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THE DETERMINATION OF ORBITS

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THE DETERMINATION OF ORBITS

Introduction

The use of modern, commercial, calculating machines for astronomical computations has introduced a transition from the older methods of logarithmic computation to methods that permit the most efficient use of the machines. With respect to the orbits of minor planets and comets, any improvement which will facilitate deriving the orbital elements and ephemerides is especially pertinent if the increasingly large numbers of newly discovered bodies are to be followed and observed.

The problem of orbit determination has held a conspicuous position in the development of both mathematics and astronomy. Newton's discovery of the law of universal gravitation and his development of the calculus provided the necessary mathematical equipment for solving this problem. His own methods of solution, however, were not highly satisfactory, as is implied by his statement, "This being a problem of very great difficulty, I tried many methods of resolving it." In collaboration with Newton, Halley computed the parabolic orbits of twenty four comets, by a "prodigious amount of labor." Improvements were made by Euler in 1744 and Lambert in 1761;

each discovered, independently, the equation of parabolic motion in terms of two radii, the chord, and the time interval. Lambert also gave the generalization to any conic section. The mathematical theory of the motion of celestial bodies was developed extensively by La Grange and La Place, tho neither greatly improved the actual methods for obtaining an orbit. In 1797 Olbers published his unsurpassed method for a parabolic orbit. It remained for Gauss, stimulated by the loss of the first minor planet, Ceres, in 1801, only a few months after its discovery, to produce a practical and general method of deriving an orbit from three observations. This splendid work clearly bears the imprint of his genius, and practically marks the end of an epoch in the history of orbit determination.

Within recent times Harzer and Leuschner have worked the La Placian method into a practical form; Cowell has introduced numerical integration of the coordinates for highly disturbed bodies, such as Halley's comet and the outer satellites of Jupiter; and Gibbs, using vectors, has given a valuable approximate formula for the triangle ratios. Current writers have, in various ways, transformed these methods in order to make them more effective for mechanical computing.*

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*The papers of G. Merton (Monthly Notices of R.A.S. 85,693), R. T. A. Innes (Monthly Notices of R.A.S. 89,422), and K. Stumpff (Astr. Nachrichten, 5828 and 5855) may be cited.

The present paper has a double purpose: to develop a method of orbit determination, especially adapted to machine computation; and to apply the method to the determination of an improved orbit of the minor planet, Biarmia (1146).

Part I

A New Method of Orbit Determination

The Fundamental Equation

The determination of the orbit of a celestial object moving under the sun's gravitational attraction involves the simultaneous satisfaction of two sets of conditions, the dynamical and the geometric. The geometric conditions provide that the theoretical positions of the object computed from the orbit shall agree with its observed positions on the celestial sphere. The dynamical conditions, which impose the law of gravitation, are contained in the vector equation*

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{-k^2(1 + m)\,\mathbf{r}}{r^3} \qquad (1)$$

*This equation, with a slight modification of notation, is given by Brand: Vectorial Mechanics, Art. 182. The heavily inked letters ($\mathbf{r}$) denote vectors.

Since this is a differential equation of the second order, two constants of integration must appear in the solution. The initial conditions, which determine the values of these constants, are obtained from the observed positions of the object, i.e. the geometric conditions.

The solution is readily obtained by means of Taylor's series. The nature of the problem leaves no question as to the continuity of the functions involved. Let the time interval be counted from some origin of time, t_1, in units of 1/k mean solar days (k is the Gaussian constant), and neglect m, the mass of the object relative to the sun. Then at any other epoch of time, t_2, the position vector of the object, referred to the sun as origin of coordinates, is

$$\mathbf{r}_2 = \mathbf{r}_1 + T\frac{d\mathbf{r}_1}{dt} + \frac{T^2}{2!}\frac{d^2\mathbf{r}_1}{dt^2} + \frac{T^3}{3!}\frac{d^3\mathbf{r}_1}{dt^3} + \dots$$

where $T = k(t_2 - t_1)$. The solution is completed by substituting for the differential coefficients their values as derived from the differential equation and its successive derivatives. Thus

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{-\mathbf{r}_1}{r_1^3}, \quad \frac{d^3\mathbf{r}_1}{dt^3} = -\frac{1}{r_1^3}\frac{d\mathbf{r}_1}{dt} + \frac{3}{r_1^4}\frac{dr_1}{dt}\mathbf{r}_1, \text{ etc.}$$ The result is

$$\mathbf{r}_2 = f\,\mathbf{r}_1 + g\,\frac{d\mathbf{r}_1}{dt}$$

where $f = 1 - T^2/2r_1^3 + T^3/2r_1^4 - \dots$, and $g = T - T^3/6r_1^3 + \dots$ The constant vectors, $\mathbf{r}_1$ and $\frac{d\mathbf{r}_1}{dt}$, which are the two constants of integration required in the solution, are the position vector and velocity vector, respectively, of the object at the epoch t_1. When these are known, the solution is complete, and the position of the object may be determined for any epoch of time, past or future.

It is apparent that the solution is also complete and the position is determined for any time, if the position vectors at two epochs are given, except when these vectors are parallel, for the solution may be written as

$$\frac{d\mathbf{r}_1}{dt} = \frac{\mathbf{r}_2 - f\,\mathbf{r}_1}{g}$$

Then, if $\mathbf{r}_2$ and $\mathbf{r}_1$ are known, the velocity, $\frac{d\mathbf{r}_1}{dt}$, required at t_1 to produce $\mathbf{r}_2$ is uniquely determined, except when $g = 0$. In this exceptional case the position vectors are parallel, and the sector-triangle ratio is indeterminate. In the development which follows, instead of the position and velocity vectors at the same epoch, the solution is based upon the position vectors at two epochs.

Comparison of Observations

The method here adopted for comparing observed and computed positions is one which has only recently come into use. It consists in dividing the two smaller geocentric rectangular coordinates of the object by the largest. These quotients are simple functions of the observed right ascension and declination, and when the computed values agree with those derived from the observations, the solution is complete.

This procedure is equivalent to using the ratios of the direction cosines. In order that the ratios shall never exceed unity, the following criteria determine which coordinate is to be taken as the divisor.

Case 1: If $-1 < \tan\alpha < 1$ and $-1 < \sec\alpha \tan\delta < 1$, let

$A_x = \tan\alpha = (y + Y)/(x + X)$, $A'_x = \sec\alpha \tan\delta = (z + Z)/(x + X)$,

$P_x = \sec\alpha \sec\delta = \rho/(x + X)$, $B_x = A_x X - Y$, $B'_x = A'_x X - Z$.

Case 2: If $-1 < \cot\alpha < 1$ and $-1 < \csc\alpha \tan\delta < 1$, let

$A_y = \cot\alpha = (x + X)/(y + Y)$, $A'_y = \csc\alpha \tan\delta = (z + Z)/(y + Y)$,

$P_y = \csc\alpha \sec\delta = \rho/(y + Y)$, $B_y = A_y Y - X$, $B'_y = A'_y Y - Z$.

Case 3: If neither of the above conditions ~~are~~ is satisfied, let

$A_z = \cos\alpha \cot\delta = (x + X)/(z + Z)$, $A'_z = \sin\alpha \cot\delta = (y+Y)/(z+Z)$,

$P_z = \csc\delta = \rho/(z + Z)$, $B_z = A_z Z - X$, $B'_z = A'_z Z - Y$.

The solar coordinates (X, Y, Z) should include the topocentric corrections, so that parallax is completely eliminated from the start.

The effect of using these A's and B's is to eliminate the third order determinants which are inevitable when the direction cosines are used. The evaluation of a third order determinant by means of a computing machine is, at best, four complete operations. The usual method is to expand by minors, which involves the evaluation of the three separate second order minors, and the sum of the products of these by their respective coefficients. Innes' method (loc. cit. p 429) requires the evaluation of ten third order determinants. In Merton's method (loc. cit. p 695) the solution of his equations (9) is equivalent to a determinant solution. His own discussion which follows the equations bears testimony to their undesirable form. Furthermore, the evaluation of a set of direction cosines requires four entries into tables of the trigonometric functions and two multiplications by a common multiplier. Exactly the same operations will yield the A's (these are all that are necessary for the comparison of an observation), and, in addition, an explicit expression for ρ.

The following development is given for the first case only, the other two being analagous. For convenience, the subscript x is omitted. In terms of the adopted notation:

$$r^2 = (1 + A^2 + A'^2)x^2 + 2(AB + A'B')x + B^2 + B'^2, \qquad y = Ax + B, \quad z = A'x + B' \tag{2}$$

so that, at the instant of an observation, the radius vector and its components are explicit functions of the x-coordinate alone, and the observation is always exactly represented. Since two radius vectors are sufficient to determine the position of the object for all times, the present method will depend upon the determination of an x_1 and x_2 as the independent variables at the epochs t_1 and t_2, respectively, such that the position at any third epoch, t_i, is satisfactorily represented.

Properties of the Coordinates

Consider the difference tables in which the three heliocentric rectangular coordinates of the object are given at equal intervals of the time. Let the notation be:

Argument	Coordinate	2nd Diff	4th Diff
t_0	x_0	Δ_0''	
t_1	x_1	Δ_1''	Δ_1^{iv}
t_2	x_2	Δ_2''	Δ_2^{iv}
t_3	x_3	Δ_3''	

and similarly for y and z. The differences may be evaluated from the fundamental differential equation of motion by a method similar to that used by Gibbs, and later by Cowell and Crommelin.* Since x is a continuous function of the time,

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J. W. Gibbs, "On the Determination of Elliptic Orbits", Nat. Acad. of Science, IV 8th mem., 1888;
Cowell and Crommelin, "The Orbit of Halley's Comet from 1759 to 1910", Greenwich Observations, Appendix, 1909.

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let $x = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4 + \ldots$

where the unit of T is $k(t_2 - t_1) = k(t_1 - t_0)$, etc. Then, if $x = x_1$ when $T = 0$, when $T = -1$, $x = x_0$, and when $T = 1$, $x = x_2$.

