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I hereby recommend that the thesis prepared under my supervision by Ian J. Wong

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APPLICATION OF THE THEORY OF ELASTICITY TO THIN WINGS

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by

Lan. J. Wong

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Nomenclature

symbols

$B(x)$	a boundary value function used in the flexure problems
$C_i (i=1,2,3\ldots)$	coefficients in the approximate solutions
D	torsional rigidity
e_{yz}, e_{zx}	components of the strain tensor $e_{ij}(i,j=z,y,x)$ defined on page ten
I	energy integral in the torsional problem
I_x, I_y	Minimum moment of inertia of the cross section about axes parallel to the x and y directions respectively.
$J(w)$	an approximate energy integral for the torsion problem which provides an estimate of the upper bound for the rigidity.
$L(w)$	an approximate energy integral forming part of $J(w)$
l,m,n	direction cosines of the normal to the surface with the x,y,z directions respectively.
M	terminal moment or couple acting on the beam (wing)
P	the component of load parallel to either the x or y axes as the case may be.
t	boundary value of the y coordinate for the section.
U	strain energy of an elastic body
u,v,w	strain displacements in the x,y,z directions respectively
V	potential energy of an elastic body
W	potentail energy of the applied torque
w	an arbitrary function used in obtaining an estimate of error in the torsion problem
α	angle of twist of the beam(wing) in torsion

$\delta_n(x,y)$	an error function used in the torsional problem
ϵ	an error function
θ	a stress quantity defined on page eight
μ	modulus of torsional rigidity
σ	Poisson's ratio
$\sigma_x, \sigma_y, \sigma_z$	normal components of stress in the x,y,z directions respectively
τ_{ij} (i,j=x,y,z)	compenent of shear stress acting in a plane, normal to the i^{th} axis and directed parallel to the j^{th} axis.
φ	a stress function used in the flexure problem
Ψ	torsional shear stress function
Ω	shear stress function used in the flexure problem
∇^2	indicates the operation $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Application of the Theory of Elasticity to Thin Wings

Introduction

The increasing use of thin wings on airplanes with operating speeds in the transonic and supersonic ranges has brought forward the problem of how to conduct the stress analysis on the structures. As the wings become thinner the ratio of skin thickness to wing section depth increases and there is possibility that the formulas in current use may not be valid since one of the assumptions employed in the process of developing them was that the skin thickness was small in comparison with the dimensions of the section and the shear was constant across the skin thickness. In order to derive new formulas it is necessary to refer to the principles of the theory of elasticity. This paper presents an analysis of the shear and normal stresses in a cantilever beam under load. Some specifications on the beam will be made in order to simplify the developement.

(1) The beam is solid and is the limiting case for a very thin wing.

(2) The beam has a constant cross section.

(3) The load is concentrated at the free end of the beam and lies in aplane which is normal to the longitudinal central axis of the beam.

(4) The cross section of the beam has at least one axis of symmetry. In this particular case a lenticular bi-parabolic shape was chosen.

Statement of the Problem

Consider an elastic beam formed from homogeneous and isotropic material which is bounded by a cylindrical surface. The cylindrical surface is known as the "lateral surface" of the beam. Two planes form the "bases" of the cylinder. One base is fixed and the other is free but with an applied load on it. The cylinder is assumed to be of length "l" with the origin of coordinates at the centroid of the fixed base. The z-axis is directed along the length of the cylinder parallel to the generators of the cylinder. (see figure 1)

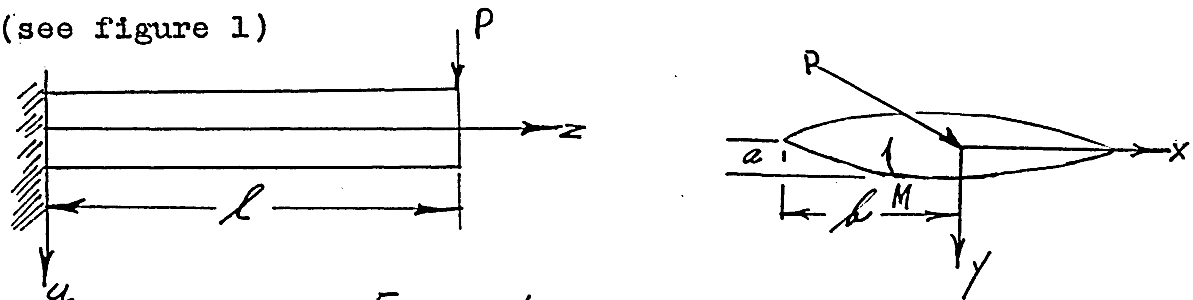

Figure 1.

Then at the fixed end $z=0$, and $z = l$ at the free end. The x and y axis are chosen as shown so that they are directed along the principal axes of inertia of the cross section. It is assumed that the beam is straight so that the z axis intersects the centroids of all sections.

The beam is considered free of body forces (i.e. weightless) and the surfaces of the cylinder are free of external loads. The applied load on the beam lies in the plane of the end $z = l$ and will be so distributed that the equilibrium conditions of a rigid body will be fulfilled. The exact stress distribution at the ends due to this load is not generally known, but this fact will not hinder the development of a solution to the problem if the usual procedure is followed of applying Saint-Venant's principle which states that the actual stress distribution over the ends of a beam, whose length is large in comparison with the linear dimensions of its section, has no appreciable influence on the character of the solution at sections sufficiently far removed from the ends. Thus any stress distribution may be prescribed at the ends of the beam as long as the resultant forces and moments reduce to those given in setting up the problem.

The applied load may be considered as being composed of a force, P, acting on the centroid of the tip end section and a couple, M, acting about the centroid whose moment is directed along the z axis. This couple results when the line of action of the applied load does not pass through elastic axis of the section which in this case is also the centroid by reason of symmetry and its magnitude M is equal to the magnitude of the applied load times the normal distance from its line of action to the centroidal point. This couple produces twisting of the beam. It will be convenient to resolve the force P into two components acting in the directions of the x and y axes. The component along the y axis may be called the "lift" force and the component along the x axis the "drag" force.

The main pr oblem can now be decomposed into three elemental parts if the principle of superposition is accepted These elementary problems are as follows:

(1) Solution for the shear stresses resulting from the torsion of the beam by the couple M.

(II) Solution for the shear and normal stresses resulting from the flexure of the cylinder by the lift force applied at the free end of the beam while at the other end there acts an equal but opposite force and also a couple of such magnitude as to equilibrate the moment produced by the lift force.

(III) Solution for the shear and normal stresses resulting from the flexure of the cylinder by the drag force applied at the free end of the beam with the same conditions as in (II) above.

The solutions are developed along the lines of the Saint Venant's "semi-inverse" method wherein certain assumptions are made concerning the components of stress, strain, and displacements, yet leaving sufficient freedom so that they may satisfy the basic equations of equilibrium and compatibility.[1,2,3] In all of the present cases the basic assumptions are

$$\sigma_x = 0, \quad \sigma_y = 0, \quad \tau_{xy} = 0 \tag{1}$$

where σ_x denotes the normal component of stress in the x direction

σ_y denotes the normal component of stress in the y direction

τ_{xy} denotes the component of shear stress acting in a plane which is normal to the x direction and directed parallel to the y axis. Due to symmetry of the stress tensor $\tau_{yz} = \tau_{xy}$

With these assumptions the equations of equilibrium become

$$\frac{\partial \tau_{zx}}{\partial z} = 0 \qquad (2)$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (3)$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{P_x}{I_y} \qquad (4)$$

when the load acts along the x direction.

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{P_y}{I_x} \qquad (4')$$

when the load acts al ong the y direction.

Where τ_{zx} and τ_{yz} are defined in a similar fashion to that above and

P is the component of the applied load along the x and y directions as the case may be.

I_x is the moment of inertia of the cross section about the axis through the centroid parallel to the x axis.

I_y is the moment of inertai of the cross section about the axis through the centroid parallel to the y axis.

The boundary conditions become

$$\tau_{zx} n = 0 \qquad (5)$$

$$\tau_{yz} n = 0 \qquad (6)$$

$$\tau_{zx} \ell + \tau_{yz} m = 0 \qquad (7)$$

Where l,m, and n are the direction cosines of the normal to the surface with respect to the x,y,z,axes respectively

The equations for the compatibility of strain may be expressed

$$\frac{\partial^2\theta}{\partial x^2} = 0 \tag{8}$$

$$\frac{\partial^2\theta}{\partial x^2} = 0 \tag{9}$$

$$(1+\sigma)\,\nabla^2\sigma_z + \frac{\partial^2\theta}{\partial_z^{\;2}} = 0 \tag{10}$$

$$(1+\sigma)\,\nabla^2\tau_{xy} + \frac{\partial^2\theta}{\partial_x\,\partial_y} = 0 \tag{11}$$

$$\left.\begin{aligned} \nabla^2\tau_{yz} &= 0 && (12)\\ \nabla^2\tau_{zx} &= -\frac{P}{I_y(1+\sigma)} && (13) \end{aligned}\right\}\ \text{(when the load acts along the } x \text{ direction)}$$

$$\left.\begin{aligned} \nabla^2\tau_{zx} &= 0 && (12')\\ \nabla^2\tau_{yz} &= -\frac{P}{I_x(1+\sigma)} && (13') \end{aligned}\right\}\ \text{(when the load acts along the } y \text{ direction)}$$

where σ_z is normal stress component acting along the z direction

$\theta = \sigma_x + \sigma_y + \sigma_z$

σ is the value of Poisson's ratio

∇^2 indicates the operation $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The three elementary problems will each be treated differently using a variational method to obtain an approximate solution whenever an exact solution is not easily obtained. The proposed treatment is as follows:

(I) Use of the Rayleigh-Ritz, Friedricks and Duncan methods to obtain a solution to the torsional problem with estimates of the range of errors.

(II) Use of the Galerkin and Duncan methods to obtain a solution to the flexure problem due to the lift force with some estimate of the range of error.

(III) Use of a method developed by Timoshenko to obtain a solution to the flexure problem due to the drag force. In as much as the solution is exact there will be no errors.

I. The Torsion Problem

Theory:

While a complete development of the theory need not be considered it may be helpful to outline the development of the variational method from the Saint- Venant theory.

