MAGNET COILS OF MAXIMUM EFFICIENCY

AND

THEIR INTERNAL TEMPERATURE RISE

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SUMMARY

It is shown that varying the wire diameter in a magnet coil winding leads to coil designs of greater efficiency. Data are given to obtain such non-uniform windings with commercially available wire sizes. It is shown that the maximum internal temperature rise in such windings is not greater than for similar uniform windings wound on rectangular cores.

The internal, steady-state temperatures are given for windings in the shape of hollow cylinders when the resistivity of the wire is constant, and when the resistivity of the wire is a linear function of the temperature.

For a winding made up of only two sections of different wire size, the difference in the maximum internal temperatures is calculated when the sections of the winding are interchanged.

ACKNOWLEDGMENT

Grateful acknowledgment is made to the Stephen H. Wilder foundation which made work on this project possible, to the director of the Applied Science Research Laboratory, and to its staff, for the cooperation given in the theoretical and experimental evaluation of this problem.

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I. INTRODUCTION

Among the fundamental components of electrical structures of all types are wire-wound coils. The purpose of such coils is to convert energy from the form of an electric current to the form of a magnetic field. This conversion is necessarily accompanied by certain losses, which, in the case of direct current or alternating current of low frequencies, are confined primarily to I^2R losses in the wire of the coil. The I^2R losses appear in the form of Joule heat in the coil and are objectionable for two principal reasons: a) the energy expended as Joule heat serves no useful purpose, and b) the temperature in the coil may rise to a point dangerous to the insulation. It is therefore desirable to reduce the I^2R losses to a minimum. This end can be accomplished by minimizing the resistance of the wire, subject to certain conditions that must necessarily be imposed in order to retain the properties of the coil which cause it to convert electrical to magnetic energy.

At first glance some applications may appear for which minimum resistance does not represent the optimum condition. For example, the time constant of a coil is given by the ratio of inductance to resistance, hence a low resistance would apparently correspond to a large time constant. However, upon closer examination of the facts [1], it is found that in these cases, too, the energy spent for increased I^2R loss is wasted, and only requires a more powerful source of electric energy to supply the same magnetic effect. In any case, if it were true that minimum resistance does not yield the best designs, then one should add external resistance to the winding, or use cheap iron wire or light-weight aluminum in place of the commonly employed annealed copper wire.

In general, all coils can be identified as belonging to either of two classifications: Coils with magnetic path of high permeability and air-core coils. The coils with magnetic path of high permeability lend themselves readily to a generalized treatment because the magnetic effect desired depends principally upon the magnetic path, which may be varied through a wide range without depending upon the configuration of the winding [2]. Thus a winding of high efficiency for one particular application is also efficient for another as long as the magnetic path consists of materials with high permeability. Such coils will be considered primarily in this paper.

For coils of this type, we shall assume that the flux lost through leakage is negligible (this assumption will be discussed further in section II). That is, regardless of how large the radius of a turn of wire surrounding the core once completely may be, its effect in adding to the magnetic flux in the core remains the same, provided the current is constant. Thus the coil has the property that every one of its turns contributes the same amount of magnetic flux.

In an ordinary coil, wound throughout with wire of uniform diameter, a turn near the outside, although contributing no more to the magnetic effect than a turn on the core, has greater length, thus greater resistance. The most efficient coil, then, would be one whose turns are all "inside", that is, one consisting of a single layer of wire wound immediately upon the core. Also, considering the relation between wire diameter and resistance, the most efficient coil would consist of wire of infinite diameter. A practical winding could not possibly satisfy the latter requirement; and, in many instances the long, single-layer coil is also impractical.

By means of the calculus of variations, however, it is possible to calculate the best type of winding specifications which are subject to practical conditions. Furthermore, approximations to the variational method may be made to satisfy additional practical considerations.

Many of the familiar formulas in coil design are based on expressions involving the volume or the cross-sectional area of the winding only, no regard being given to the manner in which this volume is actually filled with wire. The common rule is that it does not make any difference whether a small wire size with low current is used or fewer turns of a larger wire size with a correspondingly larger current — in other words, the wire diameter cancels out in many cases. Hence, one might think that it makes no difference whether the wire size is changed within the winding or not.

It is perfectly true that the wire size cancels out provided it is a constant; that is, we are free to choose any constant we wish in these formulas, but we cannot expect the wire diameter to cancel if it is not constant throughout the winding. Parts of this paper are concerned with the effects that such changes in wire size within the winding may have upon the magnetic effect and the temperature distribution of the coil.

So far, only the magnetic effect of a coil in relation to its resistance has been considered. Obtaining the maximum magnetic effect with given resistance, or a given magnetic effect with minimum resistance, is a useful design criterion: for example, in electronic circuits operating with low signal strengths or in testing equipment where extreme accuracy is difficult to attain because of the presence of unnecessary resistance. Many other cases, however, require not such a critical turns to resistance ratio, but rather an improved temperature distribution. For example, a very powerful magnetic effect can be produced by large currents in a small winding for a very brief moment. The Joule heat developed by the current is generated at such a high rate inside the coil that the coil temperature would rise rapidly to a dangerous value if the current were kept flowing too long. The winding most efficient in relation to the temperature rise is the one which has the largest number of turns within a given space for a fixed maximum temperature rise at constant current.

Since a constant flow of current generates heat at a constant rate, the flow of heat (and the temperature gradient) must proceed from the inside of the coil towards some heat sink outside. Thus, certain boundary conditions are imposed which dictate the nature of the solution. In brief, the maximum temperature rise within the coil can be computed if the solution of Poisson's equation is known for the special conditions that arise in practical cases.

The problem of finding the maximum temperature and the temperature distribution in a winding has gained importance recently by the introduction of silicone insulating materials that are able to withstand much higher temperatures than were commonly employed before. It is important to have correct design data at hand in order to take full advantage of these new developments. Most design formulas are still based on the assumption that the temperature rise within the coil is not very high; hence, the local changes of resistance caused by the temperature field are negligible, except perhaps for the measurement of the average temperature in the winding. Professor M. Jakob [3)] has made the calculations which do take into account local changes of resistance for windings shaped like flat plates (on rectangular cores) and like solid cylinders (on circular cores of vanishingly small diameters) with constant thermal conductivity. Jakob's work has been extended in this paper to the practical cases of windings in the shape of hollow cylinders, to windings which are built up of sections of different wire sizes to give the minimum resistance property outlined in section III, and to include the possibility of variations in thermal conductivity.

II. MAGNETIC FIELD DISTRIBUTION

In order to have a clear understanding of the assumptions made in the treatment of the problems to be presented later, we shall discuss briefly how the magnetic effect produced by a coil is related to the current distribution in the winding. The general case has been worked out by F. Bitter [4] for extremely strong magnetic fields, including heat dissipation problems which need not be repeated here.

By "magnetic effect" we shall understand some manifestation of the presence of a magnetic field: be it the deflection of a beam of charged particles, the attraction of an armature to the core of the magnet, or the induction of an electric current when the field changes. Quantitatively, the magnetic effect is measured in terms of the solenoidal vector $\overline{B}$, called the magnetic induction. This vector arises from the combination of the vectors $\overline{H}$, the magnetizing force, and $\overline{I}$, the surface distribution of magnetic poles. When no ferromagnetic materials are present in the system, then the vector $\overline{I}$ is zero; hence, the magnetic effect is purely that produced by the magnetizing force $\overline{H}$, which depends upon the current distribution in space.

For most practical applications, an effort is made to obtain as large a magnetic induction as possible with a small magnetizing force. This is done by making use of ferromagnetic materials which exhibit a large susceptibility; that is, materials which cause a strong surface distribution of poles to appear when only a small magnetizing force is impressed. In effect, the magnetic induction $\overline{B}$ inside a cavity or small airgap of the circuit made of magnetic material is so great compared with that over the rest of the vicinity of the coil, that the so-called leakage flux may be neglected entirely. Assuming, then, that practically the entire flux passes through the path of ferromagnetic material, the magnetizing force $\overline{H}$ is constant along the magnetic path; and since the circuit integral of the magnetizing force taken around a complete circuit enclosing each turn of wire is a constant for constant current, it follows that the electric current distribution in space has absolutely no effect on the degree of magnetization of the magnetic path. Thus, it has been proved that each and every turn of the coil has the same effect in adding to the magnetic field regardless of its distance from the core of the coil [a].

For air-core coils, the magnetic field has in general a non-uniform distribution in space. It is therefore necessary to consider the variations in the magnetic path as well as the current distribution in space for each individual case. Some of the cases most likely to arise in practice are discussed at the end of section III A, where the design criterion is to maximize the magnetic field on the axis of a long coil and on the axis of a short coil with the resistance of the winding held constant. It is there assumed that the law of Biot and Savart holds; that is, the field contributed at a point by a current element $I \cdot ds$ at distance x from the point is

$$dH = \frac{I \sin \theta \, ds}{x}$$

For a circular turn of wire of radius x, the field at the center of the circle is therefore

$$H = \frac{2 \pi I}{x} \qquad (1)$$

III. THE MAXIMUM AMPERE TURNS OR MINIMUM RESISTANCE PROBLEM

A. CONTINUOUS SOLUTION

The Problem

The first case to be considered is the coil of given dimensions which is to have a maximum number of turns for a given resistance, or a minimum resistance for a given number of turns. These two reciprocal problems have the same solution, as is well known from the general theory of the calculus of variations; but the solution is expressed in different terms for each. The details of the minimum resistance problem will be given here.

Analytical Statement

The physical dimensions of the coil are shown in Fig. 1.

a = internal radius
b = external radius
L = length
x = radius of a layer of wire in the coil
y = wire diameter = $1/\sqrt{s}$
s = number of turns per square-inch of cross section

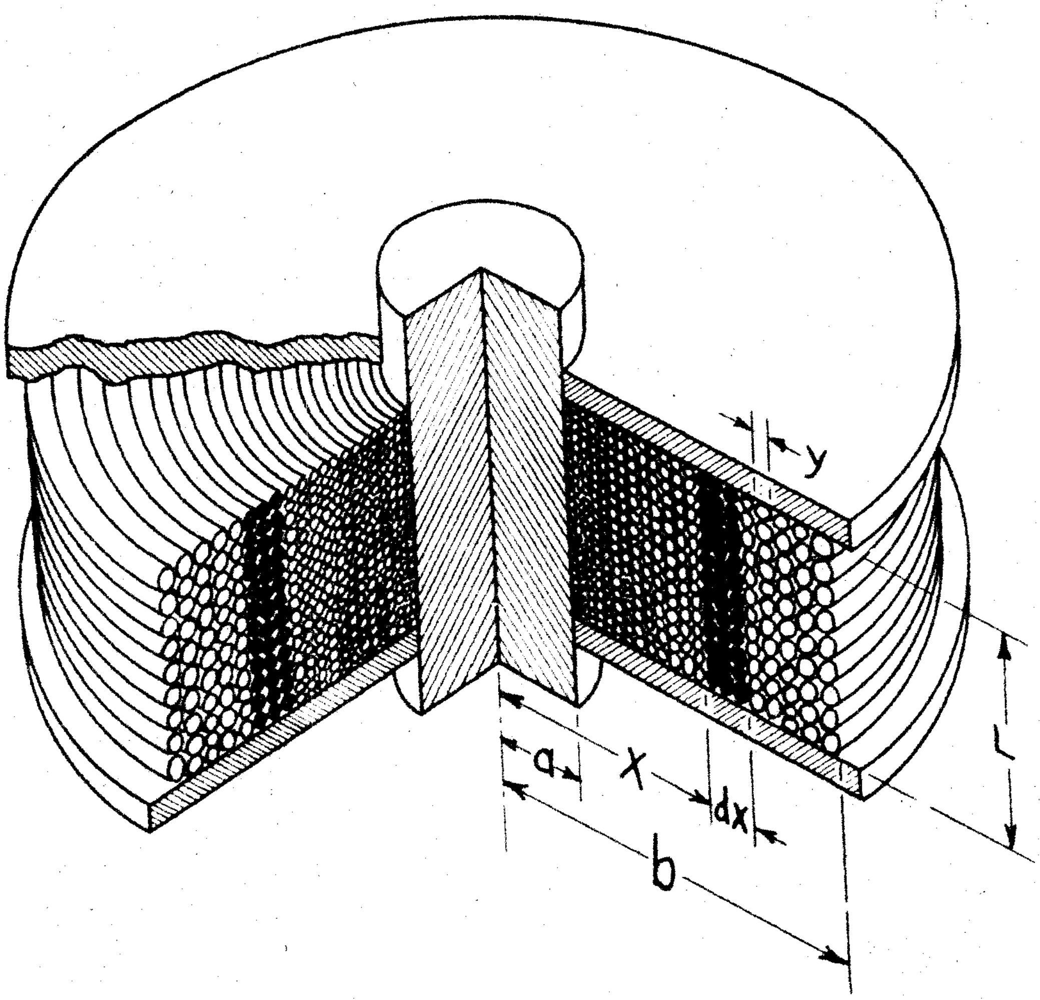
The diameter of the wire y is assumed so small that there is always an integral number of turns in each layer of length L. Relative changes in the wire size are assumed to be so small that s can be considered as a continuous function of x: $s = s(x)$. Then the total number of turns of the coil is

$$N = \int_a^b L\, s(x)\, dx \tag{2}$$

If ρ is the resistivity of the metal in the wire of which the coil is wound and α the space factor (area of metal per square-inch of winding cross section), the total resistance of the coil is

$$R = \int_a^b \frac{2\pi \rho L x s^2(x)\, dx}{\alpha} \tag{3}$$

Fig. 1 - Magnet Coil



It will be assumed that α and ρ are constant throughout the winding.

Necessary Conditions

The integral to be made a minimum is (3), subject to the condition that integral (2) is to be held constant.

Applying the method of multipliers to this isoperimetric problem and noting that N, L, λ are constants, the problem is reduced to finding the minimum of

$$(4) \quad I = \int_a^b (x\, s^2 - \lambda s)\, dx = \int_a^b F(x,s)\, dx = \frac{\alpha R}{2\pi\rho L} - \lambda \frac{N}{L}$$

This integral differs from those usually encountered in the calculus of variations because the integrand F is a function of x and s only, rather than a function of x, s, and s'. Thus, the Euler condition

$$\frac{\partial F}{\partial s} - \frac{d}{dx}\left(\frac{\partial F}{\partial s'}\right) = 0$$

becomes simply $\partial F/\partial s = 0$, and the restrictions on the endpoints of the minimizing function $s = s(x)$ are removed [5]. This particularly simple result can be considered as an extension of the problem involving higher derivatives s'', s''', ... $s^{(n)}$ to the case in which the zeroth derivative alone is present [6]. The 2n constants of integration, which correspond to the boundary conditions on the minimizing function, are here reduced to zero in number.

While the problem is therefore simpler than those ordinarily considered in the calculus of variations, its solution still does require methods more stringent than the theory of ordinary maxima and minima allows. Professor C. W. Mendel, University of Illinois, has applied the theory of ordinary maxima and minima to this problem [7] and obtained a set of parameters, yielding the same result as Euler's necessary condition when the number of parameters is increased without limit [8].

Other necessary conditions for a "strong" minimum [9] are satisfied by the function $F(x,s)$ in equation (4) in a trivial sense, i.e. the Weierstrass E function and the Legendre condition $F_{s's'} = 0$ are identically zero; and the Jacobi condition is not applicable since $F_{s's'} = 0$.

Thus the minimizing function satisfying all necessary conditions is $\partial F/\partial s = 0$, which gives the following result when solved explicitly for s:

(5) $$s(x) = \frac{\lambda}{2\ x}$$

Sufficient Conditions

For the problem of minimizing the integral

$$I = \int_a^b F(x,s)\ dx$$

Bolza [10] gives the sufficient conditions

(6) $$\frac{\partial F}{\partial s} = 0$$

(7) $$\frac{\partial^2 F}{\partial s^2} > 0$$

Equation (6) is the necessary condition giving the solution (5). To satisfy (7), the following relation must hold

$$2\ x > 0$$

This is certainly true inasmuch as the radius of a layer of wire in the coil is essentially a positive quantity. Equation (7) is therefore satisfied.

The Solution

It has thus been shown that both necessary and sufficient conditions are satisfied by equation (5) for minimizing the integral (3) under the condition that the integral (2) is constant. The multiplier λ is readily determined, yielding the exact form of the functional relation between wire size and radius of the coil

$$(8) \qquad s = \frac{N}{x\, L \log_e \frac{b}{a}}$$

The total resistance of the coil, wound with wire of diameter as given by equation (8), is then

$$(9) \qquad R_{min} = \frac{2\pi\rho N^2}{\alpha L \log_e \frac{b}{a}}$$

This is the minimum resistance obtainable for a coil of given specifications.

For the maximum turns problem, the computations are carried out in an analogous manner, except that the maximum of integral (2) is sought while (3) is held constant. The solution is given by

$$(10) \qquad s = \frac{1}{x} \sqrt{\frac{\alpha R}{2\pi\rho L \log_e \frac{b}{a}}}$$

yielding the maximum number of turns obtainable

$$(11) \qquad N_{max} = \sqrt{\frac{\alpha R L}{2\pi\rho} \log_e \frac{b}{a}}$$

In these solutions, the wire size is determined as a continuous function of x, the radius at which the wire is wound. For any physically existing coil, it is essentially impossible to have the wire size vary as a continuous function because the wire diameter is a constant in the range of radii which is bounded by the inside and outside tangents of the wire itself. This difficulty leads to the paradox pointed out by Professor Mendel [7]) in that an infinite number of cells are apparently filled by a finite number of turns of wire. Of course, every theoretically established paradox must show its effect in a practical application. As will be shown below, however, the discrepancy caused by this paradox can be neglected in most actual coil windings.

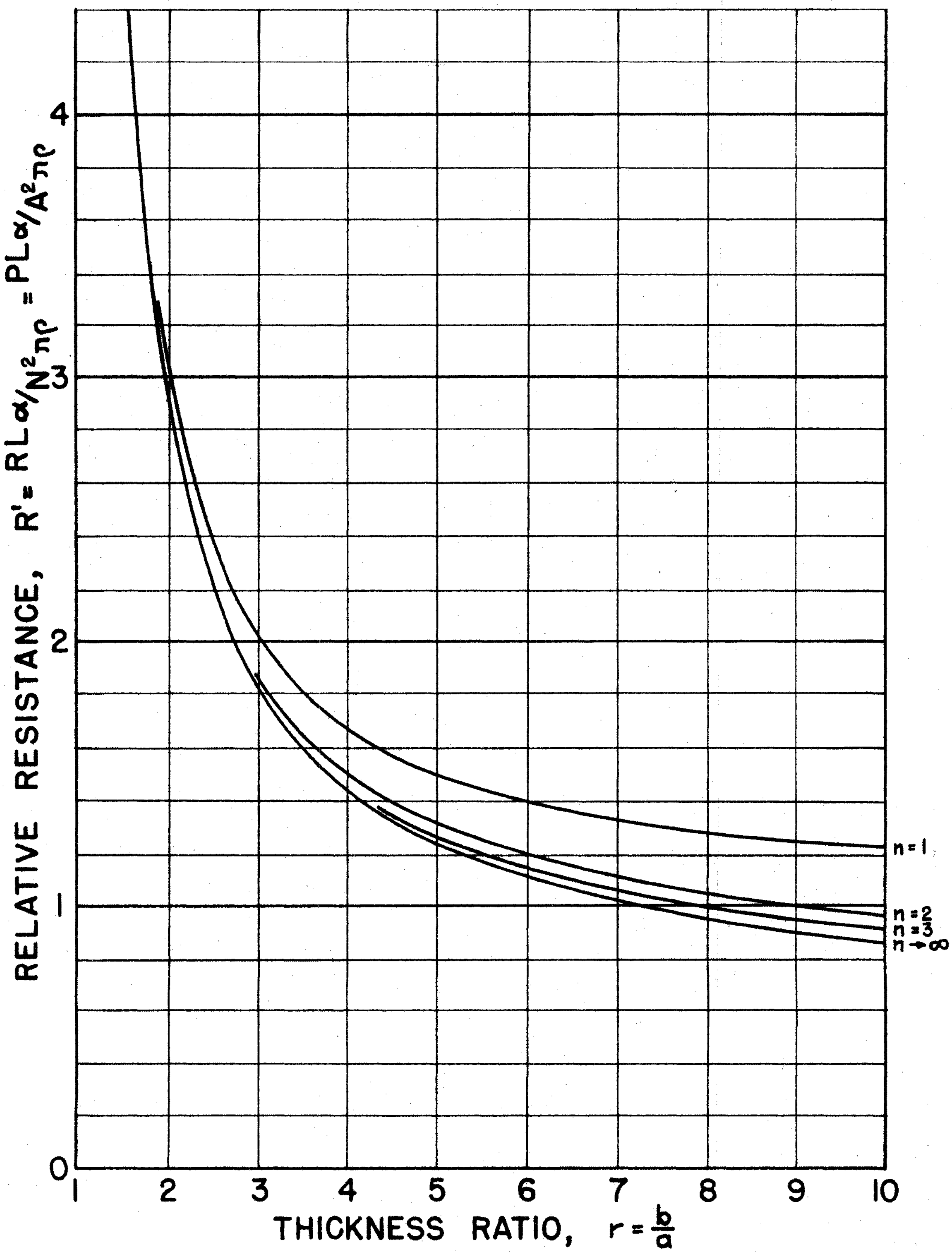
Equations (8) and (10) show that the wire diameter varies proportionally with the square root of the radius. The closest approximation to this theoretical method of winding the coil most efficiently would be to wind each layer with a different size of wire. While this is, of course, possible, the following considerations show that it is by no means necessary to go to such extremes. In Fig. 2 the resistance of windings is plotted vs. the ratio of outside to inside diameters for different degrees of approximation of the wire sizes. The approximating functions are the best step-functions that can be found to replace the ideal continuous function given by equation (8) or (10). The curve marked $n = 1$ represents a single wire size (uniform winding), $n = 2$ represents a winding composed of wire of two different sizes, and so on. The curve marked $n \to \infty$ represents the condition of continuous variation of the wire size as required for the ideal case. It is readily seen that only four or five changes of wire size are required in most practical applications to obtain a very close approximation to the theoretical maximum efficiency. It is therefore not necessary to change wire size with every layer (unless the wire diameter is unusually large); hence, it is not necessary to take into consideration the inaccuracy introduced by the paradox of the discontinuous function mentioned above.

The data for Fig. 2 were obtained by computing the best step-functions which satisfy all the necessary conditions to yield a minimum resistance for a given number of turns as described in section III B.

Air-Core Coils

The field at the center of a long solenoid may be considered as being directed uniformly along the axis of the winding; hence, free of leakage flux. Thus, the same equations apply to this point as to all the points in the high-permeability magnetic circuits already discussed.

Fig. 2 - I^2R Loss vs. Thickness Ratio for Multiple Windings



For a short solenoid, however, the non-uniformity of the field must be given special attention. From equation (1) we derive the expression for the magnetic field at the center of the winding:

$$H = 2 \pi I L \int_a^b \frac{s(x)}{x} dx$$

The resistance of the winding remains as given by equation (3).

