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I hereby recommend that the thesis prepared under my supervision by KAISER SCHOEN KUNZ

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ON THE CLASSICAL THEORY OF RADIATING ELECTRONS

This paper is an attempt to overcome some of the natural objections to Dirac's classical theory of the radiating electron and still preserve its many attractive features. The change necessary to achieve this will be shown to be slight, but effective in bringing about a treatment more in line with what one would expect in an extension of classical theory.

From classical electrodynamics one obtains the equation of motion of a test-particle in an electromagnetic field, namely the Lorentz Force equation

$$\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{H}$$

where $\vec{E}$ is the electric, $\vec{H}$ the magnetic vector, and e the charge of the particle. By a test-particle is meant a particle with a charge so small that it will not appreciably affect the field. Since the electron is a finite charge and does modify the existing field, one cannot apply to it the above equation of motion without modification.

One method for finding the motion is to consider the electron to be located at a point, and assume that it moves in the first approximation as a test-particle,

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i. e., under the action of the external field alone. The energy and momentum radiated by an electron with this motion is then calculated, and a damping force added to account for it. The motion with such a damping force can then be found, and this motion will in general more closely approximate the required motion. The approximation can be taken further by finding the radiation from this corrected motion, correcting the damping force to account for it, and calculating the motion under this new force. Thus one can find the motion to any degree of approximation, provided the method is valid; thich is only the case if the correction terms are small. This means that the method can only be applied when the acceleration is not too large.

The second method is to consider the electron to be extended in space, but the situation is not thereby materially improved. Here the equation of motion is obtained by requiring that the electron moves in such a way that the total force acting on it due to its own field and the external field is zero. Large accelerations are again excluded, in this case because the electron is unstable for such accelerations. Thus both treatments break down as soon as the acceleration is large.

There are several reasons for preferring the point model for an electron. In the first place, being an elementary particle it should not be necessary to consider the motion of its parts relative to one another. This means that it should either be considered as a point charge, or at least if it is to be given extension, it should be rigid. On the other hand one cannot have a perfectly rigid body in relativity theory, because a signal could be sent through such a body with a speed greater than the speed of light. Secondly, if one does consider the electron as made up of parts which can change their configuration, one still has to make a large number of ad hoc assumptions about the charge density throughout the electron. Moreover, such a model, being made up of like charges, must be held together by non-electromagnetic forces. This will complicate the picture, since the electron will then absorb and give out energy whenever it is distorted.

It may be well to inquire as to the form of a satisfactory equation of motion for a radiating electron. In the first place it should have only the external field coming in explicitly, since with a definite mass and charge for the electron the motion is completely determined by this field. Also since the effect of the electron on the field is proportional to the charge e, and the effect of this field on a charge is proportional to e; it is obvious that the reaction of the electron on itself should be proportional to e^2. Therefore it would seem that the equation of motion should be the regular Lorentz Force equation plus terms in e^2 to account for this reaction. These terms will drop out in comparison to the other terms when e is sufficiently small, and the equation will become that for a test-particle.

Dirac's theory will be seen to have these properties, and will be seen to involve relatively few assumptions in its derivation.

Consider an electron of charge e in the four-dimensional space of special relativity, and let

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict$$

be the coordinates of points in this space. Let $z_\mu = z_\mu(s)$ be the coordinates of the electron, s being the interval along the world-line of the electron, that is

$$ds^2 = - dz_\mu^2$$

Also let

$$u_\mu = \dot{z}_\mu = \frac{dz_\mu}{ds} \qquad \dot{u}_\mu = \frac{du_\mu}{ds}$$

