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I hereby recommend that the thesis prepared under my supervision by _____ Nelson Paul Yeardley _____

entitled _____ Representation of an Analytic Function by a _____
_____ General Laguerre Series _____

be accepted as fulfilling this part of the requirements for the degree of _____ Doctor of Philosophy _____

Approved by:

_____ *W. H. Gary* _____
_____ *Charles A. Moore* _____

REPRESENTATION OF AN ANALYTIC FUNCTION
BY A GENERAL LAGUERRE SERIES

A dissertation submitted to the
Graduate School of Arts and Sciences
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1949

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It is only a small measure of honesty which impels me to mention the important roles which my inspiring teachers played in the unfolding of this work: that of Dr. Otto Szasz, my major professor, without whose continuous help and advice I could not have rung down the final curtain and that of Professor Charles H. Moore whose timely encouragement and wise counsel kept things moving at certain moments when it seemed the play could not go on.

TABLE OF CONTENTS

1. Statement of the Problem	Page	1
2. Lemmas and Definitions	Page	6
3. The Necessity of Theorem \bar{B} : The Fundamental Inequality	Page	34
4. The Necessity of Theorem \bar{B} : Conclusion	Page	51
5. The Sufficiency of Theorem \bar{B} : Lemmas	Page	59
6. The Sufficiency of Theorem \bar{B} : The Fundamental Inequality	Page	70
7. The Sufficiency of Theorem \bar{B} : Conclusion	Page	77
8. Theorem: Theorem \bar{A} and Theorem \bar{B} and Equivalent	Page	85
9. Bibliography	Page	90

1. Statement of The Problem

We define the Laguerre polynomials, (6) p.97, (5.1.6), in z , with z in the complex plane, as follows:

$$(1.1) \quad L_n^\alpha(z) \equiv \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-z)^k}{k!} \quad (\alpha > -1; z = x+iy = re^{i\theta})$$

where the general binomial coefficient $\binom{n+\alpha}{n}$ is defined as follows:

$$\binom{n+\alpha}{n} \equiv (\alpha+1)(\alpha+2)\dots(\alpha+n)/n!$$

Such a polynomial is called a Laguerre polynomial of order α and degree n .

To an analytic function, $f(z)$ we let correspond the following series, which we shall call a Laguerre series of order α , or simply a general Laguerre series. ((6)p.238, (9.1.2))

$$(1.2) \quad f(z) \sim \sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(z)$$

where

$$(1.3) \quad a_n^\alpha = \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} \int_0^{\infty} e^{-x} x^\alpha L_n^\alpha(x) f(x) dx$$

In this investigation the following parabola plays an important role:

$$(1.4) \quad p(d_\alpha): y^2 = 4d^2 (x + d_\alpha^2)$$

where for a given parabola d_α is constant and in which we define ((6) p.358; (1.3))

$$(1.5) \quad d_\alpha \equiv - \limsup_n (2n^{\frac{1}{2}})^{-1} \log |a_n^\alpha|$$

DEFINITION(1.1)

By the symbol:

$$(1.6) \quad z \in p(b) \quad (b > 0)$$

we shall mean that x and y satisfy the inequality:

$$(1.7) \quad y^2 < 4b^2(x-b^2)$$

DEFINITION (1.2)

By the symbol:

$$(1.8) \quad z \in s(b) \quad (b > 0)$$

we shall mean that the real and imaginary parts of z (i.e. x and y respectively) satisfy the inequality:

$$(1.9) \quad |y| < b \quad (-\infty < x < \infty)$$

Replacing z by w^2 in (1.2) we get:

$$(1.10) \quad f(w^2) = g(w) = f(z) \sim \sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(w^2)$$

Pollard (5) considered the Laguerre series (1.10) for the case $\alpha = 0$, so that

$$(1.11) \quad g(w) \sim \sum_{n=0}^{\infty} a_n^0 L_n^0(w^2)$$

where

$$a_n^0 = \int_0^{\infty} e^{-x} L_n^0(x) f(x) dx$$

He then proceeds to prove (5) the following theorem for the series (1.10)

THEOREM A

In order that $g(w)$ possess a Laguerre series (1.10) which converges to it for $w \in s(d)$; $d \equiv d_0 > 0$ (1.5) p.2 it is necessary and sufficient that $g(w)$ be analytic and even in this strip and that to every b , $0 < b < d$, there corresponds a positive number $B(b)$ such that

$$(1.12) \quad g(w) \leq B(b) \exp(u^2/2 - |v|(b^2 - v^2)^{\frac{1}{2}}) \quad (w \in s(b))$$

It immediately occurs to one to try to generalize theorem A as follows:

THEOREM A'

In order that $g(w)$ possess a Laguerre series (1.10) of order \mathcal{L} ($\mathcal{L} > -1$) which converges to it for $w \in s(d_{\mathcal{L}})$ ($d_{\mathcal{L}} > 0$) it is necessary and sufficient that $g(w)$ be analytic and even ($z \in p(d_{\mathcal{L}})$) and that to every $b_{\mathcal{L}}$, ($0 \leq b_{\mathcal{L}} < d_{\mathcal{L}}$) there correspond a positive number $B(\mathcal{L}, b)$ such that condition (1.12) holds for $w \in s(b_{\mathcal{L}})$.