Proceeding in a manner analogous to that followed in the solution of the minimum resistance problem for the high-permeability coils, we find the function

$$s(x) = \frac{1}{x^2} \frac{H a^2 b^2}{\pi I L (b^2 - a^2)}$$

which yields the minimum resistance obtainable for a given field

$$R_{min} = \frac{H^2 \rho a^2 b^2}{\alpha \pi I^2 L (b^2 - a^2)}$$

It is noteworthy that the wire diameter increases proportionally with the radius rather than with the square root of the radius. A change in this direction is, of course, to be expected because the turns near the axis of the coil have greater magnetic effect in addition to the benefits derived from their decreased length. In the high-permeability coils, the magnetic effect of all the turns is the same.

Other special cases may be solved in a similar fashion, and it will be found, in general, that the most efficient windings are those that have the property that every turn contributes the same amount to the magnetic field in proportion with the amount of resistance it adds to the winding.

B. DISCONTINUOUS SOLUTIONS

Finite Number of Changes of Wire Size

The following procedure was first given by Professor C. W. Mendel, University of Illinois [7], who used it to arrive at the result stated in equation (11) by passing to the limit $n \to \infty$. We are now interested in a finite value of n, say three or four; hence, we can make use of the first part of Professor Mendel's treatment.

The notation to be used is evident from Fig. 3. Let the coil be wound in n "cells", each cell containing wire of a fixed diameter, the diameter being different for different cells. Let s_k be the number of turns per square-inch cross section in the k^{th} cell. Then the total number of turns in the coil is given by

$$N = \sum_{k=1}^{n} L\,(x_k - x_{k-1})\, s_k \tag{12}$$

The total resistance will be

$$R = \sum_{k=1}^{n} \frac{\pi \rho L}{\alpha} (x_k^2 - x_{k-1}^2)\, s_k^2$$

The problem is to hold N fixed and make R a minimum. This will be done by varying s_k first, then varying x_k. Using the method of undetermined multipliers, we find that the wire size is determined as

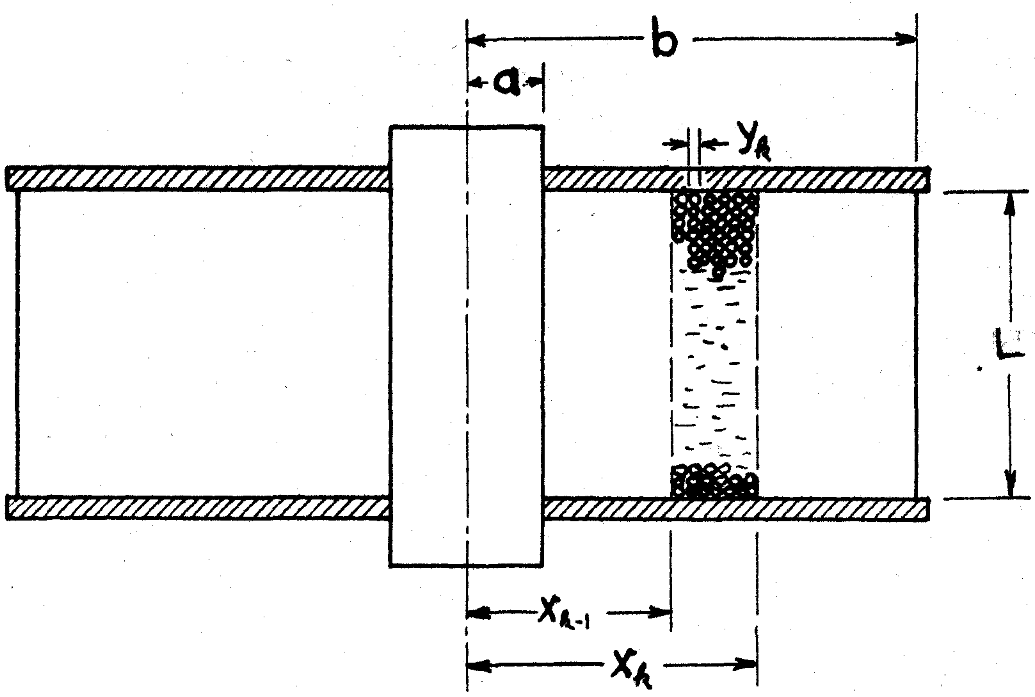
$$s_k = \frac{N}{(x_k + x_{k-1})\, L} \cdot \frac{1}{\sum\limits_{k=1}^{n} \frac{x_k - x_{k-1}}{x_k + x_{k-1}}} \tag{13}$$

which yields the total resistance

$$R = \frac{\pi \rho N^2}{\alpha L^2} \cdot \frac{1}{\sum\limits_{k=1}^{n} \frac{x_k - x_{k-1}}{x_k + x_{k-1}}} \tag{14}$$

In order to cause this expression to become a minimum, we must determine the set of x_k to make the function

Fig. 3
Coil Wound in Cells



$$h = \sum_{k=1}^{n} \frac{x_k - x_{k-1}}{x_k + x_{k-1}}$$

a maximum, subject to the conditions $a = x_0 < x_1 < \ldots\ldots < x_n = b$. We find that the x_k must be in geometric progression:

(15) $$x_k = a\, r^{\frac{k}{n}} \; ; \; k = 0,1,2,\ldots n; \quad r = \frac{b}{a} .$$

With this set, the wire sizes given by (13) become

(16) $$s_k = \frac{N}{L\, n\, a\, r^{\frac{k-1}{n}} \left(r^{\frac{1}{n}} - 1 \right)}$$

and the resistance (14) is found to be

(17) $$R = \frac{\pi \rho N^2 \left(r^{\frac{1}{n}} + 1 \right)}{\alpha L\, n \left(r^{\frac{1}{n}} - 1 \right)}$$

In (17) we have an expression for the least resistance obtainable by using the n different wire sizes given by (16). Fig. 2 is a graph of the relation between R and r as expressed in equation (17), with various values of n as a parameter.

The ordinate $R L \alpha / \pi \rho N^2$ may be multiplied by I^2/I^2 without changing its value. The resulting dimensionless variable is $P L \alpha / \pi \rho A^2$, where P is the power consumed as I^2R loss and A represents the number of ampere-turns in the winding.

There are several other ways in which this information may be plotted. For example, Fig. 4 shows the resistance of various coils as compared with the theoretically perfect winding. That is, the ordinate shows what the resistance will be if the same size coil, wound with continuously changing wire size, has a resistance of 100 ohms. In Fig. 5, the comparison is made with the uniform winding; i.e. the resistances are plotted of various coils having the same size as a 100-ohm uniform winding.

Fig. 4 - Resistance of Multiple Winding
Compared with Ideal Non-Uniform Winding

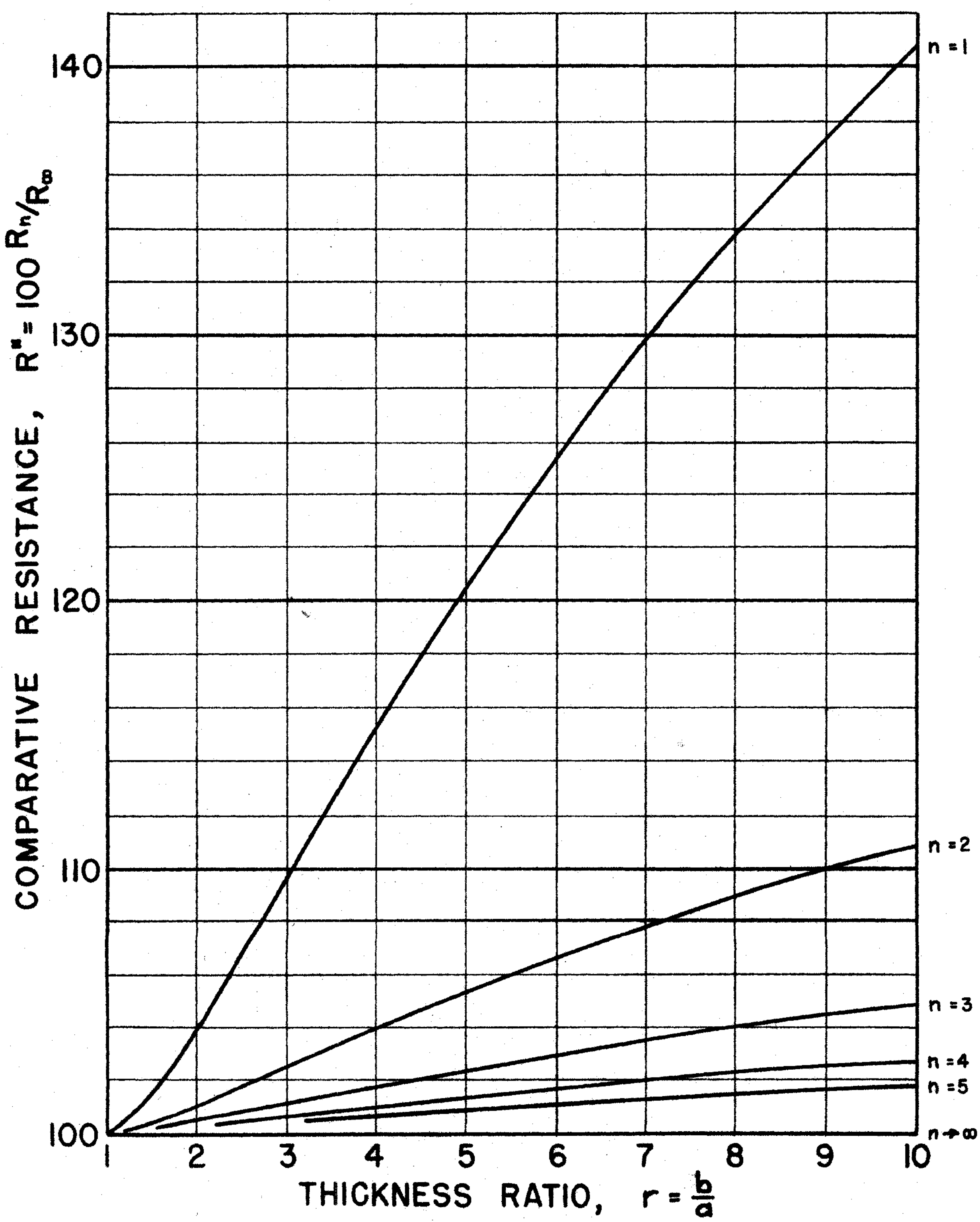
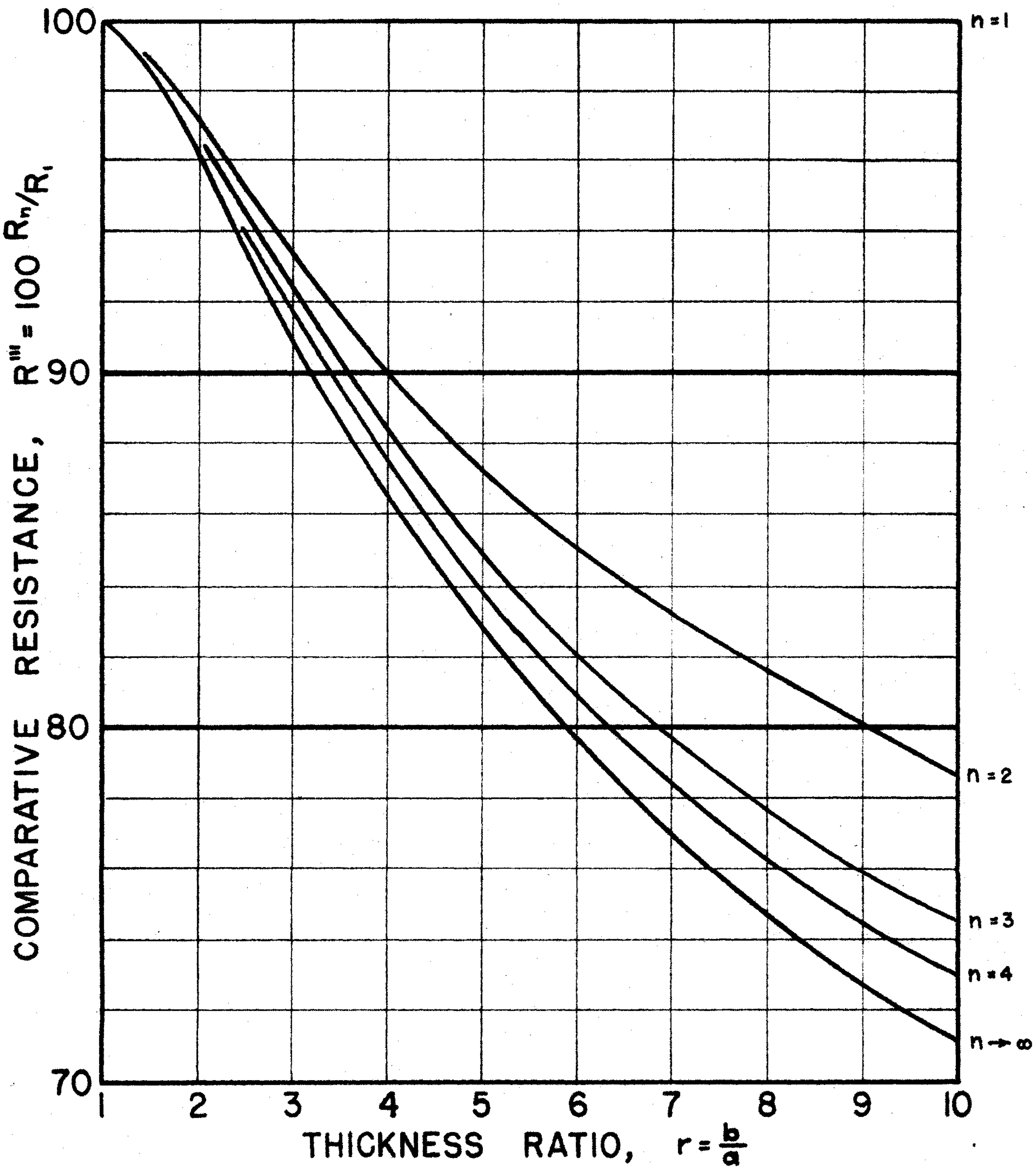


Fig. 5 - Resistance of Multiple Winding
Compared with Uniform Winding



Equations (15) and (16) permit calculation of the number of turns per cell:

$$N_k = L(x_k - x_{k-1})\, s_k = \frac{N}{n}$$

Thus, we find that in the best step-function approximations the number of turns is the same in every cell.

For the maximum turns problem, the wire sizes and total number of turns are given by the following formulas:

$$s_k = \frac{1}{a\, r^{\frac{k-1}{n}}} \sqrt{\frac{\alpha R}{\pi \rho L n (r^{\frac{2}{n}} - 1)}} \tag{18}$$

$$N = \sqrt{\frac{\alpha L R n (r^{\frac{1}{n}} - 1)}{\pi \rho (r^{\frac{1}{n}} + 1)}} \tag{19}$$

For this reciprocal problem also, it may be of interest to compare the results contained in equation (19) graphically. Corresponding to Fig. 2, Fig. 6 gives the effect that a certain number of changes of wire size has on the total number of turns of a winding. In Fig. 7 the winding efficiency is plotted; that is, the number of turns of a winding with n changes of wire size divided by the number of turns the theoretically perfect winding would have. Finally, in Fig. 8 the non-uniform windings are compared with the uniform winding; i.e. the number of turns of a winding with n changes of wire size divided by the number of turns of the uniform winding of the same resistance and same physical dimensions.

Design for Size

The curves plotted in Fig. 6 may also be used to determine the physical size required for a winding of given number of turns and given resistance. Fig. 9 is suited especially well for obtaining the information in this particular form. It is apparent that for values of

Fig. 6 - Number of Turns vs. Thickness Ratio for Multiple Windings

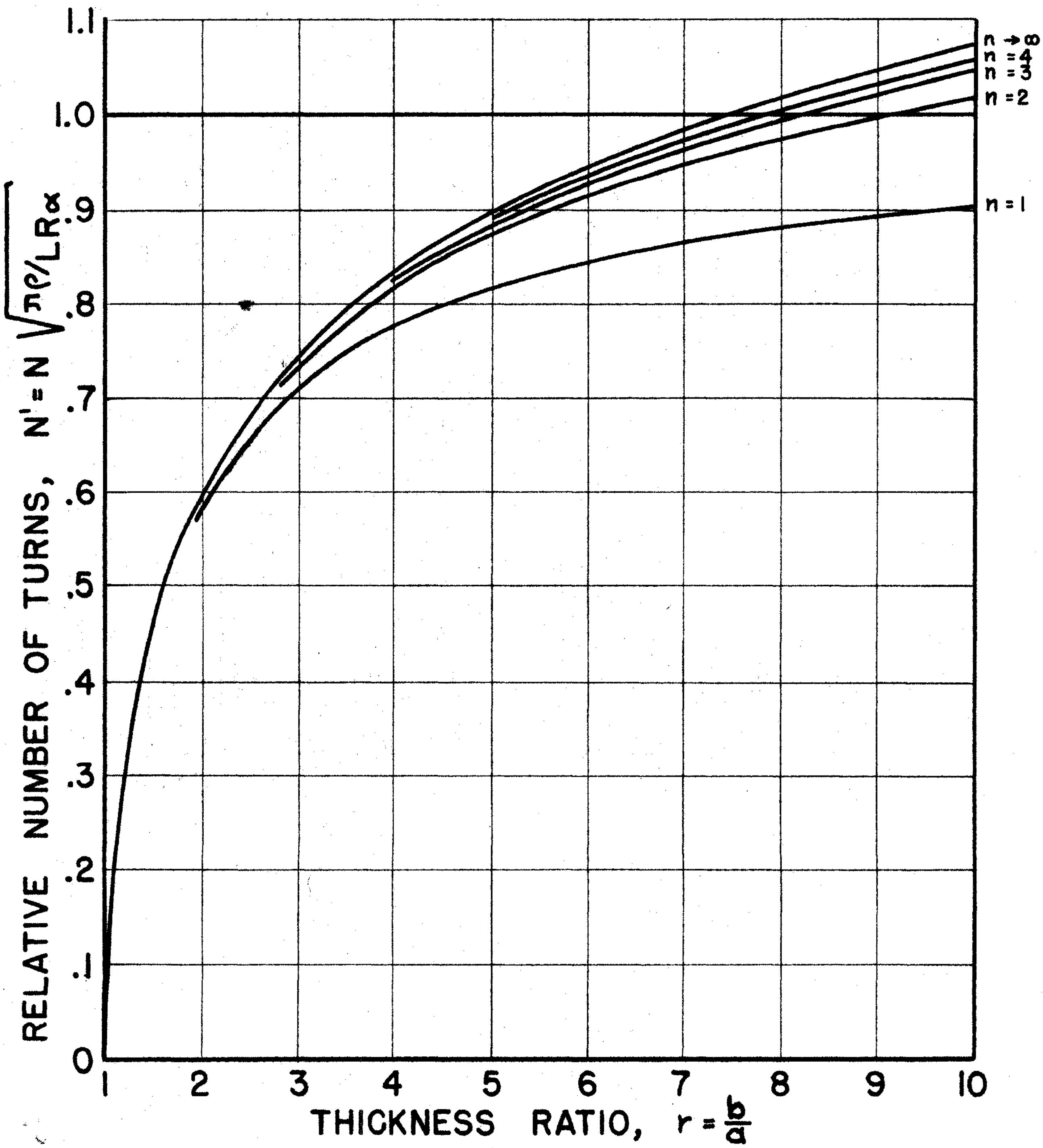


Fig. 7 - Number of Turns of Multiple Winding
Compared with Ideal Non-Uniform Winding

