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INVARIANTS OF A SYSTEM OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Introduction.

E. J. Wilczynski, in his treatise on "Projective Differential Geometry of Curves and Ruled Surfaces", (Leipzig, B. G. Teubner, 1906), Chapter IV, discusses the seminvariants and invariants of a system of linear homogeneous differential equations under a projective transformation.[1] He finds first the induced transformation of the coefficients of a system of n linear homogeneous differential equations (in n dependent variables) of the m-th order, the transformation affecting only the dependent variables, and then proceeds to calculate the seminvariants for the particular case of $m=2$, $n=2$. This calculation, as presented there, requires the solution of a complete system which is quite laborious even for the special case considered by Wilczynski. This method becomes more difficult if we try to calculate the seminvariants in the case of $m=2$, $n=3$; and the calculations become more and more involved as the number of variables becomes larger.

[1] We shall speak of the functions of the coefficients of the system of differential equations which remain unchanged under the transformation affecting only the dependent variables as <u>seminvariants</u>, while those remaining unchanged under the transformation of the dependent and independent variables as <u>invariants</u>.

It is our purpose to present an easier and more elegant solution of the problem of finding the seminvariants of a system of n linear homogeneous differential equations of the second order under a linear transformation. Consider the system

$$y_i'' + \sum_{j=1}^{n} L_{ij}(x)\, y_j' + \sum_{j=1}^{n} M_{ij}(x)\, y_j = 0, \qquad (i=1,2,\dots,n),$$

where

$$y_i' = \frac{dy_i}{dx}, \quad y_i'' = \frac{d^2 y_i}{dx^2},$$

under the transformation

$$y_i = \eta_i + \sum_{j=1}^{n} K_{ij}(x)\,\eta_j, \qquad (i = 1, 2, \ldots, n).$$

In a previous paper[2] we have computed the seminvariants depending

[2] H. Reingold, Thesis (M. A.), University of Cincinnati, 1934.

only upon the arguments L_{ij}, $L'_{ij} = \frac{dL_{ij}}{dx}$, M_{ij}. It is the purpose of the present paper to find the seminvariants depending also on the higher derivatives of the coefficients of the system of differential equations, namely on $L''_{ij} = \frac{dL'_{ij}}{dx}$ and $M'_{ij} = \frac{dM_{ij}}{dx}$. Although the number of such seminvariants and the seminvariants themselves have been found, in this paper, for the general n, the independence of this system of seminvariants has so far been shown only for n=2,3,4.

In the last two sections of this paper some invariants are also calculated and the number of all functionally independent invariants involving the arguments L_{ij}, L'_{ij}, L''_{ij}, M_{ij}, and M'_{ij} is found.

Just as Wilczynski applied the theory for the case n=2 to the study of ruled surfaces, we could also apply the results given in this paper to the study of a certain type of projective configuration represented by the system of differential equations here considered. However, this will be reserved for another work.

1. The Equations and the Transformation.

Consider the system of linear homogeneous differential equations of the second order

(1.1) $$y_i'' + \sum_{j=1}^{n} L_{ij}(x)\, y_j' + \sum_{j=1}^{n} M_{ij}(x)\, y_j = 0 \,, \quad (i=1,2,\dots,n),$$

where

$$y_i' = \frac{dy_i}{dx}\,, \quad y_i'' = \frac{dy_i'}{dx}\,,$$

and where we assume the quantities $L_{ij}(x)$ and $M_{ij}(x)$ to be continuous and possessing derivatives of all orders on the interval $a \leqq x \leqq b$. The most general point-transformation affecting only the dependent variables, which converts the system (1.1) into another of the same form and order, is given[3] by

[3] Wilczynski, loc. cit., chpt. I .

(1.2) $$y_i = \eta_i + \sum_{j=1}^{n} K_{ij}(x)\, \eta_j \,, \quad (i=1,2,\dots,n),$$

where the $K_{ij}(x)$ are arbitrary continuous functions having derivatives of all orders on $a \leqq x \leqq b$, and such that the determinant

(1.3) $$\begin{vmatrix} 1+K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & 1+K_{22} & \dots & K_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ K_{n1} & K_{n2} & \dots & 1+K_{nn} \end{vmatrix}$$

is not identically zero.

The subscripts i and j, unless otherwise specified, will have throughout all this work the range of values 1,2, ...,n. All primed letters will denote differentiation with respect to the variable x.

2. <u>The Finite Transformations Induced on the Coefficients L_{ij} and M_{ij}.</u>

In order to transform (1.1) by means of (1.2), we compute y_i' and y_i'' in terms of η_i, η_i' and η_i'' and substitute the resulting expressions in (1.1). We obtain

$$\eta_i'' + \sum_{j=1}^{n} K_{ij}\eta_j'' + z_i = 0, \tag{2.1}$$

where

$$z_i = \sum_{j=1}^{n}\left(2K_{ij}' + L_{ij} + \sum_{m=1}^{n} L_{im}K_{mj}\right)\eta_j' + \sum_{j=1}^{n}\left(K_{ij}'' + M_{ij} + \sum_{m=1}^{n} L_{im}K_{mj}' + \sum_{m=1}^{n} M_{im}K_{mj}\right)\eta_j .$$

To find the transformations induced on the coefficients L_{ij}, M_{ij}, we write (2.1) in the form

$$z_i = -\eta_i'' - \sum_{j=1}^{n} K_{ij}\eta_j'' . \tag{2.2}$$

Since the determinant (1.3) is different from zero, the system (2.2) may be solved and we obtain

$$\eta_i'' = -z_i - \sum_{j=1}^{n} H_{ij} z_j , \tag{2.3}$$

where the H_{ij} satisfy the relations

$$H_{ij} + K_{ij} + \sum_{m=1}^{n} K_{im}H_{mj} = 0 , \tag{2.4}$$

$$H_{ij} + K_{ij} + \sum_{m=1}^{n} H_{im}K_{mj} = 0 . \tag{2.5}$$

After substituting in (2.3) the expression for z_i we obtain

$$\eta_i'' + \sum_{j=1}^{n} \bar{L}_{ij}\eta_j' + \sum_{j=1}^{n} \bar{M}_{ij}\eta_j = 0, \tag{2.6}$$

where

$$\bar{L}_{ij} = L_{ij} + 2K_{ij}' + \sum_{m=1}^{n} L_{im}K_{mj} + \sum_{m=1}^{n} H_{im}L_{mj} + 2\sum_{m=1}^{n} H_{im}K_{mj}' + \sum_{m=1}^{n}\sum_{s=1}^{n} H_{im}L_{ms}K_{sj}, \tag{2.7}$$

and

$$\begin{aligned} \bar{M}_{ij} = {} & M_{ij} + K_{ij}'' + \sum_{m=1}^{n} L_{im}K_{mj}' + \sum_{m=1}^{n} M_{im}K_{mj} + \sum_{m=1}^{n} H_{im}M_{mj} \\ & + \sum_{m=1}^{n} H_{im}K_{mj}'' + \sum_{m=1}^{n}\sum_{s=1}^{n} H_{im}L_{ms}K_{sj}' + \sum_{m=1}^{n}\sum_{s=1}^{n} H_{im}M_{ms}K_{sj}. \end{aligned} \tag{2.8}$$

3. The Infinitesimal Transformations Induced on the Coefficients L_{ij} and M_{ij}.

In order to obtain the infinitesimal transformations induced on the coeffcients L_{ij} and M_{ij} we now regard the $K_{ij}(x)$ as depending not only on the variable x but also on the single parameter a; so that for a fixed x the equations (1.2) may be considered as a one-parameter group of transformations taking the point y_i into the point $\bar{y}_i$.

Let $a=a_0$ be the para-meter giving the identity transformation, so that $K_{ij}(x,a_0)\underset{x}{\equiv} 0$, and hence from (2.4) $H_{ij}(x,a_0)\underset{x}{\equiv} 0$. Writing (1.2) in the solved form, we have as in (2.3):

$$\bar{y}_i = y_i + \sum_{j=1}^{n} H_{ij}y_j,$$

and hence

$$\left(\frac{\partial \bar{y}_i}{\partial a}\right)_{a=a_0} \equiv \frac{\partial y_i}{\partial a} = \left(\sum_{j=1}^{n} \frac{\partial H_{ij}}{\partial a} y_j\right)_{a=a_0}.$$

Denoting $\left(\frac{\partial H_{ij}}{\partial a}\right)_{a=a_0}$ by $-\mathcal{H}_{ij}(x)$, we have

$$\frac{\partial y_i}{\partial a} = -\sum_{j=1}^{n} \mathcal{H}_{ij}(x)\, y_j .$$

Differentiating the last equation with respect to x we obtain

$$\frac{\partial y_i'}{\partial a} = -\sum_{j=1}^{n}\left(\mathcal{H}_{ij}' y_j + \mathcal{H}_{ij} y_j'\right), \tag{3.1}$$

$$\frac{\partial y_i''}{\partial a} = -\sum_{j=1}^{n}\left(\mathcal{H}_{ij}'' y_j + 2\mathcal{H}_{ij}' y_j' + \mathcal{H}_{ij} y_j''\right). \tag{3.2}$$

In order to find the infinitesimal transformations $\frac{\partial L_{ij}}{\partial a}$, $\frac{\partial M_{ij}}{\partial a}$, we differentiate the transformed equation (1.1) with respect to the parameter a and set $a = a_0$. We find

$$\frac{\partial y_i''}{\partial a} + \sum_{j=1}^{n}\left(\frac{\partial L_{ij}}{\partial a} y_j' + L_{ij}\frac{\partial y_j'}{\partial a} + \frac{\partial M_{ij}}{\partial a} y_j + M_{ij}\frac{\partial y_j}{\partial a}\right) = 0,$$

which in view of the equations (3.1), (3.2) and (1.1) becomes

$$\sum_{j=1}^{n}\left(-\mathcal{H}_{ij}'' + \sum_{m=1}^{n}\mathcal{H}_{im}M_{mj} - \sum_{m=1}^{n} L_{im}\mathcal{H}_{mj}' + \frac{\partial M_{ij}}{\partial a} - \sum_{m=1}^{n} M_{im}\mathcal{H}_{mj}\right) y_j$$
$$+ \sum_{j=1}^{n}\left(-2\mathcal{H}_{ij}' + \sum_{m=1}^{n}\mathcal{H}_{im}L_{mj} + \frac{\partial L_{ij}}{\partial a} - \sum_{m=1}^{n} L_{im}\mathcal{H}_{mj}\right) y_j' = 0,$$

which is true for arbitrary y_j and y_j'. Thus

$$\frac{\partial L_{ij}}{\partial a} = 2\mathcal{H}_{ij}' + \sum_{m=1}^{n}\left(L_{im}\mathcal{H}_{mj} - \mathcal{H}_{im}L_{mj}\right), \tag{3.3}$$

$$\frac{\partial M_{ij}}{\partial a} = \mathcal{H}_{ij}'' + \sum_{m=1}^{n}\left(L_{im}\mathcal{H}_{mj}' + M_{im}\mathcal{H}_{mj} - \mathcal{H}_{im}M_{mj}\right), \tag{3.4}$$

are the infinitesimal transformations of the coefficients L_{ij} and M_{ij} in (1.1).

We shall also need the infinitesimal transformations of

L'_{ij}, L''_{ij}, M'_{ij}, which by differentiating (3.3) and (3.4) are found to be given by

$$(3.5)\qquad \frac{\partial L'_{ij}}{\partial a} = 2\mathcal{K}''_{ij} + \sum_{m=1}^{n}\left(L'_{im}\mathcal{K}_{mj} - \mathcal{K}_{im}L'_{mj} + L_{im}\mathcal{K}'_{mj} - \mathcal{K}'_{im}L_{mj}\right);$$

$$(3.6)\qquad \frac{\partial L''_{ij}}{\partial a} = 2\mathcal{K}'''_{ij} + \sum_{m=1}^{n}\left(L''_{im}\mathcal{K}_{mj} - 2\mathcal{K}'_{im}L'_{mj} + 2L'_{im}\mathcal{K}'_{mj} - \mathcal{K}''_{im}L_{mj} + L_{im}\mathcal{K}''_{mj} - \mathcal{K}_{im}L''_{mj}\right);$$

$$(3.7)\qquad \frac{\partial M'_{ij}}{\partial a} = \mathcal{K}'''_{ij} + \sum_{m=1}^{n}\left(L'_{im}\mathcal{K}'_{mj} + M'_{im}\mathcal{K}_{mj} - \mathcal{K}'_{im}M_{mj} + L_{im}\mathcal{K}''_{mj} + M_{im}\mathcal{K}'_{mj} - \mathcal{K}_{im}M'_{mj}\right).$$

4. The Calculation of the Seminvariants.

In this section we exhibit the seminvariants depending only upon arguments L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij}. To do this define the functions $G_{ij}(x)$ and $U_{ij}(x)$ by

$$(4.1)\qquad G_{ij} = 2L'_{ij} - 4M_{ij} + \sum_{m=1}^{n} L_{im}L_{mj},$$

$$(4.2)\qquad U_{ij} = 2G'_{ij} + \sum_{m=1}^{n}\left(L_{im}G_{mj} - G_{im}L_{mj}\right).$$

Thus G_{ij} and U_{ij} are two matrices depending upon the matrices L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij}. We shall also need the powers of G_{ij} and of U_{ij}, defined by the relations

$$(4.3)\qquad G^{(l)}_{ij} = \sum_{m=1}^{n} G^{(l-1)}_{im}G_{mj}, \qquad (l = 0, 1, 2, \ldots),$$

$$(4.4)\qquad U^{(k)}_{ij} = \sum_{m=1}^{n} U^{(k-1)}_{im}U_{mj}, \qquad (k = 0, 1, 2, \ldots),$$

so that $G^{(l)}_{ij}$ is the l-th power of the matrix G_{ij} and $U^{(k)}_{ij}$ is the κ-th power of the matrix U_{ij}. The zeroth power of every matrix is by definition the identity matrix I.

