On the Construction of Doubly Periodic Functions Which Have Singular Points (Polar and Essential) in the Period Parallelogram.

A Dissertation in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy at the University of Cincinnati.

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Presented June., 1911.

Biographical.

The writer was born in Ohio. He received his elementary education in the public schools of that state. He attended nearly four years at Lebanon University, first with the intention of becoming a civil engineer, but later with the purpose of preparing for the teaching profession. This change has never been a source of regret. At Lebanon he studied collegiate and professional branches, and received the inspiration that made study a fixed habit and carried him through several years of encouragingly successful experience in teaching in public school, college and normal work. He took up his studies again in a formal way at the Ohio State University, studying mathematics, physics and electrical engineering. After several months he was called to teach mathematics, physics and normal branches at Defiance College/. Here he continued his studies until he received the degree Master of Arts, in 1903. He remained the following year as professor of mathematics and physical science. In 1904 he accepted the offer of a position as professor of mathematics and physical science in Lebanon University.

In September 1906 he obtained the opportunity for which he had waiting and entered the Graduate School of the University of Cincinnati to resume his studies.

The present discussion will begin with a statement of others in this field and will be followed by a classification of singular points. Then will be given the reasoning which leads to the results of the writer.

The theorem of Laurent which gives the development of a function in positive and negative powers of the variable is of fundamental importance in the study of functions having singular points. It appears that Weierstrass was the first to make extensive use of the theorem for this purpose.(1) It is well known that there were other strong contributors to the theory of essential singular points. In In a memoir9(2)?Weierstrass used the classification referred to a bove and showed the difference in the forms of the function in th neighborhood of the two kinds of points. he also proved a theorem of which mention will be made later.

Mittag-Leffler(3), disciple of Weierstrass, extended the work of his master to the case of functions having an infinite number of essential singular points but only a finite number in a finite part of the plane. He gave a classification of functions and their singular points which has been used by later writers.

Picard (4) discussed linear differential equations having doubly periodic coefficients and arrived at an expression representing a function which is doubly periodic and has an essential singular point in the period parallelogram. His method in brief was as follows: Take the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

(1) See Guichard, Theorie des Points Singuliers Essentiels, Introduction, Paris, 1883.
(2) Abhandlung, Berlin, 1876; Ecole Noranal, 1879.
(3) Acta Math. IV; Comp. rend., 1882.
(4) Comp. rend., 1880.

where P,Q are doubly periodic functions in x. Its solution is

$$y = \psi(x)\int\frac{dx}{(\psi(x))^2}e^{-\int P\,dx},$$

where $\psi(x)$ is doubly periodic. The expression on the right is a function of the kind indicated.

Appell (5) takes up the question in connection with the abelian by the aid of $\Theta(u_1, u_2, ---, u_p)$ which is a function of p variables having the periods of the normal abelian integrals u(x,y). Let $\Theta_1(u)$ be the above function where p equals unity and write with Hermite,

$$Z_1(w) = \frac{d}{dw}\log\theta_1(w), \quad Z_1^{(1)}(w) = \frac{d^2}{dw^2}\log\theta_1(w), \text{-------}$$

$$\text{----}, \quad Z_1^{(n)}(w) = \frac{d^n}{dw^n}\log\theta_1(w), \text{-------}$$

Then he shows that a doubly periodic function $\phi(w)$ having the points a_ν, as, zeros of degree n_ν $(\nu = 1,2, \text{-----} \infty)$, no poles and one essential singular point a in the period parallelogram, where $\lim. a_\nu = a$ and A any constant, may be written in the form

$$\phi(w) = A\prod_{\nu=1}^{\nu=\infty}\left\{\frac{\theta_1(w-a_\nu)}{\theta_1(w-a)}\right\}^{n_\nu}\cdot e^{-n_\nu\sum_{m=1}^{m=m_\nu}\frac{(a-a_\nu)^{m+1}}{\underline{|m+1}}\cdot Z^{(m)}(w-a)}.$$

Guichard, in his Theorie des Points Singuliers Essentials (6), begins with the theorem of Laurent mentioned above as the starting point of his treatment of the general theory of singular points. He first takes up (in Part I) the general theory and gives two methods of expressing the function in the region of a singular point of any kind. By one of the methods the function is expressed as the sum of two others, one containing negative powers of the variable and the other positive powers of the variable

(5) Comp. rend, 1882; Acta Math. I.

(6). Thesis, Paris, 1883; Referred to hereafter as "Guichard."

This he calls the decomposition in sums. Such a decomposition may always be made for each singular point of the function. By the other method the function is expressed as the product of two others in the region of a singular point. One factor contains the negative powers of the variable and the other the positive powers of the variable (1). This he calls the decomposition in products. This decomposition is not always unique but serves to define the singular points of the function. In Part. II. he takes up simply periodic functions and shows how to express them by the aid of the exponential. The third and last Part of his work is concerned with doubly periodic functions (2). He shows by the use of the ζ-function of Hermite that one may form an auxilliary functions (called the intermediary function, having an assigned singular point and which is doubly periodic except as to an added constant for one of the periods. Functions of this type are designated by Π. Having constructed a set of these functions corresponding to any set of singular points he then shows that

$$F(\omega) = \Pi_1(\omega - a_1) + \Pi_2(\omega - a_1) + \cdots + \Pi_m(\omega - a_m),$$

($a_1, a_2, ----, a_m$) being the singular points in the period parallelogram) is doubly periodic if the sum of the residues of the singula points is zero.

According to the product method the intermediary or Π -functions are so constructed as be doubly periodic except as to a constant factor for one of the periods.

(1) Guichard, pp. 11-16.

(2) " pp. 67 et seq.

Having obtained a set of thes functions corresponding to any set of singular points he forms

$$F(\omega) = \Pi_1(\omega - a_1) \cdot \Pi_2(\omega - a_2) \cdot \cdots\cdots \cdot \Pi_m(\omega - a_m)$$

where a_1, a_2, ---, a_m are the singular points. This expression is the doubly periodic function if the following conditions are satisfied; viz.:

1. The sum of the orders of both the singular points and the zeros in the period parallelogram must vanish.

2. The sum of the products of the singular points by their affixed zeros shall differ from the remaining singular points only by multiples of the periods. These conditions constitute a generalization of the theorems of Liouville on the elliptic functions (1).

Guichard then discusses two expressions, viz.:

$$\sum\sum P\left(\frac{1}{\omega - m\omega - n\omega'}\right) \text{ and } \prod\prod P\left(\frac{1}{\omega - m\omega - n\omega'}\right), \text{ where } \begin{cases} m, n \text{ are integers} \\ \omega, \omega' \text{ are periods} \\ \sum \text{ denotes infinite sum.} \\ \prod \text{ " " product.} \end{cases}$$

which he says may be employed to construct directly the doubly periodic functions.

It remains to mention the works of Pascal and Cazzaniga in this field (2). Both men studied substantially the same problem. Pascal stated his problem thus: On a Riemann surface of deficiency one, to construct in the form of an infinite product a doubly periodic function F(u) having the periods 2ω, $2\omega'$ so that in the fundamental parallelogram it admits an infinite number of zeros $u_1, u_2, ---, u_n, ---$ arranged in a simple series having a single limit point, for simp-

(1) Guichard, pp. 79-80.

