} e^{-x} \sum_{\nu < x + w\sqrt{x}} \frac{x^\nu}{\nu!} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^w e^{-t^2/2} dt \equiv p(w).$$

The following remarks concerning the function are easily verified:

- (B)
1. $p(-\infty) = \lim_{w \rightarrow -\infty} p(w) = 0$,
 2. $p(0) = 1/2$,
 3. $p(\infty) = \lim_{w \rightarrow \infty} p(w) = 1$,
 4. $p(w) \uparrow$ continuously as $w \uparrow$.

Lemma The condition

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{x_n - n}{\sqrt{n}} = -\infty, \quad \lim_{n \rightarrow \infty} x_n = \infty$$

is necessary and sufficient that

$$(2.1) \quad g_n(x_n) = e^{-x_n} \sum_{\nu=0}^n \frac{x_n^\nu}{\nu!} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Proof (a) Sufficiency: If $\frac{x_n - n}{\sqrt{n}} \rightarrow -\infty$

as $n \rightarrow \infty$, we have for all positive K , however large

$$x_n - n < -K\sqrt{n}, \quad n > n_0 = n(K)$$

$$x_n + \left(\frac{K}{2}\right)^2 < n - K\sqrt{n} + \left(\frac{K}{2}\right)^2 = \left(\sqrt{n} - \frac{K}{2}\right)^2.$$

Taking now n so large that $\sqrt{n} - \frac{K}{2} > 0$, $n > n_0$,

we have

$$\sqrt{x_n + \frac{K^2}{4}} + \frac{K}{2} < \sqrt{n},$$

$$x_n + K\sqrt{x_n + \frac{K^2}{4}} + \frac{K^2}{2} < n.$$

It therefore follows that for n sufficiently large

$$x_n + K\sqrt{x_n} < n.$$

From this last inequality, we get

$$e^{-x_n} \sum_{\nu \leq x_n + K\sqrt{x_n}} \frac{x_n^\nu}{\nu!} \leq e^{-x_n} \sum_{\nu \leq n} \frac{x_n^\nu}{\nu!} = g_n(x_n) < 1.$$

The assumption that $x_n \rightarrow \infty$ as $n \rightarrow \infty$ and (A) now give

$$p(K) \leq \liminf g_n(x_n) \leq \limsup g_n(x_n) \leq 1.$$

Inasmuch as K may be chosen arbitrarily large, it follows from (B) that

$$\lim g_n(x_n) = 1,$$

thus proving the sufficiency of (2.2) for (2.1).

(b) Necessity: Suppose, contrary to (2.2),

$$\lim \frac{x_n - n}{\sqrt{n}} = a > -\infty.$$

We take first the case when a is finite. Then for any positive ϵ and with a sequence of integers, $\{n_i\}_{i=0}^{\infty}$, depending on ϵ , we have

$$x_{m_i} - n_i \geq (a - \epsilon) \sqrt{n_i}, \quad i = 0, 1, 2, \dots, \quad x_{m_i} \uparrow \infty.$$

Writing $\alpha = a - \epsilon$, it follows that

$$x_{m_i} + \frac{\alpha^2}{4} \geq n_i + \alpha \sqrt{n_i} + \frac{\alpha^2}{4} = \left(\sqrt{n_i} + \frac{\alpha}{2} \right)^2.$$

Taking i large enough so that $\sqrt{n_i} + \frac{\alpha}{2}$ is positive, we get

$$n_i \leq x_{m_i} - \alpha \sqrt{x_{m_i} + \frac{\alpha^2}{4}} + \frac{\alpha^2}{2}.$$

Hence, since $x_{m_i} \rightarrow \infty$ as $n_i \rightarrow \infty$, with $b = \alpha - \epsilon = a - 2\epsilon$

$$n_i \leq x_{m_i} - b \sqrt{x_{m_i}}, \quad i > i_0 = i_0(\epsilon).$$

From this last inequality, it follows that

$$g_{m_i}(x_{m_i}) = \sum_{\nu=m_i}^{\nu} \frac{x_{m_i}^{\nu}}{\nu!} \leq \sum_{\nu=x_{m_i}-b\sqrt{x_{m_i}}}^{\nu} \frac{x_{m_i}^{\nu}}{\nu!}.$$

(A) and (B) now show that

$$\lim g_{m_i} \leq p(-b) = p(-a + 2\epsilon) < 1$$

which is inconsistent with (2.1).

If a is infinite, the above procedure remains the same except that α can now be any finite number. This completes the proof of the necessity of (2.2) for (2.1).

Q.E.D.

(Note: It is not difficult to see that $\lim g_n = f(-a)$.

Theorem 2.1

A necessary and sufficient

condition that the summability method defined by (1.6)

be regular is that

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{\chi_n - n}{\sqrt{n}} = -\infty, \quad \chi_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

The theorem follows immediately from the previous lemma and the fact that (1.3) and (1.4) are satisfied.

As has already been noted, the summability method (1.6) is boundedness preserving for all sequences $\{\chi_n\}$, $\chi_n \rightarrow \infty$. Theorem 2.1 gives a necessary and sufficient condition for the regularity of the method. Nevertheless, if (2.2) is not satisfied by a sequence of points $\{\chi_n\}$, we may still have a method which is convergence preserving. In fact, the method may then be considered almost regular except for a constant factor, independent of the sequence being summed and dependent only on $\{\chi_n\}$. We proceed to demonstrate these assertions.

Lemma

A necessary and sufficient condition that

$$(2.3) \quad g_n(\chi_n) = e^{-\chi_n} \sum_{\nu=0}^n \frac{\chi_n^\nu}{\nu!} \rightarrow \lambda \quad \text{as } n \rightarrow \infty$$

is that

$$(2.4) \quad \lim_{\chi_m \rightarrow \infty} \frac{\chi_m^{-n}}{\sqrt{n}} = -\rho, \quad \chi_m \rightarrow \infty$$

where $p(\rho) = \lambda$, i.e. $\rho = p_1(\lambda)$.

(Note: If $0 < \lambda < 1$, ρ is finite.)

