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I hereby recommend that the thesis prepared under my supervision by Stanley J. Lawwill *entitled* Concerning Harmonic Functions, Especially with Reference to the Theory of Approximation

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CONCERNING HARMONIC FUNCTIONS,

ESPECIALLY WITH REFERENCE TO

THE THEORY OF APPROXIMATION

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CHAPTER I

Harmonic Rational Functions

A rational function, $R_n(x,y)$, of the real variables x and y, of degree n is a function which can be expressed as the quotient of two polynomials, $p(x,y)$ and $q(x,y)$, of maximum degree n:

$$(1) \qquad R_n(x,y) \equiv \frac{p_n(x,y)}{q_n(x,y)} \equiv \frac{\sum_{r+s\leq n} a_{rs}\, x^r y^s}{\sum_{j+k\leq n} b_{jk}\, x^j y^k}.$$

The rational function, $R_n(x,y)$, is said to be harmonic in a region if at every point of the region it satisfies identically the Laplace Differential Equation:

$$(2) \qquad \frac{\partial^2 R_n}{\partial x^2} + \frac{\partial^2 R_n}{\partial y^2} = 0.$$

In this chapter we shall derive some of the elementary properties of harmonic rational functions, properties which we can, however, find displayed nowhere in the literature.

The discussion of a special case affords a simple, yet particularly fruitful, introduction to the subject.

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We consider first harmonic rational functions in which the numerator polynomial, p, is a constant. We may choose the constant to be unity without loss of generality* and in this case we call the rational function an <u>inverse polynomial</u>.

Let $V \equiv \frac{1}{q(x,y)} \equiv q^{-1}$, where q(x,y) is a polynomial, not identically zero. We seek the conditions on the coefficients of q(x,y) such that V is harmonic -- that is to say, we first seek to satisfy the Laplace Differential Equation (2) for an inverse polynomial. We find successively,

$$V_x = -q^{-2} q_x ,$$

$$V_y = -q^{-2} q_y ,$$

$$V_{xx} = 2q^{-3} q_x^2 - q^{-2} q_{xx} ,$$

$$V_{yy} = 2q^{-3} q_y^2 - q^{-2} q_{yy} .$$

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*If V(x,y) satisfies condition (2), it is evident that, if k is a constant, kV(x,y) also satisfies condition (2), and conversely. Since such a multiplicative constant, k, adds no essential generality, we suppress it.

Hence, in this case, the Laplace Equation becomes

$$2q^{-3}(q_x^2 + q_y^2) - q^{-2}(q_{xx} + q_{yy}) \equiv 0.$$

Since q is not identically zero, we may write the last identity in the convenient form

$$(3) \qquad 2(q_x^2 + q_y^2) - q(q_{xx} + q_{yy}) \equiv 0.$$

This, then is a necessary and sufficient condition that an inverse polynomial, q^{-1}, be harmonic.

Let us now assume that the inverse polynomial, q^{-1}, is harmonic and consider the polynomial, q, itself. There are two cases to be discussed: q is harmonic, q is not harmonic; we postpone the latter case and consider the former.

If the inverse polynomial, q^{-1}, is harmonic, then q satisfies condition (3). If, further, we assume that q itself is harmonic, q must also satisfy condition (2). Since condition (2) implies the vanishing of the second member of condition (3), condition (3) reduces to

$$q_x^2 + q_y^2 \equiv 0 \quad ,$$

which yields

$$(4) \qquad q_y \equiv \pm i\, q_x \,, \qquad i = \sqrt{-1}\,.$$

It is to be noted that, since we are discussing functions harmonic in a region, conditions (2), (3) and (4) are identities in (x,y) -- they must hold, and do hold, for all points of the region.

An immediate conclusion from condition (4) is the

Theorem: If q^{-1} is a harmonic inverse polynomial, then q is a harmonic polynomial with real coefficients if and only if q is a constant.

The converse theorem is equally evident from conditions (3) and (4).

Since harmonic inverse polynomials with real coefficients are thus uninteresting, we admit complex coefficients and attempt to find a necessary form for harmonic polynomials whose reciprocals are also harmonic.

Let $q^{(1)} \equiv b_{10}x + b_{01}y + b_{00}$. It is evident that $q^{(1)}$ is harmonic - any linear form satisfies condition (2) - and its reciprocal will be harmonic if $q^{(1)}$ satisfies condition (4). In this case condition (4) yields

$$b_{01} = \pm i b_{10} .$$

Without loss of generality* we may take b_{10} to be unity and express $q^{(1)}$ as

$$q^{(1)} \equiv x \pm i y + b_1 .$$

The corresponding harmonic rational function is

$$h^{(1)}(x,y) \equiv \frac{1}{x \pm iy + b_1} .$$

We investigate next the general second degree polynomial,

$$q^{(2)} \equiv b_{20} x^2 + b_{11} xy + b_{02} y^2 + b_{10} x + b_{01} y + b_{00}.$$

If $q^{(2)}$ is to be harmonic, it must satisfy condition (2) which, in this case, is

$$2 b_{20} + 2 b_{02} = 0 .$$

Hence $b_{02} = - b_{20}$ and we have the harmonic polynomial,

$$q^{(2)} \equiv b_{20} x^2 + b_{11} xy - b_{20} y^2 + b_{10} x + b_{01} y + b_{00} .$$

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*See footnote pg. 2

If, further, the inverse polynomial $1/q^{(2)}$ is to be harmonic, $q^{(2)}$ must satisfy condition (4).

We compute

$$q^{(2)}_y = b_{11}x - 2b_{20}y + b_{01} \ ,$$

$$q^{(2)}_x = 2b_{20}x + b_{11}y + b_{10} \ .$$

These quantities are the ones which appear in condition (4); we substitute them in (4) and have

$$b_{11}x - 2b_{20}y + b_{01} \equiv \pm i(2b_{20}x + b_{11}y + b_{10}) .$$

Since we have here an identity in (x,y),* the constant term and the coefficients of x and y on the left must equal respectively the constant term and the coefficients of x and y on the right; that is,

$$b_{01} = \pm i b_{10}$$

and

$$b_{11} = \pm 2i b_{20} \ .$$

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*Condition (4) holds for all points of a region.

The corresponding harmonic polynomial is

$$q^{(2)} \equiv b_{20}x^2 \pm 2ib_{20}xy - b_{20}y^2 \quad + b_{10}x \pm ib_{10}y + b_{00}.$$

We note, however, that the last three terms of $q^{(2)}$ are precisely $b_{10}q^{(1)}$ - indeed, this was to have been expected: since differentiation is a distributive process, the linear terms in $q^{(2)}$ have to satisfy conditions (2) and (4) independently of the second degree terms. That is to say, after we have computed $q_x^{(2)}$ and $q_y^{(2)}$ and substituted them in (4), we have an identity in (x,y). Hence the coefficients on the left must equal the corresponding coefficients on the right. But the terms, b_{10} and b_{01}, which now appear as constants in (4) are the coefficients of the linear terms in $q^{(2)}$. If identity (4) holds, these terms must be equal independently of the other terms - but this is merely a repetition of the argument used in the derivation of $q^{(1)}$. So we find that the inclusion of linear terms in $q^{(2)}$ has added nothing new to our discussion.* Hence, having already obtained the necessary coefficients of the linear terms in $q^{(1)}$, we may here suppress them in $q^{(2)}$. We remove the constant multiplier,

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*Indeed, we shall later show that a sum of constant multiples of harmonic polynomials of distinct degrees whose reciprocals are also harmonic is a harmonic polynomial with harmonic reciprocal.

b_{20}, and have the characteristic second degree polynomial,

$$q^{(2)} \equiv x^2 \pm 2ixy - y^2,$$

which is harmonic, and its harmonic inverse,

$$h^{(2)}(x,y) \equiv \frac{1}{x^2 \pm 2ixy - y^2}.$$

We now consider the third degree polynomial,

$$q^{(3)}(x,y) \equiv b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.$$

Here, again there is no loss of generality incurred by the omission of second degree and linear terms - since they are of distinct degrees, they must separately satisfy the differential identities (2) and (4).* We have already discussed $q^{(2)}$ and $q^{(1)}$; we omit them from $q^{(3)}$.

Since $q^{(3)}$ is assumed harmonic, it must satisfy condition (2), which becomes in this case

$$(3b_{30} + b_{12})x + (b_{21} + 3b_{03}) \equiv 0.$$

This is an identity in (x,y); the coefficients must separately vanish; we have,

$$b_{12} = -3b_{30} \quad ; \quad b_{21} = -3b_{03}.$$

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*See pp. 5-7 for a typical argument.

The third degree harmonic polynomial is, then,

$$q^{(3)} \equiv b_{30}\,x^3 - 3b_{03}\,x^2 y - 3b_{30}\,x y^2 + b_{03}\,y^3 .$$

Since, further, the inverse polynomial, $1/q^{(3)}$, is assumed harmonic, $q^{(3)}$ must also satisfy condition (4). We compute

$$q_x^{(3)} = 3b_{30}x^2 - 6b_{03}xy - 3b_{30}y^2$$

and

$$q_y^{(3)} = -3b_{03}x^2 - 6b_{30}xy + 3b_{03}y^2 .$$

These quantities are the ones which appear in condition (4); we substitute them in (4) and have

$$-3b_{03}x^2 - 6b_{30}xy + 3b_{03}y^2 \equiv \pm 3i\,b_{30}x^2 \mp 6i\,b_{03}xy \mp 3i\,b_{30}y^2 .$$

Hence, as before, corresponding coefficients must be equal, and we have

$$b_{03} = \mp i\,b_{30} .$$

We remove the constant multiplier, $-b_{30}$, and have the harmonic polynomial

$$q^{(3)} \equiv x^3 \pm 3i\,x^2 y - 3xy^2 \mp i\,y^3$$

and its corresponding harmonic rational function,

$$h^{(3)}(x,y) \equiv \frac{1}{x^3 \pm 3i\,x^2 y - 3xy^2 \mp i\,y^3} .$$

Similarly we find the harmonic polynomial

$$g^{(4)} \equiv x^4 \pm 4ix^3y - 6x^2y^2 \mp 4ixy^3 + y^4$$

whose reciprocal is also harmonic.

We collect and exhibit together these first four harmonic polynomials whose reciprocals are also harmonic:

$$g^{(1)} \equiv x \pm iy + b_1 ,$$

$$g^{(2)} \equiv x^2 \pm 2ixy - y^2 ,$$

$$g^{(3)} \equiv x^3 \pm 3ix^2y - 3xy^2 \mp iy^3 ,$$

and $g^{(4)} \equiv x^4 \pm 4ix^3y - 6x^2y^2 \mp 4ixy^3 + y^4 .$

We observe that, except for the constant b_1 which is of no importance in connection with conditions (2) and (4), we have

$$g^{(2)} \equiv [g^{(1)}]^2 ,$$

$$g^{(3)} \equiv [g^{(1)}]^3 ,$$

$$g^{(4)} \equiv [g^{(1)}]^4 .$$

The obvious conclusion is that, in general, we have

$$(5) \qquad q^{(k)} = [q^{(1)}]^k = (x \pm iy)^k.$$

As a matter of fact, we shall later prove that if a polynomial q satisfies both conditions (2) and (4), it must be of the characteristic form (5). First, there are several pertinent remarks concerning $q^{(k)}$.

If we define $q^{(k)}$ by (5), we have

$$q_x^{(k)} = k(x \pm iy)^{k-1},$$

$$q_y^{(k)} = \pm ik(x \pm iy)^{k-1},$$

$$q_{xx}^{(k)} = k(k-1)(x \pm iy)^{k-2},$$

and
$$q_{yy}^{(k)} = -k(k-1)(x \pm iy)^{k-2}.$$

It is evident that $q^{(k)}$, so defined, does satisfy both conditions (2) and (4).

We note that if a polynomial, $q^{(n)}$, and its inverse, $1/q^{(n)}$, are both harmonic, it is a necessary consequence that $q^{(n)}$ be completely* homogeneous of degree n. For, as was evident in the derivation, each coefficient of $q^{(n)}$ must be a non-zero multiple of the leading coefficient; hence if just one coefficient vanishes, the polynomial is identically zero.