For the general Laguerre series it is appropriate to use (1.2) rather than (1.10). Accordingly we shall prove the following theorem:

THEOREM B

In order that $f(z)$ possess a Laguerre series of order α ($\alpha > -1$) (1.2) which converges to it for $z \in p(d_\alpha)$ ($d_\alpha > 0$) it is necessary and sufficient that $f(z)$ be analytic ($z \in p(d_\alpha)$) and that to every b_α , $0 \leq b_\alpha < d_\alpha$, there correspond a positive number $B(\alpha, b)$ such that:

$$(1.13) \quad |f(z)| \leq B(\alpha, b) \exp\left(\frac{1}{2}x - |x|^{\frac{1}{2}}(b_\alpha^2 - \frac{1}{2}(r-x))\right)^{\frac{1}{2}} \\ (z \in p(b_\alpha); r \equiv (x^2 + y^2)^{\frac{1}{2}}) \quad \text{d. (1.1) p.1.}$$

The purpose of this paper is to prove theorem B. We will then show that theorems A and B are equivalent.

In part 2, we prove some lemmas and give definitions which we will need to prove the necessity and sufficiency of Theorem B.

In part 3, we derive an inequality which is fundamental for the proof of the necessity of theorem B, and by means of which the necessity of theorem B is readily obtained.

In part 4, we use the fundamental identity established in part 3 to establish the necessity of theorem B.

In part 5, we establish some lemmas which we will use to prove the sufficiency of theorem B.

Note: For the sake of brevity we use the symbol $B(\alpha, b)$ for $B(\alpha, b_\alpha)$

In part 6, we obtain an inequality on which the proof of the sufficiency of theorem \bar{B} is based.

In part 7, we use the inequality of part 6 to complete the proof of the sufficiency of theorem \bar{B} .

And finally in part 8, we prove that theorem \bar{A} and theorem \bar{B} are equivalent.

2. Lemmas and Definitions

In this section we list some definitions and prove lemmas which will be of use in proving the necessity and sufficiency of theorem \bar{B} p.4.

DEFINITION (2.1)

The symbol

$$(2.1) \quad A \Rightarrow B$$

is to mean: Proposition A implies proposition B.

DEFINITION (2.2)

By the symbol

$$(2.2) \quad z \notin p(b)$$

we shall mean that the real and imaginary parts of z (i.e. x and y respectively) satisfy the following inequalities.

$$(2.3) \quad \frac{1}{2}(r-x) > b \quad (x > -b^2)$$

DEFINITION (2.3)

By the symbol

$$(2.4) \quad z \in p^*(b)$$

where $b > 0$, we shall mean that $z \in p(b)$ exclusive of the non-negative real axis.

DEFINITION (2.4)

$$\text{expn}(a) \equiv n^a$$

DEFINITION (2.5)

By the symbol

$$z \in \bar{C}(b)$$

we mean that x and y satisfy the inequality

$$(2.5) \quad z \leq b$$

DEFINITION (2.6)

By the symbol

$$(2.6) \quad z \in S$$

we mean that the point z is an element of the set of points S .

DEFINITION (2.7)

By the symbol

$$(2.7) \quad \bar{S}^*(b)$$

we mean any closed finite region such that

$$(2.8) \quad z \in \bar{S}^*(b) \Rightarrow z \in p^*(b)$$

(For the definition of region and closed region see (3) p, 18 section 2)

LEMMA (2.2)

It can be shown from an asymptotic formula of Perron,

(6) p. 193 formula (8.22.3), that the following limit relation holds uniformly in any finite closed domain S in the complex z -plane cut along the non-negative real axis:

$$(2.10) \quad n^{-\frac{1}{2}} \log |L_n^\alpha(z)| \rightarrow 2R(-z)^{\frac{1}{2}} \quad (n \rightarrow \infty)$$

and we select that branch of $(z)^{\frac{1}{2}}$ for which $(-z)^{\frac{1}{2}}$ is real and positive when $z < 0$. See Szego (6) p.197 formula (8.23.3).

LEMMA (2.2)

The following limit relation holds uniformly in any finite closed domain S in the complex z -plane cut along the non-negative real axis:

$$(2.11) \quad n^{-\frac{1}{2}} \log |L_n^{\alpha}(z)| \rightarrow 2 \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} = r^{\frac{1}{2}} \sin \frac{1}{2} \theta \quad (n \rightarrow \infty)$$

Proof:

$$(2.12) \quad -z = r \exp(i(\theta - \pi))$$

We want to use lemma (2.1) so that we exclude z from the non-negative real axis. That is θ is restricted to the range:

$$(2.13) \quad 0 < \theta < 2\pi$$

From (2.12)

$$(2.14) \quad -z = r \exp(i(\theta - \pi))$$

$$(2.15) \quad (-z)^{\frac{1}{2}} = r^{\frac{1}{2}} \exp(i(\frac{1}{2}\theta - \frac{1}{2}\pi))$$