Proof (a) Sufficiency: We consider first the case where ρ is finite. It follows from (2.4) that for every positive ϵ , however small, there exists an integer $N_0 = N_0(\epsilon)$ so that

$$n - (\rho + \epsilon)\sqrt{n} \leq \chi_m \leq n - (\rho - \epsilon)\sqrt{n}, \quad n > N_0.$$

Transforming the above inequalities by the same method as was used in the proof of the previous lemma in this section and choosing n so large that $\sqrt{n} > \frac{\rho + \epsilon}{2}$, $n > N_0$, we obtain

$$\chi_m + (\rho - \epsilon)\sqrt{\chi_m + \frac{(\rho - \epsilon)^2}{4}} + \frac{(\rho - \epsilon)^2}{2} \leq n \leq \chi_m + (\rho + \epsilon)\sqrt{\chi_m + \frac{(\rho + \epsilon)^2}{4}} + \frac{(\rho + \epsilon)^2}{2}$$

Inasmuch as $\chi_m \rightarrow \infty$ as $n \rightarrow \infty$, we have for n large enough

$$\chi_m + (\rho - 2\epsilon)\sqrt{\chi_m} \leq n \leq \chi_m + (\rho + 2\epsilon)\sqrt{\chi_m}$$

From this it immediately follows that

$$e^{-\chi_m} \sum_{\nu \leq \chi_m + (\rho - 2\epsilon)\sqrt{\chi_m}} \frac{\chi_m^\nu}{\nu!} \leq g_n \leq e^{-\chi_m} \sum_{\nu \leq \chi_m + (\rho + 2\epsilon)\sqrt{\chi_m}} \frac{\chi_m^\nu}{\nu!};$$

letting n become infinite, we get from (A)

$$p(\rho - 2\epsilon) \leq \liminf g_n \leq \limsup g_n \leq p(\rho + 2\epsilon)$$

From the continuity of the function $f(\omega)$, the fact that ϵ may be chosen arbitrarily close to zero, and the well known squeeze principle, we see at once that

$$\lim g_n = f(\rho) = \lambda$$

The case of $\rho = +\infty$ was discussed in the first lemma in this section. If $\rho = -\infty$, the above proof remains the same except that $\rho + \epsilon$ is to be replaced by any negative number, large in absolute value. This completes the proof of the sufficiency of (2.4) for (2.3).

(b) Necessity: The necessity of (2.4) for (2.3) follows easily from (a), the sufficiency; for, by (a), the sequence $\left\{ \frac{\chi_m - n}{\sqrt{m}} \right\}$ could not have more than one limit point, finite or infinite, in order that (2.3) be satisfied. Again by (a), the one limit point must be $-\rho$.

Q.E.D.

Theorem 2.2 A necessary and sufficient condition that the summability method defined by (1.6) be convergence preserving is that

$$\lim_{m \rightarrow \infty} \frac{\chi_m - n}{\sqrt{m}} \text{ exist } (= -\rho), \quad \chi_m \rightarrow \infty$$

The theorem follows immediately from the preceding lemma and the fact that (1.3) and (1.4) are here satisfied.

If ρ is finite or equal to $+\infty$, we easily see that the triangular summability method defined by the equation:

$$\text{gen. lim. } s_\nu = \lim_{m \rightarrow \infty} \frac{1}{p(\rho)} e^{-\chi_m} \sum_{\nu=0}^m s_\nu \frac{\chi_m^\nu}{\nu!}$$

is regular. If $\rho = -\infty$, the method defined by the equation (1.6) sums every bounded sequence to the value zero.

We next consider two applications of these triangular matrix transforms.

3. The first application discussed is for analytic functions of a complex variable defined by a power series within a finite radius of convergence. It is the purpose of this section to show that, with certain conditions on the sequence of points $\{\chi_m\}$, the summability method defined by (1.6) defines the analytic continuation of the function within the Borel polygon when applied to the power series.

Given an analytic function defined by a power series in a finite and non-zero radius of convergence:

$$(3.1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < r,$$

let

$$(3.2) \quad s_m(z) = \sum_{\nu=0}^m a_\nu z^\nu, \quad m=0,1,2,\dots$$

Dependent on the sequence $\{\lambda_n\}$, we define now the function

$$(3.3) \quad F(z, \lambda_n) \equiv F_n(z) = e^{-\lambda_n} \sum_{\nu=0}^n \frac{s_\nu(z) \lambda_n^\nu}{\nu!} ;$$

that is (3.3) gives the application of the summability method (1.6) to the sequence of partial sums defined in (3.2). When $\{\lambda_n\}$ satisfies the condition (2.2), theorem 2.1 implies that $F_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$ for $|z| < r$. As has been anticipated, the object of this section is to determine conditions on $\{\lambda_n\}$ so that $F_n(z) \rightarrow f(z)$ in a larger domain for z , namely the Borel polygon for $f(z)$. We first set up several formulae relating more directly to $f(z)$ before stating the main theorem in that connection.

Having started with the analytic function $f(z)$, defined by (3.1), let Γ be any closed Jordan curve, containing the origin, interior to and on which the function is regular. Then the following formula is classical:

$$(3.4) \quad a_m = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{m+1}} dz, \quad m=0, 1, 2, \dots$$

From this it follows that

$$s_\nu(z) = \sum_{m=0}^{\nu} a_m z^m = \sum_{m=0}^{\nu} \frac{1}{2\pi i} z^m \int_{\Gamma} \frac{f(\zeta)}{\zeta^{m+1}} d\zeta = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta} \sum_{m=0}^{\nu} \left(\frac{z}{\zeta}\right)^m d\zeta$$

$$s_\nu(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta} \left\{ \frac{1 - \frac{z^{\nu+1}}{\zeta^{\nu+1}}}{1 - \frac{z}{\zeta}} \right\} d\zeta$$

Letting z be interior to Γ , we obtain

$$s_\nu(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} \left\{ 1 - \frac{z^{\nu+1}}{s^{\nu+1}} \right\} ds = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} ds - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} \frac{z^{\nu+1}}{s^{\nu+1}} ds$$

$$(3.5) \quad s_\nu(z) = f(z) - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} \frac{z^{\nu+1}}{s^{\nu+1}} ds.$$

Substituting now the formula of (3.5) into

(3.3), we get

$$F_n(z) = e^{-\chi_n} \sum_{\nu=0}^n \frac{f(z) \chi_n^\nu}{\nu!} - \frac{e^{-\chi_n}}{2\pi i} \sum_{\nu=0}^n \frac{\chi_n^\nu}{\nu!} \int_{\Gamma} \frac{f(s)}{s-z} \frac{z^{\nu+1}}{s^{\nu+1}} ds$$

$$(3.6) \quad F_n(z) = e^{-\chi_n} \sum_{\nu=0}^n \frac{f(z) \chi_n^\nu}{\nu!} - \frac{e^{-\chi_n}}{2\pi i} \int_{\Gamma} \frac{f(s) z}{(s-z) s} \left\{ \sum_{\nu=0}^n \frac{\chi_n^\nu z^\nu}{s^{\nu+1} \nu!} \right\} ds$$

and so by (2.1)

$$(3.6') \quad F_n(z) \equiv f(z) g_n(\chi_n) - I(\chi_n, z)$$

where

$$(3.7) \quad I(\chi_n, z) \equiv I_n = \frac{e^{-\chi_n}}{2\pi i} \int_{\Gamma} \frac{f(s) z}{(s-z) s} \left\{ \sum_{\nu=0}^n \frac{\chi_n^\nu z^\nu}{s^{\nu+1} \nu!} \right\} ds.$$

The problem now is to determine when $g_n \rightarrow 1$ and $I_n \rightarrow 0$ as $n \rightarrow \infty$; clearly $F_n(z) \rightarrow f(z)$ under these conditions. The fundamental stepping stone to the main theorem of this section is the following result.