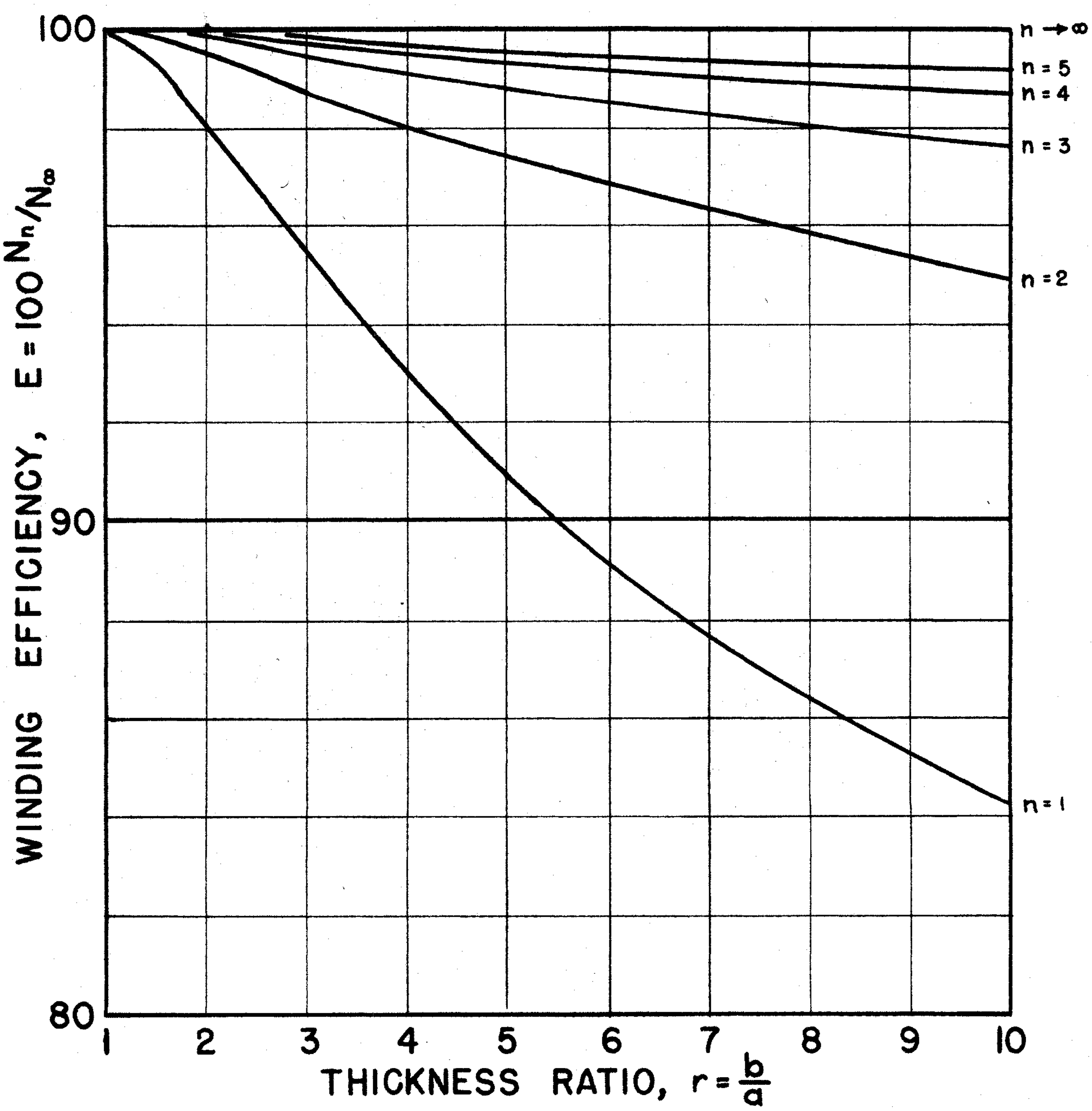


Fig. 8 - Number of Turns of Multiple Winding
Compared with Uniform Winding

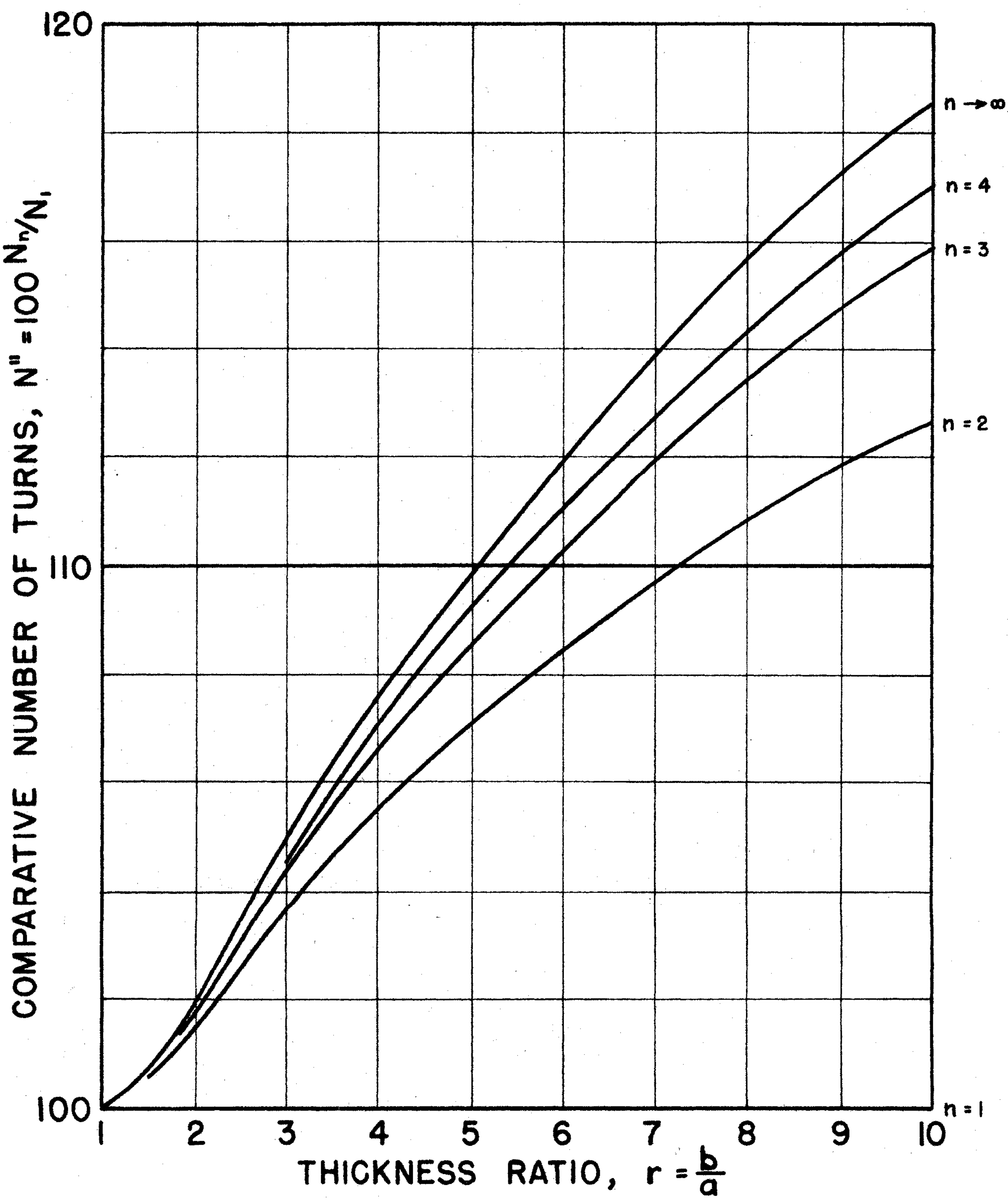
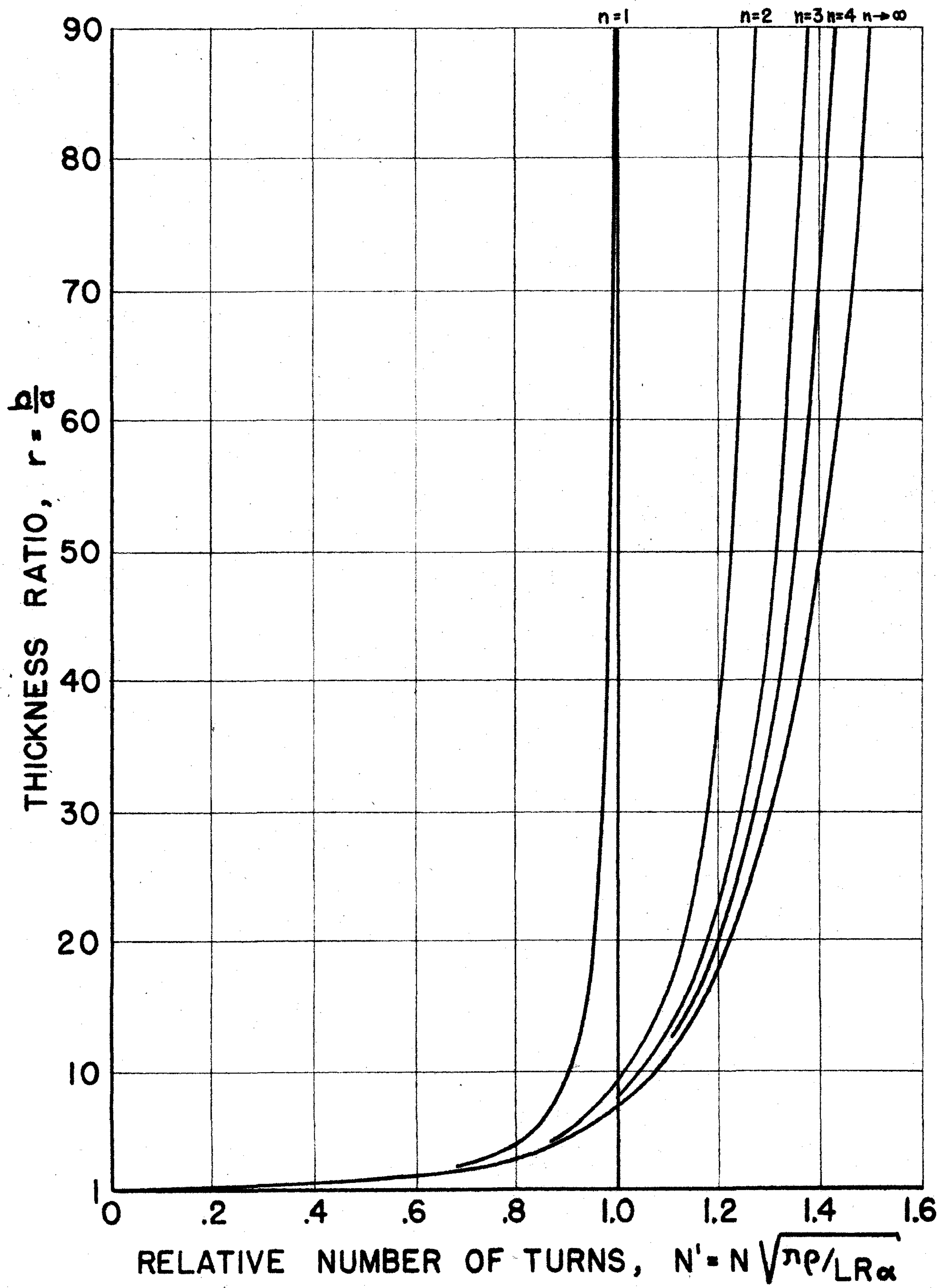


Fig. 9 - Thickness Ratio vs. Number of Turns for Multiple Windings



$4N^2\rho/LR$ in the neighborhood of one, the ratio of outside to inside diameters is of a critical nature for the uniform winding. From the family of curves represented in Fig. 9, namely

$$r = \left(\frac{n + \frac{4N^2\rho}{LR}}{n - \frac{4N^2\rho}{LR}} \right)^n$$

it is possible to deduce the critical points for multiple windings which correspond to the point $4N^2\rho = LR$ for the uniform winding. These critical points are readily seen to occur at the values $4N^2\rho/LR = n$ where n, as before, represents the number of different wire sizes used in the winding. For practically useful coils, in the range of thickness ratios 2 to 10, these points are of no significance.

Standard Wire Sizes

The equations giving s_k (16) and (18) are of little practical value because the wire sizes calculated from them are in most cases not exactly those commercially available. We are therefore faced with a new problem: Given a required number of turns N and a set of available wire sizes s_k, how is a coil of given dimensions to be wound to have the least resistance?

Following the method of multipliers as before, we write

$$R = \frac{\pi\rho L}{\alpha} \sum_{k=1}^{n} (x_k^2 - x_{k-1}^2)\, s_k^2 \quad + \mu\left[N - L\sum_{k=1}^{n}(x_k - x_{k-1})\, s_k\right]$$

The s_k are known constants, and we have decided on the number of cells n, Hence, we wish to find the set x_k which will minimize R; that is, make the partial derivative of R with respect to x_k vanish. Thus, x_k must satisfy the following set of equations:

$$x_k = \frac{\mu\alpha}{2\pi\rho(s_k + s_{k+1})} \; ; \qquad k = 1,2,3,\ldots\ldots n-1$$

The index k ranges only from 1 to n-1 because we have already defined $x_0 = a$ and $x_n = b$.

Determining the multiplier μ, we find that in order to satisfy equation (12), x_k must have the values

$$x_k = \frac{W}{D\,(s_k + s_{k+1})} \quad ; \quad k = 1,2,3,\ldots n-1$$

where $W = \frac{N}{L} + as_1 - bs_n$

and
$$D = \sum_{k=1}^{n} \left(\frac{s_k}{s_k + s_{k+1}} - \frac{s_k}{s_k + s_{k-1}} \right) \quad ; \quad \frac{1}{s_0} = \frac{1}{s_{n+1}} = 0.$$

It may be helpful for practical applications to determine the number of turns per cell. We find that the innermost cell has the number of turns

$$N_1 = \frac{L\,W\,s_1}{D(s_1 + s_2)} - L\,a\,s_1$$

that each of the other cells has the number of turns

$$N_k = \frac{L\,W}{D} \left(\frac{s_k}{s_k + s_{k+1}} - \frac{s_k}{s_k + s_{k-1}} \right) \quad ; \quad k = 2,3,4,\ldots n-1$$

and that the outer cell has the number of turns

$$N_n = L\,b\,s_n - \frac{L\,W\,s_n}{D\,(s_n + s_{n-1})}$$

These equations are rather complicated in the form given above for any arbitrary set of wire sizes. Fortunately, commercial wire sizes are standardized in a geometric progression; that is, the bare wire diameters of successive AWG numbers are in the same ratio. By assuming that the space factor is the same for all the wire sizes to be used in the winding, we find that the following equation may be written:

$$\frac{s_{k-1}}{s_k} = v \quad ; \qquad k = 2,3,4,\ldots n \tag{20}$$

where v is a constant.

Thus we find

$$x_k = \frac{E}{q} v^{(k+1)} \quad ; \quad k = 1,2,3,\ldots n-1; \quad \text{where} \quad q = s_1 v \tag{21}$$

$$\begin{cases} N_1 = L\,(v\,E - a\,s_1) \\ N_k = L\,E\,(v-1) \quad ; \quad k = 2,3,4,\ldots n-1 \\ N_n = L\,(b\,s_n - E) \end{cases} \tag{22}$$

$$\text{where} \quad E = \frac{W}{(n-1)(v-1)} = \frac{1}{(n-1)(v+1)} \left(\frac{N}{L} + as_1 - bs_n\right)$$

The total resistance of the winding will then be

$$R = \frac{\pi \rho L}{\alpha} \left[E^2\,(v^2 - 1)(n-1) - a^2 s_1^2 + b^2 s_n^2\right] \tag{23}$$

Equations (21), (22), and (23) give the design data necessary for solving the minimum resistance problem, using standard wire sizes.

The following equations are the corresponding results for the maximum turns problem:

$$x_k = \frac{B}{q} v^{(k+1)} \quad ; \quad k = 1,2,3,\ldots n-1$$

$$N_1 = L\,(v\,B - a\,s_1)$$

$$N_k = L\,B\,(v-1) \quad ; \quad k = 2,3,4,\ldots n-1$$

$$N_n = L\,(b\,s_n - B)$$

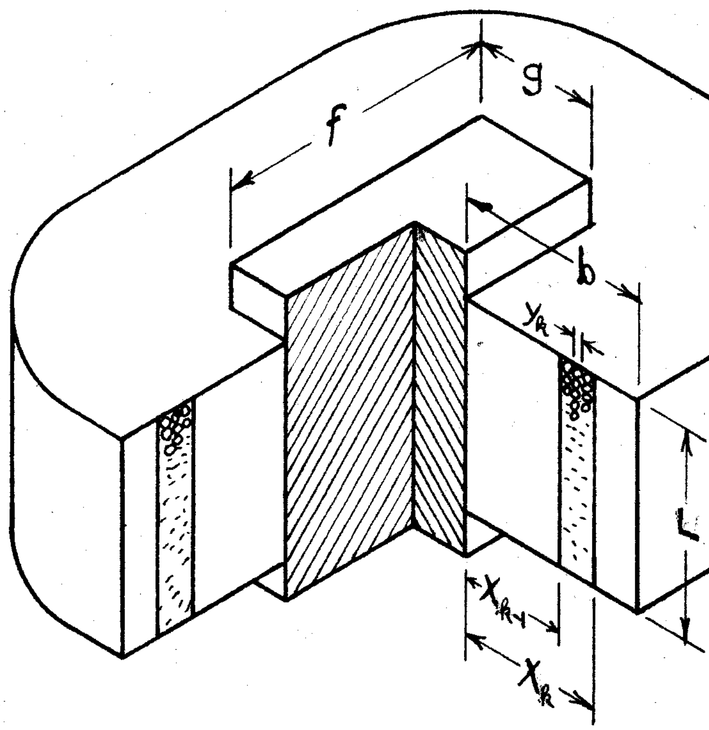
$$N = L\left[B(v-1)(n-1) - as_1 + bs_n\right]$$

$$\text{where} \quad B = \sqrt{\left(\frac{\alpha R}{\pi \rho L} + a^2 s_1^2 - b^2 s_n^2\right) \frac{1}{(n-1)\,(v^2-1)}}$$

__Rectangular Cores__

In practical applications, coils are frequently wound on rectangular cores, as shown in Fig. 10. The increase in length of the outer layers is not as great as it is in the case of windings on a cylindrical core.

Fig. 10
Winding on Rectangular Core



The results of computations made in a manner analogous to those above are as follows:

Minimum resistance problem, continuous solution ($x = 0$ refers to turns on the core, $x = b$ to turns on the outside of the winding)

$$s(x) = \frac{1}{j + x} \cdot \frac{N}{L \log_e \left(1 + \frac{b}{j}\right)} \quad ; \quad j = \frac{f + g}{\pi}$$

resulting in the minimum resistance

$$R_{min} = \frac{2 \pi \rho N^2}{\alpha L \log_e \left(1 + \frac{b}{j}\right)}$$

For the maximum turns problem

$$s(x) = \frac{1}{j + x} \sqrt{\frac{\alpha R}{2 \pi \rho L \log_e \left(1 + \frac{b}{j}\right)}}$$

yielding the maximum number of turns

$$N_{max} = \sqrt{\frac{\alpha R L}{2 \pi \rho} \log_e \left(1 + \frac{b}{j}\right)}$$

Using a finite number of wire sizes in geometric progression according to equation (20), the results of the minimum resistance problem for a winding on a rectangular core are as follows:

$$x_k = \frac{F}{q} v^{(k+1)} - j \quad ; \quad k = 1,2,3,\ldots n-1$$

$$N_1 = L (v F - j s_1)$$

$$N_k = L F (v - 1) \quad ; \quad k = 2,3,4,\ldots n-1$$

$$N_n = L \left[(b + j) s_n - F\right]$$

$$R = \frac{\pi \rho L}{\alpha} \left[F^2 (v^2 - 1)(n - 1) - j^2 s_1^2 + (b + j)^2 s_n^2\right]$$

where $$F = \frac{1}{(n-1)(v-1)} \left[\frac{N}{L} + j s_1 - (b + j) s_n\right]$$

Similarly, the results for the maximum turns problem with wire sizes in geometric progression for a winding on a rectangular core are:

$$x_k = \frac{H}{q} v^{(k+1)} - j \; ; \qquad k = 1,2,3,\ldots n-1$$

$$N_1 = L\,(v\,H - j\,s_1)$$

$$N_k = L\,H\,(v - 1) \; ; \qquad k = 2,3,4,\ldots n-1$$

$$N_n = L\left[(b + j)\,s_n - H\right]$$

$$N = L\left[H(v-1)(n-1) - j\,s_1 + (b + j)\,s_n\right]$$

$$\text{where } H = \sqrt{\left[\frac{\alpha R}{\pi \rho L} + j^2 s_1^2 - (b + j)^2 s_n^2\right]\frac{1}{(n-1)(v^2-1)}}$$

Sample Design

Core diameter...........	1",	a = 0.5
Outside diameter........	6",	b = 3.0
Number of turns.........		N = 2400
Length..................	3.5",	L = 3.5
Insulation on wire: Enamel SCC.		

Design for minimum resistance, using as many standard wire sizes as practical.

First it is necessary to obtain the approximate range of wire sizes to be used. To do this, we substitute in equation (8) to find the value of s at x=a (inside) and x=b (outside). We find s(a) = 764; s(b) = 127.

A table of wire sizes (Fig. 11) shows that there are nine sizes in this range. All nine could, of course, be used; however, as Fig. 4 shows, three or four different wire sizes are sufficient to obtain a good approximation to the ideal winding at the thickness ratio r = 6. We shall try four wire sizes: n = 4.

If we were not restricted to standard sizes, we would now use equation (16) to determine the optimum wire sizes for this winding. Thus, we would find s_1 = 606, s_2 = 388, s_3 = 248, and s_4 = 158. The standard wires actually available, as listed in Fig. 11, that come closest to this set are AWG Nos. 19, 17, 15, and 13, with an average

Fig. 11

ENAMEL SCC WIRE TABLE

AWG No.	Turns per square-inch s	Diameter inch y	Space Factor α	Size Ratio v
12	131	.08081		
13	162	.07196	.659	
14	198	.06408		1.54
15	250	.05707	.640	
16	306	.05082		1.49
17	372	.04526	.599	
18	454	.04030		1.49
19	553	.03589	.560	
20	725	.03196		

value of $v = 1.51$. Formulas (22) now give the winding schedule: $N_1 = 679$ turns of No. 19 wire, $N_2 = N_3 = 555$ turns of No. 17 and No. 15 wire each, $N_4 = 612$ turns of No. 13 wire.

From equation (23) we find that the resistance of this winding will be $R = 6.38$ ohms. If only one wire size had been used, the winding would either consist of 2400 turns of No. 15 wire, with an outside diameter of 6.48" and a resistance of 7.47 ohms, or of No. 16 wire, giving an outside diameter of 5.48" with a resistance of 8.14 ohms.

The theoretically perfect winding would have a resistance of 6.29 ohms, as computed from equation (9).

IV. TEMPERATURE RISE CALCULATIONS

A. INTRODUCTION

In the present work, only steady state conditions are included. It is assumed in all cases that equilibrium has been established; that is, the derivative of the temperature with respect to time is zero at every point in the winding. This restriction is believed to be not unreasonable, inasmuch as the criteria of maximum safe temperatures in magnet coils are usually based on continuous duty. For intermittent operation, the established practice is to reduce the operating conditions to the equivalent conditions at continuous duty. (See, for example, data given by Moses [11]).

Since transient temperature changes are eliminated from consideration, it is not necessary to have any information about the specific heat of the coil. The only data required are the rate with which heat is generated and the rate with which heat is conducted away, at every point within the winding.

From the microscopic viewpoint, the structure of the coil exhibits drastic discontinuities as far as both heat generation and heat conduction are concerned; for the copper alone is the source of heat, none being generated in the insulating spaces, and the electrical insulation represents a resistance to heat conduction several hundred times as great as that of the copper in the wire. However, considering the coil as a macroscopic entity, it can be supposed to have a well-defined average value of heat generated per unit volume per unit time and also of heat conducted per unit cube per degree temperature difference per unit time. The former is a function of current, wire size, and resistivity, and is found to be

$$Q = \frac{I^2 s^2 \rho}{\alpha} \tag{24}$$

where I = current (amperes)

s = number of turns per square-inch of cross section = $1/y^2$

ρ = resistivity (ohm-inch)

α = space factor (cross section area of copper divided by area of winding).

Expressing this quantity as a function of the ampere-turns A of the winding, we can write for a uniform winding (if s is constant):

$$Q = \frac{\rho \quad A^2}{\alpha \; L^2 \; (b - a)^2} \tag{25}$$

The heat conductivity of a body composed of round copper bars imbedded in insulating material has been the subject of a theoretical investigation made by Professor A.D. Moore [12]. Professor Moore states that the curves shown in Fig. 27 of his book were derived by graphical field mapping of an adequate number of cases and using the assumption that copper is an infinitely good heat conductor. This assumption holds very well for ordinary insulating materials, as the following list of heat conductivities shows:

Relative Heat Conductivities

Air................	1.0	= 0.000566 watts/°C-inch at 0°C [13]
Cotton.............	1.9	
Silk...............	2.0	
Paper..............	5.8	
Paraffin...........	11.2	
Glass..............	39	
Copper.........	17400	

In any case, some effective value of heat conductivity K may be assigned to a given coil determined by experimental or theoretical methods.

Thus, the steady-state temperature distribution in a coil must satisfy Poisson's equation:

(26) $$\nabla^2 T = -\frac{Q}{K}$$

Simple as the equation appears, its solution for a body of general shape, satisfying a set of general boundary conditions, is extremely complicated; and no attempt will be made here to establish such a general solution.

However, when certain special cases are considered, in particular cases that are apt to arise in practical applications, very useful particular soluctions can be found readily. The special cases to be considered are: (Illustrations in Fig. 12)

Thin, long, hollow cylinder (or thin, short, hollow cylinder with ends thermally insulated).

Thick, long, hollow cylinder (or thick, short, hollow cylinder with ends thermally insulated). This includes as a special case the long, solid cylinder.

Thick, short, hollow cylinder with inside thermally insulated, ends and outside face at temperature zero.

Ventilated winding, built up of thin, long, hollow cylinders with ventilating ducts separating sections of the winding [14]).

Two sets of boundary conditions will be applied to the first two cases: a) Inside of winding thermally insulated, outside face at temperature zero, i.e.

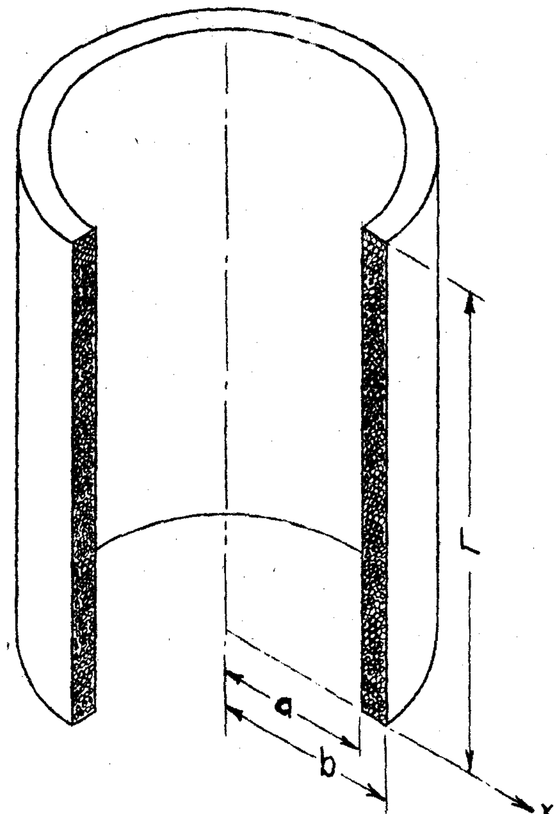
(a) $\left(\frac{dT}{dx}\right)_{x=a} = 0\ ; \quad T(b) = 0.$

and b) both inside and outside faces at temperature zero, i.e.

