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ON HYPERGEOMETRIC SUMMABILITY

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by

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H.David Lipsich

Introduction

In the general theory of summability the notion of a sequence to sequence transformation arising out of an infinite matrix plays a central role. Thus, if we let

$$\| a_{i,j} \| , \quad i,j = 0, 1, 2, \ldots$$

be an infinite matrix there is defined a transformation of any sequence $\{ s_v \}$ as follows:

$$\sigma_n = \sum_{v=0}^{\infty} a_{nv} s_v$$

According to the well known theorem of Silverman and Toeplitz (Szasz,1,p.16)*, the transform is a regular transform, that is, transforms any sequence $\{s_v\}$ converging to the value s into a sequence $\{\sigma_v\}$ converging to the value s if and only if the following three conditions are satisfied:

$$\text{(a)} \quad \lim_{i \to \infty} a_{i,j} = 0 , \quad j = 1, 2, 3, \ldots$$

0.1 $$\text{(b)} \quad \sum_{j=0}^{\infty} | a_{i,j} | \leq C \quad i = 1, 2, 3, \ldots$$

$$\text{(c)} \quad \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{i,j} = 1$$

Thus, in order to define Cesaro summability of order α we consider the function

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* The numbers refer to the references in the bibliography.

$$(1-x)^{-\alpha} = \sum_{\nu=0}^{\infty}(-1)^{\nu}\binom{-\alpha}{\nu}x^{\nu} = \sum_{\nu=0}^{\infty}\frac{\alpha(\alpha+1)\ldots(\alpha+\nu-1)}{\nu!}\,x^{\nu} = \sum_{\nu=0}^{\infty}A_{\nu}^{\alpha-1}x^{\nu}$$

and hence $A_{n}^{\alpha} = \sum_{\nu=0}^{n} A_{\nu}^{\alpha-1}$.

We then choose

$$a_{ij} = \begin{cases} \dfrac{A_{i-j}^{\alpha-1}}{A_i}, & i \geq j \\ 0, & i < j. \end{cases}$$

so that

$$\sigma_{n}^{\alpha} = \frac{\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_{\nu}}{A_{n}^{\alpha}}.$$

Thus the terms in the matrix defining (C,α) summability are related in a very specific manner to the coefficients of the binomial function.

Now a natural generalization of the binomial function is the Hypergeometric function defined as follows:

$$F(\alpha,\beta,\gamma,x) = F(\beta,\alpha,\gamma,x) = \sum_{\nu=0}^{\infty} b_{\nu}x^{\nu}$$

where

$$b_{\nu} = \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+\nu-1)\beta(\beta+1)\ldots(\beta+\nu-1)}{\gamma(\gamma+1)\ldots(\gamma+\nu-1)\,\nu!}, \quad \nu \geq 1, \quad b_0 = 1.$$

By analogy with (C,α) summability, therefore, we consider the triangular matrix

$$a_{ij} = \begin{cases} \dfrac{b_{i-j}}{B_i}, & i \geq j \\ 0, & i < j, \end{cases}$$

where

$$B_i = \sum_{\nu=0}^{i} b_\nu$$

Thus we shall say that the sequence $\{s_\nu\}$ is summable $H(\alpha,\beta,\gamma)$ if $\lim_{m\to\infty} \sigma_m$ exists, where

$$\sigma_m = \sum_{\nu=0}^{m} \frac{b_{m-\nu} s_\nu}{B_m}$$

We thus define a three parameter family of summability methods. In the sequel we shall study the properties of these methods, concentrating on their relationships to the Cesaro scale of methods and harmonic summability, and to the application of the methods to the summation of Fourier series We shall compare our methods with Abel summability and thence deduce an elementary Tauberian theorem. Finally in the last section we shall set down some as yet unsolved problems and conjectures with regard to the method $H(\alpha,\beta,\gamma)$.

In connection with the summability method under consideration it is important to note that summability $H(\alpha,\gamma,\gamma)$ coincides with (C,α) summability and that all of the methods fall under the general classification of Norlund summability methods.

<u>Article 1</u>. Behaviour of the coefficients b_ν and the regularity theorem.

In this article we shall investigate monotonicity properties of the coefficients b_ν in the Hypergeometric function. We shall furthermore establish a regularity theorem for summabity $H(\alpha,\beta,\gamma)$. This latter will be arrived at through a simple application of the asymptotic estimates for the quantities b_ν and B_ν. These estimates were first given by Hill (1). We shall establish these estimates as simple consequences of a theorem of Cauchy. We shall have occasion to use the following three results:

<u>Theorem 1.1</u> (Zygmund 1, p.42).

If $A_n^{\alpha-1} = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!}$ then $A_n^{\alpha-1} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}$.

<u>Theorem 1.2.</u> (Polya and Szego 1,p.11)

If the two sums

$$|a_1 + a_2 + \dots a_n|, \quad |a_1| + |a_2| + \dots |a_n|$$

both tend to infinity in such a way that

$$|a_1| + |a_2| + \dots |a_n| \leq k|a_1 + a_2 + \dots a_n|$$

where k is independent of n and if $\lim_{n\to\infty} \frac{c_n}{a_n}$ exists, then

$$\lim_{n\to\infty} \frac{c_1+c_2+\cdots c_n}{a_1+a_2+\cdots a_n} = \lim_{n\to\infty} \frac{c_n}{a_n}.$$

Theorem 1.3. (Copson, 1.).

The series

$$F(\alpha,\beta,\gamma,1) = \sum_{\nu=0}^{\infty} b_\nu$$

converges when $\gamma > \alpha + \beta$ and diverges when $\gamma \leq \alpha+\beta$.

We suppose now, and subsequently, that α, β, γ are positive real numbers.

If we put

$$b_n = \frac{\alpha_{n-1}\beta_{n-1}}{\gamma_{n-1}\, n!}$$

we find

$$b_{n+1} - b_n = \frac{\alpha_{n-1}\beta_{n-1}}{\gamma_n (n+1)!}\left[\alpha\beta - \gamma + (\alpha+\beta-\gamma-1)n\right]$$

which is positive or negative according as the term in brackets is positive or negative.

We consider three cases:

(a). $\alpha + \beta - \gamma - 1 > 0$

It is clear in this case, by considering the linear function $\alpha\beta - \gamma + (\alpha+\beta-\gamma-1)x$

that $b_{n+1} - b_n > 0$ for all sufficiently large n.

(b). $\alpha + \beta - \gamma - 1 < 0$

Here similar considerations show that $b_{n+1} - b_n < 0$ for all sufficiently large n.

(c). $\alpha + \beta - \gamma - 1 = 0$

In this case the term in brackets becomes

$$\alpha\beta - \gamma = \alpha\beta - \alpha - \beta + 1 = (\alpha - 1)(\beta - 1)$$

which is positive for $\alpha \gtrless 1, \beta \gtrless 1$ and negative for $\alpha > 1,\ 0 < \beta < 1,$ or $0 < \alpha < 1,\ \beta > 1$.