(2) Pascal in Rom. Acc. L, Rend. (5) 5, 319-323, 1896; Cazzaniga's paper in Atti Acc. Torino, 33, 1898.

plicity say at the origin, and which admits only one essential singular point. He obtained a solution as the product of primary factor of the form

$$\prod_{m,n}\left\{\frac{\sigma(w-w_{m,n})}{\sigma(w)}\cdot e^{Q_k(w)}\right\},\ \left\{\begin{matrix}\text{where } u_{m,n}=\dfrac{1}{2(m\omega+n\omega')}, \\ m=n=0 \text{ excluded,}\end{matrix}\right\} \text{ and}$$

where $Q_k(w) = \dfrac{\zeta(w_{m,n})}{p(w_{m,n})}\zeta(w) + \dfrac{1}{2}\dfrac{1}{p(w_{m,n})}p(w) + \tfrac{1}{2}p'(w_{m,n}) + \dfrac{\zeta(w_{m,n})}{p(w_{m,n})}p(w_{m,n})w$

~~and where~~

ζ, σ, p being the Weierstrassian functions.

Cazzaniga obtained a solution in a way which he claims is more natural and which posseses more points of symmetry. His primary factors are of the form

$$\frac{\sigma(w-u_n)}{\sigma(w)}\cdot e^{\left\{u_n\zeta(w)+\frac{1}{2}u_n^2 p(w)-\frac{1}{3}u_n^3 p'(w)+\dots+\frac{(-1)^m}{\lfloor m}u_n^m p^{(m-1)}(w)\right\}}$$

where $u_1, u_2, \dots, u_n, \dots$ is a simple series having the origin as a limit point.

Art. II. Limit Points; Classification of Singular Points.

If any set of points constitute[1] a single infinity of individual points, i.e. if a set of points can be put into one to one correspondence with the series of natural integers, we shall call such an assemblage a simply infinite assemblage or set. If such a set have a limit point as defined below we shall call it the limit point of a simply infinite set. If a set contain a doubly infinite number of individual points, we shall call it a doubly infinite set or assemblage. Such a set is obtained if to each integer of the natural series we assign a simply infinite set of individual points as defined above

(1) Cantor, Crelle, Vol. 84, pp. 242-258; Acta Math, Vol. II pp. 311-328.

and take the aggregate of points thus obtained. The limit point of such an assemblage or set will be called the limit point of a doubly infinite set .

A limit point may be defined as a point such that within a circle of arbitrarily small radius having the point as a center there is always at least one other point belonging to the set. (1).

If at a point a a function has an infinity of finite multiplicity we say a is a pole of the function. In such a case a multiplier of the form $(u-a)^n$ where n is a finite positive integer will render the function finite at a, n being equal to the order of the pole. If the function is such that it become infinite at an isolated point a and that a multiplier of the form $(u-a)^n$ where n is infinite is required to render the function finite at a we shall call the point a an infinity of simply infinite multiplicity. If a function is such that at an isolated point a a multiplier of the form $(u-a)^{m\cdot m}$ where $m = n = \infty$ is required to render the function finite we shall say the function has an infinity of doubly infinite multiplicity at a and so on. The absolute value of the function becomes infinite for some path of approach of the variable to the singular point when it is an essential singular point. (2).

If a function is defined by an expression of the form

$$\phi(w) = Aw^n + A_1 w^{n-1} + \cdots + A_{n-1} w + B + B_1 \frac{1}{w-a_1} + B_2 \frac{1}{(w-a_2)^2} + \cdots + B_m \frac{1}{(w-a_m)^m} + \cdots$$

where n is a positive integer and $a_1, a_2, \text{----}, a_m, \text{---}$ is an infinite set having a single limit point a, then a is a non-isolated essential singular point, for it is a limit point of poles of $\phi(u)$ (3).

(1) Osgood, Lehrbuch der Funktionen Theorie, I, p 32; Goursat, Cours D'Analyse, II, p. 110. Young, Theory of Sets, p. 170.

(2) Forsyth, Theory of Functions, second ed., p. 6-8.

(3) Forsyth, pp. 78, 79, 166.

The function $f(u) = A u^m + A_1 u^{m-1} + \cdots + A_{m-1} u + B + B_1 \frac{1}{u-a} + B_2 \frac{1}{(u-a)^2} + \cdots + B_n \frac{1}{(u-a)^n} + \cdots$ where m is a positive integer has an isolated singular point at u=a, For $f(u)$ has an infinity of infinite multiplicity at u=a. Both ϕ and f are wholly indeterminate at u=a. Fuller explanation will be give this subject later. Sufficient has been suggested to indicate the meaning of the following classification of the singular points of functions.

Classification of Singular Points

I. A singular point of class one is a polar singularity, i.e. an infinity of finite multiplicity.

II. A singular point of class two is either a limit point of singular points of class one or an infinity of simply infinite multiplicity.

III. A singular point of class three is either a limit point of singularities of class two or an infinity of doubly infinite multiplicity

N. A singular point of class n is either a limit point of singularities of class n-1 or an infinity of (n-1)-ply infinite multiplicity.

All singular points for which the integer n is finite belong to the first genre of singular points. When n becomes infinite there is begun anew, the second, genre. see Art. X. In the second genre the classes are related to one another in the same way as in the first genre.

Functions are classified according to their singular points of highest class and genre in the plane as follows:

I. Functions of class zero have no singular points.

II. Functions of class one have only polar singularities.

III. functions of class two have singular points of class two, but no higher, and may have singular points of class one.

N. Functions of class n have singular points of class n , and may have singular points of lower classes, but no higher.

This classification does not have reference to any particular part of the plane but rather the whole plane. If a limited region is specified the function may be of lower class than for the whole plane, according to the nature of the singular points in the region considered. In all cases the class of a singular point corresponds to that of it characteristic function, (see Art. III). A correspondence between the class of a singular point and the transcendental character of its characteristic function will next be established.

Art. III. Orders of Transcendentals.

It is clear from the preceding definitions that either a limit point of a simply infinite assemblage of singular points of class one or an infinity of simply infinite multiplicity is necessary and sufficient to constitute a singular point of class two $\Longleftrightarrow$. This assumes that transcendental character of the function in the neighborhood of both kinds of singular points is the same . The first case is the non-isolated singular point , the second the isolated singular point.

Weierstrass proved the following theorem: In the neighborhood of an isolated essential singular point the function approaches arbitrarily near to every assigned value, (1). In other words if a circle of arbitrarily small radius be drawn about the isolated singular point as a center, there exist always points within this circle for which the relation $f(x)-A<\epsilon$ holds, where $f(x)$ is the function, A a constant value and ϵ an arbitrarily small quantity (2). The following more definite theorem was obtained by Picard, which states that in the circle there exist an infinite number of points for which the equation $f(x)=A$ is rigorously true, save for two values at most of A, viz. $A=0$ and $A=\infty$, (3). THe exceptional case where $A=\infty$ correspond to the multiplicity clause of the foregoing definitions, Art. II.