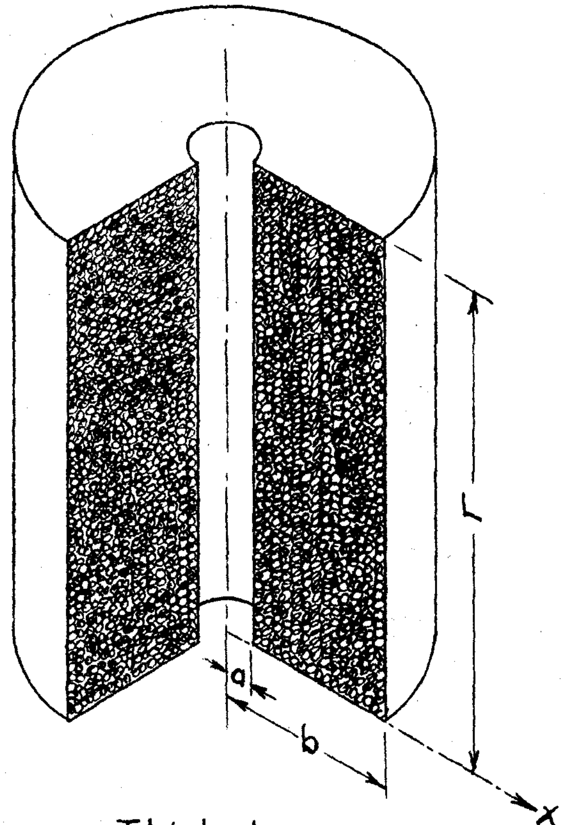
(b) $T(a) = T(b) = 0.$

For the boundary conditions to be used, the assumptions made are not well justified in all cases. As experiments show, and Professor Moore's work indicates, no serious errors are made in assuming that

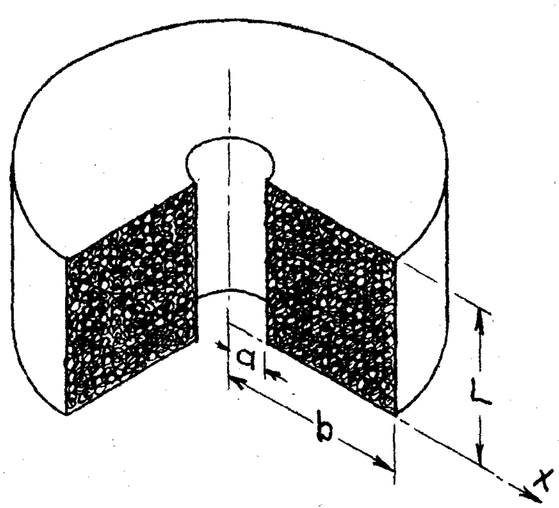
Fig. 12
Types of Windings



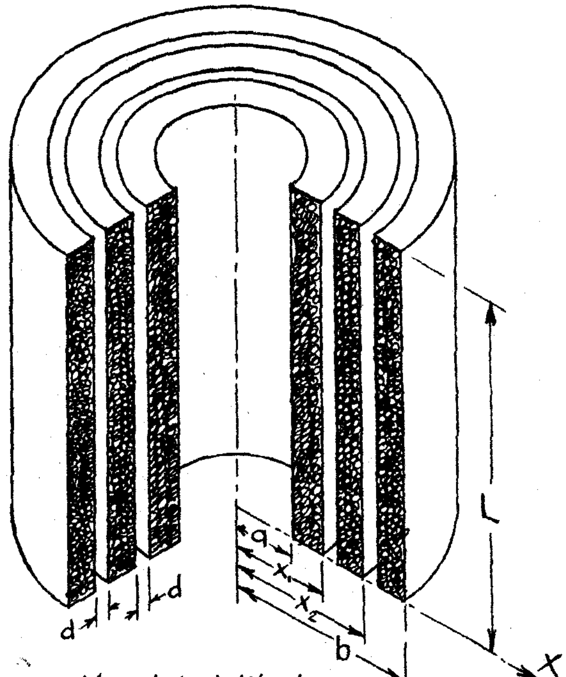
Thin, Long
Hollow Cylinder



Thick, Long
Hollow Cylinder



Thick, Short
Hollow Cylinder



Ventilated Winding

ordinary insulating materials represent the case of perfect thermal insulation at the ends of an enclosed coil, or at the inside of the winding when the coil is wound upon an insulating spool, even when these parts are relatively thin.

At the outside surface of the coil, the temperature is assumed to be uniform (the temperature zero is chosen merely as a convenient reference level for small temperature rise). This assumption holds only under very special conditions. In the experiments conducted at the Applied Science Laboratory, special care was taken to insure adequate circulation in the bath in which the windings were tested. However, as the data given by Stewart and Whitman [15] indicate, the surface temperature of a coil is in general not easy to predict. When the surface temperature is known, the calculations of internal temperature rise above the surface temperature will, of course, give the actual internal temperatures. It will therefore be assumed here that the surface temperature is known by experiment [16] or can be calculated from data as given, for example, by G.L. Moses [17].

Every type of winding listed on page 36 will be discussed first for relatively small internal temperature rise, then for large temperature rise, the distinction being made on the basis of the following considerations:

The heat generated per unit volume per unit time, as given by equations (24) and (25), is a function of the resistivity of the wire. As used in the equations in this paper, the quantity ρ is the resistivity of the metal in the wire, a constant at constant temperature. For small temperature rise, ρ is assumed to remain constant as far as its effect on Q in Poisson's equation is concerned, but it is assumed to be a linear function of the temperature as far as resistance measurements on the winding are concerned. In other words, in the treatment of small temperature rise, ρ is given the quality of being both constant and variable, an inherently contradictory hypothesis.

For large temperature rise, this assumption becomes untenable, and the curves in Fig. 17 and Fig. 18 show exactly to what degree the temperature calculations based on this simplifying, but contradictory, hypothesis are in error.

As experimental results show (see section V), it also becomes necessary to include another effect at large temperature rise under certain conditions: The temperature coefficient of thermal conductivity. This effect is difficult to treat in a general manner because the thermal conductivity K is an average conductivity for the heterogeneous winding, thus being a function of the materials used in the winding as well as of the arrangement of these materials in space. One possible direction of attack is to assume that the thermal conductivity is a linear function of the temperature, and thus enters the equations in a manner similar to the electrical conductivity of the wire.

A further classification to be made in the following calculations is whether the winding is uniform (i.e. wound of a single wire size) or non-uniform in the sense of having variable wire size according to equation (8).

B. TEMPERATURE DISTRIBUTIONS

MAXIMUM AND AVERAGE TEMPERATURE RISE

Thin Long Hollow Cylinder

α) SMALL TEMPERATURE RISE. The solution of Poisson's equation (26) for the various cases listed on page 36 gives the point-by-point temperature distribution throughout the coil. This is more information than the designer requires ordinarily because the design criterion is usually one of maximum temperature rise; hence, it is only necessary to find an expression for the relation between the maximum internal temperature rise and the physical characteristics of the winding.

Another quantity of practical importance is the average temperature. The average temperature is defined as the temperature to which the entire coil would have to be raised in order to have the same increase in resistance as it actually has in the temperature field created by the Joule heat. The average temperature is a quantity that can be measured easily on an existing coil. Thus, one of the results of practical significance derived from this work is the relation between maximum temperature, a quantity difficult to measure, and the average temperature, easily obtained experimentally.

The winding contained in the space of a thin, long, hollow cylinder represents one of the applications found frequently in coil design. It approaches the ideal condition of having all the turns "inside" (see page 2). The flow of heat in this type of winding takes place in the radial direction only. Since the radius of every turn of wire in the winding is large compared to the thickness of the winding, the temperature distribution is the same as that in the straight portions of the winding on a rectangular core (with ends thermally insulated); hence, the same as in a flat plate of the same thickness $(b - a)$ as the thickness of the hollow cylinder.

For this case Poisson's equation (26) reduces to the ordinary differential equation

$$(27) \qquad \frac{d^2T}{dx^2} = -\frac{Q}{K}$$

Equation (24) shows that Q is independent of x for the winding of uniform wire size (s = constant). K is assumed to be constant; hence, the general solution of (27) is given by

$$(28) \qquad T(x) = -\frac{1}{2}\frac{Q}{K}x^2 + C'x + C''$$

Applying the set of boundary conditions a) to determine the constants C' and C", we obtain the temperature distribution in a thin, long, hollow cylinder, in which heat is being generated uniformly:

$$(29) \qquad T(x) = \frac{1}{2}\frac{Q}{K}(b^2 + 2\,a\,x - x^2 - 2\,a\,b)$$

The maximum temperature rise will occur at the inside of the winding, where

$$(30) \qquad T_m = T(a) = \frac{1}{2}\frac{Q}{K}(b - a)^2$$

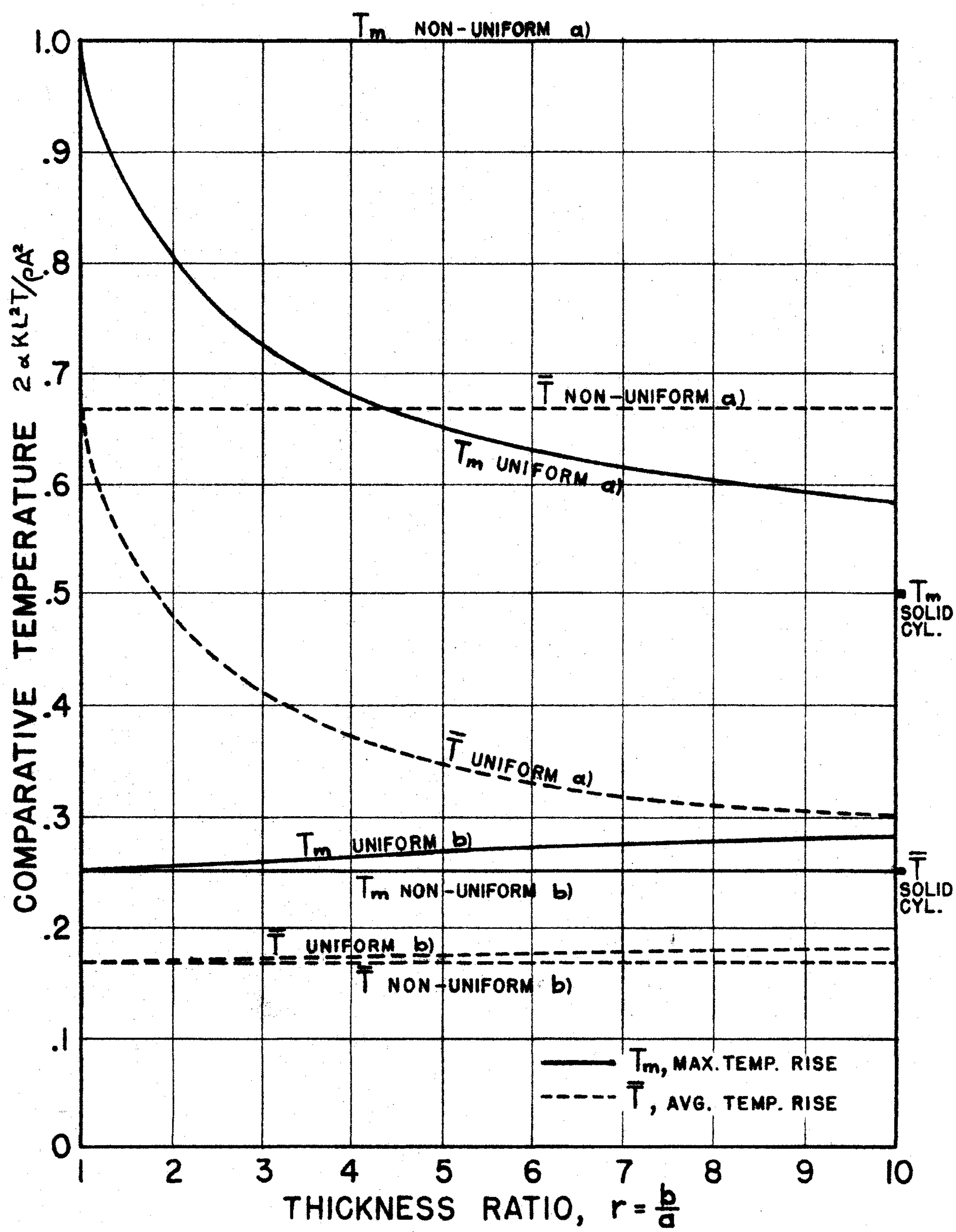
The rate of heat generation depends upon the current and the number of turns in the winding according to equation (25). It is possible to substitute the number of ampere-turns A in place of the current and some of the other constants in equation (30). As a result we find the maximum temperature rise as a function of the number of ampere turns, independent of the wire size or the thickness of the winding:

$$(31) \qquad T_m = \frac{\rho A^2}{2\alpha K L^2}$$

This is plotted in terms of the dimensionless variable $2\alpha K L^2 T/\rho A^2 = 1$ in Fig. 13.

To obtain the average temperature rise, we shall proceed from the basic ideas involved in the experimental measurement of this quantity.

Fig. 13 - Small Maximum and Average Temperature Rise for Uniform and Ideal Non-Uniform Windings



The resistance of a metallic conductor is a linear function of the temperature in the range of ordinary temperatures for electrical equipment. This fact is used to deduct the changes in temperature of a winding by measuring the changes in its resistance. Let us assume that the resistance of a winding is R ohms at $\overline{T}$ °C and R_o ohms at the temperature T_o °C. Then the following equation is found to be true for copper wire in the temperature range 0 to 100 °C:

$$\frac{R - R_o}{R_o} = \frac{\overline{T} - T_o}{234.5 + T_o} \tag{32}$$

The difference $\overline{T} - T_o$ is the average internal temperature rise of the winding.

We may think of a given winding as being subdivided into a number of sections, each having its own resistance dR and its own temperature T. Thus we may write an equation like (32) for every one of the elemental sections that make up the winding.

$$\frac{dR - dR_o}{dR_o} = \frac{T - T_o}{234.5 + T_o} \tag{33}$$

where T_o is the same for all sections (i.e., the temperature of the cold coil), but T varies from section to section according to the distribution of the temperature field. When the coil is cold, its resistance is given in the limit by the integral

$$R_o = \int dR_o$$

But when current is flowing in the winding, its total resistance will be

$$R = \int dR$$

Hence, substituting in equation (32) and using the relation established by equation (33), we have

$$\overline{T} - T_o = \frac{\int (T - T_o)\, dR_o}{\int dR_o}$$

If we assume that the reference temperature at the surface of the coil, T_o, is constant, then the expression for the average internal temperature rise reduces to the following:

$$(34) \qquad \overline{T} = \frac{\int T\, dR}{\int dR}$$

For a winding which approximates the shape of a flat plate, the resistance of an elemental section is proportional to the thickness of the section:

$$dR = p\, dx\ ; \qquad p = \text{constant}$$

Substituting this value for dR and the expression in (29) for T in equation (34), we obtain the result

$$\overline{T} = \frac{1}{3}\frac{Q}{K}(b-a)^2$$

Expressed in terms of the ampere-turns of the winding, this is

$$(35) \qquad \overline{T} = \frac{\rho A^2}{3\alpha K L^2}$$

This is plotted as $2\alpha KL^2T/\rho A^2 = 2/3$ in Fig. 13.

By comparing equations (31) and (35), we find that the maximum internal temperature rise exceeds the average temperature rise by

$$T_m - \overline{T} = \frac{1}{6}\frac{Q}{K}(b-a)^2 = \frac{\rho A^2}{6\alpha K L^2}$$

and that they are in the ratio 3:2.

It is standard practice at the present time to allow a fixed temperature difference of 10°C between T_m and $\overline{T}$ for many design calculations [18]. This arbitrary figure may be compared with the actual temperature difference to be expected from the above calculations, and from those to be applied to the other special cases that follow.

If the inside of the thin hollow cylinder is cooled to the same temperature as the outside face, boundary conditions b) are applied to determine the constants C' and C" in the general solution (28):

$$T(x) = \frac{1}{2}\frac{Q}{K}(ax + bx - x^2 - ab)$$

The maximum temperature will be the temperature at the center of the coil $x = \frac{1}{2}(b + a)$, where

(36) $$T_m = \frac{1}{8}\frac{Q}{K}(b - a)^2 = \frac{\rho A^2}{8 \alpha K L^2}$$

In Fig. 13 this is plotted as $2\alpha KL^2T/\rho A^2 = 1/4$.

Substituting the proper values in equation (34), we obtain the average temperature rise in the winding:

(37) $$\bar{T} = \frac{1}{12}\frac{Q}{K}(b - a)^2 = \frac{\rho A^2}{12 \alpha K L^2}$$

which is shown in Fig. 13 as $2\alpha KL^2T/\rho A^2 = 1/6$. The difference between the maximum and average temperature rises will be

$$T_m - \bar{T} = \frac{1}{24}\frac{Q}{K}(b - a)^2 = \frac{\rho A^2}{24 \alpha K L^2}$$

β) LARGE TEMPERATURE RISE. As was pointed out on page 38, the change of electrical conductivity with temperature must be taken into account if the internal temperature rise is large. As a good approximation, we may use the relation given by equation (32). Thus the rate of heat generation can be written as a linear function of the temperature rise:

(38) $$\frac{Q}{K} = u^2 T + w^2$$

where $w^2 = \dfrac{I^2 s^2 \rho}{\alpha K} = \dfrac{\rho A^2}{\alpha K L^2 (b - a)^2}$

$$u^2 = \frac{w^2}{T_0 + 234.5}$$

In these expressions, α is the space factor of the winding, ρ the resistivity of the copper in the wire at temperature T_0 °C, and T_0 is the temperature at the surface of the coil.

Thus Poisson's equation reduces to the following form;

$$\frac{d^2T}{dx^2} + u^2T = - w^2$$

The solution of this equation is given by trigonometric functions, as shown by Jakob [3]. It will be shown later that the ideal non-uniform winding satisfies the same equation; thus, all the data plotted in Fig. 18 can be applied to this case.

When one side of the winding is thermally insulated (boundary condition a), it is evident from the symmetry of the system that the temperature distribution in the winding is the same as in half the winding twice as thick with boundary condition b).

For an actually existing coil, the thermal conductivity K cannot always be assumed to be a constant, as indicated by experiments (see section V). Assuming $1/K$ to be a linear function of the temperature, equation (38) takes the form of a second order differential equation of the second degree:

$$\frac{d^2T}{dx^2} = -u\,T^2 - v\,T - w \tag{39}$$

where
$$w = \frac{I^2 s^2 \rho_0}{\alpha K_0}$$

$$v = \frac{2T_0 + 234.5 - \beta}{(T_0 + 234.5)(T_0 - \beta)}\,w$$

$$u = \frac{1}{(T_0 + 234.5)(T_0 - \beta)}\,w$$

In these expressions, K_0 and ρ_0 are the thermal conductivity and electrical resistivity of the coil at the surface temperature T_0 °C; β is the temperature at which $1/K$ would become zero, if extrapolated, corresponding to -234.5 °C, the temperature at which ρ would become zero for copper.

The solution of equation (39) is the same that would be obtained if the electrical conductivity were a quadratic function of the temperature and the thermal conductivity a constant. This solution has been found to contain elliptic functions, as published by E.S. Davies [19].

It is not a simple solution, and in view of the fact that the temperature coefficient of the thermal conductivity of the heterogeneous winding is not generally known in the first place, there is little gain in working out the theoretical solution over an empirical answer found from measurements on actual coils.

Thick Long Hollow Cylinder

α) SMALL TEMPERATURE RISE.

1. Uniform Winding. For the thick hollow cylinder, it becomes necessary to consider the changes introduced by the increasing radii of the turns in the winding. Again we will assume heat flow in the radial direction only (long cylinder, or ends thermally insulated). Thus equation (26) may be replaced by the ordinary differential equation

$$\frac{d^2T}{dx^2} + \frac{1}{x}\frac{dT}{dx} = -\frac{Q}{K} \tag{40}$$

For the uniform winding (constant wire size), the general solution of this equation is

$$T(x) = -\frac{1}{4}\frac{Q}{K}x^2 + C' \log_e x + C'' \tag{41}$$

Applying boundary conditions a) to determine the constants, the temperature distribution in the winding is found to be

$$T(x) = \frac{1}{4}\frac{Q}{K}(b^2 - x^2 - 2a^2 \log_e \frac{b}{x})$$

The maximum temperature will appear at the inside of the winding:

$$T_m = \frac{1}{4}\frac{Q}{K}(b^2 - a^2 - 2a^2\log_e \frac{b}{a}) = \frac{\rho A^2}{4\alpha K L^2} \cdot \frac{r^2 - 1 - 2\log_e r}{(r-1)^2} \tag{42}$$

where $r = b/a$. The relation given by equation (42) is plotted in Fig. 13 as the curve marked "uniform a)".

The average temperature can be computed again from equation (34); however, the quantities dR and dx are no longer directly proportional. They satisfy the relation

$$dR = p\,x\,dx; \qquad \text{where } p = \text{constant.}$$

Thus the average temperature is found to be

$$\overline{T} = \frac{1}{8}\frac{Q}{K}\left(b^2 - 3a^2 + \frac{4\,a^4}{b^2 - a^2}\log_e \frac{b}{a}\right) = \frac{\rho\,A^2}{8\,\alpha\,K\,L^2} \cdot \frac{r^2 - 3 + \frac{4}{r^2 - 1}\log_e r}{(r-1)^2}$$

Fig. 13 shows this curve, marked "uniform a)".

By making the change of variable $z = a/b$ and letting the inside radius a approach zero, we arrive at the special case of the thick solid cylinder, with maximum temperature rise

$$T_m = \frac{1}{4}\frac{Q}{K}b^2 = \frac{\rho\,A^2}{4\,\alpha\,K\,L^2}$$

and the average temperature rise

$$\overline{T} = \frac{1}{8}\frac{Q}{K}b^2 = \frac{\rho\,A^2}{8\,\alpha\,K\,L^2}$$

Both these temperature rises are indicated at the right-hand margin of Fig. 13.

The fact that the maximum and average temperatures are in the ratio of 3:2 for the thin cylinder and in the ratio 4:2 for the long, solid cylinder, is stated by Jakob [8].