We have thus proved the following theorem:

<u>Theorem 1.4.</u>

If α, β, γ are positive real numbers then a necessary and sufficient condition that $\{b_n\}$ be ultimately monotonic increasing is that one of the following hold:

(a) $\alpha + \beta - \gamma - 1 > 0$

(b) $\alpha + \beta - \gamma - 1 = 0$ and $\alpha > 1, \beta > 1$ or $\alpha < 1, \beta < 1$.

A necessary and sufficient condition that $\{b_n\}$ be ultimately monotonic decreasing is that one of the following hold:

(a) $\alpha + \beta - \gamma - 1 < 0$

(b) $\alpha + \beta - \gamma - 1 = 0$ and either $0 < \alpha < 1, \beta > 1,$ or $\alpha > 1, 0 < \beta < 1.$

We proceed now to the proof of the theorem which gives us the desired central asymptotic estimates for b_ν and B_n.

Theorem 1.5.

(a). $b_n \sim n^{\alpha+\beta-\gamma-1} \dfrac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}$

(b). $$B_n \sim \begin{cases} \dfrac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-\gamma)}\, n^{\alpha+\beta-\gamma}, & \gamma<\alpha+\beta. \\[2ex] \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \log n, & \gamma=\alpha+\beta. \end{cases}$$

Proof:

The first part of the theorem follows immediately from theorem 1.1 when we note that

$$b_n = \frac{A_n^{\alpha-1} A_n^{\beta-1}}{A_n^{\gamma-1}} \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\, n^{\alpha+\beta-\gamma-1} .$$

For the proof of the second part of the theorem we use theorem 1.2, setting

$$c_n = b_n \;, \quad a_n = n^{\alpha+\beta-\gamma-1} .$$

Thus

$$\lim_{n\to\infty} \frac{b_1+b_2+\cdots b_n}{1+2^{\alpha+\beta-\gamma-1}+\cdots n^{\alpha+\beta-\gamma-1}} = \begin{cases} \dfrac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-\gamma)}\, n^{\alpha+\beta-\gamma}, & \gamma<\alpha+\beta. \\[2ex] \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \log n, & \gamma=\alpha+\beta. \end{cases}$$

where we have made use of the well known facts that

$$\sum_{\nu=1}^{n} \nu^{\alpha+\beta-\gamma-1} \sim \Gamma(\alpha+\beta-\gamma)\, n^{\alpha+\beta-\gamma}, \quad \gamma < \alpha+\beta$$

$$\sum_{\nu=1}^{n} \frac{1}{\nu} \sim \log n.$$

This completes the proof of the theorem.

We turn now to the proof of the following important regularity theorem.

<u>Theorem 1.6.</u>

If α, β, γ are all positive real numbers and if $\gamma \leq \alpha+\beta$, then the method $H(\alpha, \beta, \gamma)$ is regular.

Proof:

Here we need only verify that the conditions 0.1 are satisfied. This however, is an immediate consequence of theorem 1.5 and our theorem is thus established.

In connection with theorem 1.6 it is important to remark that in accordance with theorem 1.3 the method $H(\alpha, \beta, \gamma)$ is regular even when $\gamma > \alpha+\beta$. This is an interesting case from the point of view of consideration of the question of when the method thus defined will sum any divergent sequences. In this connection see Piranian (1). We shall content ourselves with a study of the method $H(\alpha, \beta, \gamma)$ for $\gamma \leq \alpha+\beta$ Thus in the sequel we assume $\gamma \leq \alpha+\beta$ unless otherwise specified.

Article 2. $H(1,\beta,\gamma)$ Summability.

In this article we shall concern ourselves with the two parameter family of summability methods $H(1,\beta,\gamma)$. We shall, in particular, be concerned with a comparison between $(C,1)$ and $H(1,\beta,\gamma)$ summability. Since these two methods coincide when $\beta=\gamma$ it is natural to expect an inclusion theorem centering about $\beta=\gamma$. In this direction we have the following:

Theorem 2.1.

If $\beta\geq\gamma$ then any sequence $\{s_n\}$ which is summable $(C,1)$ is summable $H(1,\beta,\gamma)$. On the other hand if $\beta<\gamma$ then every sequence summable $H(1,\beta,\gamma)$ is summable $(C,1)$.

Proof:

We note first according to theorem 1.4 that in this case

$$b_n \uparrow \quad \text{if } \beta>\gamma \; ; \quad b_n \downarrow \quad \text{if } \beta<\gamma \; .$$

Now let

$$F(1,\beta,\gamma,x) = \sum_{\nu=0}^{\infty} b_\nu x^\nu$$

$$T_n = \frac{\sum_{\nu=0}^{n} s_\nu}{n+1}$$

$$\sigma_n = \frac{\sum_{\nu=0}^{n} b_{n-\nu} s_\nu}{B_n}$$

We have

$$s_\nu = (\nu+1)\tau_\nu - \nu\,\tau_{\nu-1} \quad , \quad \tau_{-1}=0$$

and hence

$$B_\mu \sigma_\mu = \sum_{\nu=0}^{\mu} b_{\mu-\nu}\{(\nu+1)\tau_\nu - \nu\,\tau_{\nu-1}\} \; , \; \tau_{-1}=0$$

Therefore

$$\sigma_\mu = \sum_{\nu=0}^{\mu} \frac{(b_{\mu-\nu} - b_{\mu-\nu-1})(\nu+1)\tau_\nu}{B_\mu} \; , \quad \tau_{-1} = b_{-1} = 0$$

We have thus expressed the sequence $\{\sigma_\mu\}$ as a transform of the sequence $\{\tau_\mu\}$. Hence in order to prove the first part of the theorem we need only show that the Silverman Toeplitz conditions 0.1 are satisfied.

For $\beta > \gamma$ the first of these conditions is readily verified and the remaining two are equivalent since $b_{\mu-\nu} - b_{\mu-\nu-1} > 0$

But

$$\sum_{\nu=0}^{\mu} \frac{(b_{\mu-\nu} - b_{\mu-\nu-1})(\nu+1)}{B_\mu} = \frac{B_\mu}{B_\mu} = 1$$

which establishes the result.

On the other hand if $\beta < \gamma$ we have

$$\sum_{\mu=0}^{\infty} \tau_\mu (\mu+1)\, x^\mu = \frac{1}{1-x} \sum_{\mu=0}^{\infty} s_\mu x^\mu$$

$$\sum_{\mu=0}^{\infty} \sigma_\mu B_\mu x^\mu = F(1,\beta,\gamma,x) \sum_{\mu=0} s_\mu x^\mu$$

Hence

$$\sum_{\mu=0}^{\infty} T_\mu (\mu+1) x^\mu = \frac{\frac{1}{1-x}}{F(1,\beta,\gamma,x)} \sum_{\mu=0}^{\infty} \sigma_\mu B_\mu x^\mu$$

Setting

$$\frac{1}{(1-x)F(1,\beta,\gamma,x)} = \sum_{\mu=0}^{\infty} p_\mu x^\mu ,$$

we have

$$T_\mu = \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} p_{\mu-\nu} B_\nu \sigma_\nu .$$

Now

$$\sum_{\mu=0}^{\infty} x^\mu = F(1,\beta,\gamma,x) \sum_{\mu=0}^{\infty} p_\mu x^\mu$$