Functions which have essential singular points are called transcendental. Complete indetermination of the function is characteristic of all essential singular points, (4). The function becomes infinite at an essential singular point for at least one path of approach of the variable to the point, (5). It appears that for a singular point of class two the following conditions exist simultaneously, Viz: (a) Either a is a limit point of singular points of class one or a is an infinity of simply infinite multiplicity. (b) A simply infinite assemblage of roots for the equation $f(x)=A$ in a small circle of center a, A arbitrary except as to value mentioned above. (c) Complete indetermination of the function within this circle. The same conditio exist at a singular point of class three, where a simple infinity is

(1) Weierstrass, Ges. Werke, t. II. pp. 122. Osgood, p. 264, Forsyth, p 59.
(2) Goursat, II., p. 111.
(3) Picard, Comp. Rend., 1882; Osgood, p. 265; Goursat, p. 113; Forsyth, pp. 59-60.
(4) Osgood, p. 264; Goursat, p. 113; Forsyth, p. 56.

to be replaced by a double infinity and the indetermination is of the second order, and. These may be briefly incorporated in the following theorem . If $f(x)=A$ has an n-ply infinite assemblage of roots aboute a as a limit point for arbitrary values of A it has the same infinity of roots for all values of A , is completely indeterminate at a and has a for a singular point of class n-1.

We may give a characteristic property of transcendental functions which will be made fundamental in further developments. For brevity shall call this property the ~~order~~ of the transcendental the term referring solely to the transcendental character of the function . For this reason we shall write the term trans-order to distinguish it from the ordinary use of the word. It is this transcendentalism that is insisted upon in what follows

Consider (1) $F(u)=K$,where F is an algebraic polynimial of degree m in u. This equation has m roots to which there correspond m points in the plane. Again suppose (2) $f(u)=K$,where f is of degree n in u, This equation has n roots to which there correspond n points in the plane. Write again (3) $f(F(u))=K$ formed by putting F(u) for u in f(u). This equation(3) has m·n roots to which there correspond m·n points in the plane. In all these equations K is supposed to be any definite value. Now if ϕ denotes a polynomial of degree p. Then the equation (4) $\phi(f(F(u)))=K$ admits m·n·p roots corresponding to m·n·p points in the plane. Further these relations hold however great the integers m , n , p become. We shall assume them to hold when m ,n ,p become infinite if F ,f ,ϕ represent definite functions, i.e. when they are transcendental integral functions.

Under this hypothesis when $m=\infty$ Eq.(1) admits a simple infinity of roots in the whole plane. There must be a limit point somewhere and in this case it is known to be at infinity, (1[x]). The point at infinity is said to be a singular point of class two. The exponential is one of the simplest examples of such a function. When $m=n=\infty$ eq.(3) admits a double infinity of roots in the whole plane. The limit point of this assemblage is at infinity and infinity is a singular point of class three. A simple example of such a function is $e^{(e^{\omega})}$. Finally when $m=n=p=\infty$ eq. (4) admits a triply infinite assemblage of roots in the whole plane having its limit point at infinity which is a singular point of class four. A simple example of such a function is $e^{(e^{(e^{\omega})})}$, (2[x]). This method of reasoning may be extended to functions having singular points of any class at infinity.

The function represented by eq.(1) when $m=\infty$ is taken as the type of transcendental functions of trans-order one. Such a function belong to the class of simple transcendental functions. The funftion of eq. (3) when $m=n=\infty$ is doubly transcendental and is typical of functions of tráns-order two, and so on.

It is now seen that the class of singular point is one greater than the trans-order of the function in the immediate neighborhood of the singular point.

It is evidently possible to place the singular point at any finite point a by the transformation $u=\frac{1}{\omega'-a}$. This transformation cannot affect the the degree or the trans-order of the function. It follows that $F(\frac{1}{\omega'-a})$. has a singular point of class two at $u'=a$.

(1[x]) Weierstrass, Werke, Bd.III. p.100; Forsyth. pp.84 et. seq.

(2[x]) This form will be used without the curves, but is always to be interpreted as written above.

Similarly $f(F(\frac{1}{u-a}))$ has a singular point of class three at $u' = \underline{a}$. $\phi(f(F(\frac{1}{u-a})))$ has a singular point of class four at $u = \underline{a}$.

Let $\phi_1(u)$, $\phi_2(u)$, $\phi_3(u)$, ----$\phi_m(u)$ denote simple transcendental functio i.e. functions of trans-order one and class two at $u' = a$. They may be alike or different at pleasure. The functions,

$$\begin{array}{ll} (1) & \phi_1(\frac{1}{u-\underline{a}_1}) \\ (2) & \phi_2(\frac{1}{u-\underline{a}_2}) \\ \vdots & \\ (n-1) & \phi_{m-1}(\frac{1}{u-\underline{a}_{n-1}}) \end{array}$$

all have singular points of class two at $\underline{a}_1$, $\underline{a}_2$, ----, $\underline{a}_{m-1}$ respectively. From the above reasoning and the definitions already given we infer that any function like

$$F_3(u) = \phi_p(\phi_q(\frac{1}{u-\underline{a}}))$$

will have a singular point of class ~~two~~ three at $u = \underline{a}$, for F_3 is of trans-order two, p,q being any of the integers 1,2,3,------,n . Any function like

$$F_4(u) = \phi_p(\phi_q(\phi_r(\frac{1}{u-\underline{a}})))$$

will have a singular point of class four at $u = \underline{a}$, for its trans-order is three. We may now form a function having any given distributio of singular points as follows: Write

$$\phi_p(\frac{1}{u-\underline{a}_1}) = F_2(u)$$

$$\phi_q(\phi_r(\frac{1}{u-\underline{a}_2})) = F_3(u)$$

$$\vdots$$

$$\phi_l(\phi_m(\text{-----}(\phi_t(\frac{1}{u-\underline{a}_{m-1}}))\text{-----} = F_n(u)$$

having singular points of class one ,two,----- n at the points $\underline{a}_1$,$\underline{a}_2$, ------,$\underline{a}_{n-1}$ respectively .Now form any integral polynomial involving these functions,say

$$F(\omega) = R\left\{ \frac{1}{\omega-\underline{a}_0},\ F_2(\omega),\ F_3(\omega), ------, F_m(\omega) \right\}$$

This is the general expression required.THe simplest examples of it are

$$_sF(\omega) = \frac{1}{\omega-\underline{a}_0} + F_2(\omega) + F_3(\omega) + \cdots + F_m(\omega)$$

$$_pF(\omega) = \frac{1}{\omega-\underline{a}_0} \cdot F_2(\omega) \cdot F_3(\omega) \cdot ----- \cdot F_m(\omega)$$

where s, p denote sum and product respectively.We shall now put each ϕ equal to the exponential function of the corresponding argument and change F to $\overline{F}$. Write

$$\overline{F}(\omega) = \overline{R}\left\{ \frac{1}{\omega-\underline{a}_{\alpha,1}},\ e^{\frac{1}{\omega-\underline{a}_{\beta,2}}},\ -----,\ e^{e^{\cdot^{\cdot^{\cdot^{e^{\frac{1}{\omega-\underline{a}_{\kappa,n}}}}}}}} \right\}$$

$$_s\overline{F}(\omega) = \frac{1}{\omega-\underline{a}_{\alpha,1}} + e^{\frac{1}{\omega-\underline{a}_{\beta,2}}} + \cdots\cdots + e^{e^{\cdot^{\cdot^{\cdot^{e^{\frac{1}{\omega-\underline{a}_{\kappa,n}}}}}}}}$$

$$_p\overline{F}(\omega) = \frac{1}{\omega-\underline{a}_{\alpha,1}} \cdot e^{\frac{1}{\omega-\underline{a}_{\beta,2}}} \cdot ------- \cdot e^{e^{\cdot^{\cdot^{\cdot^{e^{\frac{1}{\omega-\underline{a}_{\kappa,n}}}}}}}}$$

In these equations the first subscript distinguishes the a's and the second indicates the class of singular point.