Thus:

$$\begin{array}{rl} x_0 = & a_0 - a_1 + a_2 - a_3 + a_4 - \ldots \\ -2x_1 = & -2a_0 \\ x_2 = & a_0 + a_1 + a_2 + a_3 + a_4 + \ldots \\ \hline \Delta_1'' = & 2a_2 + 2a_4 + \ldots \end{array}$$

Similarly $\Delta_1^{iv} = 24\, a_4 + 120\, a_6 + \ldots$ Also, by comparing the coefficients with Taylor's series and the differential equation of motion, $a_2 = \frac{1}{2!}\left[\frac{d^2 x}{dt^2}\right]_{x=a_0} = \frac{-T^2 a_0}{2!\; r^3}$. Writing x_1 for a_0, and substituting for a_2, a_4, etc. gives:

$$\Delta_1'' = -T^2 x_1 / r_1^3 + 1/12\, \Delta_1^{iv} - \ldots \qquad (3)$$

For Δ_1'' write: $\Delta_1'' = \Delta_2'' + \Delta_0'' - 2\Delta_1''$

$$= \frac{-T^2 x_2}{[r_1 + (r_2 - r_1)]^3} + \frac{1}{12}\Delta_2^{iv} - \ldots + \frac{-T^2 x_0}{[r_1 + (r_0 - r_1)]^3} + \frac{1}{12}\Delta_0^{iv} - \ldots$$

$$+ \frac{2\, T^2 x_1}{r_1^3} - \frac{1}{6}\Delta_1^{iv} + \ldots$$

Expand the brackets by the binomial theorem, and collect. Then

$$\Delta_1^{IV} = -(x_2 + x_0 - 2x_1)T^2/r_1^3 + (\Delta_2^{IV} + \Delta_0^{IV} - 2\Delta_1^{IV})/12 + 3[\quad]T^2/r_1^4 + \ldots$$

$$\Delta_1^{IV} = T^4_4/r_1^6 + \Delta_1^{VI}/12 + 3\left[(r_2 - r_1)x_2 - (r_1 - r_0)x_0\right]T^2/r_1^4 + \ldots \qquad (4)$$

Similar expressions for the corresponding differences in the y and z tables, and the differences opposite t_2 in each table, are derived in exactly the same manner.

In the case of three observations separated by sufficiently small time intervals, let t_1 be the time of the first observation, t_2 of the third, and t_i of the intermediate one. Let $(t_i - t_1)/(t_2 - t_1) = n$, $1 - n = m$, $k(t_2 - t_1) = T$. Using Everett's interpolation formula* and the values of the dif-

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* This is a recent, and not well known, formula, given by Everett in the British Association Report for 1900. It is easily derived by substituting for the odd differences in Bessel's formula in terms of the even differences of the two adjoining rows. Putting m 1 - n, the result is:

$$x_m = m\,x_0 - E_2(m)\,\Delta_0^{II} + E_4(m)\,\Delta_0^{IV} - \ldots +$$
$$+ n\,x_1 - E_2(n)\,\Delta_1^{II} + E_4(n)\,\Delta_1^{IV} - \ldots,$$

where $E_{2i}(m) = m(m^2 - 1^2)(m^2 - 2^2)\ldots(m^2 - i^2)/(2i + 1)!$

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ferences derived in (3) and (4), the value of the x-coordinate at the intermediate epoch, t_i, is given by

$$x_i = \left\{m + \frac{m(1 - m^2)}{6} K_1 \left[1 + \frac{(7 - 3m^2)}{60} K_1\right]\right\} x_1$$
$$\left\{n + \frac{n(1 - n^2)}{6} K_2 \left[1 + \frac{(7 - 3n^2)}{60} K_2\right]\right\} x_2 \qquad (5)$$

and similarly for y and z, where $K_j = T^2/r_j^3$, $(j = 1, 2)$. Terms of the sixth order have been neglected, and also the term, $3\left[(r_2 - r_1)x_2 - (r_1 - r_0)x_0\right]T^2/r_1^4$, in the fourth difference. In nearly circular orbits or in the neighborhood of perihelion, this term is inappreciable, for r is nearly constant, so that the formula includes most of the effect of terms to the fifth*

Both Δ_0^{iv} and Δ_1^{iv} include part of $\Delta_{1/2}^{v}$ in Bessel's formula.

order of differences. The formula is especially convenient in practise, for the Everett interpolation coefficients may be taken from tables* (these will change in value only once, when the correction for light time is applied), $1/r^3$ may be taken from a table* with the argument r^2, and $(7 - 3m^2)/60$ is tabu-

Chappell: A Table of Coefficients to facilitate Interpolation by means of the Formulae of Gauss, Bessel, and Everett. London, 1931.
Thompson: Table of the Coefficients of Everett's Central-Difference Interpolation Formula. London, 1921.
Comrie: Planetary Coordinates for the Years 1800 - 1940. London, 1933. Table X gives $1/r^3$.

lated in the adjoining TABLE I. All the terms in the formula are positive and additive.

TABLE I

m	$\frac{7 - 3m^2}{60}$	
.0	.1167	- 5
.1	.1162	-15
.2	.1147	-25
.3	.1122	-35
.4	.1087	-45
.5	.1042	-55
.6	.0987	-65
.7	.0922	-75
.8	.0847	-85
.9	.0762	-95
1.0	.0667	

Triangle Ratios

Let the expression (5) for the coordinate x_i, and the similar expressions for y_i and z_i, be written as

$$\begin{aligned} x_i &= M\,x_1 + N\,x_2, \\ y_i &= M\,y_1 + N\,y_2, \\ z_i &= M\,z_1 + N\,z_2. \end{aligned} \qquad (6)$$

These equations are equivalent to $\mathbf{r}_i = M\,\mathbf{r}_1 + N\,\mathbf{r}_2$. However, when a vector, $\mathbf{r}_i$, is expressed in terms of two other vectors by such a relationship, then M and N are the ratios of the areas of the triangles, $\frac{[r_2,\ r_i]}{[r_2,\ r_1]}$ and $\frac{[r_i,\ r_1]}{[r_2,\ r_1]}$, respectively, for the vector product $\mathbf{r}_2 \times \mathbf{r}_i = M\, \mathbf{r}_2 \times \mathbf{r}_1$, or

$$M = \frac{r_2 r_i \sin(v_2 - v_i)}{r_2 r_1 \sin(v_2 - v_1)}, \text{ and similarly } N = \frac{r_i r_1 \sin(v_i - v_1)}{r_2 r_1 \sin(v_2 - v_1)}.$$

THUS, M AND N ARE EXPANSIONS OF THE TRIANGLE RATIOS, EACH IN TERMS OF THE TIME AND ONLY ONE EXPLICIT INDEPENDENT VARIABLE.

The simultaneous use of two radii implicitly includes the effect of the derivative of a single radius, so that the higher powers of the time may be included directly in the expressions for M and N, instead of having to derive approximate values or add correction terms. The usual expressions, to terms of the fourth powers of the time, but involving only one radius, may be found in various standard works.* In the first approximation only terms to the second power of the time may be included, and the complexity of subsequently including higher power terms is evident from the formulae. The Gibb's ratios offer a slightly higher degree of approximation, but they involve three radii and do not admit of such simple differential coefficients as will be given for M and N. The writer has been unable to find expressions for the triangle ratios which prove to be equally as effective in the actual determination of orbits from short arcs as the expressions given in (5).

* See Merton (loc. cit.) p 694.

General Solution

By the formulae thus far developed, x_1 and x_2 remain as unknowns, and two observations are always exactly satisfied. It is to be noted that this is the least possible number of unknowns, since a differential equation of the second order is being solved. The efficacy of the completed method will show that the choice of unknowns to which the problem has been reduced is a very expeditious one.

The intermediate observation will now be employed to determine the values of x_1 and x_2. Combining the geometric conditions of each observation, as expressed in (2), with the dynamical relationships among the three positions, as expressed in (6), gives:

$$\begin{aligned} y_i &= A_i(Mx_1 + Nx_2) + B_i = M(A_1x_1 + B_1) + N(A_2x_2 + B_2) \\ z_i &= A_i'(Mx_1 + Nx_2) + B_i' = M(A_1'x_1 + B_1') + N(A_2'x_2 + B_2'), \end{aligned}$$

or

$$\begin{aligned} M(A_1 - A_i)x_1 + N(A_2 - A_i)x_2 &= B_i - M B_1 - N B_2 \\ M(A_1' - A_i')x_1 + N(A_2' - A_i')x_2 &= B_i' - M B_1' - N B_2'. \end{aligned} \qquad (7)$$

ALL OF THE CONDITIONS OF THE GENERAL SOLUTION ARE CONTAINED IN THIS SIMPLE PAIR OF EQUATIONS. The M and N provide that the heliocentric rectangular coordinates of the middle place are derived from those of the first and third place according to the law of gravitation, and that the three radius vectors shall be coplanar; while the A's and B's insure that all three observations are exactly represented. It will be shown later that, for minor planets, the determinant of the coefficients can not vanish. While it may appear that the x_1 and x_2 are involved in the M and N, respectively, in a complicated manner, nevertheless, the solution may be readily derived by the following iteration process. One rational guess will displace the entire preliminary solution of other orbit methods. For a minor planet, assume initial values of $r_1 = 2.5 = r_2$, with which approximate values of M and N may be computed. Solve the equations (7) for x_1 and x_2, then the equation (2) provides improved values of r_1 and r_2, with which the process is repeated. In general, this method will converge rapidly, for the errors in the approximate values of the r's enter the expressions for M and N with very small coefficients, since $\frac{\partial M}{\partial r_1} = \frac{m(1 - m^2)}{6} \frac{T^2(-3)}{r_1^4}$, $\frac{\partial N}{\partial r_2} = \frac{n(1 - n^2)}{6} \frac{T^2(-3)}{r_2^4}$.

This iteration process neglects the effect of the variations of x_1 and x_2 on M and N, respectively, in the successive approximations. Since M and N are each explicit functions of a single variable, these effects may be easily taken into account by performing a differential correction upon the first approximations of x_1 and x_2, obtained as above. This greatly increases the rapidity of the convergence of the solutions, and is usually the most practical method of solution when the series expansions for M and N can be used. From the direct differentiation of equations (2) and (5)

$$dr_1 = \frac{(1+A_1^2+A_1'^2)x_1 + (A_1B_1 + A_1'B_1')}{r_1}\, dx_1,$$

$$dr_2 = \frac{(1+A_2^2+A_2'^2)x_2 + (A_2B_2 + A_2'B_2')}{r_2}\, dx_2.$$

$$dM = \frac{-m(1-m^2)\,K_1}{2}\left[\frac{(1+A_1^2+A_1'^2)x_1+(A_1B_1+A_1'B_1')}{r_1^2}\right] dx_1$$

$$dN = \frac{-n(1-n^2)\,K_2}{2}\left[\frac{(1+A_2^2+A_2'^2)x_2+(A_2B_2+A_2'B_2')}{r_2^2}\right] dx_2$$

In order to simplify the notation, these expressions will be written as: $dM = -\overline{M}\, dx_1$, and $dN = -\overline{N}\, dx_2$. The corrected values of x_1 and x_2 must satisfy the equations

$$A_i\left[(M+dM)(x_1+dx_1)+(N+dN)(x_2+dx_2)+X_i\right] = (M+dM)(y_1+dy_1)+(N+dN)(y_2+dy_2)+Y_i$$

$$A_i'\left[(M+dM)(x_1+dx_1)+(N+dN)(x_2+dx_2)+X_i\right] = (M+dM)(z_1+dz_1)+(N+dN)(z_2+dz_2)+Z_i$$

Now $y = Ax + B$, $dy = A\,dx$; $z = A'x + B'$, $dz = A'\,dx$. Substitute and transpose:

$$\left[(A_1-A_i)(M-\overline{M}x_1)-\overline{M}B_1\right]dx_1+\left[(A_2-A_i)(N-\overline{N}x_2)-\overline{N}B_2\right]dx_2 = -\left[(A_1-A_i)Mx_1+(A_2-A_i)Nx_2+MB_1+NB_2-B_i\right]$$

$$\left[(A_1'-A_i')(M-\overline{M}x_1)-\overline{M}B_1'\right]dx_1+\left[(A_2'-A_i')(N-\overline{N}x_2)-\overline{N}B_2'\right]dx_2 = -\left[(A_1'-A_i')Mx_1+(A_2'-A_i')Nx_2+MB_1'+NB_2'-B_i'\right] \quad (8)$$

where terms of the form $(A_1-A_i)\overline{M}(dx)^2$ have been neglected.