Hence

$$1 = \sum_{\nu=0}^{\mu} b_{\mu-\nu} p_\nu$$

and

$$p_\mu = 1 - \sum_{\nu=0}^{\mu-1} b_{\mu-\nu} p_\nu$$

Moreover

$$0 = 1 - \sum_{\nu=0}^{\mu-1} b_{\mu-\nu-1} p_\nu$$

Subtracting these last two equations we find,

$$p_\mu = \sum_{\nu=0}^{\mu-1} \left[b_{\mu-\nu-1} - b_{\mu-\nu} \right] p_\nu .$$

Since, however, $\beta \geq 1$ and in this case $b_{\mu-\nu-1} > b_{\mu-\nu}$, we find by mathematical induction that

$$p_\mu \geq 0 , \quad \mu = 0, 1, 2, \ldots$$

Since in addition

$$p_\mu = 1 - \sum_{\nu=0}^{\mu-1} b_{\mu-\nu} p_\nu$$

it follows that

$$0 \le p_n \le 1, \quad n = 0, 1, 2, \ldots$$

Hence

$$\sum_{\nu=0}^{n} \frac{|p_\nu| B_{n-\nu}}{n+1} = \sum_{\nu=0}^{n} \frac{p_\nu B_{n-\nu}}{n+1} = 1$$

and

$$\frac{p_n}{n+1} \to 0 \quad .$$

These, however, constitute verification of the Silverman Toeplitz conditions which completes the proof of the theorem.

Article 3. A comparison of $H(q,\beta,\gamma)$ summability with (C,α) summability.

In this article we wish to extend the theorem of the previous article to integral values of α greater than 1. In this connection we note that the essential feature of the proof of theorem 2.1 was the monotonicity of the b_ν. An analagous result is contained in the following lemma.

Lemma 3.1.

If k is any positive integer and if $\beta > \gamma$ then the k'th differences of the coefficients in

$$F(k,\beta,\gamma,x) = \sum_{\nu=0}^{\infty} b_\nu x^\nu$$

are all positive with the exception of a finite number at most.

Proof:

We write

$$b_n = \frac{K_{n-1}\beta_{n-1}}{\gamma_{n-1}\, n!} = A_n^{k-1}\frac{\beta_{n-1}}{\gamma_{n-1}}, \quad b_0 = 1,$$

Set

$$v_n = A_n^{k-1}$$

$$u_n = \frac{\beta_{n-1}}{\gamma_{n-1}}$$

$$\Delta^0 v_n = A_n^{k-1}$$

$$\Delta^1 v_n = A_n^{k-1} - A_{n-1}^{k-1} = A_n^{k-2}$$

$$\vdots$$

$$\Delta^\mu v_n = A_n^{k-\mu-1}$$

$$\Delta^0 u_n = \frac{\beta_{n-1}}{\gamma_{n-1}}$$

$$\Delta^1 u_n = \frac{\beta_{n-1}}{\gamma_{n-1}} - \frac{\beta_{n-2}}{\gamma_{n-2}} = \frac{\beta_{n-2}}{\gamma_{n-1}}(\beta-\gamma)$$

$$\Delta^2 u_n = \frac{\beta_{n-3}}{\gamma_{n-1}}(\beta-\gamma)(\beta-\gamma-1)$$

$$\vdots$$

$$\Delta^\nu u_n = \frac{\beta_{n-\nu-1}}{\gamma_{n-1}}(\beta-\gamma)(\beta-\gamma-1)\ldots(\beta-\gamma-\nu+1).$$

Now according to a formula of Bosanquet (1,p.41),

$$\Delta^k (u_n, v_n) = \sum_{\nu=0}^{k} \binom{k}{\nu} \Delta^\nu u_n \, \Delta^{k-\nu} v_{n-\nu} .$$

Hence in our case

$$\Delta^k b_n = \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\beta_{n-\nu-1}}{\gamma_{n-1}} (\beta-\gamma)\cdots(\beta-\gamma-\nu+1) A_{n-\nu}^{\nu-1} .$$

Setting $\beta-\gamma=\mu>0$ and noting that $A_n^{-1}=0,\ n>0,$

the above sum becomes

$$\frac{\beta_{n-2}}{\gamma_{n-1}} \sum_{\nu=1}^{k} \binom{k}{\nu} \frac{\beta_{n-\nu-1}}{\beta_{n-2}} \mu(\mu-1)\cdots(\mu-\nu+1) A_{n-\nu}^{\nu-1} .$$

But

$$\frac{\beta_{n-\nu-1}}{\beta_{n-2}} \mu(\mu-1)\cdots(\mu-\nu+1) A_{n-\nu}^{\nu-1} = \frac{\mu\cdots(\mu-\nu+1)}{\nu!} \frac{(n-1)(n-2)\cdots(n-\nu}{(\beta+n-2)\cdots(\beta+n}$$

and as n becomes infinite, this tends to

$$\frac{\mu(\mu-1)\cdots(\mu-\nu+1)}{\nu!} = \binom{\mu}{\nu} .$$

Hence as n becomes infinite,

$$\sum_{\nu=1}^{k} \binom{k}{\nu} \frac{\beta_{n-\nu-1}}{\beta_{n-2}} \mu(\mu-1)\cdots(\mu-\nu+1) A_{n-\nu}^{\nu-1} \longrightarrow \sum_{\nu=1}^{k} \binom{k}{\nu}\binom{\mu}{\nu} .$$

But

$$\sum_{\nu=1}^{k} \binom{k}{\nu}\binom{\mu}{\nu} = \sum_{\nu=1}^{k} \binom{k}{k-\nu}\binom{\mu}{\nu}$$

and

$$\sum_{n=0}^{\infty} \binom{\mu+k}{n} x^n = (1+x)^\mu (1+x)^k = \sum_{n=0}^{\infty} \binom{\mu}{n} x^n \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \binom{k}{n-\nu}\binom{\mu}{\nu} x^n .$$

Hence

$$\sum_{\nu=0}^{k} \binom{k}{k-\nu} \binom{\mu}{k} = \binom{\mu+k}{k},$$

and therefore, if $\mu > 0$

$$\sum_{\nu=1}^{k} \binom{k}{k-\nu} \binom{\mu}{k} = \binom{\mu+k}{k} - 1 > \frac{k!}{k!} - 1 = 0$$

Hence

$$\sum_{\nu=1}^{k} \binom{k}{\nu} \frac{\beta_{n-\nu-1}}{\beta_{n-2}} \mu(\mu-1)\ldots(\mu-\nu+1) A^{\nu-1}_{n-\nu}$$

tends to a positive limit as n becomes infinite and therefore may be negative for at most a finite number of values of n. This completes the proof of the lemma.

We are now in a position to give the following generalization of theorem 2.1.

Theorem 3.2.

If k is any positive integer then any sequence $\{s_n\}$ summable (C, k) is summable $H(k, \beta, \gamma)$ for any $\beta \geq \gamma$.

Proof: If $\beta = \gamma$ then the method $H(k, \beta, \gamma)$ is identical with the method (C, k) and we therefore suppose $\beta > \gamma$.