The four-potential is given by

$$A_\mu = (\vec{A}, i\varphi)$$

$\vec{A}$ being the vector potential, and φ the scalar potential. By requiring them to satisfy the condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0$$

each component satisfies the wave-equation

$$\nabla^2 A_\mu - \frac{1}{c}\frac{\partial^2 A_\mu}{\partial t^2} = -4\pi j_\mu \qquad (2.)$$

where j_μ is the charge-density four-vector, that is

$$j_\mu = \left(\frac{\rho \vec{v}}{c}, \ i\rho \right)$$

A general solution of this linear non-homogeneous equation is given by a particular solution plus the general solution of the homogeneous equation, i. e., with the right-hand side set equal to zero. A particular solution of this equation is given by the retarded potentials, which in a given coordinate system are expressed as

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\rho \vec{v}(\vec{r}\,', t')}{|\vec{r} - \vec{r}\,'|}\, dV \qquad t' = t - \frac{|\vec{r} - \vec{r}\,'|}{c}$$

$$\varphi(\vec{r}) = \int \frac{\rho(\vec{r}\,', t')}{|\vec{r} - \vec{r}\,'|}\, dV$$

where $\vec{r}$ and $\vec{r}\,'$ are position vectors and the integration is to be taken over the whole three-dimensional space. We see that the charge and current densities at $\vec{r}\,'$ are taken at a time t' which is such that the disturbance from $\vec{r}\,'$ will arrive at $\vec{r}$ at the time t. Since c comes into equation (2.) as c^2, if we change c to $-c$ in the above equations we will also have a solution. The potentials thus obtained are called the advanced potentials, since

$$t' = t + \frac{|\vec{r} - \vec{r}\,'|}{c}$$

indicating that the charge and current density is to be taken at a time t' later than t by an amount equal to the time for light to travel the distance $|\vec{r} - \vec{r}'|$.

Corresponding to the potentials A_μ, there is the field-strength tensor $F_{\mu\nu}$, defined by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

where $F_{\mu\nu}$ is related to $\vec{E}$ and $\vec{H}$ thus

$$F_{4k} = i\, E_k, \quad k = 1, 2, 3$$

$$F_{12} = H_3, \quad F_{23} = H_1, \quad F_{31} = H_2$$

also the retarded field, $F_{\mu\nu,ret}$, can be defined as

$$F_{\mu\nu,ret} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

The actual field can therefore be expressed as

$$F_{\mu\nu,\ act} = F_{\mu\nu,\ ret} + F_{\mu\nu,\ in}$$

where the incident field, $F_{\mu\nu,in}$, is the field that would be there without the particle, and hence might be thought of as the external field in which the particle finds itself. It is therefore the $F_{\mu\nu}$ corresponding to certain potentials satisfying the homogeneous equation.

Also one can use the advanced potentials as a particular solutions and add a suitable solution of the homogeneous equation. In this case the actual field is given by.

$$F_{\mu\nu,\ act} = F_{\mu\nu,\ adv} + F_{\mu\nu,\ out}$$

which defines $F_{\mu\nu,\ out}$, termed the outgoing field.

Dirac then defines $F_{\mu\nu,\ rad}$ by

$$F_{\mu\nu,\ rad} = F_{\mu\nu,\ out} - F_{\mu\nu,\ in} \qquad (3.)$$

or

$$F_{\mu\nu,\ rad} = F_{\mu\nu,\ ret} - F_{\mu\nu,\ adv}$$

which he interprets as being the radiation field; but as this assumption is not necessary in his derivation of his equation of motion, one can consider it merely as a mathematical definition of $F_{\mu\nu,\ rad}$. He also defines $f_{\mu\nu}$ as

$$f_{\mu\nu} = \frac{1}{2}\ (F_{\mu\nu,\ in} + F_{\mu\nu,\ out})$$

which by the equations above can be transformed as follows

$$f_{\mu\nu} = F_{\mu\nu,\ in} + \frac{1}{2}\ (F_{\mu\nu,\ out} - F_{\mu\nu,\ in})$$

$$= F_{\mu\nu,\ in} + \frac{1}{2}\ F_{\mu\nu,\ rad}$$

Calculating the $F_{\mu\nu,\ rad}$ by taking the difference between the retarded and advanced fields he finds it to be finite on the world-line and having the value

$$F_{\mu\nu,\ rad} = - \frac{4e}{3}\ (\ddot{u}_\mu u_\nu - \ddot{u}_\nu u_\mu)$$

From this $f_{\mu\nu}$ on the world-line becomes

$$f_{\mu\nu} = F_{\mu\nu,\ in} - \frac{2e}{3}\ (\ddot{u}_\mu u_\nu - \ddot{u}_\nu u_\mu) \qquad (4.)$$