Now, since each polynomial $q^{(k)} = (x \pm iy)^k$, $(k = 0, 1, 2, \ldots, n)$, is analytic and harmonic, we may express the most general harmonic rational function whose numerator is a constant and whose denominator is itself harmonic as

$$(6) \qquad h^{(n)}(x,y) \equiv \frac{a_n}{\sum_{k=0}^{n} b_k (x \pm iy)^k}$$

There are here two propositions to be proved:

1° If a function satisfies the conditions of the preceding paragraph, it must be of form (6).

2° If a function is of form (6), it does satisfy the specified conditions.

We prove these in order.

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*We shall say a polynomial is <u>completely</u> homogeneous of degree n if every term is of total degree n and if no term of total degree n is wanting.

1° We have seen that if a polynomial satisfies both conditions (2) and (4), then it must be completely homogeneous. Hence we may write the denominator of the corresponding harmonic rational function in the form

$$q^{(n)} \equiv \sum_{j=0}^{n} a_{n-j,j}\, x^{n-j} y^{j}.$$

We find now the conditions under which q, so defined, is harmonic; we apply condition (2), and compute

$$q_{xx}^{(n)} = \sum_{j=0}^{n-2} (n-j)(n-j-1)\, a_{n-j,j}\, x^{n-j-2} y^{j}$$

and

$$q_{yy}^{(n)} = \sum_{j=2}^{n} j(j-1)\, a_{n-j,j}\, x^{n-j} y^{j-2}.$$

In $q_{yy}^{(n)}$ we set $r = j-2$, $j = r+2$, and have

$$q_{yy}^{(n)} = \sum_{r=0}^{n-2} (r+2)(r+1)\, a_{n-r-2,r+2}\, x^{n-r-2} y^{r}.$$

The summations which appear in (2) have now the same index range; we change the index in $q_{yy}^{(n)}$ from r to j and add these expressions for $q_{xx}^{(n)}$ and $q_{yy}^{(n)}$, obtaining

$$q_{xx}^{(n)} + q_{yy}^{(n)} = \sum_{j=0}^{n-2} \left[(n-j)(n-j-1)\, a_{n-j,j} + (j+1)(j+2)\, a_{n-j-2,j+2}\right] x^{n-j-2} y^{j}.$$

Condition (2) implies the identical vanishing of this sum and hence the vanishing of each coefficient. That is,

$$(n-j)(n-j-1)\, a_{n-j,j} + (j+1)(j+2)\, a_{n-j-2,j+2} = 0,$$

$$(j = 0, 1, 2, \ldots, n).$$

We recast this equation into the convenient form

$$a_{n-j-2,j+2} = -\frac{(n-j)(n-j-1)}{(j+1)(j+2)}\, a_{n-j,j}, \qquad (7)$$

$$(j = 0, 1, 2, \cdots, n).$$

This is the condition that a polynomial be harmonic. We examine a few of the coefficients of $q^{(n)}$ in more detail. Using (6) we have, successively,

$$a_{n-2,2} = -\frac{n(n-1)}{1 \cdot 2}\, a_{n,0},$$

$$a_{n-3,3} = -\frac{(n-1)(n-2)}{2 \cdot 3}\, a_{n-1,1},$$

$$a_{n-4,4} = -\frac{(n-2)(n-3)}{3 \cdot 4}\, a_{n-2,2},$$

$$= \frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}\, a_{n,0},$$

$$a_{n-5,5} = -\frac{(n-3)(n-4)}{4\cdot 5}\, a_{n-3,3}$$

$$= \frac{(n-1)(n-2)(n-3)(n-4)}{2\cdot 3\cdot 4\cdot 5}\, a_{n-1,1} \;\ldots ,$$

We note that each coefficient is a non-zero multiple* of the second preceding coefficient: that each coefficient with an even second subscript is a multiple of the leading coefficient, $a_{n,0}$, and that all the other coefficients are multiples of the second coefficient, $a_{n-1,1}$. It is clear, then, that a homogeneous harmonic polynomial is uniquely determined when the coefficients of any two terms of unequal parity are given.#

We now insist that q^{-1} also be harmonic -- that q satisfy condition (4). We compute

$$q_x^{(n)} = \sum_{j=0}^{n-1} (n-j)\, a_{n-j,j}\, x^{n-j-1} y^{j}$$

and

$$q_y^{(n)} = \sum_{j=1}^{n} j\, a_{n-j,j}\, x^{n-j} y^{j-1} .$$

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*Non-zero because j ranges only up to n - 2.

#For example, to find the homogeneous harmonic polynomial of fourth degree with the two prescribed coefficients: $a_{3,1} = 3$ and $a_{0,4} = 2$, we find successively, using (7), $a_{1,3} = -3$, $a_{2,2} = -12$, $a_{4,0} = 2$; and the polynomial is:

$$q^{(4)} = 2x^4 + 3x^3y - 12x^2y^2 - 3xy^3 + 2y^4.$$

In $g_y^{(n)}$ we set $r = j-1$, $j = r+1$ and obtain

$$g_y^{(n)} = \sum_{r=0}^{n-1} (r+1)\, a_{n-r-1,\, r+1}\, x^{n-r-1}\, y^{r} .$$

We now change the summation index in $g_y^{(n)}$ from r to j and substitute these expressions for $g_x^{(n)}$ and $g_y^{(n)}$ in condition (4). Again the identity of two sums of products of discreet powers of x and y implies the identity of coefficients of corresponding products. Hence we have

$$(j+1)\, a_{n-j-1,\, j+1} = \pm i\, (n-j)\, a_{n-j,\, j} .$$

This we write in the convenient form

$$(8) \qquad a_{n-j-1,\, j} = \frac{\pm i\, (n-j)}{(j+1)}\, a_{n-j,\, j} , \qquad (j = 0, 1, 2, \cdots, n-1),$$

which is the condition that, for a harmonic polynomial q, q^{-1} be harmonic.

We note that condition (8) implies condition (7), which is to be expected since condition (7) was an essential simplification in the derivation of condition (8) -- condition (7) permitted the transition from condition (3) to condition (4).

An exhibition of the first few coefficients computed using (8) shows immediately that (8) is a binomial-expansion coefficient relationship. We have:

$$a_{n-1,1} = \pm \frac{n}{1} a_{n,0} (i),$$

$$a_{n-2,2} = \pm \frac{n-1}{2} a_{n-1,1} (i),$$

$$= \frac{n(n-1)}{1 \cdot 2} a_{n,0} (i^2),$$

$$a_{n-3,3} = \cdots = \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a_{n,0} (i^3).$$

These are evidently coefficients in the expansion of $(x \pm iy)^n$. Our conclusion is that if a polynomial, q, and its reciprocal, q^{-1}, are harmonic, then q is essentially an integral power of $(x \pm iy)$. Hence in the general case q must be a sum of constant multiples of integral powers of $(x \pm iy)$; that is, of the form (5).

2° We now show explicitly that the function defined by (5) is harmonic.

If
$$q = \sum_{k=0}^{n} b_k (x \pm iy)^k,$$

then
$$q_{xx} = \sum_{k=0}^{n-2} k(k-1) b_k (x \pm iy)^{k-2},$$

and
$$q_{yy} = -\sum_{k=0}^{n-2} k(k-1) b_k (x \pm iy)^{k-2},$$

hence q is evidently harmonic. Further,

$$q_x = \sum_{k=0}^{n-1} k\, b_k (x \pm iy)^{k-1}$$

and

$$q_y = \pm i \sum_{k=0}^{n-1} k\, b_k (x \pm iy)^{k-1}$$

so that $q_y = \pm i q_x$ and condition (4) is satisfied, hence $h_n(x,y) \equiv q^{-1}$ is harmonic.

We turn now to the postponed case,* in which q^{-1} is harmonic and q is not harmonic, and attempt to find a necessary form for q. Since q^{-1} is assumed harmonic, it is evident that condition (3) is the identity which will be the basis of our investigation.

Let $q^{(1)} = b_{10} x + b_{01} y + b_{00}$. Since $q_{xx} = q_{yy} = 0$, condition (3) reduces to

$$q_x^2 + q_y^2 = 0$$

which yields $b_{01} = \pm i b_{10}$, so that $q^{(1)}$ becomes

$$q^{(1)} = x \pm iy + b_1 \, .$$

But this is precisely the linear polynomial we derived in Case 1!#

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*Cf. pg. 3

#As it must be, since any linear form is harmonic.

Let $$g^{(2)} = b_{20}x^2 + b_{11}xy + b_{02}y^2 . \,*$$

We compute successively,

$$g_x^{(2)} = 2b_{20}x + b_{11}y \,,$$

$$g_y^{(2)} = b_{11}x + 2b_{02}y \,,$$

$$(g_x^{(2)})^2 = 4b_{20}^2x^2 + 4b_{20}b_{11}xy + b_{11}^2y^2 \,,$$

$$(g_y^{(2)})^2 = b_{11}^2x^2 + 4b_{02}b_{11}xy + 4b_{02}^2y^2 \,,$$

$$g_{xx}^{(2)} = 2b_{20} \,,$$

$$g_{yy}^{(2)} = 2b_{02} \,.$$

We substitute these quantities into identity (3) and equate corresponding coefficients, obtaining

$$(b_{20}+b_{02})\,b_{20} = 4b_{20}^2 + b_{11}^2 \,,$$

$$(b_{20}+b_{02})\,b_{11} = 4b_{11}(b_{20}+b_{02}) \,,$$

$$(b_{20}+b_{02})\,b_{02} = b_{11}^2 + 4b_{02}^2 \,.$$

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*Again, we may suppress the linear terms without loss of generality. Cf. pg. 7.

The second of these conditions yields either $b_{11} = 0$, or $b_{02} = -b_{20}$. The latter of these two possibilities implies that $q^{(2)}$ is harmonic; hence, we reject it and continue with $b_{11} = 0$. But the first equation then yields either $b_{02} = 3b_{20}$, which we must reject because it fails to satisfy the last equation, or $b_{20} = 0$. Now $b_{11} = b_{20} = 0$ implies $b_{02} = 0$, hence $q^{(2)}$ is identically zero, or else (returning to $b_{02} = -b_{20}$), q is harmonic! We find moreover, that if $b_{02} = -b_{20}$, then $b_{11} = \pm 2i b_{20}$ so that $q^{(2)}$ has the necessary form

$$q^{(2)} = (x \pm iy)^2.$$

We shall not include here the similar computations for the polynomials $q^{(3)}$and $q^{(4)}$. The derivation, although straight-foreward, is quite burdensome. Our conclusions were pointed: we found that necessarily

$$q^{(3)} = (x \pm iy)^3,$$

$$q^{(4)} = (x \pm iy)^4.$$

These polynomials are quite evidently harmonic -- in each case we have found the assumption that the inverse polynomial q^{-1} is harmonic implies that the polynomial q is necessarily harmonic!

We conjecture that, in every case, if an inverse polynomial is harmonic then so is the polynomial itself.

We have found no further evidence to support this conjecture, nor have we been able to prove or disprove it in general. Essentially, the problem is readily phrased in the question,

"Is there any non-harmonic polynomial solution of the differential identity

$$g(x,y) \equiv \frac{2(g_x^2 + g_y^2)}{(g_{xx} + g_{yy})} \quad ?"$$

We may now dismiss the subject of general harmonic functions

$$h_n(x,y) = \frac{p_n(x,y)}{g_n(x,y)}$$

with a few remarks.

We observe, first, that if we separate into real and pure imaginary parts any inverse polynomial in the complex variable, we obtain a pair of harmonic rational functions. That the derived functions are harmonic follows from the analyticity of an inverse polynomial; that the derived functions are rational is a consequence of the separation process.

A few examples illustrate the technique.*

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*To separate a complex rational function into real and pure imaginary parts, multiply numerator and denominator by the conjugate of the denominator.