It can be shown [2] that the torsion problem can be completely solved if a shearing stress function Ψ can be found which satisfies the following conditions

$$\Psi = 0 \text{ on the boundary C of the cross section} \quad (14)$$

$$\nabla^2\Psi = -2 \text{ in the region R of the cross section} \quad (15)$$

The torsional stiffness of the beam is then expressed by the relation

$$D = 2\mu \iint_R \psi \, dx\, dy \tag{16}$$

Where D is the torsional rigidity

μ is the modulus of rigidity

$\iint_R$ indicates integration over the region, R, of the cross section

The shearing stresses may be expressed as

$$\tau_{zx} = \mu\alpha \frac{\partial \psi}{\partial y} \tag{17}$$

$$\tau_{zy} = -\mu\alpha \frac{\partial \psi}{\partial x} \tag{18}$$

Where α is the angle of twist evaluated

from $\alpha = \frac{M}{D}$ (19)

M being the applied moment

The displacements and strains corresponding to the stresses are

$$u = -\alpha\, y\, z \tag{20}$$

$$v = \alpha\, z\, x \tag{21}$$

$$e_{zx} = \alpha \frac{\partial \psi}{\partial y} \tag{22}$$

$$e_{zy} = -\alpha \frac{\partial \psi}{\partial x} \tag{23}$$

Where u is the strain displacement in the x direction

v is the strain displacement in the y direction

w is the strain displacement in the z direction

e_{yz}, e_{zx} (the components of the strain tensor) are defined as

$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

In many cases an exact expression for the stress function ψ is not readily obtainable and recourse is had to one of the Variational methods which will provide an approximate solution for the stress function. The development used here is that due to Rayleigh and Ritz.[2] It is necessary to make use of the " Theorem of Minimum Potential Energy " which states: " Of all displacements satisfying given boundary conditions, those that satisfy given boundary conditions make the total potential energy, V, a minimum."

Since body forces are assumed to be non-existant the total potential energy is

$$V = U + W \tag{24}$$

Where U is the strain energy

W is the potential energy of the applied torque

$$\text{Now } U = \int_T (\tau_{zx} e_{zx} + \tau_{zy} e_{zy})\, dT$$

$$U = \tfrac{1}{2}\mu\, \alpha^2 \ell \iint_R (\nabla \psi)^2\, dx\, dy \tag{25}$$

Where $\int_T$ denotes integration over the volume in the beam

$(\nabla\psi)^2$ indicates $\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$

$$\text{And } W = -\iint_R (\tau_{zx} u + \tau_{zy} v)\, dx\, dy$$

$$= -2\mu\, \alpha^2 \ell \iint_R \psi\, dx\, dy \tag{26}$$

at the end z = 1

$$\text{So} \quad V = \tfrac{1}{2}\mu\, \alpha^2 \ell \iint_R \left[(\nabla\psi)^2 - 4\psi\right] dx\, dy \tag{27}$$

Inasmuch as the quantity $\left(\tfrac{1}{2}\mu\, \alpha^2 \ell\right)$ is constant it will be convenient to drop it and consider the minimizing of the "energy integral"

$$I = \iint_R \left[(\nabla\Psi)^2 - 4\Psi\right] dx\, dy \qquad \text{instead.} \qquad (28)$$

Now let the approximate torsion stress function be of the form $\Psi_n = \sum_{i=1}^{N} c_i f_i(x,y)$

Where c_i are constant coefficients

$f_i(x,y)$ are arbitrary functions which must vanish on the boundary C

Substitutingthis approximate function into (28) yields

$$I_n(c_1, c_2, \ldots c_n) = \iint_R \left[(\nabla\Psi_n)^2 - 4\Psi_n\right] dx\, dy \qquad (30)$$

In practise the expression is minimized through evaluating the coefficients c_i by means of the relations

$$\frac{\partial I_n}{\partial c_i} = 0 \qquad (i=1,2,-----N) \qquad (31)$$

Once the c_i's are determined Ψ_n is established.

Since Ψ_n is an approximate function it will be in order to investigate the magnitude of the errors involved in the approximation. The development follows along the lines set out by Friedricks and Duncan.[4] Estimates will be obtained of the error in the torsional rigidity D_n and in the torsional stressfunction Ψ_n.

The relation between the energy integral and the torsional rigidity may be seen from the following

$$I = \iint_R \left[(\nabla\Psi)^2 - 4\Psi\right] dx\, dy$$

whichcan be written, by application of Green's theorm, as

$$I = \int_C \Psi \frac{\partial\Psi}{\partial n} ds \; - \iint_R \nabla^2\Psi\, dx\, dy \; - \; 4\iint_R \Psi\, dx\, dy \qquad (32)$$

Since $\nabla^2\Psi = -2$ in the region R of the section

And $\Psi = 0$ on the boundary C of the section the result is

$$I = -2\iint_R \Psi \, dx\, dy \tag{33}$$

however $D = 2\mu \iint_R \Psi \, dx\, dy$

so that $\underline{D = -\mu I}$ (34)

It is possible to show that the approximate value I_n always exceeds the true value I. First the error function $\delta_n(x,y)$ must be defined

$$\delta_n(x,y) = \Psi(x,y) - \Psi_n(x,y) \tag{35}$$

whence $\quad I_n = \iint_R \quad \nabla(\Psi - \delta_n) \quad ^2 - 4(\Psi - \delta_n) \quad dx\, dy$

Rearranging terms

$$I_n = \iint_R \left\{ (\nabla\Psi)^2 - 4\Psi \, dx\, dy - 2\iint_R \nabla\Psi \cdot \delta_n \right\} dx\, dy$$
$$+4\iint_R \delta_n \, dx\, dy + \iint_R (\nabla\delta_n)^2 \, dx\, dy \tag{3 6}$$

Again using Green's theorem

$$I_n = I - 2\left[\oint_C \delta_n \frac{\partial \Psi}{\partial n} ds - \iint_R \delta_n \nabla^2 \Psi \right] dx\, dy$$
$$+ 4\iint_R \delta_n \quad dx\, dy + \iint_R (\nabla\delta_n)^2 \, dx\, dy$$

However δ_n vanishes on the boundary C and

$\nabla^2\Psi = -2$ in the region R

Therefore $I_n = I + \iint_R (\nabla\delta_n)^2 \, dx\, dy$

or $\quad I_n \geq I$

It is seen from this that $D_n \doteq -\mu\, I_n$ will always fall short of the exact value, D, and may be considered a lower bound to the true value. To estimate the range of error an upper bound to the true value must be obtained.

Consider the inequality

$$(\nabla\Psi)^2 - 4\Psi + (\nabla w)^2 - 4x\frac{\partial w}{\partial y} + 4x^2 + 4\frac{\partial}{\partial x}(x\Psi)$$
$$+ 2 \quad \frac{\partial}{\partial x}\left(w\frac{\partial \Psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(w \quad \frac{\partial \Psi}{\partial x}\right) \geq 0 \tag{38}$$

The equality will hold if and only if,

$$\frac{\partial \Psi}{\partial x} = \frac{\partial w}{\partial y} - 2x \text{ and } \frac{\partial \Psi}{\partial y} = -\frac{\partial w}{\partial x} \tag{39}$$

If the expression is integrated over the region R, the last three terms become

$$\iint_R 4\,\frac{\partial}{\partial x}(x\,\bar{\Psi})\,dx\,dy + 2\iint_R\left[\frac{\partial}{\partial x}\left(w\,\frac{\partial\bar{\Psi}}{\partial y}\right) - \frac{\partial}{\partial y}\left(w\frac{\partial\bar{\Psi}}{\partial x}\right)\right]dx\,dy$$
$$= 4\int_C x\,\bar{\Psi}dy + 2\int_C w\left(\frac{\partial\bar{\Psi}}{\partial x}dx + \frac{\partial\bar{\Psi}}{\partial y}dy\right)$$
$$= 4\int_C x\,\bar{\Psi}dy + 2\int_C w\,d\bar{\Psi} = 0$$

Since $\bar{\Psi}$ vanishes on the boundary C the remaining terms are (40)

$$\iint_R\left[(\nabla\bar{\Psi})^2 - 4\bar{\Psi}\right]dx\,dy + \iint_R\left[(\nabla w)^2 - 4x\frac{\partial w}{\partial y} + 4x^2\right]dx\,dy \geq 0$$

Define $J(w) = -\iint_R\left[(\nabla w)^2 - 4x\frac{\partial w}{\partial y} + 4x^2\right]dx\,dy$ (41)

then $I(\bar{\Psi}) - J(w) \geq 0$

and $I(\bar{\Psi}) \geq J(w)$ (42)

The equality holds only if theconditions (39) are satisfied. If $J_n(w) = -\iint_R\left[(\nabla w_n)^2 - 4x\frac{\partial w_n}{\partial y} = 4x^2\right]dx\,dy$ is the approximate function, it must be maximized so that it will approach the minimum of $I(\bar{\Psi})$ in value. Noting that only the first two terms involve $w_n(x,y)$ let

$$L_n(W) = -\iint_R\left[(\nabla w_n)^2 - 4x\frac{\partial w_n}{\partial y}\right]dx\,dy \qquad (43)$$

Then it will only be necessary to maximize $L_n(w)$.

Assume $w_n = \sum_{i=1}^{N} K_i\, g_i\,(x,y)$ (44)

Where $g_i(x,y)$ may be the real and imaginary parts of $(x + iy)^n$. It is not necessary to inguire into the boundary conditions on $g_i(x,y)$ since only bounds on the er ror in $J_n(w)$ are involved.

If a term of the form $w = Kx^r y^s$ be substituted in (43) and the maximizing condition

$$\frac{\partial L_n(w)}{\partial K} = 0 \qquad (45)$$

be imposed, it will be found that K = 0 unless both r and s are odd. Accordingly the terms for $g_1(x,y)$ will be of the type xy, x^3y-y^3x, etc.

Once the value for the maximized function $J_n(w)$ is found the true minimum $I(\psi)$ can be located between the upper bound I_n and the lower bound $J_n(w)$. In turn the bounds on the true value, D , of the torsional rigidity will be determined.

The limits to the error in ψ_n may be found as follows: define an error function $\epsilon(x,y)$ by the expression

$$\epsilon(x,y) = \nabla^2 \psi_n + 2 \qquad (46)$$

Let $\psi' = \dfrac{2}{2-\epsilon} \psi_n$

then $\nabla^2 \psi' = -2$ so that $\psi' = \psi$

and $\underline{\psi_n = \dfrac{2-\epsilon}{2} \psi}$ (47)

Now let ϵ_M and ϵ_m be the maximum and minimum values for $\epsilon(x,y)$ respectively either in the region R or on the boundary C.

define $\psi_{n_M} = \dfrac{2-\epsilon_M}{2} \psi$

then $\nabla^2(\psi_{n_M} - \psi_n) = \epsilon_M - \epsilon \geqq 0$ (48)

However the following may be used:

Lemma: If z vanishes on a closed curve α and if $\nabla^2 z$ is everywhere positive (negative) within α, then z is everywhere negative (positive) within α, so that $\psi_n \geqq \psi_{n_M}$ everywhere. (49)

In a similar fashion if $\psi_{n_m} = \dfrac{2-\epsilon_m}{2} \psi$

then $\psi_{n_m} \geqq \psi_n$ everywhere (50)

And combining (49) and (50)

$$(1 - \frac{\epsilon_M}{2})\bar{\Psi} \leq \Psi_n \leq (1 - \frac{\epsilon_m}{2})\bar{\Psi} \qquad (51)$$

And the upper and lower bounds to the error in the approximation Ψ_n can be assigned whehever ϵ_M and ϵ_m have been determined. This does not provide information as to the exact location of $\bar{\Psi}_n$ within the bounds nor does it give any indications as to the limits of errors on the stresses.