Using the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi i c}(F_{\mu\varrho} F_{\nu\varrho} - \frac{1}{4} F^2_{\alpha\beta}\ \delta_{\mu\nu})$$

Dirac calculates the flow of energy and momentum through a cylinder about the world-line of the electron. The cylinder is such that any cross-section taken perpendicular to the proper-time is a sphere about the electron of radius ϵ, ϵ being taken very small. The flow of energy and momentum out of the tube between the points s_1 and s_2 is given by

$$\int_{s_1}^{s_2} \left(\frac{1}{2} e^2 \epsilon^{-1} \dot{u}_\mu - e f_{\mu\nu} u_\nu \right) ds \qquad (5.)$$

provided one keeps only the terms that do not vanish with ϵ.

In order for this integral to depend solely on the conditions at the end points, i. e., on the energy and momentum at the times t_1 and t_2 corresponding to s_1 and s_2, the integrand must be a total derivative of s, say $\dot{B}_\mu$. If the momentum and energy in a sphere about the electron of radius ϵ, ϵ approaching zero, is to depend linearly on the four-velocity u_μ, B_μ must be given by

$$B_\mu = k\, u_\mu$$

From this follows

$$k\, \dot{u}_\mu = 1/2\ e^2 \epsilon^{-1} \dot{u}_\mu - e\, f_{\mu\nu} u_\nu$$

or

$$\frac{e\, f_{\mu\nu}\, u_\nu}{\dot{u}_\mu} = 1/2\ e^2 \epsilon^{-1} - k$$

In order for the right-hand side to be finite as ϵ approaches zero, k must be an infinite constant. The difference fo these terms may be designated by a constant **b**, of dimensions of energy and characteristic of the particle. Therefore the left-hand side of the equation must be constant; thus requiring the four-acceleration, $\dot{u}_\mu$, to be proportional to the Lorentz Force due to the field $f_{\mu\nu}$. Hence

$$b\,\dot{u}_\mu = e\, f_{\mu\nu}\, u_\nu \qquad (6.)$$

Since the equation of motion should be in terms of the incident field, equation (4.) will be used to substitute for $f_{\mu\nu}$

$$b\dot{u}_\mu = e\left(F_{\mu\nu,\,in} - \frac{2e}{3}\,(\ddot{u}_\mu u_\nu - \ddot{u}_\nu u_\mu)\right) u_\nu$$

$$= e\, F_{\mu\nu,\,in}\, u_\nu - \frac{2}{3}\, e^2 \ddot{u}_\mu u_\nu^2 + \frac{2}{3}\, e^2 (\ddot{u}_\nu u_\nu)\, u_\mu$$

It is convenient to introduce the auxillary equations for the u_μ, $\dot{u}_\mu$, and $\ddot{u}_\mu$ from the definition of the four-velocity u_μ

$$u_\mu^2 = \frac{dz_\mu^2}{ds^2} = -1 \quad \text{since } ds^2 = -dz_\mu^2 \qquad (7.)$$

Differentiating with respect to s, gives

$$u_\mu \dot{u}_\mu = 0 \qquad (8.)$$

and differentiating again, gives one

$$\dot{u}_\mu^2 + u_\mu \ddot{u}_\mu = 0 \qquad (9.)$$

Thus by substituting for u^2 and $u_\mu \ddot{u}_\mu$ and transposing terms, the equation becomes

$$b\dot{u}_\mu - \frac{2}{3}\, e^2\, \ddot{u}_\mu + \frac{2}{3}\, e^2\, \dot{u}_\alpha^2 u_\mu = e\, u_\nu\, F_{\mu\nu,\,in}$$

If b is taken as mc^2, one gets Dirac's equation

$$mc^2 \dot{u}_\mu - \frac{2}{3} e^2 \ddot{u}_\mu + \frac{2}{3} e^2 \dot{u}_\alpha^2 u_\mu = e\, u_\nu F_{\mu\nu}, \text{ in} \quad (10.)$$

This equation has the beautiful feature of involving only the external field acting on the electron and taking into account the effect of radiation upon the electron by terms in e^2.