This completes the development of the method to be used when the observations are separated by sufficiently small time intervals. For minor planets it will be applicable to observations extending over an interval of about forty days.

Comet Orbits

The case of comet orbits, however, involves additional considerations. During the period of observation, the heliocentric distance of a comet is more probably one unit than 2.5 units, as was assumed for minor planets; and in the large majority of comet orbits, the eccentricity is very nearly unity. The smaller heliocentric distance not only reduces the time interval in which the series for M and N will converge, but it introduces a more serious difficulty, as will be shown from a theorem of Lambert on the curvature of the apparent path. Because of the nearly parabolic nature of the orbit, it is customary to derive a preliminary orbit that is exactly parabolic, and thereby simplify the ephemeris computation. This is attained by imposing the Euler-Lambert equation of parabolic motion as one of the conditions of the solution, and neglecting one of the observed coordinates of the middle observation.* The resulting representation of

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Leuschner's method forces a parabolic velocity, and neglects the declination of the first observation.

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this neglected coordinate is an index to the validity of the parabolic hypothesis.

Lambert's theorem on the curvature of the apparent path states that the heliocentric distance of a celestial object is greater than, less than, or equal to the earth's heliocentric distance, according as the apparent path is convex toward the sun, concave toward the sun, or at an inflection point. Now the determinant $\begin{vmatrix} 1 & A_i & A_i' \\ 1 & A_1 & A_1' \\ 1 & A_2 & A_2' \end{vmatrix}$ is positive, zero, or negative, according as the middle observation lies above, on, or below the great circle on the celestial sphere joining the two outer observations. Also, the determinant $\begin{vmatrix} X_i & Y_i & Z_i \\ 1 & A_1 & A_1' \\ 1 & A_2 & A_2' \end{vmatrix}$ is positive, zero, or negative, according as the sun lies above, on, or below the same great circle at the time of the middle observation. Therefore, if the two determinants have the same sign, the object is farther from the sun than the earth is; if they have opposite signs, the object is closer to the sun than the earth is; and if the first determinant vanishes, the object is at the same distance from the sun as the earth is, or moves in the ecliptic.* But the first deter-

*This last statement is not apparent from the theorem, but the proof of the theorem breaks down if the object moves in the ecliptic. From geometric considerations alone, it is obvious that the statement is true.

minant is easily reduced to the determinant of the coefficients of the equations (7) of the general solution, except for the non-zero factor, MN. Since, with only one known exception, the minor planets all lie outside of the earth's orbit, the equations of the general solution cannot be indeterminate, unless the object moves in the ecliptic, in which case the problem cannot be solved by any method.*

*This is due to the fact that the determination of the plane of the orbit eliminates only two of the six unknown elements, while each observation then furnishes only one polar coordinate in the plane, so that a fourth observation is necessary in order to derive the remaining four elements.

None of the methods for determining a parabolic orbit are at all comparable in elegance, simplicity, and utility to the general solution which is developed above. This is due to the fact that the parabolic hypothesis imposes a different type of analytic condition which must be used simultaneously with the condition of gravitation. In his revision of Olber's method,* B. Stroemgren has constructed a

* Publikationer og mindre Meddelelser fra Københavns Observatorium. No. 66.

nomogram which gives a first approximation to ρ. This crude graphical value is probably as satisfactory as one which is obtained from computation. Merton (loc. cit. page 702) writes down five equations, numerically eliminates all but ρ_2 and r_2, and then solves by trials for the first approximation.

Parabolic Solution

A method of determining a parabolic orbit, following the pattern of the general solution above, will now be developed. First apply Lambert's test to the observations, as this indicates the approximate magnitude of the comet's heliocentric distance, and whether or not the general equations (7) are indeterminate.* The determinants are easily evaluated in the

* It may be noted that, since the A's and B's were derived from the solar coordinates, when the heliocentric distance of the object approaches one astronomical unit, the solution is necessarily indeterminate, because the real solution is confounded with the extraneous solution of the earth's orbit. This is not a fault of the present method; the same difficulty occurs in other methods also, although the reason is not so apparent.

following forms: $(A_1 - A_i)(A_2' - A_i') - (A_1' - A_i')(A_2 - A_i)$, and $(A_1A_2' - A_2A_1')X_i + (A_1' - A_2')Y_i + (A_2 - A_1)Z_i$, respectively. If the general equations (7) are not indeterminate, their solution will yield approximate values of x_1 and x_2, just as in the case of minor planets, except that the assumed values of r should be in accord with the indications of Lambert's test. The difficulty arises when the equations (7) are indeterminate and the apparent path is nearly along a great circle, for "it is the curvature of the apparent path upon which one chiefly relies for the determination of the geocentric distance"* or the rectangular coordinate. In this case

* Crawford, Determination of Orbits etc. page 117.

approximate values of x_1 and x_2 are obtained by putting $r = 1$ in the equation (2) which connects r and x, and solving the quadratic for x, rejecting the value which is the negative of the solar coordinate. A better approximation may be obtained by solving the equations (7) for x_1 and x_2, using several arbitrary values of r, and interpolating the results for the values of x_1 and x_2 which will reproduce themselves.

These approximate values must then be improved in such a way as to satisfy the Euler-Lambert equation of parabolic motion and one of the equations (7), the choice being made on the basis of which equation has the larger coefficients. The orbit is exactly parabolic if

$$0 = S_d^2 - S_g^2 = R(x_1,x_2)$$

where $R(x_1,x_2)$ represents the residual which results from the substitution of approximate values of x_1 and x_2, $S_d^2 = (\eta\zeta)^2(r_1 + r_2)^2$, and $S_g^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$.

The subscripts, d and g, indicate the dynamical and geometric values of the chord, respectively. S_d is Encke's modified form of the Euler-Lambert equation.
c.f. Bauschinger, Bahnbestimmung, page 194.

The increments, dx_1 and dx_2, to be added to x_1 and x_2, respectively, are obtained from the simultaneous solution of

$$R(x_1,x_2) + \frac{\partial R}{\partial x_1} dx_1 + \frac{\partial R}{\partial x_2} dx_2 = 0$$

and one of the equations (8), depending upon which of the equations (7) is used. By partial differentiation

$$\frac{\partial R}{\partial x_1} = 2(\eta\zeta)^2(\lambda_1+\lambda_2)\frac{d\lambda_1}{dx_1} + (\lambda_1+\lambda_2)^2\frac{\partial(\eta\zeta)^2}{\partial x_1} + 2\left[(x_2-x_1)+(y_2-y_1)\frac{dy_1}{dx_1} + (z_2-z_1)\frac{dz_1}{dx_1}\right]$$

Letting $\zeta = 1$, $\frac{\partial(\eta\zeta)^2}{\partial x_1} = \frac{\partial[4T^2(\lambda_1+\lambda_2)^{-3}]}{\partial x_1} = \frac{-12T^2}{(\lambda_1+\lambda_2)^4}\frac{d\lambda_1}{dx_1}$

Substitute this and the values of the differentials obtained from equations (2).

$$\frac{\partial R}{\partial x_1} = (\lambda_1+\lambda_2)\left[2(\eta\zeta)^2 - 3\eta^2\right]\left[\frac{(1+A_1^2+A_1'^2)x_1+(A_1B_1+A_1'B_1')}{\lambda_1}\right] + 2\left[(x_2-x_1)+(y_2-y_1)A_1+(z_2-z_1)A_1'\right]$$

$$\frac{\partial R}{\partial x_2} = (\lambda_1+\lambda_2)\left[2(\eta\zeta)^2 - 3\eta^2\right]\left[\frac{(1+A_2^2+A_2'^2)x_2+(A_2B_2+A_2'B_2')}{\lambda_2}\right] - 2\left[(x_2-x_1)+(y_2-y_1)A_2+(z_2-z_1)A_2'\right]$$

The computation is facilitated by the use of existing tables.*

*Stracke, Tafeln zur Theoretischen Astronomie von J. Bauschinger. Table 26 gives $(\eta\zeta)$ with the argument η. η is derived with the aid of Comrie's Table X, loc. cit.

Long Time Intervals

The only case not yet considered is when the time interval is so long that the series expansions for M and N do not converge to the desired degree of accuracy. In such a case, approximate values of x_1 and x_2 are almost invariably known from a preliminary orbit. Therefore, regardless of whether the final orbit is to be a general solution which satisfies both coordinates of the middle position, or a parabolic solution, the preliminary orbit will leave small residuals of the form

A_i(observed) - A_i(computed), and A_i'(observed) - A_i'(computed), or $S_d^2 - S_g^2$. Two equations of the form

$$R(w_1, w_2) + \frac{\partial R}{\partial w_1} dw_1 + \frac{\partial R}{\partial w_2} dw_2 = 0$$

determine the corrections, dw_1 and dw_2, which will remove the residuals. The observations will often fall into more than one of the three classes defined on page 6, so that the independent variables will not always be the same coordinate, therefore w may be x, y, or z. For the general solution, the equations reduce to

$$[A_i(w_1+h, w_2) - A_i(w_1, w_2)]\,dw_1 + [A_i(w_1, w_2+h) - A_i(w_1, w_2)]\,dw_2 = h\,[A_i(\text{observed}) - A_i(w_1, w_2)],$$
$$[A_i'(w_1+h, w_2) - A_i'(w_1, w_2)]\,dw_1 + [A_i'(w_1, w_2+h) - A_i'(w_1, w_2)]\,dw_2 = h\,[A_i(\text{observed}) - A_i(w_1, w_2)].$$

The term h is an arbitrary small increment, say .01 or .001, and from the form of the coefficients it is apparent that three separate solutions have to be carried thru, and then the partial differential coefficients are determined numerically. The same method is applicable to a parabolic orbit.

The use of differential correction equations analagous to equations (8) would require a knowledge of closed forms for $dM = \frac{\partial M}{\partial w_1} dw_1 + \frac{\partial M}{\partial w_2} dw_2$ and $dN = \frac{\partial N}{\partial w_1} dw_1 + \frac{\partial N}{\partial w_2} dw_2$, since the series expansions (of which $\overline{M}$ and $\overline{N}$ are only the first terms) are no longer valid. But the closed forms of M and N (page 11) are obviously very complicated functions of w_1 and w_2, so that the partial differential coefficients would be cumbersome. However, the following method may be employed for obtaining the numerical values of the partial differential coefficients in any particular case. The development is given in full generality.