Set

$$T_n = \frac{\sum_{\nu=0}^{n} A^{k-1}_{n-\nu} s_\nu}{A^k_n}$$

$$\sigma_n = \frac{\sum_{\nu=0}^{n} b_{n-\nu} s_\nu}{B_n}$$

Hence

$$\sum_{n=0}^{\infty} A_n^k \tau_n x^n = \frac{1}{(1-x)^k} \sum_{n=0}^{\infty} s_n x^n$$

and

$$\sum_{n=0}^{\infty} \sigma_n B_n x^n = F(k,\beta,\gamma,x) \sum_{n=0}^{\infty} s_n x^n$$

i.e.

$$\sum_{n=0}^{\infty} \sigma_n B_n x^n = (1-x)^k F(k,\beta,\gamma,x) \sum_{n=0}^{\infty} A_n^k \tau_n x^n .$$

Therefore

$$\sigma_n = \sum_{\nu=0}^{n} \frac{\Delta^k b_{n-\nu} A_\nu^k \tau_\nu}{B_n}$$

and we nedd only show that σ_n is a regular transform of $\{\tau_n\}$. First since $\frac{b_n}{B_n} \to 0$, it follows that for each fixed ν

$$\lim_{n\to\infty} \frac{\Delta^k b_{n-\nu} A_\nu^k}{B_n} = 0$$

Moreover

$$\sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \Delta^k b_{n-\nu} A_\nu^k x^n = (1-x)^k F(k,\alpha,\beta,x)(1-x)^{-k-1}$$

$$= \frac{F(k,\beta,\gamma,x)}{1-x} = \sum_{n=0}^{\infty} B_n x^n .$$

and therefore

$$\sum_{\nu=0}^{n} \frac{\Delta^{k} b_{n-\nu} A_{\nu}^{k}}{B_{n}} = 1 \quad .$$

It remains only to verify that

$$\sum_{\nu=0}^{n} \frac{|\Delta^{k} b_{n-\nu}| A_{\nu}^{k}}{B_{n}} = \sum_{\nu=0}^{n} \frac{|\Delta^{k} b_{\nu}| A_{n-\nu}^{k}}{B_{n}} = O(1).$$

According to the lemma $\Delta^{k} b_{\nu} > 0$ for all but a finite number of values at most. Hence for n large we write

$$\sum_{\nu=0}^{n} \frac{|\Delta^{k} b_{\nu}| A_{n-\nu}^{k}}{B_{n}} = \sum{}' + \sum{}''$$

where the first sum is over those values for which $\Delta^{k} b_{\nu} \geq 0$ and the second over those for which $\Delta^{k} b_{\nu} < 0$.

We have

$$\begin{aligned} \sum_{\nu=0}^{n} \frac{|\Delta^{k} b_{\nu}| A_{n-\nu}^{k}}{B_{n}} &= \sum_{\nu=0}^{n} \frac{\Delta^{k} b_{\nu} A_{n-\nu}^{k}}{B_{n}} - 2 \sum{}'' \\ &= \frac{B_{n}}{B_{n}} - 2 \sum{}'' \\ &= 1 + o(1) = O(1). \end{aligned}$$

since $\frac{A_{n-\nu}^{k}}{B_{n}} \to 0$ for $\beta > \gamma$ and the last sum contains a finite number of terms independent of n.

This completes the proof of the theorem.

Article 4. Comparison with Abel Summability. A Tauberian theorem.

In this article we shall compare the relative strength of Abel and $H(\alpha,\beta,\gamma)$ summability, and prove an elementary Tauberian theorem. We shall need to use the following theorem.

Theorem 4.1. (Szasz,1,p.20).

Let

$$g(x) = \sum_{n=0}^{\infty} u_n x^n \quad , \qquad h(x) = \sum_{n=0}^{\infty} v_n x^n$$

be convergent for $|x|<1$. Let $v_n > 0$ and $\sum_{n=0}^{\infty} v_n$ divergent and suppose $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$ exists. Then

$$\lim_{x\to 1-0} \frac{g(x)}{h(x)} = \lim_{n\to\infty} \frac{u_n}{v_n}.$$

Using this result we can now prove the following theorem.

Theorem 4.2.

Any sequence $\{s_n\}$ summable $H(\alpha,\beta,\gamma)$, $\gamma\leq\alpha+\beta$ is summable by Abel's method.

Proof

We have

$$\sigma_n = \frac{\sum_{v=0}^{n} b_{n-v} s_v}{B_n}$$

and hence

$$\sum_{n=0}^{\infty} \sigma_n B_n x^n = F(\alpha,\beta,\gamma,x) \sum_{n=0}^{\infty} s_n x^n$$

i.e.

$$\sum_{n=0}^{\infty} s_n x^n = \frac{\sum_{n=0}^{\infty} \sigma_n B_n x^n}{\sum_{n=0}^{\infty} b_n x^n}$$

Therefore

$$(1-x)\sum_{n=0}^{\infty} s_n x^n = \frac{\sum_{n=0}^{\infty} \sigma_n B_n x^n}{\sum_{n=0}^{\infty} B_n x^n}$$

If now, in theorem 4.1 we choose $u_n = \sigma_n B_n$, $v_n = B_n$ we can then easily prove convergence of the series involved by using the ordinary ratio test and the fact that $\lim_{n\to\infty} \sigma_n$ exists. The other hypotheses are satisfied in virtue of our fundamental estimates for B_n , and the theorem is thus proved.

The following Tauberian theorem is now an immediate consequence of the corresponding theorem for Abel summability and theorem 4.2.

Theorem 4.3.

If $\{s_n\}$ is summable $H(\alpha,\beta,\gamma)$, $\gamma \le \alpha+\beta$ and if $s_n - s_{n-1} = O\left(\frac{1}{n}\right)$ then $\{s_n\}$ is convergent.

Article 5. Application of $H(\alpha,\beta,\gamma)$ summability to Fourier series

In this article we shall consider the question of whether the Fourier series of a continuous is summable $H(\alpha,\beta,\gamma)$. To this end let $f(x)$ be a continuous periodic function of period 2π ,

$$f(x) \sim \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + d_\nu \sin \nu x)$$

$$a_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos \nu t \, dt$$

$$d_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin \nu t \, dt$$

$$s_n(x) = \frac{1}{2} + \sum_{\nu=1}^{n}(a_\nu \cos \nu x + d_\nu \sin \nu x)$$

It is well known then that

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\, D_n(t)\, dt$$

where

$$D_n(t) = \frac{1}{2} + \sum_{\nu=1}^{n} \cos \nu(t-x) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}$$

Writing

$$\sigma_n(x) = \frac{\sum_{\nu=0}^{n} b_{n-\nu} s_\nu(x)}{B_n}$$

we have

$$\sigma_n(x) = \frac{1}{\pi B_n}\int_{-\pi}^{\pi} \sum_{\nu=0}^{n} b_{n-\nu} D_\nu(t)\, f(t+x)\, dt = \frac{1}{\pi B_n}\int_{-\pi}^{\pi} K_n(t)\, f(t+x)\, dt$$

and

$$\sigma_n(x) - f(x) = \frac{1}{\pi B_n}\int_{0}^{\pi} \varphi(t)\, K_n(t)\, dt$$

where

$$\varphi(t) = f(x+t) + f(x-t) - 2f(x).$$

We now introduce the notions of a positive or quasi-positive kernel.