Consider now the product of the two matrices $G_{ij}^{(\ell)}$ and $U_{ij}^{(\kappa)}$, namely the matrix

$$\sum_{m=1}^{n} G_{im}^{(\ell)} U_{mj}^{(\kappa)} \quad . \tag{4.5}$$

For $\kappa=0$ this last matrix becomes $G_{ij}^{(\ell)}$, and for $\ell=0$ it reduces to $U_{ij}^{(\kappa)}$.

Introduce now the expression

$$\sum_{m=1}^{n} \sum_{i=1}^{n} G_{im}^{(\ell)} U_{mi}^{(\kappa)} \quad , \tag{4.6}$$

which is the trace of the matrix (4.5). Next, we want to prove in this section that the traces given in (4.6) are seminvariants. In order to do this we derive the infinitesimal transformations induced on $G_{ij}^{(\ell)}$ and $U_{ij}^{(\kappa)}$ by (1.2). These transformations are found to be

$$\frac{\partial G_{ij}^{(\ell)}}{\partial a} = \sum_{m=1}^{n} \left(G_{im}^{(\ell)} \mathcal{H}_{mj} - \mathcal{H}_{im} G_{mj}^{(\ell)} \right), \tag{4.7}$$

$$\frac{\partial U_{ij}^{(\kappa)}}{\partial a} = \sum_{m=1}^{n} \left(U_{im}^{(\kappa)} \mathcal{H}_{mj} - \mathcal{H}_{im} U_{mj}^{(\kappa)} \right). \tag{4.8}$$

These formulas are readily proved by the method of mathematical induction. We shall restrict ourselves to the derivation of (4.7), since that of (4.8) is similar. We first show that

$$\frac{\partial G_{ij}}{\partial a} = \sum_{m=1}^{n} \left(G_{im} \mathcal{H}_{mj} - \mathcal{H}_{im} G_{mj} \right) \tag{4.9}$$

holds. Differentiating (4.1) and then using (3.3), (3.4) and (3.5) we obtain

$$\frac{\partial G_{ij}}{\partial a} = 2\frac{\partial L'_{ij}}{\partial a} - 4\frac{\partial M_{ij}}{\partial a} + \sum_{m=1}^{n}\left(L_{im}\frac{\partial L_{mj}}{\partial a} + \frac{\partial L_{im}}{\partial a}L_{mj}\right)$$
$$= \sum_{m=1}^{n}\left(2L'_{im} - 4M_{im} + \sum_{s=1}^{n} L_{is}L_{sm}\right)\mathcal{H}_{mj}$$
$$- \sum_{m=1}^{n}\mathcal{H}_{im}\left(2L'_{mj} - 4M_{mj} + \sum_{s=1}^{n} L_{ms}L_{sj}\right)$$
$$= \sum_{m=1}^{n}\left(G_{im}\mathcal{H}_{mj} - \mathcal{H}_{im}G_{mj}\right),$$

which establishes (4.9).

To complete the induction we assume

$$(4.10) \qquad \frac{\partial G_{ij}^{(p)}}{\partial a} = \sum_{m=1}^{n}\left(G_{im}^{(p)}\mathcal{H}_{mj} - \mathcal{H}_{im}G_{mj}^{(p)}\right)$$

to be true. From (4.10) and the definition of $G_{ij}^{(p+1)}$ we see that

$$\frac{\partial G_{ij}^{(p+1)}}{\partial a} = \frac{\partial}{\partial a}\sum_{m=1}^{n} G_{im}^{(p)}G_{mj}$$
$$= \sum_{m=1}^{n}\left(\frac{\partial G_{im}^{(p)}}{\partial a}G_{mj} + G_{im}^{(p)}\frac{\partial G_{mj}}{\partial a}\right)$$
$$= \sum_{m=1}^{n}\left[\sum_{s=1}^{n}\left(G_{is}^{(p)}\mathcal{H}_{sm} - \mathcal{H}_{is}G_{sm}^{(p)}\right)G_{mj} + \sum_{s=1}^{n} G_{im}^{(p)}\left(G_{ms}\mathcal{H}_{sj} - \mathcal{H}_{ms}G_{sj}\right)\right]$$
$$= \sum_{m=1}^{n}\left(G_{im}^{(p+1)}\mathcal{H}_{mj} - \mathcal{H}_{im}G_{mj}^{(p+1)}\right).$$

Using (4.7) and (4.8) it is seen at once that the expressions in (4.6) are seminvariants, for

$$\frac{\partial \sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)}U_{mi}^{(k)}}{\partial a} = \sum_{i=1}^{n}\sum_{m=1}^{n}\left(\frac{\partial G_{im}^{(\ell)}}{\partial a}U_{mi}^{(k)} + G_{im}^{(\ell)}\frac{\partial U_{mi}^{(k)}}{\partial a}\right)$$
$$= \sum_{i=1}^{n}\sum_{m=1}^{n}\left[\sum_{s=1}^{n}\left(G_{is}^{(\ell)}\mathcal{H}_{sm} - \mathcal{H}_{is}G_{sm}^{(\ell)}\right)U_{mi}^{(k)}\right.$$
$$\left. + \sum_{s=1}^{n} G_{im}^{(\ell)}\left(U_{ms}^{(k)}\mathcal{H}_{si} - \mathcal{H}_{ms}U_{si}^{(k)}\right)\right] = 0.$$

We have thus proved the following

Theorem 4.1 . If G_{ij} and U_{ij} are defined as

$$G_{ij} = 2L'_{ij} - 4M_{ij} + \sum_{m=1}^{n} L_{im}L_{mj} \quad ,$$
$$U_{ij} = 2G'_{ij} + \sum_{m=1}^{n}\left(L_{im}G_{mj} - G_{im}L_{mj}\right) ,$$

then all the traces

$$\sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)} U_{mi}^{(k)} \quad , \qquad (\ell, k = 0, 1, 2, \ldots),$$

will remain invariant under transformations (2.7) and (2.8) induced on the coefficients L_{ij} , M_{ij} by the projective transformation

$$y_i = \eta_i + \sum_{j=1}^{n} K_{ij}(x)\eta_j \; .$$

5. The Complete System.

The problem which presents itself next is to find the number of all functionally independent seminvariants, depending only upon the arguments L_{ij} , L'_{ij} , L''_{ij} , M_{ij} and M'_{ij} .

If[4] f is a seminvariants depending only upon the argu-

[4] Wilczynski, loc. cit., p. 95 .

ments L_{ij} , L'_{ij} , L''_{ij} , M_{ij} and M_{ij}, the expression

$$(5.1) \quad \frac{\partial f}{\partial a} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial f}{\partial L_{ij}}\frac{\partial L_{ij}}{\partial a} + \frac{\partial f}{\partial L'_{ij}}\frac{\partial L'_{ij}}{\partial a} + \frac{\partial f}{\partial L''_{ij}}\frac{\partial L''_{ij}}{\partial a} + \frac{\partial f}{\partial M_{ij}}\frac{\partial M_{ij}}{\partial a} + \frac{\partial f}{\partial M'_{ij}}\frac{\partial M'_{ij}}{\partial a}\right),$$

which represents the increment which the infinitesimal transformation gives to f , must vanish for all values of the arbitrary functions $\mathcal{H}_{ij}$, $\mathcal{H}'_{ij}$, $\mathcal{H}''_{ij}$, $\mathcal{H}'''_{ij}$. Substituting the values from (3.3), (3.4), (3.5), (3.6) and (3.7), and equating to zero the coefficients of these $4n^2$ arbitrary functions, we obtain a system of $4n^2$ partial differential equations for f ..By the general theory of Lie we know, that this set of equations will form a complete system of which every solution is a seminvariant, and conversely every seminvariant depending upon L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} is a solution of this complete system.

For brevity let us introduce the permanent notation

$$\frac{\partial f}{\partial L_{ij}} = \mathcal{L}_{ij}\,, \quad \frac{\partial f}{\partial L'_{ij}} = \mathcal{L}'_{ij}, \quad \frac{\partial f}{\partial L''_{ij}} = \mathcal{L}''_{ij}, \quad \frac{\partial f}{\partial M_{ij}} = \mathcal{M}_{ij}, \quad \frac{\partial f}{\partial M'_{ij}} = \mathcal{M}'_{ij}\,,$$

so that (5.1) yields the set of equations

$$\text{(a)}\quad 2\mathcal{L}''_{ij} + \mathcal{M}'_{ij} = 0\,,$$

$$\text{(b)}\quad 2\mathcal{L}'_{ij} + \mathcal{M}_{ij} + \sum_{m=1}^{n}\left(L_{mi}\mathcal{L}''_{mj} - L_{jm}\mathcal{L}''_{im} + L_{mi}\mathcal{M}'_{mj}\right) = 0,$$

$$\text{(5.2)}\quad \text{(c)}\quad 2\mathcal{L}_{ij} + \sum_{m=1}^{n}\Big(L_{mi}\mathcal{L}'_{mj} - L_{jm}\mathcal{L}'_{im} - 2L'_{jm}\mathcal{L}''_{im} + M_{mi}\mathcal{M}'_{mj} - M_{jm}\mathcal{M}'_{im} + L_{mi}\mathcal{M}_{mj}\Big) = 0,$$

$$\text{(d)}\quad \sum_{m=1}^{n}\Big(L_{mi}\mathcal{L}_{mj} - L_{jm}\mathcal{L}_{im} + L'_{mi}\mathcal{L}'_{mj} - L'_{jm}\mathcal{L}'_{im} + L''_{mi}\mathcal{L}''_{mj} - L''_{jm}\mathcal{L}''_{im} + M_{mi}\mathcal{M}_{mj} - M_{jm}\mathcal{M}_{im} + M'_{mi}\mathcal{M}_{mj} - M'_{jm}\mathcal{M}'_{im}\Big) = 0.$$

This system contains $4n^2$ equations with the $5n^2$ variables L_{ij}, L'_{ij}, L''_{ij}, M_{ij}, M'_{ij}. Denoting the n^2 equations of (5.2d) by $X_{ij}[f]=0$, we readily obtain the relation

(5.3) $$\sum_{i=1}^{n} X_{ii} \equiv 0 .$$

In the sequel it will be shown that the remaining $4n^2-1$ equations of the complete system (5.2) are independent.

We shall now show from equations (5.2a) and (5.2b) that our seminvariants are functions of the $3n^2$ arguments L_{ij}, G_{ij}, G'_{ij}. By (4.1) we have

(5.4) $$G'_{ij} = 2L''_{ij} - 4M'_{ij} + \sum_{m=1}^{n}\left(L_{im}L'_{mj} + L'_{im}L_{mj}\right).$$

Denoting the left members of (5.2a) by $Y_{rs}[f]$, $(r,s=1,2,\ldots,n)$, we readily find

$$Y_{rs}[L_{ij}]=0 , \quad Y_{rs}[G_{ij}]=0, \quad Y_{rs}[G'_{ij}]=0 .$$

Similarly denoting the left members of (5.2b) by $Z_{rs}[f]$ we have

$$Z_{rs}[L_{ij}]=0, \quad Z_{rs}[G_{ij}]=0, \quad Z_{rs}[G'_{ij}]=0 .$$

We shall now show from equations (5.2c) that our seminvariants are functions of G_{ij} and U_{ij}. To this end denote the left members of (5.2c) by Ω_{rs}. Making use of (4.1) we find

$$\Omega_{rs}[G_{ij}]=0.$$

Similarly using the definition of U_{ij} given in (4.2) and keeping in mind (5.4) and (4.1) we obtain after some calculations

$$\Omega_{rs}[U_{ij}] = 0.$$

If as before $X_{ij}[f]$ denote the left-hand members of (5.2d) and if the variables G_{ij} and U_{ij} are introduced as independent variables, the system of equations $X_{ij}[f]=0$, becomes[5]

[5] L. E. Dickson, Differential Equations from the Group Standpoint, Annals of Mathematics, vol. 25, No, 4, p. 322.

$$(5.5) \qquad \sum_{r=1}^{n}\sum_{s=1}^{n} X_{ij}[G_{rs}]\frac{\partial f}{\partial G_{rs}} + \sum_{r=1}^{n}\sum_{s=1}^{n} X_{ij}[U_{rs}]\frac{\partial f}{\partial U_{rs}} = 0.$$

We proceed to find the form of this last set of equations, when expressed in terms of the new variables G_{ij} and U_{ij}.