Computation

The section boundaries are expressed as shown in figure 2.

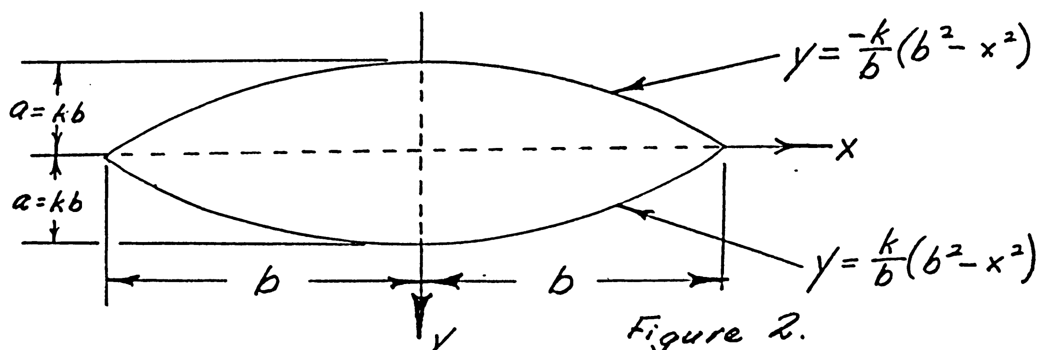

Figure 2.

As the first approximation consider the stress function

$$\Psi_1 = c_1(y^2 - k^2b^2 + 2k^2x^2 - \frac{k^2x^4}{b^2}) \qquad (52)$$

then $(\nabla\Psi_1)^2 = c_1^2(4y^2 + 16k^4x^2 + \frac{16k^4x^6}{b^4} - \frac{32k^4x^4}{b^2})$

$$\text{and } I_1 = \iint_R \left[c_1^2(4y^2 + 16k^4x^2 + \frac{16k^4x^6}{b^4} - \frac{32k^4x^4}{b^2}) - 4c_1(y^2 - k^2b^2 + 2k^2x^2 - \frac{k^2x^4}{b^2})\right] dx\, dy \qquad (53)$$

$$\frac{I_1}{2} = \int_0^b \Big\{ C_1^2 \Big[\frac{4k^3}{3b^3}(b^2-x^2)^3 + \frac{10k^5}{b} x^2(b^2-x^2) + \frac{16k^5}{b^3} x^6(b^2-x^2) - \frac{32k^5}{b^3} x^4(b^2-x^2) \Big] -4C_1 \Big[\frac{k^3}{3b^3}(b^2-x^2)^3 - k^3 b(b^2-x^2) + \frac{2k^3}{b} x^2(b^2-x^2) - \frac{k^3}{b^3} x^4(b^2-x^2) \Big] \Big\} dx$$

$$\frac{I_1}{4k^3b^4} = C_1^2 \left[\frac{64}{105} + \frac{256}{315} k^2 \right] + \frac{128}{105} C_1 \tag{54}$$

Minimizing and dropping the constant coefficients

$$\frac{\partial I_1}{\partial C_1} = C_1 \left[\frac{192+256k^2}{315} \right] + \frac{64}{105} = 0$$

$$\text{or } C_1 = -\frac{3}{3+4k^2} \tag{55}$$

$$\therefore \underline{\Psi_1 = \left(\frac{3}{3+4k^2}\right)\left(k^2b^2 - y^2 + \frac{k^2x^4}{b^2} - 2k^2x^2\right)} \tag{56}$$

To estimate the error in the rigidity compute

$$D_1 = 2\mu \iint_R \Psi_1 \, dx\, dy$$

$$= 2\mu\left(\frac{3}{3+4k^2}\right) \iint_R \left(k^2b^2 - y^2 + \frac{k^2x^4}{b^2} - 2k^2x^2\right) dx\, dy \tag{57}$$

$$= 8\mu\left(\frac{3}{3+4k^2}\right) \int_0^b \left[k^3 b(b^2-x^2) - \frac{k^3}{3b^3}(b^2-x^2)^3 + \frac{k^3}{b^3} x^4(b^2-x^2) - \frac{2k^3}{b}(b^2-x^2) \right] dx$$

$$= 8\mu\left(\frac{3}{3+4k^2}\right) \left[\frac{32}{105} k^3 b^4 \right]$$

$$= \frac{256}{105}\left(\frac{3}{3+4k^2}\right) \mu k^3 b^4 \quad \text{which is the lower bound on D.} \tag{58}$$

In computing J(w) choose as a first approximation

$$w_1 = K_1(xy) \tag{59}$$

$$\text{and then } J_1(w) = -\iint_R \left[K_1^2(x^2+y^2) - 4K_1x^2 + 4x^2 \right] dx\, dy \tag{60}$$

in this instance it will be most convenient to treat $J_1(w)$ as a whole rather than separating it into the expression $L_1(w)$ and the term in $4x^2$

$$J_1(w) = -4\int_0^b \Big\{ K_1^2 \Big[\frac{k}{b} x^2(b^2-x^2) + \frac{k^3}{3b^3}(b^2-x^2)^3 \Big] - (4K_1 - 4)\frac{k}{b} x^2(b^2-x^2) \Big\} dx$$

$$= -4kb^4\left[K_1{}^2\left(\frac{14 + 16k^2}{105}\right) - \frac{8}{15}(K_1 - 1)\right] \tag{61}$$

Maximizing

$$\frac{\partial J_1(w)}{\partial K_1} = 2K_1\left(\frac{14 + 16k^2}{105}\right) - \frac{8}{15} = 0$$

and $$K_1 = \frac{14}{7 + 8k^2} \tag{62}$$

Substituting in and reducing

$$J_1(w) = -\frac{256}{105}\left(\frac{7}{7 + 8k^2}\right)k^3b^4 \text{ which is the upper bound for D.} \tag{63}$$

In order to see the possible error in D, let $k = .1$ and evaluate. It is seen that

$$2.40601\ \mu k^3b^4 \leq D \leq 2.41055\ \mu k^3b^4$$ and the maximum error in the estimate of D will be .0094% (64)

To estimate the error in ψ_1 consider

$$\epsilon = \nabla^2\psi_1 + 2 = 2 - \left(\frac{3}{3 + 4k^2}\right)\left(2 + 4k^2 - \frac{12k^2x^2}{b^2}\right) \tag{65}$$

In the region

$$\frac{\partial \epsilon}{\partial y} = 0 \quad \text{and } \epsilon \text{ is independent of } y$$

$$\frac{\partial \epsilon}{\partial x} = \left(\frac{3}{3 + 4k^2}\right)\left(\frac{24k^2}{b^2}x\right) = 0 \text{ or } x = 0$$

$\therefore$ the maximum ϵ in the region occurs at $x=0$

On the boundary, ϵ will be a maximum again at $x = 0$ since the expression for ϵ is independent of y. In the evaluation the vertices must be considered.

The values for ϵ become

In the region when $x = 0$, $$\epsilon = 2 - \left(\frac{3}{3 + 4k^2}\right)\left(4k^2 + 2\right) \tag{66}$$

At the vertices $(\pm b, 0)$, $$\epsilon = 2 - \left(\frac{3}{3 + 4k^2}\right)\left(2 - 8k^2\right) \tag{67}$$

then if $k = .1$ the values are as follows

$\epsilon_M = +.10526$ at the vertices

$\epsilon_m = -.01316$ in the region

so that $$(1 - .0526)\psi \leq \psi_1 \leq (1 + .0066)\psi \tag{68}$$

It may be seen from (66) and (67) that as the wing section becomes thinner ($k \to 0$) the error approaches zero.

Two other functions were investigated to determine whether or not a better approximation could be obtained by the use of more terms. The functions and the estimates of errors for ($k = .1$) are as follows

$$\text{(a)} \quad \Phi_{2x} = (C_1 + C_2 x^2)\left(y^2 - k^2b^2 + 2k^2x^2 - \frac{k^2x}{b^2}\right) \qquad (69)$$

$$\text{where } C_1 = -\left[\frac{3}{3+4k^2} + \frac{k^2(11 + 4k^2)}{33(34.9935k^4 + 25.9421k^2 - \frac{1}{6})}\right]$$

$$C_2 = \frac{k^2(3-4k^2)}{b^2(34.9935k^4 + 25.9421k^2 - \frac{1}{6})}$$

$$2.40520\ \mu k^3b^4 \leq D \leq 2.41055\ \mu k^3b^4 \qquad (70)$$

$$(1-.3225)\Phi \leq \Phi_{2x} \leq (1+.0452)\Phi \qquad (71)$$

$$\text{(b)} \quad \Phi_{2y} = (C_1 + C_2 y^2)\left(y^2 - k^2b^2 + 2k^2x^2 - \frac{k^2x^4}{b^2}\right) \qquad (72)$$

$$\text{where } C_1 = \frac{A_3A_5 - 2A_2A_4}{4A_1A_2 - A_3A_3}$$

$$C_2 = \frac{1}{k^2b^2}\left[\frac{A_3A_4 - 2A_1A_5}{4A_1A_2 - A_3A_3}\right]$$

$$\text{and } A_1 = \frac{64}{105} + \frac{256}{315}k^2$$

$$A_2 = \frac{90212}{675675} + \frac{229376k^2}{3828825}$$

$$A_3 = \frac{2048}{10395} + \frac{40960k^2}{135135}$$

$$A_4 = \frac{128}{105}$$

$$A_5 = \frac{2048}{10395}$$

$$\text{and} \quad 2.406016\ \mu k^3b^4 \leq D \leq 2.41055\ \mu k^3b^4 \qquad (73)$$

$$(1 - .05238)\Phi \leq \Phi_{2y} \leq (1 - .03314)\Phi \qquad (74)$$

From these results it may be seen that in this particular case no benefits are derived from the additional tedious calculations required to solve for the higher approximation functions.