Theorem 3.1

If $\chi_n = o(n)$ and if $\chi_n \rightarrow \infty$, and if

for all z on Γ

$$(3.8) \quad R\left(\frac{z}{s}\right) < 1,$$

then

$$I_n \equiv \frac{e^{-\chi_n}}{2\pi i} \int_{\Gamma} \frac{f(\zeta) \zeta}{(\zeta-3)\zeta} \cdot \sum_{\nu=0}^n \frac{\chi_n^\nu \zeta^\nu}{\zeta^\nu \nu!} \cdot d\zeta = o(1), n \rightarrow \infty.$$

Proof (a) The first step in preparing to estimate $I(\chi_n, \zeta)$ is to set up an integral representation for the kernel $\sum_{\nu=0}^n \frac{\chi_n^\nu \zeta^\nu}{\zeta^\nu \nu!}$. As has become ^{standard} in this case, the starting point for the conversion is the equation:

$$k! = \int_0^\infty t^k e^{-t} dt, \quad k=0, 1, 2, \dots$$

It follows from this equation that

$$\begin{aligned} \sum_{\nu=0}^n \frac{p^\nu}{\nu!} &= \frac{1}{n!} \sum_{\nu=0}^n \frac{n!}{\nu! (n-\nu)!} p^\nu \int_0^\infty t^{n-\nu} e^{-t} dt \\ &= \frac{1}{n!} \int_0^\infty e^{-t} \left\{ \sum_{\nu=0}^n \frac{n!}{\nu! (n-\nu)!} p^\nu t^{n-\nu} \right\} dt \\ (3.9) \quad \sum_{\nu=0}^n \frac{p^\nu}{\nu!} &= \frac{1}{n!} \int_0^\infty e^{-t} (p+t)^n dt, \quad n \geq 0. \end{aligned}$$

Substituting $p = \frac{\chi_n \zeta}{\zeta}$ in (3.9) and then setting (3.9) into (3.7), we obtain

$$(3.10) \quad I_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta) \zeta}{(\zeta-3)\zeta} \cdot \left\{ \frac{e^{-\chi_n}}{n!} \int_0^\infty \left(\frac{\chi_n \zeta}{\zeta} + t \right)^n e^{-t} dt \right\} d\zeta.$$

Write now

$$(3.11) \quad J_n = \frac{e^{-\chi_n}}{n!} \int_0^\infty \left(\frac{\chi_n \zeta}{\zeta} + t \right)^n e^{-t} dt.$$

Clearly, it is sufficient to show, with the hypothesis, that $J_n \rightarrow 0$ uniformly as to ζ , ζ on Γ , as $n \rightarrow \infty$, in order to prove the theorem. It

is to the estimate of J_m that we now turn our attention.

(b) We proceed to prove that if $\chi_m = d/n$, $m \rightarrow \infty$, and if (3.8) is satisfied for all ζ on Γ , then $J_m \rightarrow 0$ as $n \rightarrow \infty$, uniformly as to ζ .

Firstly let us notice that since Γ is a closed Jordan curve containing the origin and the point ζ , the hypothesis on the real part of ζ/χ_m implies that there exists an ϵ , $\epsilon > 0$, such that for all ζ on Γ we have

$$(3.12) \quad \Re(\zeta/\chi_m) \leq 1 - \epsilon$$

Write

$$(3.13) \quad \zeta/\chi_m = \omega = \alpha + i\beta, \quad \alpha \leq 1 - \epsilon$$

Then

$$J_m = \frac{e^{-\chi_m}}{n!} \int_0^{\infty} (t + \alpha\chi_m + i\chi_m\beta)^n e^{-t} dt$$

By a change of variable, $u = t + \alpha\chi_m$, we obtain

$$J_m = \frac{e^{-\chi_m}}{n!} \int_{\alpha\chi_m}^{\infty} (u + i\chi_m\beta)^n e^{-u + \chi_m\alpha} du$$

$$(3.14) \quad J_m = \frac{e^{-\chi_m(1-\alpha)}}{n!} \int_{\alpha\chi_m}^{\infty} (u + i\chi_m\beta)^n e^{-u} du$$

$$(3.14') \quad J_m = \frac{e^{-\chi_m(1-\alpha) + i\chi_m\beta}}{n!} \int_{\alpha\chi_m}^{\infty} (t + i\chi_m\beta)^n e^{-(t + i\chi_m\beta)} dt$$

With another change of variable, we now interpret the formula (3.14') for J_m as a complex integral. Let

$s = \sigma + i\tau$ be a complex variable; set $s = t + i\chi_m \beta$. Then

(3.14') becomes

$$(3.15) \quad J_m = \frac{e^{-\chi_m(1-\alpha) + i\chi_m\beta}}{m!} \int_{\alpha\chi_m + i\chi_m\beta}^{\infty + i\chi_m\beta} s^m e^{-s} ds$$

$$J_m = \frac{e^{-\chi_m(1-\alpha) + i\chi_m\beta}}{m!} \int_C s^m e^{-s} ds$$

where C is the straight line path in the s -plane, parallel to the axis of reals; $C: \alpha\chi_m + i\chi_m\beta \rightarrow \infty + i\chi_m\beta$.

Let $C_1 = C_1(U)$ be a portion of this path; $C_1: \alpha\chi_m + i\chi_m\beta \rightarrow U + i$ and define

$$(3.16) \quad J_m(U) = \frac{e^{-\chi_m(1-\alpha) + i\chi_m\beta}}{m!} \int_{C_1(U)} s^m e^{-s} ds$$

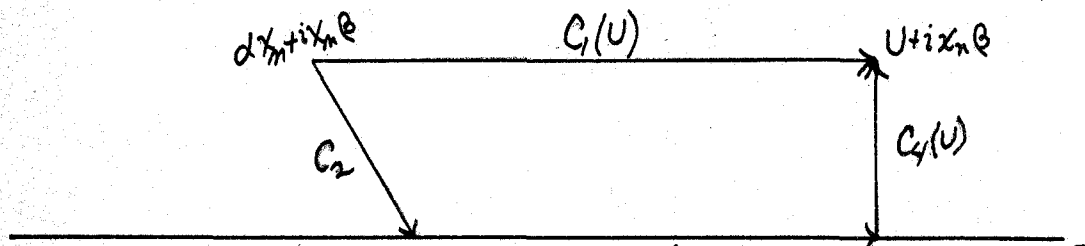
It is quite evident that

$$(3.17) \quad J_m = \lim_{U \rightarrow \infty} J_m(U)$$

Let us define now the following paths of integration: (see also the figure below)

$$\begin{aligned} C_2 &: \alpha\chi_m + i\chi_m\beta \rightarrow 0 \\ C_3 = C_3(U) &: 0 \rightarrow U \\ C_4 = C_4(U) &: U \rightarrow U + i\chi_m\beta \end{aligned}$$