Let w_1 ($w = x$, y, or z) be the rectangular coordinate which is the independent variable of the first observation, similarly w_2 for the second observation, and, to avoid any ambiguity, let $\overline{w}$ in each of the three observations be the coordinate which is the independent variable of the third or i th observation. The i th observation is no longer necessarily intermediate in time between t_1 and t_2. It is best to attach the subscripts 1 and 2 to the two of the three observations whose radius vectors from the sun are most nearly perpendicular, and attach i to the remaining observation.

The following operations involve only differentiations and substitutions. If $M = \frac{pq}{rs}$,

$$\frac{\partial M}{\partial t} = \frac{q}{rs}\frac{p}{p}\frac{\partial p}{\partial t} + \frac{p}{rs}\frac{q}{q}\frac{\partial q}{\partial t} - \frac{pq}{s}\frac{1}{r^2}\frac{\partial r}{\partial t} - \frac{pq}{r}\frac{1}{s^2}\frac{\partial s}{\partial t}$$

$$= M\left(\frac{1}{p}\frac{\partial p}{\partial t} + \frac{1}{q}\frac{\partial q}{\partial t} - \frac{1}{r}\frac{\partial r}{\partial t} - \frac{1}{s}\frac{\partial s}{\partial t}\right).$$

From page 11, $M = \frac{r_2 r_i \sin(v_2 - v_i)}{r_2 r_1 \sin(v_2 - v_1)}$, and $N = \frac{r_i r_1 \sin(v_i - v_1)}{r_2 r_1 \sin(v_2 - v_1)}$.

Noting that the coordinates of the first observation are independent of those of the second, it follows immediately that

$$\frac{\partial M}{\partial w_1} = M\left[\frac{1}{r_i}\frac{\partial r_i}{\partial w_1} + \frac{1}{\sin(v_2-v_i)}\frac{\partial \sin(v_2-v_i)}{\partial w_1} - \frac{1}{\sin(v_2-v_1)}\frac{\partial \sin(v_2-v_1)}{\partial w_1} - \frac{1}{r_1}\frac{dr_1}{dw_1}\right]$$

$$\frac{\partial M}{\partial w_2} = M\left[\frac{1}{r_i}\frac{\partial r_i}{\partial w_2} + \frac{1}{\sin(v_2-v_i)}\frac{\partial \sin(v_2-v_i)}{\partial w_2} - \frac{1}{\sin(v_2-v_1)}\frac{\partial \sin(v_2-v_1)}{\partial w_2}\right] \qquad (9)$$

$$\frac{\partial N}{\partial w_1} = N\left[\frac{1}{r_i}\frac{\partial r_i}{\partial w_1} + \frac{1}{\sin(v_i-v_1)}\frac{\partial \sin(v_i-v_1)}{\partial w_1} - \frac{1}{\sin(v_2-v_1)}\frac{\partial \sin(v_2-v_1)}{\partial w_1}\right]$$

$$\frac{\partial N}{\partial w_2} = N\left[\frac{1}{r_i}\frac{\partial r_i}{\partial w_2} + \frac{1}{\sin(v_i-v_1)}\frac{\partial \sin(v_i-v_1)}{\partial w_2} - \frac{1}{\sin(v_2-v_1)}\frac{\partial \sin(v_2-v_1)}{\partial w_2} - \frac{1}{r_2}\frac{dr_2}{dw_2}\right]$$

For $\frac{\partial r_i}{\partial w_k}$ substitute $\frac{dr_i}{d\overline{w}_i}\frac{\partial \overline{w}_i}{\partial w_k}$, (k = 1 or 2). Since $\overline{w}_i = M\overline{w}_1 + N\overline{w}_2$,

$$\frac{\partial \overline{w}_i}{\partial w_k} = M\frac{\partial \overline{w}_1}{\partial w_k} + \overline{w}_1\frac{\partial M}{\partial w_k} + N\frac{\partial \overline{w}_2}{\partial w_k} + \overline{w}_2\frac{\partial N}{\partial w_k}, \qquad (10)$$

where one term always vanishes because of the independence of w_1 and w_2.

Since $\sin V = (1 - \cos^2 V)^{\frac{1}{2}}$, $\frac{\partial \sin V}{\partial w_k} = \frac{1}{2}\frac{(-2\cos V)}{(1-\cos^2 V)^{1/2}}\frac{\partial \cos V}{\partial w_k}$

or $\frac{\partial \sin V}{\partial w_k} = -\cot V \frac{\partial \cos V}{\partial w_k}$. Let j = 1 and k = 2, or vise versa.

Then $\cos(v_j - v_k) = \frac{x_j x_k + y_j y_k + z_j z_k}{r_j r_k}$, and

$$\frac{\partial \cos(v_j-v_k)}{\partial w_k} = \frac{\left(x_j\frac{dx_k}{dw_k} + y_j\frac{dy_k}{dw_k} + z_j\frac{dz_k}{dw_k}\right) - r_j\cos(v_j-v_k)\frac{dr_k}{dw_k}}{r_j r_k}$$

where $\frac{dr_k}{dw_k}$, $\frac{dx_k}{dw_k}$, $\frac{dy_k}{dw_k}$, and $\frac{dz_k}{dw_k}$ are obtained directly from equations (2) or the analagous equations in the other two cases defined on page 6.

$$\frac{\partial \cos(v_i - v_j)}{\partial w_k} = \frac{\left(x_j \frac{\partial x_i}{\partial w_k} + y_j \frac{\partial y_i}{\partial w_k} + z_j \frac{\partial z_i}{\partial w_k}\right) - r_j \cos(v_i - v_j) \frac{dr_i}{dw_k}}{r_i r_j} = \frac{\left(x_j \frac{dx_i}{d\overline{w}_i} + y_j \frac{dy_i}{d\overline{w}_i} + z_j \frac{dz_i}{d\overline{w}_i}\right) - r_j \cos(v_i - v_j) \frac{dr_i}{d\overline{w}_i}}{r_i r_j} \left(\frac{\partial \overline{w}_i}{\partial w_k}\right)$$

$$\frac{\partial \cos(v_i - v_k)}{\partial w_k} = \frac{\left(x_i \frac{dx_k}{dw_k} + y_i \frac{dy_k}{dw_k} + z_i \frac{dz_k}{dw_k}\right) + \left(x_k \frac{\partial x_i}{\partial w_k} + y_k \frac{\partial y_i}{\partial w_k} + z_k \frac{\partial z_i}{\partial w_k}\right) - \cos(v_i - v_k)\left(r_i \frac{dr_k}{dw_k} + r_k \frac{\partial r_i}{\partial w_k}\right)}{r_i r_k}$$

$$= \frac{\left(x_i \frac{dx_k}{dw_k} + y_i \frac{dy_k}{dw_k} + z_i \frac{dz_k}{dw_k}\right) - r_i \cos(v_i - v_k) \frac{dr_k}{dw_k}}{r_i r_k} + \frac{\left(x_k \frac{dx_i}{d\overline{w}_i} + y_k \frac{dy_i}{d\overline{w}_i} + z_k \frac{dz_i}{d\overline{w}_i}\right) - r_k \cos(v_i - v_k) \frac{dr_i}{d\overline{w}_i}}{r_i r_k} \left(\frac{\partial \overline{w}_i}{\partial w_k}\right)$$

$$\frac{\partial M}{\partial w_1} = M\left[\left\{\frac{1}{r_i}\frac{dr_i}{d\overline{w}_i} - \frac{\cot(v_2 - v_i)}{\sin(v_2 - v_i)}\left(\frac{\left(x_2 \frac{dx_i}{d\overline{w}_i} + y_2 \frac{dy_i}{d\overline{w}_i} + z_2 \frac{dz_i}{d\overline{w}_i}\right) - r_2 \cos(v_2 - v_i)\frac{dr_i}{d\overline{w}_i}}{r_i r_2}\right)\right\}\left(\frac{\partial \overline{w}_i}{\partial w_1}\right) + \frac{\cot(v_2 - v_1)}{\sin(v_2 - v_1)}\left[\frac{\left(x_2 \frac{dx_1}{dw_1} + y_2 \frac{dy_1}{dw_1} + z_2 \frac{dz_1}{dw_1}\right) - r_2 \cos(v_2 - v_1)\frac{dr_1}{dw_1}}{r_1 r_2}\right] - \frac{1}{r_1}\frac{dr_1}{dw_1}\right]$$

$$\frac{\partial N}{\partial w_1} = N\left[\left\{\frac{1}{r_i}\frac{dr_i}{d\overline{w}_i} - \frac{\cot(v_i - v_1)}{\sin(v_i - v_1)}\left(\frac{\left(x_1 \frac{dx_i}{d\overline{w}_i} + y_1 \frac{dy_i}{d\overline{w}_i} + z_1 \frac{dz_i}{d\overline{w}_i}\right) - r_1 \cos(v_i - v_1)\frac{dr_i}{d\overline{w}_i}}{r_i r_1}\right)\right\}\left(\frac{\partial \overline{w}_i}{\partial w_1}\right) - \right.$$

$$\left. - \frac{\cot(v_i - v_1)}{\sin(v_i - v_1)}\left(\frac{\left(x_i \frac{dx_1}{dw_1} + y_i \frac{dy_1}{dw_1} + z_i \frac{dz_1}{dw_1}\right) - r_i \cos(v_i - v_1)\frac{dr_1}{dw_1}}{r_i r_1}\right) + \frac{\cot(v_2 - v_1)}{\sin(v_2 - v_1)}\left(\frac{\left(x_2 \frac{dx_1}{dw_1} + y_2 \frac{dy_1}{dw_1} + z_2 \frac{dz_1}{dw_1}\right) - r_2 \cos(v_2 - v_1)\frac{dr_1}{dw_1}}{r_2 r_1}\right)\right]$$

and the similar expressions for $\frac{\partial M}{\partial w_2}$ and $\frac{\partial N}{\partial w_2}$. While this may appear to be an imposing array of terms, yet everything is readily reduced to its numerical value, except $\frac{\partial \bar{w}_i}{\partial w_1}$. Substituting for this the expression given by (10) produces a pair of simultaneous, linear, algebraic equations in $\frac{\partial M}{\partial w_1}$ and $\frac{\partial N}{\partial w_1}$, from which their numerical values may be obtained. Similarly, the values of $\frac{\partial M}{\partial w_2}$ and $\frac{\partial N}{\partial w_2}$ are obtained from the other pair of equations.