We have:

$$\frac{1}{x+iy} = \left(\frac{1}{x+iy}\right)\left(\frac{x-iy}{x-iy}\right)$$

$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}.$$

Hence each of $\frac{x}{x^2+y^2}$, $\frac{y}{x^2+y^2}$ is harmonic.

$$\frac{1}{(x+iy)^2} = \left(\frac{1}{(x+iy)^2}\right)\left(\frac{(x-iy)^2}{(x-iy)^2}\right)$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} - 2i\,\frac{xy}{(x^2+y^2)^2}.$$

Hence each of $\frac{x^2-y^2}{(x^2+y^2)^2}$, $\frac{xy}{(x^2+y^2)^2}$ is harmonic.

The utility and importance of this process of deriving harmonic rational functions from rational functions of the complex variable will appear in the next chapter when we discuss approximations by harmonic functions.

Chapter II

The Theory and Method of Approximation

In this section we shall discuss critically the theories and methods of approximation by sequences of polynomials and rational functions in the complex variable and by harmonic polynomials, in order to show their relationship to the corresponding theory of approximation by harmonic rational functions. The theories in all these cases are to a large extent analogous. However, each type of approximation has its own peculiarities and it is not at all, much less _a priori_, true that a theorem in one of these fields of approximation has necessarily a complete analogue in another field; or, if it has, that the corresponding theorem has content; or, that it may be proved by the same, or even by similar, methods. Illustrations for these remarks will appear later.

In any discussion of the theory of approximation (whether by sequences of polynomials, harmonic polynomials, rational functions, or harmonic rational functions) there are three general topics to be considered:

1° the possibility of approximation,

2° the degree of approximation, and

3° the possible overconvergence of the sequence of approximating functions.

We shall center our remarks, then, on these topics and indicate the analogies and contrasts existing among the various fields of approximation.

Because the development of the theory of approximation by sequences of harmonic rational functions has been a relatively new project, that theory is much less complete than is the corresponding theory for other types of functions. Except for early, widely scattered, and highly specialized results, there has been a noticeable trend in the development of the theory of approximation, as will be evident from the following remarks. It is natural, because of its inherent simplicity and general utility, that the theory of polynomial approximation should have been first considered by many of the early workers in the field of approximation and that the theory of polynomial approximation should have been rather fully developed well in advance of the more general theory. Results concerning approximation by sequences of polynomials have been extended to more general results concerning approximation by sequences of rational functions. On the other hand, general results concerning approximation in terms of polynomials have been modified to more special results concerning approximation by sequences of harmonic polynomials. There are thus two obvious avenues of attack on the problem of approximation by harmonic rational functions: we can specialize the theorems on approximation by rational functions or we can generalize the theorems on approximation by harmonic polynomials.

In the first place, we can explode the common, but naive, notion that all properties of approximating harmonic functions can ensue immediately from the use of so simple a device as the separation, into real and pure imaginary parts, of an approximating function of a complex variable. A most startling illustration of this fact has been furnished by Walsh and Merriman.* They have shown that the orthogonality of a set of polynomials $\{p_k(z)\}$ in the complex variable simultaneously on an infinite family of level curves C_R does not necessarily imply a similar orthogonality for the set of harmonic polynomials obtained by separating the complex polynomials into real and pure imaginary parts. In fact, some of the sets of harmonic polynomials so obtained are orthogonal on <u>no</u> level curve of the family! The further condition necessary here to insure the subsistence of the orthogonality

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*Walsh and Merriman, Note on the simultaneous orthogonality of harmonic polynomials on several curves, Duke Mathematical Journal, Vol. 3, No. 2, June, 1937, pp. 279-288.

property is that the integral

$$J \equiv \int_C p_k(z)\, p_l(z)\, n(z)\, |dz|$$

vanish for $k \neq l$ and be real for $k = l$. We note that this further condition is in no respect a consequence of the assumption of the orthogonality on C of the set $\{p_k(z)\}$.*

Thus in many cases new methods are needed in the study of approximation by sequences of harmonic functions. An excellent illustration of the changed techniques required to establish new results concerning other types of approximation -- results obtained by analogy from a theorem already established in the field of polynomial approximation -- is afforded by the comparison of the following three theorems: the first (and original) dealing with polynomials,

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*More than incidentally, they also establish the Theorem: _If the set of polynomials $\{p_k(z)\}$ is cannonical on C with respect to a norm function n(z), and if the integral J vanishes for $k \neq l$ but not for $k = l$, then an arbitrary function f(z) analytic on and within C may be uniformly approximated on and within C by a series_

$$f(z) \sim \sum_{j=0}^{\infty} a_j\, p_j(z)$$

where

$$a_j = \int_C f(z)\, p_j(z)\, n(z)\, dz \Big/ \int_C [p_j(z)]^2\, n(z)\, dz .$$

the latter two (analogies of the first) dealing with harmonic polynomials and rational functions respectively.

Theorem A_p:* "Let C be an arbitrary closed limited point set (not a single point) of the z-plane, whose complementary set with respect to the entire plane is simply connected. Let $w = \varphi(z)$ be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve $|\varphi(z)| = R > 1$, that is, the transform in the z-plane of the circle $|w| = R$.

If the function $f(z)$ is analytic on and within C_R, then there exist polynomials $p_n(z)$ of respective degrees n, ($n = 0,1,2,\ldots$), such that the inequalities

$$(1) \qquad |f(z) - p_n(z)| \le \frac{M}{R^n}, \qquad R > 1,$$

where M is independent of n and of z, are valid for every z in C.

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*Walsh, On the overconvergence of sequences of polynomials of best approximation, Transactions of the American Mathematical Society, Vol. 32, No. 4, pp. 794-795.

If the polynomials $p_n(z)$ are given so that (1) is valid for every z in C, then the sequence $\{p_n(z)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, and thus $f(z)$ is analytic throughout the interior of C_R."

Several proofs of this theorem may be found in the literature: Walsh* credits S. Bernstein, M. Riesz, Faber, Fejer, and Szegö with various parts of the theorem. Essentially, the proofs involve a generous use of Cauchy's integral formula. Since we are here interested in a comparison of general methods, we shall accept Theorem A_p and turn to its analogue for harmonic polynomials.

Theorem A_{hp}:# "Let C be an arbitrary closed Jordan region of the (x,y)-plane, and let $w = \varphi(z)$, $z = x + iy$, be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve $|\varphi(z)| = R$, $R > 1$, that is, the trans-

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*For detailed references see Walsh, Sitzungsberichte der Bayerischen Akadamie der Wissenschaften, 1926, pp. 223-229.

#Walsh, On the degree of approximation to a harmonic function, Bulletin of the American Mathematical Society, Sept.-Oct. 1927, pg. 591.

form onto the z-plane of the circle $|w| = R$.

A necessary and sufficient condition that an arbitrary function $u(x,y)$, defined in C, be harmonic in (the closed region) C is that there should exist harmonic polynomials $p_n(x,y)$ of degree n, ($n = 0, 1, \ldots$), and numbers M, $R > 1$, such that the inequalities

$$(2) \qquad |u(x,y) - p_n(x,y)| \leqq \frac{M}{R^n}$$

where M and R are constants, that is, independent of n and of x,y, and where $R > 1$, should be valid for every point (x,y) in C.

If the polynomials $p_n(x,y)$ are given so that (2) is satisfied for every (x,y) in C, the sequence $\{p_n(x,y)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, so the function $u(x,y)$ is harmonic throughout the interior of C_R.*

If $u(x,y)$ is given harmonic in the closed region interior to C_ρ, the polynomials $p_n(x,y)$ can be chosen to satisfy (2) with $R = \rho$ for (x,y) in C."

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*If $u(x,y)$ were not originally defined in the whole region being considered, we now define $u(x,y)$ in the new points by harmonic extension - that is, by the convergent series of harmonic polynomials.

The first part of Theorem A_{hp} is an immediate consequence of Theorem A_p if we employ the following device: u(x,y) being harmonic in the closed region interior to C, there exists there a function v(x,y) conjugate harmonic to u(x,y)* and hence a function

$$F(z) = u(x,y) + i\,v(x,y)$$

analytic in the same closed region. Now, by Theorem A_p, there exist polynomials $p_n(z)$ such that (1) holds for z in C and R = ρ . The real part of $F(z) - p_n(z)$, which is $u(x,y) - q_n(x,y)$ where q is a harmonic polynomial of degree n, cannot fail to satisfy (1), hence (2) is established. The proof of the rest of Theorem A_{hp} may parallel precisely the proof of Theorem A_p with the one important change in fundamental tool from

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*In fact, we may define v(x,y) by

$$v(x,y) = \int_{(0,0)}^{(x,y)} \left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right) + c,$$

where C is an arbitrary constant.

Cauchy's integral formula to Green's integral formula.*
Since Green's integral formula does require the existence of a derivative along the normal to the boundary of the region at every point, it is clear that we need here a definite restriction on the region C.# However, this restriction is not so severe as might be expected, as is evident from the following illustration.%

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*For comparison, we exhibit together Cauchy's integral formula (1) and Green's integral formula (2):

(1) $$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\,dt, \quad z \text{ on } C, \quad t \text{ on } \Gamma$$

(2) $$u(x,y) = \frac{1}{2\pi}\int_\Gamma u(\xi,\eta)\frac{\partial G}{\partial N}\,d\sigma, \quad (x,y) \text{ on } C, \quad (\xi,\eta) \text{ on } \Gamma,$$

where G(x,y) is the Green's function for the region C bounded by Γ . We note that the simple rational function, $1/(t-z)$, in Cauchy's integral corresponds to the normal derivative of Green's function, $\partial G/\partial N$, in Green's integral. In this latter case, it is certain that we have a formidable complication: not only does a Green's function not exist for some regions, but whenever it does exist, it is not an elementary function even for the simplest region. To require further that the normal derivative of the Green's function exists at every point of the boundary of C certainly does not ease the situation!

#Such as, for example, that the boundary of C have a continuously turning tangent. However, this condition could be lightened if conditions on u(x,y) were correspondingly strengthened. Cf. Kellogg, Transactions of the American Mathematical Society, Vol. 9, 1908, pp. 39,51.

%Cf. Walsh, Journal fur Mathematik, Vol. 159, 1928, pg. 203.

Let C be the <u>most</u> <u>general</u> closed region such that an arbitrary function, <u>analytic</u> interior to C and continuous in the corresponding closed region, can be approximated uniformly in the closed region as closely as desired by <u>polynomials in the complex variable</u>.* It has been shown that C is <u>not</u> the most general region such that an arbitrary function <u>harmonic</u> interior to C and continuous in the corresponding closed region can be approximated as closely as desired in the closed region by <u>harmonic polynomials</u>.# In brief, uniform approximation

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*It is not yet known what this most general region C is.

#In this connection it is interesting to note that Walsh has shown by the following example that it is <u>not</u> true that an arbitrary function continuous in a limited closed region and harmonic interior to the region can be uniformly approximated in that region as closely as desired by a harmonic rational function. "Consider the region C formed from the circle $x^2 + y^2 < 1$ by cutting out the line segment $y = 0$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$; let $f(x,y)$ be the function harmonic interior to this region, continuous in the closed region, zero on the circumference, and unity on the line segment. The function $f(x,y)$ cannot be uniformly approximated in the closed region. For the approximating functions have only isolated singularities, are continuous without exception in the closed region (else are not uniformly bounded), hence are harmonic in the closed region. These approximating functions can be uniformly approximated as closely as desired in the closed region C by harmonic polynomials in (x,y), so $f(x,y)$ can also be uniformly approximated in C as closely as desired by a harmonic polynomial in (x,y). But if such a polynomial differs from $f(x,y)$ by less than ϵ for points on $x^2 + y^2 = 1$, that polynomial differs from zero by less than ϵ for all points $x^2 + y^2 < 1$, which is a contradiction for $x = y = 0$ if $\epsilon < \frac{1}{2}$. Cf. Walsh, Bulletin of the American Mathematical Society, July-August, 1929, pp.520,521.

to harmonic functions by harmonic polynomials is "less restricted" than uniform approximation to analytic functions by polynomials in the complex variable.