Making use of (5.2d) and (4.1) we have after some calculations

$$\sum_{r=1}^{n}\sum_{s=1}^{n} X_{ij}[G_{rs}]\frac{\partial f}{\partial G_{rs}} = \sum_{m=1}^{n}\left(G_{mi}\frac{\partial f}{\partial G_{mj}} - G_{jm}\frac{\partial f}{\partial G_{im}}\right).$$

Similarly making use of (5.2d) and of

$$U_{ij} = 4\left[L''_{ij} - 2M'_{ij} + \sum_{m=1}^{n}\left(L_{im}L'_{mj} - L_{im}M_{mj} + M_{im}L_{mj}\right)\right],$$

we obtain

$$\sum_{r=1}^{n}\sum_{s=1}^{n} X_{ij}[U_{rs}]\frac{\partial f}{\partial U_{rs}} = \sum_{m=1}^{n}\left(U_{mi}\frac{\partial f}{\partial U_{mj}} - U_{jm}\frac{\partial f}{\partial U_{im}}\right).$$

So that in terms of the new variables G_{ij} and U_{ij} the system (5.5) takes on the form

$$(5.6) \qquad \sum_{m=1}^{n}\left(U_{mi}\frac{\partial f}{\partial U_{mj}} - U_{jm}\frac{\partial f}{\partial U_{im}} + G_{mi}\frac{\partial f}{\partial G_{mj}} - G_{jm}\frac{\partial f}{\partial G_{im}}\right) = 0.$$

The equations (5.6) form a complete system of n^2 partial differential equations in the $2n^2$ independent variables G_{ij}, U_{ij}. Any solution of (5.6) is a seminvariant, and conversely, every seminvariant involving L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} is a solution of (5.6). Hence the number of all functionally independent seminvariants involving L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} is equal to the number of independent solutions of (5.6), which in turn is equal to $2n^2$ minus the number of independent equations of (5.6).

The problem of finding the number of all functionally independent seminvariants involving L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} reduces itself then to the problem of finding the number of independent equations in the system (5.6). It is seen from (5.3) that there exists a relation among the equations of (5.6). We want to prove now that the complete system (5.6) consists of n^2-1 independent equations, leaving (5.3) as the only relation among them, hence it will follow that the number of all independent seminvariants is $2n^2-(n^2-1)=n^2+1$. We do this in the next section, first for the special case $n=3$ and then for the general n.

6. The Number of Indepenndnet Solutions of the Complete System.

We consider first the complete system (5.6) in the case $n=3$, namely

$$(6.1)\quad \sum_{m=1}^{3}\left(U_{mi}\frac{\partial f}{\partial U_{mj}}-U_{jm}\frac{\partial f}{\partial U_{im}}+G_{mi}\frac{\partial f}{\partial G_{mj}}-G_{jm}\frac{\partial f}{\partial G_{im}}\right)=0\quad,\ (i,j=1,2,3).$$

We shall now show that this complete system consists of only

$8(3^2-1)$ independent equations. To do this consider its matrix of coefficients

$$
\left\| \begin{array}{cccccccccccccccccc}
0 & -U_{12} & -U_{13} & U_{21} & 0 & 0 & U_{31} & 0 & 0 & 0 & -G_{12} & -G_{13} & G_{21} & 0 & 0 & G_{31} & 0 & 0 \\
-U_{21} & U_{11}-U_{22} & -U_{23} & 0 & U_{21} & 0 & 0 & U_{31} & 0 & -G_{21} & G_{11}-G_{22} & -G_{23} & 0 & G_{21} & 0 & 0 & G_{31} & 0 \\
-U_{31} & -U_{32} & U_{11}-U_{33} & 0 & 0 & U_{21} & 0 & 0 & U_{31} & -G_{31} & -G_{32} & G_{11}-G_{33} & 0 & 0 & G_{21} & 0 & 0 & G_{31} \\
U_{12} & 0 & 0 & U_{22}-U_{11} & -U_{12} & -U_{13} & U_{32} & 0 & 0 & G_{12} & 0 & 0 & G_{22}-G_{11} & -G_{12} & -G_{13} & G_{32} & 0 & 0 \\
0 & U_{12} & 0 & -U_{21} & 0 & -U_{23} & 0 & U_{32} & 0 & 0 & G_{12} & 0 & -G_{21} & 0 & -G_{23} & 0 & G_{32} & 0 \\
0 & 0 & U_{12} & -U_{31} & -U_{32} & U_{22}-U_{33} & 0 & 0 & U_{32} & 0 & 0 & G_{12} & -G_{31} & -G_{32} & G_{22}-G_{33} & 0 & 0 & G_{32} \\
U_{13} & 0 & 0 & U_{23} & 0 & 0 & U_{33}-U_{11} & -U_{12} & -U_{13} & G_{13} & 0 & 0 & G_{23} & 0 & 0 & G_{33}-G_{11} & -G_{12} & -G_{13} \\
0 & U_{13} & 0 & 0 & U_{23} & 0 & -U_{21} & U_{33}-U_{22} & -U_{23} & 0 & G_{13} & 0 & 0 & G_{23} & 0 & -G_{21} & G_{33}-G_{22} & -G_{23} \\
0 & 0 & U_{13} & 0 & 0 & U_{23} & -U_{31} & -U_{32} & 0 & 0 & 0 & G_{13} & 0 & 0 & G_{23} & -G_{31} & -G_{32} & 0
\end{array} \right\| , \tag{6.2}
$$

and out of this matrix consider the following eighth order determinant

$$
\begin{vmatrix}
U_{11}-U_{22} & -U_{23} & 0 & 0 & 0 & U_{31} & G_{11}-G_{22} & -G_{23} \\
-U_{32} & U_{11}-U_{33} & 0 & U_{21} & 0 & 0 & -G_{32} & G_{11}-G_{33} \\
0 & 0 & U_{22}-U_{11} & -U_{13} & U_{32} & 0 & 0 & 0 \\
0 & U_{12} & -U_{31} & U_{22}-U_{33} & 0 & 0 & 0 & G_{12} \\
0 & 0 & U_{23} & 0 & U_{33}-U_{11} & -U_{12} & 0 & 0 \\
U_{13} & 0 & 0 & 0 & -U_{21} & U_{33}-U_{22} & G_{13} & 0 \\
-U_{12} & -U_{13} & U_{21} & 0 & U_{31} & 0 & -G_{12} & -G_{13} \\
U_{12} & 0 & -U_{21} & -U_{23} & 0 & U_{32} & G_{12} & 0
\end{vmatrix} .
$$

Setting $U_{ij}=0$, when $i \neq j$, we readily find that this determinant reduces to

$$-(U_{11}-U_{22})^2(U_{11}-U_{33})^2(U_{22}-U_{33})^2 G_{12}G_{13} \neq 0 .$$

We have proved, then, that the matrix (6.2) is of rank eight and hence that eight equations in the complete system are independent, so that (5.3), for $n=3$, is the only relation among the equations of (6.1).

Consider finally the general case, namely the complete system (5.6). We shall now prove that n^2-1 of the n^2 equations of (5.6) are independent, so that we shall show that the matrix of the coefficients of this complete system is of rank n^2-1. The matrix of the coefficients of (5.6) is analogous to the matrix (6.2) of the coefficients of (6.1), and may be described in the following manner. It consists of n^2 rows and $2n^2$ columns, n^2 columns of which have as their elements the same expressions in U_{ij} as the remaining n^2 columns have in G_{ij}. The variables U_{11} , U_{22} , ..., U_{nn} appear only in the binomial form $U_{ii}-U_{jj}$ in the column of coefficients of $\frac{\partial f}{\partial U_{ij}}$. Out of this matrix pick a determinant of order n^2-1 in the following way: Omit the row of coefficients of X_{nn} ; out of the n^2 columns associated with the coefficients of $\frac{\partial f}{\partial U_{ij}}$ take the n^2-n columns when $i \neq j$, and out of the n^2 columns associated with the coefficients of $\frac{\partial f}{\partial G_{ij}}$ take the n-1 columns of coefficients of $\frac{\partial f}{\partial G_{1p}}$, $(p=2,3,\ldots,n)$. By a suitable rearrangement of rows and columns we find that this determinant, apart from sign, is equal to

$$\begin{vmatrix} -G_{12} & -G_{13} & -G_{14} & \cdots & -G_{1\,n-1} & -G_{1n} \\ G_{12} & 0 & 0 & \cdots & 0 & 0 \\ 0 & G_{13} & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & G_{1\,n-1} & 0 \end{vmatrix} \prod_{\substack{i,j=1 \\ i<j}}^{n} (v_{ii}-v_{jj})^2 =$$

$$= \pm G_{12}\, G_{13} \cdots G_{1\,n-1}\, G_{1n} \prod_{\substack{i,j=1 \\ i<j}}^{n} (v_{ii}-v_{jj})^2 \not\equiv 0 .$$

We have thus proved

Theorem 6.1 . If the system of n linear homogeneous differential equations of the second order

$$y_i'' + \sum_{j=1}^{n} L_{ij}\, y_j' + \sum_{j=1}^{n} M_{ij}\, y_j = 0,$$

be transformed by means of

$$y_i = \eta_i + \sum_{j=1}^{n} K_{ij}\, \eta_j \;,$$

then there are exactly n^2+1 functionally independent seminvariants involving the arguments L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij}.

7. The Number of Seminvariants Depending on the Arguments $L_{ij}, L'_{ij}, \ldots, L^{v}_{ij}, M_{ij}, M'_{ij}, \ldots, M^{v-1}_{ij}$.

Although it is not the purpose of this paper to consider seminvariants depending on the derivatives higher than L''_{ij} and M'_{ij}, it will not be without interest to determine the number of seminvariants depending on higher derivatives, especially since this can be made to follow from the discussion just given.

From (3.3) and (3.4) we have by Leibniz' Theorem

(7.1) $$\frac{\partial \overset{\rho}{L}_{ij}}{\partial a} = 2\mathcal{K}^{\rho+1}_{ij} + \sum_{r=0}^{\rho}\sum_{m=1}^{n}\binom{\rho}{r}\left(\overset{r}{L}_{im}\mathcal{K}^{\rho-r}_{mj} - \mathcal{K}^{\rho-r}_{im}\overset{r}{L}_{mj}\right),$$

(7.2) $$\frac{\partial \overset{\rho}{M}_{ij}}{\partial a} = \mathcal{K}^{\rho+2}_{ij} + \sum_{r=0}^{\rho}\sum_{m=1}^{n}\binom{\rho}{r}\left(\overset{r}{L}_{im}\mathcal{K}^{\rho-r+1}_{mj} + \overset{r}{M}_{im}\mathcal{K}^{\rho-r}_{mj} - \mathcal{K}^{\rho-r}_{im}\overset{r}{M}_{mj}\right),$$

where $\binom{\rho}{r}$ is the binomial coefficient and where $\overset{\kappa}{L}_{ij} = \frac{d^{\kappa}L_{ij}}{dx^{\kappa}}$ and $\overset{0}{L}_{ij} = L_{ij}$, etc.

If f is a seminvariant depending upon the arguments $L_{ij}, \overset{1}{L}_{ij}, \ldots, \overset{\nu}{L}_{ij}, M_{ij}, \overset{1}{M}_{ij}, \ldots, \overset{\nu-1}{M}_{ij}$, the expression

(7.3) $$\frac{\partial f}{\partial a} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{\kappa=0}^{\nu}\frac{\partial f}{\partial \overset{\kappa}{L}_{ij}}\frac{\partial \overset{\kappa}{L}_{ij}}{\partial a} + \sum_{\ell=0}^{\nu-1}\frac{\partial f}{\partial \overset{\ell}{M}_{ij}}\frac{\partial \overset{\ell}{M}_{ij}}{\partial a}\right),$$

which represents the increment which the infinitesimal transformation gives to f, must vanish for all values of the arbitrary functions $\mathcal{K}_{ij}, \mathcal{K}'_{ij}, \ldots, \mathcal{K}^{\nu+1}_{ij}$ after expressions (7.1) and (7.2) have been substituted, respectively, for $\frac{\partial \overset{\kappa}{L}_{ij}}{\partial a}$ and $\frac{\partial \overset{\ell}{M}_{ij}}{\partial a}$. Equating to zero the coefficients of these $(\nu+2)n^2$ arbitrary functions we obtain a system of $(\nu+2)n^2$ partial differential equations for f. By the general theory of Lie we know, that this set of partial differential equations will form a complete system, and that every solution of it is a seminvariant and conversely every seminvariant depending on the above stated arguments is a solution of this complete system.