Dividing the equation by mc^2 and letting

$$a = \frac{2e^2}{3mc^2} \qquad F_{\mu\nu} = F_{\mu\nu}, \text{ in}$$

Dirac's equation can be written

$$\dot{u}_\mu - a\ddot{u}_\mu + a u_\mu \dot{u}_\alpha^2 = \frac{e}{mc^2} F_{\mu\nu} u_\nu$$

Thus for field-free space

$$\dot{u}_\mu - a(\ddot{u}_\mu - \dot{u}_\alpha^2 u_\mu) = 0$$

If a coordinate system is chosen in which the initial velocity and acceleration are along the x-axis, this equation has the solution

$$\dot{x} = \sinh (b e^{\pm \frac{s}{a}} + h)$$

$$\dot{t} = \frac{1}{c} \cosh (b e^{+\frac{s}{a}} + h)$$

or

$$v = \frac{dx}{dt} = \frac{\dot{x}}{\dot{t}} = c \tanh (b e^{+\frac{s}{a}} + h)$$

Thus since 1/a is a very large number -- the reciprocal of the classical radius of the electron --, the electron will speed up very rapidly to the velocity of light except in those solutions in which b = 0. Since Dirac's equation is of the third degree, three independent parameters must be specified; and so one has, after specifying the initial position and velocity of the electron, still a one-parameter family of solutions. In this case he chooses from this family the solution that has a zero final acceleration. This means that b = 0, which makes $\vec{v}$ a constant. In general, Dirac states, one must always specify, together with the initial position and velocity, the final acceleration.

It would seem then that the equation of motion is incomplete and has to be supplemented by one's experience or judgement as to the correct final acceleration in each problem. Besides the obvious theoretical objections to such a precedure, there arises also the practical difficulty of deciding between two or more possible behaviors corresponding to satisfactory final accelerations.

The question then arises as to whether it is possible to modify Dirac's equation so that the electron's behavior will depend only on its past and immediate condition and not on what may lie ahead for it -- as will be the case if we choose that motion which has a desired final acceleration. Such an equation would be complete in that the motion would be fully determined by the initial conditions. The necessity of specifying the initial acceleration would then be seen as a reasonable generalization from the classical equation of a test-particle to the equation of motion of the electron, which is influenced by its own acceleration.

Pryce in a paper* following Dirac's shows that by modifying the energy-momentum tensor but keeping it in line with the requirement for this tensor in classical electrodynamics, he comes out with a finite value for the energy of the electromagnetic field of an electron. By choosing a special form of this tensor and equating the flow of energy through an infinitesimal sphere about the electron to the decrease in energy and momentum within, he arrives at Dirac's equation of motion.

He states however, that the modified tensor is arbitrary to the extent of an added term with a coefficient λ. If this term is introduced, the rate of flow of energy and momentum, R_μ, in through the sphere is found to be

$$R_\mu = \left(\frac{2}{3} - \frac{\lambda}{12}\right) e^2 \ddot{u}_\mu - \left(\frac{2}{3} - \frac{\lambda}{6}\right) e^2 \dot{u}_\alpha^2 u_\mu + e\, F_{\mu\nu,\ in}\, u_\nu$$

and therefore if we are to have conservation of energy and momentum, this must be equal to the rate of change of these within the sphere. The electric part goes to

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* Proceedings of the Royal Society, Vol. 169, pp. 389-401, Nov. 7, 1938.