BI-PARABOLIC SECTION

CONTOURS OF THE SHEAR STRESS FUNCTION
DUE TO TORSION

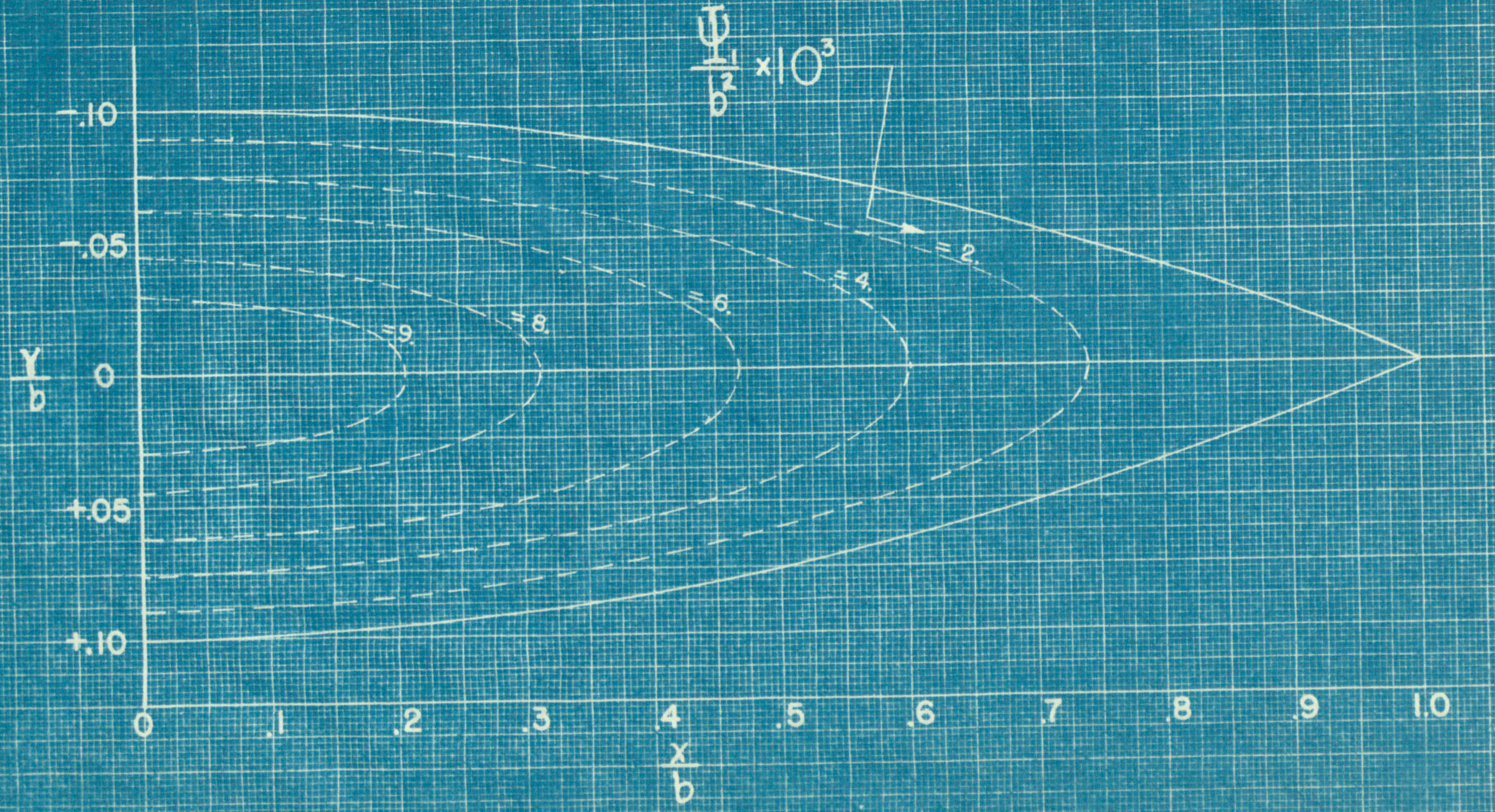

Figure 3

BI-PARABOLIC SECTION
VARIATION OF SHEAR STRESSES DUE TO TORSION

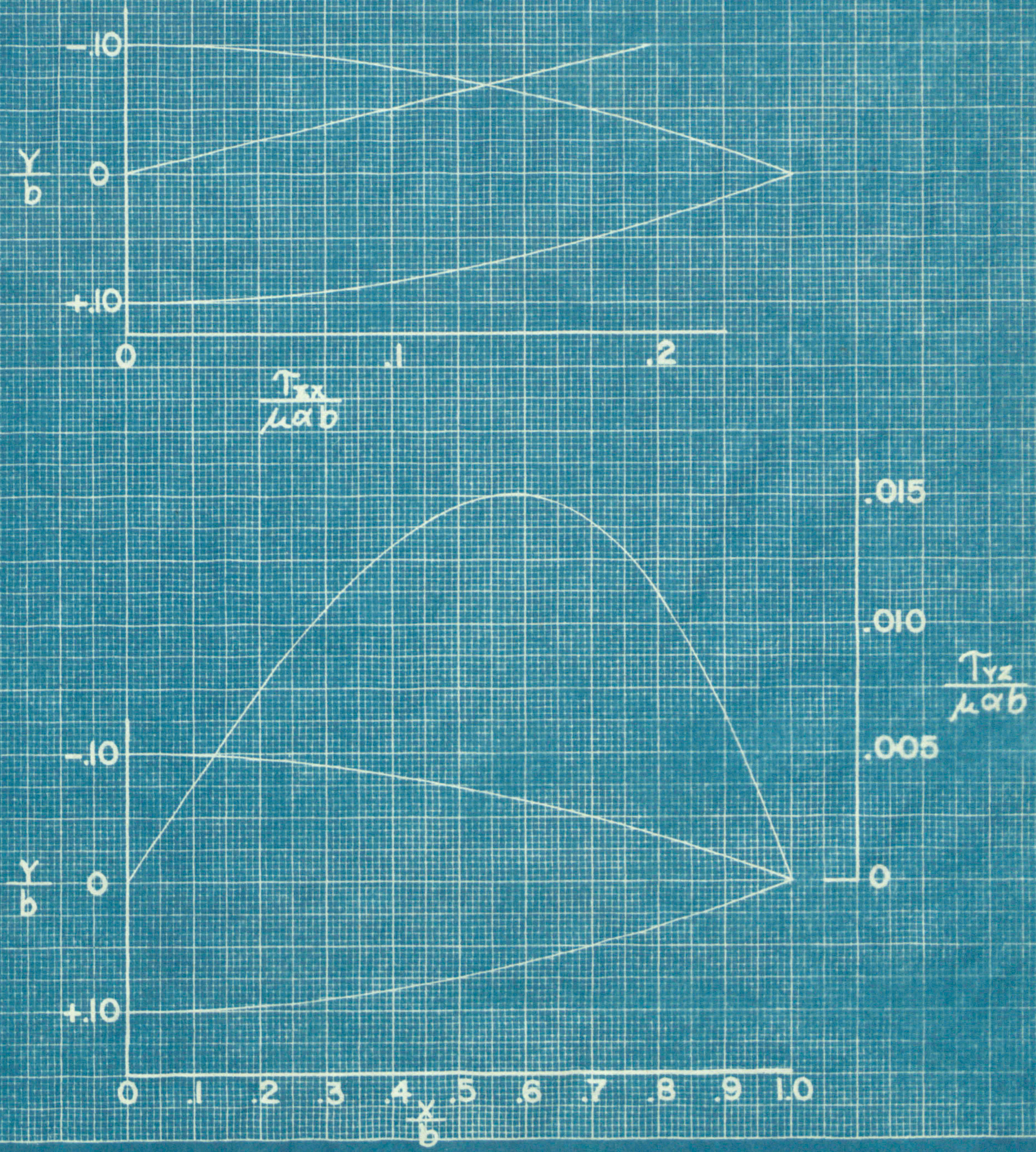

Figure 4.

Returning to the first result the shear stresses become

$$\tau_{zx} = -\left(\frac{6}{3 + 4k^2}\right)\mu\alpha y \tag{75}$$

$$\tau_{yz} = \left(\frac{12}{3 + 4k^2}\right)\mu\alpha \left[k^2 x - \frac{k^2}{b^2}x^3\right] \tag{76}$$

Acontour of the stress function ψ_1 is shown in figure ~~three~~ for a value of $k = .1$. The stress functions are shown in figure ~~four~~ for values of $k = .1$ and $.05$.

II. Flexure due to Lift Force

Theory

In addition to the basic assumptions (1) set forth previously the terminal load is presumed to act so that at any section the resulatant of the shear stress distribution will be statically equal to the applied load while the normal stress will behave in the same manner as if the beam were bent by a couple of magnitude equal to the product of the load times the distance from the load to the section under consideration. The latter behavior is simply expressed as

$$\sigma_z = -\frac{P(\ell - z)y}{I_x} \tag{77}$$

The solution therefore is concerned mainly with the behavior of the shear stresses which together with the expression above must satisfy the equations of equilibrium and compatibility set forth in the beginning. It is seen that owing to the assumptions made the relations (2), (3), (5), (6), (8), (9), (10) and (11) are identically satisfied

Consider the following relations[4,6]

$$\tau_{zx} = \frac{P}{I_x}\frac{\partial \psi}{\partial y} \tag{78}$$

$$\tau_{yz} = \frac{P}{I_x}\left\{-\frac{\partial\psi}{\partial x} + \frac{1}{2}y^2 - \frac{\sigma x^2}{2(1+\sigma)}\right\} \tag{79}$$

Where $\psi(x,y)$ is some stress function.

If these relations are substituted into (7) the result is

$$\frac{d\psi}{ds} = \frac{dx}{ds}\left[\frac{1}{2}y^2 - \frac{\sigma x^2}{2(1+\sigma)}\right] \tag{80}$$

using the relations

$$\ell = -\frac{dy}{ds}, \quad m = \frac{dx}{ds}$$

Since the origin of coordinates was chosen at the centroid of the section it may be shown by use of Green's theorem that the integral $\int y^2\,dx$ taken cyclically around the complete boundary is zero. Therefore it is possible to integrate equation (80) to get a single-valued boundary value for which may be deignated as ψ_B

$$\psi_B = \frac{1}{2}\int_A^p y^2\,dx - \frac{\sigma x^3}{6(1+\sigma)} + \text{constant} \tag{81}$$

where the limits A and P are arbitrary.

If equations (78) and (79) are substituted into equations (12') and (13') it is seen equations of compatibility require that

$$\nabla^2\psi = 0 \text{ everywhere within the boundary} \tag{82}$$

So that if a function $\psi(x,y)$ is found that will satisfy equations (81) and (82) the flexure problem is solved.

Consider the section as shown in figure 5. If p and p'

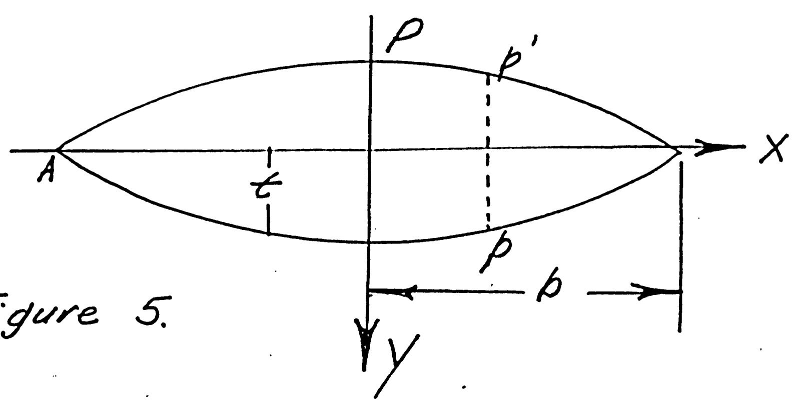

Figure 5.

be symmetrically placed points on the boundary then

$$\int_A^p y^2\left(\frac{dx}{ds}\right)ds = -\int_{p'}^A y^2\left(\frac{dx}{ds}\right)ds$$

and from the equation (81) it is seen that $\Psi_p = \Psi_{p'}$ (83)

where Ψ_p and $\Psi_{p'}$ represent values of Ψ on the boundary at points p and p' respectively. Now let t be the value of y at the boundary and let

$$B(x) = \frac{1}{2}\int_A^x t^2\,dx - \frac{\sigma(x^3 - b^3)}{6(1+\sigma)} \tag{84}$$

so that $\Psi_p = \Psi_{p'} = B(x)_{x=p}$ if the constant of integration (85) is equal to $\frac{\sigma b^3}{6(1+\sigma)}$.