Then, since $s^m e^{-s}$ is a regular function of s in the



finite plane, we have by the classical Cauchy theorem

$$J_n(u) = \frac{e^{-\gamma_n(1-\alpha) + i\gamma_n\beta}}{n!} \left(\int_{C_2} + \int_{C_3} + \int_{C_4} \right) s^m e^{-s} ds$$

$$\equiv P + Q + R$$

Now

$$|P| \leq \frac{e^{-\gamma_n(1-\alpha)}}{n!} |\alpha\gamma_n + i\beta\gamma_n|^m e^{|\alpha|\gamma_n} |\alpha\gamma_n + i\beta\gamma_n|$$

$$= \frac{\gamma_n^{m+1} e^{|\alpha|\gamma_n - \gamma_n(1-\alpha)}}{n!} |\alpha + i\beta|^{m+1} ;$$

$$|Q| = \frac{e^{-\gamma_n(1-\alpha)}}{n!} \int_0^U s^m e^{-s} ds ;$$

$$|R| \leq \frac{e^{-\gamma_n(1-\alpha)}}{n!} |U + i\gamma_n\beta|^m e^{-U} |\gamma_n\beta| = o(1) \text{ as } U \rightarrow \infty$$

Hence, letting $U \rightarrow \infty$, we have

$$|J_n| \leq \lim_{U \rightarrow \infty} |J_n(u)| \leq \frac{\gamma_n^{m+1} e^{|\alpha|\gamma_n - \gamma_n(1-\alpha)}}{n!} |\alpha + i\beta|^{m+1} + \epsilon$$

$$\equiv V + W$$

Using now our assumption that $\gamma_n = o(n)$, and the Stirling formula:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

we have for all positive ϵ , however small

$$\gamma_n^{m+1} |\alpha + i\beta|^{m+1} \leq \epsilon^{n+1} n^{n+1}, \quad n > n_0 = n_1$$

$$V \leq \frac{\epsilon^{n+1} n^{n+1} e^{An}}{n!} = o(1), \quad n \rightarrow \infty, \text{ for } \epsilon \text{ small.}$$

Inasmuch as $\alpha + i\beta = \frac{1}{z}$ is bounded for all z on Γ , it is obvious that the above estimate for V holds uniformly as to z, γ on Γ . Now by (3.12) and (3.13) we also have

$$W \leq e^{-\chi_m \epsilon} = o(1)$$

uniformly as to z, γ on Γ . Hence we have proven that $J_m \rightarrow 0$ as $n \rightarrow \infty$ uniformly for all z on Γ .

By (3.10) and (3.11) we have $I_m \rightarrow 0$ as $n \rightarrow \infty$, and this completes the proof of the theorem.

Q.E.D.

Having established theorem 3.1, it is an immediate consequence that we have $I_m \rightarrow 0$ in the whole Borel polygon for the function $f(z)$; the proof is completed in the same manner as the Borel polygon theorem.

Theorem 3.2

If $\chi_m = o(n)$, then the summability method (1.6) sums a power series, $\sum_0^{\infty} a_n z^n$, having a finite radius of convergence, inside the Borel polygon for the function represented by the series and defines in this polygon the analytic continuation of the function.

Formula (3.6) gives the expression obtained when the summability method (1.6) was applied to the power series (3.1). Theorem 3.1 gives $I_m \rightarrow 0$.

Inasmuch as $\chi_m = o(n)$ implies that the set of points $\{\chi_m\}$ satisfies the condition (2.2), it follows that $g_m \rightarrow 1$ as $m \rightarrow \infty$. This completes the proof of theorem 3.2.

We shall subsequently return to some of the formulae in this section in order to compare the methods defined by (1.6) with Borel's exponential method. In the meanwhile we consider another application of the triangular summability methods defined by (1.6).

4. The purpose of this section is to apply the summability methods (1.6) to Fourier series and to determine when their power, in this case, is the same as that of Borel's method. That the methods (1.6) are equipowerful with Borel's method in this case, if $\chi_m = o(n)$, will follow from the results of section 5, where a more general theorem is obtained using the formulae of the preceding section. Nevertheless, we pursue here an investigation independent of that theorem and shall obtain the kernel and Lebesgue constants involved. The main results of this section are embodied in theorems 4.1 and 4.2.

For the Fourier series of a Lebesgue integrable function, $f(x)$, of period 2π , we have

$$(4.1) \quad f(x) \sim \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_0^{\infty} A_n(x) \quad .$$

$$(4.2) \quad A_n(x) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ uniformly as to } x \quad .$$

$$(4.3) \quad \begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi} \int_0^{2\pi} f(x) \begin{matrix} \cos nx \\ \sin nx \end{matrix} dx, \quad n \geq 0 \quad .$$

Setting

$$S_n(x) = \sum_{\nu=0}^n A_{\nu}(x), \quad S_n^*(x) = S_n(x) - \frac{1}{2} A_n(x),$$

it follows that

$$(4.4) \quad S_n^*(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi(x,t) \cot \frac{1}{2}t \sin \nu t dt,$$

where

$$(4.5) \quad \varphi(x,t) = \frac{1}{2} \{ f(x+t) + f(x-t) - 2f(x) \} \quad .$$

From (4.2) and the fact that the methods (1.6) are boundedness preserving and linear (therefore preserving convergence to zero and uniform convergence to zero), we obtain the same results by applying (1.6) to $\{S_n^*(x)\}$ as from applying it to $\{S_n(x)\}$. Write

$$(4.6) \quad T_m = T_m(x_m, x, f) = e^{-x_m} \sum_{\nu=0}^m \frac{S_{\nu}^*(x) x_m^{\nu}}{\nu!}$$

$$(4.7) \quad \begin{aligned} T_m - f(x) &= T_m - e^{-x_m} \sum_0^{\infty} \frac{f(x) x_m^{\nu}}{\nu!} = e^{-x_m} \sum_{\nu=0}^m \frac{\{S_{\nu}^*(x) - f(x)\} x_m^{\nu}}{\nu!} - e^{-x_m} f \\ &\equiv U_m - f(x) e^{-x_m} \sum_{m+1}^{\infty} \frac{x_m^{\nu}}{\nu!}, \end{aligned}$$

where

$$(4.8) \quad U_m = e^{-x_m} \sum_{\nu=0}^m \frac{\{S_{\nu}^*(x) - f(x)\} x_m^{\nu}}{\nu!}$$

If $\{x_n\}$ satisfies the condition (2.2), it follows immediately from the lemma on page 4 that

$$e^{-x_n} \sum_{\nu=n+1}^{\infty} \frac{x_n^{\nu}}{\nu!} = o(1) \quad \text{as } n \rightarrow \infty.$$

It therefore remains to investigate U_n . By (4.4)

$$U_n = e^{-x_n} \sum_{\nu=0}^n \frac{x_n^{\nu}}{\nu!} \frac{1}{\pi} \int_0^{\pi} \varphi(x, t) \cos \frac{1}{2} t \sin \nu t \, dt$$

$$(4.9) \quad U_n = \frac{e^{-x_n}}{\pi} \int_0^{\pi} \varphi(x, t) \cos \frac{1}{2} t \left\{ \sum_{\nu=0}^n \frac{\sin \nu t \cdot x_n^{\nu}}{\nu!} \right\} dt.$$

We now prove the theorem :

Theorem 4.1 If the Fourier series in (4.1)

is summable by Borel's exponential method, and if

$$x_n \leq \frac{n}{e} + k, \quad x_n \rightarrow \infty, \quad n \rightarrow \infty, \quad k \text{ fixed},$$

then $U_n = o(1)$ as $n \rightarrow \infty$.