zero as ϵ approaches zero; hence we have that R_μ equals the increase in the mechanical energy and momentum of the particle, that is

$$R_\mu = \dot{p}_\mu \qquad (11.)$$

Now

$$u_\mu R_\mu = (\tfrac{2}{3} - \tfrac{\lambda}{12})e^2 u_\mu \ddot{u}_\mu - (\tfrac{2}{3} - \tfrac{\lambda}{6})e^2 \dot{u}_\alpha^2 u_\mu^2 + e u_\mu F_{\mu\nu,\text{in}}\, u_\nu$$

or since

$$u_\mu^2 = -1, \quad \dot{u}_\mu^2 = u_\mu u_\mu, \quad F_{\mu\nu} = -F_{\nu\mu}$$

one has that

$$u_\mu R_\mu = \frac{\lambda}{12} u_\mu \ddot{u}_\mu = u_\mu \dot{p}_\mu$$

which requires that

$$\dot{p}_\mu = \frac{\lambda}{12} \ddot{u}_\mu - L_\mu$$

where $u_\mu L_\mu = 0$. The simplest choice of L_μ is $L_\mu = h\,\dot{u}_\mu$, h being a constant; for which

$$p_\mu = h\,u_\mu + \frac{\lambda}{12}\dot{u}_\mu$$

and setting $\dot{p}_\mu = R_\mu$, one obtains the equation of motion

$$h\dot{u}_\mu - (\tfrac{2}{3} - \tfrac{\lambda}{6})e^2 \ddot{u}_\mu + (\tfrac{2}{3} - \tfrac{\lambda}{6})e^2 \dot{u}_\alpha^2 u_\mu = eF_{\mu\nu,\text{in}}\, u_\nu \qquad (12.)$$

This should reduce to the test-particle equation when the terms in e^2 are dropped. Since the test-particle equation can be written as

$$mc^2 \dot{u}_\mu = e\,F_{\mu\nu,\text{ in}}\, u_\nu$$

h must be equal to mc^2. Now if instead of putting $\lambda = 0$ and getting Dirac's equation, one puts $\lambda = 8$

$$mc^2\dot{u}_\mu + \frac{2}{3} e^2\ddot{u}_\mu - \frac{2}{3} e^2\dot{u}_\alpha^2 u_\mu = eu_\nu F_{\mu\nu}, \text{ in} \qquad (13.)$$

It is this equation that will be shown to be without some of the unsatisfactory features pointed out in Dirac's equation.

First it should be observed that the electron is hereby imbued not only with a velocity-inertia, but also with an acceleration-inertia, which causes the acceleration to change continuously. This can be seen from the expression for the momentum

$$p_\mu = mc^2 u_\mu + \frac{2}{3} e^2\dot{u}_\mu = mc^2(\, u_\mu + a\dot{u}_\mu) \qquad (14.)$$

from which it is also clear that the acceleration-inertia is very small compared to the velocity-inertia, since a is very small. The problem of an electron deflected by an electric field, which is worked out later, will illustrate the effect of this acceleration-inertia on the motion.

To show that this equation does improve the treatment of an electron in field-free space, divide equation (13.) by mc^2, then

$$\dot{u}_\mu + a\,(\,\ddot{u}_\mu - \dot{u}_\alpha^2 u_\mu) = 0$$

and the general solution is

$$\frac{dx}{dt} = c \tanh\left(\, b\, e^{-\frac{s}{a}} + h\,\right)$$

Thus all the solutions are allowed and correspond simply to different initial accelerations; and for all of them the electron quickly settles down to uniform velocity.

With this new equation, one gets that the electron spirals outward in the field of a positive nucleus, instead of inward as in Dirac's equation. Dirac considered the fact that the electron spiralled inward for his equation a check on its validity. But it must be remembered that in actual experience it is not observed to spiral in either direction; its behavior is satisfactorily accounted for only by quantum mechanics. What is more if the electron spirals in toward the nucleus, the total energy of the system finally becomes negative. This is due to the fact that the absolute value of the potential energy comes out to be twice the kinetic energy, so that as the particles approach one another, the potential energy being negative, the sum of these two energies approaches negative infinity and finally exceeds the rest energies of the particles. The total energy thus becomes negative, which probably has no physical meaning.