It may be shown as follows that the singular points defined by a function like $e^{e^{\frac{1}{\omega-\underline{a}}}}$

is isolated: The function must take the value infinity for at least one path of the variable to the singular point. The equation

$$e^{\frac{1}{\omega-\underline{a}}} = \infty$$

has no roots near $u=\underline{a}$. It follows that there is no singular point near $u=\underline{a}$. Call $e^{\frac{1}{u-a}}=U$. Then the equation $e^{U}=\infty$ has no roots for $U\neq\infty$, i.e. for $u\neq a$. It follows that the point for which the function takes the value infinity is isolated; and consequently for both the functions $e^{\frac{1}{u-a}}$ and $e^{e^{\frac{1}{u-a}}}$,

the point $\underline{a}$ is an isolated singular point, (1).

We shall next explain the meaning of the term characteristic function of a singular point. There are two methods of defining singular points by characteristic functions, (2). By the first method the function is expressed as the sum of two others in the region of a singular point a say in the form

$$F(u) = P(u-\underline{a}) + Q\left(\frac{1}{u-\underline{a}}\right),$$

where Q defines the singular point $\underline{a}$ of the function F and is called the characteristic function of the singular point, (3) while P has no singular points near $\underline{a}$. It is seen that Q is that part of the Laurent series which contains the negative powers of the variable, (4). The trans-order of Q determines the class of the singular point. According to the other method the funstion is expressed as the product of two others in the neighborhood of the singular point. Write

$$F(u) = \bar{P}(u-\underline{a}) \cdot \bar{Q}\left(\frac{1}{u-\underline{a}}\right)(u-\underline{a})^n \quad ,(5)$$

Now $\bar{Q}\frac{1}{u-\underline{a}}$ defines the singular point $\underline{a}$. $\bar{P}$ has no zeros near $u=a$.

THe class of the characteristic function is the same as the class of the singular point which it defines.

(1) Forsyth pp. 78-79.
(2) Guichard, pp. 11-15; Forsyth, p. 165.
(3) Goursat, pp 111-112; Forsyth, pp. 108 and 165.
(4) Called also principal part as to the singular point $\underline{a}$.
(5) Since no special use is made of the factor $(u-\underline{a})^n$, we shall include it in $\bar{P}$.

It is this function the $Q(\frac{1}{u-a})$ which determines the number of roots of f in the neighborhood of the singular point a. For $P(u-a)$ becomes vanishingly small for u near a, or approaches a constant not zero. We have then virtually $Q = K$ where K is wholly arbitrary. The other singular points of F must be singular points of P not of Q. The function

$$F(u) = A_0 u^n + A_1 u^{n-1} + \dots + A_{m-1} u + B_0 + B_1 \frac{1}{u} + B_2 \frac{1}{u^2} + \dots + B_m \frac{1}{u^m},$$

where m, n are positive integers, has a singular point of class one at $u=0$. For the series of negative powers is finite and of trans-order zero in $\frac{1}{u}$. The series positive powers is characteristic of a singular point of class one at infinity. For this series is of trans-order zero in u. Again suppose $F(u) = e^{\frac{1}{u} + A_0 + A_1 u + \dots + A_n u^n + \dots}$

where the series $A_0 + A_1 u + \dots + A_n u^n + \dots$, is an infinite series (convergent) of trans-order one at $u=\infty$. We may write F as the product of two factors, thus:

$$F(u) = e^{\frac{1}{u}} \cdot e^{A_0 + A_1 u + \dots + A_n u^n + \dots} = \bar{Q}(u) \cdot \bar{P}(u)$$

Now $e^{\frac{1}{u}}$ is of trans-order one at $u=0$/ and is characteristic of a singular point of two at $u=0$. The other factor is of trans-order two and is characteristic of a singular point of class three at infinity.

The derivative of a function is of the same trans-order as the function itself. Now consider X and θ two functional operations such that $X(\theta(u)) = u$. X and θ are each to be regarded as the inverse of the other. Suppose θ is of trans-order one, we shall define X as being of trans-order minus one. If θ is of trans-order two, X must be of trans-order minus two, etc. Denote by ${}^nX$ the result of n successive operations of X on an argument, and by $^{m}\theta$ the result of m successive operations of θ on the argument. Then if (2) $f(u) = {}^{r}\theta(\phi(u))$

it follows that: (3) $^{p}X(f(u)) = {}^{r-p}\theta(\phi(u))$

where p ,q are positive integers.If $\phi(u)$ in (2) is of trans-order zero,f(u) is of trans-order r and X(f(u)) is of trans-order r-p.

When the trans-order of a function F is known, and if

$F(u) = {}^{p}\theta(\phi(u))$ then $^{p}X(F(u)) = \phi(u)$,

where $\phi(u)$ is of trans-order p lower than F(u).For illustration suppose F(u) is a function of trans-order five in u and suppose also θ_1 ,θ_2 are each of trans-order one but different from each other and from θ. We may write $F(u) = {}^{3}\theta(\theta_1(\theta_2(u)))$.

Then $^{3}X(F(u)) = \theta_1(\theta_2(u))$ is a function of trans-order two. The application of X to any function reduces its trans-order by one for each successive application.Simple examples of such functions are the exponential and logarithm, and the trigonometric functions and their inverses, the circular functions . Suppose again that $f(u) = e^{\theta_1(\theta_2(u))}$,

Then $\log f(u) = \theta_1(\theta_2(u))$,

where θ_1 ,θ_2 are defined as above.This makes the trans-order of f(u) three. Then log f(u) is a function of trans-order two .

In view of these considerations the logarithmic function will be regarded as of trans-order minus one,since we have already defined its inverse the exponential as of trans-order one,(plus one).

Functions whose fundamental construction is indicated by:-

Simply infinite summation of powers of the variable,

Simply infinite product of factors of finite degree,

Any set of equivalent operations on the variable,

~~are to be regarded as functions of trans-order one. Functions whose fundamental construction as indicated by~~

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are to be regarded as functions of trans-order one. Functions whose fundamental construction is indicated by

Doubly infinite summation of powers of the variable,

Doubly infinite product of factors of finite degree,

Any set of equivalent operations on the variable,

are to be regarded as functions of trans-order two, etc. When two functions are related by an inverse relationship in the sense explained above one may chosen as positive trans-order when no criterion is decisive. The other will then be regarded as of negative trans-order. When a function has no known inverse a similar choice may be made tentatively. A function which is composed of other functions in a finite number of operations takes its trans-order from that constituent which is of highest trans-order. So far as this paper is concerned the last statement applies only to integral algebraic functions of the arguments, no assumption being made as to other forms.