Proof (a) In estimating U_n , we shall first transform the kernel $e^{-x_n} \sum_{\nu=0}^n \frac{\sin \nu t \cdot x_n^{\nu}}{\nu!}$. We have

$$e^{-x_n} \sum_{\nu=0}^n \frac{\sin \nu t \cdot x_n^{\nu}}{\nu!} = \mathcal{I} \left\{ e^{-x_n} \sum_{\nu=0}^n \frac{e^{i\nu t} x_n^{\nu}}{\nu!} \right\}.$$

Hence, by (3.9), with $p = e^{it} x_n$, we have

$$(4.10) \quad e^{-x_n} \sum_{\nu=0}^n \frac{\sin \nu t \cdot x_n^{\nu}}{\nu!} = \mathcal{I} \left\{ \frac{e^{-x_n}}{n!} \int_0^{\infty} (v + x_n e^{it})^n e^{-v} \, dv \right\} = \mathcal{I} \{ Z_n \}$$

$$(4.11) \quad Z_n = \frac{e^{-x_n}}{n!} \int_0^{\infty} (v + x_n e^{it})^n e^{-v} \, dv.$$

Writing $e^{it} = \cos t + i \sin t$, and making the substitution $u = v + x_n \cos t$ in the integral defining Z_n , we obtain

$$Z_m = \frac{e^{-\gamma_m}}{n!} \int_{\gamma_m \cos t}^{\infty} (u + i \gamma_m \sin t)^n e^{-u + \gamma_m \cos t} du$$

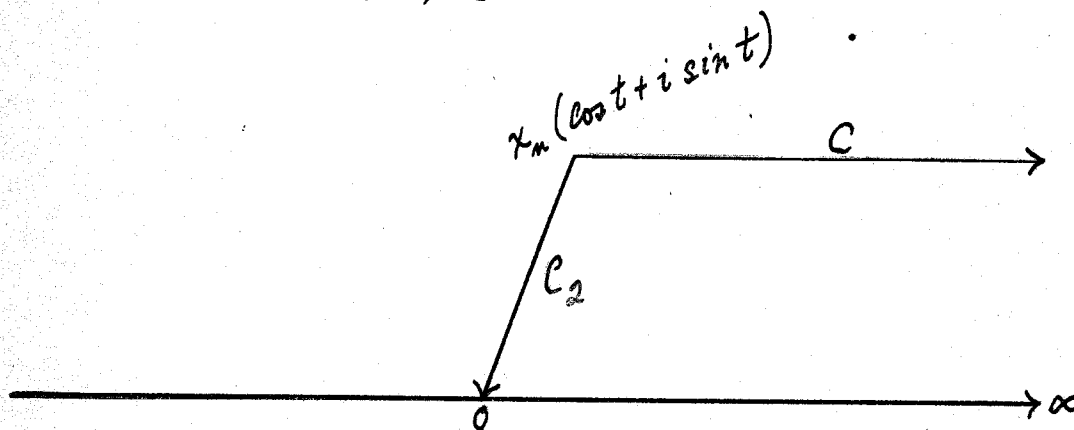
With another change of variable, we now interpret the above formula for Z_m as a complex integral. Let $s = \sigma + i\tau$ be a complex variable; set $s = u + i \gamma_m \sin t$. Then

$$(4.12) \quad Z_m = \frac{e^{-\gamma_m(1-\cos t) + i \gamma_m \sin t}}{n!} \int_C s^n e^{-s} ds$$

where C is the straight line path: $\gamma_m \cos t + i \gamma_m \sin t \rightarrow \infty + i \gamma_m \sin t$

Following through the analysis of pages 15 and 16 with $\alpha = \cos t$, $\beta = \sin t$, we replace the path of integration C in (4.12) with the straight line paths (see figure below)

$$C_2: \gamma_m \cos t + i \gamma_m \sin t \rightarrow 0, \text{ and} \\ 0 \rightarrow \infty$$



We thus obtain

$$Z_m = \frac{e^{-\gamma_m(1-\cos t) + i \gamma_m \sin t}}{n!} \left(\int_{C_2} + \int_0^{\infty} \right) s^n e^{-s} ds$$

$$Z_m = e^{-\gamma_m(1-\cos t) + i \gamma_m \sin t} + \frac{e^{-\gamma_m(1-\cos t) + i \gamma_m \sin t}}{n!} \int_{C_2} s^n e^{-s} ds$$

$$(4.13) \quad A_m = e^{-\chi_m(1-\cos t)} \sin(\chi_m \sin t) + \mathcal{I}\{D_m\},$$

where

$$(4.14) \quad D_m = \frac{e^{-\chi_m(1-\cos t) + i\chi_m \sin t}}{m!} \int_{\mathcal{C}_2} s^m e^{-s} ds$$

(b) With the above expression for A_m , we have

$$(4.15) \quad U_m = \frac{1}{\pi} \int_0^\pi \varphi(x,t) \cos \frac{1}{2} t \cdot e^{-\chi_m(1-\cos t)} \sin(\chi_m \sin t) dt + \mathcal{I}\left\{ \frac{1}{\pi} \int_0^\pi \varphi(x,t) \right.$$

From the formula (4.14) for D_m , it follows that with the change of variable $S = \lambda(\cos t + i \sin t)$, we have

$$D_m = \frac{e^{-\chi_m(1-\cos t) + i\chi_m \sin t}}{m!} \int_{\chi_m}^0 \lambda^m e^{-\lambda(\cos t + i \sin t)} d\lambda \cdot e^{i(n+1)t}$$

$$D_m = \frac{e^{i(\chi_m \sin t + (n+1)t)}}{m!} \int_{\chi_m}^0 \lambda^m e^{-(\lambda \cos t + \chi_m - \chi_m \cos t)} e^{-i\lambda \sin t} d\lambda$$

Now $\chi_m \rightarrow \infty$; therefore we may assume that $\chi_m \geq 0$.

Hence, it is easily seen that if $0 \leq \lambda \leq \chi_m$, then

$$\lambda \cos t + \chi_m - \chi_m \cos t \geq 0$$

Therefore

$$(4.16) \quad |D_m| \leq \frac{1}{m!} \int_0^{\chi_m} \lambda^m d\lambda = \frac{\chi_m^{m+1}}{(m+1)!} \sim \frac{e^{m+1} \chi_m^{m+1}}{(m+1)^{m+1} \sqrt{2\pi(m+1)}}$$

Hence, if

$$(4.17) \quad \left(\frac{e\chi_m}{m+1} \right)^{m+1} = o(\sqrt{m+1}) \quad \text{as } m \rightarrow \infty,$$

it follows that $D_m = o(1)$, as $m \rightarrow \infty$, uniformly as to t .