Using boundary conditions b) to determine C' and C" in equation (41), we find that the temperature distribution is

$$(43) \qquad T(x) = \frac{1}{4}\frac{Q}{K}\left(b^2 - x^2 - \frac{b^2 - a^2}{\log_e \frac{b}{a}}\log_e \frac{b}{x}\right)$$

The radius of the maximum temperature level surface is readily determined by setting the derivative of T with respect to x equal to zero. The value of x at the point of maximum temperature is given by the formula

$$x_m^2 = \frac{b^2 - a^2}{2\log_e \frac{b}{a}}$$

Rather than trying to substitute this value in algebraic form in equation (43), it is usually easier to evaluate x_m numerically, and to substitute the number in (43) to obtain the maximum temperature rise T_m. The relation between T_m and r is plotted in Fig. 13, marked "uniform b)".

To calculate the average temperature rise, we make use of equations (34) and (43) and find

$$\overline{T} = \frac{1}{8}\frac{Q}{K}\left(b^2 + a^2 - \frac{b^2 - a^2}{\log_e \frac{b}{a}}\right) = \frac{\rho\ A^2}{8\ \alpha\ K\ L^2} \cdot \frac{r^2 + 1 - \frac{r^2 - 1}{\log_e r}}{(r - 1)^2} ,$$

also plotted in Fig. 13, "uniform b)".

2. Non-uniform Winding. In the ideal solution of the minimum resistance problem, given by equation (5), the wire diameter is found to vary in proportion to the square root of the radius. Substituting this result in the expression for the rate of heating (24), we find that Poisson's equation for the thick long cylinder reduces to the ordinary differential equation

$$\frac{d^2T}{dx^2} + \frac{1}{x}\frac{dT}{dx} + \frac{\lambda^2\ I^2\ \rho}{4\ \alpha\ K\ x^2} = 0$$

which has the general solution

$$T(x) = -\frac{\lambda^2\ I^2\ \rho}{4\ \alpha\ K} \cdot \frac{1}{2}(\log_e x)^2 + C'\log_e x + C''$$

With boundary conditions a), the particular solution becomes

$$(44) \qquad T(x) = \frac{\rho\ A^2}{2\ \alpha\ K\ L^2\ (\log_e \frac{b}{a})^2}\ \log_e \frac{b}{x}\ \log_e \frac{b\,x}{a^2}$$

This formula was obtained by substituting the value of λ from equations (5) and (8). It yields the expression for the maximum temperature rise, at the inside of the winding:

$$(31) \qquad T_m = T(a) = \frac{\rho\ A^2}{2\ \alpha\ K\ L^2}$$

This temperature is seen to be independent of the thickness ratio and is represented in Fig. 13 by a straight horizontal line, "non-uniform a)".

To compute the average temperature rise, the relation between dR and dx must be known for the non-uniform winding. Since the wire diameter increases proportionally with the square root of the radius, we find that this relation is

$$dR = p \frac{1}{x} dx \; ; \quad p = \text{constant}$$

Hence, the average temperature is found to be

(35) $$\bar{T} = \frac{\rho A^2}{3 \alpha K L^2}$$

The graph of this equation is also shown in Fig. 13.

For boundary conditions b), the temperature distribution is given by the equation

$$T(x) = \frac{\lambda^2 I^2 \rho}{8 \alpha K} \log_e \frac{b}{x} \log_e \frac{x}{a}$$

The level surface of maximum temperature has the radius

$$x_m = \sqrt{a b}$$

where the maximum temperature is

(36) $$T_m = \frac{\rho A^2}{8 \alpha K L^2}$$

The average temperature is found to be

(37) $$\bar{T} = \frac{\rho A^2}{12 \alpha K L^2}$$

These equations, too, are plotted as straight lines in Fig. 13.

From the graphs representing these quantities in Fig. 13, it is evident that the difference in temperature rise of the non-uniform winding as compared with the uniform winding is insignificant for the winding cooled both inside and outside; however, for the practically important case of cooling from the outside only, there appears a distinct advantage for the uniform winding. Nevertheless, the ideal non-uniform winding is in no case hotter than a uniform winding of the same number of ampere-turns wound on a rectangular core.

Although only the maximum and average temperatures are of practical importance, it may be illuminating to compare some of the complete temperature distributions discussed so far. For the purpose of comparison, we shall choose windings of the same thickness $(b - a)$, but not necessarily of the same thickness ratio $r = b/a$. Thus, all the windings to be compared contain the same number of turns, but the turns differ in curvature. One might form the mental picture of a shifting axis: Close to the inside turns for large values of r; removed to infinity for $r = 1$.

Fig. 14 shows the temperature distribution for uniform windings with boundary conditions a). Fig. 15 is a graph of the distribution in non-uniform windings, also with boundary conditions a). In Fig. 16 the curves of Fig. 14 and Fig. 15 are superposed for the purpose of comparison, and the positions of the axes of the windings have been indicated. The equation for the flat plate is the member common to both families of curves shown in Fig. 16. It represents the highest space average temperature. The distributions in Fig. 15 all have the same average temperature, in spite of the fact that they differ considerably in appearance. The reason for this unusual behavior may be found in the fact that the average temperature is not a space average but is weighted with the distribution of resistance in space.

In the discussion of the minimum resistance problem (section III), it was found that only four or five changes in wire size are necessary to obtain a close approximation to the ideal case of continuously varying wire size. From the experiments performed on a coil with four different wire sizes and a thickness ratio of about $5\frac{1}{2}$, it appears that the same rule applies to the temperature distributions; that is, the data plotted in Fig. 20 show that a coil of four wire sizes exhibits substantially the same temperature distribution as the theory predicts

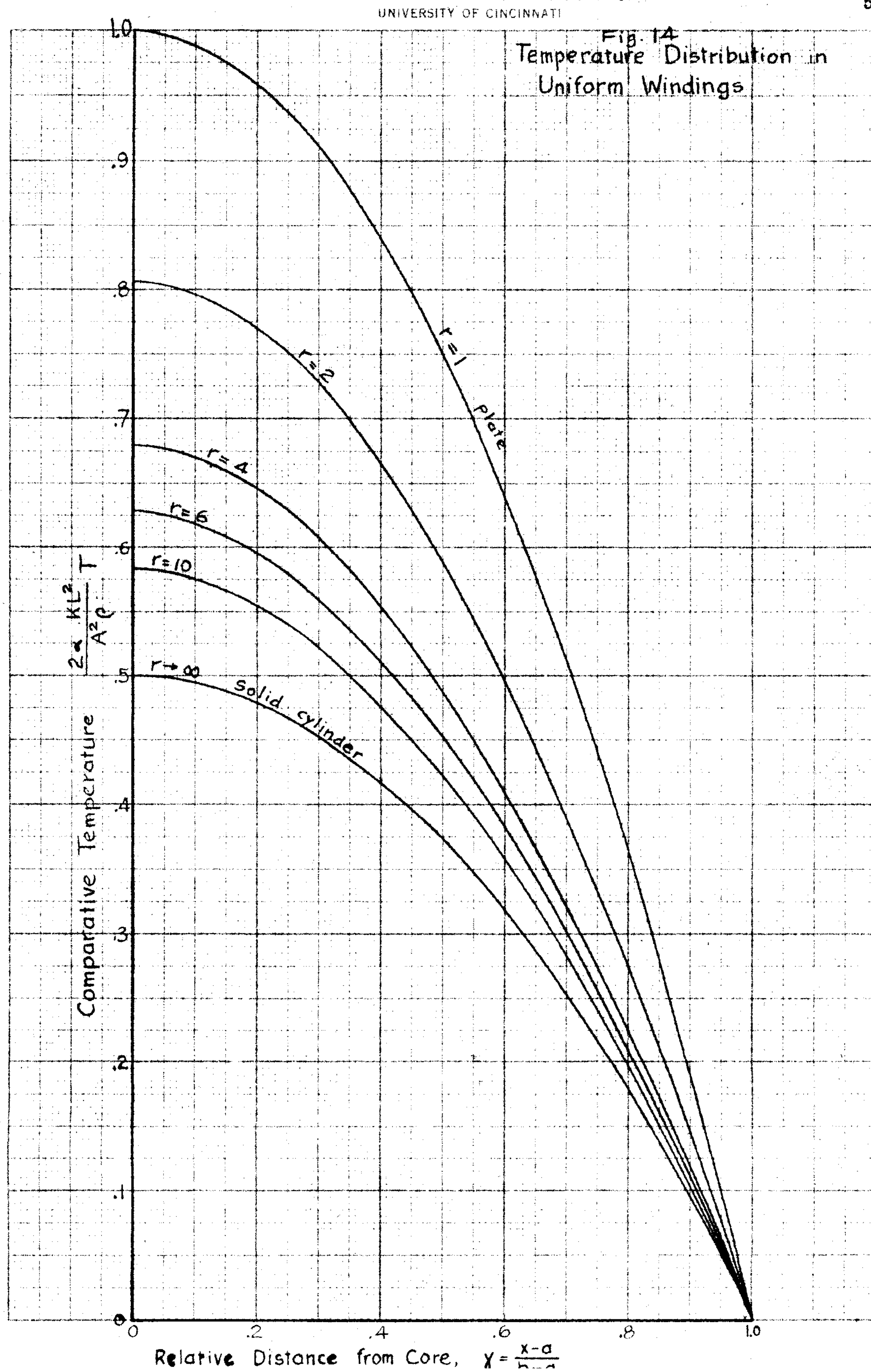
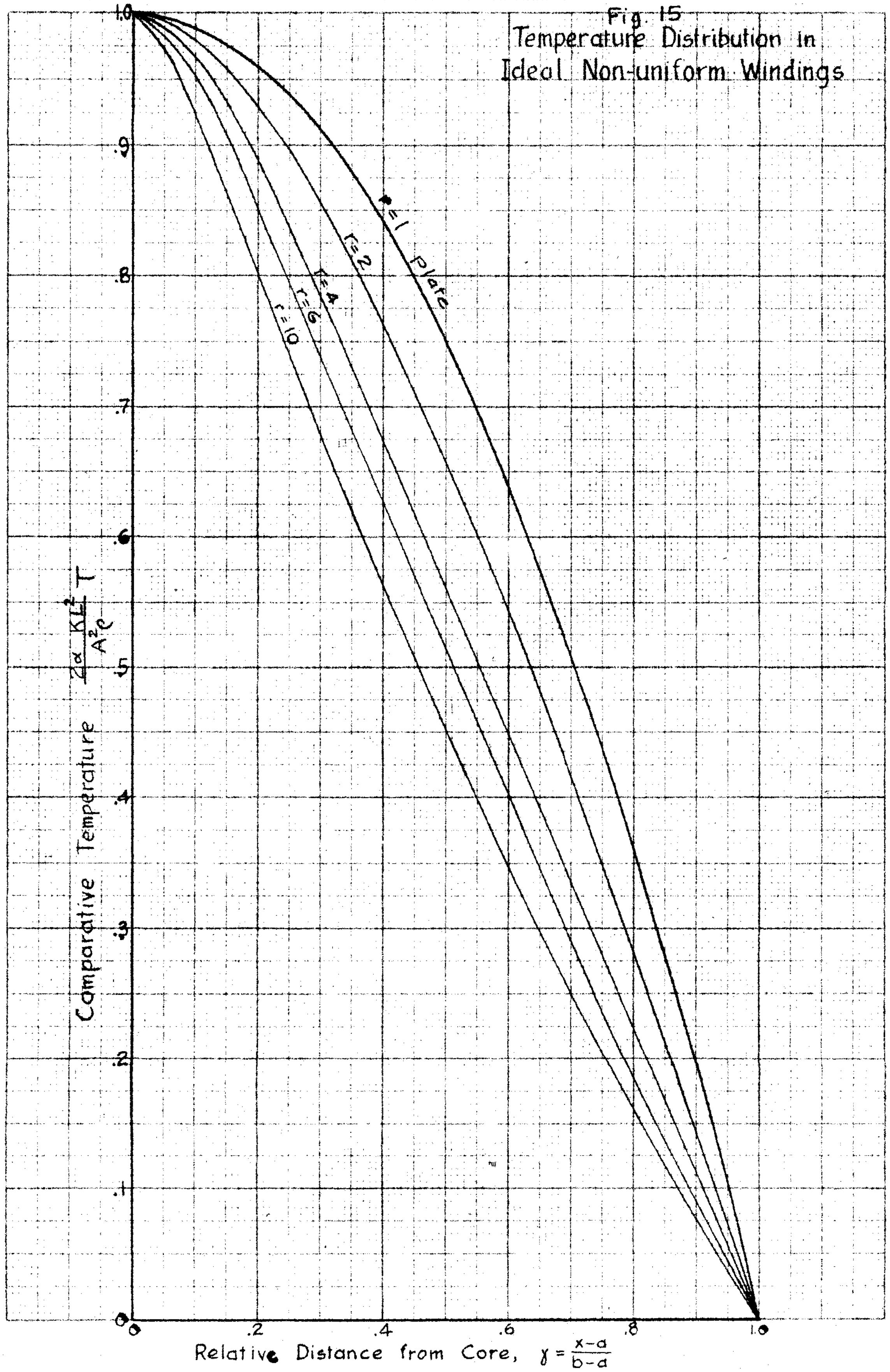


Fig. 14
Temperature Distribution in Uniform Windings

Fig. 15
Temperature Distribution in
Ideal Non-uniform Windings



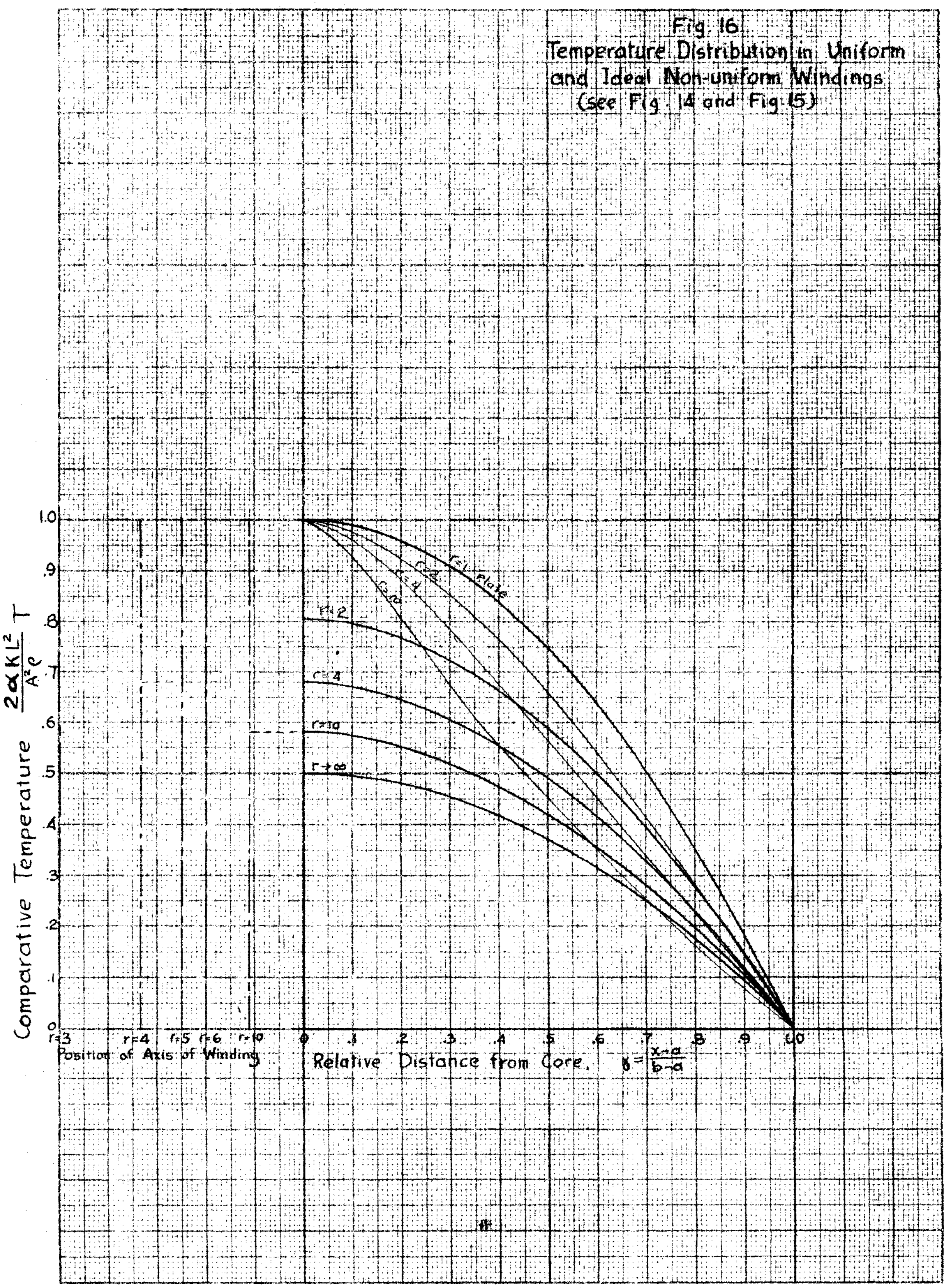


Fig. 16
Temperature Distribution in Uniform and Ideal Non-uniform Windings (see Fig. 14 and Fig. 15)
r=1 Plate
r=2
r=4
r=10
r→∞
Comparative Temperature $\frac{2\alpha K L^2}{A^2 \rho} T$
Position of Axis of Winding
Relative Distance from Core. $\gamma = \frac{x-a}{b-a}$
r=3
r=4
r=5
r=6
r=10
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

for a winding with continuously varying wire size. Hence, the curves in Fig. 15 represent the temperature distribution to be expected in a practically useful approximation to the winding of minimum resistance.

(β) LARGE TEMPERATURE RISE.

1. Uniform Winding. For cylindrical windings, Jakob [8] gives the results applicable only to the solid cylinder (for which the thickness ratio r is infinite). We shall compute here the solutions that apply to long hollow cylinders with thickness ratios between one and infinity.

Poisson's equation in cylindrical coordinates, with heat generated as a linear function of the temperature, becomes (equation 38)

$$\frac{d^2T}{dx^2} + \frac{1}{x}\frac{dT}{dx} + u^2 T = - w^2$$

A particular solution is given by $T = - w^2/u^2$, and the solution of the reduced equation

$$\frac{d^2T}{dx^2} + \frac{1}{x}\frac{dT}{dx} + u^2 T = 0$$

is expressible in terms of Bessel functions of the first kind of zero order (J_0), and of the second kind of zero order (Y_0). The complete solution is therefore

$$T(x) = C' J_0(xu) + C'' Y_0(xu) - \frac{w^2}{u^2} \tag{45}$$

For boundary conditions a), the constants C' and C" are such that the temperature distribution in the coil may be written in the following form:

$$T(x) = \frac{w^2}{u^2}\left[\frac{J_1(au)\, Y_0(xu) - J_0(xu)\, Y_1(au)}{J_1(au)\, Y_0(bu) - J_0(bu)\, Y_1(au)} - 1\right]$$

The point of maximum temperature rise is at the inside of the winding, where

$$T_m = T(a) = (T_0 + 234.5)\left\{\frac{2}{\pi\, a\, u\,[J_1(au)\, Y_0(bu) - J_0(bu)\, Y_1(au)]} - 1\right\}$$

In this expression let

$$(46) \qquad \sigma = a\,u = \frac{A\,\sqrt{\rho}}{L\,(r-1)\,\sqrt{\alpha K\,(T_0 + 234.5)}}$$

Then bu can be replaced by $r\sigma$, where r is the thickness ratio $r = b/a$. The formula for the maximum temperature rise may thus be written

$$(47) \qquad T_m = (T_0 + 234.5)\left[\frac{2}{\pi\,\sigma\{J_1(\sigma)\,Y_0(r\sigma) - J_0(r\sigma)\,Y_1(\sigma)\}} - 1\right]$$

Since all of the physical constants of the system appear in the arguments of the Bessel functions, further simplification is difficult; hence, the following steps will be taken to demonstrate the dependence of maximum temperature rise on the physical constants of the winding.

In the discussion of small temperature rise, the maximum temperature appeared on the ordinate in Fig. 13 in the dimensionless variable $2\alpha KL^2T/\rho A^2$. Thus to each value of T_m there corresponds a set of physical constants of the winding for each thickness ratio. For example, for a uniform winding with boundary conditions a) and thickness ratio $r = 4$, the constants of the winding must be such that the equation $2\alpha KL^2T'_m/\rho A^2 = 0.68$ is satisfied. The maximum temperature rise T'_m is primed to indicate that it is not the actual maximum temperature but the maximum temperature as computed from equation (42) - with the assumption that the temperature rise is small. Thus we find that σ is given by the expression

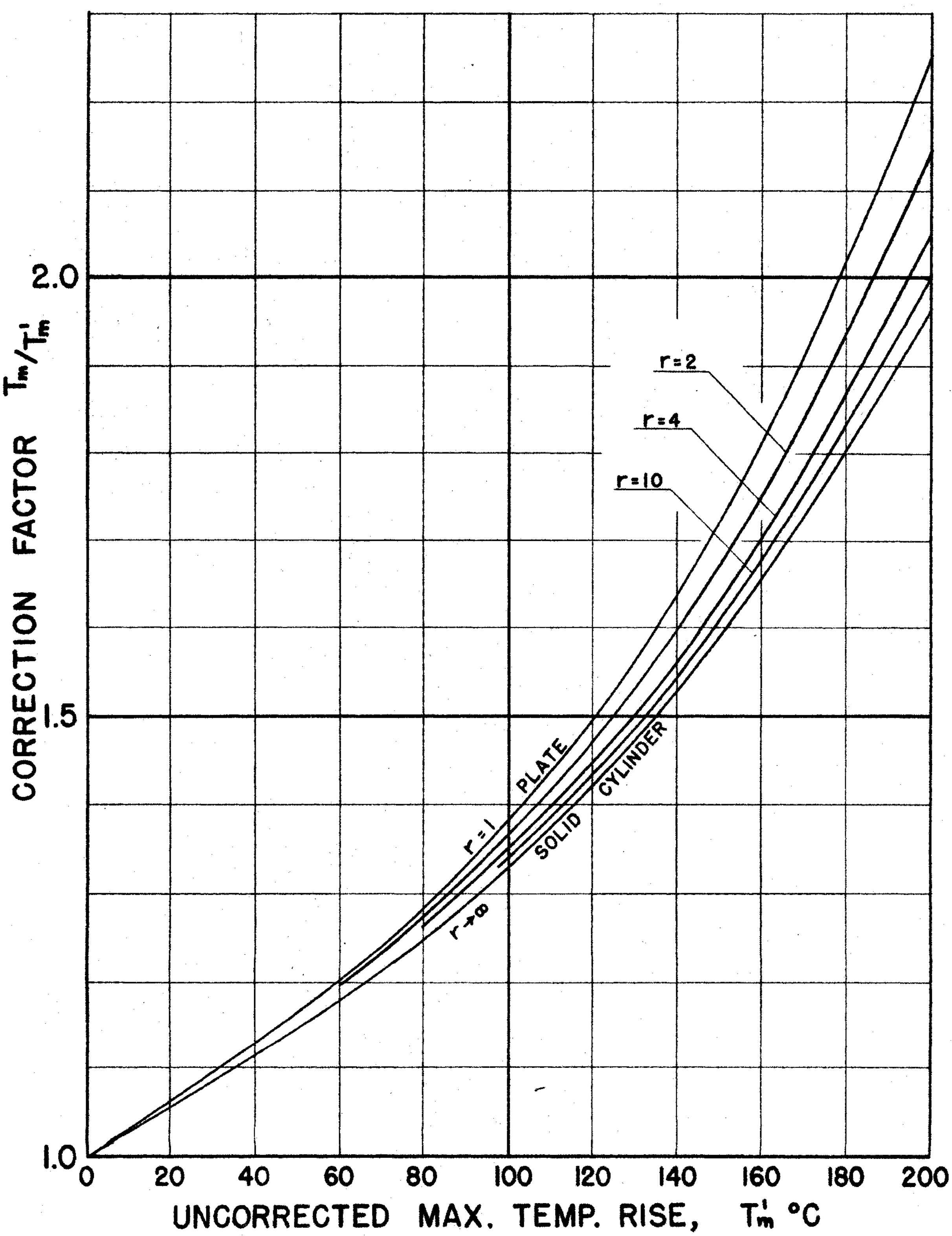
$$\sigma = \frac{1}{3}\sqrt{\frac{2\,T'_m}{0.68\,(T_0 + 234.5)}}$$

σ is therefore a function of T'_m and T_0, where T'_m is the maximum temperature rise in the winding when the effect of local changes in resistance is neglected, and T_0 is the temperature at the surface of the coil. If we let T_m be the actual maximum temperature in the winding, a graph can be drawn of T_m vs. T'_m, provided T_0 is known. Usually in electrical apparatus, the value of T_0 is specified. It will be assumed here as 40 °C above an ambient temperature of 20 °C. Hence, we can substitute $T_0 = 60$ in equations (46) and (47). The results of these computations are plotted in a set of graphs in Fig. 17, each curve representing a different thickness ratio of the winding.