We consider now the extension by analogy of Theorem A_p, concerning approximation by sequences of polynomials, to Theorem A_r, concerning approximation by sequences of rational functions. We may dismiss the proof of Theorem A_r with the remark that, since a polynomial is a rational function all of whose poles lie at infinity, the only complication which might arise would be connected with the location of the poles of the approximating rational functions. In general, we assume the poles all lie in a region having no point in common with the region S in which the approximation occurs;* then, in S, Cauchy's integral may be employed and the proof follows that of Theorem A_p. Thus we may establish the

<u>Theorem A_r</u>:# "<u>Suppose $f(z)$ is an analytic function whose singularities form a set E one of whose derivatives $E^{(k)}$ is empty. Suppose C is a closed point set with no point in common with E. Then there exists a sequence of rational functions $r_n(z)$ of respective degrees n whose</u>

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*An assumption which effectively clears the problem.

#Walsh, Acta Mathematica, Vol. 57, pp. 415-419.

poles lie on E such that for an arbitrary R we have

$$(3) \qquad |f(z) - r_n(z)| \leq \frac{M}{R^n}, \qquad z \text{ on } C,$$

where M depends on R but not on n."

If the sequence $r_n(z)$ converges so that (3) is valid for all z on C, where M depends on R but not on n, then the sequence $r_n(z)$ converges and f(z) is analytic at every point of the extended plane except the limit points of the poles of the functions $r_n(z)$ and except points separated from C by such limit points. Convergence is uniform in any closed region C' containing no such limit point, and for z on C' an inequality of form (3) holds for an arbitrary R provided that M (depending on R) is suitably chosen.

Even though theorems A_p, A_{hp}, and A_r have appeared separately and at different times in the literature, the correspondences among them are not at all subtle: each theorem is a recast of the polynomial theorem into the vocabulary and concepts of the new approximation field. The slight changes in the essential hypotheses appear quite incidental in these cases. In more general results,*

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*That is, in theorems based on weaker hypotheses.

however, the changes in hypotheses necessary for adapting polynomial approximation theorems to other fields of approximation are much more pronounced — it is in such cases, in fact, that the analogy fails to exist - or if it does, to be amenable to the same proof.

In this same connection,* it is most gratifying to note that in approximating a function f(z), analytic in a closed, limited region C whose complement f(z) has just one singularity, sequences of polynomials behave quite differently from sequences of rational functions. Naturally, the singularity is a definite bound to the extent of the region of convergence of the approximating polynomials. On the other hand, if we pick the poles of the approximating rational functions to lie on the singularity of f(z),# then the sequence of approximating rational functions converges on the entire plane with the exception of the singular point.

Each of the theorems A_p, A_{hp}, and A_r contains an existence theorem - each asserts that under certain rather general conditions there does exist a sequence of approximating functions - these conditions are, however,

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*Cf. Walsh, Acta Mathematica, Vol. 57, pg. 414.

#As we may.

not the most general of their kind, as we shall see later. The existence of a sequence of approximating functions depends, in general, on two types of conditions: the properties of the function* being approximated and the properties of the point set# on which the approximation is to occur.

There are many different existence theorems for each of the fields of approximation - the theorems differ essentially because of the slight differences in their respective hypotheses. It is not desirable here to exhibit a list of such theorems. A single example, concerning rational functions, is typical of them all.

Theorem: *If a function $f(z)$ is analytic in k mutually exclusive regions $S_1, S_2, \ldots, S_k$, of the extended plane each bounded by a finite number of non-intersecting contours, then there exists a sequence of rational functions whose poles lie on the boundaries of the S_j, which converges to $f(z)$ interior to the S_j, uniformly on any closed set interior to the S_j.*%

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*Such as analyticity, harmonicity, continuity, existence and continuity of derivatives of the first and higher orders, location of zeros and poles, etc.

#Such as boundedness, closure, connectedness, compactness, etc; types of boundary if a region: Jordan curve, analytic Jordan curve, etc; characteristics of point set if a curve: Jordan arc, analytic curves, etc.

%Walsh, Interpolation and Approximation, American Mathematical Society Colloquium Publications, Vol. XX, 1935, pg. 12.

The problem of determining the most general conditions under which there exists a sequence of approximating functions of one of these types has not yet been solved -- indeed, it is not even known that such a solution exists.

We indicate now several of the various measures of the degree of approximation of a sequence of functions -- the various measures of the discrepancy between any one of the approximating functions and the function being approximated; for example, the inequality which appears in Theorems A_p, A_{hp}, and A_r is a type of measure of approximation for the functions being considered -- at no point in the region does the value of the approximating function differ from that of the approximated function by more than M/R^n. We may, in brief, classify the types of "best approximation" by measures of the degree of approximation 1) in the sense of Tchebycheff, 2) by line integrals over the boundary of the region on which the approximation occurs, 3) by surface integrals over the region being considered, and 4) by line integrals taken over the unit circle in a new plane, after conformal mapping of the original region onto the interior of this circle. We define these concepts in turn.

1) The Tchebycheff approximating function $\pi_n(z)$ of degree n is such that

$$\max |f(z) - \pi_n(z)|, \quad z \text{ on } C,$$

is not greater than

$$\max |f(z) - p_n(z)|, \quad z \text{ on } C,$$

for any other approximating function $p_n(z)$ of degree n of the type being considered.* It is known that the Tchebycheff polynomial exists and is unique. <u>The Tchebycheff rational function likewise exists but is not unique.</u>#

2) We may exhibit together the functions of best approximation in the sense of measure by line and surface integrals. A sequence $\pi_n(z)$ shall be said to be a sequence of best approximation in the sense of least p^{th} powers (as measured by a line integral) if

$$\int_\Gamma |f(z) - \pi_n(z)|^p n(s)\, ds \tag{4}$$

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*A generalization of this concept by the inclusion of a positive continuous norm function n(z) concerns

$$\max |n(z)||f(z) - \pi_n(z)|, \quad z \text{ on } C.$$

#Illuminating examples of this lack of uniqueness may be found in Walsh, The existence of rational functions of best approximation, Transactions of the American Mathematical Society, Vol. 33, No. 3, pp. 676-679; and in Walsh, Interpolation and Approximation, American Mathematical Society Colloquium Publications, Vol. XX, 1935, pg. 356.

is not greater than the corresponding expression

$$\int_{\Gamma} | f(z) - q_n(z) |^p n(s)\, ds$$

formed for any other approximating function $q_n(z)$ of degree n where p is an integer greater than unity, and n(s) is an arbitrary function positive and continuous on Γ, the boundary of the region of approximation.

3) Similarly the sequence $\pi_n(z)$ of approximating functions is said to be a sequence of best approximation to f(z) in the sense of least p^{th} powers (as measured by a surface integral) if

$$(5) \qquad \iint_C | f(z) - \pi_n(z)| \; n(z)\, dS$$

is not greater than the corresponding expression

$$\iint_C | f(z) - q_n(z) | \; n(z)\, dS$$

formed for any other approximating function $q_n(z)$ of degree n of the type being considered, where C is the region of approximation; n(z) is any positive continuous function, and S is the area of the region C.

In each of these cases it has been shown that the polynomial of best approximation exists and is unique; <u>the rational functions of best approximation exist</u>*

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*That is, provided there is at least one rational function for which the defining integral (4) or (5) exists.

but are not unique.

It is to be noted that measurement of degree of approximation by a line integral taken over the boundary Γ of the region, implies that the boundary Γ is a rectifiable curve. This is a restriction which we may circumvent by mapping the interior of Γ onto the interior of the unit circle in the w-plane and then measuring the degree of approximation by a line integral over the circumference of the unit circle. Here again, the polynomial of best approximation exists and is unique; *here, again, the rational function of best approximation exists* but is not unique*.

It sometimes happens that not only does a sequence of approximating functions converge to a given function with a certain degree of approximation in a given region, but that the sequence converges to the function in a larger region containing the given region in its interior. We have already had several examples of this phenomenon: Theorems A_p, A_{hp}, and A_r contain specific conditions under

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*As before, the rational function of best approximation exists only if there is at least one rational function for which the defining integral exists. There is a further necessary condition here — that the poles of the rational functions lie in a closed point set which contains no point of the boundary corresponding to more than a finite number of points of $|w| = 1$.

which overconvergence always occurs; we have seen that a sequence of rational functions, for example, may converge uniformly throughout the entire plane except for one point. On the other hand, it is evident that a sequence of polynomials can approximate uniformly a continuous function, not a constant, on no unlimited point set, for on any such set the polynomial becomes infinite.

There are many scattered theorems relating to overconvergence of sequences of polynomials, harmonic polynomials, and rational functions. Here again, we shall exhibit just one example which is typical of the others.

Theorem: "Let $V_n(z)$ be a sequence of rational functions of respective degrees n such that we have for z in an arbitrary closed Jordan region C

$$|f(z) - V_n(z)| \leq \frac{M}{\rho^n}, \quad \rho > 1. \tag{6}$$

If the poles of $V_n(z)$ lie in the closed set D, where D has no point in common with C, then the sequence $V_n(z)$ converges uniformly for z in C_R, where $R < \rho^{1/2}$.

If (6) is valid for z in C and if the function $V_{n+1}(z) - V_n(z)$ is of degree $n + 1$, with its poles in D, then the sequence $V_n(z)$ converges uniformly for z in C_R, where $R < \rho$.

If (6) is valid for z in C and if there exists a sequence $V'_n(z)$ of functions such that $V'_n(z) - V_n(z)$ is a rational function of degree n with its poles in D, if we have

$$(7) \qquad |f(z) - V'_n(z)| \leq \frac{M}{\rho^n}, \quad z \text{ in } C,$$

and if the sequence $V'_n(z)$ converges uniformly for z in C_R, where $R < \rho$ (R may or may not depend on the sequence $V'_n(z)$), then the sequence $V_n(z)$ also converges uniformly for z in C_R."*

We include this particular theorem on approximation for two reasons. First, the theorem indicates one of the directions in which Theorem A_r (and A_p and A_{hp}) may be extended: to obtain successively more and more general results on overconvergence, we postulate more and more stringent conditions on the approximated and approximating functions.# Second, the theorem is basic in the corresponding discussion of the possibility of overconvergence of sequences of harmonic rational functions, which we shall consider in greater detail in the next chapter.

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*Walsh, American Journal of Mathematics, Vol. LIV, No. 3, 1932, pg. 563.

#Cf. pg. 36

We have not included theorems concerning approximation by sequences of harmonic rational functions for the reason that existing results are rather unsatisfactory. Though they are, in the main, analogous to those concerning approximation by sequences of rational functions and approximation by sequences of harmonic polynomials, they are in no sense complete, nor are they as yet unified.

For the sake of completeness, however, we include a theorem, proved by Osgood, which furnishes a very powerful tool in the study of approximation by harmonic rational functions, especially in its relation to the corresponding study for rational functions.

Theorem:* Given $u(x,y)$ a harmonic rational function, define

$$v(x,y) = \int_{(0,0)}^{(x,y)} \left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \right) + C;$$

then

$$f(z) = u(x,y) + i\,v(x,y)$$

is a rational function of z.

Using this result, we have for the theory of approximation by harmonic rational functions a tool completely analogous to that used in the study of approximation by harmonic polynomials (compare pg. 30 and footnote).

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*Osgood, Funktionentheorie, Vol. I, Leipzig, 1912, pg. 680.

Chapter III

On the Overconvergence of Sequences of Harmonic Rational Functions

Walsh* has considered the general problem of the overconvergence of sequences of rational functions and has suggested the application of similar techniques to the consideration of the overconvergence of sequences of harmonic rational functions. In this chapter we shall carry out the proposed investigation in detail.

We consider a sequence of harmonic rational functions $h_n(x,y)$ which converges, with a certain degree of approximation, to a given harmonic function $H(x,y)$ on a given set C. We show that the sequence may necessarily converge to $H(x,y)$, or to its analytic extension, on a larger point set containing C in its interior.