Inasmuch as (4.17) implies that $\frac{x_n}{n+1}$ is bounded as $n \rightarrow \infty$ it follows that we may replace (4.17) with

$$(4.17') \quad \left(\frac{e^{x_n}}{n}\right)^m = o(\sqrt{n}) \quad \text{as} \quad n \rightarrow \infty.$$

It is clear that this condition is satisfied if $e^{x_n} \leq n+1$ $n > n_0$, which is the hypothesis of the theorem.

Returning now to the formula (4.15) for U_n , we have

$$(4.18) \quad U_n = \frac{1}{\pi} \int_0^\pi \varphi(x, t) \cdot \cos \frac{1}{2} t \cdot e^{-x_n(1-\cos t)} \sin(x_n \sin t) dt + o(1), \text{ uniformly}$$

The integral occurring in (4.18) is essentially the transform of Fourier series by Borel's exponential method; hence, by assumption, it is $o(1)$ as $n \rightarrow \infty$. This completes the proof of the theorem.

Q.E.D.

Formulae (4.15) and (4.16) also prove the following theorem concerning the Lebesgue constants involved in applying the methods (1.6) to Fourier series.

Theorem 4.2 If the sequence of points satisfies the condition (4.17'), then the Lebesgue constants determined by applying the summability method (1.6) to Fourier series are asymptotic to those obtained by applying Borel's exponential method thereto.

We conclude this section with some remarks concerning the application of (1.6) to the Fourier series

of continuous functions, when the sequence of points $\{\lambda_n\}$ satisfies the condition (4.17').

i. Theorem 4.2 implies that the Lebesgue constants obtained by applying (1.6) to Fourier series tend to infinity. It follows therefore from a general theorem due to Lebesgue that there exists a continuous function whose Fourier series is not summed by (1.6).

ii. C. N. Moore has given an example of a continuous function whose Fourier series is divergent but which can nevertheless be summed by Borel's exponential method to the value of the function. By theorem 4.1, this series is also summable by the method (1.6) to the value of the function.

iii. In the paper referred to in the preceding paragraph, Professor Moore also notes that Borel's method does not sum the example given by Fejér of a divergent Fourier series of a continuous function. Using formula (4.18), we see that the method (1.6) likewise will not sum Fejér's example, for certain sequences $\{\lambda_n\}$ satisfying (4.17'). For if (1.6) did sum the series for all sequences $\{\lambda_n\}$ satisfying (4.17'), then it would follow from (4.18) that the series were also Borel summable.

5. We now return to the formulae of section three in order to obtain some comparison results regarding the methods (1.6) and other summability methods, when applied to power series with a non-zero radius of convergence.

Theorem 5.1 If $\chi_n = o(n)$, $\chi_n \rightarrow \infty$, then the triangular method (1.6) is at least as powerful as Borel's exponential method in summing power series with a non-zero circle of convergence.

Proof Let $B(z, \chi)$ represent the expression resulting from the application of Borel's exponential method to the power series

$$(3.1) \quad \sum a_n z^n = f(z), \quad |z| < R.$$

Let Γ be any closed Jordan curve containing the origin and the point z , interior to and on which the function $f(z)$ is regular. Then

$$(5.1) \quad B(z, \chi) = f(z) - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} \cdot \frac{z}{s} \cdot e^{-\chi(1-\frac{z}{s})} ds.$$

Recalling now that $F(z, \chi_n)$ represents the application of (1.6) to (3.1), we have

$$(5.2) \quad F(z, \chi_n) = g_n(\chi_n) f(z) - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} \cdot \frac{z}{s} \cdot e^{-\chi_n(1-\frac{z}{s})} ds - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z}$$

where

$$P = e^{-\chi_n(1-\frac{z}{s})} \int_{\rho} s^n e^{-s} ds, \quad C_2: \chi_n \frac{z}{s} \rightarrow 0.$$

Now from the estimates on page 16, $P = o(1)$ as $n \rightarrow \infty$, uniformly as to z , γ on Γ . By the lemma on page 4, $g_n \rightarrow 1$ as $n \rightarrow \infty$. Therefore it is immediate that if $\lim_{x \rightarrow \infty} B(z, x)$ exists, then $\lim_{n \rightarrow \infty} F(z, x_n)$ exists and has the same value. This completes the proof if z is a regular point of the function $f(z)$.

If z is a singular point, replace $f(z), \Gamma$ in (5.1) and (5.2) with

$$0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\gamma)}{\gamma - z} d\gamma$$

where Γ now is a curve containing the origin, interior to and on which the function $f(\gamma)$ is regular. The remainder of the proof now follows easily.

Q.E.D.

(Remark: We cannot conclude from formulae (5.1) and (5.2) that if $\lim_{n \rightarrow \infty} F(z, x_n)$ exists for a particular sequence $\{x_n\}$, then (3.1) is Borel summable at the point z . But $\lim_{n \rightarrow \infty} F(z, x_n)$ existing for all sequences $\{x_n\}$, $x_n = o(n)$, implies that (3.1) is Borel summable.)

Corollary If $\sum a_n$ is summable by Borel's exponential method, and if for some $K > 0$, $a_n = O(K^n)$, then $\sum a_n$ is summable by (1.6) when $x_n = o(n)$.

The corollary can be considered as a Tauberian paraphrase of theorem 5.1. The Tauberian condition

$a_n = O(K^n)$ implies that $\sum a_n z^n$ has a non-zero radius

of convergence; hence by theorem 5.1, if $\lim_{x \rightarrow \infty} B(1, x)$ exists, $\lim_{m \rightarrow \infty} F(1, x_m)$ exists and has the same value, $x_m \rightarrow \infty$.

We next proceed to demonstrate that the summability method (1.6) does not include the method of arithmetic means, regardless of the sequence $\{x_n\}$.

Theorem 5.2 (C,1) summability is not included in the method (1.6) for any sequence $\{x_n\}$.

Proof For certain sequences $\{x_n\}$, the theorem is immediate from the results of section 4 concerning Fourier series. However, we use here a general method invoking the Silverman-Toeplitz conditions for regularity.

Let

$$t_n = e^{-x_n} \sum_{\nu=0}^n \frac{s_\nu x_n^\nu}{\nu!}, \quad n \geq 0,$$

$$\sigma_n = \frac{1}{n+1} \{s_0 + s_1 + \dots + s_n\}, \quad n \geq 0.$$

Clearly

$$s_\nu = (\nu+1)\sigma_\nu - \nu\sigma_{\nu-1}, \quad \nu \geq 0, \quad \sigma_{-1} = 0$$

$$e^{x_n} t_n = \sum_{\nu=0}^n \frac{x_n^\nu}{\nu!} [(\nu+1)\sigma_\nu - \nu\sigma_{\nu-1}].$$