The usefulness of this set of data may be illustrated by the following example: A uniform winding (boundary conditions a) of thickness ratio $r = 3$ has such a configuration that the maximum temperature rise as determined from equation (42), or Fig. 13, is $T'_m = 67$ °C. Corresponding to the abscissa $T'_m = 67$ in Fig. 17, the correction factor T_m/T'_m is seen to fall between 1.20 for the solid cylinder, and 1.24^+ for the flat plate. The winding of thickness ratio $r = 3$ should therefore have a correction factor of approximately 1.22. Thus the actual maximum temperature rise in the winding will be $1.22 \cdot 67 = 82$ °C. Since the surface temperature was assumed to be 60 °C, the "hot spot" in the coil will reach a temperature of $60 + 82 = 142$ °C. If the change in resistivity of the wire had been neglected, the computed maximum temperature in the winding would have been $60 + 67 = 127$ °C.

Jakob [3]) has shown that the ratio of maximum to average temperature rise is practically the same in large temperature rise problems as was found for small temperature rises (equations 31 and 35, 36 and 37, etc.). This is true because the shape of the temperature distribution curve is only slightly steeper for large temperature rise than it is for small temperature rise. Thus we shall assume for practical purposes that the fraction $\overline{T}/T_m$ is the same for large and small temperature rise problems.

Fig. 17 - Correction Factor for Large Maximum Temperature Rise in Uniform Windings



Assuming variable thermal conductivity in thick-cylinder problems, as was done for the thin-cylinder winding on page 46, would lead to an even more complicated solution than was found by Davies [19] in the case of the thick plate. No computations will be made here for variable thermal conductivity. A few experimental data for this case are given in section V.

If the inside of the winding is cooled to the same temperature T_0 as the outside, boundary conditions b), applied to the general solution (45), give the temperature distribution

$$T(x) = \frac{w^2}{u^2}\left\{\frac{\left[Y_0(bu) - Y_0(au)\right] J_0(xu) + \left[J_0(au) - J_0(bu)\right] Y_0(xu)}{J_0(au)\, Y_0(bu) - J_0(bu)\, Y_0(au)} - 1\right\}$$

To find the radius of the surface at which the maximum temperature will occur, it is necessary that the derivative of T with respect to x vanish. The resulting equation must be solved for x_m

$$\left[Y_0(bu) - Y_0(au)\right] J_1(x_m u) = -\left[J_0(au) - J_0(bu)\right] Y_1(x_m u)$$

2. Non-uniform Winding. For the thick, long, hollow cylinder, equation (40) applies; however, Q/K is now given by the function

$$\frac{Q}{K} = \frac{u^2}{x^2} T + \frac{w^2}{x^2}$$

where
$$w^2 = \frac{\rho A^2}{\alpha K L^2 (\log_e \frac{b}{a})^2}$$

$$u^2 = \frac{w^2}{T_0 + 234.5}$$

Thus Poisson's equation becomes

$$x^2 \frac{d^2T}{dx^2} + x \frac{dT}{dx} + u^2 T = - w^2$$

A particular solution is again $T = - w^2/u^2$. Added to the solution of the reduced equation, which happens to be Euler's equation

$$x^2 \frac{d^2T}{dx^2} + x \frac{dT}{dx} + u^2 T = 0,$$

the complete solution is

$$T(x) = C' \sin (u \log_e x) + C'' \cos (u \log_e x) - \frac{w^2}{u^2}$$

Boundary conditions a) serve to determine the constants:

$$T(x) = (T_0 + 234.5) \left\{ \frac{\cos (u \log_e \frac{x}{a})}{\cos (u \log_e \frac{b}{a})} - 1 \right\}$$

The point of maximum temperature rise is at x = a, where

$$T_m = (T_0 + 234.5) \left[\sec (u \log_e r) - 1 \right] = (T_0 + 234.5) \left\{ \sec \frac{A \sqrt{\rho}}{L \sqrt{\alpha K (T_0 + 234.5)}} - 1 \right\}$$

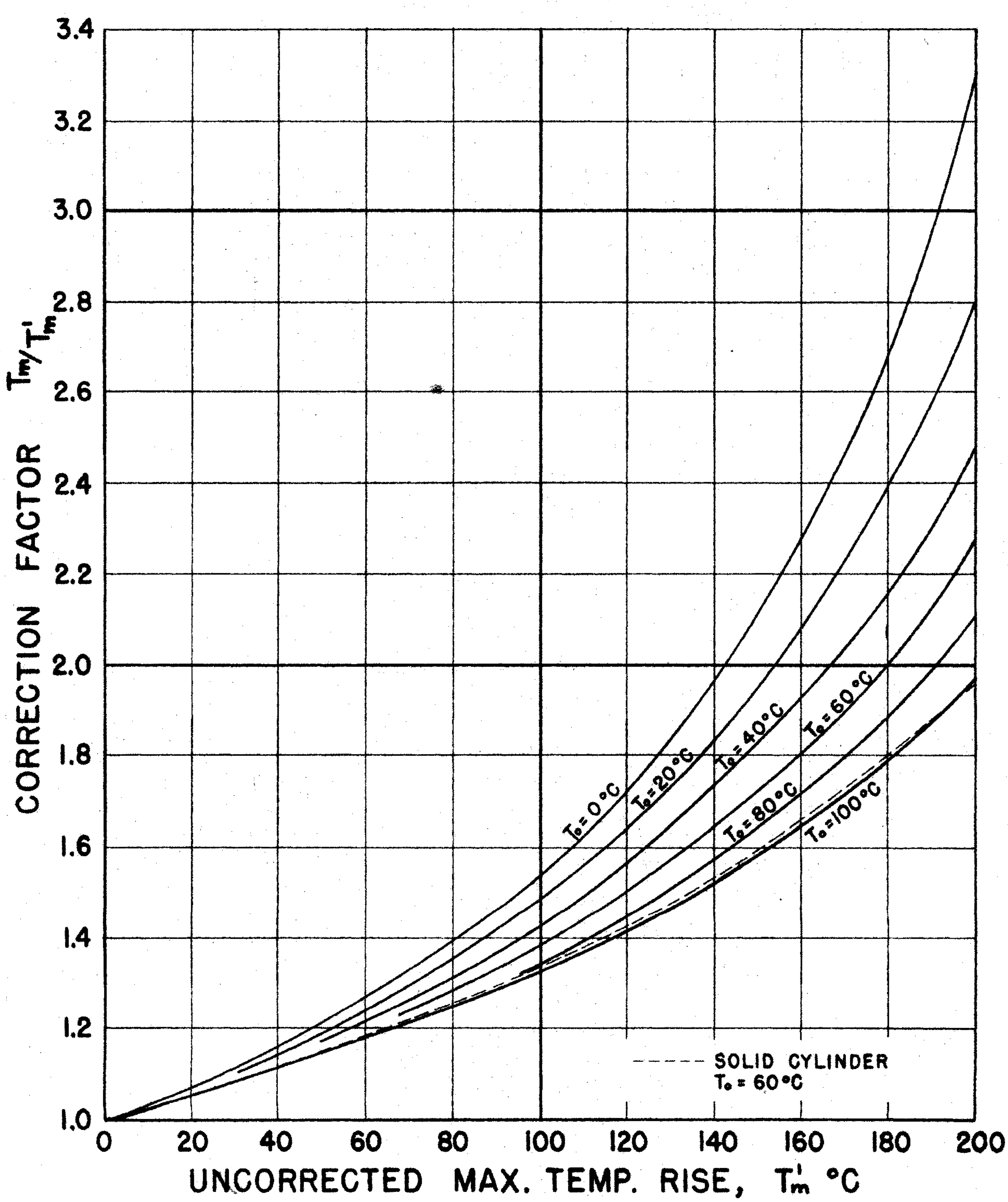
Hence, it is seen that the maximum temperature in the ideal non-uniform winding is independent of the thickness ratio, that it is only a function of the surface temperature T_0. Fig. 18 is a graph of the ratio T_m/T'_m for various values of T_0, where T'_m is the maximum temperature as computed from equation (31).

Since the temperature characteristics of a cylindrical non-uniform winding and of a uniform winding in the shape of a flat plate are alike, the data given in Fig. 18 apply to both types of windings, as was noted already on page 46. Thus, for example, if the maximum temperature rise of a uniform winding on a rectangular core is computed to be 80 °C according to equation (31), the correction factor for a surface temperature of T_0 = 60 °C is seen to be 1.30. Hence, the actual temperature rise will be 1.30 · 80 = 104 °C. In computing T'_m from equation (31), it is necessary to use the resistivity ρ which corresponds to the surface temperature of the coil, 60 °C in this example.

For the purpose of comparison, the curve for a solid cylinder wound of uniform wire size, with T_0 = 60 °C, has been drawn in Fig. 18.

Boundary conditions b) lead again to equations somewhat more complicated. The temperature distribution is found to be

Fig. 18 - Correction Factor for Large Maximum Temperature Rise
in Ideal Non-Uniform Windings



$$T(x) = \frac{w^2}{u^2}\left\{\frac{[\sin(u \log_e b) - \sin(u \log_e a)]\cos(u \log_e x)}{\sin(u \log_e \frac{b}{a})} + \frac{[\cos(u \log_e a) - \cos(u \log_e b)]\sin(u \log_e x)}{\sin(u \log_e \frac{b}{a})} - 1\right\}$$

The point of maximum temperature, x_m, is given by the solution of the following equation:

$$\left[\sin(u \log_e b) - \sin(u \log_e a)\right]\sin(u \log_e x_m) = \left[\cos(u \log_e a) - \cos(u \log_e b)\right]\cos(u \log_e x_m)$$

Thick, Short, Hollow, Cylinder

α) SMALL TEMPERATURE RISE.

1. Uniform Winding. For a coil with ends held at temperature zero, Poisson's equation (26) must be solved in two dimensions since heat may flow in an axial direction as well as radially outward.

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{x}\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial z^2} = -\frac{Q}{K}$$

The solution of this equation in closed form is not usually given because separation of the variables is impossible. However, M. Jakob and K. Homburg [20]) have listed some data relating the average and maximum temperatures of the winding. In the notation of this paper, Jakob [3]) gives the following table for windings shaped like flat plates:

$L/(b-a)$	=	∞	10	5	2.5	1.5	1.0
$\bar{T}/T_m$	=	2/3	.625	.58	.52	.485	.475

The complete temperature distribution for any specific case may be obtained by an application of the relaxation method or similar approximating device used in the solution of partial differential equations.

2. Non-uniform Winding. The fraction Q/K takes the form

$$\frac{Q}{K} = \frac{\rho \quad A^2}{\alpha \; K \; L^2 \; (\log_e \frac{b}{a})^2} \cdot \frac{1}{x^2}$$

for ideal non-uniform windings; thus Poisson's equation may be written

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{x}\frac{\partial T}{\partial x} + \frac{k}{x^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{48}$$

where k is a constant. In this equation, too, separation of the variables is impossible; hence, an approximating device must be used to obtain a solution.

Since an experiment was performed to check the temperature distribution in a non-uniform winding (see section V), we shall compare the experimental results with those determined by the relaxation method as follows.

The difference equation corresponding to the differential equation (48) is given by

$$T_{i,j} = \frac{1}{4 + \frac{h}{x}} \left[(1 + \frac{h}{x}) \; T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} + k \frac{h^2}{x^2} \right] \tag{49}$$

where i is the index of the gridpoints in the radial (x) direction, and j the index in the axial (z) direction, with h as the separation of the gridpoints. The grid of experimental data is shown in Fig. 21, and it is possible to show that the experimental results agree fairly well with the difference equation (49). It is thus evident that the relaxation method yields the correct representation of the temperature field, although in many cases the amount of time spent on the theoretical solution might be used better in performing experiments on the physical system.

β) LARGE TEMPERATURE RISE.

The only difference between the equations for small and large temperature rise is in the form taken by Q, which is now to be considered as a function of T. The partial differential equation in this form is still non-homogeneous; hence, separation of the variables is not possible. Special cases can be solved again by the relaxation method, but there are probably too few applications to warrant working out a sufficient number of examples to plot the results that could be useful in general for either uniform or non-uniform windings.

C. THE MINIMUM TEMPERATURE PROBLEM

Continuous Solution

In section III A the minimum resistance problem was stated and solved. It was found that when the wire size is a certain function of the radius, a winding is obtained which has a lower resistance than any other winding of the same size and same number of ampere-turns.

The question now arises whether it is possible to determine a functional relationship between wire size and radius which will make the internal temperature rise smaller than that of any other winding of the same size and same number of ampere-turns.

We shall discuss this problem for the thick, long, hollow cylindrical winding, with boundary conditions a). To this winding we may apply equation (40). The discussion of the data given by Moore (page 35) assures us that the thermal conductivity K is a constant if wire sizes with the same space factor are used. From equation (24) it is seen that Q depends on x implicitly through the function $s = s(x)$. Substituting the value of Q from equation (24) and performing the first integration on equation (40), we find

$$\frac{dT}{dx} = - \frac{\rho I^2}{\alpha K x} \int x s^2 \, dx + \frac{C'}{x}$$

Applying the condition that the inside face of the winding is thermally insulated, we obtain the particular solution of the temperature gradient

$$\frac{dT}{dx} = \frac{\rho I^2}{\alpha K x} \left[G(a) - G(x) \right]$$

where $G(x) = \int x s^2 \, dx$

The second integration gives the temperature distribution after a constant of integration has been determined from the boundary conditions:

$$T(x) = \frac{\rho I^2}{\alpha K} \left[\int_x^b \frac{G(x)}{x} \, dx - G(a) \log_e \frac{b}{x} \right]$$

From this solution, the maximum temperature rise is found to be

$$T_m = T(a) = \frac{\rho I^2}{\alpha K} \left[\int_a^b \frac{G(x)}{x}\, dx - G(a) \log_e \frac{b}{a} \right]$$

As a matter of incidental interest, we may ask what the function $s(x)$ must be in order to make this maximum temperature rise an unconditional minimum. A simple calculation shows that the only solution would be $s = 0$; i.e. an infinitely large wire diameter.

To include the condition that the number of ampere-turns is to be A, we make use of Lagrangian multipliers again to write

$$T_m = \frac{\rho I^2}{\alpha K} \left[\int_a^b \frac{G(x)}{x}\, dx - G(a) \log_e \frac{b}{a} \right] + \lambda A - \lambda I L \int_a^b s\, dx$$

By applying the methods of the calculus of variations already justified in section III A, it is found that the following equation must be satisfied by the function $s(x)$ as a necessary condition:

$$(50) \qquad \int_a^b \frac{1}{t} \int^{t} x\, s(x)\, dx\, dt - \left(\log_e \frac{b}{a}\right) \int^{a} x\, s(x)\, dx = \frac{\lambda \alpha K L (b-a)}{2 \rho I}$$

Whether any solutions of this equation exist has not been determined so far.

D. TEMPERATURE DISTRIBUTION IN MULTIPLE WINDINGS

Two Wire Sizes

In section III B it was found that the ideal continuous solution can be approximated with good accuracy by a discontinuous step-function (Fig. 2) in the minimum resistance problem.

Since the search for the continuous function $s(x)$ which satisfies equation (50) is proving to be a difficult task, it may be more fruitful to investigate the possibility of finding step-function approximations for the temperature problem before the continuous solution has been established.

As a practical digression, we shall discuss later the ventilated coil, in which the effect of the temperature rise has been separated from the magnetic properties of the winding by local cooling. Any method of reducing the temperature in the coil, by radial or axial vents, or by cooling from the ends, will naturally serve to loosen the ties between temperature and magnetic effect: As was stated on pages 3 and 4, we could produce powerful magnets very easily if it were possible to carry away the Joule heat.

The most extreme condition that is liable to be found in practical cases is the long coil which is cooled only at the outer surface; that is, the coil's core and ends are thermally insulated. This set of boundary conditions will be applied to a thick winding on a rectangular core and to a thick winding on a cylindrical core. In the former, the various sections of the coil, wound of different wire sizes, possess temperature distributions like those in thick, flat plates.

The simplest case for which a step function may be a solution is a winding made up of two sections or cells, the one next to the core with s turns per square inch, and the outer section with

s/v turns per square-inch, where v is the size ratio as defined in equation (20).

We can determine the temperature distribution within each section by the use of equations of the type derived in section III B if the boundary conditions for each section were known.

Thus, the general procedure indicated for finding the temperature distribution in the entire coil is to set up the particular solutions for the two sections and then to connect them by virtue of the fact that the temperature is a continuous function and the rate of heat flow must be continuous across the boundary between two sections; that is, T as well as dT/dx is the same at points in the immediate neighborhood on both sides of a boundary.

For a thick plate we have the temperature distribution given by equation (28)

$$T(x) = -\frac{\rho I^2 s^2}{2 \alpha K} x^2 + C' x + C''$$

and the temperature gradient

$$\frac{dT}{dx} = -\frac{\rho I^2 s^2}{\alpha K} x + C'$$

For the inner section, the temperature gradient is zero at the boundary $x = a$. Thus constant C_1' is immediately determined, and we can write the expression for both the temperature and the temperature gradient at the boundary $x = x_1$:

$$T(x_1) = \frac{\rho I^2 s^2}{2 \alpha K} (2 a x_1 - x_1^2) + C_1''$$

$$\left(\frac{dT}{dx}\right)_{x_1} = -\frac{\rho I^2 s^2}{\alpha K} (x_1 - a)$$

Since temperature and heat flow across the boundary $x = x_1$ are continuous, we can express these quantities in terms of the constants of the outer section:

$$T(x_1) = -\frac{\rho I^2 s^2}{2 \alpha K v^2} x_1^2 + C_2' x_1 + C_2''$$

$$\left(\frac{dT}{dx}\right)_{x_1} = -\frac{\rho I^2 s^2}{\alpha K v^2} x_1 + C_2'$$

These equations serve to determine C_2' explicitly and C_1'' in terms of C_2''

$$C_2' = \frac{\rho I^2 s^2}{\alpha K} \left(\frac{x_1}{v^2} - x_1 + a \right)$$

$$C_1'' = C_2'' - \frac{\rho I^2 s^2}{2 \alpha K} \left(x_1^2 - \frac{x_1^2}{v^2} \right)$$

Now we make use of the condition that the temperature at the outside $x = b$ is zero, to determine C_2''

$$C_2'' = \frac{\rho I^2 s^2}{2 \alpha K} \left(\frac{b^2}{v^2} + 2x_1 b - 2ab - \frac{2x_1 b}{v^2} \right)$$

Thus, we can find C_1'' and have therefore a complete knowledge of the temperature distribution in the two-section winding.

In this discussion we are mainly interested in the maximum temperature rise, which is the temperature at the inside face of the winding. It is noteworthy that we cannot find this temperature unless C_1'' is known, which in turn depends upon the boundary condition at the outside face. If we were to follow through the step-by-step solution as above for a coil of many sections, the work involved in the computations would become too cumbersome to be presented here in detail. However, since we are only interested in the maximum temperature, the general case of n sections can be expressed in the formula

$$T_m = \frac{\rho I^2 s^2}{2 \alpha K} \left[a^2 - 2ab + \frac{b^2}{v^{2(n-1)}} + (v^2 - 1) \sum_{k=1}^{n-1} \frac{2 b x_k - x_k^2}{v^{2k}} \right]$$

For the case of the two-section winding (n=2), this becomes

$$(51) \qquad T_m = \frac{\rho I^2 s^2}{2 \alpha K} \left[(b - a)^2 + \frac{1 - v^2}{v^2} (b - x_1)^2 \right]$$

If $v = 1$ (uniform winding), T_m reduces to the value derived in equation (30). It is readily apparent that $v > 1$ results in a smaller value of T_m, and $v < 1$ in a greater value of T_m than for the uniform winding. This condition is to be expected inasmuch as $v > 1$ represents a wire of lower resistance per unit length in the outer section; hence, less power expended in the entire coil for the same current.

In order to examine the relation between the maximum temperature rise and the magnetic effect of the winding, we make use of the expression for the number of ampere-turns of a coil wound in two sections:

$$(52) \qquad A = I\,L\,s\,(x_1 - a + \frac{b - x_1}{v})$$

If A is to be held constant, then x_1 and v must satisfy the relation established by (52), hence we may eliminate either x_1 or v between equations (51) and (52), with the result that T_m can be found as a function of x_1:

$$(53) \qquad T_m = \frac{\rho\, I^2\, s^2}{2\,\alpha\, K}\left[2(x_1 - a)(b - a - \frac{A}{ILs}) + (\frac{A}{ILs})^2\right]$$

When we try to determine what value of x_1 yields the lowest maximum temperature rise, we arrive at the conclusion that the minimum value of T_m occurs at one of the endpoints of the interval $a \leq x_1 \leq b$. In practical terms, this means that a single wire size will always give the lowest maximum temperature rise for a long coil wound on a rectangular core.

It may still be desirable to use two different wire sizes in a coil, perhaps to avoid having a special wire size manufactured for a particular application. As has been stated qualitatively by Moore [21]), it is best to wind the thick wire near the core in order to have a lower maximum temperature. Equation (53) may be employed to determine quantitatively the difference in maximum temperature to be expected when the thin wire is near the core as compared with the winding which has the thick wire near the core. For a winding on the flat part of a rectangular core, the ampere-turns as well as the power loss are the same whether the thin or the thick wire is placed near the core; hence, for this case we may choose the configuration of lowest maximum temperature without affecting the power loss or number of ampere-turns in any way.

We find that the difference of the two maximum temperature rises is given by the following expression

$$(54) \qquad T_m - T_{mi} = \frac{\rho I^2 s^2}{\alpha K v^2} (b - x_1)(x_1 - a)(v^2 - 1)$$

where T_m is the maximum temperature rise in the two-section winding as computed in other cases in this paper, and T_{mi} is the maximum temperature rise in a winding obtained by interchanging the two sections. It is obvious from this equation that the temperature rise is greater when the thin wire is wound near the core ($v > 1$), as would be expected.