Let C be an arbitrary closed Jordan region of the extended# (x,y)-plane and let D be an arbitrary closed point set having no point in common with C. Let $w = \varphi(z)$, $z = x + iy$, be a function which maps conformally the complement of C onto the exterior of the unit circle, $|w| = 1$, in the w-plane so that the point (x_0,y_0) of D

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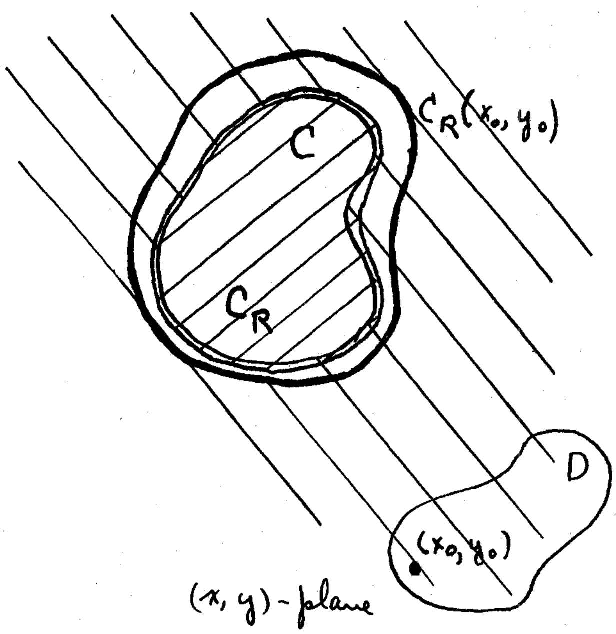
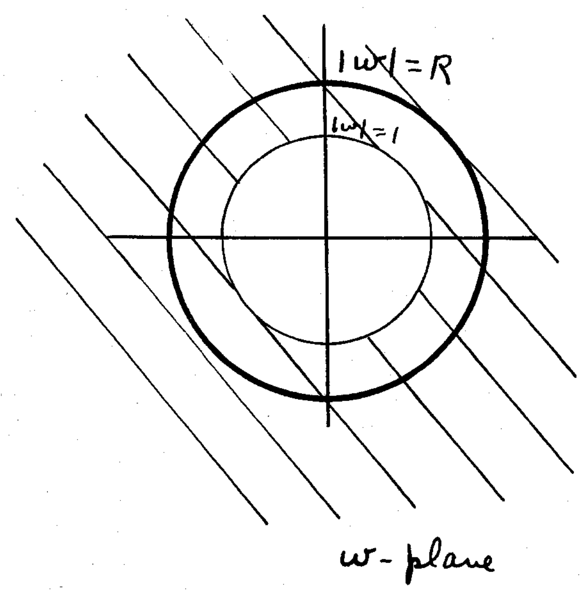
*American Journal of Mathematics, Vol. LIV, No. 3, July, 1932, pp. 559-570.

#By the extended plane we shall mean the whole finite plane closed by the addition of the point at infinity.

corresponds to the point $w = \infty$.* Let $C_R(x_o,y_o)$ denote the curve in the (x,y)-plane whose image in the w-plane is $|w| = R > 1$. Since the boundary of C corresponds to the unit circle, $|w| = 1$, it is clear that the closed regions bounded by the various curves $C_R(x_o,y_o)$ have the region C in common when the point (x_o,y_o) varies over D, and may have an even larger point set in common. We denote this common point set by C_R.





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*In as much as a Jordan region is simply-connected and we have established the correspondence of a pair of points, one in each plane, the function $w = \varphi(z)$ has a single valued inverse $z = \psi(w)$ which maps conformally the exterior of the unit circle, $|w| = 1$, in the w-plane onto the complement of C in the (x,y)-plane so that the point $w = \infty$ corresponds to (x_o,y_o). (C. Caratheodory "Conformal Representation", Cambridge Tracts in Mathematics and Mathematical Physics, No. 28, pg. 70.)

In establishing the principal theorem, we shall have need of two lemmas, which we now state and prove.

Lemma 1. If $h_n(x,y)$ is a harmonic rational function of degree n* whose poles lie in D and if we have

(1) $$|h_n(x,y)| \leq M, \text{ for } (x,y) \text{ on } C,$$

then we have also

(2) $$|h_n(x,y)| \leq MR^n, \text{ for } (x,y) \text{ on } C_R.$$

Let the points (x_k,y_k), $(k = 1,2, \ldots, n)$, be the roots of $\sum_{r+s \leq n} b_{rs} x^r y^s = 0$; these points are the poles of $h_n(x,y)$, lie in D, and need not be distinct. Further, since some of these points (x_k,y_k) may also be roots of $\sum_{l+m \leq n} a_{lm} x^l y^m = 0$ and hence are only apparent poles of $h_n(x,y)$, $h_n(x,y)$ need not have n poles; if there are fewer, the reasoning is simplified.

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*We express $h_n(x,y)$ as

$$h_n(x,y) = \frac{\sum_{l+m \leq n} a_{lm} x^l y^m}{\sum_{r+s \leq n} b_{rs} x^r y^s}$$

Let $w = \varphi(z)$, $z = x + iy$, be a function which maps the complement of C onto the exterior of the unit circle, $|w| = 1$, so that the point $z_k = x_k + iy_k$ of D corresponds to $w = \infty$.

Except for this pole of the first order at (x_k, y_k), the function $\varphi_K(z)$ is analytic exterior to C and continuous in the corresponding closed region if it is suitably defined on C itself. Consider, now, the function

$$\frac{h_n(x,y)}{\varphi_1(z)\varphi_2(z)\cdots\varphi_n(z)}, \qquad z = x + iy,$$

which is analytic exterior to C, except possibly for branch points, and continuous in the corresponding closed region. Hence, the modulus of this function cannot be a maximum exterior to C, unless the modulus is everywhere constant.* Further, as the point (x,y) approaches C, the modulus approaches a limit not greater than M, for the modulus of each $\varphi_K(z)$ approaches unity and on C inequality

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*According to the Principle of Maximum, if a function f(z) is analytic in a region R of the extended plane, then the modulus $|f(z)|$ cannot have a proper maximum interior to R and can have an improper maximum interior to R only if f(z) is identically constant in R.

(1) holds. That is

$$(3) \qquad \left| \frac{h_n(x,y)}{\varphi_1(z)\,\varphi_2(z)\cdots\varphi_n(z)} \right| \leqq M \, ,$$

for (x,y) on the boundary of C. Now, applying the Principle of Maximum we have successively, $|\varphi_k(z)| = 1$ on C, inequality (3) is valid for (x,y) on C, and inequality (3) is valid for (x,y) exterior to C. Since the curve $C_R(x_k,y_k)$ corresponds to $|w| = |\varphi_k(z)| = R$, we have

$$|\varphi_k(z)| \leqq R$$

throughout C_R exterior to C. Hence from (3) we have

$$|h_n(x,y)| \leqq M R^n$$

for (x,y) on C_R exterior to C. But C_R contains C in its interior and $h_n(x,y)$ has no pole in C_R, so (2) is valid for (x,y) everywhere in C_R.

Using the same notation, we now state and prove a general mapping theorem.

Lemma II. If C is a Jordan region and if the point set D (having no point in common with C) is a Jordan region bounded by a curve which can be denoted by $C_\rho(\zeta)$,

$\zeta = x + iy$, (where ζ is some point of D), then the point set C_R is the Jordan region which contains C and is bounded by the curve $C_\nu(\zeta)$, where

$$\nu = \frac{(R\rho + 1)}{(R + \rho)} .$$

Let $w = \varphi(z)$, $z = x + iy$, be a function which maps the complement of C onto the exterior of $|w| = 1$ so that the point $z = \zeta$ corresponds to $w = \infty$. Then the function

$$w' = \frac{(1 - \overline{w_1}\, w)}{(w - w_1)} = \frac{[1 - \overline{\varphi(z_1)}\, \varphi(z)]}{[\varphi(z) - \varphi(z_1)]}$$

maps the exterior of C onto the exterior of $|w'| = 1$, so that the point $z_1 = x_1 + iy_1$ corresponds to $w' = \infty$.*

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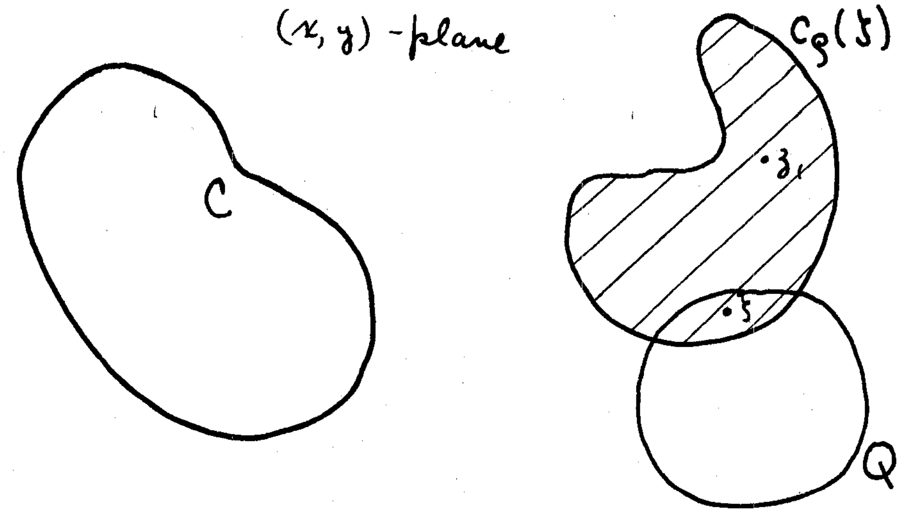
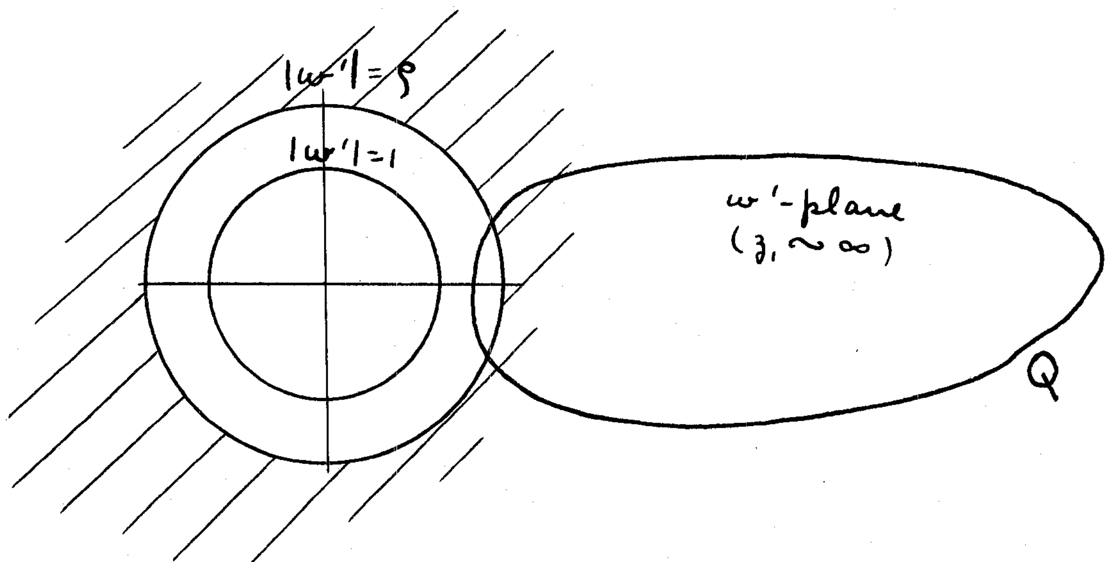
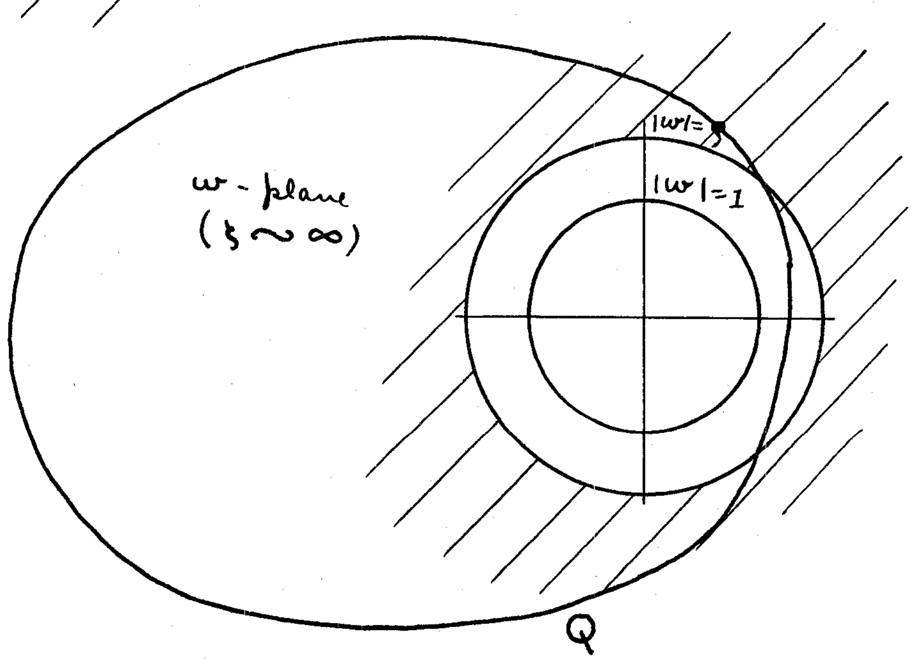
*Cf. illustration on next page. An auxiliary closed curve Q has been exhibited in the complement of C in the (x,y)-plane and its images have been exhibited after mapping. Note that there is an essential distinction between the exterior of $|w| = 1$ and the exterior of $|w'| = 1$ - unless, of course, the points z_1 and ζ coincide.