To illustrate the use of equation (13.), consider the problem of the deflection of an electron on passing through a uniform electric field. For this purpose we need in general only the non-relativistic equation, since the velocities will not be high.

Using the non-relativistic approximations for the u_μ, $\dot{u}_\mu$, and $\ddot{u}_\mu$ given in the appendix, the first component of (13.) becomes

$$mv_x' + \frac{2e^2}{3}\left(\frac{v_x''}{c^3} + \frac{3(v\cdot v')v_x' + v'^2 v_x}{c^5}\right) - \frac{2}{3} e^2 \frac{v'^2}{c^4}\frac{v}{c} =$$

$$e(-iE_x)i + e\left(-H_y\frac{v_z}{c} + H_z\frac{v_y}{c}\right)$$

or

$$mv_x' + \frac{2e^2}{3}\left(\frac{v_x''}{c^3} + \frac{3(v\cdot v')v_x'}{c^5}\right) = eE_x + \frac{e}{c}(v \times H)_x$$

The first three components will therefore give us the vector equation

$$m\vec{v}' + \frac{2e^2}{3}\left(\frac{\vec{v}''}{c^3} + \frac{3(v\cdot v')\ \vec{v}'}{c^5}\right) = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{H}$$

and the fourth will be a consequence of the other three. For fields up to the ultra-strong field produced by the nucleus at the classical electron radius distance, the last term on the left is very small compared to the second. Therefore one may drop this term and use as the non-relativistic equation

$$m\vec{v}' + \frac{2}{3} e^2 \frac{\vec{v}''}{c^3} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{H} \qquad (15.)$$

since an ultra-strong field would necessitate a relativistic treatment anyway.

Having this equation we can return to the problem of finding the motion of an electron which having been selected from a stream of electrons by a pair of collimating holes, passes through the uniform field of a parallel-plate condenser. The origin of the coordinate system is put at the point at which the electron enters the field. The positive y-axis is taken in the direction of the field, and the x-axis perpendicular to the field and in the plane of motion of the electron. For simplicity the electric field is assumed to be uniform and to end abruptly at the edges of the condenser.

The path of the electron can be broken up into three regions: regionl, the part of the path before entering the field; region 2, the path in the uniform field between the plates of the condenser; region 3, the part of the path after leaving the field. The electron's behavior in each of the three regions is found; and by fitting up these solutions at the boundaries so that the position, velocity, and acceleration will be continuous, the total path is determined in terms of the initial conditions.

Setting $\vec{H} = 0$ and dividing through by m, equ. (15.) becomes

$$\vec{v}' + \frac{a}{c}\vec{v}'' = \frac{e}{m}\vec{E}$$

having the solution

$$\vec{v}' = \vec{d}\, e^{-\frac{c}{a}t} + \frac{e\vec{E}}{m}$$

Integrating this equation one gets

$$\vec{v} = -\frac{\vec{\alpha}a}{c} e^{-\frac{c}{a}t} + \frac{e\vec{E}}{m} t + \vec{\beta}$$

and finally

$$\vec{r} = \frac{\vec{\alpha}\, a^2}{c^2} e^{-\frac{c}{a}t} + \frac{1}{2}\frac{e\vec{E}}{m} t^2 + \vec{\beta} t + \vec{\gamma} \qquad (16.)$$

where $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$ are arbitrary vectors.