Let Ω be another stress function defined by

$$\Omega = \Psi - B(x) \tag{86}$$

It is seen that Ω vanishes everywhere at the boundary by virtue of equation (85). Also by equations (82) and (86) there is obtained the relation

$$\nabla^2\Omega + \frac{d^2B}{dx^2} = 0 \tag{87}$$

And now the flexure problem resolves itself into finding a function Ω which vanishes on the boundary and satisfies equation (87) everywhere inside the boundary. The expressions for the shear stresses become

$$\tau_{zx} = \frac{P}{I_x}\frac{\partial\Omega}{\partial y} \tag{88}$$

$$\tau_{yz} = -\frac{P}{2I_x}\left[t^2 - y^2 + 2\frac{\partial\Omega}{\partial x}\right] \tag{89}$$

In practice an exact expression for the function Ω is difficult to obtain in many cases so that a variational procedure offers the most expedient method of solution. Assume a series function of the form $\Omega_N = (t^{2n} - y^{2n})\sum_{i=1}^{N} c_i f_i(x,y)$

where n is any integer

It is seen that this satisfies the condition that Ω vanishes on the boundary. However, the function will in general not exactly satisfy the condition specified by equation (87). If the difference due to substitution of the exact and the approximate functions in equation (87) is called the error function $\epsilon(x,y) = \nabla^2\Omega_n + \frac{d^2B}{dx^2}$ (91)

In the method proposed by Galerkin[2,5] it is possible to evaluate the coefficients c_i such that the potential energy is minimized by means of the set of N conditions

$$\iint_R \epsilon(x,y)\, f_i(x,y)\, dx\, dy = 0 \quad (i = 1,2,\ldots\ldots..N) \tag{92}$$

This method eliminates the necessity for the evaluation of an energy integral and its derivatives.

Computation:

Consider the equation $\Omega_2=(c_1x+c_2x^3)(k^2b^2-2k^2x^2+\frac{k^2x^4}{b^2}-y^2)$ (93)

$$\text{then}\quad \epsilon=\nabla^2\Omega_2+\frac{d^2B}{dx^2}$$

$$=C_1\left[20\frac{k^2}{x^2}x^3-(2k^2 12k^2)\,x\right]$$

$$+C_2\left[42\frac{k^2}{b^2}x^5-(2+40k^2)x^3+6k^2b^2x-6xy^2\right]$$

$$+2\frac{k^2}{b^2}x^3-(2k^2+\frac{\sigma}{1+\sigma})\,x$$

$$=A_1x^5+A_2x^3+A_3x+A_4xy^2 \qquad (94)$$

where

$$A_1=42\frac{k^2}{b^2}C_2$$

$$A_2=20\frac{k^2}{b^2}C_1-(2+40k^2)C_2+2\frac{k^2}{b^2}$$

$$A_3=-(2+12k^2)C_1+6k^2b^2C_2-(2k^2+\frac{\sigma}{1+\sigma})$$

$$A_4=-6C_2$$

The first Galerkin equation is

$$0=\iint_R\left\{\begin{aligned}&A_1\left[k^2b^2x^6-2k^2x^8+\frac{k^2}{b^2}x^{10}-x^6y^2\right]\\&+A_2\left[k^2b^2x^4-2k^2x^6+\frac{k^2x^8}{b^2}-x^4y^2\right]\\&+A_3\left[k^2b^2x^2-2k^2x^4+\frac{k^2x^6}{b^2}-x^2y^2\right]\\&+A_4\left[k^2b^2x^2y^2-2k^2x^4y^2+\frac{k^2x^6y^2}{b^2}-x^2y^4\right]\end{aligned}\right\}dx\,dy \qquad (95)$$

$$0=\int_0^b\left\{\begin{aligned}&A_1\left[k^3bx^6(b^2-x^2)-2\frac{k^3}{b}x^8(b^2-x^2)+\frac{k^3}{b^3}x^{10}(b^2-x^2)-\frac{k^3}{3b^3}x^6(b^2-x^2)^3\right]\\&+A_2\left[k^3bx^4(b^2-x^2)-2\frac{k^3}{b}x^6(b^2-x^2)+\frac{k^3}{b^3}x^8(b^2-x^2)-\frac{k^3}{3b^3}x^4(b^2-x^2)^3\right]\\&+A_3\left[k^3bx^2(b^2-x^2)-2\frac{k^3}{b}x\,(b^2-x^2)+\frac{k^3}{b^3}x\,(b^2-x^2)-\frac{k^3}{3b^3}x^2(b^2-x^2)^3\right]\\&+A_4\left[\frac{k^5}{3b}x^2(b^2-x^2)^3-\frac{2k^5}{3b^3}x^4(b^2-x^2)^3+\frac{k^5}{3b^5}x^6(b^2-x^2)^3-\frac{k^5}{5b^5}x^2(b^2-x^2)^3\right]\end{aligned}\right\}dx$$

Upon further integration, clearing, and dropping of common facto factors the expression results as

$$0=15A_1b^4+39A_2b^2+143A_3+10A_4k^2b^2 \qquad (96)$$

The second Galerkin Equation is

$$0 = \iint_R \left\{ \begin{aligned} & A_1\left[k^2b^2x^8 - 2k^2x^{10} + \frac{k^2}{b^2}x^{12} - x^8y^2\right] \\ & + A_2\left[k^2b^2x^6 - 2k^2x^8 + \frac{k^2}{b^2}x^{10} - x^6y^2\right] \\ & + A_3\left[k^2b^2x^4 - 2k^2x^6 + \frac{k^2}{b^2}x^8 - x^4y^2\right] \\ & + A_4\left[k^2b^2x^4y^2 - 2k^2x^6y^2 + \frac{k^2}{b^2}x^8y^2 - x^4y^4\right] \end{aligned} \right\} dx\,dy \qquad (97)$$

$$0 = \int_0^b \left\{ \begin{aligned} & A_1\left[k^3bx^8(b^2-x^2) - 2\frac{k^3}{b}x^{10}(b^2-x^2) + \frac{k^3}{b^3}x^{12}(b^2-x^2) - \frac{k^3}{3b^3}x^8(b^2-x^2)^3\right] \\ & + A_2\left[k^3bx^6(b^2-x^2) - 2\frac{k^3}{b}x^8(b^2-x^2) + \frac{k^3}{b^3}x^{10}(b^2-x^2) - \frac{k^3}{3b^3}x^6(b^2-x^2)^3\right] \\ & + A_3\left[k^3bx^4(b^2-x^2) - 2\frac{k^3}{b}x^6(b^2-x^2) + \frac{k^3}{b^3}x^8(b^2-x^2) - \frac{k^3}{3b^3}x^4(b^2-x^2)^3\right] \\ & + A_4\left[\frac{k^5}{3b}x^4(b^2-x^2)^3 - \frac{2k^5}{3b^3}x^6(b^2-x^2)^3 + \frac{k^5}{3b^5}x^8(b^2-x^2)^3 - \frac{k^5}{5b^5}x^4(b^2-x^2)^5\right] \end{aligned} \right\}$$

Again integrating, clearing and dropping common factors

$$0 = 35A_1b^4 + 75A_2b^2 + 195A_3 + 16A_4k^2b^2 \qquad (98)$$

Before using equations (96) and (98) to obtain values for the coefficients C_1 and C_2 it will be convenient to let

$$\left. \begin{aligned} A_1 &= \frac{B_1}{b^2}C_2 \\ A_2 &= \frac{B_2}{b^2}C_1 + B_3C_2 + \frac{B_4}{b^2} \\ A_3 &= B_5C_1 + B_6b^2C_2 + B_7 \\ A_4 &= B_8C_2 \end{aligned} \right\} \qquad (99)$$

Where $B_1 = 42k^2$

$B_2 = 20k^2$

$B_3 = -(2+40k^2)$

$B_4 = 2k^2$

$B_5 = -(2+12k^2)$

$B_6 = 6k^2$

$B_7 = -(2k^2 + \frac{\sigma}{1+\sigma})$

$B_8 = -6$

Then equation (96) becomes

$$0 = \left[39B_2 + 143B_5\right]C_1 + \left[15B_1 + 39B_3 + 143B_6 + 16k^2B_8\right] b^2C_2 + \left[39B_4 + 143B_7\right] \qquad (100)$$

or further condensing

$$0 = G_1C_1 + G_2b^2C_2 + G_3 \qquad (101)$$

Where $G_1 = -(286 + 936k^2)$

$G_2 = -(78 + 108k^2)$

$G_3 = -(208k^2 + 143\frac{\sigma}{1+\sigma})$

In a similar fashion it is found that equation (98) may be condensed into the following

$$0 = H_1C_1 + H_2b^2C_2 + H_3 \qquad (102)$$

Where $H_1 + -(390 + 840k^2)$

$H_2 = -(150 + 456k^2)$

$H_3 = -(240k^2 + 195\frac{\sigma}{1+\sigma}$

From the equations (101) and (102) it is seen that

$$C_1 = -\left[\frac{H_3 + H_2\left(\frac{G_1H_3 - G_3H_1}{G_2H_1 - G_1H_2}\right)}{H_1}\right]$$

$$C_2 = \frac{1}{b^2}\left(\frac{G_1H_3 - G_3H_1}{G_2H_1 - G_1H_2}\right) \tag{104}$$

In order to estimate the error involved in using the approximate stress function the maximum and minimum values of ϵ must be found. Again the investigation is carried out in the region, on the boundary and at the vertices.