Substituting (7.1) and (7.2) into (7.3) we obtain

(7.4) $$\frac{\partial f}{\partial a} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\sum_{\kappa=0}^{\nu}\frac{\partial f}{\partial \overset{\kappa}{L}_{ij}}\left[2\mathcal{K}^{\kappa+1}_{ij} + \sum_{r=0}^{\kappa}\sum_{m=1}^{n}\binom{\kappa}{r}\left(\overset{r}{L}_{im}\mathcal{K}^{\kappa-r}_{mj} - \mathcal{K}^{\kappa-r}_{im}\overset{r}{L}_{mj}\right)\right]\right.$$
$$\left. + \sum_{\ell=0}^{\nu-1}\frac{\partial f}{\partial \overset{\ell}{M}_{ij}}\left[\mathcal{K}^{\ell+2}_{ij} + \sum_{r=0}^{\ell}\sum_{m=1}^{n}\binom{\ell}{r}\left(\overset{r}{L}_{im}\mathcal{K}^{\ell-r+1}_{mj} + \overset{r}{M}_{im}\mathcal{K}^{\ell-r}_{mj} - \mathcal{K}^{\ell-r}_{im}\overset{r}{M}_{mj}\right)\right]\right\}.$$

The set of the n^2 partial differential equations, of the complete system, obtained by equating to zero the coefficients of $\mathcal{K}_{ij}$ is

$$(7.5)\quad \sum_{m=1}^{n}\left\{\sum_{k=0}^{\nu}\left(\frac{\partial f}{\partial L^{k}_{mj}}L^{k}_{mi}-\frac{\partial f}{\partial L^{k}_{im}}L^{k}_{jm}\right)+\sum_{l=0}^{\nu-1}\left(\frac{\partial f}{\partial M^{l}_{mj}}M^{l}_{mi}-\frac{\partial f}{\partial M^{l}_{im}}M^{l}_{jm}\right)\right\}=0.$$

Denoting the left hand member of (7.5) by $V_{ij}[f]$ we readily obtain

$$(7.6)\qquad \sum_{i=1}^{n}V_{ii}[f]\equiv 0.$$

We now shall prove that (7.6) is the only relation among the equations of the complete system. To do that we find that the set of the n^2 equations of the complete system, obtained by equating to zero the coefficients of $\mathcal{K}^{\nu+1}_{ij}$ in (7.4), is given by

$$(7.7)\qquad 2\frac{\partial f}{\partial L^{\nu}_{ij}}+\frac{\partial f}{\partial M^{\nu-1}_{ij}}=0.$$

When $\nu=\alpha$ the complete system consists of $(\alpha+2)n^2$ equations obtained by equating to zero the coefficients of $\mathcal{K}_{ij}$, $\mathcal{K}'_{ij}$, ..., $\mathcal{K}^{\alpha+1}_{ij}$ in (7.4). Now, when $\nu=\alpha-1$ the corresponding complete system consists of $(\alpha+1)n^2$ equations, obtained by equating to zero the coefficients of $\mathcal{K}_{ij}$, $\mathcal{K}'_{ij}$, ..., $\mathcal{K}^{\alpha}_{ij}$ in (7.4). From (7.4) it is obvious that the complete system corresponding to $\nu=\alpha-1$, may be obtained from the complete system corresponding to $\nu=\alpha$ by omitting all terms involving L^{α}_{ij} and $M^{\alpha-1}_{ij}$. By this omission the n^2 equations (7.7), of the complete system corresponding to $\nu=\alpha$, will not appear, while the remaining $(\alpha+1)n^2$ equations of this sytem will become the $(\alpha+1)n^2$ equations of the complete system corresponding to $\nu=\alpha-1$. Hence every relation among the equations of the complete system corresponding to $\nu=\alpha$, except a relation involving only the equations of (7.7), will hold among the equations of the complete system corresponding to $\nu=\alpha-1$. But, obviously, equations (7.7) are independent. Now for $\nu=2$ the equation (7.6) reduces to (5.3), so it follows that for $\nu\geqq 2$, (7.6) is the only relation among the $(\nu+2)n^2$ equations of the complete system. Thus we have proved

Theorem 7.1. <u>If the system of n linear homogeneous differential equations of the second order</u>

$$y''_i+\sum_{j=1}^{n} L_{ij}(x)\, y'_j+\sum_{j=1}^{n} M_{ij}(x)\, y_j=0,$$

<u>be transformed by means of</u>

$$y_i=\eta_i+\sum_{j=1}^{n} K_{ij}(x)\,\eta_j ,$$

<u>then the number of all functionally independent seminvariants involving the arguments $L_{ij}, L'_{ij}, \ldots, L^{(\nu)}_{ij}, M_{ij}, M'_{ij}, \ldots, M^{(\nu-1)}_{ij}$, for[6] $\nu\geqq 2$, is $(\nu-1)n^2+1$</u>.

---

[6] For the case $\nu=1$ we have shown that there are exactly n seminvariants. See H. Reingold, Thesis (M. A.), University of Cincinnati, 1934.

8. Several Lemmas.

It was proved in Theorem 4.1 that the traces (4.6) are seminvariants involving the coefficients L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij}. In Theorem 6.1 it was shown that the number of all such functionally independent seminvariants is n^2+1. Our problem of finding the complete system of seminvariants involving the arguments L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} will then be solved if we can exhibit n^2+1 functionally independent traces of (4.6). We consider then the following n^2+1 traces

$$(8.1)\qquad \begin{aligned} &\sum_{m=1}^{n}\sum_{i=1}^{n} G^{(l)}_{im} U^{(k)}_{mi}, \quad (l=1,2,\dots,n;\ k=0,1,\dots,n-1),\\ &\sum_{i=1}^{n} U^{(n)}_{ii}, \end{aligned}$$

which we want to prove to be functionally independent. We do this in this paper for the cases $n=2$, $n=3$ and $n=4$ and are as yet unable to do it for the general n.

For these considerations it will be found convenient to introduce some preliminary matters concerning the derivatives of the traces of composite matrices as well as properties of certain determinants which arise here. This will be done in this section and the next.

We first prove

Lemma 8.1 [7].

$$(8.2)\qquad \frac{\partial \sum_{i=1}^{n}\sum_{m=1}^{n} G^{(l)}_{im} U^{(k)}_{mi}}{\partial U_{rs}} = \sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{p=1}^{k} U^{(k-p)}_{sm} G^{(l)}_{mi} U^{(p-1)}_{ir} .$$

[7] H. W. Turnbull, "A Matrix Form of Taylor's Theorem", Proceedings of the Edinburgh Mathematical Society, 2-nd series, vol. 2, 1930-31, p.39, Theorem IV . Turnbull obtains this lemma as a corollary of another theorem proved in his paper.

From the relation (4.4) and by direct differentiation it follows that

$$\frac{\partial \sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)} U_{mi}^{(k)}}{\partial U_{rs}} = \sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)} \frac{\partial U_{mi}^{(k)}}{\partial U_{rs}}$$

$$= \sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)} \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n}\sum_{\alpha_3=1}^{n}\cdots\sum_{\alpha_{k-1}=1}^{n} \frac{\partial}{\partial U_{rs}}\left(U_{m\alpha_1}U_{\alpha_1\alpha_2}U_{\alpha_2\alpha_3}\cdots U_{\alpha_{k-2}\alpha_{k-1}}U_{\alpha_{k-1}i}\right)$$

$$= \sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)} \left(\sum_{\alpha_1=1}^{n} \frac{\partial U_{m\alpha_1}}{\partial U_{rs}} U_{\alpha_1 i}^{(k-1)} + \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n} U_{m\alpha_1}\frac{\partial U_{\alpha_1\alpha_2}}{\partial U_{rs}} U_{\alpha_2 i}^{(k-2)}\right.$$

$$\left. + \cdots + \sum_{\alpha_{k-2}=1}^{n}\sum_{\alpha_{k-1}=1}^{n} U_{m\alpha_{k-2}}^{(k-2)} \frac{\partial U_{\alpha_{k-2}\alpha_{k-1}}}{\partial U_{rs}} U_{\alpha_{k-1}i} + \sum_{\alpha_{k-1}=1}^{n} U_{m\alpha_{k-1}}^{(k-1)} \frac{\partial U_{\alpha_{k-1}i}}{\partial U_{rs}}\right).$$

But obviously

$\frac{\partial U_{ij}}{\partial U_{rs}} = 1$ when $i=r, j=s$, and $\frac{\partial U_{ij}}{\partial U_{rs}} = 0$ otherwise.

Hence

$$\frac{\partial \sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)} U_{mi}^{(k)}}{\partial U_{rs}} = \sum_{i=1}^{n}\sum_{m=1}^{n}\left(U_{si}^{(k-1)} G_{ir}^{(\ell)} + U_{si}^{(k-2)} G_{im}^{(\ell)} U_{mr} + \cdots \right.$$

$$\left. \cdots + U_{si} G_{im}^{(\ell)} U_{mr}^{(k-2)} + G_{sm}^{(\ell)} U_{mr}^{(k-1)}\right),$$

which establishes (8.2).

For $\ell=0$, (8.2) reduces to

$$\frac{\partial \sum_{i=1}^{n} U_{ii}^{(k)}}{\partial U_{rs}} = k\, U_{sr}^{(k-1)} . \tag{8.3}$$

The n^2+1 seminvariants given in (8.1) may be regarded as composed of two sets of functions . One set is given by the n seminvariants

$$\sum_{i=1}^{n} G_{ii}^{(\ell)} , \quad (\ell = 1, 2, \ldots, n) , \tag{8.4}$$

which are functions of the variables G_{11} , G_{12} , ..., G_{nn} only, and the other set by

$$\begin{aligned} &\sum_{m=1}^{n} \sum_{i=1}^{n} G_{im}^{(\ell)} U_{mi}^{(k)} , \quad (\ell = 1, 2, \ldots, n ; k = 1, 2, \ldots, n-1), \\ &\sum_{i=1}^{n} U_{ii}^{(n)} , \end{aligned} \tag{8.5}$$

where we have n^2-n+1 seminvariants which are functions of G_{11} , G_{12}, ..., G_{nn} , U_{11} , U_{12}, ..., U_{nn}.

We can show now that the traces (8.4) are independent functions of the n^2 variables G_{ij} . To do that it is sufficient to show that they are independent when considered as functions of the n variables G_{11} , G_{22} , ..., G_{nn} only. Making use of (8.3) we see that the Jacobian of the n functions (8.4) with respect to the n variables G_{11} , G_{22} , ..., G_{nn} is

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ 2G_{11} & 2G_{22} & \ldots & 2G_{nn} \\ \cdot & \cdot & \cdot & \cdot \\ nG_{11}^{(n-1)} & nG_{22}^{(n-1)} & \ldots & nG_{nn}^{(n-1)} \end{vmatrix} .$$

Setting $G_{ij}=0, (i \neq j)$, so that $G_{ii}^{(\ell)} = G_{ii}^{\ell}$ we find that the last determinant reduces to

$$n!\begin{vmatrix} 1 & 1 & \cdots & 1 \\ G_{11} & G_{22} & \cdots & G_{nn} \\ \cdot & \cdot & \cdot & \cdot \\ G_{11}^{n-1} & G_{22}^{n-1} & \cdots & G_{nn}^{n-1} \end{vmatrix} = n!\prod_{\substack{i,j=1 \\ i>j}}^{n}(G_{ii}-G_{jj}) \not\equiv 0.$$

We have thus proved

Lemma 8.2 . The first n successive traces

$$\sum_{i=1}^{n} G_{ii}^{(\ell)}\ , \quad (\ell = 1, 2, \ldots, n),$$

are functionally independent in the variables G_{11} , G_{12} , . . . , G_{nn}.

We have now the following result

Lemma 8.3 . If the seminvariants given in (8.5) are independent when considered as functions of the variables U_{11} , U_{12}, . . . , U_{nn} only, then the n^2+1 seminvariants given in (8.1) are independent in the variables G_{11} , G_{12} , . . . , G_{nn} , U_{11} , U_{12} , . . . , U_{nn}.

For suppose that the seminvariants (8.1) are not independent, then there exists a functional relationship among them with G_{ij} and U_{ij} as independent variables. Because of Lemmma 8.2 some of the seminvariants of (8.5) must enter into this relationship, which gives us a functional relationship among the semivariants (8.5) in the independent variables U_{ij} , contrary to our assumption.