(x, y) - plane
C
$C_\rho(\zeta)$
z_1
ζ
Q


$|w'| = \rho$
$|w'| = 1$
w'- plane
$(z_1 \sim \infty)$
Q


w - plane
$(\zeta \sim \infty)$
$|w| =$
$|w| = 1$
Q

It is clear from the map that the conditions

$|w'| \geqq \rho$, $|\varphi(z_1)| \geqq \rho$, z_1 in the closed region not containing C bounded by $C_{\rho}(\gamma)$, z_1 in D, are all equivalent. It is further evident that all points z not in C_R can be expressed by the conditions

$$(4) \qquad \left| \frac{1 - \overline{\varphi(z_1)}\,\varphi(z)}{\varphi(z) - \varphi(z_1)} \right| \geqq R , \qquad |\varphi(z_1)| \geqq \rho ,$$

or by the equivalent conditions

$$(5) \qquad \left| \frac{1 - \overline{w_1}\,w}{w - w_1} \right| \geqq R , \qquad |w_1| \geqq \rho ,$$

where $w = \varphi(z)$, $w_1 = \varphi(z_1)$.

Conditions (5) considered as restrictions on w, are not difficult to transform. We have

$$w' = \frac{1 - \overline{w_1}\,w}{w - w_1} ,$$

hence

$$w'w - w'w_1 = 1 - \overline{w_1}\,w ,$$

and $$w(w' - \overline{w}_1) = 1 + w'w_1 \;,$$

so that $$w = \frac{1 + w'w_1}{w' + \overline{w}_1} \;.$$

We may now write (5) in the form

$$(6) \qquad w = \frac{w_1 w' + 1}{w' + \overline{w}_1} \;, \qquad \begin{array}{l} |w'| \geqq R > 1 \;, \\ |w_1| \geqq \rho > 1 \;. \end{array}$$

We note in passing that these really are the conditions that a point z lie exterior to C_R, for the point z_1 does correspond to a point lying <u>outside</u> or on $|w| = \rho$ since z_1 lies inside or on $C_\rho(\zeta)$ which is the map of $|w| = \rho$, hence $|w_1| = |\varphi(z_1)| \geqq \rho$. Further, the region $|w'| > R > 1$ is the map of the complement of C_R and hence does contain all points corresponding to those which are exterior to C_R.

We return now to (6); w' and w_1 are arbitrary, subject to the conditions indicated, and we seek the locus of w. The transformation in (6) maps the unit circle $|w'| = 1$ and its exterior onto $|w| = 1$ and its exterior, so we obviously have $|w| > 1$, since we are concerned only with $|w'| > 1$. Let us set $|w'| = R_1 \geq R$, $|w_1| = \rho_1 \geq \rho$. By (6) we have

$$\bar{w} = \frac{(\bar{w}_1 \bar{w}' + 1)}{(\bar{w}' + w_1)},$$

hence

$$w\bar{w} = \left(\frac{w_1 w' + 1}{w' + \bar{w}_1}\right)\left(\frac{\bar{w}_1 \bar{w}' + 1}{\bar{w}' + w_1}\right)$$

$$= \frac{w_1 \bar{w}_1 \bar{w}' w' + w_1 w' + \bar{w}_1 \bar{w}' + 1}{w' \bar{w}' + \bar{w}_1 \bar{w}' + w' w_1 + w_1 \bar{w}_1}$$

$$= \frac{R_1^2 \rho_1^2 + w_1 w' + \bar{w}_1 \bar{w}' + 1}{R_1^2 + w_1 w' + \bar{w}_1 \bar{w}' + \rho_1^2}.$$

But $\omega_1 \omega' + \bar{\omega}_1 \bar{\omega}'$ is real, for if

$$\omega_1 = u_1 + i v_1 \quad , \quad \bar{\omega}_1 = u_1 - i v_1 \; ,$$

and

$$\omega' = u' + i v' \quad , \quad \bar{\omega}' = u' - i v' \; ,$$

then

$$\omega_1 \omega' + \bar{\omega}_1 \bar{\omega}' = (u_1 + i v_1)(u' + i v') + (u_1 - i v_1)(u' - i v')$$

$$= 2 \, (u_1 u' - v_1 v') .$$

Hence we may write

$$\omega \bar{\omega} = \frac{R_1^2 \rho_1^2 + 2x + 1}{R_1^2 + 2x + \rho_1^2} .$$

Let us consider the function

(7) $$y = \frac{R_1^2 \rho_1^2 + 2x + 1}{R_1^2 + 2x + \rho_1^2} .$$

The numerator is greater than the denominator, for since $\rho_1^2 > 1$ and $R_1^2 - 1 > 0$, we have

$$\rho_1^2(R_1^2 - 1) > R_1^2 - 1 ,$$

which yields

$$R_1^2 \rho_1^2 + 1 > R_1^2 + \rho_1^2 ,$$

so that

$$R_1^2 \rho_1^2 + 2x + 1 > R_1^2 + 2x + \rho_1^2 .$$

For $-\rho_1 R_1 \leqq x \leqq \rho_1 R_1$, the denominator of (7) is positive, and vanishes only for $x = -\frac{1}{2}(R_1^2 + \rho_1^2)$ which does not exceed $-R\rho$. Since $(R_1 - \rho_1)^2 \geqq 0$, we have

$$R_1^2 - 2R_1\rho_1 + \rho_1^2 \geqq 0 ,$$

hence

$$R_1^2 + \rho_1^2 \geqq 2R_1\rho_1 \geqq 2R\rho ,$$

which implies the results just stated. We now show that the curve represented by (7) is an equilateral hyperbola. We have

$$y(R_1^2+2x+\rho_1^2) = R_1^2\rho_1^2+2x+1 \ ,$$

$$2xy-2x+(R_1^2+\rho_1^2)y = R_1^2\rho_1^2+1 \ ,$$

$$2\left[xy-x+\left(\frac{R_1^2+\rho_1^2}{2}\right)y\right] = R_1^2\rho_1^2+1 \ ,$$

$$2\left[x(y-1)+\left(\frac{R_1^2+\rho_1^2}{2}\right)(y-1)\right] = R_1^2\rho_1^2+1-\rho_1^2-R_1^2 \ ,$$

$$2\left(x+\frac{R_1^2+\rho_1^2}{2}\right)(y-1) = R_1^2\rho_1^2+1-\rho_1^2-R_1^2 \ ,$$

If we set

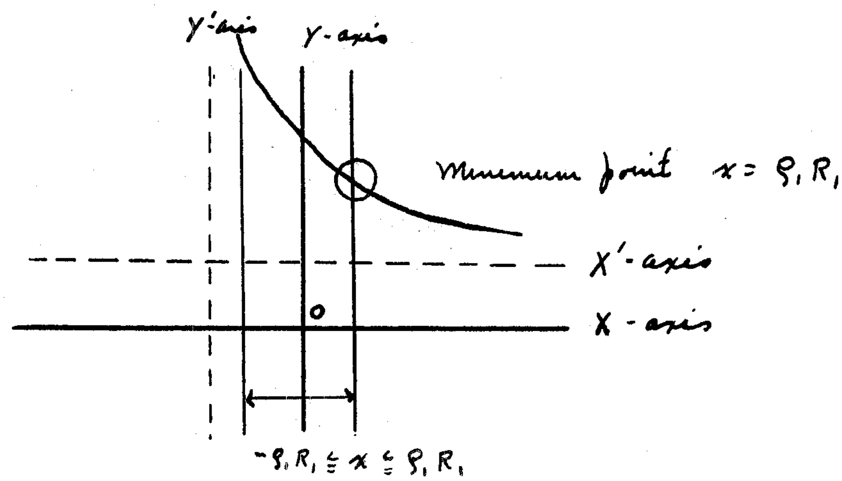
$$x' = x+\frac{R_1^2+\rho_1^2}{2} \ , \quad y' = y-1 \ ,$$

we have the normal form of the equilateral hyperbola

$$2x'y' = K \ .$$

Hence (7) represents the equilateral hyperbola whose asymptotes are the lines:

$$x + \frac{R_1^2 + \rho_1^2}{2} = 0 \quad , \qquad y - 1 = 0 .$$



For $x = R_1 \rho_1$ the function (7) has the value

$$(R_1 \rho_1 + 1)^2 / (R_1 + \rho_1)^2$$

which is the minimum in the strip under consideration. Since $R_1 \geqq R$ and $\rho_1 \geqq \rho$, we have

$$\frac{R_1 \rho_1 + 1}{R_1 + \rho_1} \geqq \frac{R \rho + 1}{R + \rho} .$$

Returning, now, to the expression just preceding (7), we find

$$|w| \geq \frac{R\rho + 1}{R + \rho} ;$$

hence we have shown that the complement of C_R lies in the Jordan region not containing C bounded by the curve $C_\nu(\zeta)$, $\nu = (R\rho + 1)/(R + \rho)$ and the Lemma is established. Reciprocally, let z be given in the Jordan region not containing C bounded by this curve $C_\nu(\zeta)$; we shall show that z belongs to the complement of C_R. When this situation is transformed to the w-plane it reduces to the following: given $|w| \geq (R\rho + 1)(R + \rho)$, find w and w' which satisfy

$$w = \frac{w_1 w' + 1}{w' + \overline{w_1}} \tag{6}$$

If $|w| \geq R$, (6) is evidently satisfied by $w' = w$, $w_1 = \infty$. In the remaining case $R > |w| \geq (R\rho + 1)(R + \rho)$, we define the constant ρ_1 , by the equation

$$|w| = \frac{(R\rho_1 + 1)}{(R + \rho)} .$$

Since $\dfrac{R\rho_1+1}{R+\rho_1} \geqq \dfrac{R\rho+1}{R+\rho}$, we have in turn

$$\frac{R\rho_1+1}{R+\rho_1} - \frac{R\rho+1}{R+\rho} \geqq 0,$$

$$\frac{(R\rho_1+1)(R+\rho) - (R\rho+1)(R+\rho_1)}{(R+\rho_1)(R+\rho)} \geqq 0,$$

$$\frac{(R^2-1)(\rho_1-\rho)}{(R+\rho_1)(R+\rho)} \geqq 0.$$

Now, since $R^2-1>0$ and $(R+\rho_1)(R+\rho) > 0$, we have $\rho_1 \geqq \rho$. Further, (6) is satisfied if we set

$$w_1 = \rho_1 w \frac{(R+\rho_1)}{(R\rho_1+1)},$$

and

$$w' = \frac{R(R\rho_1+1)}{w(R+\rho_1)},$$

for on substituting these values in the right member of (6), we have

$$\frac{\rho_1 \omega \dfrac{(R+\rho_1)}{(R\rho_1+1)} \dfrac{R(R\rho_1+1)}{\omega(R+\rho_1)} + 1}{\dfrac{R(R\rho_1+1)}{\omega(R+\rho)} + \dfrac{\rho_1\overline{\omega}(R+\rho_1)}{(R\rho_1+1)}},$$

which yields

$$\frac{\omega(R\rho_1+1)^2(R+\rho_1)}{R(R\rho_1+1)^2+\rho_1(R+\rho_1)^2|\omega|^2}.$$

We now use the definition of ρ_1, we have

$$\frac{\omega(R\rho_1+1)^2(R+\rho_1)}{R(R\rho_1+1)^2+\rho_1(R+\rho_1)^0(R\rho_1+1)^2} = \omega$$

Finally we have $|\omega_1| = \rho_1$ and $|\omega'| = R$,
and Lemma II is complete.