In the region the maximizing conditions are

$$0 = \frac{\partial\epsilon}{\partial y} = 2A_4xy \quad \text{so that } y = 0 \text{ is a solution} \tag{105}$$

$$0 = \frac{\partial\epsilon}{\partial x} = 5A_1x^4 + 3A_2x^2 + A_3 + A_4y^2$$

and solving $\quad x^2 = \dfrac{-3A_2 \pm \sqrt{9A_2^2 - 20A_1A_3}}{10A_1}$

or in more convenient form

$$\left(\frac{x}{b}\right)^2 = \frac{-3A_2b^2 \pm \sqrt{9A_2^2b^4 - 20A_1b^4A_3}}{10A_1b^4} \tag{106}$$

On the boundary

$$\epsilon = A_1x^5 + A_2x^3 + A_3x + A_4x\left(k^2b^2 - 2k^2x^2 + \frac{k^2x^4}{b^2}\right) \tag{107}$$

and maximizing

$$0 = \frac{\partial\epsilon}{\partial x} = 5A_1x^4 + 3A_2x^2 + A_3 + A_4\left(k^2b^2 - 6A_4k^2x^2 + \frac{5k^2}{b^2}x^4\right)$$

which yields

$$x^2 = \frac{-(3A_2 - 6A_4k^2) \pm \sqrt{(3A_2 - 6A_4k^2)^2 - 20\left(A_1 + \frac{k^2}{b^2}A_4\right)(A_3 + A_4k^2b^2)}}{10\left(A_1 + \frac{k^2}{b^2}A_4\right)}$$

or more conveniently

$$\left(\frac{x}{b}\right)^2 = \frac{-b^2(3A_2 - 6A_4k^2) \pm \sqrt{(3A_2b^2 - 6A_4k^2b^2)^2 - 20(A_1b^4 + k^2b^2A_4)(A_3 + A_4k^2b^2)}}{10(A_1b^4 + A_4k^2b^2)} \tag{108}$$

To obtain explicit values for the error estimate it is necessary to specify the parameter k and the value for σ.

Let $k = .1$, $\sigma = .28$. Then it is found that the maximizing conditions yield

In the region

$$y = 0$$

$$\frac{x}{b} = \pm .2374377 \text{ and } \pm .0053766$$

so that the values for the errror are

$$\frac{\epsilon}{b} = \pm 7.99011 \times 10^{-6} \text{ and } \pm 6.307956 \times 10^{-6} \quad \text{respectively}$$

On the boundary

$$\frac{x}{b} = \mp .567446 \text{ and imaginary values}$$

with the result that

$$\frac{\epsilon}{b} = \mp 3.28776 \times 10^{-5}$$

At the vertices $(\pm b, 0)$ the error is

$$\epsilon = \pm \left[A_1 b^5 + A_2 b^3 + A_3 b \right]$$

$$\frac{\epsilon}{b} = \pm \left[(2-8k^2)(c_1+b^2c_2) + \frac{\sigma}{1+\sigma} \right]$$

$$= \pm .0002009$$

It is seen that $\epsilon_M = .0002b$ and $\epsilon_m = -.0002b$. In general it is not expected that the shear stresses will be maximum at the vertices so that the error in Ω will be in most cases less at the critical points in the section.

As a matter of interest the results obtained for the first approximation to the correct function may be noted.

The function was $\Omega_1 = c_1 x(k^2b^2 - 2k^2x^2 + \frac{k^2}{b^2}x^4 - y^2)$ (109)

On solving $c_1 = - \left[\dfrac{16k^2 + 11\frac{\sigma}{1+\sigma}}{72k^2 + 22} \right]$ (110)

and $\epsilon = - \left[(2 + 12k^2)c_1 + (2k^2 + \frac{\sigma}{1+\sigma}) \right] x + \left[20\frac{k^2}{b^2}c_1 + 2\frac{k^2}{b^2} \right] x^3$ (111)

maximizing $\dfrac{x}{b} = \pm \sqrt{\dfrac{(2+12k^2)c_1 + (2k^2 + \frac{\sigma}{1+\sigma})}{6k^2(10c_1 + 1)}}$ (112)

Evaluating for $k = .1$ and $\sigma = .28$

In the region

$$\frac{\theta}{b} = \pm .0002059$$

At the vertices

$$\frac{\theta}{b} = \pm .018447$$

So that $\theta_M = .018447b$ and $\theta_m = -.018447b$.

In this case the use of a higher approximation function was very helpful in increasing the accuracy of the solution. It may be noted, at this point, that other functions were investigated of the form

$$\Omega_0 = c(t^2 - y^2)$$

$$\Omega'_2 = (c_1 + c_2 x)(t^2 - y^2)$$

$$\Omega''_2 = (c_1 + c_2 x^2)(t^2 - y^2)$$

$$\Omega'''_2 = (c_1 x + c_2 y^2)(t^2 - y^2)$$

The results indicated that the only suitable type of function for use in this method of solution for the section was one of the form $\Omega = (t^2 - y^2) \sum_{i=1}^{n} c_i x^i$ where i is odd. (113) This should also hold true for other sections which are symmetrical about the x-axis.

The complete solution may be summarized as follows

$$\Omega_2 = (c_1 x + c_2 x^3)\left(k^2 b^2 - 2k^2 x^2 + \frac{k^2}{b^2} x^4 - y^2\right) \qquad (114)$$

Where

$$c_1 = -\left[\frac{\left(16k^2 + 13\frac{\sigma}{1+\sigma}\right)}{26 + 56k^2} + \frac{(25 + 76k^2)\left(65k^2 - 260k^4 - 325k^2\frac{\sigma}{1+\sigma}\right)}{(65 + 140k^2)(65 + 728k^2 + 1488k^4)}\right]$$

$$c_2 = \frac{1}{b^2}\left[\frac{65k^2 - 260k^4 - 325k^2\frac{\sigma}{1+\sigma}}{65 + 728k^2 + 1488k^4}\right]$$

and for $k = .1$, $\sigma = .28$ the error in the approximate solution is related to the exact solution Ω, by the relation