There is nothing which precludes allowing w_1, w_2, and $\bar{w}_i$ to be ρ_1, ρ_2, and ρ_i, respectively, so that this procedure is also applicable to the classical method of the variation of geocentric distances. The only differences occur in the expression for $\frac{\partial \bar{w}_i}{\partial w_k}$. Instead of $\bar{w}_i = M\, w_1 + N\, w_2$, now $(h_i \rho_i - W_i) = M(h_1 \rho_1 - W_1) + N(h_2 \rho_2 - W_2)$, where $h = l$, m, or n (the direction cosines) and $W = X$, Y, or Z (the solar coordinates). Then, corresponding to equation (10),

$$\left(\frac{\partial \rho_i}{\partial w_k}\right) = M\frac{h_1}{h_i}\frac{\partial \rho_1}{\partial w_k} + \frac{(\rho_1 h_1 - W_1)}{h_i}\frac{\partial M}{\partial w_k} + N\frac{h_2}{h_i}\frac{\partial \rho_2}{\partial w_k} + \frac{(\rho_2 h_2 - W_2)}{h_i}\frac{\partial N}{\partial w_k}, \qquad (11)$$

which is to be substituted for $\left(\frac{\partial \bar{w}_i}{\partial w_k}\right)$. Obviously h_i will be the largest of the three direction cosines of the i th observation. The derivatives $\frac{dx}{dw}$, $\frac{dy}{dw}$, and $\frac{dz}{dw}$ are merely the direction cosines, and $\frac{dr}{dw} = \frac{\rho - lX - mY - nZ}{r}$.

So far as the present writer has been able to ascertain, no means of obtaining the partial differential coefficients, other than that of the numerical method described on page 20, has ever before been given.

COLLECTION OF FORMULAE

and

SUMMARY

A. General Solution (Short Time Interval)

(A I) Observations

The data furnished by the three observations to be used are t_1, t_i, t_2; $\alpha_1, \alpha_i, \alpha_2$; $\delta_1, \delta_i, \delta_2$; where the t's are reduced to decimal parts of a mean solar day, and the right ascensions and declinations are referred to some convenient mean equinox. The geocentric solar coordinates at the instants t_1, t_i, and t_2 must be interpolated from the values tabulated in the ephemeris. These are most readily obtained by using Everett's modified formula and Table XVIII of the British Nautical Almanac. Parallax is completely eliminated by applying the topocentric corrections to the geocentric solar coordinates, thus obtaining the solar coordinates referred to the point of observation. These corrections are most readily obtained by the method which Bower describes in the Lick Observatory Bulletin No. 445. For the Standard Equinox of 1950.0, let t be expressed in decimals of a julian day, and let L be the east longitude of the observatory, expressed in decimals of a circle. Then $\Theta = L + 1.0027\ 3780\ 3094\ (t - 2428250.0)$, $\Delta X = A \cos \Theta$, $\Delta Y = A \sin \Theta$; ΔZ and A are constants for the observatory.

For each observation,* $A = \tan\alpha$, $A' = \sec\alpha \tan\delta$, $P = \sec\alpha \sec\delta$,

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* These are for case 1, defined on page 6. For the other two cases it is necessary, in what follows, to interchange x and y, or x and z, respectively.

$B = AX - Y$, $B' = A'X - Z$.

$$r_k^2 = (1 + A_k^2 + A_k'^2)x_k^2 + 2(A_k B_k + A_k' B_k')x_k + (B_k^2 + B_k'^2), \quad (k = 1, 2)$$

$$D = (A_1 - A_i)(A_2' - A_i') - (A_1' - A_i')(A_2 - A_i).$$

(A II) Preliminary Solution

$$n = (t_i - t_1)/(t_2 - t_1), \quad m = 1 - n, \quad T = k(t_2 - t_1).$$

$k = .01720210$ (Gaussian constant), or if the masses of the four inner planets are supposed to be centered in the sun, $k = .01720215$. $K_1 = T^2 r_1^{-3}$, $K_2 = T^2 r_2^{-3}$; use Comrie's Table X. Assume $r_1 = 2.5 = r_2$, for the initial values of the K's.

$$M = \left[K_1 \frac{(7 - 3m^2)}{60} + 1\right] K_1 \frac{m(1 - m^2)}{6} + m, \quad C = B_i - M B_1 - N B_2$$

$$N = \left[K_2 \frac{(7 - 3n^2)}{60} + 1\right] K_2 \frac{n(1 - n^2)}{6} + n, \quad C' = B_i' - M B_1' - N B_2'$$

$$x_1 = \frac{C(A_2' - A_i') - C'(A_2 - A_i)}{M D}, \quad x_2 = \frac{C'(A_1 - A_i) - C(A_1' - A_i')}{N D}.$$

(A III) Improved Solution, Method (a)

From the formulae of (A I) derive new values of r_1^2 and r_2^2, and repeat the solution of (A II) until a pair of values of M and N reproduces itself.

Method (b)

Alternatively, differential corrections of the first values of x_1 and x_2 may be determined.

$$\overline{M} = \frac{(1 + A_1^2 + A_1'^2)x_1 + (A_1B_1 + A_1'B_1')}{r_1} \, \underline{K_1} \, \frac{m(1 - m^2)}{2},$$

$$\overline{N} = \frac{(1 + A_2^2 + A_2'^2)x_2 + (A_2B_2 + A_2'B_2')}{r_2} \, \underline{K_2} \, \frac{m(1 - m^2)}{2};$$

$$\left[(A_1 - A_i)(M - \overline{M}x_1) - B_1\overline{M}\right]dx_1 + \left[(A_2 - A_i)(N - \overline{N}x_2) - B_2\overline{N}\right]dx_2$$

$$= B_i - (A_1 - A_i)Mx_1 - (A_2 - A_i)Nx_2 - MB_1 - NB_2,$$

$$\left[(A_1' - A_i')(M - \overline{M}x_1) - B_1'\overline{M}\right]dx_1 + \left[(A_2' - A_i')(N - \overline{N}x_2) - B_2'\overline{N}\right]dx_2$$

$$= B_i' - (A_1' - A_i')Mx_1 - (A_2' - A_i')Nx_2 - MB_1' - NB_2'.$$

Apply the correction for planetary aberration,

$$t(\text{true}) = t(\text{observed}) - .005771\rho.$$

Then,using the corrected values of M and N, the right members of the above equations will need to be recomputed to determine the final corrections, but the coefficients on the left will not need revision unless the corrections have been very large. Thus are determined the x_1, y_1, z_1, x_2, y_2, and z_2 which satisfy the three observations.

(A IV) Conversion of Elements

To transform to the usual orbital elements, refer the rectangular coordinates to the ecliptic:

$$\overline{x} = x, \ \overline{y} = y \cos\epsilon + z \sin\epsilon, \ \overline{z} = z \cos\epsilon - y \sin\epsilon.^*$$

ϵ is the mean obliquity of the ecliptic; its trigonometric functions are tabulated in Comrie, Table I.

The components of a vector normal to the orbit plane are:

$$x_n = \overline{y}_1\overline{z}_2 - \overline{z}_1\overline{y}_2, \ y_n = \overline{z}_1\overline{x}_2 - \overline{x}_1\overline{z}_2, \ z_n = \overline{x}_1\overline{y}_2 - \overline{y}_1\overline{x}_2.$$

$x_n^2 + y_n^2 + z_n^2 = S^2$, where S is the double-area of the triangle formed by the two outer radii and their chord.

The sector-triangle ratio is needed, and it is obtained by the classical Gaussian method.*

* Except for some rearrangement, the formulae which follow may be found in various standard treatises. c.f Gauss, Theoria Motus Corporum Coelestium, section 88; Bauschinger, Bahnbestimmung, page 154; Watson, Theoretical Astronomy, page 247; also a short paper by Innes, R. A. S. Monthly Notices, vol. 90, page 810.

$\cos(v_2 - v_1) = (x_1x_2 + y_1y_2 + z_1z_2)/r_1r_2$,

$H^2 = 2(r_1r_2 + x_1x_2 + y_1y_2 + z_1z_2)$, $5/6 + j = 1/3 + (r_1 + r_2)/2H$,

$m^2 = T^2H^{-3}$, $h = m^2/(5/6 + j + \xi)$, where $\xi = 0$ as a first approximation and thereafter it may be taken from tables with the argument $x = m^2/y_0^2 - j$. (Stracke, loc. cit. Table 19b)

The sector-triangle ratio, y_0, satisfies the equation

$y_0^2(y_0 - 1) - hy_0 - h/9 = 0$. For small values of h, this may be solved by using Smiley's table (it is printed in Innes' paper) or Stracke's Table 20 or 21. In the absence of suitable tables, the equation must be solved by Newton's method of successive approximations. After y_0 is found, $p = (Sy_0/T)^2$,

$e \cos v_1 = p/r_1 - 1$, $e \cos v_2 = p/r_2 - 1$,

$$e \sin v_1 = \frac{e \cos v_1 \cos(v_2 - v_1) - e \cos v_2}{\sin(v_2 - v_1)}$$

$$e \sin v_2 = \frac{e \cos v_1 - e \cos v_2 \cos(v_2 - v_1)}{\sin(v_2 - v_1)}$$

$a = p/(1 - e^2)$, $k\, a^{-3/2} = \mu$ (in radians per day)

$$\sin E_1 = \frac{(1 - e^2)^{\frac{1}{2}} e \sin v_1}{e(1 + e \cos v_1)}, \quad \cos E_1 = \frac{e \cos v_1 + e^2}{e(1 + e \cos v_1)}$$

$$E_1 - e \sin E_1 = M_1; \quad \sin i = (x_n^2 + y_n^2)^{\frac{1}{2}}/S, \quad \cos i = z_n/S$$

$$\sin \Omega = x_n/S \sin i, \quad \cos \Omega = -y_n/ S \sin i, \quad \sin u_1 = \bar{z}_1/r_1 \sin i,$$

$$\cos u_1 = (\bar{x}_1 \cos \Omega - \bar{y}_1 \sin \Omega)/r_1, \quad \sin \omega = \frac{\sin u_1 \, e \cos v_1 - \cos u_1 \, e \sin v_1}{e}$$

$$\cos \omega = (e \cos v_1 \cos u_1 - e \sin v_1 \sin u_1)/e.$$

In these forms, the formulae exhibit the extent of the indeterminacy which is inevitable from the nature of the orbit, or the available data.

B. General Solution (Long Time Interval)

When the series expressions for M and N do not provide a sufficient degree of accuracy, it is necessary to use closed forms. (B I) is the same as (A I), and preliminary values of the coordinates are generally known, so that (B II) is unnecessary.