9. Generalized Determinants of Vandermonde.

Consider an arbitrary n by n square matrix $\|M_{ij}\|$, the determinant of which is denoted by m. As usual M_{ij} is the element in the i-th row and j-th column of the square matrix. Let $M^{(k)}_{ij}$ stand for the element in the i-th row and j-th column of the k-th power of the given matrix $\|M_{ij}\|$. So that

$$M^{(k)}_{ij} = \sum_{s=1}^{n} M^{(k-1)}_{is} M_{sj} \;, \quad M^{(0)}_{ij} = \delta_{ij} \,.$$

Since every square matrix satisfies its own characteristic equation, we have

$$(9.1) \qquad a_n M^{(n)}_{ij} + a_{n-1} M^{(n-1)}_{ij} + \ldots + a_1 M_{ij} + a_0 M^{(0)}_{ij} = 0,$$

for every pair of integers i, j where $a_n = (-1)^n$, $a_0 = m$.

Consider now determinants of the form

$$(9.2) \qquad \begin{vmatrix} M_{r_1 s_1} & M_{r_2 s_2} & \ldots & M_{r_n s_n} \\ M^{(2)}_{r_1 s_1} & M^{(2)}_{r_2 s_2} & \ldots & M^{(2)}_{r_n s_n} \\ \cdot & \cdot & \ldots & \cdot \\ M^{(n)}_{r_1 s_1} & M^{(n)}_{r_2 s_2} & \ldots & M^{(n)}_{r_n s_n} \end{vmatrix} ,$$

where $r_1, r_2, \ldots, r_n$ and $s_1, s_2, \ldots, s_n$ are arbitrary integers on the range 1 to n, equal or unequal. For the sake of simplicity we introduce the permanent notation to denote a determinant of the type (9.2) by exhibiting its i-th row, i.e. by writing

$$(9.3) \qquad \left| M^{(i)}_{r_1 s_1} \quad M^{(i)}_{r_2 s_2} \quad \ldots \quad M^{(i)}_{r_n s_n} \right| .$$

We call a determinant of the type (9.3) a <u>generalized determinant of Vandermonde</u>, since for the particular square matrix $\|M_{ij}\| \equiv \|x_i \delta_{ij}\|$ it reduces to the determinant

$$\left| x_1^i \quad x_2^i \quad \ldots \quad x_n^i \right| ,$$

which is known as the determinant of Vandermonde.

We want to factor now the determinant (9.3). In order to do that operate on this determinant as follows: Multiply the last row by a_n and add to it a_{n-1} times the (n-1)-st row, a_{n-2} times the (n-2)-nd row, ..., a_2 times the second row and a_1 times the first row. Then making use of equation (9.1) we find that the last row of this determinant is equal to

$$-a_0 M^{(0)}_{r_1 s_1} \quad , -a_0 M^{(0)}_{r_2 s_2} , \ldots, -a_0 M^{(0)}_{r_n s_n} ,$$

while the other rows are unchanged. The value of our new determinant is clearly $a_n \begin{vmatrix} M^{(i)}_{r_1 s_1} & M^{(i)}_{r_2 s_2} & \ldots & M^{(i)}_{r_n s_n} \end{vmatrix}$. Now by n-1 interchanges of pairs of adjacent rows the last row may be shifted into the first row and then if we factor $-a_0$ out of the first row we have

$$a_n \begin{vmatrix} M^{(i)}_{r_1 s_1} & M^{(i)}_{r_2 s_2} & \ldots & M^{(i)}_{r_n s_n} \end{vmatrix} = (-1)^{n-1}(-a_0) \begin{vmatrix} M^{(i-1)}_{r_1 s_1} & M^{(i-1)}_{r_2 s_2} & \ldots & M^{(i-1)}_{r_n s_n} \end{vmatrix},$$

or

$$a_n \begin{vmatrix} M^{(i)}_{r_1 s_1} & M^{(i)}_{r_2 s_2} & \ldots & M^{(i)}_{r_n s_n} \end{vmatrix} = (-1)^{n} a_0 \begin{vmatrix} M^{(i-1)}_{r_1 s_1} & M^{(i-1)}_{r_2 s_2} & \ldots & M^{(i-1)}_{r_n s_n} \end{vmatrix}.$$

Since $a_n = (-1)^n$ and $a_0 = m$ we have

Lemma 9.1

$$\begin{vmatrix} M^{(i)}_{r_1 s_1} & M^{(i)}_{r_2 s_2} & \ldots & M^{(i)}_{r_n s_n} \end{vmatrix} = m \begin{vmatrix} M^{(i-1)}_{r_1 s_1} & M^{(i-1)}_{r_2 s_2} & \ldots & M^{(i-1)}_{r_n s_n} \end{vmatrix}. \tag{9.4}$$

This lemma gives us the factorization of the determinant (9.3).

If no pair of the n sets (r_1, s_1), (r_2, s_2), ..., (r_n, s_n)

have equal elements, i.e. if $r_1 \neq s_1$, $r_2 \neq s_2$, ..., $r_n \neq s_n$ simultaneously, then the first row of the determinant on the right hand side of (9.4) has every element zero. Consequently we have

Lemma 9.2. If $r_1 \neq s_1$, $r_2 \neq s_2$, ..., $r_n \neq s_n$ simultaneously, then the value of the determinant (9.3) is zero.

10. The Independence of the Seminvariants for the Case $n=3$.

As pointed out at the beginning of section eight, the problem of finding the complete system of seminvariants involving the arguments L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} will be solved if we can prove that the n^2+1 seminvariants given in (8.1) are functionally independent. We do this in the present section for the case $n=3$, so that i and j will have the range of values 1,2,3. In view of Lemma 8.3 it will suffice to prove that the seven functions

$$
(10.1) \qquad \begin{aligned} &\sum_{i=1}^{3}\sum_{m=1}^{3} G^{(l)}_{im} U^{(k)}_{mi} \equiv [l,k], \quad (l=1,2,3;\ k=1,2), \\ &\sum_{i=1}^{3} U^{(3)}_{ii} \equiv [0,3], \end{aligned}
$$

are independent in the variables U_{11}, U_{12}, ..., U_{33}.

We are thus lead to the consideration of the following seven by nine matrix

$$(10.2)\qquad \begin{Vmatrix}
\frac{\partial[1,1]}{\partial U_{11}} & \frac{\partial[1,1]}{\partial U_{12}} & \cdots & \frac{\partial[1,1]}{\partial U_{sr}} & \cdots & \frac{\partial[1,1]}{\partial U_{33}} \\
\frac{\partial[2,1]}{\partial U_{11}} & \frac{\partial[2,1]}{\partial U_{12}} & \cdots & \frac{\partial[2,1]}{\partial U_{sr}} & \cdots & \frac{\partial[2,1]}{\partial U_{33}} \\
\frac{\partial[3,1]}{\partial U_{11}} & \frac{\partial[3,1]}{\partial U_{12}} & \cdots & \frac{\partial[3,1]}{\partial U_{sr}} & \cdots & \frac{\partial[3,1]}{\partial U_{33}} \\
\frac{\partial[1,2]}{\partial U_{11}} & \frac{\partial[1,2]}{\partial U_{12}} & \cdot & \cdots & & \cdot \\
\frac{\partial[2,2]}{\partial U_{11}} & \frac{\partial[2,2]}{\partial U_{12}} & \cdot & \cdots & & \cdot \\
\frac{\partial[3,2]}{\partial U_{11}} & \frac{\partial[3,2]}{\partial U_{12}} & \cdots & \frac{\partial[3,2]}{\partial U_{sr}} & \cdots & \frac{\partial[3,2]}{\partial U_{33}} \\
\frac{\partial[0,3]}{\partial U_{11}} & \frac{\partial[0,3]}{\partial U_{12}} & \cdots & \frac{\partial[0,3]}{\partial U_{sr}} & \cdots & \frac{\partial[0,3]}{\partial U_{33}}
\end{Vmatrix}.$$

Applying Lemma 8.1 and formula 8.3 and setting $U_{ij}=0$ when $i \neq j$, we find that (10.2) takes on the form

$$\begin{Vmatrix}
G_{11} & G_{21} & \cdots & G_{rs} & \cdots & G_{33} \\
G_{11}^{(2)} & G_{21}^{(2)} & \cdots & G_{rs}^{(2)} & \cdots & G_{33}^{(2)} \\
G_{11}^{(3)} & G_{21}^{(3)} & \cdots & G_{rs}^{(3)} & \cdots & G_{33}^{(3)} \\
(U_{11}+U_{11})G_{11} & (U_{22}+U_{11})G_{21} & \cdots & (U_{rr}+U_{ss})G_{rs} & \cdots & (U_{33}+U_{33})G_{33} \\
(U_{11}+U_{11})G_{11}^{(2)} & (U_{22}+U_{11})G_{21}^{(2)} & \cdots & (U_{rr}+U_{ss})G_{rs}^{(2)} & \cdots & (U_{33}+U_{33})G_{33}^{(2)} \\
(U_{11}+U_{11})G_{11}^{(3)} & (U_{22}+U_{11})G_{21}^{(3)} & \cdots & (U_{rr}+U_{ss})G_{rs}^{(3)} & \cdots & (U_{33}+U_{33})G_{33}^{(3)} \\
3U_{11}^{2} & 0 & \cdots & 3U_{rs}^{2}\,\delta_{rs} & \cdots & 3U_{33}^{2}
\end{Vmatrix}.$$

Employing the notation introduced in section nine for the representation of generalized determinants of Vandermonde the last matrix can be written in the more compact form

$$\left\| \begin{matrix} G_{11}^{(i)} & G_{21}^{(i)} & \cdots & G_{rs}^{(i)} & \cdots & G_{33}^{(i)} \\ (U_{11}+U_{11})G_{11}^{(i)} & (U_{22}+U_{11})G_{21}^{(i)} & \cdots & (U_{rr}+U_{ss})G_{rs}^{(i)} & \cdots & (U_{33}+U_{33})G_{33}^{(i)} \\ 3U_{11}^2 & 0 & \cdots & 3U_{rs}^2\,\delta_{rs} & \cdots & 3U_{33}^2 \end{matrix} \right\| .$$

Now let $U_{11} = -U_{22} = 1$ and $U_{33} = 0$, so that the last matrix becomes

$$\left\| \begin{matrix} G_{11}^{(i)} & G_{21}^{(i)} & G_{31}^{(i)} & G_{12}^{(i)} & G_{22}^{(i)} & G_{32}^{(i)} & G_{13}^{(i)} & G_{23}^{(i)} & G_{33}^{(i)} \\ 2G_{11}^{(i)} & 0 & G_{31}^{(i)} & 0 & -2G_{22}^{(i)} & -G_{32}^{(i)} & G_{13}^{(i)} & -G_{23}^{(i)} & 0 \\ 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \end{matrix} \right\| .$$

To show that this matrix is of rank seven consider the following seventh order determinant

$$(10.3) \qquad \left| \begin{matrix} G_{11}^{(i)} & G_{21}^{(i)} & G_{31}^{(i)} & G_{12}^{(i)} & G_{22}^{(i)} & G_{32}^{(i)} & G_{33}^{(i)} \\ 2G_{11}^{(i)} & 0 & G_{31}^{(i)} & 0 & -2G_{22}^{(i)} & -G_{32}^{(i)} & 0 \\ 3 & 0 & 0 & 0 & 3 & 0 & 0 \end{matrix} \right| .$$

By the Laplace development this determinant is equal to

$$3 \left| \begin{matrix} 2G_{11}^{(i)} & G_{31}^{(i)} & -2G_{22}^{(i)} & G_{32}^{(i)} \\ 1 & 0 & 1 & 0 \end{matrix} \right| \left| G_{21}^{(i)} \quad G_{12}^{(i)} \quad G_{33}^{(i)} \right| =$$

$$= -6 \left| G_{33}^{(i)} \quad G_{21}^{(i)} \quad G_{12}^{(i)} \right| \left[\left| G_{22}^{(i)} \quad G_{31}^{(i)} \quad G_{32}^{(i)} \right| + \left| G_{11}^{(i)} \quad G_{31}^{(i)} \quad G_{32}^{(i)} \right| \right] .$$

Denoting by g the determinant of the matrix $\|G_{ij}\|$ and applying Lemma 9.1 we see that (10.3) reduces finally to

$$-12g^2 \left| G_{33}^{(i-1)} \quad G_{21}^{(i-1)} \quad G_{12}^{(i-1)} \right| \cdot \left| G_{22}^{(i-1)} \quad G_{31}^{(i-1)} \quad G_{32}^{(i-1)} \right| .$$

But this can be readily verified not to be identically zero by taking

$$\|G_{ij}\| \equiv \begin{Vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{Vmatrix} .$$

We have thus proved our final result for $n = 3$, namely

Theorem 10.1 . Let

$$y_i'' + \sum_{j=1}^{3} L_{ij}(x)\, y_j' + \sum_{j=1}^{3} M_{ij}(x)\, y_j = 0 ,$$

be a system of three linear homogeneous differential equations of the second order. Suppose that the variables y_i are transformed by means of

(10.4) $$y_i = \eta_i + \sum_{j=1}^{3} K_{ij}(x)\, \eta_j .$$

Then, if we define

$$G_{ij} = 2L_{ij}' - 4M_{ij} + L_{ij}^{(2)} ,$$

and

$$U_{ij} = 2G_{ij}' + \sum_{m=1}^{3} \left(L_{im} G_{mj} - G_{im} L_{mj} \right) ,$$

the ten traces

$$\sum_{i=1}^{3} \sum_{m=1}^{3} G_{im}^{(l)}\, U_{mi}^{(k)} , \quad (l = 1, 2, 3 ; \; k = 0, 1, 2) ,$$

$$\sum_{i=1}^{3} U_{ii}^{(3)} ,$$

are a system of ten indeoendent functions of $G_{11}, G_{12}, \ldots, G_{33}$, $U_{11}, U_{12}, \ldots, U_{33}$ which remain unchanged by the transformation (10.4); and any seminvariant depending upon L_{ij}, L_{ij}', L_{ij}'', M_{ij}, M_{ij}' may be expressed functionally in terms of these ten traces.