Using Lemma II, we prove

Theorem I. Given a harmonic function $H(x,y)$, let $h_n(x,y)$ be a sequence of harmonic rational functions of respective degrees n such that we have for (x,y) in an arbitrary closed Jordan region C

$$(8) \qquad |H(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}, \quad \rho > 1.$$

A. If the poles of the functions $h_n(x,y)$ lie in the closed set D, where D has no point in common with C, then the sequence $h_n(x,y)$ converges uniformly for (x,y) in C_R, where $R < \rho^{1/2}$.

B. If (8) is valid for (x,y) in C and if the function $h_{n+1}(x,y) - h_n(x,y)$ is of degree $n+1$ with its poles in D, then the sequence $h_n(x,y)$ converges uniformly for (x,y) in C_R, where $R < \rho$.

C. If (8) is valid for (x,y) in C and if there exists a sequence $h'_n(x,y)$ of harmonic rational functions such that $h'_n(x,y) - h_n(x,y)$ is a harmonic rational function of degree n with its poles in D, if we have

$$(9) \qquad |H(x,y) - h'_n(x,y)| \leq \frac{M}{\rho^n},$$

for (x,y) in C, and if the sequence $h'_n(x,y)$ converges uniformly for (x,y) in C_R, where $R < \rho$ (R may or may not depend on the sequence $h'_n(x,y)$), then the sequence $h_n(x,y)$ also converges uniformly for (x,y) in C_R.

We prove the three parts of Theorem I in order.

A. We have the inequalities, valid for (x,y) on C

$$|H(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}$$

and

$$|H(x,y) - h_{n+1}(x,y)| \leq \frac{M}{\rho^{n+1}}.$$

Hence

$$|H(x,y) - h_n(x,y)| + |H(x,y) - h_{n+1}(x,y)| \leq \frac{M}{\rho^n}\left(1 + \frac{1}{\rho}\right).$$

The left side of this inequality is not less than

$$|H(x,y) - h_n(x,y) + h_{n+1}(x,y) - H(x,y)| = |h_{n+1}(x,y) - h_n(x,y)|$$

So we have also

$$|h_{n+1}(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}\left(1 + \frac{1}{\rho}\right),$$

for (x,y) on C.

The harmonic function $h_{n+1}(x,y) - h_n(x,y)$, when expressed as the quotient of two polynomials is of degree $2n + 1$, so Lemma I yields, for (x,y) on C_R,

$$|h_{n+1}(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}\left(1+\frac{1}{\rho}\right)R^{2n+1} \leq MR\left(1+\frac{1}{\rho}\right)\left(\frac{R^2}{\rho}\right)^n$$

If, now, $R < \rho^{1/2}$, we have $R^2 < \rho$ and $\left(R^2/\rho\right)^n < K < 1$ for $n > l$, hence the sequence $h_n(x,y)$ does converge uniformly for (x,y) on C_R, and part A of the theorem is established.

B. It follows from the proof just given that if $h_{n+1}(x,y) - h_n(x,y)$ is of degree $n + 1$, with its poles in D,* we have for (x,y) on C_R,

$$|h_{n+1}(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}\left(1+\frac{1}{\rho}\right)R^{n+1}.$$

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*Consider, for example, the sequence of harmonic rational functions in which each approximation differs from the preceding by the addition of a single harmonic rational function of degree corresponding to the index of approximation:

$$\begin{aligned}
h_0(x,y) &= a_0, \\
h_1(x,y) &= a_0 + a_1\frac{p_1}{q_1}, \\
&\vdots \\
h_n(x,y) &= a_0 + a_1\frac{p_1}{q_1} + \cdots + a_n\frac{p_n}{q_n}, \\
h_{n+1}(x,y) &= a_0 + a_1\frac{p_1}{q_1} + \cdots + a_n\frac{p_n}{q_n} + a_{n+1}\frac{p_{n+1}}{q_{n+1}}.
\end{aligned}$$

It is clear that the difference

$$h_{n+1}(x,y) - h_n(x,y) = a_{n+1}\frac{p_{n+1}}{q_{n+1}}$$

is a harmonic rational function of degree $n + 1$.

The right member is not greater than

$$M R\left(1+\frac{1}{\rho}\right)\left(\frac{R}{\rho}\right)^{n} < M R\left(1+\frac{1}{\rho}\right) K ,$$

for $R < \rho$ and $n > \ell$; hence $h_n(x,y)$ does converge uniformly under the prescribed conditions and part B of the theorem is established.

C. If both (8) and (9) are valid, we obtain similarly for (x,y) on C,

$$|h_n(x,y) - h_n'(x,y)| \leqq \frac{2M}{\rho^n}$$

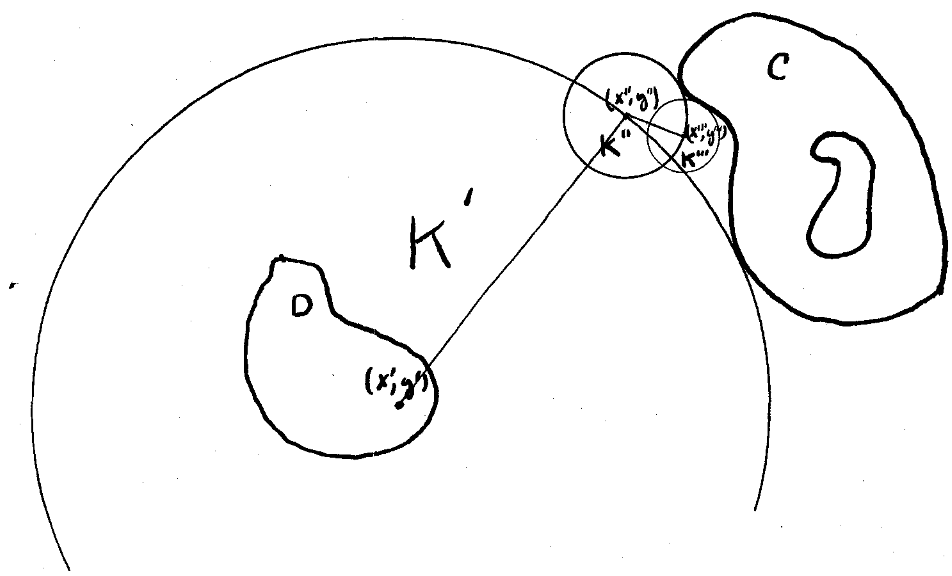
and Lemma I yields for (x,y) on C_R

$$|h_n(x,y) - h_n'(x,y)| \leqq \frac{2M}{\rho^n} R^n = 2M\left(\frac{R}{\rho}\right)^n$$

Thus if $R < \rho$, then $\left(\frac{R}{\rho}\right)^n < K$, for all $n > \ell$, and the sequence $h_n(x,y) - h'_n(x,y)$ converges uniformly for (x,y) on C_R. By hypothesis, the sequence $h'_n(x,y)$ converges uniformly for (x,y) on C_R, hence $h_n(x,y)$ also converges uniformly for (x,y) on C_R, and Theorem I is complete.

We turn now to the more complicated situation in which the complement of C is not simply connected.

Again, let C and D be arbitrary closed point sets with no common point. We now make the further assumption that the Dirichlet problem (for arbitrary boundary values) has a solution for every region $S(x',y')$ defined to be the totality of points which can be joined to the point (x',y') of D by broken lines not meeting C. Such a totality of points forms an open region whose boundary consists entirely of points of C. For, since C and D are closed and have no common point, (x',y') cannot be a boundary point of C. Hence there exists about (x',y') an open circular region K' of which no inner point is a point of C. Every inner point of K' can be joined to (x',y') by a line segment not touching C and so must belong to $S(x',y')$. Consider now each boundary point (x'',y'') of K': either it is a boundary point of C or there exists about it an open circular region K'' each of whose inner points can be joined to (x'',y'') by a line segment not touching C, and hence can be joined to (x',y') by a broken line segment not touching C. Evidently each of these open regions K'' belongs to $S(x',y')$. Now consider each boundary point (x''',y''') of each region K'': either it is a boundary point of C or there exists about it a region K''', - - - . It is clear that the region $S(x',y')$ does result from the continuation of this process, that it is open, and all of its boundary points are points of C.



Let $w = \varphi(z)$, $z = x + iy$, be a function which maps conformally the region $S(x',y')$ onto the exterior of the unit circle, $|w| = 1$, in the w-plane so that the point (x',y') corresponds to $w = \infty$.* Let $C_R(x',y')$ denote the curve or curves in the (x,y)-plane whose image in the w-plane is $|w| = R > 1$.# Let $S_R(x',y')$ denote the complement in the (x,y)-plane of the set $|\varphi(z)| > R > 1$, so that $S_R(x',y')$ is closed, bounded by $C_R(x',y')$, and

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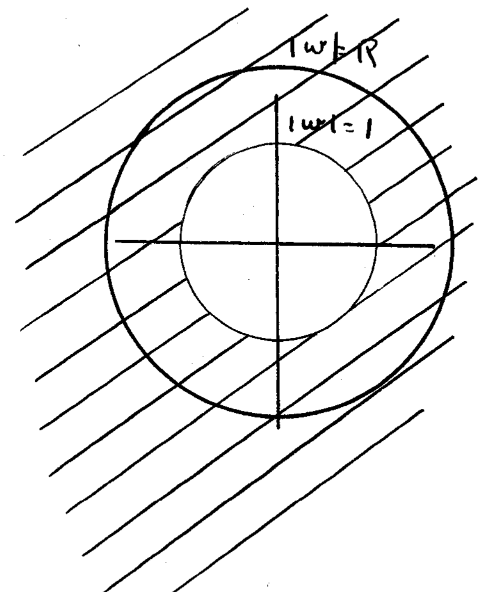
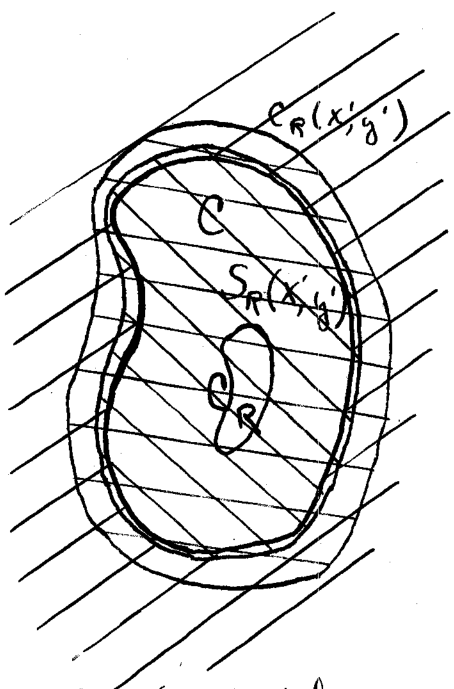
*The hypothesis on $S(x',y')$ is sufficient to insure the existence of such a mapping function even though it is not uniquely defined. However, the conformal mapping may not be smooth.