(B III) Improved Solution

Choose as t_1 and t_2 the observations which will make $\cos(v_2 - v_1)$ most nearly zero. With the preliminary values of the coordinates, derive ψ_0, p, e, μ, M_1, $(\cos E_1 - e)$, $\sin E_1$, $(\cos E_2 - e)$, $\sin E_2$. The last two quantities should agree with the corresponding values obtained by solving Kepler's equation: $E_2 - e \sin E_2 = M_1 + \mu(t_2 - t_1)$. This is practically a perfect check, as almost every type of error which can be made in the computation up to this stage will show its effect upon one or both of these quantities. Also derive $(\cos E_i - e)$ and $\sin E_i$ from $E_i - e \sin E_i = M_1 - \mu(t_i - t_1)$.

$$M = \frac{(\cos E_i - e)\sin E_2 - (\cos E_2 - e)\sin E_i}{(\cos E_1 - e)\sin E_2 - (\cos E_2 - e)\sin E_1},$$

$$N = \frac{(\cos E_1 - e)\sin E_i - (\cos E_i - e)\sin E_1}{(\cos E_1 - e)\sin E_2 - (\cos E_2 - e)\sin E_1}.$$

Simultaneously with this solution, compute two other solutions, one in which w_1 is given a small arbitrary increment, say $.01 = h$ or $.001 = h$, while w_2 is held fixed, and another in which w_2 is given the increment while w_1 is held fixed. In each of the three sets, compute $A = \frac{My_1 + Ny_2 + Y_i}{Mx_1 + Nx_2 + X_i}$ and $A' = \frac{Mz_1 + Nz_2 + Z_i}{Mx_1 + Nx_2 + X_i}$. The corrections to w_1 and w_2 are then determined by:

$$[A_i(w_1+h, w_2) - A_i(w_1, w_2)]\,dw_1 + [A_i(w_1, w_2+h) - A_i(w_1, w_2)]\,dw_2 = h[A_i(\text{observed}) - A_i(w_1, w_2)],$$

$$[A'_i(w_1+h, w_2) - A'_i(w_1, w_2)]\,dw_1 + [A'_i(w_1, w_2+h) - A'_i(w_1, w_2)]\,dw_2 = h[A'_i(\text{observed}) - A'_i(w_1, w_2)].$$

The conversion to the usual orbital elements is the same as in (A IV).

C. Parabolic Solution

If a parabolic orbit is to be derived, (C I) will be the same as (A I).

(C II) Preliminary Solution

Compare D with $\overline{D} = (A_1A'_2 - A_2A'_1)X_i + (A'_1 - A'_2)Y_i + (A_2 - A_1)Z_i$. If they are of opposite sign, r is less than 1; if they are of the same sign, r is greater than 1; and if D is nearly zero, r is nearly 1. In the first two cases, x_1 and x_2 may be found as in (A II), using values of r which are in accordance with the indications of the test. In the last case, x_1 and x_2 may be obtained from

$$(1 + A_1^2 + A_1'^2)x_1^2 + 2(A_1 B_1 + A_1' B_1')x_1 + (B_1^2 + B_1'^2 - 1) = 0$$

$$(1 + A_2^2 + A_2'^2)x_2^2 + 2(A_2 B_2 + A_2' B_2')x_2 + (B_2^2 + B_2'^2 - 1) = 0,$$

reject the root which is the negative of the solar coordinate. A closer approximation may be obtained by solving as in (A II) with several arbitrary values of r; two or three will usually be sufficient. In general, some of the resulting values of r will be greater than, and others less than, the assumed values upon which they are based. Interpolate to the point at which the assumed value would also be the value resulting from the solution, and use these as the initial values of x_1 and x_2. This method is used in the illustration which is given below.

(C III) Improved Solution

With the initial values of x_1 and x_2, compute $y = Ax + B$, $z = A'x + B'$; $\eta = 2T(r_1 + r_2)^{-3/2}$, and take $(\eta\xi)$ from Stracke's Table 26. The differential corrections of x_1 and x_2 are determined by the equation of (A III b) which has the larger coefficients and the following equation:

$$\left\{(r_1+r_2)\left[2(\eta\xi)^2-3\eta^2\right]\left[\frac{(1+A_1^2+A_1'^2)x_1+(A_1B_1+A_1'B_1')}{r_1}\right]+2\left[(x_2-x_1)+(y_2-y_1)A_1+(z_2-z_1)A_1'\right]\right\}dx_1 +$$

$$\left\{(r_1+r_2)\left[2(\eta\xi)^2-3\eta^2\right]\left[\frac{(1+A_2^2+A_2'^2)x_2+(A_2B_2+A_2'B_2')}{r_2}\right]-2\left[(x_2-x_1)+(y_2-y_1)A_2+(z_2-z_1)A_2'\right]\right\}dx_2 =$$

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-(\eta\xi)^2(r_1+r_2)^2.$$

(C IV) Conversion of Elements

This part of the work is the same as (A IV), except that for a parabolic orbit the following simplifications obtain: $p = 2q = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2 + z_1 z_2)}{r_1 + r_2 - H}$, and ψ_0 is not needed. $\tan\frac{1}{2}v_1 = (r_1/q - 1)^{\frac{1}{2}}$, and the time of perihelion passage is given by $t_1 - \frac{\sqrt{2}\, q^{3/2}}{k}(\tan\frac{1}{2}v_1 + \frac{1}{3}\tan^3\frac{1}{2}v_1)$.

D. Nearly-parabolic Solution

When the orbit is nearly, but not exactly, parabolic, some of the formulae given in (A IV) are of little value, or even meaningless. Various methods* for dealing with this class of orbits have been given. Over a century of time, however, does not seem to have affected the general superiority of the method devised by Gauss.

The quantities, ψ_0, p, and e, are derived just as in (A IV). Then, using the formulae which accompany Table , find T, the time of perihelion passage, and $\tan\frac{1}{2}v_1$, $\tan\frac{1}{2}v_2$.

$$M = \frac{(\tan\frac{1}{2}v_2 - \tan\frac{1}{2}v_i)(1 + \tan\frac{1}{2}v_2\tan\frac{1}{2}v_i)\,\nu_1^2}{(\tan\frac{1}{2}v_2 - \tan\frac{1}{2}v_1)(1 + \tan\frac{1}{2}v_2\tan\frac{1}{2}v_1)\,\nu_i^2}$$

$$N = \frac{(\tan\frac{1}{2}v_i - \tan\frac{1}{2}v_1)(1 + \tan\frac{1}{2}v_i\tan\frac{1}{2}v_1)\,\nu_2^2}{(\tan\frac{1}{2}v_2 - \tan\frac{1}{2}v_1)(1 + \tan\frac{1}{2}v_2\tan\frac{1}{2}v_1)\,\nu_i^2}.$$

*The classical methods of Bessel, Brünnow, and Gauss are given in Oppolzer, Lehrbuch zur Bahnbestimmung I, page 55. Two new methods have appeared in R. A. S. Monthly Notices, vol 93, page 33 and page 777. The Marth modification of Gauss' method is given in Astronomische Nachrichten, vol. 43, page 115. The present writer has computed tables of the natural values of the quantities used in the Gauss-Marth method, Table

These are also the closed forms of M and N for the parabola if the ν's are set equal to unity.

Alternatively, Lambert's generalized equation may be used, with the consequent elimination of ψ_0. The equation is

$$6k(t_2 - t_1) = \sum_{n=0}^{n=\infty} C_n a^{-n}\left[(r_1 + r_2 + s)^{n+\frac{3}{2}} \mp (r_1 + r_2 - s)^{n+\frac{3}{2}}\right],$$

where $C_0 = 1$, $C_1 = 3/40$, $C_2 = 9/896, \ldots$ $C_n = \frac{24}{1}\,\frac{1}{2}\,\frac{3}{4}\,\frac{5 \ldots (2n-1)}{6\,(2n)(2n+3)2^{2n+3}}$.

The upper or lower sign is to be used according as $\sin(v_2-v_1) > 0$ or $\sin(v_2-v_1) < 0$. When the rectangular coordinates at two epochs are known, the equation reduced to a power series in $1/a$ which may be solved by Newton's method of approximation.

Then $G_1 = 1 - \frac{r_1 + r_2 - s}{2a}$, $G_2 = 1 - \frac{r_1 + r_2 + s}{2a}$,

$\cos(E_2 - E_1) = G_1 G_2 \pm \left\{(1 - G_1^2)(1 - G_2^2)\right\}^{\frac{1}{2}}$, where the upper or lower sign is to be used according as $1/a$ is positive or negative (corresponding to an ellipse or hyperbola, respectively).

$$p = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2 + z_1 z_2)}{r_1 + r_2 - \sqrt{r_1 r_2\left[1 + \cos(v_2 - v_1)\right]\left[1 + \cos(E_2 - E_1)\right]}}, \quad e^2 = 1 - \frac{p}{a},$$

and the rest of the work proceeds as before.

ILLUSTRATIONS

The general method for short time intervals will now be illustrated by observations of the minor planet 1909 HC. These have been chosen in order that the present method may be directly compared with the classical methods that have been in use. The same example is given by Williams*, using both the La Placian and the Gaussian methods, and by Crawford,* using Leuschner's method. The latter, however, is only a differential correction of a previous solution. For an illustration of the parabolic orbit method, the observational material of Merton's example* will be used, again for the

Williams, Calculation of Orbits etc. pages 132 and 162.
Crawford, loc. cit. page 170. Merton, loc. cit. page 713.

purpose of comparison. The work is arranged to follow the order of the Collection of Formulae.

Orbit of 1909 HC

(A I) Observations

1910 G.M.T.	Nov. 7.8205	Nov. 26.7480	Nov. 48.6262
α (1910.0)	3° 50' 24".3	3° 13' 03".0	4° 54' 19".5
δ (1910.0)	25 11 10.5	22 29 31.3	20 14 51.9
X	-.7000687	-.4306907	-.0628371
Y	-.6429399	-.8143496	-.9007098
Z	-.2789211	-.3532745	-.3907417

(These include the topocentric reductions, the computation is omitted.)

A	.067123	.056215	.085826
A'	.471329	.414703	.370230
P	1.107545	1.084038	1.069784
B	.5959492	.7901383	.8953167
B'	-.0510416	.1746658	.3674775

$$r_1^2 = 1.2266565\ x_1^2 + .0318890\ x_1 + .3577607$$

$$r_2^2 = 1.1444364\ x_2^2 + .4257853\ x_2 + .9366317$$

$A_1 - A_i$	.010908	$A_2 - A_i$	.029611
$A_2' - A_i'$	.056626	$A_2' - A_i'$	-.044473

D -.00216186

(A II) Preliminary Solution

T	.7019437	T^2	.4927250
m	.5361555	n	.4638445
$m(1-m^2)/6$	.063672	$n(1-n^2)/6$	.060675
$(7-3m^2)/60$	.1022	$(7-3n^2)/60$	.1058
K_1	.031534	K_2	.031534
M	.5381698	N	.4657642
C	.0524100	C'	.0309770
x_1	2.791774	x_2	2.611798

(A III) Improved Solution, Method (a)

r_1^2	10.007351	r_2^2	9.858757		
K_1	.015564	K_2	.015917		
M	.5371481	N	.4648119		
C	.0538715	C'	.0312748		
x_1	2.860654	x_2	2.696278	x_i	2.789857

Correct for planetary aberration.