Applying now the method of partial summation to the above formula, we get

$$(5.3) \quad t_n = e^{-x_n} \left\{ \sum_{\nu=0}^{n-1} (\nu+1)\sigma_\nu \left[\frac{x_n^\nu}{\nu!} - \frac{x_n^{\nu+1}}{(\nu+1)!} \right] + \frac{(n+1)\sigma_n}{n!} x_n^n \right\} \equiv \sum_{\nu=0}^n \sigma_\nu$$

where

$$(5.4) \quad \begin{aligned} a_{mv} &= e^{-\chi_m} (v+1) \left[\frac{\chi_m^v}{v!} - \frac{\chi_m^{v+1}}{(v+1)!} \right], \quad v \leq m-1 \\ a_{mn} &= e^{-\chi_m} \frac{(m+1)}{m!} \chi_m^n \end{aligned}$$

In order that the transformation (5.3) of $\{\sigma_m\}$ be regular, it is necessary and sufficient that

$$(a) \quad \sum_{v=0}^n a_{mv} = e^{-\chi_m} \left\{ \sum_{v=0}^{m-1} (v+1) \left[\frac{\chi_m^v}{v!} - \frac{\chi_m^{v+1}}{(v+1)!} \right] \right\} + e^{-\chi_m} \frac{(m+1)}{m!} \chi_m^n = e^{-\chi_m} \sum_{v=0}^n \frac{\chi_m^v}{v!}$$

$$(b) \quad \lim_{m \rightarrow \infty} a_{mv} = \lim_{m \rightarrow \infty} e^{-\chi_m} (v+1) \left[\frac{\chi_m^v}{v!} - \frac{\chi_m^{v+1}}{(v+1)!} \right] = 0, \quad v=0,1,2,\dots$$

$$(c) \quad f_n \equiv \sum_{v=0}^n |a_{mv}| = e^{-\chi_m} \sum_{v=0}^{m-1} (v+1) \left| \frac{\chi_m^v}{v!} - \frac{\chi_m^{v+1}}{(v+1)!} \right| + \frac{e^{-\chi_m} (m+1) \chi_m^n}{m!} = O(1)$$

(a) is satisfied if $\{\chi_m\}$ satisfies the condition (2.2). In fact, let us admit those sequences $\{\chi_m\}$ satisfying (2.4) with $\beta > -\infty$. Then, for some K, $\chi_m < n + K\sqrt{n}$

(b) is easily seen to be satisfied for any sequence $\{\chi_m\}$, $\chi_m \rightarrow \infty$.

Contrary to the first condition, (c) cannot be satisfied if there exists a constant K, such that $\chi_m \leq n + K\sqrt{n}$, $n > 0$. To prove this, we consider two cases:

(1) Suppose $m \leq \chi_m$. Then

$$\begin{aligned} f_m &= e^{-\chi_m} \sum_{v=0}^{m-1} \frac{\chi_m^v}{v!} |(v+1) - \chi_m| + e^{-\chi_m} \frac{(m+1) \chi_m^m}{m!} \\ &= -e^{-\chi_m} \sum_{v=0}^{m-1} \frac{\chi_m^v}{v!} [\chi_m - (v+1)] + e^{-\chi_m} \frac{(m+1) \chi_m^m}{m!} \\ &= - \left\{ e^{-\chi_m} \sum_{v=0}^{m-1} \frac{\chi_m^v}{(v+1)!} [\chi_m^{v+1} - \chi_m^{v+1}] \right\} + e^{-\chi_m} \frac{(m+1) \chi_m^m}{m!} \end{aligned}$$

$$f_m = \frac{2e^{-\chi_m} (m+1) \chi_m^m}{m!} - e^{-\chi_m} \sum_{\nu=0}^m \frac{\chi_m^\nu}{\nu!}$$

$$\geq \frac{2e^{-\chi_m} \cdot m \cdot \chi_m^m}{m!} - 1$$

It is easy to see that the function $e^{-x} x^p$ ($\chi, p > 0$) attains its maximum when χ is equal to p and is a monotonically decreasing function of χ , $\chi > p$. Now for some fixed K , we have $\chi_m \leq m + K\sqrt{m}$.

Therefore

$$f_m \geq \frac{2e^{-m-K\sqrt{m}} \cdot m \cdot (m+K\sqrt{m})^m}{m!} - 1$$

From Stirling's formula, we get

$$\frac{e^{-m-K\sqrt{m}} \cdot m \cdot (m+K\sqrt{m})^m}{m!} \sim \sqrt{\frac{m}{2\pi}} \cdot e^{-K\sqrt{m}} \cdot \left(1 + \frac{K}{\sqrt{m}}\right)^m$$

Inasmuch as

$$e^{-K\sqrt{m}} \left(1 + \frac{K}{\sqrt{m}}\right)^m \rightarrow e^{-\frac{K^2}{2}}, \quad m \rightarrow \infty$$

we see that, for the case being considered, f_m is of the order \sqrt{m} as $m \rightarrow \infty$.

(ii) Suppose $m > \chi_m$. Let $\mu = [\chi_m]$. Then

$$f_m = e^{-\chi_m} \sum_{\nu=0}^{\mu-1} \frac{\chi_m^\nu}{\nu!} \{\chi_m - (\nu+1)\} + e^{-\chi_m} \sum_{\nu=\mu}^{m-1} \frac{\chi_m^\nu}{\nu!} \{(\nu+1) - \chi_m\} + e^{-\chi_m} \frac{(m-1)!}{(m-1)!}$$

$$f_m = 2e^{-\chi_m} \sum_{\nu=0}^{\mu-1} \frac{\chi_m^\nu}{\nu!} \{\chi_m - (\nu+1)\} + e^{-\chi_m} \sum_{\nu=0}^{m-1} \frac{\chi_m^\nu}{\nu!} \{(\nu+1) - \chi_m\} + e^{-\chi_m} \frac{(m-1)!}{(m-1)!}$$

$$f_m = -2e^{-\chi_m} \sum_{\nu=0}^{\mu-1} (\nu+1) \left\{ \frac{\chi_m^\nu}{\nu!} - \frac{\chi_m^{\nu+1}}{(\nu+1)!} \right\} + e^{-\chi_m} \sum_{\nu=0}^m \frac{\chi_m^\nu}{\nu!}$$

$$\begin{aligned}
 f_m &= -2e^{-x_m} \left\{ \sum_{v=0}^{m-1} \frac{x_m^v}{v!} - \frac{m x_m^m}{m!} \right\} + g_m(x_m) \\
 &= \frac{2m x_m^m e^{-x_m}}{m!} - 2e^{-x_m} \sum_{v=0}^{m-1} \frac{x_m^v}{v!} + g_m(x_m) \\
 f_m &\geq \frac{2m e^{-x_m} x_m^m}{m!} - 2
 \end{aligned}$$

We have already noted that for $x > p > 0$, $x^p e^{-x}$ is a decreasing function of x . From the fact that $m \leq x_m < m+1$, it now follows that

$$f_m \geq \frac{2m(m+1)^m e^{-m-1}}{m!} - 2$$

Again using Stirling's formula, we have

$$\frac{m(m+1)^m e^{-m-1}}{m!} \sim \sqrt{\frac{m}{2\pi}} \left(1 + \frac{1}{m}\right)^m e^{-1} \sim \sqrt{\frac{m}{2\pi}}$$

Hence, for this case, we see that f_m is of the order $\sqrt{x_m}$, $x_m \rightarrow \infty$.