#The point set $C_R(x',y')$ is uniquely defined.

contains C. Let C_R denote the point set common to all the point sets $S_R(x',y')$ when the point (x',y') takes on all possible positions in D; the given set C evidently belongs to C_R.



w-plane



(x, y) -plane.

In establishing the generalized theorem we shall have need of a lemma more general than Lemma I. We now state and prove

Lemma III. If $h_n(x,y)$ is an harmonic rational function of the real variables (x,y) of degree n^* whose poles lie in D and if we have

$$(1) \qquad |h_n(x,y)| \leq M, \text{ for } (x,y) \text{ on } C,$$

then we have also

$$|h_n(x,y)| \leq MR^n, \text{ for } (x,y) \text{ on } C_R.$$

Let the points $(x_k,y_k),(k = 1,2,\ldots,n)$ be the poles of $h_n(x,y)$, all lying in D, but not necessarily all distinct. Let $(x_k,y_k),(k = 1,2,\ldots,\underline{m})$ be the poles of $h_n(x,y)$ which lie in $S(x_1,y_1)$. Let $w = \varphi_k(z)$, $z = x + iy$, $(k = 1,2,\ldots,m)$ be the functions which map the region $S(x_1,y_1)$ onto the exterior of the unit circle, $|w| = 1$, so that the respective points $(x_k,y_k),(k = 1,2,\ldots,m)$,

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*Cf. footnote pg. 46

of D correspond to $w = \infty$.* Consider, now, the function

$$\frac{h_n(x,y)}{\varphi_1(z)\,\varphi_2(z)\cdots\varphi_m(z)}, \quad z = x+iy,$$

which is analytic in $S(x_1,y_1)$, except possibly for branch points, and which has a single valued modulus in $S(x_1,y_1)$. Hence the modulus of this function cannot be a maximum in $S(x_1,y_1)$, unless the modulus is everywhere constant.# Further, as the point (x,y) approaches the boundary of $S(x_1,y_1)$ the modulus approaches a limit not greater than M, for the modulus of each $\varphi_k(z)$ approaches unity and on C, and hence on the boundary of $S(x_1,y_1)$, inequality (1) holds. That is,

$$\left|\frac{h_n(x,y)}{\varphi_1(x,y)\cdots\varphi_n(x,y)}\right| \leqq M \tag{10}$$

for (x,y) on the boundary of $S(x_1,y_1)$. To be sure, $\varphi_k(z)$ is not properly defined on the boundary of $S(x_1,y_1)$, but as (x,y) approaches that boundary, $\varphi_k(z)$ approaches unity and $h_n(x,y)$ is continuous: this is the sense in which we

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*The functions $w = \varphi_k(z)$ need not be single valued.

#Cf. footnote pg. 47

interpret the inequality (10). It follows that (10) is valid throughout the region $S(x_1,y_1)$, a region in which the function is analytic except possibly for branch points.* Since the curve $C_R(x_k,y_k)$ corresponds to $|w| = |\varphi_k(z)| = R$, we have

$$|\varphi_k(z)| \leqq R$$

throughout the region common to $S(x_1,y_1)$ and C_R. Hence from (10) we have

$$|h_m(x,y)| \leqq M R^m$$

for (x,y) in the region common to $S(x_1,y_1)$ and C_R.

Since $m \leqq n$, we have established inequality (2) for the region common to C_R and $S(x_1,y_1)$. In a similar manner we show that (2) is valid in the region common to C_R and <u>any</u> $S(x_k,y_k)$ containing poles of $h_n(x,y)$.#

We denote by Σ the set complementary to the totality of the regions $S(x_k,y_k)$ containing poles of $h_n(x,y)$. Since inequality (2) is valid on the boundary of each region $S(x_k,y_k)$, it is valid at each boundary point of Σ. Again, since the function $\varphi_k(z)$, $(k = 1,2,\ldots,n)$, is

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*Compare pg. 47

#In the simpler case in which the complement of C was simply connected, there was only one such region $S(x_k,y_k)$ - namely, the complement of C.

analytic throughout Σ , inequality (2) is valid at each point of Σ , and in particular is valid at each point common to C_R and Σ . But we have already shown that inequality (2) is valid at each point of C_R not in Σ (that is, at each point of C_R lying in a region $S(x_k,y_k)$ containing poles of $h_n(x,y)$), so the proof of Lemma III is complete.

Using Lemma III we can now establish a result analogous to Theorem I but much more general. In this case C and D are arbitrary closed point sets with no common point, such that the Dirichlet problem (for arbitrary continuous boundary values) has a solution for every region $S(x',y')$ defined as the totality of points which can be joined to a given point (x',y') of D by broken lines not meeting C.

Theorem II. Given a harmonic function $H(x,y)$, let $h_n(x,y)$ be a sequence of harmonic rational functions of respective degrees n such that we have for (x,y) on C

$$(8) \qquad \left| H(x,y) - h_n(x,y) \right| \leq \frac{M}{\rho^n}, \quad \rho > 1.$$

A. *If the poles of the functions $h_n(x,y)$ lie in D, then the sequence $h_n(x,y)$ converges uniformly for (x,y) on C_R, where $R < \rho^{1/2}$.*

B. *If (8) is valid for (x,y) in C and if the function $h_{n+1}(x,y) - h_n(x,y)$ is of degree $n + 1$, with its poles in D, then the sequence $h_n(x,y)$ converges uniformly for (x,y) in C_R, where $R < \rho$.*

C. *If (8) is valid for (x,y) on C and if there exists a sequence $h'_n(x,y)$ of harmonic rational functions such that $h'_n(x,y) - h_n(x,y)$ is a harmonic rational function of degree n with its poles in D, if we have*

$$(9) \qquad |H(x,y) - h'_n(x,y)| \leq \frac{M}{\rho^n}, \quad (x,y) \text{ in } C,$$

and if the sequence $h'_n(x,y)$ converges uniformly for (x,y) in C_R, where $R < \rho$, (R may or may not depend on the sequence $h'_n(x,y)$), then the sequence $h_n(x,y)$ also converges uniformly for (x,y) on C_R.

We prove the three parts of Theorem II in order.

A. We have the inequalities, valid for (x,y) on C

$$|H(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}$$

and

$$|H(x,y) - h_{n+1}(x,y)| \leq \frac{M}{\rho^{n+1}} .$$

Hence

$$|H(x,y) - h_n(x,y)| + |h_{n+1}(x,y) - H(x,y)| \leq \frac{M}{\rho^n}\left(1+\frac{1}{\rho}\right).$$

The left side of this inequality is not less than

$$|H(x,y) - h_n(x,y) + h_{n+1}(x,y) - H(x,y)| = |h_{n+1}(x,y) - h_n(x,y)|.$$

So we have also

$$|h_{n+1}(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}\left(1+\frac{1}{\rho}\right), \text{ for } (x,y) \text{ on } C.$$

The harmonic function $h_{n+1}(x,y) - h_n(x,y)$, when expressed as the quotient of two polynomials is of degree $2n + 1$, so Lemma III yields, for (x,y) on C_R,

$$|h_{n+1}(x,y) - h_n(x,y)| \leq \frac{M}{\rho^n}\left(1+\frac{1}{\rho}\right) R^{2n+1}$$

$$\leq MR\left(1+\frac{1}{\rho}\right)\left(\frac{R^2}{\rho}\right)^n .$$

If, now, $R < \rho^{1/2}$, we have $R^2 < \rho$ and $\left(\frac{R^2}{\rho}\right)^n < K < 1$ for $n > \ell$, hence the sequence $h_n(x,y)$ does converge uniformly for (x,y) on C_R, and part A of the theorem is established.

B. It follows from the proof just given that if $h_{n+1}(x,y) - h_n(x,y)$ is of degree $n + 1$, with its poles in D,* we have for (x,y) on C_R,

$$|h_{n+1}(x,y) - h_n(x,y)| \leqq \frac{M}{\rho^n}\left(1 + \frac{1}{\rho}\right) R^{n+1}$$

The right member is not greater than

$$MR\left(1+\frac{1}{\rho}\right)\left(\frac{R}{\rho}\right)^n < MR\left(1+\frac{1}{\rho}\right)K,$$

for $R < \rho$ and $n > \ell$; hence $h_n(x,y)$ does converge uniformly under the prescribed conditions and part B of the theorem is established.

C. If both (8) and (9) are valid, we obtain similarly for (x,y) on C,

$$|h_n(x,y) - h_n'(x,y)| \leqq \frac{2M}{\rho^n}$$

and Lemma III yields for (x,y) on C_R

$$|h_n(x,y) - h_n'(x,y)| \leqq \frac{2M}{\rho^n} R^n = 2M\left(\frac{R}{\rho}\right)^n$$

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*Cf. footnote pg. 63

Thus if $R < \rho$, then $\left(\frac{R}{\rho}\right)^n < K$, for all $n > \ell$, and the sequence $h_n(x,y) - h'_n(x,y)$ converges uniformly for (x,y) on C_R. By hypothesis, the sequence $h'_n(x,y)$ converges uniformly for (x,y) on C_R, hence $h_n(x,y)$ also converges uniformly for (x,y) on C_R, and Theorem II is complete.

We conclude our discussion of the overconvergence of harmonic rational functions with a demonstration that a theorem obtained by analogy may be much less potent than its source. Walsh, in discussing approximation by rational functions $r_n(z)$ has established lemmas of which our lemmas II and III are analogues. He says,* "If the point set C of Lemma III actually separates the plane, and if the various regions into which C separates the plane are known to contain respectively specific numbers of poles of $r_n(z)$ less than n, then it may occur that a sharper result than Lemma III can be established, with a corresponding application more general than Theorem III.#" He gives the following simple illustration of his remark.

Lemma IV. Let C be an arbitrary Jordan curve in whose interior the origin lies. If $r_{2n}(z)$ is a rational function of z of the form

$$r_{2n}(z) = a_{-n} z^{-n} + a_{-n+1} z^{-n+1} + \cdots + a_{-1} z^{-1} + a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n,$$

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*Walsh, American Journal of Mathematics, Vol. LIV, No. 3, July 1932, pg. 567.

#Cf. pg. 36

<u>and if we have for z on C</u>

$$|r_{2n}(z)| \leq M,$$

<u>then we have also</u>

$$|r_{2n}(z)| \leq MR^{n}$$

<u>for z on C_R, where D consists of the origin and the point at infinity</u>.

We shall not concern ourselves with the proof of this lemma; let us consider the corresponding theorem for harmonic rational functions. Certainly the analogue exists - we merely set $z = x + iy$ and $a_k = a_k' + i a_k''$ and separate $r_{2n}(z)$ into real and pure imaginary parts,

$$r_{2n}(z) = h_{2n}'(x,y) + i\, h_{2n}''(x,y)$$

, thus obtaining two sequences of harmonic rational functions for which the analogy of Lemma IV holds. But before we draw any further conclusions let us examine one of the harmonic rational functions of this sequence, say $h_4'(x,y)$; we have

$$h_4'(x,y) = \frac{a_{-2}'(x^2-y^2) - 2a_{-2}''xy}{(x^2+y^2)^2} + \frac{a_{-1}'x + a_{-2}''y}{x^2+y^2} + a_0 + a_1'x - a_1''y + a_2'(x^2-y^2) + 2a_2''xy.$$

It is surely evident that the complexity of the functions $h_{2n}(x,y)$ for large n is tremendous - and yet the source, $r_{2n}(z)$, was so simple! It might be thought that perhaps some other analogy exists - that here in the field of harmonic rational functions we might have a function, similar to $r_{2n}(z)$, involving negative powers of x and y.

Such is not the case: there is no finite sum of constant, non-zero, multiples of negative powers of x and y which is harmonic. For if there were such a finite sum, there would be a term containing the largest negative power of x, say x^{-k}; after differentiation twice with respect to x, this term would contain x^{-k-2} and no other term could contain this factor. Hence if the sum is to satisfy the Laplace Equation for all points of a point set, the coefficient of the term involving the highest negative power of x must vanish. Hence there is no greatest negative power of x in the sum, likewise there is no greatest power of y, the sum is infinite and so does not satisfy the conditions of the theorem.

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