The function $K_n(t)$ is said to be a <u>positive kernel</u> if the following conditions are satisfied:

5.1 $K_n(t) \geq 0$, $-\pi \leq t \leq \pi$.

5.2 $M_n(\delta) \to 0$ where $M_n(\delta) = \max_{0<\delta\leq t\leq\pi} |K_n(t)|$

5.3 $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = 1$

The kernel is said to be quasi-positive if 5.2, 5.3, are satisfied and in addition

5.4. $\int_{-\pi}^{\pi} |K_n(t)|\,dt = O(1)$.

The essential result concerning such kernels is given in the following theorem.

Theorem 5.5 (Zygmund,1,p45).

If $f(x)$ is continuous at the point x and if $K_n(t)$ is positive or quasi-positive, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \varphi(t)\, K_n(t)\,dt = 0$$

In the case of $H(\alpha,\beta,\delta)$ summability we have the following initial easily proved theorem:

Theorem 5.6.

If the coefficients b_γ of $F(\alpha,\beta,\gamma,x)$, $\gamma<\alpha+\beta$ are monotonic increasing then the Fourier series of any continuous function $f(x)$ is summable $H(\alpha,\beta,\gamma)$ to the value $f(x)$.

Proof:

We need merely show that the kernel $K_n(t)$ is positive. Condition 5.3 is clearly satisfied. In order to verify the remaining two conditions we write,

$$K_n(t) = \frac{1}{B_n}\sum_{\nu=0}^{n} b_{n-\nu} D_\nu(t) = \frac{1}{B_n}\sum_{\nu=0}^{n}(b_\nu - b_{\nu-1})\,\mathcal{D}_{n-\nu}(t), \quad b_{-1}=0,$$

where

$$\mathcal{D}_n(t) = \sum_{\nu=0}^{n} D_\nu(t) = \frac{1}{2}\left(\frac{\sin(n+1)\frac{t}{2}}{\sin\frac{t}{2}}\right)^2 \ge 0$$

Hence since $b_\nu \ge b_{\nu-1}$ it follows that $K_n(t) \ge 0$.

Next since

$$\mathcal{D}_{n-\nu}(t) \le \frac{1}{2\sin^2\frac{\delta}{2}}, \quad 0<\delta\le t\le\pi,$$

We have

$$K_n(t) \le \frac{1}{B_n}\,\frac{1}{2\sin^2\frac{\delta}{2}}\sum_{\nu=0}^{n}(b_\nu - b_{\nu-1}) = \frac{b_n}{2B_n\sin^2\frac{\delta}{2}} = o(1),$$

and condition 5.2 is satisfied. This completes the proof of the theorem.

The situation when the b_ν are monotonic decreasing is not quite so simple, but the result remains true. In order to establish the result we shall need to use the following lemma.

Lemma 5.7.

If $\gamma<\alpha+\beta$, $b_\nu\downarrow$, then

$$|F(\alpha,\beta,\gamma,e^{-it})| \le C\,|1-e^{-it}|^{\gamma-\alpha-\beta}, \quad t\ne 0.$$

Proof:

We make use of the following easily verified identity:

$$F(\alpha,\beta,\gamma,x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma,x)$$

Since, however $\gamma-(\gamma-\alpha)-(\gamma-\beta) = -(\gamma-\alpha-\beta) > 0$, the series for $F(\gamma-\alpha,\gamma-\beta,\gamma,x)$ is absolutely convergent for $|x| \leq 1$ and hence

$$|F(\alpha,\beta,\gamma,x)| \leq C|1-x|^{\gamma-\alpha-\beta}.$$

The lemma then follows from the monotonicity of the b_ν and continuity considerations.

As a second tool we shall need the following result on power series, due to Szego (1).

<u>Theorem 5.8</u>.

If $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ is convergent for $|z|<1$, if $c_\nu > 0$, if $\sum_{\nu=0}^{\infty} c_\nu$ is divergent, and if $\frac{c_{\mu+1}}{c_\mu} \uparrow$, then

$$|c_0 + c_1 z + \cdots c_\mu z^\mu| \leq 2|f(z)|, \quad |z| \leq 1, \quad z \neq 1.$$

We proceed now to the proof of the following theorem:

<u>Theorem 5.9</u>.

If $\gamma < \alpha+\beta$ and $b_\nu \downarrow 0$, then the Fourier series of any continuous function $f(x)$ is summable $H(\alpha,\beta,\gamma)$ to the value $f(x)$.

<u>Proof</u>:

We show that in this case the kernel $k_\mu(t)$ is quasi-positive.

Conditions 5.2, 5.3, are satisfied as in the proof of the previous theorem. We need only show that condition 5.4 holds.

Now in this case

$$K_n(t) = \frac{1}{\pi B_n} \sum_{\nu=0}^{n} b_{n-\nu} D_\nu(t)$$

and hence

$$\pi K_n(t) \leq \sum_{\nu=0}^{n} \frac{b_{n-\nu} |D_\nu(t)|}{B_n} \leq (n+1) \sum_{\nu=0}^{n} \frac{b_{n-\nu}}{B_n}$$

$$\leq 2n .$$

On the other hand

$$\sum_{\nu=0}^{n} b_{n-\nu} D_\nu(t) = \mathscr{I} \left\{ \frac{\sum_{\nu=0}^{n} b_{n-\nu} e^{i(\nu+\frac{1}{2})t}}{2 \sin \frac{1}{2} t} \right\}$$

$$= \mathscr{I} \left\{ \frac{e^{i(n+\frac{1}{2})t} \sum_{\nu=0}^{n} b_\nu e^{-i\nu t}}{2 \sin \frac{1}{2} t} \right\} .$$

But since $|\mathscr{I} z| \leq |z|$ we have

$$\left| \sum_{\nu=0}^{n} b_{n-\nu} D_\nu(t) \right| \leq \frac{1}{2 \sin \frac{1}{2} t} \left| \sum_{\nu=0}^{n} b_\nu e^{-i\nu t} \right| .$$

We now notice that the conditions of theorem 5.8 are satisfied, for

$$b_n^2 - b_{n-1} b_{n+1} = \frac{\alpha_{n-1}^2 \beta_{n-1}^2}{\gamma_{n-1}^2 (n!)^2} - \frac{\alpha_{n-2} \beta_{n-2}}{\gamma_{n-2} (n-1)!} \frac{\alpha_n \beta_n}{\gamma_n (n+1)!}$$

Thus we need only show that

$$(n+1)(\alpha+n-1)(\beta+n-1)(\gamma+n) - n(\gamma+n-1)(\alpha+n)(\beta+n) \leq 0.$$

On simplification the left hand side reduces to

$$n(\alpha+n)(\beta-\gamma) + (\alpha-1)(\beta+n-1)(\gamma+n)$$

If we consider

$$g(u) = u(\alpha+u)(\beta-\gamma) + (\alpha-1)(\beta+u-1)(\gamma+u)$$

we find

$$g''(u) = 2(\beta-\gamma+\alpha-1)$$

Since the coefficients are monotonic decreasing, we have, in accordance with theorem 1.4,

$$\beta-\gamma+\alpha-1 \leqq 0$$

In case inequality holds we conclude that the quadratic polynomial $g(u)$ has a maximum value and must therefore be negatively ultimately. Szego's theorem then holds with the constant 2 possibly replaced by a larger constant. If equality holds the function $g(u)$ is linear and has the form:

$$g(u) = (1-\alpha)(1-\beta)(2u+\gamma) < 0 \quad , u > 0$$

where we have again used theorem 1.4.