In the light of the preceding discussion we shall now consider the Weierstrassian functions $\sigma(u)$ and $p(u)$. We know that

$$\sigma(u) = u \prod \prod \left(1 - \frac{u}{w}\right) \cdot e^{\frac{u}{w} + \frac{1}{2}\left(\frac{u}{w}\right)^2}, \text{ where } \begin{cases} w = 2m\omega + 2n\omega' \\ \left.\begin{matrix} m \\ n \end{matrix}\right\} = 0, \pm 1, \pm 2, \cdots \\ w \neq 0. \end{cases}$$

This function is of trans-order three, (1). For it is determined by the doubly infinite product of the exponential of a finite expression in u. Log $\sigma(u)$ is of trans-order two. Now $\frac{d}{du} \log \sigma(u) = \frac{\sigma'(u)}{\sigma(u)}$ is also of trans-order two. It follows that

$$-\frac{d^2}{du^2} \log \sigma(u) = \wp(u)$$

is a function of trans-order two. We know that $\frac{1}{w^2}$ is characteristic

(1) This function has only a doubly infinite system of zeros, but zero is a value which the exponential does not take. The equation $\sigma(u) = K$ in general admits a triply infinite system of roots.

of the singular point of p(u) at the origin. It is of trans-order zero, indicating a singular point of class one. This agrees with the known facts. The other part of p(u), viz.: $p(u)-\frac{1}{u^2}$ is of trans-order two and characterizes a singular point of class three at infinity. The function $e^{p(u)}$ has a singular point at the origin whose characteristic function is $e^{\frac{1}{u^2}}$. This is of class two. The expression $e^{p(u)-\frac{1}{u^2}}$ characterizes a singular point of class four at infinity.

The general formula of Mittag-Leffler, $f(u) = G(u) + \sum_1^\infty G_k\left(\frac{1}{u-c_k}\right) - f_k(u)$ (1) is of trans-order one in G_k which defines the singular points of f; but G_k may be so chosen as to make the singular points of any class. G has no singular points and f_k ensures convergence of $\sum\{\cdots\cdots\}$. For this purpose it is necessary to construct G_k of trans-order one lower than is desired of f(u). As an example, consider $G_k(u) = \frac{e^{\frac{1}{u_k}}}{\lfloor k}$ a function of trans-order one. It is convergent and $\sum_1^\infty G_k$ is convergent so f_k is not needed in the development. If u_k has a limit point any where in the plane, it is a singular point of class three for f(u). For f(u) is of trans-order two in the neighborhood of such a point.

The functions in Art. I. belong to the second class of the two parted classification, which appears to be the intent of their authors, no attempt at further classification being made by them. Guichard discussed the general theory of all classes of singular points, but did not particularize with regard to any definite set of corresponding functions.

Art. IV. Discussion Extended to Doubly Periodic Functions.

Let $f_1(u)$ be any doubly periodic function whose only singular points in the finite part of the plane are poles. We may write

$$f_1(w) = Q_1\left(\frac{1}{w-a}\right) + P_1(w-a) \quad \text{and} \quad \bar{f}_1 = \bar{Q}_1\left(\frac{1}{w-a}\right) \cdot \bar{P}(w-a)$$

where Q_1, $\bar{Q}_1$ are of trans-order zero and <u>a</u> is the pole in the fundamental parallelogram. Now consider any functional operation θ_k of trans-order one and write

$$\theta_k(f_1(w)) = \theta_k\left\{Q_1\left(\frac{1}{w-a}\right) + P_1(w-a)\right\} \quad \text{and} \quad \bar{\theta}_k(f_1(w)) = \bar{\theta}_k\left\{\bar{Q}_1\left(\frac{1}{w-a}\right) \cdot \bar{P}_1(w-a)\right\},$$

These functions may be put in the form respectively

$$f_2(w) = Q_2\left(\frac{1}{w-a}\right) + P_2(w-a) \quad \text{and} \quad \bar{f}_2(w) = \bar{Q}_2\left(\frac{1}{w-a}\right) \cdot \bar{P}_2(w-a)$$

where Q_2, $\bar{Q}_2$ are of trans-order one in $\frac{1}{w-a}$ indicating a singular point of class two at <u>a</u>. Let θ_l be another functional operation of trans -order one and write

$$f_3(w) = \theta_l\left(\theta_k\left\{Q_1\left(\frac{1}{w-a}\right) + P_1(w-a)\right\}\right) \quad \text{and} \quad \bar{f}_3(w) = \theta_l\left(\theta_k\left\{\bar{Q}_1\left(\frac{1}{w-a}\right) \cdot \bar{P}_1(w-a)\right\}\right),$$

which can be put in the form respectively

$$f_3(w) = Q_3\left(\frac{1}{w-a}\right) + P_3(w-a) \quad \text{and} \quad \bar{f}_3(w) = \bar{Q}_3\left(\frac{1}{w-a}\right) \cdot \bar{P}_3(w-a).$$

The functions Q_3, $\bar{Q}_3$ are of trans-order two indicating a singular point of class three at <u>a</u>.

Since the θ's transcendental integral operations on their respective arguments it is evident that f_2, f_3 are doubly periodic. By this method we may gradually construct a function that is doubly periodic and has a singular point of any class at the point <u>a</u>. From now on we shall be concerned with the special case where each θ is replaced by the exponential function. The question of convergence does not require special treatment, for the exponential series is convergent and one valued. For these reasons the following discussion will be very simple and the results quite symmetrical. It is evidently possible to make use of other simple transcendental functions. It has been shown that the singular points of the exponential are isolated.

I. Write $W_2(u) = e^{p(\omega)} - 1$ It follows without difficulty that:

1. $W_2(u)$ is doubly periodic in the same periods as $p(u)$.
2. $W_2(u)$ has zeros at the zeros of $p(u)$.
3. $W_2(u)$ has singular points of class two at the poles of $p(u)$.

~~$W_2(u)$~~ For $e^{\frac{1}{\omega^2}}$ is of trans-order one in u.

4. $W_2(u)$ has no poles intthe period parallelogram.
5. $W_2(u)$ is finite one valued and continuous except at the singular points mentioned in (3).
6. $W_2(u)$ has a singular point of class four at infinity. for $e^{p(\omega)-\frac{1}{\omega^2}}$ is of trans-order three in u.

II. Next write $W_3(u) = e^{e^{p(\omega)}-1} - 1$. As above we may assert:

1. $W_3(u)$ is doubly periodic in the same periods as $p(u)$.
2. $W_3(u)$ has zeros at the zeros of $p(u)$.
3. $W_3(u)$ has singular points of class three at the poles of $p(u)$. For $e^{e^{\frac{1}{\omega^2}}}$ is of trans-order two in u.
4. $W_3(u)$ has no poles inythe period parallelogram.
5. $W_3(u)$ is finite one valued and continuous except at the singular points mentioned in (3).
6. $W_3(u)$ has a singular point at infinity of class five. For $e^{e^{p(\omega)-\frac{1}{\omega^2}}}$ is of trans-order four in u.

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N. Finally write $W_n(u) = e^{e^{\cdots^{e^{p(\omega)}-1}}\cdots -1} - 1$. As above we have:

1. $W_n(u)$ is doubly periodic in the same periods as $p(u)$.
2. $W_n(u)$ has zeros at the zeros of $p(u)$.
3. $W_n(u)$ has singular points of class n at the poles of $p(u)$. For $e^{e^{\cdots^{e^{\frac{1}{\omega^2}}}}}$ is of trans-order n-1 in u.
4. $W_n(u)$ has no poles in the period parallelogram.

5. $W_n(u)$ is finite one valued and continuous except at the singular points mentioned in (3).

6. $W_n(u)$ has a singular point of class n-2 at infinity . FOR

$e^{e^{\cdot^{\cdot^{\cdot^{e^{p(\omega-\frac{1}{\omega^2}}}}}}}$. is of trans-order n-1 in u.