Combining the results of (i) and (ii), we see that (c) and (a) are incompatible for the same sequence of points $\{x_m\}$. This completes the proof of the theorem.

Q.E.D.

We next show, by means of an example, that the method (1.6) is not included in convergence, for any sequence $\{x_m\}$.

Example The method (1.6) sums the sequence $S_v = (-1)^v$, $v = 0, 1, 2, \dots$, to the value zero, $x_m \rightarrow \infty$.

Proof One method of demonstration would be to use

formula (3.9). However, we use here a more direct method independent of that formula. Let

$$t_m = e^{-\chi_m} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \chi_m^\nu}{\nu!}$$

We proceed to show that $t_m \rightarrow 0$ as $m \rightarrow \infty, \chi_m \rightarrow \infty$.

We recall that the maximum of the function $\frac{e^{-x} x^p}{p!}$ ($x > 0$), is attained when $x = p$. By Stirling's formula, it is easy to see that this maximum is $O\left(\frac{1}{\sqrt{p}}\right) = o(1)$ as $p \rightarrow \infty$. Therefore

$$(5.5) \quad \lim_{\substack{x \rightarrow \infty \\ p \rightarrow \infty}} \frac{e^{-x} x^p}{p!} = 0, \quad x, p \text{ independent.}$$

Consider now two cases:

(i) Assume that $\chi_m \geq m$. Then the terms of the summation

$$e^{\chi_m} t_m = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \chi_m^\nu}{\nu!}$$

are increasing in absolute value. Inasmuch as the terms are also alternating in sign, we have

$$\left| \sum_{\nu=0}^m \frac{(-1)^\nu \chi_m^\nu}{\nu!} \right| \leq \frac{\chi_m^m}{m!},$$

$$|t_m| \leq e^{-\chi_m} \frac{\chi_m^m}{m!}, \quad \chi_m \geq m.$$

(ii) Assume that $\chi_m < m$. Then

$$e^{\chi_m} t_m = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \chi_m^\nu}{\nu!} = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \chi_m^\nu}{\nu!} - \sum_{\nu=m+1}^{\infty} \frac{(-1)^\nu \chi_m^\nu}{\nu!} = e^{-\chi_m} - \sum_{\nu=m+1}^{\infty} \frac{(-1)^\nu \chi_m^\nu}{\nu!}$$

The terms of the alternating series in the right hand

member of the above equation are decreasing in absolute value; therefore

$$\left| \sum_{\nu=m+1}^{\infty} \frac{(-1)^{\nu} \chi_m^{\nu}}{\nu!} \right| \leq \frac{\chi_m^{m+1}}{(m+1)!} \leq \frac{\chi_m^m}{m!},$$

$$|t_m| \leq e^{-2\chi_m} + e^{-\chi_m} \frac{\chi_m^m}{m!}, \quad \chi_m < m.$$

Combining the estimates of (i) and (ii) with (5.5), it follows immediately that $t_n = o(1)$, $n \rightarrow \infty$, $\chi_n \rightarrow \infty$.

Q.E.D.

NOTES

- 1.(p.2, 1.10) O. Szász, Some New Summability Methods with Applications, Annals of Math., v. 43(1942).
- 2.(p.3, 1.4) In this paper, the problem was considered for Abel and related summability methods.
- 3.(p.3, 1.5) It will be noted, however, that if the first method is boundedness preserving, then the associated triangular methods are also boundedness preserving for all sequences $\{\chi_n\}$, $\chi_n \rightarrow \xi$.
- 4.(p.4, 1.9) A short proof of this limit theorem may be given using the theory of Fourier transforms. We also refer to: (a) the 'central limit theorem' in the book of Uspensky on the theory of probability, (b) Pólya-Szegő, Aufgaben und Lehrsätze aus der Analysis, (c) M. Kac, Note on the partial sums of the exponential series, Univ. Nac. Tucumán, Revista A.3(1942).
- 5.(p.5, 1.8) The purpose of the preceding lines was to transform the inequality with which we started, thus obtaining μ in terms of χ_n . This will be the method of attack used throughout section 2.
- 6.(p.10, 1.17) For an example of analytic continuation by triangular summability methods see R. G. Cooke, On lower semi-matrices, Journal of the London Math. Soc., 14(1939).
- 7.(p.14, 1.8) As a matter of fact, the hypothesis that $R\left(\frac{2}{y}\right) < 1$ and the fact that Γ contains the origin

imply that Γ must also contain γ . We understand by the interior of Γ , the finite region bounded by Γ .

8.(p.17, 1.14) See Borel, Lecons sur les séries divergentes

9.(p.19, 1.7) The use of the modified partial sums instead of the ordinary partial sums has become quite common in the literature.

10.(p.23, 1.20) For the asymptotic expansion of the Borel-Lebesgue constants see L. Lorch, The Lebesgue Constants for Borel Summability, Duke Math. Journal, 11(1944).

11.(p.24, 1.6) H. Lebesgue, Sur les Intégrales Singulières, Annales de Toulouse, 1(1909).

12.(p.24, 1.8) C. N. Moore, On the Application of Borel's Method to the summation of Fourier Series, Proceedings of the National Academy of Sciences, 11(1925). See also Hardy, Remarks on some points in the theory of divergent series, Annals of Math., 36(1935), Hardy and Littlewood, Some new convergence criteria for Fourier series, Ann. della Scuola Norm. di Pisa, 3(1934).

13.(p.25, 1.20) This formula follows from the formula on line 2 of page 16 and formulae (3.6), (3.7), (3.10), (3.11).

14.(p.26, 1.7) It is possible to give power series which are Borel summable at singular points.

PROBLEMS

1. The comparison of Borel's exponential method with (1.6) in the case of power series with a zero radius of convergence. In other words, the comparison for a series $\sum a_n$, where $\lim_{n \rightarrow \infty} \frac{|a_n|}{k^n} = \infty$ for all $k > 0$.
2. The development of Tauberian theorems for the methods (1.6). It is not hard to see that the convergence condition $a_n = o(n^{-k})$ on Borel's method may be extended to (1.6).
3. The 'triangulation' of the class of Borel-Sannia methods, in particular Borel's integral method written as a series-to-function transform. There is a marked difference between the resulting methods when the process is applied to Abel summability written first as a series and then as a sequence transform.¹
4. The comparison of the triangular methods obtained by the application of the process to the Borel-Sannia methods.
5. The comparison of (1.6) with Euler's method. The open case would be the series $\sum a_n$ where $\lim_{n \rightarrow \infty} \frac{|a_n|}{k^n} = \infty$ for all $k > 0$. A general formula relating (1.6) to Euler summability would be desirable. Such a formula exists between Borel's exponential method

and Euler summability.

6. The application of Borel's methods and the triangular methods to asymptotic series.
7. The study of Borel's and the triangular methods in connection with the Gibbs phenomenon for Fourier series.

Some unpublished results have been obtained for the above problems. It is the intention of Dr. Szász and the writer, either individually or jointly, to thoroughly consider them.