ρ_1	2.3930	ρ_2	2.8172	ρ_i	2.5574
t_1	7.8067	t_2	48.6099	t_i	26.7332

T	.7019007	T^2	.4926646
m	.5361516	n	.4638484
$m(1-m^2)/6$	.063672	$n(1-n^2)/6$	.060675
$(7-3m^2)/60$	.1022	$(7-3n^2)/60$	.1058
r_1^2	10.487137	r_2^2	10.404623
K_1	.014507	K_2	.014680
M	.5370767	N	.4647405
C	.0539779	C'	.0321974
x_1	2.865700	x_2	2.702459
r_1^2	10.522738	r_2^2	10.445444
K_1	.014433	K_2	.014594
M	.5370719	N	.4647353
C	.0539855	C'	.0312990
x_1	2.866056	x_2	2.702897

These values are final, for, with the series expansions of M and N, they represent the intermediate observation exactly. With the values of M and N given by Williams, the residuals are only $0''.4$ and $0''.1$. The alternative method of improving the solution will also be illustrated.

(A III) Improved Solution, Method (b)

x_1	2.791774	x_2	2.611798
r_1^2	10.007351	r_2^2	9.858757
K_1	.015564	K_2	.015917
$\underline{M}$	.5371481	$\underline{N}$	.4648119
$\overline{M}$	.001022	$\overline{N}$	.000941

$$.005219\, dx_1 + .012848\, dx_2 = .0015663; \quad -.0000079$$

$$\text{(F)} \quad .030307\, dx_1 - .020908\, dx_2 = .0003486; \quad -.0000021$$

(F = .614502, the elimination factor)

$$.023843\, dx_1 = .0017805; \quad -.0000092$$

$$dx_1 = .074677; \quad -.000385$$

$$dx_2 = .091574; \quad -.000458$$

x_1	2.866451	x_2	2.703372	x_i	2.796268

Correct for planetary aberration.

ρ_1	2.3994	ρ_2	2.8248	ρ_i	2.5644
t_1	7.8067	t_2	48.6099	t_i	26.7332

T	.7019007	T^2	.4926646
m	.5361516	n	.4638484
$m(1-m^2)/6$	.063672	$m(1-m^2)/6$	.060675
$(7-3m^2)/60$	.1022	$(7-3m^2)/60$	.1058

r_1^2	10.528043	r_2^2	10.451481
K_1	.014422	K_2	.014581
M	.5370712	N	.4647345

Recompute the right members of the above equations and solve for the final corrections to x_1 and x_2.

x_1	2.866066	x_2	2.702914

The discrepancies among the various solutions indicate the range of values which will give a satisfactory representation of the three observations. The results are

x_1	x_2	
2.866056	2.702897	Method (a)
2.866066	2.702914	Method (b)
2.866145	2.703045	Williams, La Placian method
2.866029	2.703042	Williams, Gaussian method
2.866001	2.702814	Crawford, Leuschner's method.

The conversion to the elliptic elements involves nothing new, so that these computations are not given.

From the number of figures employed, it is evident that, compared with the other methods of solution cited above, the present method reduces **very** materially the amount of computation necessary to obtain the solution. For instance, the work of (A III) is equivalent to the entire example given by Crawford, and the residuals here removed are even larger than those of his problem. A more careful inspection emphasizes the fact that the operations involved in the computations are also much simpler, so that the solution is obtained with greater ease and less probability of errors.

Orbit of Comet 1925-c

(C I) Observations

1925 April	5.0000	9.5000	14.0000
α (1925.0)	336° 39' 42"9	337° 47' 54"4	339° 03' 16"3
δ (1925.0)	16 30 08.1	21 17 30.8	26 24 27.5
X	.9678457	.9465962	.9197381
Y	.2329943	.3011409	.3675075
Z	.1010649	.1306259	.1594120
A	-.431456	-.408124	-.382773
A'	.322654	.420928	.531705
P	1.135896	1.159201	1.195501
B	-.6505771	-.6874695	-.7195584
B'	.2112144	.2678229	.3296173

$$r_1^2 = 1.2902599\, x_1^2 + .6976891\, x_1 + .4678621$$

$$r_2^2 = 1.4292254\, x_2^2 + .9013734\, x_2 + .6264119.$$

$A_1 - A_i$	-.023332	$A_2 - A_i$	.025351
$A_1' - A_i'$	-.098274	$A_2' - A_i'$	.110777
D	-.0000933	$\overline{D}$	-.156843

(C II) Preliminary Solution

From Lambert's theorem, it is obvious the r exceeds 1, but D is very small; therefore derive preliminary solutions with r = 1.0, 1.1, 1.2, respectively. The results are:

r (assumed)	1.0	1.1	1.2
x_1	-1.031	.292	1.213
x_2	-.976	.201	1.021
r_1		.884	1.792
r_2		.930	1.743

The result of using r = 1 is approximately the Earth's coordinate, and the real solution lies between r = 1.1 and r = 1.2. Using simple proportions:

$$x_1 = .292 + \frac{(1.100 - .884)(1.213 - .292)}{(1.1 - .884) - (1.2 - 1.792)} = .538$$

$$x_2 = .201 + \frac{(1.100 - .930)(1.021 - .201)}{(1.1 - .930) - (1.2 - 1.743)} = .397.$$

(C III) Improved Solution

T	.154819	T^2	.0239689		
m	.500000	n	.500000		
$m(1-m^2)/6$	.0625	$n(1-n^2)/6$	.0625		
$(7-3m^2)/60$	.1042	$(7-3n^2)/60$	.1042		
x_1	.538000	x_2	.397000	$x_2 - x_1$	-.141000
y_1	-.8827000	y_2	-.871519	$y_2 - y_1$	.011181
z_1	.384802	z_2	.540704	$z_2 - z_1$	.155902
r_1^2	1.216676	r_2^2	1.209515	S_g^2	.0443114
r_1	1.103030	r_2	1.099780		

$r_1 + r_2$	2.202810	$\sqrt{r_1 + r_2}$	1.484187	η	.0947084
K_1	.017860	K_2	.018019	$(\eta\xi)$	.0947439
M	.501118	N	.501128		
$\overline{M}$	.002871	$\overline{N}$	.002844	S_d^2	.0435569

$$-.049701\, dx_1 + .054451\, dx_2 = .001255$$

$$\text{(F)}\quad -.209699\, dx_1 + .106508\, dx_2 = .000755$$

(F = -.511239, the elimination factor)

$$.057505\, dx_1 = .000869$$

$$dx_1 = .015112$$

$$dx_2 = .036842$$

Correct for planetary aberration.

x_1	.553112	x_i	.494585	x_2	.433842
ρ_1	1.7276	ρ_i	1.6706	ρ_2	1.6182
t_1	4.99003	t_i	9.49036	t_2	13.99066

T	.154830	T^2	.0239723
m	.499998	n	.500002
$m(1-m^2)/6$	.0625	$n(1-n^2)/6$	.0625
$(7-3m^2)/60$	.1042	$(7-3n^2)/60$	.1042

x_1	.553112	x_2	.433842	$x_2 - x_1$	-.119270
y_1	-.889220	y_2	-.885621	$y_2 - y_1$	.003599
z_1	.389678	z_2	.560293	$z_2 - z_1$	.170615
r_1^2	1.248494	r_2^2	1.286472	S_g^2	.0433478
r_1	1.117360	r_2	1.134227		
$r_1 + r_2$	2.251587	$\sqrt{r_1 + r_2}$	1.500529	η	.0916541
K_1	.017184	K_2	.016429	$(\eta\xi)$	.0916863
M	.501074	N	.501031		
$\overline{M}$	.002742	$\overline{N}$	.002564	S_d^2	.0426174

$$\text{(F)}\quad -.049673\, dx_1 + .054534\, dx_2 = -.000002$$

$$-.149507\, dx_1 + .042031\, dx_2 = .000730$$

(F = -.770727, the elimination factor)

$$-.111223\, dx_1 = .000732$$

$$dx_1 = -.006577$$

$$dx_2 = -.006027$$

The final values are $x_1 = .546535$ and $x_2 = .427815$; the residuals are entirely removed.

Part II

The Orbit of Biarmia (1146)

The minor planet, Biarmia (1146), was discovered by Dr. G. Neujmin at the Simeis Observatory on May 7, 1929. At the present time, the available observations extend over an entire revolution around the sun, and the object has deviated considerably from its preliminary orbit. At the suggestion of the Astronomisches Rechen-Institut of Berlin, an improvement of the orbit, including all the observations and the perturbations by Jupiter and Saturn, was undertaken. The observations, reduced to the Standard Equinox of 1950.0, are given in Table II.

Elements of the preliminary orbit were furnished by the Rechen-Institut. These elements represented the positions of 1929 very well. The first step in the work was to compute approximate perturbations from 1929 to 1930, and adjust the orbit to represent the 1930 observations along with those of 1929. Then this adjusted orbit would be sufficiently accurate to allow the computation of the complete perturbations for the whole interval of time involved. Also, at the time that the work was done, there were only three known observations after 1930, so that the final result would depend largely upon the single observation of 1934.

TABLE II Observations

No.	1929 U. T.	α (1950.0)	δ (1950.0)	Place	Authority	$\cos\delta\,\Delta\alpha$	$\Delta\delta$
1	May 7.99000	$16^h52^m15^s.59$	$-9^\circ25'58''.6$	Simeis	Pulkova Bulletin No. 116	$-2''.3$	$1''.2$
2	May 9.98556	16 51 20.28	-9 02 15.7	"		0.3	1.6
3	May 11.01472	16 50 49.46	-8 50 03.9	"	"	-0.8	0.0
4	May 12.99097	16 49 46.80	-8 26 39.3	"	"	1.4	-0.1
5	May 13.96410	16 49 14.32	-8 15 10.2	"	"	-1.0	-0.3
6	May 15.95017	16 48 04.68	-7 51 52.4	"	"	-0.8	-1.9
7	Jun 1.90368	16 36 11.67	-4 48 35.3	"	"	-1.2	0.6
8	Jun 5.90556	16 33 12.37	-4 12 03.9	"	"	0.6	0.0
9	Jun 7.84236	16 31 46.96	-3 55 37.6	"	"	-1.6	0.0
10	Jun 8.84382	16 31 03.54	-3 47 27.1	"	"	0.5	0.8
11	Jun 11.87861	16 28 54.92	-3 24 13.5	"	"	1.7	-1.1
12	Jul 6.84174	16 17 20.39	-1 42 20.2	"	"	0.3	4.0
13	Jul 9.85438	16 16 55.97	-1 40 33.7	"	"	0.4	3.0
14	Aug 2.85222	16 22 26.61	-2 27 00.0	"	"	1.8	2.7

(cont.)

TABLE II Observations (cont.)