For the initial conditions the electron is required to be moving originally with uniform velocity in the positive x-direction, and the time origin is taken so that $t = 0$ when the electron enters the field. Taking the x-component of (16.)

$$x = \frac{\alpha_x a^2}{c^2} e^{-\frac{c}{a}t} + \beta_x t + \gamma_x$$

one sees that it is the same for all three regions and in order to satisfy the initial conditions, $\alpha_x = 0$ and $\gamma_x = 0$. Hence the motion in the x-direction is uniform -- the velocity being β_x --, and one need only consider the y-component of the motion.

$$y = \frac{\alpha_y a^2}{c^2} e^{-\frac{c}{a}t} + \frac{1}{2}\frac{eE_y}{m} t^2 + \beta_y t + \gamma_y$$

Dropping the subscript y and breaking up the solution into that for the three regions, which may be denoted by subscripts, one has for the first region

$$y_1 = \frac{\alpha_1 a^2}{c^2} e^{-\frac{c}{a}t} + \beta_1 t + \gamma_1$$

Here again the initial conditions require that $\alpha_1 = \beta_1 = 0$;

and also since the electron must pass through the origin, $\gamma_1 = 0$. Thus

$$y_1 = 0, \quad v_1 = y_1' = 0, \quad \text{and } v_1' = 0$$

For the second region

$$y_2 = \frac{\alpha_2 a^2}{c^2} e^{-\frac{c}{a}t} + 1/2 \frac{eE}{m} t^2 + \beta_2 t + \gamma_2$$

$$v_2 = \frac{\alpha_2 a}{c} e^{-\frac{c}{a}t} + \frac{eE}{m} t + \beta_2$$

$$v_2' = \alpha_2 e^{-\frac{c}{a}t} + \frac{eE}{m}$$

where the α_2, β_2, and γ_2 are to be chosen so that at $t = 0$ these are equal to zero and thus match up with the solution for region 1. Therefore, beginning with the last equation, one gets

$$\alpha_2 = -\frac{eE}{m}$$

$$\beta_2 = -\frac{a}{c}\frac{eE}{m}$$

$$\gamma_2 = \frac{a^2}{c^2}\frac{eE}{m}$$

Substituting these back in the equations, one has that

$$y_2 = \frac{a^2}{c^2}\frac{eE}{m}\left(1 - e^{-\frac{c}{a}t}\right) + \frac{1}{2}\frac{eE}{m}t^2 - \frac{a}{c}\frac{eE}{m}t$$

$$v_2 = -\frac{a}{c}\frac{eE}{m}\left(1 - e^{-\frac{c}{a}t}\right) + \frac{eE}{m}t$$

$$v_2' = \frac{eE}{m}\left(1 - e^{-\frac{c}{a}t}\right)$$

Now if t is set equal to t_1, where $t_1 = l/\beta_x$ or the time at which the electron emerges from the field, one has the initial conditions that must be satisfied by the solution for region 3. Here the solution is

$$y_3 = \frac{\alpha_3 a^2}{c^2}\, e^{-\frac{c}{a}t} + \beta_3 t + \gamma_3$$

$$v_3 = -\frac{\alpha_3 a}{c}\, e^{-\frac{c}{a}t} + \beta_3$$

$$v_3' = \alpha_3 e^{-\frac{c}{a}t}$$

and these must have the same value as those for region 2 when t is put equal to t_1, which demands that

$$\alpha_3 = \frac{eE}{m}\left(e^{-\frac{c}{a}t_1} - 1 \right)$$

$$\beta_3 = \frac{eE}{m}\, t_1$$

$$\gamma_3 = -\frac{1}{2}\,\frac{eE}{m}\, t_1^2 - \frac{a}{c}\,\frac{eE}{m}\, t_1$$

and so

$$y_3 = \frac{a^2 eE}{c^2 m}\left(e^{+\frac{c}{a}t_1} - 1\right) e^{-\frac{c}{a}t} + \frac{eE}{m}\, t_1 t - \frac{1}{2}\,\frac{eE}{m}\, t_1^2 - \frac{a}{c}\,\frac{eEt_1}{m}$$