THe W-functions are to be regarded as fundamental in the present discussion. They will be used in the construction of more general functions. If in p(u), u-a is put for u, the poles of p(u) and all the singular points of the W-functions are placed at the point a in the fundamental parallelogram a being any arbitrary point. Further since any elliptic function having the same periods as p(u) may be expressed rationally through p(u) and p′(u), it follows that functions may be formed in an analogous manner which have singular points distributed with no more restriction than the poles of the elliptic functions. This fact will be symbolized as follows:

$$ {}_gW_2(\omega) = e^{R(p(\omega),p'(\omega))} - 1 = e^{f(\omega)} - 1, $$

$$ {}_gW_3(\omega) = e^{e^{R(p(\omega),p'(\omega))}-1}-1 = e^{e^{f(\omega)}-1}-1, \text{ etc.,} $$

where f(u) is any elliptic function having the periods of p(u) and where R is a rational function of its arguments. The index g denotes a more general function than W. ${}_gW_2(u)$ will have zeros at the zeros of f and singular points of class two at the poles of f, but no pole. It is noted that $W_2(u)$ takes every value a simple infinity of times in the period parallelogram, in virtue of the essential singular point.

This is analogous to the property of elliptoc functions. With the latter the number is finite and equal to the number or multiplicity of poles. It is in all cases determined by the characteristic functions of the singular points in the period parallelogram.

Art. V. Derivatives of the W-functions.

For use in the derivatives of the W-functions the following derivatives of p(u) are inserted.

1. $p'^2 = 4p^3 - g_2 p - g_3$
2. $p'' = 6p^2 - \frac{1}{2} g_2$
3. $p''' \quad 12p \cdot p'$, g_2, g_3 being the well known invariants.

where p is used for brevity for p(u). W will also be used for W(u).

Write $W_2 = e^p - 1$

$$W_2' = e^p \cdot p' = (W_2 + 1) p'$$

$$W_2'' = \frac{d}{du} W_2' = (W_2 + 1)(p'^2 - 6p^2 - \tfrac{1}{2} g_2)$$

$$W_2''' = (W_2 + 1)\{18p^2 \cdot p' - 12p'(p - p' - \tfrac{1}{2} g_2) - g_2\}.$$

In general the n-th derivative may be expressed in the form

$$\frac{d^n}{du^n} W_2 = (W_2 + 1) R(p, p'),$$

where R denotes a rational integral function of its arguments.

Write $W_3 = e^{e^{p(u)} - 1} - 1 = e^{W_2} - 1$

$$W_3' = (W_3 + 1)(W_2 + 1) p'$$

$$W_3'' = (W_3 + 1)(W_2 + 1)[(W_2 + 2) p'^2 - 6p^2 - \tfrac{1}{2} g_2]$$

$$W_3''' = (W_3 + 1)(W_2 + 1)[(W_2 + 1)(p'^2 - 6p^2 - \tfrac{1}{2} g_2) - p'^3 + 18p^2 p' - \tfrac{3}{2} g_2 p' + 12pp']$$

In general it is seen that

$$\frac{d^n}{du^n} W_m = (W_m + 1) R\{W_{m-1}, W_{m-2}, \cdots W_2, p, p'\},$$

where R is a rational integral function of its arguments.

Art. VI. Expressions of W_2 through Infinite series and Infinite Products.

Write $W_2 = e^p - 1$ It is known that at an ordinary point by Taylor's (1) theorem

$$p = p(a) + p'(a)(w-a) + \cdots\cdots p^{(m)}(a)\frac{(w-a)^n}{\lfloor n} + \cdots\cdots$$

Hence $W_2 + 1 = e^{p(a) + p'(a)(w-a) + \cdots\cdots + p^{(m)}(a)\frac{(w-a)^n}{\lfloor n} + \cdots} = \prod_0^\infty e^{p^{(m)}(a)\frac{(w-a)^n}{\lfloor n}}$ (1)

Expanding e^p as an exponential, we obtain

$$W_2 = p + \frac{p^2}{\lfloor 2} + \frac{p^3}{\lfloor 3} + \cdots\cdots + \frac{p^m}{\lfloor n} + \cdots = \sum_1^\infty \frac{p^n}{\lfloor n} \qquad (2)$$

W_2 expanded by Taylor's theorem gives

$$W_2 = W_2(a) + W_2'(a)(w-a) + \cdots\cdots + W_2^{(m)}(a)\frac{(w-a)^n}{\lfloor n} + \cdots = \sum_0^\infty W_2^{(m)}(a)\frac{(w-a)^n}{\lfloor n}. \qquad (3)$$

Write $p = \frac{1}{w^2} + \sum_{m=-\infty}^{m=\infty}\sum_{m=-\infty}^{m=\infty}\left\{\frac{1}{(w-2m\omega-2n\omega')^2} - \frac{1}{(2m\omega+2n\omega')^2}\right\}, n = m \neq 0$ (4)

Whence $W + 1 = e^{\frac{1}{w^2}} \cdot \prod_{m=-\infty}^{m=\infty}\prod_{m=-\infty}^{m=\infty}\left\{e^{\left(\frac{1}{(w-2m\omega-2n\omega')^2} - \frac{1}{(2m\omega+2n\omega')^2}\right)}\right\}, n = m \neq 0.$ (5)

Substituting the values of p from (4) in (2) gives

$$W_2 = \sum_{k=1}^{k=\infty}\left[\frac{\frac{1}{w^2} + \sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{m=\infty}\left\{\frac{1}{(w-2m\omega-2n\omega')^2} - \frac{1}{(2m\omega+2n\omega')^2}\right\}}{\lfloor K}\right]^K \qquad (6)$$

Equations (4), (5), (6) reveal the trans-order of the function but equation (3) disguises the trans-order by the character of the coefficients. We will now show how W_2 may be expanded directly in a series from the definitions of Art. IV. We know that

$$p(w) = \frac{1}{w^2} + A_1 w^2 + A_2 w^4 + \cdots\cdots + A_m w^{2n} + \cdots\cdots$$

where the A's are known functions of the invariants g_2, g_3.

(1) This refers the Cauchy-Taylor theorem.

Taking the equation $W_2 = e^{p(\omega)} - 1$

and using the above series we may write

$$W_2 + 1 = e^{\frac{1}{\omega^2}} \cdot e^{A_1\omega^2} \cdot \cdots\cdots \cdot e^{A_m\omega^{2m}} \cdot \cdots\cdots$$

which may be expanded in an infinite series of positive and negative powers of u .It is evident from this series that W_2 is an even function. In fact all the W-functions are even.This series enables us to express W_2 in the form

$$W_2(\omega) = Q\left(\frac{1}{\omega^2}\right) + P(\omega^2) ,$$

The function Q is characteristic of the singular point at the origin P is characteristic of the singular point at infinity. In the above development in series the coefficients are known functions of the invariants g_2 , g_3: According to the decomposition in products the expressions $e^{\frac{1}{\omega^2}}$, $e^{A_1\omega^2+A_2\omega^4+\cdots}$ (8) and (9) constitute a set of factors defining the singular points. For we may write

$$W_2(\omega) = e^{\frac{1}{\omega^2}} \cdot e^{A_1\omega^2 + A_2\omega^4 + \cdots} = \bar{Q}\left(\frac{1}{\omega^2}\right) \cdot \bar{P}(\omega^2) ,$$

Art. VII. Relations among the W-functions.