For a winding which is ventilated equally inside and outside, the maximum temperature rise is of course the same whether the sections are interchanged or not; and for a winding which is cooled at its ends, the temperature rise is less than for one not cooled in that manner. Hence equation (54) may be used as the criterion for the upper limit of temperature rise difference to be expected when the sections are interchanged.

The ratio of the magnitudes of these maximum temperatures illustrates perhaps a little better what the effect of interchanging the two sections is. This ratio of the temperature rises is given by the following expression

$$\frac{T_{mi}}{T_m} = \frac{(b-a)^2 + (x_1 - a)^2 (v^2 - 1)}{(b-a)^2 v^2 - (b - x_1)^2 (v^2 - 1)}$$

For the case in which the two sections are of equal thickness, that is $x_1 = (b+a)/2$, the formula reduces to the following value

$$\frac{T_{mi}}{T_m} = \frac{v^2 + 3}{3 v^2 + 1}$$

If $v = 1.5$, this ratio is $1/1.48$ or approximately $2/3$. Thus, the effect of interchanging the two sections is to reduce the maximum temperature rise by a factor of 1.48. It is readily apparent that for large values of v, the temperature reduction obtained by interchanging two sections of equal thickness on a rectangular core approaches the factor 3.

We can expect the cylindrical winding to have somewhat different properties, because an interchange of the sections does alter the power loss in the coil if the number of ampere-turns is held constant. In particular, if the thin wire is near the core, then the total power consumed by the coil is less than when the thick wire is near the core. Thus we may expect to find less difference in the maximum temperatures when the sections are interchanged; and under certain conditions, the winding with thin wire near the core may be the cooler one.

Proceeding as before, with the exception that the differential equation now is given by (40), we find the maximum temperature rise at the core of the two-section winding as a function of v and x_1:

$$(55)\quad T_m = \frac{\rho I^2 s^2}{4\alpha K}\left[\frac{b^2 - x_1^2}{v^2} + x_1^2 - a^2 + 2x_1^2\frac{v^2-1}{v^2}\log_e\frac{b}{x_1} - 2a^2\log_e\frac{b}{a}\right]$$

The number of ampere-turns is the same as in equation (52). Hence, eliminating v between (52) and (55), the maximum temperature rise as a function of x_1 is given by

$$(56)\quad T_m = \frac{\rho I^2 s^2}{4\alpha K}\left\{\frac{\left[\frac{A}{ILs} - x_1 + a\right]^2\left[b^2 - x_1^2\left(1 + 2\log_e\frac{b}{x_1}\right)\right]}{(b - x_1)^2} + x_1^2\left(1 + 2\log_e\frac{b}{x_1}\right) - a^2\left(1 + 2\log_e\frac{b}{a}\right)\right\}$$

The derivative of this expression with respect to x_1 yields a complicated equation when set equal to zero, and we shall not attempt to solve it for the value of x_1 which results in the lowest maximum temperature rise. However, we may use equation (56) to find the difference in the maximum temperature rises when the sections of the winding are interchanged. Here, too, we are lead to very unwieldy equations, but by making the assumption that the two sections are of equal thickness, we can derive the following formula:

$$T_m - T_{mi} = \frac{\rho I^2 s^2}{8 \alpha K} \cdot \frac{v^2 - 1}{v^2} \left[(a+b)^2 \log_e \frac{2b}{a+b} - 4a^2 \log_e \frac{b}{a} - (b-a)^2 \right]$$

The ratio of T_{mi} to T_m has also been computed for the special case of two sections of equal thickness. It is given by the following equation:

$$\frac{T_{mi}}{T_m} = \frac{4 v^2 r^2 - (r+1)^2 (v^2 - 1) \left[1 + 2 \log_e \frac{2r}{r+1}\right] - 4 \left[1 + 2 \log_e r\right]}{4 r^2 + (r+1)^2 (v^2 - 1) \left[1 + 2 \log_e \frac{2r}{r+1}\right] - 4 v^2 \left[1 + 2 \log_e r\right]}$$

where $r = b/a$ is the thickness ratio of the winding.

As a numerical example shows, for $v = 1.5$, and a thickness ratio of $r = 3$, interchanging the sections of the winding reduces the maximum temperature at the core by the factor 1.36, if each of the sections occupies half of the available cross-section of the winding. This is smaller than the factor found for a similar interchange on a rectangular core (page 72), as was expected.

Ventilated Winding

If one has no particular applications in mind, it is only possible to derive solutions of a rather general nature, as those found above. Possibly one application arising frequently among the cases that have not been discussed so far is the ventilated coil [14]).

We assume that a winding is built up of a series of thin, long, hollow cylindrical sections of several diameters, one inside another (see Fig. 12). The spaces between the cylinders, the ventilating ducts, are to be filled with circulating fluid so that the temperature may be assumed constant at both inner and outer surface of each section. Since it is assumed that the individual cylinders are thin and long, the solution to this case applies equally well to a ventilated winding on a rectangular core (i.e. the thick plate solution).

The design criterion arises from the following considerations: There exists a maximum safe temperature to which the winding may be heated. For maximum efficiency, each section of the winding should be heated to that temperature — if it remained cooler, more wire could be put into the available space and thus the magnetic effect could be increased, if the temperature would rise above the maximum safe limit, the winding might be harmed.

As an example, we will assume that a winding of four sections is to be constructed, with wire sizes that are in geometric progression. Following the notation of Fig. 12, and observing that equations (24) and (30) apply to each section, we can write four equations

$$T_m = \frac{\rho I^2 s_1^2}{8 \alpha K} (x_1 - a)^2 \qquad T_m = \frac{\rho I^2 s_2^2}{8 \alpha K} (x_2 - x_1 - d)^2$$

$$T_m = \frac{\rho I^2 s_3^2}{8 \alpha K} (x_3 - x_2 - d)^2 \qquad T_m = \frac{\rho I^2 s_4^2}{8 \alpha K} (b - x_3 - d)^2$$

These equations contain the seven unknowns x_1 x_2 x_3 s_1 s_2 s_3 s_4. With the assumption that the wire sizes are in a geometric progression of known ratio v, we can write three more equations, thus completing the set of the required seven equations. The width of the ventilating ducts d is assumed known, being defined by the flow requirements of the cooling fluid. The solution of the set of equations is then given by the following formulas:

$$x_1 = \frac{(v^3 + v^2 + v)\, a + b - 3\, d}{(v + 1)\,(v^2 + 1)}$$

$$x_2 = \frac{(v^3 + v^2)\, a + (v + 1)\, b + (v^3 + v^2 - 2\, v - 2)\, d}{(v + 1)\,(v^2 + 1)}$$

$$x_3 = \frac{v^3\, a + (v^2 + v + 1)\, b + (2\, v^3 - v^2 - v - 1)\, d}{(v + 1)\,(v^2 + 1)}$$

$$s_1 = \frac{2\,(v+1)\,(v^2+1)}{I\,(b-a-3\,d)}\sqrt{\frac{2\,\alpha\,K\,T_m}{\rho}}\;;\quad s_2 = \frac{s_1}{v}\;;\quad s_3 = \frac{s_1}{v^2}\;;\quad s_4 = \frac{s_1}{v^2}\,.$$

For a ventilated winding composed of just two sections, the solution is given by the following equations:

$$x_1 = \frac{v\,a+b-d}{v+1}\;;\quad s_2 = \frac{s_1}{v}\;;\quad s_1 = \frac{2\,(v+1)}{I\,(b-a-d)}\sqrt{\frac{2\,\alpha\,K\,T_m}{\rho}}\;.$$

These formulas give the winding schedules that insure the same maximum temperature rise in each section, but nothing has been said about the relation between magnetic effect and maximum temperature rise.

Assuming that the duct width d is unalterable, we still have a choice in selecting the ratio of the wire sizes v. If v is equal to one, the sections of the winding are all wound with the same wire size, $v > 1$ represents the case of thicker wire for the outer sections, and $v < 1$ means that thinner wire is used in the outer sections.

When we try to compute the number of ampere-turns of the winding, we find that it is independent of the wire size, being given by equation (36) for each of the sections of the winding. Since the values of T_m are the same for all the sections of the winding, the values of A must be the same also; hence, the total number of ampere-turns of the winding is equal to the value of A found from equation (36) multiplied by the number of sections of the winding.

The fact that the equations are independent of the wire size leaves us free to choose any size we wish, and we are therefore able to make use of the results obtained in section III of this paper; that is, we can select the wire sizes which make the resistance of the winding a minimum for a given number of turns.

Probably the simplest way to obtain the winding schedule in its final form is to work with the design criterion that every section of the winding should have the same resistance.

V. EXPERIMENTS

Introduction

The practical value of the equations presented in this paper rests on the validity of the assumptions made in their derivations. The logic of the mathematical steps taken is simple to check; and unless errors have slipped in, there is little doubt that the equations obtained in section IV are the correct solutions of Poisson's equation with the appropriate boundary conditions.

A large number of assumptions must be made in order to connect the mathematical structure with an existing physical system, both in the premises and in the results, and these connecting links are not sharply defined, being broadened by limits of accuracy of the experimental measurements.

There is, of course, little doubt that Poisson's equation applies to the flow of heat, even in the heterogeneous medium represented by a coil wound of insulated wire, so that if results are obtained which do not agree with the theory, the investigation of the cause for this discrepancy should be focused on the manner in which the mathematical structure is linked to the physical system.

Consider, for example, the quantity Q. The heat generated per unit volume per unit time in the winding as given by equation (24) depends upon the current, the number of turns in the unit volume, the resistivity of the metal in the wire, and the space factor.

An ammeter measures the current with a maximum accuracy of 0.5 °/o, thus the square of the current is known within 1 °/o. The number of turns per square-inch of winding cross-section depends for its accuracy on the count of the turns and on the measurements of inside and outside diameters and length of the winding. Since the outside surface of a layer-wound winding is not circular in cross section, an average value of the diameter must be chosen, thus giving the value of s with a probable accuracy of not more than 1 %. The resistivity may be assumed to be that of high-grade copper, with a value of ρ of 0.669 microhm-inch at 20 °C. α is the ratio of metal cross-section to total cross-section, which depends again on the length and diameters of the coil, as well as on the assumption that the wire has the correct AWG diameter, and is circular in cross section. In view of the fact that some stretching of the wire takes place in winding a coil, it is likely that the value of α is off by a few percent, and that even the value of ρ might be affected by the cold-working of the metal.

Thus the quantity Q alone cannot be determined very accurately — and it is among the variables considered adequately defined in this work. The other variables which are defined with considerable accuracy are measurements of length, and measurements of temperature by thermocouples. The assumptions made in adopting the various boundary conditions have already been discussed on pages 36 and 38.

The weakest link appears to reside in the average thermal conductivity K. This quantity depends on the materials used in the winding as well as on their distribution in space, and one of the purposes of the experiments performed at the Applied Science Laboratory was to determine the range of K under several conditions — with the assumption that all the other variables were controlled to a reasonable degree, and that the theoretical work done was correct.

Temperature Distributions

In order to establish that the theory does give correct results, it was decided to check the temperature distribution in a winding.

Coil No. 1 is a thin, long, hollow cylindrical winding. It consists of 1391 turns of AWG No. 26 Formex insulated wire in ten layers wound upon an aluminum tube $2\frac{1}{4}$" in diameter. The winding is 2.60" long and

0.161" thick. Thus, the thickness ratio is $r = 1.13$, which is sufficiently close to 1.00 for a winding with boundary conditions b) so that the temperature distribution is essentially the same as in a flat plate.

Nine thermocouples are placed in the winding, one each between successive layers. The couples are located at different distances from the ends of the winding in order that the temperature distribution in the coil will not be disturbed too much near the points where the temperature is measured. The thermocouples are made of 0.001" copper and 0.001" constantan wire. The soldered junctions average 0.0045" in diameter, the smallest being 0.003", and the largest 0.005", except one, which measures 0.008" in diameter.

This coil was tested with a current of 1.80 amperes at a surface temperature of 66.8 °C in a kerosene bath. The temperature distribution indicated by the nine thermocouples is shown in Fig. 19. The curve drawn in Fig. 19 is the best (least squares) fit to the experimental data.

Since the internal temperature rise was small, no correction factor was applied. Since the space factor of the winding is $\alpha = 0.660$, and the resistivity of copper at 66.8 °C is $\rho = 0.802$ microhm-inch, the average thermal conductivity of Coil No. 1 is found from equation (36) to be $K = 0.0242$ watts/°C-inch.

In section II it was found that the optimum number of turns was approximated rather closely by a few changes of wire size. The same approximation appears to hold for the temperature distribution also, as is shown by the measurements on Coil No. 2:

Coil No. 2 is a thick, short, hollow cylindrical winding with the following specifications:

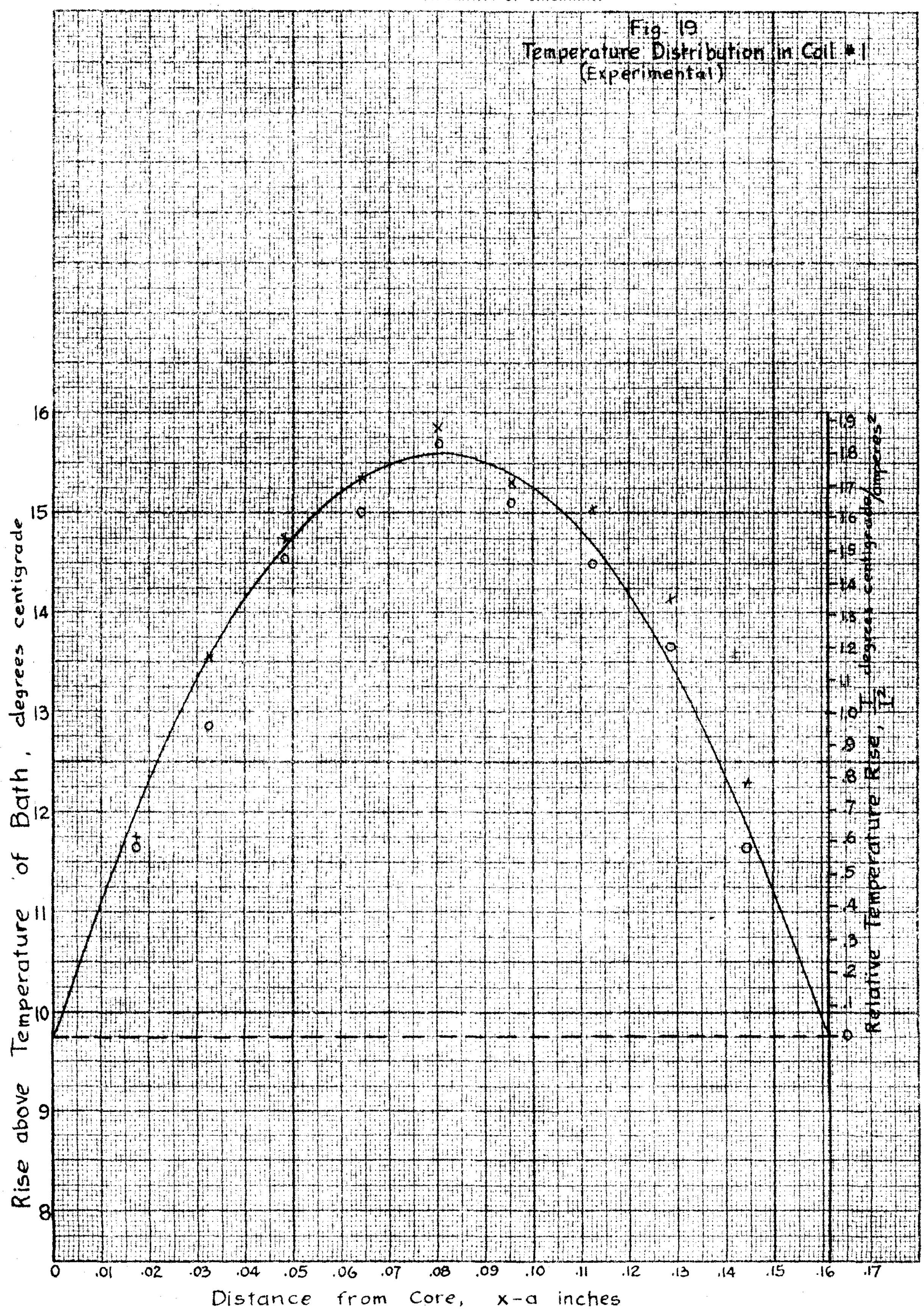


Fig. 19
Temperature Distribution in Coil #1
(Experimental)
Rise above Temperature of Bath, degrees centigrade
Relative Temperature Rise, $\frac{T}{I^2}$ degrees centigrade/amperes^2
Distance from Core, x−a inches
16
15
14
13
12
11
10
9
8
0
.01
.02
.03
.04
.05
.06
.07
.08
.09
.10
.11
.12
.13
.14
.15
.16
.17

$a = 0.376"$ $b = 2.060"$ $r = 5.48$ $L = 2.00"$

$N_1 = 307$ turns AWG No. 20 Formex insulated wire,

$N_2 = 348$ turns AWG No. 18 Formex insulated wire,

$N_3 = 352$ turns AWG No. 16 Formex insulated wire,

$N_4 = 316$ turns AWG No. 14 Formes insulated wire

$N = 1323$ turns.

$x_1 = 0.565"$

$x_2 = 0.880"$

$x_3 = 1.368"$

36 iron-constantan thermocouples (AWG No 30) are placed in the winding, spaced equally in both radial and axial directions in a plane containing the axis of the winding.

The coil was tested with currents of 1.5 and 2.5 amperes while thermally insulating wooden ends and core maintained heat flow in the radial direction outward only. There was some heat flow in the axial direction as indicated by a slightly lower reading of the outer thermocouples. The points plotted in Fig. 20 represent the maximum temperature in each row of six thermocouples.

The curve shown in Fig. 20 is the best (least squares) fit to the experimental data indicated by crosses, according to equation (44). It corresponds to the value $K = 0.0406$ watts/°C-inch since the space factor of the coil is $\alpha = 0.704$, and the resistivity of the copper at the surface temperature 30 °C during the 1.5 ampere test was $\rho = 0.697$ microhm-inch.

For the large temperature correction, a factor of 1.07 should be applied in order to give the proper value of the maximum temperature rise. This is seen to agree with the correction factor given in Fig. 18 at a temperature rise of about 25 °C.

Coil No. 2 was operated at 2.0 and 3.5 amperes as a short winding, that is, with ends and outside surface at bath temperature.

Fig. 21 shows the values of T/I^2 as indicated by the thermocouples. The figure above the line at each gridpoint was obtained with a current

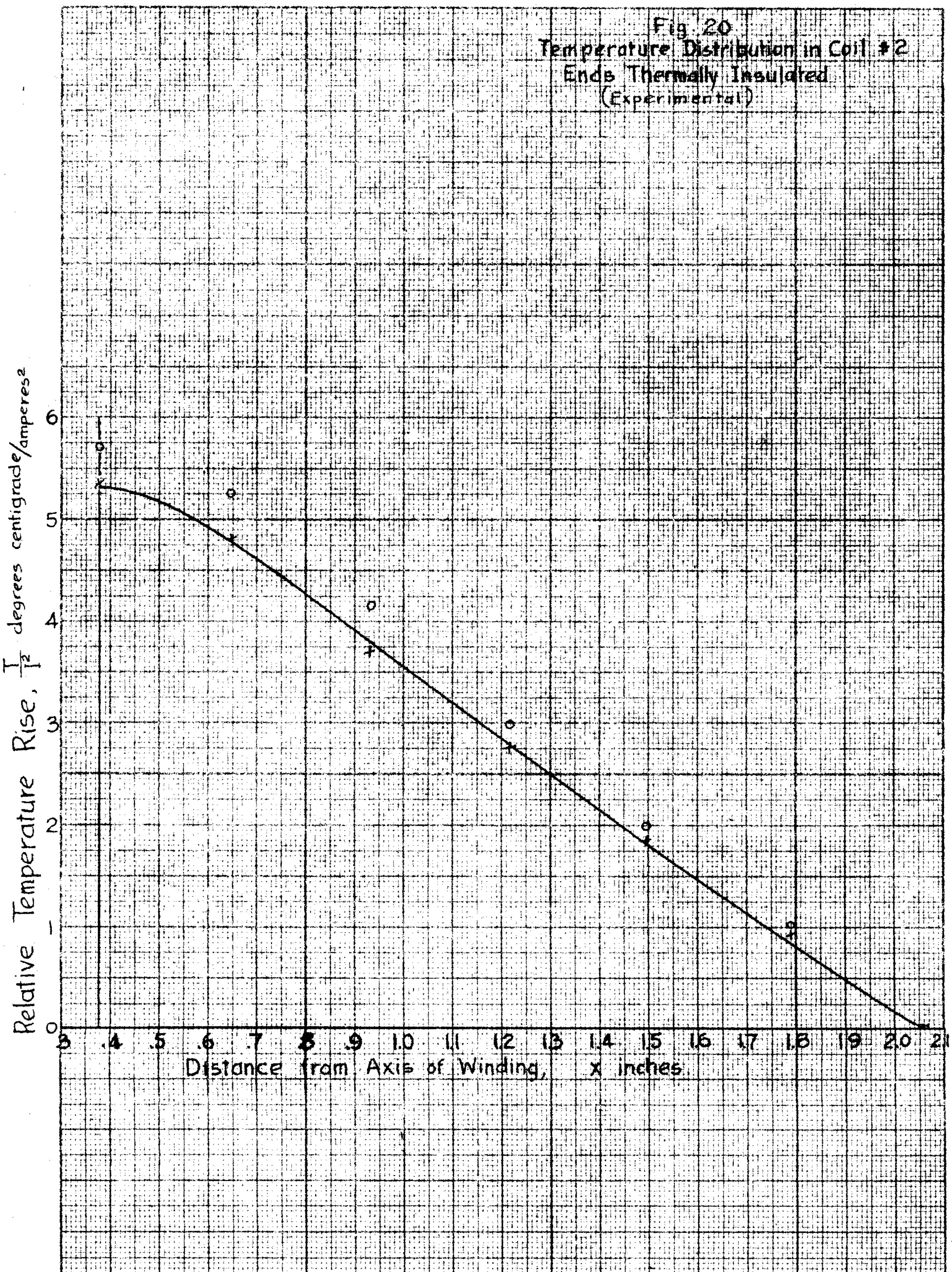


Fig. 20
Temperature Distribution in Coil #2
Ends Thermally Insulated
(Experimental)
Relative Temperature Rise, $\frac{T}{I^2}$ degrees centigrade/amperes^2
0
1
2
3
4
5
6
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.1
Distance from Axis of Winding, x inches

Fig. 21
Temperature Distribution in Coil #2
Ends at Bath Temperature
(Experimental)

j							
8	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0
7	3.19 / 2.74	2.68 / 2.38	1.97 / 1.70	1.31 / 1.13	.90 / .78	.40 / .34	0 / 0
6	4.26 / 3.61	3.60 / 3.06	2.68 / 2.31	1.77 / 1.56	1.16 / 1.04	.58 / .48	0 / 0
5		4.06 / 3.45	3.06 / 2.63	2.16 / 1.86	1.35 / 1.17	.68 / .58	0 / 0
4	4.44 / 3.73	4.08 / 3.45	3.06 / 2.63	2.16 / 1.86	1.38 / 1.17	.69 / .58	0 / 0
3	4.15 / 3.54	3.62 / 3.10	2.73 / 2.32	1.97 / 1.63	1.23 / 1.02	.59 / .50	0 / 0
2	3.15 / 2.75	2.67 / 2.30	2.11 / 1.75	1.61 / 1.26	.97 / .85	.44 / .39	0 / 0
1	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0
i =	1	2	3	4	5	6	7
x ≈	.376	.662	.948	1.234	1.520	1.806	2.092

h = .286

of 3.5 amperes. The number below the line at each gridpoint corresponds to the 2.0 amperes test.