Hence in either case, using theorem 5.8 and lemma 5.7, we have for $0 \leq t \leq \pi$

$$|\pi B_n K_n(t)| \leq C_1 \frac{|F(\alpha,\beta,\gamma,e^{-it})|}{\sin\frac{t}{2}} \leq C_2 \frac{|1-e^{-it}|^{\gamma-\alpha-\beta}}{t} \leq C_3 t^{\gamma-\alpha-\beta-1}, t$$

And therefore

$$\int_0^{\pi} |K_n(t)|\,dt = \int_0^{\frac{1}{n}} |K_n(t)|\,dt + \int_{\frac{1}{n}}^{\pi} |K_n(t)|\,dt$$

$$\leq \frac{2n}{n} C + \int_{\frac{1}{n}}^{\pi} |K_n(t)|\,dt$$

$$\leq O(1) + \frac{C_3}{\pi B_n} \int_{\frac{1}{n}}^{\pi} t^{\gamma-\alpha-\beta-1}\,dt$$

$$= O(1) + C_4 \frac{n^{\alpha+\beta-\gamma}}{B_n} = O(1).$$

This completes the proof of the theorem.

Article 6. $H(1,\beta,\beta+1)$ Summability.

Throughout the foregoing we have been considering primarily the case of summability $H(\alpha,\beta,\gamma)$ with $\gamma < \alpha+\beta$. According to theorem 1.6 however, the method is regular for $\gamma=\alpha+\beta$ We note moreover, that summability $H(1,1,2)$ coincides with harmonic summability. This suggests that a natural generalization of harmonic summability would be summability $H(1,\beta,\beta+1)$. The possibility that we can thus generate a genuine scale of summability methods is obviated by the following theorem:

Theorem 6.1.

If $\beta>0$ then each member of the one parameter family of summability methods $H(1,\beta,\beta+1)$ is equivalent to harmonic summability.

Proof:

We shall establish the theorem for $\beta \geq 1$ and indicate the necessary alterations for the case $\beta < 1$.

Note first that

$$F(1,\beta,\beta+1,x) = \sum_{\nu=0}^{\infty} \frac{\beta x^{\nu}}{\beta+\nu}$$

$$F(1,1,2,x) = \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu+1}$$

Hence for a given sequence $\{s_\mu\}$ let

$$\sigma_\mu = \frac{1}{A_\mu} \sum_{\nu=0}^{\mu} \frac{s_\nu}{\mu-\nu+1}$$

$$\tau_\mu = \frac{1}{B_\mu} \sum_{\nu=0}^{\mu} \frac{\beta s_\nu}{\mu-\nu+\beta}$$

where

$$A_\mu = \sum_{\nu=0}^{\mu} \frac{1}{\nu+1}, \quad B_\mu = \sum_{\nu=0}^{\mu} \frac{\beta}{\nu+\beta}$$

It follows then that

$$\sum_{\mu=0}^{\infty} A_\mu \sigma_\mu x^\mu = F(1,1,2,x) \sum_{\mu=0}^{\infty} s_\mu x^\mu$$

$$\sum_{\mu=0}^{\infty} B_\mu \tau_\mu x^\mu = F(1,\beta,\beta+1,x) \sum_{\mu=0}^{\infty} s_\mu x^\mu$$

and hence that

$$\sum_{\mu=0}^{\infty} A_\mu \sigma_\mu x^\mu = \frac{F(1,1,2,x)}{F(1,\beta,\beta+1,x)} \sum_{\mu=0}^{\infty} B_\mu \tau_\mu x^\mu ,$$

and

$$\sum_{\mu=0}^{\infty} B_\mu \tau_\mu x^\mu = \frac{F(1,\beta,\beta+1,x)}{F(1,1,2,x)} \sum_{\mu=0}^{\infty} A_\mu \sigma_\mu x^\mu$$

Thus if we set

$$\frac{F(1,1,2,x)}{F(1,\beta,\beta+1,x)} = \sum_{\mu=0}^{\infty} q_\mu x^\mu$$

$$\frac{F(1,\beta,\beta+1,x)}{F(1,1,2,x)} = \sum_{\mu=0}^{\infty} p_\mu x^\mu ,$$

we have

$$\sigma_\mu = \frac{1}{A_\mu} \sum_{\nu=0}^{\mu} q_{\mu-\nu} B_\nu \tau_\nu , \quad \tau_\mu = \frac{1}{B_\mu} \sum_{\nu=0}^{\mu} p_{\mu-\nu} A_\nu \sigma_\nu$$

Hence in order to establish the theorem we need only show that each of the following two sets of conditions hold:

$$\frac{1}{A_n}\sum_{\nu=0}^{n}|q_\nu|\,B_{n-\nu}=O(1), \qquad \frac{q_n}{A_n}\to 0$$

and

$$\frac{1}{B_n}\sum_{\nu=0}^{n}|p_\nu|\,A_{n-\nu}=O(1), \qquad \frac{p_n}{B_n}\to 0 .$$

We shall prove the second of the two first. To this end we have

$$F(1,\beta,\beta+1,x)=F(1,1,2,x)\sum_{n=0}^{\infty}p_n x^n .$$

Hence

$$\frac{\beta}{\beta+n}=\sum_{\nu=0}^{n}\frac{p_\nu}{n-\nu+1}$$

i.e.

$$p_n=\frac{\beta}{\beta+n}-\sum_{\nu=0}^{n-1}\frac{p_\nu}{n-\nu+1}$$

Moreover

$$0=\frac{\beta}{\beta+n-1}-\sum_{\nu=0}^{n-1}\frac{p_\nu}{n-\nu}$$

Multiplying the first of these relations by $\frac{1}{\beta+n-1}$, the second by $\frac{1}{\beta+n}$ and subtracting, we have

$$\frac{p_n}{\beta+n-1}=\sum_{\nu=0}^{n-1}p_\nu\left[\frac{1}{(\beta+n)(n-\nu)}-\frac{1}{(\beta+n-1)(n-\nu+1)}\right]$$

and hence

$$p_n=\sum_{\nu=0}^{n-1}\frac{(\beta+\nu-1)\,p_\nu}{(\beta+n)(n-\nu)(n-\nu+1)}$$

Since, however, $p_0 = 1$, $p_1 = \frac{\beta-1}{2(\beta+1)}$, it follows by induction that $p_\mu \geq 0$, $\mu = 1, 2, 3, \ldots$. Having established this it is now clear that

$$0 \leq p_\mu \leq \frac{\beta}{\beta+\mu} \leq 1, \quad \mu = 0, 1, 2, \ldots$$

Hence

$$\sum_{\nu=0}^{\mu} \frac{|p_\nu| A_{\mu-\nu}}{B_\mu} = \sum_{\nu=0}^{\mu} \frac{p_\nu A_{\mu-\nu}}{B_\mu} = 1$$

and

$$\frac{p_\mu}{A_\mu} \leq \frac{1}{A_\mu} \to 0.$$

We have thus proved that if $\{s_\mu\}$ is harmonically summable it is summable $H(1,\beta,\beta+1)$ for any $\beta \geq 1$.