From the definitions of Art. IV. we may write

$$\mathrm{Log}(W_2 + 1) = p(\omega)$$

$$\mathrm{Log}(W_3 + 1) = W_2$$

$$\vdots$$

$$\mathrm{Log}(W_m + 1) = W_{m-1}$$

Also, $p - W_2 = \log \frac{W_2 + 1}{W_3 + 1}$, $W_2 - W_3 = \log \frac{W_3 + 1}{W_4 + 1}$, etc

From the equations of Art. V . it appears that no algebraic relation corresponding to the equation (a) $p'^2 = 4p^3 - g_2 p - g_3$

exists for any of the W-functions and consequently there is no algebraic addition theorem. This is what might be expected, for with Weierstrass the study of elliptic functions was one important division of his study of all functions having algebraic addition theorems. The following theorem due to him shows that functions like the W-functions are not included in his category: Analytic functions which possess algebraic addition theorems are either,

1. Algebraic functions of the variable, say u, or
2. Algebraic functions of $e^{\frac{i\pi u}{\omega}}$, where ω is a period, or
3. Algebraic functions of $p(u)$ and $p'(u)$.

For purposes of calculation the addition theorem of $\wp(u)$ may be used for the W-functions. For

$$W_2(w+v) = e^{p(w+v)} - 1$$

$$W_3(w+v) = e^{e^{p(w+v)}-1} - 1.$$

A simply transcendental relation corresponding to eq. (a) may easily be obtained as follows: We have

$$\text{(b)} \quad W_2' = (W_2+1)\,p'.$$

From equations (a) and (b)

$$4p^3 - g_2 p - g_3 = \left(\frac{W_2'}{W_2+1}\right)^2$$

Call one of the roots of this cubic in p, $F_1(W_2, W_2')$ and note that

$$\operatorname{Log}(W_2+1) = p$$

We then have

$$\text{(c)} \quad (W_2+1) = e^{F_1(W_2, W_2')}$$

where F_1 is algebraic in its arguments. F_1 is of trans-order two in u, being a value of p. Equation (c) is a relation of trans-order one between W_2 and W_2'.

If we take

$$W_3+1 = e^{e^{p}-1}-1, \text{ we have } W_3' = (W_3+1)\log(W_3+1)\,p'.$$

By using (a) there results

$$\frac{(W_3')^2}{[(W_3+1)\log(W_3+1)]^2} = 4p^3 - g_2 p - g_3.$$

Calling one root of this cubic in p, $F_2(W_3, ?W_3')$ we obtain as before

where F is an algebraic function of its arguments. Here again F_2 Being a value of p is of trans-order two in u. The eq. (d) is a relation of trans-order two between W_3 and W_3'. In general it appears that a relation of trans-order n-1 exists between W_n and W_n'.

We note with Weierstrass that the functions

1. Rational functions of the variable, say u,
2. Rational functions of the $e^{\frac{i\pi u}{\omega}}$,
3. Rational functions of p(u) and p'(u),

are such that between each of them and its first derivative there exists an algebraic relation, i.e. of trans-order zero. The degree of this relation depends on the degree of the characteristic functions of the singular points in the region concerned. From eq. (c) it is seen that betwwen W_2 and W_2' there exists a relation of trans-order one in W_2 and W_2'. This corresponds to a singular point of class two in the period parallelogram. Similarly it is seen that a relation of trans-order n-1 exists between W_n and W_n'. corresponding to a singular point of class n in the period parallelogram.

Liouville gave a classification of functions based of their transcendental character as follows: If u, v, w, --- are algebraic functions of x, then all algebraic functions of

u, v, w, ---

Log u, Log v, Log w, ----

e^u, e^v, e^w -----

are transcendental of the first kind. If u', v', w', ---- are transcendentals of the first kind, then all algebraic functions of

u', v', w', -----

Log u', Log v', Log w', ----

$e^{u'}, e^{v'}, e^{w'}$, -----are transcendentals of the second kind, etc.(1)

A comparison of this classification with the one of Art. III. shows agreement so far as the exponential is concerned. With regard to the logarithmic function they agree except as to the sign of the trans-order, which in this paper is to be regarded as negative.

Art. VIII. Functions with Given Singular Points.

Attention will now be given to the formation of doubly periodic functions having prescribed singular points (isolated and non-isolated) in the period parallelogram. Let it be proposed to construct a function having isolated singular points, say a_1, a_2, -- of class one; b_1, b_2, --- of class two; d_1, d_2, -- of class k. Write

$p(u-a_1)$, $p(u-a_2)$, -----

$W_2(u-b_1)$, $W_2(u-b_2)$, ----

⋮

$W_k(u-d_1)$, $W_k(u-d_2)$, ----

These functions define the required singular points and are doubly periodic. It is only necessary to form integral polynomials of them The simplest will be of the form

(1) Crelle's Journal, Vols. 12 and 13.

$$(1)\quad {}_sV_k(w) = p(w-\underline{a}_1)+p(w-\underline{a}_2)+\cdots+W_2(w-\underline{b}_1)+W_2(w-\underline{b}_2)+\cdots+W_k(w-\underline{d}_1)+W_k(w-\underline{d}_2)+\cdots$$

$$(2)\quad {}_pV_k(w) = p(w-\underline{a}_1)\cdot p(w-\underline{a}_2)\cdot\cdots\cdot W_2(w-\underline{b}_1)\cdot W_2(w-\underline{b}_2)\cdot\cdots\cdot W_k(w-\underline{d}_1)\cdot W_k(w-\underline{d}_2)\cdot\cdots$$

The general elliptic function may be written

$$f(w) = C\frac{\sigma(w-a_1)\sigma(w-a_2)\cdots\cdots\sigma(w-a_m)}{\sigma(w-b_1)\sigma(w-b_2)\cdots\cdots\sigma(w-b_m)} \quad \text{or}$$

$$f(w) = C'\frac{[p(w)-p(a_1)][p(w-a_2)]\cdots\cdots[p(w)-p(a_m)]}{[p(w)-p(b_1)][p(w)-p(b_2)]\cdots\cdots[p(w)-p(b_m)]}$$

and the general V-function may be written

$${}_qV_m(w) = R\left\{f(w),\, W_2(w-c_r),\, W_3(w-e_s),\cdots,W_m(w-d_t)\right\},$$

where R denotes an integral algebraic function of its arguments and where the isolated singular points in the period parallelogram are b_1, b_2, -- ,b_m of the first class; c_1, c_2, ---,c_ℓ of the second class; d_1, d_2, ,----- d_t of class n.

A function F_2 which is doubly periodic and has a non-isolated singular point in the period parallelogram may be written as follows: Write

$$F_2(w) = K_2\sum_1^\infty \frac{p(w-a_n)}{\lfloor n},$$

where K_2 is constant and a_1, a_2, ----,a_n,---- is a simply infinite series of points having the point $\underline{a}$ as a limit point. The point $\underline{a}$ is then a non-isolated singular point of clas two, being a limit point of poles of the function. Similarly write

$$F_3(w) = K_3\sum_1^\infty \frac{W_2(w-b_r)}{\lfloor r}$$

where b_1, b_2, ---, b_r,-- have a limit point $\underline{b}$. The point $\underline{b}$ is a non-isolated singular point of class three, being a limit point of singular points of class two. F_2 ,F_3 are convergent for all points except the poles and singular points indicated above; and congruent points in the plane. A function having a non-isolated singular point of any class may be expressed as follows

$$F_m(\omega) = K_n \sum_1^\infty \frac{W_{n-1}(\omega - v_m)}{\underline{m}}$$

where v_1, v_2, ---, v_m, --have a limit point $\underline{v}$. The point $\underline{v}$ is a singular point of class n.