The proof of the functional independence of the five traces

$$\sum_{i=1}^{2} \sum_{m=1}^{2} G_{im}^{(\ell)} U_{mi}^{(k)}, \quad (\ell=1,2;\ k=0,1),$$
$$\sum_{i=1}^{2} U_{ii}^{(2)},$$

for the case $n=2$ goes through in an exactly analogous fashion, by taking the set of values $U_{ij}=0$, $i \neq j$, $U_{11}=1$, $U_{22}=0$. Thus we also have a result corresponding to Theorem 10.1 for the case $n=2$, where instead of ten independent seminvariants we now have five.

11. The Independence of the Seminvariants for the Case $n=4$.

In view of Lemma 8.3 it will suffice to prove that the 13 functions

$$\sum_{i=1}^{4} \sum_{m=1}^{4} G_{im}^{(\ell)} U_{mi}^{(k)} \equiv [\ell, k], \quad (\ell=1,2,3,4;\ K=1,2,3),$$
$$\sum_{i=1}^{4} U_{ii}^{(4)} \equiv [0,4],$$

are independent in the variables U_{11}, U_{12}, ..., U_{44}. We are thus lead to the consideration of the following 13 by 16 matrix

$$
(11.1) \qquad \left\| \begin{matrix}
\frac{\partial[1,1]}{\partial U_{11}} & \frac{\partial[1,1]}{\partial U_{12}} & \cdots & \frac{\partial[1,1]}{\partial U_{sr}} & \cdots & \frac{\partial[1,1]}{\partial U_{44}} \\
\frac{\partial[2,1]}{\partial U_{11}} & \frac{\partial[2,1]}{\partial U_{12}} & \cdots & \frac{\partial[2,1]}{\partial U_{sr}} & \cdots & \frac{\partial[2,1]}{\partial U_{44}} \\
\frac{\partial[3,1]}{\partial U_{11}} & \frac{\partial[3,1]}{\partial U_{12}} & \cdots & \frac{\partial[3,1]}{\partial U_{sr}} & \cdots & \frac{\partial[3,1]}{\partial U_{44}} \\
\frac{\partial[4,1]}{\partial U_{11}} & \frac{\partial[4,1]}{\partial U_{12}} & \cdots & \frac{\partial[4,1]}{\partial U_{sr}} & \cdots & \frac{\partial[4,1]}{\partial U_{44}} \\
\frac{\partial[1,2]}{\partial U_{11}} & \frac{\partial[1,2]}{\partial U_{12}} & \cdots & \frac{\partial[1,2]}{\partial U_{sr}} & \cdots & \frac{\partial[1,2]}{\partial U_{44}} \\
\frac{\partial[2,2]}{\partial U_{11}} & \frac{\partial[2,2]}{\partial U_{12}} & \cdots & \frac{\partial[2,2]}{\partial U_{sr}} & \cdots & \frac{\partial[2,2]}{\partial U_{44}} \\
\cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\
\frac{\partial[4,3]}{\partial U_{11}} & \frac{\partial[4,3]}{\partial U_{12}} & \cdots & \frac{\partial[4,3]}{\partial U_{sr}} & \cdots & \frac{\partial[4,3]}{\partial U_{44}} \\
\frac{\partial[0,4]}{\partial U_{11}} & \frac{\partial[0,4]}{\partial U_{12}} & \cdots & \frac{\partial[0,4]}{\partial U_{sr}} & \cdots & \frac{\partial[0,4]}{\partial U_{44}}
\end{matrix} \right\| .
$$

Applying Lemma 8.1 and formula 8.3 and letting $U_{ij} = 0$ when $i \neq j$, we find that (11.1) takes on the form

$$
\left\| \begin{matrix}
G_{rs} \\
G_{rs}^{(2)} \\
G_{rs}^{(3)} \\
G_{rs}^{(4)} \\
(U_{rr} + U_{ss})\, G_{rs} \\
(U_{rr} + U_{ss})\, G_{rs}^{(2)} \\
(U_{rr} + U_{ss})\, G_{rs}^{(3)} \\
(U_{rr} + U_{ss})\, G_{rs}^{(4)} \\
(U_{rr}^2 + U_{rr}U_{ss} + U_{ss}^2)\, G_{rs} \\
(U_{rr}^2 + U_{rr}U_{ss} + U_{ss}^2)\, G_{rs}^{(2)} \\
(U_{rr}^2 + U_{rr}U_{ss} + U_{ss}^2)\, G_{rs}^{(3)} \\
(U_{rr}^2 + U_{rr}U_{ss} + U_{ss}^2)\, G_{rs}^{(4)} \\
4U_{rs}^3\, \delta_{rs}
\end{matrix} \right\| ,
$$

where we are representing our matrix by exhibiting a column with subscripts s and r .

Employing the notation introduced for the representation of generalized determinants of Vandermonde, the last matrix can be written in the form

$$\left\| \begin{array}{ccccc} G_{11}^{(i)} & \cdots & G_{rs}^{(i)} & \cdots & G_{44}^{(i)} \\ (U_{11}+U_{11})G_{11}^{(i)} & \cdots & (U_{rr}+U_{ss})G_{rs}^{(i)} & \cdots & (U_{44}+U_{44})G_{44}^{(i)} \\ (U_{11}^2+U_{11}U_{11}+U_{11}^2)G_{11}^{(i)} & \cdots & (U_{rr}^2+U_{rr}U_{ss}+U_{ss}^2)G_{rs}^{(i)} & \cdots & (U_{44}^2+U_{44}U_{44}+U_{44}^2)G_{44}^{(i)} \\ 4U_{11}^3 & \cdots & 4U_{rs}^3\delta_{rs} & \cdots & 4U_{44}^3 \end{array} \right\| .$$

Now let $U_{11}=-U_{22}=1$, $U_{33}=2$ and $U_{44}=0$, so that the last matrix becomes (after omitting the factor 4 in the last row),

$$\left\| \begin{array}{cccccccccccccccc} G_{11}^{(i)} & G_{21}^{(i)} & G_{31}^{(i)} & G_{41}^{(i)} & G_{12}^{(i)} & G_{22}^{(i)} & G_{32}^{(i)} & G_{42}^{(i)} & G_{13}^{(i)} & G_{23}^{(i)} & G_{33}^{(i)} & G_{43}^{(i)} & G_{14}^{(i)} & G_{24}^{(i)} & G_{34}^{(i)} & G_{44}^{(i)} \\ 2G_{11}^{(i)} & 0 & 3G_{31}^{(i)} & G_{41}^{(i)} & 0 & -2G_{22}^{(i)} & G_{32}^{(i)} & -G_{42}^{(i)} & 3G_{13}^{(i)} & G_{23}^{(i)} & 4G_{33}^{(i)} & 2G_{43}^{(i)} & G_{14}^{(i)} & -G_{24}^{(i)} & 2G_{34}^{(i)} & 0 \\ 3G_{11}^{(i)} & G_{21}^{(i)} & 7G_{31}^{(i)} & G_{41}^{(i)} & G_{12}^{(i)} & 3G_{22}^{(i)} & 3G_{32}^{(i)} & G_{42}^{(i)} & 7G_{13}^{(i)} & 3G_{23}^{(i)} & 12G_{33}^{(i)} & 4G_{43}^{(i)} & G_{14}^{(i)} & G_{24}^{(i)} & 4G_{34}^{(i)} & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \end{array} \right\| .$$

To show that this matrix is of rank 13 consider the following 13-th order determinant

$$(11.2)\qquad \left| \begin{array}{ccccccccccccc} G_{11}^{(i)} & G_{21}^{(i)} & G_{31}^{(i)} & G_{41}^{(i)} & G_{12}^{(i)} & G_{22}^{(i)} & G_{32}^{(i)} & G_{42}^{(i)} & G_{13}^{(i)} & G_{23}^{(i)} & G_{33}^{(i)} & G_{14}^{(i)} & G_{44}^{(i)} \\ 2G_{11}^{(i)} & 0 & 3G_{31}^{(i)} & G_{41}^{(i)} & 0 & -2G_{22}^{(i)} & G_{32}^{(i)} & -G_{42}^{(i)} & 3G_{13}^{(i)} & G_{23}^{(i)} & 4G_{33}^{(i)} & G_{14}^{(i)} & 0 \\ 3G_{11}^{(i)} & G_{21}^{(i)} & 7G_{31}^{(i)} & G_{41}^{(i)} & G_{12}^{(i)} & 3G_{22}^{(i)} & 3G_{32}^{(i)} & G_{42}^{(i)} & 7G_{13}^{(i)} & 3G_{23}^{(i)} & 12G_{33}^{(i)} & G_{14}^{(i)} & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \end{array} \right| .$$

Denote the matrix of the first four rows of (11.2) by a , that of the second four rows by b and the third four rows by c . After performing the elementary operations indicated below we find that, except for sign, the determinant (11.2) reduces to

$$(11.3)\quad \begin{array}{c} \left|\begin{array}{ccccccccccccc} -2G^{(i)}_{11} & 0 & 0 & -2G^{(i)}_{41} & 0 & 6G^{(i)}_{22} & 0 & 2G^{(i)}_{42} & 0 & 0 & 3G^{(i)}_{33} & -2G^{(i)}_{14} & -G^{(i)}_{44} \\ 2G^{(i)}_{11} & 0 & 3G^{(i)}_{31} & G^{(i)}_{41} & 0 & -2G^{(i)}_{22} & G^{(i)}_{32} & -G^{(i)}_{42} & 3G^{(i)}_{13} & G^{(i)}_{23} & 4G^{(i)}_{33} & G^{(i)}_{14} & 0 \\ -3G^{(i)}_{11} & G^{(i)}_{21} & -2G^{(i)}_{31} & -2G^{(i)}_{41} & G^{(i)}_{12} & 9G^{(i)}_{22} & 0 & 4G^{(i)}_{42} & -2G^{(i)}_{13} & 0 & 0 & -2G^{(i)}_{14} & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \end{array}\right| \end{array} \begin{array}{l} (e-2b-a) \\ (b) \\ (c-3b) \\ \end{array}$$