No.	1930 U. T.	α(1950.0)	δ(1950.0)	Place	Authority	$\cos\delta\,\Delta\alpha$	$\Delta\delta$
15	Sep 28.94271	$1^h26^m48^s.35$	$16^\circ52'59''.5$	Heidelberg	A.N. 242:189	$0''.0$	$-2''.0$
16	Sep 29.04792	1 26 43.81	16 52 11.7	Simeis	Letter	-1.1	-0.4
17	Oct 3.01782	1 23 55.86	16 19 52.0	Heidelberg	A.N. 242:189	-0.8	1.9
18	Oct 15.86344	1 14 24.13	14 24 16.9	Heidelberg	A.N. 242:189	0.6	1.6
19	Oct 16.27083	1 14 05.85	14 20 23.7	Flagstaff	Letter	-0.9	-2.3
20	Oct 16.86622	1 13 39.64	14 14 48.1	Heidelberg	A.N. 242:189	0.6	0.4
21	Oct 17.84375	1 12 56.47	14 05 32.4	Simeis	Letter	0.3	-0.6
22	Oct 17.89014	1 12 54.50	14 05 06.3	Heidelberg	A.N. 242:189	1.2	-0.1
23	Oct 18.23958	1 12 39.14	14 01 48.8	Flagstaff	Letter	0.6	0.3
24	Oct 18.97500	1 12 06.93	13 54 49.4	Simeis	Letter	2.3	0.1
	1931 U. T.						
25	Dec 5.96742	5 22 06.80	7 11 36.8	Alger	Letter	-3.5	0.6
26	Dec 16.26860	5 13 51.64	6 41 30.8	Yerkes	A.J. 42:166	-5.5	1.4
27	Dec 31.21			Flagstaff	Letter		

(cont.)

TABLE II Observations (cont.)

No.	1932 U. T.	α(1950.0)	δ(1950.0)	Place	Authority	$\cos\delta\ \Delta\alpha$	$\Delta\delta$
		h m s	° ' "			"	"
28	Jan 5.28			Flagstaff	Letter		
	1934 U. T.						
29	Apr 14.92160	13 31 05.06	-14 49 45.8	Simeis	Letter	0.5	-0.2
30	May 24.97828			Johannesbg	Letter		

The asteroid, 1913 RO, is probably identical with (1146), (c.f. A.N. 195:128) but this point has not been considered in the present work. The writer gratefully acknowledges the cooperation of Drs. G. Neujmin (Simeis), C. O. Lampland (Flagstaff), and M. G. Reiss (Algers) in communicating their observations by letter.

In the following work, the perturbations are taken into account by using Encke's method* of computing the perturbation in each of the rectangular coordinates. These are then to be

*This method is developed in various standard works. See Watson, loc. cit., page 429. The formulae are given in Comrie, loc. cit., page 154.

added to the corresponding solar coordinates before the osculating orbit is compared with an observation. All the required data relative to the perturbing planets are contained in Comrie, loc. cit. The coordinates of the perturbed body are obtained with sufficient accuracy by the formula:

$$w = P_w X + Q_w Y, \qquad (w = x, y, z),$$

where X and Y may be taken from Tables of X and Y Elliptic Rectangular Coordinates, Appendix to Circular No. 71 of the Union Observatory of South Africa.

The following data were derived from the preliminary elements:

Epoch of Osculation = 1929 July 8.0 = J.D. 2425800.5

$P_x = 0.192662 \quad P_y = -3.009347 \quad P_z = -0.460515$

$Q_x = 2.904675 \quad Q_y = 0.104122 \quad Q_z = 0.534799$

$M_o = -7\overset{\circ}{.}64137 \quad 40\mu = 7\overset{\circ}{.}39970$

The approximate perturbations which were obtained at intervals of 40 days by numerical integration are given in Table III. The values of the perturbations at the epochs of observations Nos. 1, 14, 18, and 20 were obtained by interpolating directly

TABLE III

Date	10^7	10^7	10^7
J.D. 2425680.5	-275	-356	-165
5720.5	-118	-153	-74
5760.5	-29	-37	-19
5800.5	0	0	0
5840.5	-28	-37	-19
5880.5	-111	-147	-77
5920.5	-240	-333	-172
5960.5	-406	-588	-301
6000.5	-601	-898	-458
6040.5	-829	-1239	-638
6080.5	-1107	-1589	-838
6120.5	-1464	-1931	-1056
6160.5	-1941	-2258	-1295
6200.5	-2575	-2578	-1562
6240.5	-3408	-2914	-1862
6280.5	-4474	-3297	-2205
6320.5	-5799	-3771	-2598
Obs. No. 1	-67	-87	-43
" " 14	-12	-15	-8
" " 18	-4041	-3144	-2070
" " 20	-4052	-3148	-2073

from the integration table.*

*A table of the necessary coefficients is given in the Lick Observatory Bulletin, No. 445.

From the preliminary elements and these approximate perturbations, the following two rectangular coordinates were derived: Epoch 1929 Aug. 2.84237, $y_1 = -2.2630761$,
1930 Oct. 15.85139, $x_2 = 2.8422631$.

These were used as the basis of an orbit which represented the observation of 1929 May 7 within one second of arc, so that no adjustment of the orbit was needed. The observation of 1930 Oct. 16 was used as a check.

By choosing 1930 Oct 31.0 as the osculation date, the perturbations for the whole revolution can be conveniently represented without the necessity of a further rectification. A new integration table was computed from the following basic elements:

Epoch of Osculation = 1930 Oct 31.0 = J.D. 2426280.5

$P_x = 0.199182 \quad P_y = -3.007525 \quad P_z = -0.459238$

$Q_x = 2.996289 \quad Q_y = 0.114106 \quad Q_z = 0.552284$

$M_o = 81^\circ.15849 \quad 80\mu = 14^\circ.810881$

The 80-day interval is sufficiently accurate when the object is not in the neighborhood of perihelion.

The two 1931 observations were represented as soon as the perturbations of those epochs were derived. The representations showed a discrepancy between these two observations. Later correspondence proved this to be an error of 9".7 in the declination of observation No. 25.

The basic elements for the perturbations beyond 1931 were improved slightly, so as to give a better representation of observation No. 26. The improved values are:

Epoch of Osculation = 1930 Oct 31.0 = J.D. 2426280.5

$P_x = 0.196057 \quad P_y = -3.006267 \quad P_z = -0.459688$

$Q_x = 2.995146 \quad Q_y = 0.110997 \quad Q_z = 0.551523$

$M_o = 81^\circ.25016 \quad 80\mu = 14^\circ.820908$

After the perturbations for the whole revolution were known, it was possible to compare all the observations with the osculating orbit. The scant observational material after 1930, and the evident discrepancy between the two 1931 observations, led to the decision to base the final orbit upon one normal place of 1929, another of 1930, and the single observation of 1934. The following normal places were formed:

1929 May 12.0 (Obs. Nos. 1 - 6)	1929 June 7.0 (Obs. Nos. 7 - 11)	1930 Oct 12.0 (Obs. Nos. 15 - 24)
$A_y = .3137061$	$A_y = .4019128$	$A_x = .3506663$
$A'_y = .1592116$	$A'_y = .0761710$	$A'_x = .2840656.$

By using the second and third normal places, the corresponding radius vectors from the sun are most nearly perpendicular. The resulting elements which satisfy the observation of 1934 are:

Epoch of Osculation = 1930 Oct. 31.0 = J.D. 2426280.5

$P_x = 0.197020 \quad P_y = -3.006406 \quad P_z = -0.459528$

$Q_x = 2.995275 \quad Q_y = 0.111964 \quad Q_z = 0.551691$

$M_o = 81^\circ.230477 \quad 80\mu = 14^\circ.819509.$

In order to obtain a slightly better approximation to the perturbations, and especially to provide an adequate check upon the numerical computations, all the perturbations were recomputed from this last set of elements, they are given in Table IV.

TABLE IV

Date	10^7	10^7	10^7
J.D. 2425720.5	-5712	-2946	-2286
5760.5	-5357	-1983	-2001
5800.5	-5037	-1437	-1777
5840.5	-4632	-1192	-1579
5880.5	-4100	-1106	-1379
5920.5	-3464	-1062	-1168
5960.5	-2787	-985	-948
6040.5	-1551	-688	-536
6120.5	-673	-334	-228
6200.5	-168	- 86	- 54
6280.5	0	0	0
6360.5	-182	- 92	- 48
6440.5	-768	-404	-185
6520.5	-1816	-1029	-406
6600.5	-3353	-2105	-709
6680.5	-5356	-3798	-1090
6760.5	-7743	-6286	-1536
6840.5	-10369	-9734	-2025
6920.5	-13026	-14272	-2518
7000.5	-15446	-19966	-2958
7080.5	-17318	-26766	-3262
7160.5	-18338	-34442	-3321
7240.5	-18314	-42488	-3016
7280.5	-17936	-46388	-2693
7320.5	-17403	-50010	-2250
7360.5	-16862	-53178	-1697
7400.5	-16552	-55701	-1069
7440.5	-16825	-57408	-434
7480.5	-18170	-58201	85
7520.5	-21203	-58146	305
7560.5	-26627	-57606	-36

A final correction, using the revised values of the perturbations, gave the following rectangular coordinates. These form the basis of the adopted osculating orbit.

Epoch of Osculation = 1930 Oct. 31.0 = J.D. 2426280.5

t_1 = 1929 June 7.0 t_2 = 1930 Oct. 12.0

$x_1 = -0.7460529$ $x_2 = 2.8452414$
$y_1 = -2.1521461$ $y_2 = 0.9515113$
$z_1 = -0.4868893$ $z_2 = 0.6630990$

The residuals of Table II were obtained from this set of elements.

The epoch of osculation was then transferred to 1934 March 24.0 and a new integration table of the perturbations was started. The basic elements for this table are:

Epoch of Osculation = 1934 Mar. 24.0 = J.D. 2427520.5

$P_x = 0.177150$ $P_y = -3.012403$ $P_z = -0.462860$

$Q_x = 3.001833$ $Q_y = 0.092275$ $Q_z = 0.548344$

$M_o = -48\overset{\circ}{.}87498$ $40\mu = 7\overset{\circ}{.}3908879$

The following ephemeris for the 1935 opposition was obtained by interpolating with the equations (5) of Part I between the rigorously computed positions of Aug. 26.0 and Oct. 5.0.

EPHEMERIS

1935 U. T.	α (1925.0)	δ (1925.0)
Aug 26.0	$23^h\ 19^m\ 27\overset{s}{.}8$	$17^\circ\ 09'\ 09''$
Sep 3.0	23 14 08.3	16 17 30
Sep 11.0	23 08 30.2	15 10 32
Sep 19.0	23 03 01.6	13 51 39
Sep 27.0	22 58 08.4	12 25 09
Oct 5.0	22 54 13.8	10 55 58

The elements which were forwarded to the Rechen-Institut were obtained for the osculation epoch of 1936 Feb. 2.0, and reduced to the equinox of 1925.0. The final results are:

Epoch of Osculation = 1936 Feb. 2.0 = J.D. 2428200.5

x_0	2.8611909	x_0'	-0.0415971	(1950.0)
y_0	0.8736664	y_0'	0.5626359	
z_0	0.6516328	z_0'	0.0849139	
aP_x	0.1641772	bQ_x	2.9103789	(1925.0)
aP_y	-3.0117294	bQ_y	0.0780960	
aP_z	-0.4619993	bQ_z	0.5251379	
M	76°.761045	i	17°.221278	(1925.0)
μ	0°.18491037	ω	59.031784	
e	.2449668	Ω	215.328814	

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