In order to prove the converse we consider

$$1 - \frac{F(1,1,2,x)}{F(1,\beta,\beta+1,x)} = 1 - \sum_{\mu=0}^{\infty} q_\mu x^\mu = \sum_{\mu=1}^{\infty} r_\mu x^\mu$$

whence

$$F(1,\beta,\beta+1,x) - F(1,1,2,x) = F(1,\beta,\beta+1,x) \sum_{\mu=1}^{\infty} r_\mu x^\mu$$

and hence

$$\frac{\beta}{\beta+\mu} - \frac{1}{\mu+1} = \sum_{\nu=1}^{\mu} \frac{\beta}{\beta+\mu-\nu} r_\nu$$

Therefore

$$\frac{\mu(\beta-1)}{(\mu+\beta)(\mu+1)} = \sum_{\nu=1}^{\mu} \frac{\beta}{\beta+\mu-\nu} r_\nu.$$

and

$$r_n = \frac{n(\beta-1)}{(n+\beta)(n+1)} - \sum_{\nu=1}^{n} \frac{\beta}{\beta+n-\nu} r_\nu$$

Moreover

$$0 = \frac{(n-1)(\beta-1)}{(n+\beta-1)\,n} - \sum_{\nu=1}^{n-1} \frac{\beta}{\beta+n-1-\nu} r_\nu$$

Multiplying the first of these relations by $\frac{n-1}{n(n+\beta-1)}$, the second by $\frac{n}{(n+\beta)(n+1)}$, and subtracting we have

$$\frac{(n-1)\,r_n}{(n+\beta-1)\,n} = \sum_{\nu=1}^{n-1} \beta r_\nu \left[\frac{n}{(\beta+n-1-\nu)(\beta+n)(n+1)} - \frac{n-1}{n(n+\beta-1)(\beta+n-\nu)}\right]$$

Upon reduction the numeratior of the term in brackets is found to be

$$n^2(n+\beta-1)(n+\beta-\nu) - (n^2-1)(n+\beta)(n+\beta-1-\nu).$$

For fixed n this is a linear function of ν, $1 \le \nu \le n-1$.

When $\nu=1$ it has the value

$$2n^2 + 2n(\beta-1) + \beta(\beta-2) \ge 0, \quad n=1,2,\dots \quad \beta \ge 1.$$

whereas for $\nu=n-1$ it has the value

$$n^3 - n^2 + n\beta + \beta^2 \ge 0 \quad n=1,2,\dots$$

Hence since a simple computation shows that $r_1 = \frac{\beta-1}{2(\beta+1)} \ge 0$, $\beta \ge 1$, it follows by induction that $q_0 = 1$, $q_n = -r_n \le 0$, $n=1,2,\dots$

It is consequently clear that

$$0 \le r_n \le \frac{n(\beta-1)}{(n+\beta)(n+1)} \le 1$$

i.e.

$$|q_n| \le 1, \quad n=0,1,2,\dots$$

Hence

$$\frac{1}{A_n}\sum_{\nu=0}^{n}|q_\nu|B_{n-\nu} = \frac{1}{A_n}\left[B_n - \sum_{\nu=1}^{n} q_\nu B_{n-\nu}\right]$$

$$= \frac{1}{A_n}\left[2B_n - \sum_{\nu=0}^{n} q_\nu B_{n-\nu}\right]$$

$$= 2\frac{B_n}{A_n} - 1 = O(1)$$

the last step following since $A_n, B_n,$ are asymptotically a constant multiple of log n by theorem 1.5.

Thus any sequence summable $H(1,\beta,\beta+1),\ \beta\geq 1$ is harmonically summable and the theorem is established for $\beta\geq 1$. In case $\beta<1$ the technique of the above proof is merely reversed. We prove on the one hand that $p_n \leq 0,\ n=1,2,\cdots$ and on the other hand that $q_n \geq 0,\ n=0,1,2,\cdots$ The details are quite similar to the above and need not be repeated here. This completes the proof of the theorem.

In the remainder of this article we shall consider the relative strength of harmonic and hypergeometric summability for $\gamma<\alpha+\beta$. In this direction it is known that any sequence which is harmonically summable is (C,α) summable for any positive α .(Riesz,1). We shall generalize this result. We shall need to make use of the following lemma, stated by Kaluza (1), Szego and Davenport (1), and established independently by the author.

Lemma 6.2.

Let $\sum_{\mu=0}^{\infty} a_\mu x^\mu$ be convergent for $|x| < 1$ with $\sum_{\mu=0}^{\infty} a_\mu$ divergent, and suppose $a_\mu \geq 0$, $\frac{a_\mu}{a_{\mu-1}} \leq \frac{a_{\mu+1}}{a_\mu}$.

Then

$$\frac{1}{\sum_{\mu=0}^{\infty} a_\mu x^\mu} = 1 - \sum_{\nu=1}^{\infty} d_\nu x^\nu$$

where $d_\nu \geq 0$ and $\sum_{\nu=1}^{\infty} d_\nu = 1$.

Using the lemma we can now prove the following theorem:

Theorem 6.3.

If $\{s_\mu\}$ is harmonically summable then $\{s_\mu\}$ is summable $H(\alpha,\beta,\gamma)$ for any α,β,γ for which $\gamma < \alpha+\beta$.

Proof:

Let

$$\sigma_\mu = \frac{1}{A_\mu} \sum_{\nu=0}^{\mu} \frac{s_\nu}{\mu-\nu+1}$$

$$\tau_\mu = \frac{1}{B_\mu} \sum_{\nu=0}^{\mu} b_{\mu-\nu} s_\nu$$

where

$$A_\mu = \sum_{\nu=0}^{\mu} \frac{1}{\nu+1}, \quad B_\mu = \sum_{\nu=0}^{\mu} b_\nu .$$

Exactly as in the previous theorem we find

$$\tau_\mu = \frac{1}{B_\mu} \sum_{\nu=0}^{\mu} b_\nu A_{\mu-\nu} \sigma_{\mu-\nu}$$

where

$$\sum_{\nu=0}^{\infty} p_\mu x^\mu = \frac{F(\alpha,\beta,\gamma,x)}{F(1,1,2,x)} .$$

Since

$$\frac{\mu}{\mu+1} \leq \frac{\mu+1}{\mu+2} \qquad \mu=1,2,\ldots.$$

the function $F(1,1,2,x)$ satisfies the hypothesis of lemma 6.2.