Expanding the last determinant by the elements of its last row we find

$$\left|\begin{array}{cccccccccccc} 0 & 0 & -2G^{(i)}_{41} & 0 & 6G^{(i)}_{22} & 0 & 2G^{(i)}_{42} & 0 & 0 & 3G^{(i)}_{33} & -2G^{(i)}_{14} & -G^{(i)}_{44} \\ 0 & 3G^{(i)}_{31} & G^{(i)}_{41} & 0 & -2G^{(i)}_{22} & G^{(i)}_{32} & -G^{(i)}_{42} & 3G^{(i)}_{13} & G^{(i)}_{23} & 4G^{(i)}_{33} & G^{(i)}_{14} & 0 \\ G^{(i)}_{21} & -2G^{(i)}_{31} & -2G^{(i)}_{41} & G^{(i)}_{12} & 9G^{(i)}_{22} & 0 & 4G^{(i)}_{42} & -2G^{(i)}_{13} & 0 & 0 & -2G^{(i)}_{14} & 0 \end{array}\right|$$

$$+\left|\begin{array}{cccccccccccc} -2G^{(i)}_{11} & 0 & 0 & -2G^{(i)}_{41} & 0 & 0 & 2G^{(i)}_{42} & 0 & 0 & 3G^{(i)}_{33} & -2G^{(i)}_{14} & -G^{(i)}_{44} \\ 2G^{(i)}_{11} & 0 & 3G^{(i)}_{31} & G^{(i)}_{41} & 0 & G^{(i)}_{32} & -G^{(i)}_{42} & 3G^{(i)}_{13} & G^{(i)}_{23} & 4G^{(i)}_{33} & G^{(i)}_{14} & 0 \\ -3G^{(i)}_{11} & G^{(i)}_{21} & -2G^{(i)}_{31} & -2G^{(i)}_{41} & G^{(i)}_{12} & 0 & 4G^{(i)}_{42} & -2G^{(i)}_{13} & 0 & 0 & -2G^{(i)}_{14} & 0 \end{array}\right|$$

$$+8\left|\begin{array}{cccccccccccc} -2G^{(i)}_{11} & 0 & 0 & -2G^{(i)}_{41} & 0 & 6G^{(i)}_{22} & 0 & 2G^{(i)}_{42} & 0 & 0 & -2G^{(i)}_{14} & -G^{(i)}_{44} \\ 2G^{(i)}_{11} & 0 & 3G^{(i)}_{31} & G^{(i)}_{41} & 0 & -2G^{(i)}_{22} & G^{(i)}_{32} & -G^{(i)}_{42} & 3G^{(i)}_{13} & G^{(i)}_{23} & G^{(i)}_{14} & 0 \\ -3G^{(i)}_{11} & G^{(i)}_{21} & -2G^{(i)}_{21} & -2G^{(i)}_{41} & G^{(i)}_{12} & 9G^{(i)}_{22} & 0 & 4G^{(i)}_{42} & -2G^{(i)}_{13} & 0 & -2G^{(i)}_{14} & 0 \end{array}\right|.$$

Making use of Lemma 9.2 we may show that the last expression is equal to

$$\left|2G^{(i)}_{41}\ \ 2G^{(i)}_{42}\ \ 2G^{(i)}_{14}\ \ G^{(i)}_{44}\right| \cdot \begin{vmatrix} 0 & 3G^{(i)}_{31} & 0 & -2G^{(i)}_{22} & G^{(i)}_{32} & 3G^{(i)}_{13} & G^{(i)}_{23} & 4G^{(i)}_{33} \\ G^{(i)}_{21} & -2G^{(i)}_{31} & G^{(i)}_{12} & 9G^{(i)}_{22} & 0 & -2G^{(i)}_{13} & 0 & 0 \end{vmatrix}$$

$$-\left|2G^{(i)}_{41}\ \ 2G^{(i)}_{42}\ \ 2G^{(i)}_{14}\ \ G^{(i)}_{44}\right| \cdot \begin{vmatrix} 2G^{(i)}_{11} & 0 & 3G^{(i)}_{31} & 0 & G^{(i)}_{32} & 3G^{(i)}_{13} & G^{(i)}_{23} & 4G^{(i)}_{33} \\ -3G^{(i)}_{11} & G^{(i)}_{21} & -2G^{(i)}_{31} & G^{(i)}_{12} & 0 & -2G^{(i)}_{13} & 0 & 0 \end{vmatrix}$$

$$+8\left|2G^{(i)}_{41}\ \ 2G^{(i)}_{42}\ \ 2G^{(i)}_{14}\ \ G^{(i)}_{44}\right| \cdot \begin{vmatrix} 2G^{(i)}_{11} & 0 & 3G^{(i)}_{31} & 0 & -2G^{(i)}_{22} & G^{(i)}_{32} & 3G^{(i)}_{13} & G^{(i)}_{13} \\ -3G^{(i)}_{11} & G^{(i)}_{21} & -2G^{(i)}_{31} & G^{(i)}_{12} & 9G^{(i)}_{22} & 0 & -2G^{(i)}_{13} & 0 \end{vmatrix} .$$

Making use of the Laplace expansion and Lemma 9.2 we find that the last expression is equal to

$$\begin{aligned} 8\left|G^{(i)}_{41}\ G^{(i)}_{42}\ G^{(i)}_{14}\ G^{(i)}_{44}\right| \Big\{ & 216\left|G^{(i)}_{31}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{33}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{12}\ G^{(i)}_{13}\ G^{(i)}_{22}\right| + \\ & + 216\left|G^{(i)}_{13}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{33}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{31}\ G^{(i)}_{12}\ G^{(i)}_{22}\right| - 72\left|G^{(i)}_{31}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{33}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{12}\ G^{(i)}_{13}\ G^{(i)}_{11}\right| \\ & - 72\left|G^{(i)}_{13}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{33}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{31}\ G^{(i)}_{12}\ G^{(i)}_{11}\right| - 864\left|G^{(i)}_{31}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{11}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{12}\ G^{(i)}_{13}\ G^{(i)}_{22}\right| \\ & - 864\left|G^{(i)}_{13}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{11}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{31}\ G^{(i)}_{12}\ G^{(i)}_{22}\right| + 288\left|G^{(i)}_{31}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{22}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{12}\ G^{(i)}_{13}\ G^{(i)}_{11}\right| \\ & + 288\left|G^{(i)}_{13}\ G^{(i)}_{32}\ G^{(i)}_{23}\ G^{(i)}_{22}\right| \cdot \left|G^{(i)}_{21}\ G^{(i)}_{31}\ G^{(i)}_{12}\ G^{(i)}_{11}\right| \Big\} . \end{aligned}$$

If we now use the factorization given in Lemma 9.1 we finally obtain that the value of (11.3) is

$$\begin{aligned} -3456g^3\left|G^{(i-1)}_{41}\ G^{(i-1)}_{42}\ G^{(i-1)}_{14}\ G^{(i-1)}_{44}\right| \Big\{ & \left|G^{(i-1)}_{31}\ G^{(i-1)}_{32}\ G^{(i-1)}_{23}\ G^{(i-1)}_{33}\right| \cdot \left|G^{(i-1)}_{21}\ G^{(i-1)}_{12}\ G^{(i-1)}_{13}\ G^{(i-1)}_{22}\right| + \\ & + \left|G^{(i-1)}_{13}\ G^{(i-1)}_{32}\ G^{(i-1)}_{23}\ G^{(i-1)}_{33}\right| \left|G^{(i-1)}_{21}\ G^{(i-1)}_{31}\ G^{(i-1)}_{12}\ G^{(i-1)}_{22}\right| \Big\}, \end{aligned} \tag{11.4}$$

where g is the determinant of the matrix $\| G_{ij} \|$. That the expression (11.4) is not identically zero can be shown by taking

$$\| G_{ij} \| = \begin{Vmatrix} 1 & 1 & -1 & 2 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \end{Vmatrix} ,$$

and (11.4) becomes

$$-3456\cdot(-2)^3\cdot 2\left[0+1\cdot(-7)\right]=-3456\cdot 8\cdot 2\cdot 7\neq 0.$$

We have thus our final result for $n=4$,

Theorem 11.1 . Let

$$y_i''+\sum_{j=1}^{4}L_{ij}(x)y_j'+\sum_{j=1}^{4}M_{ij}(x)y_j=0,$$

be a system of four linear homogeneous differential equations of the second order. Suppose that the variables y_i are transformed by means of

$$y_i=\eta_i+\sum_{j=1}^{4}K_{ij}(x)\eta_j. \tag{11.5}$$

Then, if we define

$$G_{ij}=2L_{ij}'-4M_{ij}+L_{ij}^{(2)},$$

and

$$U_{ij}=2G_{ij}'+\sum_{m=1}^{4}\left(L_{im}G_{mj}-G_{im}L_{mj}\right),$$

the traces

$$\sum_{i=1}^{4}\sum_{m=1}^{4}G_{im}^{(l)}U_{mi}^{(k)},\quad (l=1,2,3,4;\ k=0,1,2,3),$$

$$\sum_{i=1}^{4}U_{ii}^{(4)},$$

are a system of seventeen independent functions of G_{11}, G_{12}, ..., G_{44}, U_{11}, U_{12}, ..., U_{44}, which remain unchanged by the transformation (11.5); and any seminvariant depending upon L_{ij}, L_{ij}', L_{ij}'', M_{ij}, M_{ij}' may be expressed functionally in terms of these seventeen traces.

12. <u>The Effect of a Transformation of the Independent Variable Upon the Seminvariants.</u>

The invariants of the system (1.1) of linear differential equations must obviously be functions of the seminvariants, namely such functions of the seminvariants as are left unchanged by an arbitrary transformation of the independent variable x. In order to determine the invariants it is necessary to find the effect of such a transformation upon the seminvariants. We do this in the present section.

Let (1.1) be the given system as before. If we introduce a new independent variable

(12.1) $$\xi = \xi(x),$$

where $\xi(x)$ is an arbitrary function having derivatives up to and including the third order on $x_1 \leqq x \leqq x_2$, then (1.1) becomes

$$\frac{d^2 y_i}{d\xi^2} + \sum_{j=1}^{n} \bar{L}_{ij} \frac{d y_j}{d\xi} + \sum_{j=1}^{n} \bar{M}_{ij} y_j = 0,$$

where

(12.2) $$\bar{L}_{ij} = \frac{1}{\xi'}\left(\alpha \delta_{ij} + L_{ij}\right), \quad \bar{M}_{ij} = \frac{1}{(\xi')^2} M_{ij},$$

and where α is given by

(12.3) $$\alpha = \frac{\xi''}{\xi'}.$$

We find from (12.2)

(12.4) $$\frac{d\bar{L}_{ij}}{d\xi} = \frac{1}{(\xi')^2}\left[L'_{ij} - \alpha L_{ij} + (\alpha' - \alpha^2)\delta_{ij}\right],$$

and hence using the definition

$$\bar{G}_{ij} = 2\frac{d\bar{L}_{ij}}{d\xi} - 4\bar{M}_{ij} + \bar{L}_{ij}^{(2)},$$

we have

(12.5) $$\bar{G}_{ij} = \frac{1}{(\xi')^2}\left(G_{ij} + 2\mu\,\delta_{ij}\right),$$

where

(12.6) $$\mu = \alpha' - \frac{\alpha^2}{2} = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2,$$

is the so-called <u>Schwarzian derivative</u>. We find at once from (12.5)

$$\bar{G}_{ij}^{(\ell)} = \frac{1}{(\xi')^{2\ell}}\Big\{G_{ij}^{(\ell)} + 2\binom{\ell}{1}\mu G_{ij}^{(\ell-1)} + 2^2\binom{\ell}{2}\mu^2 G_{ij}^{(\ell-2)} + \dots + 2^p\binom{\ell}{p}\mu^p G_{ij}^{(\ell-p)}$$
$$+ \dots + 2^{\ell-1}\binom{\ell}{\ell-1}\mu^{\ell-1}G_{ij} + 2^\ell\mu^\ell\delta_{ij}\Big\}.$$

Denoting the trace $\sum_{i=1}^{n}\bar{G}_{ii}^{(\ell)}$ by $\bar{t}_\ell$ and $\sum_{i=1}^{n}G_{ii}^{(\ell)}$ by t_ℓ we have

(12.7) $$\bar{t}_\ell = \frac{1}{(\xi')^{2\ell}}\Big\{t_\ell + 2\binom{\ell}{1}\mu t_{\ell-1} + 2^2\binom{\ell}{2}\mu^2 t_{\ell-2} + \dots + 2^p\binom{\ell}{p}\mu^p t_{\ell-p}$$
$$+ \dots + 2^{\ell-1}\binom{\ell}{\ell-1}\mu^{\ell-1}t_1 + 2^\ell\mu^\ell n\Big\}.$$

The traces t_ℓ are the seminvariants[8] of (1.1) under the transformation (1.2), depending upon the arguments L_{ij}, L'_{ij} and M_{ij}. It was also shown there, that the n traces t_ℓ, (ℓ=1,2, ..., n), are functionally independent and that any seminvariant of (1.1) involving L_{ij}, L'_{ij} and M_{ij} may be expressed in terms of these n traces.

From (12.7) we may deduce the following independent re-

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[8] <u>Reingold</u>, loc. cit.

lative invariants[9]

[9] cf. Dickson, loc. cit., p.375.

$$\theta_4 = t_1^2 - n t_2 \quad ,$$
$$\theta_6 = t_1^3 - \frac{3n}{2} t_1 t_2 + \frac{n^2}{2} t_3 \ ,$$
$$\theta_8 = t_1^4 - 2n t_1^2 t_2 + \frac{4n^2}{3} t_1 t_3 - \frac{n^3}{3} t_4 ,$$

since we may readily verify that

$$\bar{\theta}_\nu = \frac{1}{(\xi')^\nu} \theta_\nu \ , \ (\nu = 4, 6, 8).$$

From these relative invariants we obtain at once the two absolute invariants $\frac{\theta_4^2}{\theta_8}$ and $\frac{\theta_4 \theta_8}{\theta_6^2}$, which are obviously independent.