Since the maximum temperature rise during the 3.5 amperes test was about 50 °C, it is seen from Fig. 18 that a correction factor of 1.2 should be applied to the maximum temperature measurement. Comparison of the maximum temperature rises in Fig. 21 shows that the factor actually observed was 4.44/3.73 = 1.19. This shows that the general trend is predicted by the theory with fair accuracy.

Fig. 22 is a three-dimensional graph of the data obtained in the 2.0 amperes test, illustrating the general shape of the temperature field.

Having thus established the fact that the temperature distribution within a winding is represented accuractely by the theory for uniform and non-uniform windings, it is only necessary to check the maximum temperature rise in a winding for all further tests.

Average Thermal Conductivity

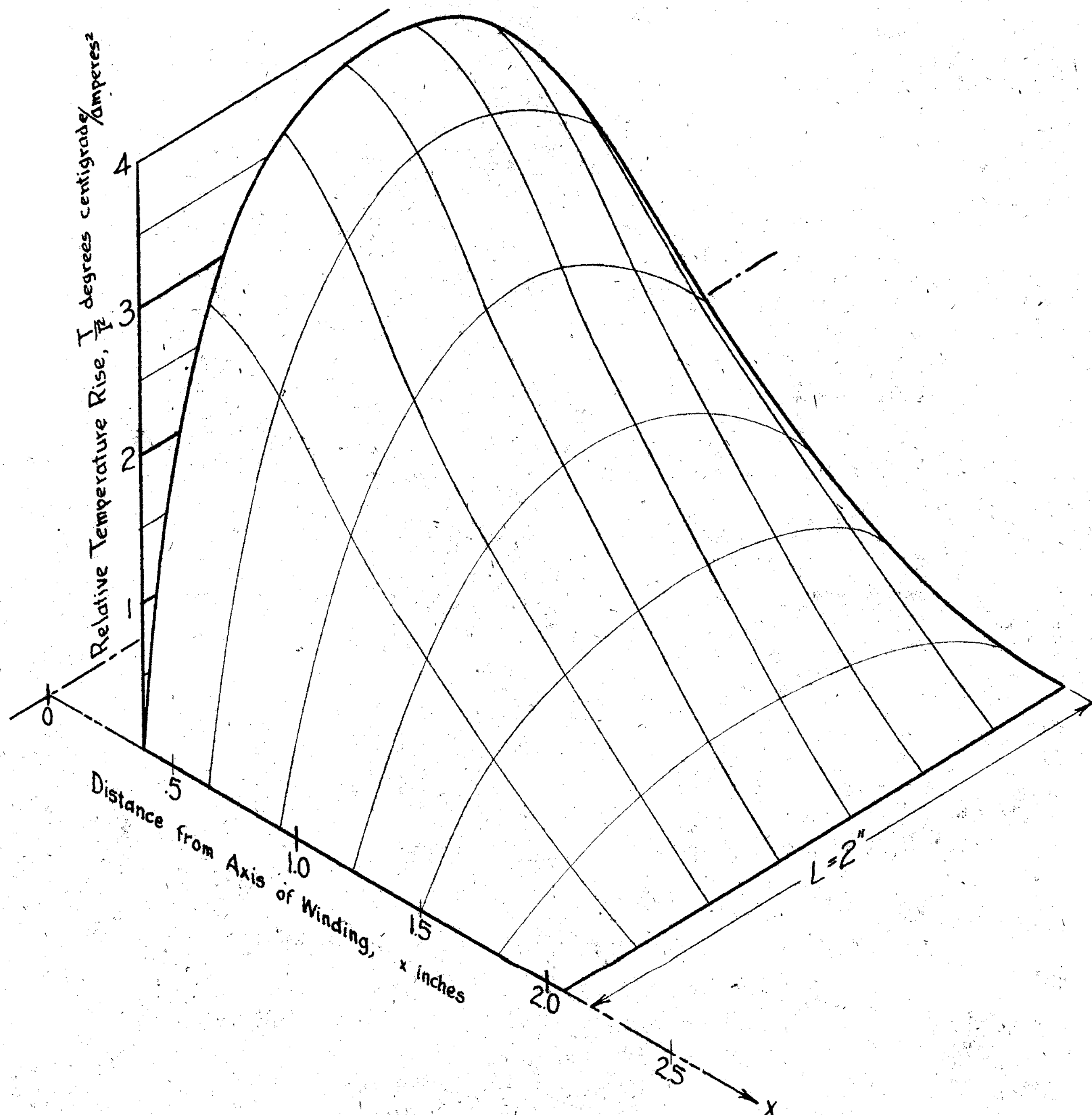
Coils No. 3, 4, and 5 were wound with only two thermocouple junctions in the winding - one at the point of maximum temperature, the other at the outer surface. The specifications of these coils are given in the following table:

Coil No	Wire Size AWG	a inch	b inch	L inch	N turns	r	α
3	20	.420	.815	2.00	672	1.94	.684
4	22	.245	.461	1.81	561	1.88	.720
5	22	.245	.555	1.69	755	2.26	.730

The coils are wound on wooden cores, with thick wooden ends, so that they satisfy boundary conditions a) for long, thick, hollow cylinders.

Fig. 23 shows the value obtained for the average thermal conductivity of these coils in a kerosene bath. The data indicated by circles and dots were obtained by solving equation (42) for K when the measured values of T_m are used. This is called the "uncorrected" set of data because

Fig. 22
Temperature Distribution in Coil #2
Ends at Bath Temperature
(Experimental)



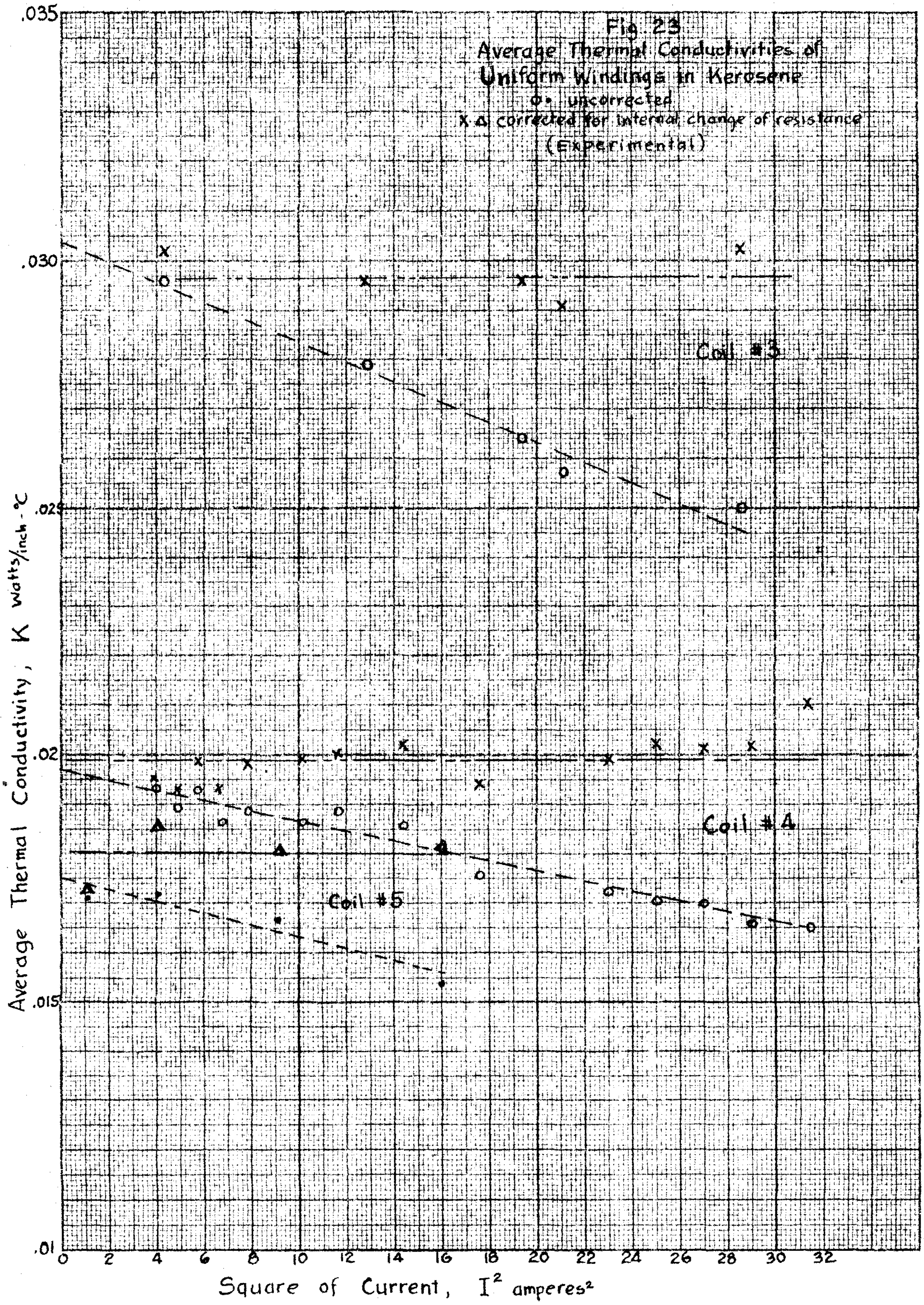


Fig. 23
Average Thermal Conductivities of Uniform Windings in Kerosene
o• uncorrected
x △ corrected for internal change of resistance
(Experimental)
Coil #3
Coil #4
Coil #5
.035
.030
.025
.02
.015
.01
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
Average Thermal Conductivity, K watts/inch-°C
Square of Current, I² amperes²

equation (42) is based on the theory of small temperature rise, and will be strictly applicable only at the origin $I^2 = 0$. The data indicated by crosses and triangles were obtained by substituting in equation (42) the measured values of T_m divided by the appropriate correction factor from Fig. 17 or Fig. 18. This "corrected" set of data is seen to yield an approximately constant value of K at all temperatures. In order to check the relation between thermal conductivity and temperature, an experiment was made with small internal temperature rise at various bath temperatures. The results, plotted in Fig. 24, confirm the theory that K is a constant when the coil is immersed in kerosene.

The scope of these experiments was limited by the fact that the heat distortion temperature of Vinyl Formal Resins is about 75 °C [32]; hence, no part of the winding was permitted to become heated above this temperature.

With the windings in air, the average thermal conductivity was computed from equation (42) with measured values of T_m substituted. The data, plotted in Fig. 25, indicate clearly that the thermal conductivity of the winding in air is not constant; hence, that equation (42) does not apply. As would be expected, the conductivity increases with increasing temperature [13]; however, the temperature coefficient of the average thermal conductivity of the coil is very difficult to predict; thus, it is not possible to substitute the correct function for $K = K(T)$ in Poisson's equation and obtain a solution. As was pointed out on page 46, even a linear relation between $1/K$ and T would result in elliptic functions.

Fig. 26 shows the experimental setup in the tests made on Coil No. 2. The thermocouple potentials were measured with a Leeds and Northrup Type K potentiometer, with a Weston 1 mV meter serving as a galvanometer (5 ohms internal resistance).

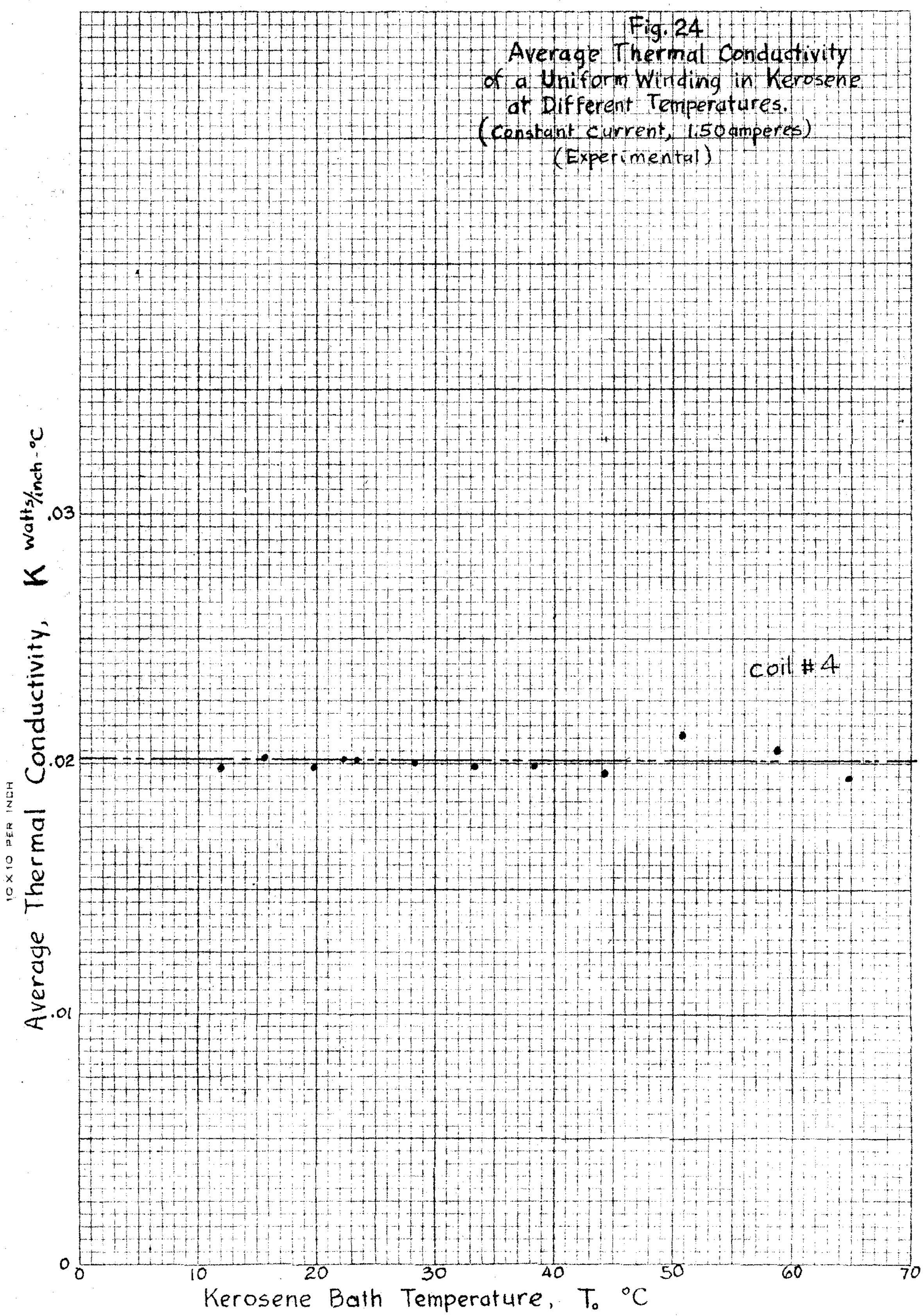


Fig. 24
Average Thermal Conductivity of a Uniform Winding in Kerosene at Different Temperatures.
(Constant current, 1.50 amperes)
(Experimental)
coil #4
Average Thermal Conductivity, K watts/inch-°C
.03
.02
.01
0
0
10
20
30
40
50
60
70
Kerosene Bath Temperature, T₀ °C
10 X 10 PER INCH

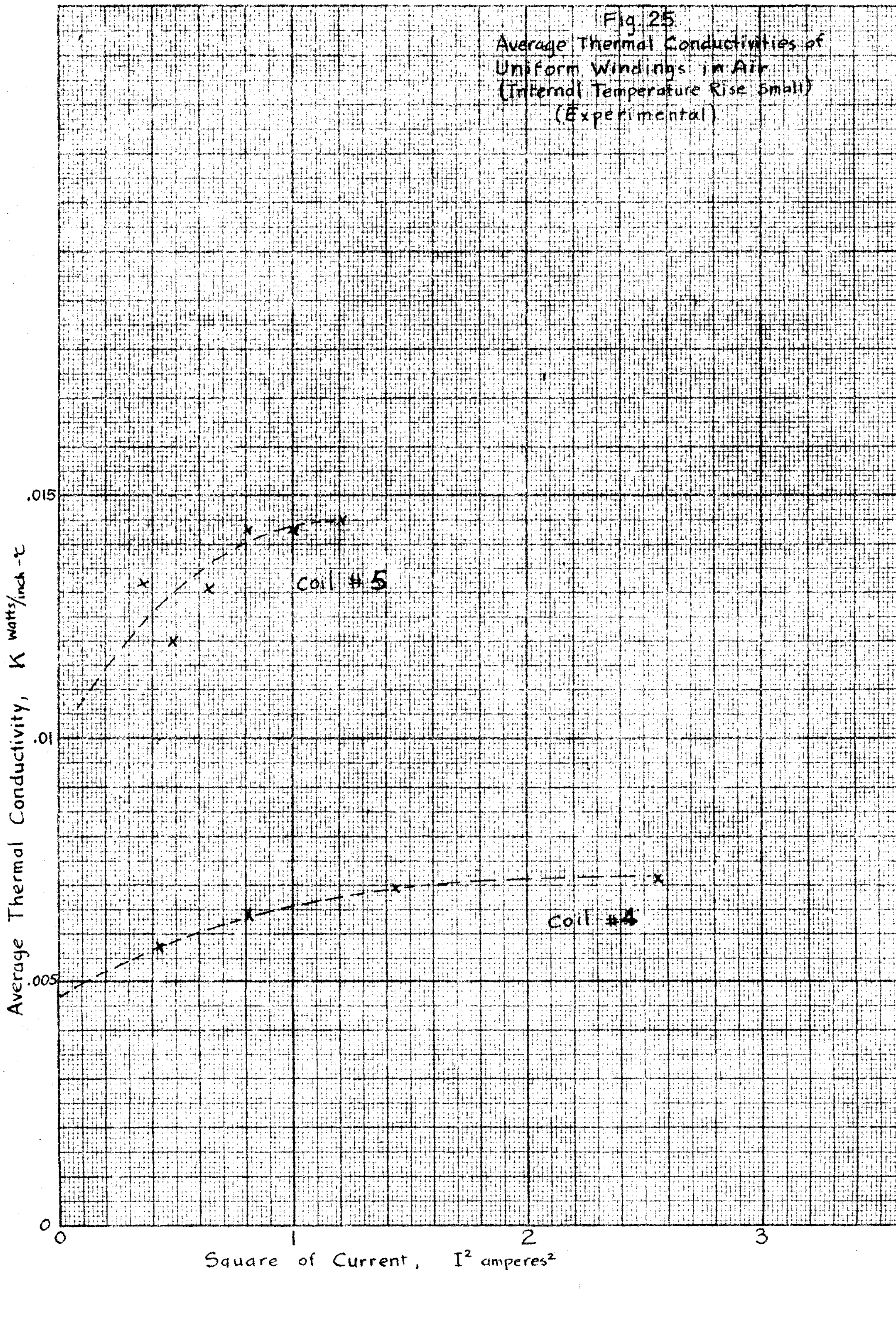
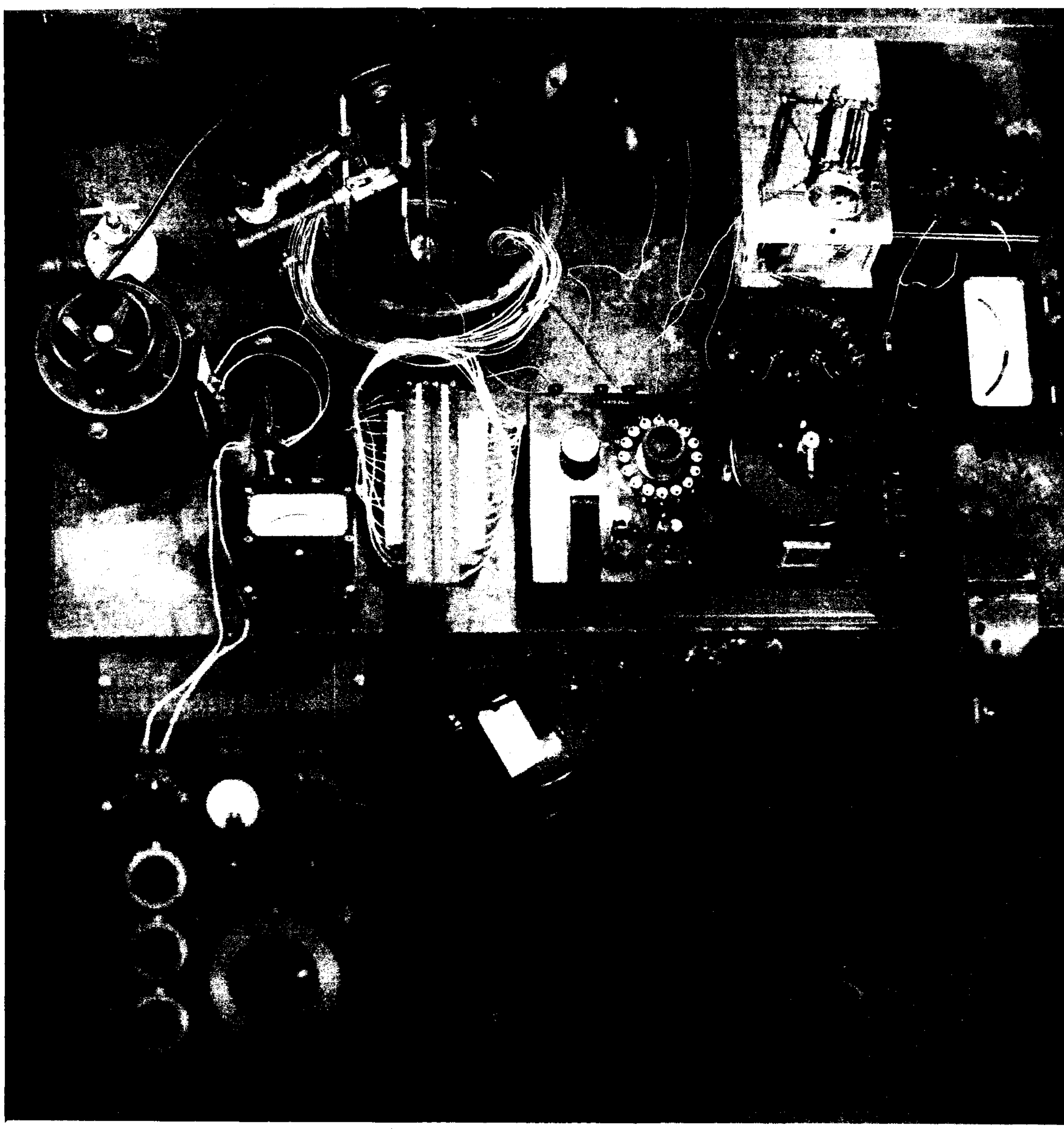


Fig. 25
Average Thermal Conductivities of Uniform Windings in Air
(Internal Temperature Rise Small)
(Experimental)
Coil #5
Coil #4
Average Thermal Conductivity, K watts/inch-°C
.015
.01
.005
0
0
1
2
3
Square of Current, I^2 amperes²

Fig. 26 - Experimental Set-up



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