BI-PARABOLIC SECTION

CONTOURS OF THE SHEAR STRESS FUNCTION DUE TO THE LIFT FORCE

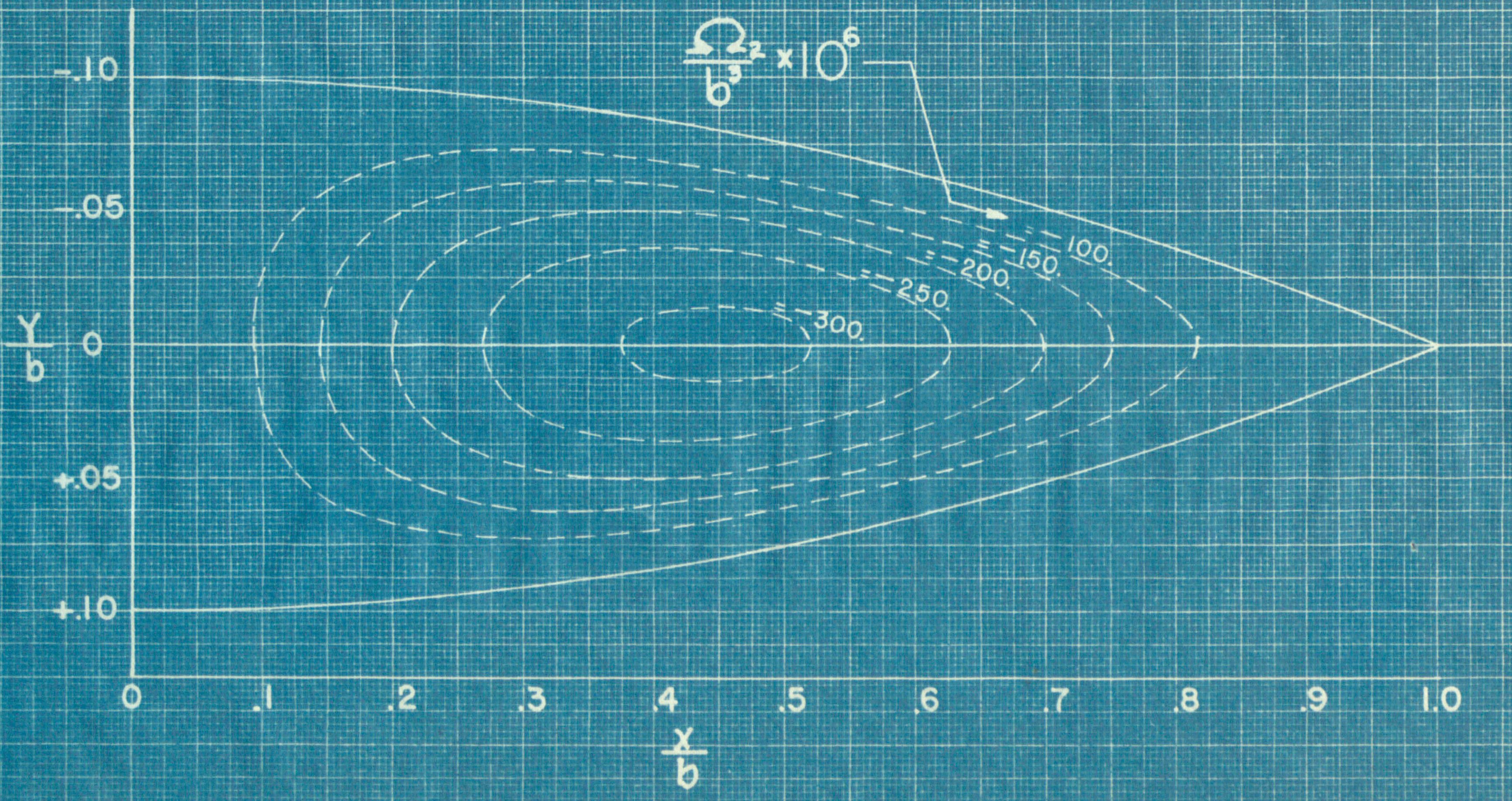

Figure 6

BI-PARABOLIC SECTION
VARIATION OF STRESSES DUE TO THE LIFT FORCE

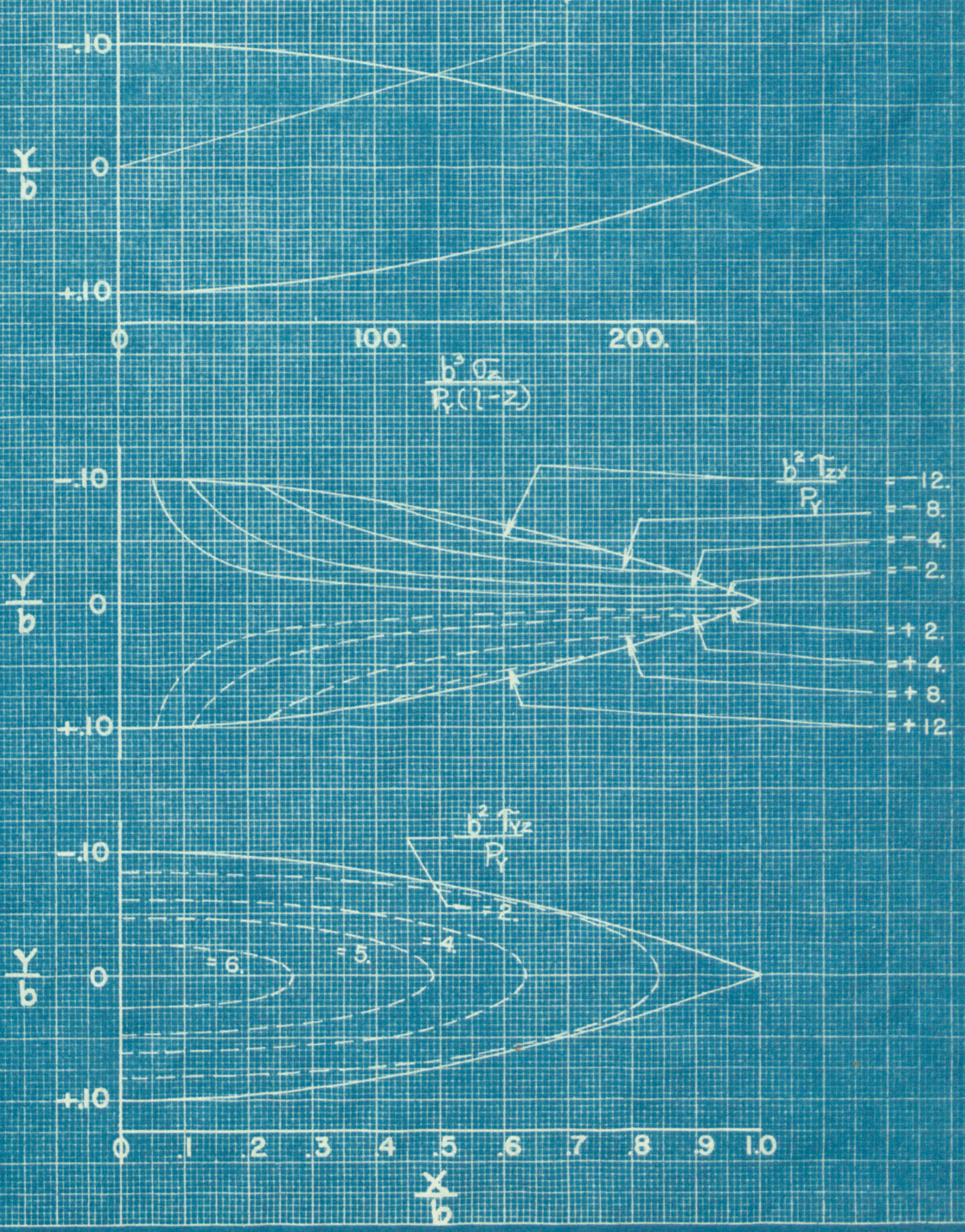

Figure 7

is related to the exact solution Ω by the relation

$$(\Omega_2 - .0001b\,\Psi) \leq \Omega \leq (\Omega_2 + .0001b\,\Psi) \tag{115}$$

Inasmuch as the exact value for Ψ was not determined values for the approximation torsion stress function Ψ_1 may be used in its place.

The stresses are as follows

$$\sigma_z = -\frac{P(1-z)x}{I_x} \tag{116}$$

$$\tau_{zx} = -\frac{P}{I_x}\left[2c_1xy + 8c_2x^3y\right] \tag{117}$$

$$\begin{aligned}\tau_{yz} = -\frac{P}{I_x}\Big[\,&\tfrac{1}{2}(k^2b^2 - 2k^2x^2 + \frac{k^2}{b^2}x^4 - y^2)\\ &+c_1(k^2b^2 - 6k^2x^2 + \frac{5k^2}{b^2}x^4 - y^2)\\ &+C_2(3k^2b^2x^2 - 10k^2x^4 + \frac{7k^2}{b^2}x^6 - 3x^2y^2)\Big]\end{aligned} \tag{118}$$

where $$I_x = \frac{64}{105}k^3b^4 \tag{119}$$

The curves for Ω, σ_z, τ_{zx} and τ_{yz} are shown in figures 5ixe and 57: for a value of k =.1

III. Flexure due to Drag Force

<u>Theory</u>

In the previous developement for flexure due to the lift force consider the effect of expressing the stresses as [3]

$$\tau_{zx} = \frac{\partial\phi}{\partial y} - \frac{Px^2}{2Iy} + f(y) \tag{120}$$

$$\tau_{yz} = -\frac{\partial\varphi}{\partial x} \tag{121}$$

where φ is a stress function and $f(y)$

is arbitrary for the case when the load acts along the x-axis again the equations (2),(3),(5),(6),(8),(9),(10) and (11) are identically satisfied. If these relations (120) and (121) are substituted into the compatibility equations (12) and (13) there results

$$\frac{\partial}{\partial x}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = 0 \qquad (122)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = \frac{P}{I_y}\left(\frac{\sigma}{1+\sigma}\right) - \frac{\partial^2 f(y)}{\partial y^2} \qquad (123)$$

and these conditions will be satisfied if

$$\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = \frac{P}{I_y}\left(\frac{\sigma}{1+\sigma}\right)y - \frac{\partial f(y)}{\partial y} + C \qquad (124)$$

It is possible to show that C represents a twist in the beam due to the applied load not passing through the elastic axis. However, in the formulation of the problem the flexure load was chosen to act through the elastic center so that there will be no twist and thus C = 0

Substituting (122) and (123) into the remaining boundary condition (7) it is seen that the condition now appears as

$$\frac{d\varphi}{ds} = \left\{\frac{P}{2I_y}x^2 - f(y)\right\}\frac{dy}{ds} \qquad (125)$$

Thus the function φ is completely specified by the equations (124) and (125). By one method of solution it will be convenient to choose the arbitrary function f(y) in such a way that

$$\frac{P}{2I_y}x^2 - f(y) = 0 \qquad (126)$$

<u>Computation</u>

The beam section now appears as in figure eight

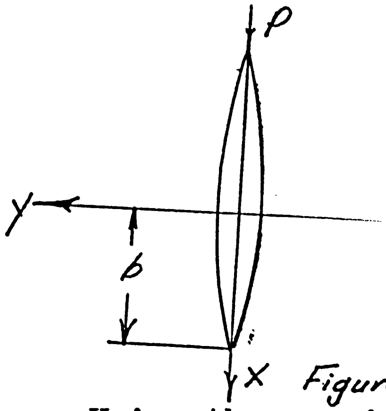

Figure 8.

The right half of the boundary is

$$x^2 - \frac{b}{k}y - b^2 = 0$$

The left half of the boundary is

$$x^2 + \frac{b}{k}y - b^2 = 0$$

Using the previous relationship as a guide choose a function

$$\varphi = \frac{\sigma P}{2(1+\sigma)I_y}\left[x^2 \pm \frac{b}{k}y - b^2\right]y \qquad (127)$$

and by equation (126) let

$$f(y) = \frac{P}{2I_y}\left[\pm\frac{b}{k}y + b^2\right] \qquad (128)$$

The shear stresses may be evaluated to give

$$\tau_{zx} = -\frac{P}{2(1+\sigma)I_y}\left[x^2 \pm \frac{(1-\sigma)b}{k}y - b^2\right] \qquad (129)$$

Where the $\pm$ sign indicates the ~~right~~ left or ~~left~~ right half of the boundary respectively

$$\tau_{yz} = -\frac{\sigma P}{(1+\sigma)I_y}(xy) \qquad (130)$$

$$\text{Where } I_y = \frac{8}{15}kb^4 \qquad (131)$$

These expressions must satisfy all the equations of equilibrium and compatibility(2)(13) if they constitute exact solutions. It is easily seen that

(2) $\frac{\partial \tau_{zx}}{\partial z} = 0$ is identically satisfied

(3) $\frac{\partial \tau_{yz}}{\partial z} = 0$ is identically satisfied

(4) $\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{P(x)}{I_y}$ is required

$$\text{or } \frac{-P}{2(1+\sigma)I_y}(2x) - \frac{\sigma P}{(1+\sigma)I_y}(x) = \frac{-P}{I_y}(x)$$

$$\frac{-(1+\sigma)P}{(1+\sigma)I_y}(x) = \frac{-P}{I_y}(x)$$

$$\therefore \frac{-P}{I_y}x = \frac{-P}{I_y}(x)$$

The boundary conditions

(5) $\tau_{zx} n = 0$ is identically satisfied

(6) $\tau_{yz} n = 0$ is identically satisfied

(7) $\tau_{zx} \ell + \tau_{yz} m = 0$

or $-\tau_{zx} \frac{dy}{ds} + \tau_{yz} \frac{dx}{ds} = 0$

and $\frac{dy}{dx} = \frac{\tau_{yz}}{\tau_{zx}}$ is required

now $y = \frac{k}{b}(x^2 - b^2)$ on the right half of the boundary

so $\frac{dy}{dx} = \frac{2kx}{b}$

but $$\frac{\tau_{yz}}{\tau_{zx}} = \frac{-\frac{\sigma P}{(1+\sigma) I_y}(x, y)}{-\frac{P}{2(1+\sigma) I_y}\left[x^2 - \frac{(1+\sigma)b}{k}y - b^2\right]}$$

$$= \frac{2\sigma(xy)}{\left[(x^2 - \frac{b}{k}y - b^2) + \sigma\frac{b}{k}y\right]}$$

$$= \frac{2kx}{b}$$

$\therefore \frac{dy}{dx} = \frac{\tau_{yz}}{\tau_{zx}}$

The compatibility equations

(8) $\frac{\partial^2 \theta}{\partial x^2} = 0$

(9) $\frac{\partial^2 \theta}{\partial y^2} = 0$

(10) $(1+\sigma) \nabla^2 \sigma_z + \frac{\partial^2 \theta}{\partial z^2} = 0$

(11) $(1+\sigma) \nabla^2 \tau_{xy} + \frac{\partial^2 \theta}{\partial x \partial y} = 0$