For the purpose of determining the number of functionally independent invariants, we set up the infinitesimal transformations. We now regard the $\xi(x)$ as depending not only on the variable x but also on a single parameter a, so that for a fixed x the equation (12.1) may be considered as a one-parameter group of transformations taking x into $\bar{x}$. Let $a = 0$ be the parameter giving the identity transformation, i.e. $\left[\xi(x)\right]_{a=0} = x$.

The infinitesimal transformation of (12.1) is defined by

$$\frac{d\,\xi(x,a)}{da} = \varphi(\xi) \quad , \quad \xi(x,0) = x ,$$

where φ is an arbitrary function having derivatives up to and including those of the fourth order. We have at once

$$\left[\xi'\right]_{a=0}=1,\ \left[\xi''\right]_{a=0}=0,\ \left[\xi'''\right]_{a=0}=0,\ \left[\frac{d\xi'}{da}\right]_{a=0}=\varphi'(x),\ \left[\frac{d\xi''}{da}\right]_{a=0}=\varphi'',\ \left[\frac{d\xi'''}{da}\right]_{a=0}=\varphi'''.$$

Hence from (12.3) and (12.6) we have

$$\left[\alpha\right]_{a=0}=0,\quad \left[\alpha'\right]_{a=0}=0,\quad \left[\mu\right]_{a=0}=0,\quad \left[\mu'\right]_{a=0}=0,$$

$$\left[\frac{d\alpha}{da}\right]_{a=0}=\varphi'',\ \left[\frac{d\alpha'}{da}\right]_{a=0}=\varphi''',\ \left[\frac{d\mu}{da}\right]_{a=0}=\varphi''',\ \left[\frac{d\mu'}{da}\right]_{a=0}=\varphi^{IV}.$$

Making use of these expressions we shall now find the infinitesimal transformations induced by (12.1) in the coefficients L_{ij} and M_{ij} and in the seminvariants.

We have then from (12.2)

$$\frac{dL_{ij}}{da}=-\varphi' L_{ij}+\varphi''\delta_{ij}\quad,\quad \frac{dM_{ij}}{da}=-2\varphi' M_{ij},$$

and from (12.4)

$$\left[\frac{d}{da}\left(\frac{d\bar{L}_{ij}}{d\xi}\right)\right]_{a=0}=-2\varphi' L'_{ij}-\varphi'' L_{ij}+\varphi'''\delta_{ij}.$$

Hence from the definition of G_{ij} we get

$$\frac{dG_{ij}}{da}=2\left(-\varphi' G_{ij}+\varphi''\delta_{ij}\right),$$

and more generally

$$\text{(12.8)}\qquad \frac{dG^{(\ell)}_{ij}}{da}=2\ell\left(-\varphi' G^{(\ell)}_{ij}+\varphi'''G^{(\ell-1)}_{ij}\right).$$

Using (12.5) we find

$$\left[\frac{d}{da}\left(\frac{d\bar{G}_{ij}}{d\xi}\right)\right]_{a=0}=-3\varphi' G'_{ij}-2\varphi'' G_{ij}+2\varphi^{IV}\delta_{ij},$$

and hence

$$\frac{dU_{ij}}{da} = -3\varphi' U_{ij} - 4\varphi'' G_{ij} + 4\varphi^{IV}\delta_{ij} ,$$

(12.9) $$\frac{dU_{ij}^{(k)}}{da} = -3k\varphi' U_{ij}^{(k)} - 4\varphi'' \sum_{p=0}^{k-1}\sum_{m=1}^{n}\sum_{s=1}^{n} U_{im}^{(p)} G_{ms} U_{sj}^{(k-p-1)} + 4k\varphi^{IV} U_{ij}^{(k-1)} .$$

From (12.8) and (12.9) it follows that

$$\frac{d}{da}\sum_{m=1}^{n} G_{im}^{(\ell)} U_{mj}^{(k)} = -(2\ell+3k)\varphi' \sum_{m=1}^{n} G_{im}^{(\ell)} U_{mj}^{(k)} - 4\varphi'' \sum_{p=0}^{k-1}\sum_{m=1}^{n}\sum_{s=1}^{n}\sum_{r=1}^{n} G_{im}^{(\ell)} U_{ms}^{(p)} G_{sr} U_{rj}^{(k-p-1)}$$
$$+ 2\ell\varphi''' \sum_{m=1}^{n} G_{im}^{(\ell-1)} U_{mj}^{(k)} + 4k\varphi^{IV} \sum_{m=1}^{n} G_{im}^{(\ell)} U_{mj}^{(k-1)} .$$

Denoting the trace $\sum_{i=1}^{n}\sum_{m=1}^{n} G_{im}^{(\ell)} U_{mi}^{(k)}$ by $T_{\ell,k}$ we have

$$\frac{dT_{\ell,k}}{da} = -(2\ell+3k)\varphi' T_{\ell,k} + 2\ell\varphi''' T_{\ell-1,k} + 4k\varphi^{IV} T_{\ell,k-1}$$

(12.10) $$- 4\varphi'' \sum_{p=0}^{k-1}\sum_{i=1}^{n}\sum_{m=1}^{n}\sum_{s=1}^{n}\sum_{r=1}^{n} G_{im}^{(\ell)} U_{ms}^{(p)} G_{sr} U_{ri}^{(k-p-1)} .$$

For $k = 0$ the last expression reduces to

(12.11) $$\frac{dt_\ell}{da} = 2\ell\left(-\varphi' t_\ell + \varphi''' t_{\ell-1}\right) .$$

The last two equations furnish us with the infinitesimal transformations induced by (12.1) in the seminvariants. Equation (12.11) gives us the transformation of the seminvariants involving only the arguments L_{ij}, L'_{ij} and M_{ij}, and (12.10) of the seminvariants depending only upon L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij}.

13. The Number of Invariants.

In the present section we first find the number of all independent invariants of (1.1) involving only the arguments L_{ij}, L'_{ij} and M_{ij} for any n. Next we proceed to find the number of invariants depending only upon L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij}. Since the independence of the seminvariants $T_{\ell,\kappa}$ has been proved only for $n=2$, $n=3$ and $n=4$ (sections 10 and 11), the determinantion of the number of independent invariants involving L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} will be confined to the same values of n.

Let any function $F(t_1, t_2, \ldots, t_n)$ be an arbitrary absolute invariant involving only L_{ij}, L'_{ij} . [redacted] and M_{ij}. Then the expression

$$\frac{dF}{d\alpha} = \sum_{\ell=1}^{n} \frac{\partial F}{\partial t_\ell} \cdot \frac{dt_\ell}{d\alpha} , \tag{13.1}$$

must vanish for all values of the arbitrary function $\varphi(x)$ and its derivatives. Using (13.1) and (12.11) and equating to zero the coefficients of $\varphi'(x)$ and $\varphi'''(x)$ we obtain

$$\begin{aligned} &\sum_{\ell=1}^{n} \ell\, t_\ell \frac{\partial F}{\partial t_\ell} = 0 , \\ &\sum_{\ell=1}^{n} \ell\, t_{\ell-1} \frac{\partial F}{\partial t_\ell} = 0 . \end{aligned} \tag{13.2}$$

The two equations (13.2) form a complete system of which every solution is an absolute invariant, and conversely every invariant depending upon L_{ij}, L'_{ij} and M_{ij} is a solution of (13.2). These equations are obviously independent and hence the number of all independent absolute invariants is n-2.

Since an absolute invariant is the quotient of two relative invariants of the same weight, we have then the following

<u>Theorem 13.1 . If a system of n linear homogeneous differential equations of the second order</u>

$$y_i'' + \sum_{j=1}^{n} L_{ij} y_j' + \sum_{j=1}^{n} M_{ij} y_j = 0,$$

<u>be transformed by means of</u>

$$y_i = \eta_i + \sum_{j=1}^{n} K_{ij} \eta_j, \quad \xi = \xi(x),$$

<u>then the number of all functionally independent relative invariants involving the arguments L_{ij} , L'_{ij} and M_{ij} is n-1.</u>

For $n=4$, these relative invariants are given by Θ_4, Θ_6 and Θ_8 of the previous section.

We now proceed to find the number of all independent invariants involving L_{ij} , L'_{ij} , L''_{ij} , M_{ij} and M'_{ij} for the case $n=4$. In view of Theorem 11.1 we know that any arbitrary invariant involving L_{ij} , L'_{ij} , L''_{ij} , M_{ij} and M'_{ij} is a function F of the seventeen traces $T_{\ell,\kappa}$ ($\ell=1,2,3,4$; $\kappa=0,1,2,3$) and T_{04}. Hence the expression

$$\frac{dF}{da} = \sum_{\ell=1}^{4}\sum_{\kappa=1}^{3} \frac{\partial F}{\partial T_{\ell,\kappa}} \frac{dT_{\ell,\kappa}}{da} + \sum_{\ell=1}^{4} \frac{\partial F}{\partial T_{\ell,0}} \frac{dT_{\ell,0}}{da} + \frac{\partial F}{\partial T_{0,4}} \frac{dT_{0,4}}{da}, \tag{13.3}$$

must vanish for all values of the arbitrary function $\varphi(x)$ and its derivatives. Substituting (12.10) into (13.3) and equating to zero the coefficients of $\varphi'(x)$, $\varphi''(x)$, $\varphi'''(x)$ and $\varphi^{IV}(x)$ we obtain

$$\sum_{\ell=1}^{4}\sum_{\kappa=1}^{3} (2\ell+3\kappa) T_{\ell,\kappa} \mathcal{T}_{\ell,\kappa} + 2\sum_{\ell=1}^{4} \ell\, T_{\ell,0} \mathcal{T}_{\ell,0} + 12\, T_{0,4} \mathcal{T}_{0,4} = 0,$$

$$\sum_{\ell=1}^{4}\sum_{\kappa=1}^{3}\sum_{p=0}^{\kappa-1}\sum_{i=1}^{4}\sum_{m=1}^{4}\sum_{s=1}^{4}\sum_{r=1}^{4} G_{im}^{(\ell)} U_{ms}^{(p)} G_{sr} U_{ri}^{(\kappa-p-1)} \mathcal{T}_{\ell,\kappa} + 4 T_{1,3} \mathcal{T}_{0,4} = 0, \tag{13.4}$$

$$\sum_{\ell=1}^{4}\sum_{\kappa=1}^{3} \ell\, T_{\ell-1,\kappa}\, \mathcal{J}_{\ell,\kappa} + \sum_{\ell=1}^{4} \ell\, T_{\ell-1,0}\, \mathcal{J}_{\ell,0} = 0,$$

$$\sum_{\ell=1}^{4}\sum_{\kappa=1}^{3} \kappa\, T_{\ell,\kappa-1}\, \mathcal{J}_{\ell,\kappa} + 4\, T_{0,3}\, \mathcal{J}_{0,4} = 0,$$

where $\mathcal{J}_{ij}$ stands for $\dfrac{\partial F}{\partial T_{i,j}}$.

The above four equations form a complete system of which every solution is an absolute invariant and conversely every invariant depending upon L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} is a solution of (13.4). We can readily show that these four equations are independent by considering some non-identically vanishing four by four determinant of its matrix of coefficients. For, we see that in this matrix the determinant of the coefficients of $\mathcal{J}_{11}$, $\mathcal{J}_{10}$, $\mathcal{J}_{20}$ and $\mathcal{J}_{04}$ is

$$\begin{vmatrix} 5T_{11} & 2T_{1,0} & 4T_{2,0} & 12T_{0,4} \\ T_{2,0} & 0 & 0 & 4T_{1,3} \\ T_{0,1} & T_{0,0} & 2T_{1,0} & 0 \\ T_{1,0} & 0 & 0 & 4T_{0,3} \end{vmatrix},$$

so that if we set $U_{ij} = \delta_{ij}$, for all i and j, and $G_{ij} = 0$, $i \neq j$, $G_{22} = G_{33} = G_{44} = 0$ and $G_{11} = 1$, we find at once that the value of this determinant is 144.

Since the equations of (13.4) are independent the number of all independent absolute invariants is $17-4=13$ (4^2+1-4). We thus have the following result for $n=4$,

<u>Theorem 13.2. If a system of four linear homogeneous differential equations of the second order</u>

$$y_i'' + \sum_{j=1}^{4} L_{ij} y_j' + \sum_{j=1}^{4} M_{ij} y_j = 0,$$

be transformed by means of

$$y_i = \eta_i + \sum_{j=1}^{4} K_{ij} \eta_j , \quad \xi = \xi(x),$$

then the number of all functionally independent relative invariants involving the arguments L_{ij}, L'_{ij}, L''_{ij}, M_{ij} and M'_{ij} is 14 ($4^2+1-4+1$).

Applying Theorem 10.1 we find in an exactly analogous fashion a corresponding result for $n=3$, where instead of 14 relative invariants we now have 7 ($3^2+1-4+1$). For the case $n=2$ the number of relative invariants[10] is 2 ($2^2+1-4+1$).

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[10] cf. Wilczynski, loc. cit., pp. 110-113.

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