To obtain functions having isolated and non-isolated singular it is only necessary to incorporate the F-functions and the V-functions in algebraic polynomials. These may be symbolized as follows:

$$\overline{W}_{m,p}(\omega) = R\left\{ {}_gV_n(\omega) \, , {}_gF_p(\omega) \right\} ,$$

where the ~~lefthand~~ subscripts n , p assume any positive integral values at will. The simplest examples of this are

$$\overline{W}_{n,p} = \sum_r {}_gV_r + \sum_s {}_gF_s \text{ and } \overline{W}_{n,p} = \prod_r {}_gV_r \cdot \prod_s {}_gF_s ,$$

where r , s assume any or all positive integral values not exceeding n , p respectively.

Art. IX. Change of Notation.

We may pass to other notations of elliptic functions. For example we have

$$p(\omega) = e_3 - \frac{e_1 - e_3}{sn^2 v} , \quad \text{where } v = \sqrt{e_1 - e_3} \cdot \omega ,$$

Now $W_2(\omega) = e^{p(\omega)} - 1 = e^{e_3} \cdot e^{\frac{e_1 - e_3}{sn^2 \sqrt{e_1 - e_3} \cdot \omega}} - 1 = C_1 e^{\frac{e_1 - e_3}{sn^2 \sqrt{e_1 - e_3} \cdot \omega}} - 1 \equiv$

$J_2(\omega)$, say ,

$J_2(u)$ has singular points at the zeros of sn u and since these are of the first order there are two such singular points in the period parallelogram. $J_2(u)$ has zeros at the poles of sn v. We have by differentiation

$$J_2(u) = \frac{d}{du} J_2(u) = -\frac{2C_1(e_1-e_3)^{3/2} \cdot e^{\frac{e_1-e_3}{sn^2\sqrt{e_1-e_3}\cdot u}} \cdot cn\sqrt{e_1-e_3}\cdot u \cdot dn\sqrt{e_1-e_3}\cdot u}{sn^3\sqrt{e_1-e_3}\cdot u} .$$

Remark: One cuold equally well begin with $e^{sn\,u}-1$ and construct a set of functions corresponding to the W-functions and in a manner similar to the above pass to the Weierstrassian notation.

Art. X. Further Generalizations.

The question now arises, how may this discussion be extended to the second genre. It is the purpose to indicate here only very briefly how this may be done. Assume $\Phi(u)$ to be the function of highest class in the first genre. Then

$$\mathbb{X}_1(u) = e^{\Phi(u)}-1$$

is the first, lowest class, function in the second genre.

A practical difficulty exists in the nature of the infinite integer but once in the realm of the infinite, these quantities may have some of the characteristics of ordinary numbers in their relations among themselves. Following this idea we may write

$$\Phi(u) = e^{e^{\cdot^{\cdot^{\cdot^{k-k_1}e^{p(u)}-1}}}-1}-1, \qquad (1)$$

(1) Here k_1 denotes the number of successive applications of the operator (exponential) to $p(u)$.

where k becomes infinite. Then, having thus defined $\phi(u)$, it plays a role in the second genre exactly similar to that of p(u) in the first genre. It is seen that

1. $X_1(u) = e^{\phi(u)}$ is doubly periodic in the same periods as $\phi(u)$ and consequently in the same periods as p(u).

2. $X_1(u)$ has zeros at the zeros of p(u).

3. $X_1(u)$ has singular points of class one genre two at the pole of p(u).

4. $X_1(u)$ has no poles.

5. $X_1(u)$ is finite one valued and continuous at all points except the singular points.

6. $X_1(u)$ has a singular point of class three genre two at infinity.

Write $$X_n(u) = \underbrace{e^{e^{\cdot^{\cdot^{e^{\phi(u)}-1}}-1}}}_{n} \; .$$

This function,

1. Is doubly periodic in the same periods as p(u).

2. Has zeros at the zeros of p(u).

3. Has singular points at the poles of p(u) of class n genre two.

4. Has no poles.

5. Is finite one valued and continuous except at the singular

points.

6.Has a singular point of class n-2 genre two at infinity.

InIn a manner exactly analogous to that employed with the W-functions to form the V-functions, we may form the functions

$${}_s\Upsilon_k(w) = \sum \left\{ \mathfrak{X}_1(w-a_n) + \mathfrak{X}_2(w-b_m) + \cdots + \mathfrak{X}_k(w-d_p) \right\},$$ where the summation extends over: $\begin{cases} n = 1, 2, \ldots\ldots n_1, \\ m = 1, 2, \ldots\ldots m_1, \\ \vdots \\ p = 1, 2, \ldots\ldots p_1, \end{cases}$

and $${}_p\Upsilon_k(w) = \prod_{n, m, \ldots p} \left\{ \mathfrak{X}_1(w-a_n) \cdot \mathfrak{X}_2(w-b_m) \cdot \ldots\ldots \cdot \mathfrak{X}_k(w-d_p) \right\}.$$

In general

$${}_g\Upsilon_m(w) = R\left\{ \mathfrak{X}_1, \mathfrak{X}_2, \ldots\ldots, \mathfrak{X}_m \right\}$$

where R is an integral algebraic function of irs arguments. We may now form

$$\Upsilon_n(w) = R\left\{ \mathfrak{X}_n, \mathfrak{X}_{n-1}, \ldots\ldots, \mathfrak{X}_1, W_m, W_{m-1}, \ldots\ldots, W_2, f \right\}$$

which has singular points of varisus classes in both the first and second genre, and where R denotes an integral algebraic function of its arguments.

From the definition equations we may write at once:

$$\Phi(w) = \log(\mathfrak{X}_1 + 1)$$
$$\mathfrak{X}_1(w) = \log(\mathfrak{X}_2 + 1)$$
$$\vdots$$
$$\mathfrak{X}_{n-1}(w) = \log(\mathfrak{X}_n + 1).$$

Adding these equations gives

$$\phi(w) + X_1(w) + \dots + X_{m-1}(w) = \log\{(X_1+1)(X_2+1)\cdots\cdots(X_m+1)\} .$$

Differentiating $X_1(u) = e^{\phi(w)} - 1$ we obtain .

$$\frac{d}{dw} X_1 = (X_1 + 1)\,\phi'$$

$$\frac{d^2}{dw^2} X_1 = X_1'' = (X_1+1)(\phi'^2 + \phi'') , \text{ etc.}$$

Similarly

$$\frac{d}{dw} X_2 = (X_2+1)(X_1+1)\,\phi'$$

$$\frac{d^2}{dw^2} X_2 = X_2'' = (X_2+1)\{(X_1+1)^2\phi'^2 + (X_1+1)\,\phi'^2 + (X_1+1)\,\phi''\}$$

In general ,just as with the W-functions we have

$$\frac{d^m}{dw^m} X_m = (X_m+1)\,R\,\{X_{m-1}, X_{m-2}, \dots, X_1, \phi', \phi'', \dots \phi^{(m)}\} ,$$

where R denotes annintegral algebraic function of its arguments.

March ,1910.