The final solution is then

$$x = \beta_x t \qquad \text{(for all three regions)}$$

$$y_1 = 0$$

$$y_2 = \frac{a^2}{c^2}\,\frac{eE}{m}\left(1 - e^{-\frac{c}{a}t}\right) + \frac{1}{2}\,\frac{eE}{m}\, t^2 - \frac{a}{c}\, t$$

$$y_3 = \frac{a^2 eE}{c^2 m}\left(1 - e^{-\frac{c}{a}t_1}\right) e^{-\frac{c}{a}(t-t_1)} + \frac{eE}{m}\, t_1 t - \frac{1}{2}\,\frac{eE}{m}\, t_1^2 - \frac{a}{c}\,\frac{eEt_1}{m}$$

The first term in y_3 is exceedingly small even at $t = t_1$, as a/c is the time it takes light to traverse the classical electron radius and is of the order of 10^{-23} seconds.

This term is also transitory since $e^{-\frac{c}{a}(t-t_1)}$ is only of the order of one, when $t - t_1$ is of the order of $\frac{a}{c}$ seconds or less; and so y_3 may be written thus

$$y_3 = \frac{eE}{m} t_1 (t - t_1 - \frac{a}{c}) + \frac{1}{2} \frac{eE}{m} t_1^2 + \cdots.$$

Thereby showing that to a very high degree of approximation the electron leaves the field and travels in a straight line with a slope $eE/m \; t_1$.

Similarly y_2 can be written as

$$y_2 = \frac{1}{2} \frac{eE}{m} (t - a/c)^2 + \cdots.$$

which is the parabolic motion that one has for a constant acceleration eE/m.

If the transient terms are dropped altogether, the equations become exactly those that we would get in the classical treatment of a test-particle if the time of entering and leaving the field were a/c and $t_1 + a/c$. Thus although the electron responds immediately to an accelerating force, as is seen from the exact equation, the total effect is as if the electron moved as a test-particle but delayed its response until the disturbance has spread out to the classical radius distance.

The transient terms are very small and come in only for a time comparable to this delay. They show the effect of the small acceleration-inertia mentioned before, and serve only to make the acceleration continuous at the boundaries.

APPENDIX

Here are given the components of the u_μ, $\dot{u}_\mu$, and $\ddot{u}_\mu$ together with their non-relativistic approximations.

$$u_4 = \frac{i}{\sqrt{1 - v^2/c^2}} \cong i$$

$$u_k = \frac{v_K}{c\sqrt{1 - v^2/c^2}} \cong v_\kappa/c \qquad k = 1, 2, 3.$$

$$\dot{u}_4 = \frac{i(v\cdot v')}{c^3(1 - v^2/c^2)^2} \cong \frac{(v\cdot v')}{c^3} i$$

$$\dot{u}_k = \frac{(v\cdot v')v_K}{c^4(1 - v^2/c^2)^2} + \frac{v'_K}{c^2(1 - v^2/c^2)} \cong v'_\kappa/c^2$$

$$\ddot{u}_4 = \frac{(v'^2 + v\cdot v'')i}{c^4(1 - v^2/c^2)^{5/2}} + \frac{4(v\cdot v')^2 i}{c^6(1 - v^2/c^2)^{7/2}} \cong 1/c^4\ (v'^2 + v\cdot v'')$$

$$\ddot{u}_k = \frac{v''_K}{c^3(1 - \frac{v^2}{c^2})^{3/2}} + \frac{(v\cdot v'')v_K + v'^2 v_K + 3(v\cdot v')v'_K}{c^5(1 - \frac{v^2}{c^2})^{5/2}} + \frac{4(v\cdot v')^2 v_\kappa}{c^7(1 - \frac{v^2}{c^2})^{7/2}}$$

$$\cong v''_\kappa/c^3 + 1/c^5(\ 3(v\cdot v')v'_K + v'v_K)$$

$$\dot{u}_\kappa^2 \cong v'^2/c^4$$

primes denoting defferentiation with respect to t, and

$v_1 = v_x$, $v_2 = v_y$, and $v_3 = v_z$