and (12) $\nabla^2 \tau_{yz} = 0$ *are identically satisfied*

(13) $\nabla^2 \tau_{zx} = -\frac{P}{I_y(1+\sigma)}$ is required

$-\frac{P}{2(1+\sigma)I_y}$ (2) =

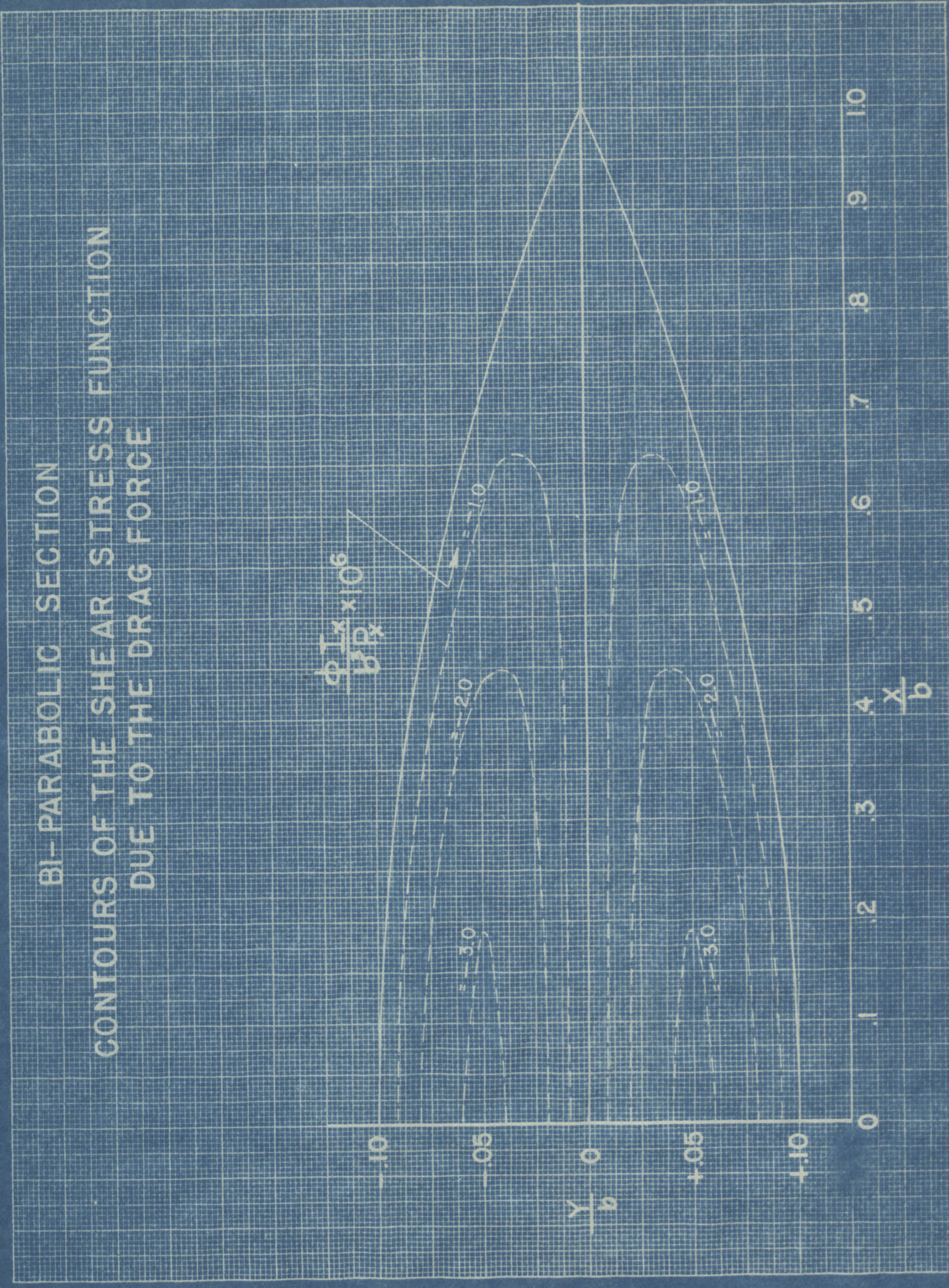
BI-PARABOLIC SECTION
CONTOURS OF THE SHEAR STRESS FUNCTION
DUE TO THE DRAG FORCE
$\frac{\phi I_x}{b^3 P_x} \times 10^6$
1.0
2.0
3.0
1.0
2.0
3.0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$\frac{x}{b}$
-.10
-.05
0
+.05
+.10
$\frac{y}{b}$

Figure 9.

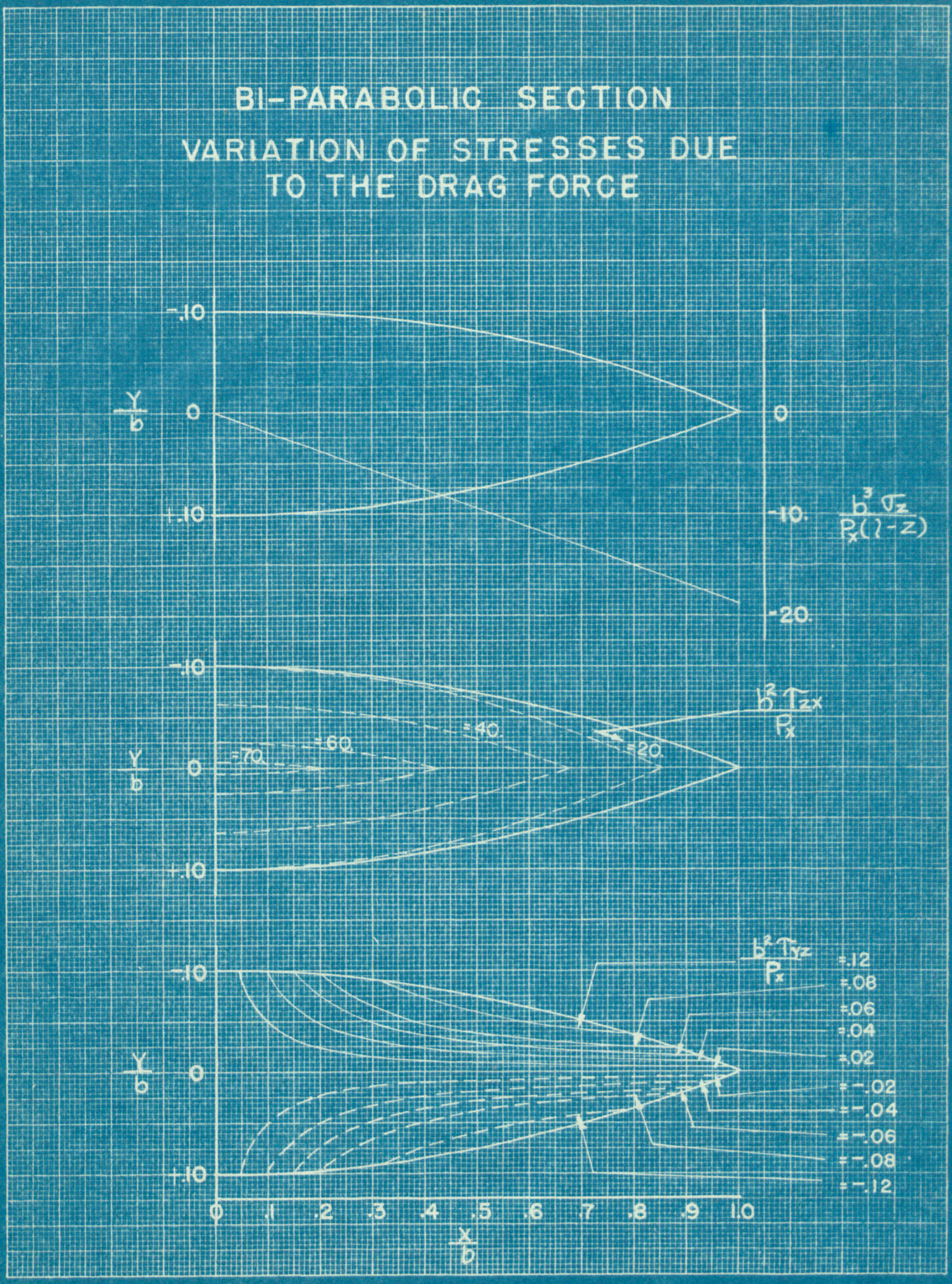
BI-PARABOLIC SECTION
VARIATION OF STRESSES DUE
TO THE DRAG FORCE
Y/b
-.10
0
+.10
$\frac{b^3 \sigma_z}{P_x(l-z)}$
0
-10.
-20.
$\frac{b^2 \tau_{zx}}{P_x}$
-.70
-.60
-.40
-.20
$\frac{b^2 \tau_{yz}}{P_x}$
±.12
±.08
±.06
±.04
±.02
±-.02
±-.04
±-.06
±-.08
±-.12
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
X/b

Figure 10.

$$-\frac{P}{(1+\sigma)I_y} = -\frac{P}{I_y(1+\sigma)}$$

In addition to the above conditions it is seen from inspection that $\iint_R \tau_{yz} dx\, dy = 0$ so that, as expected, there is no resultant side force. And

$$\iint_R \tau_{zx} dx\, dy = \left\{ 4\int_0^b \int_0^{\frac{k}{b}(b^2-x^2)} \left[x^2 + \frac{(1-\sigma)b}{k} y - b^2\right] dx\, dy \right\} \frac{-P}{2(1+\sigma)I_y}$$
$$= \frac{-2P}{(1+\sigma)I_y}\left[\frac{-4}{15}(1+\sigma)kb^4\right]$$
$$= +P$$

So the sum of all the vertical shear forces is equal to the applied load. The sum of the moments of all the shear forces should be zero and it is seen that for the right half of the section; $M = \iint_{R\,right} (\tau_{zx} y + \tau_{yz} x)\, dy\, dx$

$$\frac{-P}{(1+\sigma)I_y}\int_0^b \int_0^{\frac{k}{b}(x^2-b^2)} \left\{ \left[x^2 - \frac{(1+\sigma)b}{k} y - b^2\right] y + \left[\sigma xy\right] x \right\} dy\, dx = 0$$

and the same holds true for the left half. Thus the expressions for the shear stresses satisfy all conditions and are the exact solutions. The plots for the stress function φ and the stresses σ_z, τ_{zx}, τ_{yz} are shown in figures nine and ten for $k = .1$.

It should be mentioned that this method does not yield any worthwhile results in the problem of the flexure due to the lift force. If for that particular case

$$f(x) = \frac{P}{2I}\left[\frac{k}{b}(b^2-x^2)\right]^2$$ then the equation (125) will be satisfied if x and y are reversed.

Now let $\varphi = \frac{\sigma P}{2(1+\sigma)I_x}\left[y^2 - \frac{k^2}{b^2}(b^2-x^2)^2\right] x$

then $T_{zx} = \frac{-\sigma P}{(1+\sigma)I_x}\left[xy\right]$

and $$\tau_{yz} = \frac{1}{2(1+\sigma)} \frac{P}{I_x} \left[(k^2b^2 - y^2) - (2+4\sigma)k^2x^2 + (1-4\sigma)\frac{k^2}{b^2} x^4 \right]$$

On substitution into the equilibrium and compatibility equations it is found that equations (7) and (13') will not be satisfied.

Summary:

In cases where a more accurate knowledge of the stresses in a cantilever wing with a terminal load is required than can be ascertained by the ordinary formulas, it is necessary to use the procedure of the Theory of Elasticity in realizing a solution. The general problem is divided into three elemental problems which may be solved independently. In instances where an exact solution is not easily realized, variational methods may be employed to yield a solution with as high a degree of accuracy as may be desired. Each solution consists of expressions for the shear and normal stresses for each elemental problem. Having these it is possible by use of the principle of superposition to evaluate the combined stresses at any point in the wing so that the critical stress regions may be determined and provided for in the design.

While the discussion in this paper was concerned with a solid wing loaded by a concentrated force at its tip, it is possible to extend the treatment to the case of a hollow wing with distributed loading.

The solution for the torsion problem was developed using the assumption that cross-sections can warp, in the z-direction. However, at the center section or wing root, no warping can occur due to the symmetry present there and the center section will remain plane. Therefore the torsional stress solution fall down at or near the root so that further theoretical developement is necessary to provide a differant and

and suitable solution.

It appears that much work still remains to be done to adapt the principles of the Theory of Elasticity to the stress analysis of wings for high speed aircraft. However, it is anticipated that as the need arises the necessary development will be carried out.

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