Thus if we let

$$\frac{1}{F(1,1,2,x)} = 1-\sum_{\nu=1}^{\infty} d_\nu x^\nu$$

we have

$$\sum_{\nu=0}^{\infty} p_\nu x^\nu = \sum_{\nu=0}^{\infty} b_\nu x^\nu \left(1-\sum_{\nu=1}^{\infty} d_\nu x^\nu\right)$$

and hence

$$p_\mu = b_\mu - \sum_{\nu=1}^{\mu} d_\nu b_{\mu-\nu} .$$

Moreover by definition

$$0 = \frac{1}{\mu+1} - \sum_{\nu=1}^{\mu} \frac{d_\nu}{\mu-\nu+1}$$

Dividing the first of these relations by b_μ the second by $\frac{1}{\mu+1}$,

we have

$$\frac{p_\mu}{b_\mu} = \sum_{\nu=1}^{\mu} \left(\frac{\mu+1}{\mu-\nu+1} - \frac{b_{\mu-\nu}}{b_\mu}\right) d_\nu$$

Now consider

$$\frac{\mu+1}{\mu} - \frac{b_{\mu-1}}{b_\mu} = \frac{\mu+1}{\mu} - \frac{\mu(\gamma+\mu-1)}{(\alpha+\mu-1)(\beta+\mu-1)} .$$

On reduction the numerator of the right hand side becomes

$$\mu^2(\alpha+\beta-\gamma)+\mu(\alpha\beta-1)+(\alpha-1)(\beta-1)$$

which is positive for n sufficiently large, say for $\mu > M$.

We consider two cases:

Case I. $M=0$

In this case we have

$$\frac{n+1}{n-\nu+1}=\frac{n+1}{n}\cdot\frac{n}{n-1}\cdots\frac{n-\nu+2}{n-\nu+1}\geq\frac{b_{n-1}}{b_n}\cdot\frac{b_{n-2}}{b_{n-1}}\cdots\frac{b_{n-\nu}}{b_{n-\nu+1}}=\frac{b_{n-\nu}}{b_n}.$$

Hence since $p_0=1$ we have, by induction

$$0\leq p_n\leq b_n,\quad n=0,1,2,\ldots$$

Thus

$$\sum_{\nu=0}^{n}\frac{|p_\nu|\,A_{n-\nu}}{B_n}=\sum_{\nu=0}^{n}\frac{p_\nu A_{n-\nu}}{B_n}=1$$

and

$$\frac{p_n}{B_n}\leq\frac{b_n}{B_n}\rightarrow 0,$$

which establishes the theorem in this case.

Case II. $M>0$.

We introduce an intermediate summability method defined as follows:

Let

$$r(x)=\sum_{n=0}^{\infty}t_n x^n$$

where we shall define t_n shortly, and let

$$\omega_n=\frac{1}{T_n}\sum_{\nu=0}^{n}s_{n-\nu}t_\nu,$$

where

$$T_n=\sum_{\nu=0}^{n}t_\nu$$

If $\lim_{n\to\infty}\omega_n$ exists we shall say that the sequence is summable T. We proceed now to choose t_n. To this end let

$$t_n = \frac{1}{n+1}, \quad n = M, M+1, M+2, \dots$$

Now choose $t_{M-1} \geq 0$ so that

$$\frac{t_M}{t_{M-1}} \leq \frac{t_{M+1}}{t_M} \qquad \text{and} \qquad \frac{t_M}{t_{M-1}} \leq \frac{b_M}{b_{M-1}} .$$

This choice is clearly possible since it requires merely replacing $\frac{1}{M}$ by a larger number. Having fixed t_{M-1} choose t_{M-2} so that

$$\frac{t_{M-1}}{t_{M-2}} \leq \frac{t_M}{t_{M-1}} \qquad \text{and} \qquad \frac{t_{M-1}}{t_{M-2}} \leq \frac{b_{M-1}}{b_{M-2}} .$$

We continue the process and thus define t_n so that

$$t_n = \frac{1}{n+1} + h_n,$$

where

$$h_n \geq 0, \quad n = 0, 1, 2, \dots M-1; \quad h_n = 0, \quad n \geq M.$$

and moreover

$$\frac{t_n}{t_{n-1}} \leq \frac{t_{n+1}}{t_n}, \quad n = 1, 2, \dots$$

$$\frac{t_n}{t_{n-1}} \leq \frac{b_n}{b_{n-1}}, \quad n = 1, 2, \dots$$

It is immediately verifiable that summability T is a regular method. Moreover since by definition the function $\lambda(x)$ satisfies the hypothesis of lemma 6.2, we can prove, exactly as in case I that any sequence summable T is summable $H(\alpha, \beta, \delta)$

for $\gamma < \alpha + \beta$. Thus in order to complete the proof of the theorem we need only show that any sequence summable harmonically is summable T.

In the usual fashion we find

$$\omega_n = \frac{1}{T_n} \sum_{\nu=0}^{n} g_\nu A_{n-\nu} \sigma_{n-\nu}$$

where

$$\sum_{\nu=0}^{\infty} g_\nu x^\nu = \frac{r(x)}{F(1,1,2,x)} .$$

But

$$r(x) = \sum_{\nu=0}^{\infty} \left(\frac{1}{\nu+1} + h_\nu\right) x^\nu = F(1,1,2,x) + \sum_{\nu=0}^{M-1} h_\nu x^\nu .$$

Hence

$$\frac{r(x)}{F(1,1,2,x)} = 1 + \frac{\sum_{\nu=0}^{M-1} h_\nu x^\nu}{F(1,1,2,x)}$$

and therefore

$$\sum_{\nu=0}^{\infty} |g_\nu| \leq 1 + \sum_{\nu=0}^{M-1} h_\nu \left(1 + \sum_{\nu=1}^{\infty} d_\nu\right) = O(1)$$

Thus $\frac{g_n}{T_n} \to 0$ and

$$\frac{1}{T_n} \sum_{\nu=0}^{n} |g_\nu| A_{n-\nu} \leq \frac{A_n}{T_n} \sum_{\nu=0}^{n} |g_\nu| = O(1) ,$$

and this completes the proof of the theorem.

Article 7. Conclusion.

Several questions remain unsettled. First, a study of the relative strength of summability $H(\alpha+\epsilon, \beta, \gamma)$ and $H(\alpha, \beta, \gamma)$ would be valuable. Secondly since B_n is asymptotically a constant multiple of $n^{\alpha+\beta-\gamma}$ for $\gamma<\alpha+\beta$ it is reasonable to expect a close relationship between summability $H(\alpha,\beta,\gamma)$, $\gamma<\alpha+\beta$, and summability $(C,\alpha+\beta-\gamma)$. It may even be true that the two methods ore equivalent. Along thes lines it would be of interest to have an extension of theorem 6.1, in the more general case of summability $H(\alpha,\beta,\alpha+\beta)$. Moreover the case $\gamma>\alpha+\beta$ is of some interest as has already been mentioned in article 1. All of these questions are now under consideration by the author.

More generally the study of inclusion theorems for Norlund summability methods seems to have been somewhat neglected. It is clear that the problem is bound up with the question of the behaviour of the coefficients in the power series representing the reciprocal of a given analytic function. This latter question is of course related to the allied topic of the zeros of analytic functions.

The most recent work on the general problem of Nörlund summability has been done by Riesz (1), Hayashi and Izumi (1), and Piranian (1). Finally, since the completion of this paper a new book on divergent series by G.H.Hardy has appeared. This book contains a section on Nörlund summability in which the techniques used closely resemble those of this paper.

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