If $z < 0$ by (2.13) $\theta = \pi$ so that from (2.15)

$$(2.16) \quad (-z)^{\frac{1}{2}} = r^{\frac{1}{2}} > 0 \quad (z < 0)$$

Thus by imposing conditions (2.13) and (2.14) the conditions of lemma (2.1) are satisfied. Now from (2.15) we see that:

$$(2.17) \quad R(-z)^{\frac{1}{2}} = r \cos(\frac{1}{2}\theta - \frac{1}{2}\pi) = r^{\frac{1}{2}} \sin \frac{1}{2} \theta > 0 \quad (0 < \theta < 2\pi)$$

By the formula from trigonometry:

$$(2.18) \quad \sin \frac{1}{2}\theta = \left(\frac{1}{2}(1 - \cos\theta) \right)^{\frac{1}{2}}$$

we obtain from (2.17)

$$(2.19) \quad R(-z)^{\frac{1}{2}} = r^{\frac{1}{2}} \sin \frac{1}{2}\theta = \left(\frac{r(1 - \cos\theta)}{2} \right)^{\frac{1}{2}} = \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}}$$

Applying (2.19) to (2.10) the lemma and formula (2.11) follows:

LEMMA (2.3)

The necessary and sufficient condition—that $z \in p(b)$ is that the real and imaginary parts of z (i.e. x and y respectively) satisfy the inequalities:

$$(a) \quad \frac{1}{2}(r-x) < b^2$$

Proof:

A) Necessity: Let $z \in p(b)$ then by d(1.1) p.2, we have:

$$0 \leq y^2 < 4b^2(x-b^2)$$

$$x^2 + y^2 < x^2 + 4b^2x + 4b^4$$

$$0 \leq x^2 + y^2 < (x+2b^2)^2$$

$$(x^2 + y^2)^{\frac{1}{2}} < x + 2b^2$$

$$\frac{1}{2}(r-x) = \frac{1}{2} \left((x^2 + y^2)^{\frac{1}{2}} - x \right) < b^2 \quad \text{q.e.d.}$$

B) Sufficiency: Assume: $\frac{1}{2} \left((x^2 + y^2)^{\frac{1}{2}} - x \right) = \frac{1}{2}(r-x) < b^2$

Hence: $0 \leq (x^2 + y^2)^{\frac{1}{2}} < x + 2b^2$

$$x^2 + y^2 < (x+2b^2)^2$$

$$y^2 < 4b^2(x+b^2) \quad (z \in p(b)) \quad \text{by d(1.1) p.2.q.e.d.}$$

LEMMA (2.4)

Let d_α be defined by (1.5) p.2. If $d_\alpha > 0$, then for every b_α , $0 \leq b_\alpha < d_\alpha$ the series (1.2) p.1 converges absolutely and uniformly for $z \in \overline{S}^*(b_\alpha)$ and diverges for $s \notin \overline{p}(d_\alpha)$.

Proof: From (1.5) p.2 and the definition of limit superior we have:

For any $\epsilon > 0$ there exists an n_ϵ , independent of z , such that:

$$\begin{aligned} (2n^{\frac{1}{2}})^{-1} \log |a_n^\alpha| &< -d_\alpha + \epsilon & (n \geq n_\epsilon) \\ (2n^{\frac{1}{2}})^{-1} \log |a_n^\alpha| &> -d_\alpha - \epsilon & (n = n_k; k = 1, 2, \dots) \end{aligned}$$

From which we have:

$$\begin{aligned} \log |a_n^\alpha| &< -2n^{\frac{1}{2}}(d_\alpha - \epsilon) & (n \geq n_\epsilon) \\ \log |a_n^\alpha| &> -2n^{\frac{1}{2}}(d_\alpha + \epsilon) & (n = n_k; k = 1, 2, \dots) \end{aligned}$$

Hence for $\epsilon > 0$ independent of z

$$(2.20) \quad |a_n^\alpha| = \exp(\log |a_n^\alpha|) < \exp(-2n^{\frac{1}{2}}(d_\alpha - \epsilon)) \quad (n \geq n_\epsilon)$$

$$(2.21) \quad |a_n^\alpha| = \exp(\log |a_n^\alpha|) > \exp(-2n^{\frac{1}{2}}(d_\alpha + \epsilon)) \quad (n = n_k; k = 1, 2, \dots)$$

For any point z exclusive of the non-negative real axis we obtain from (2.11) p.8 that given an $\epsilon(z) > 0$, there exist an n_ϵ such that:

$$\begin{aligned} n^{-\frac{1}{2}} \log |L_n^\alpha(z)| &< 2 \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} + \epsilon(z) & (n \geq n_\epsilon) \\ n^{-\frac{1}{2}} \log |L_n^\alpha(z)| &> 2 \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} - \epsilon(z) & (n \geq n_\epsilon) \end{aligned}$$

-From which we obtain:

$$\begin{aligned} \log |L_n^a(z)| &< 2n^{\frac{1}{2}} \left\{ \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} + \frac{1}{2}\epsilon(z) \right\} & (n \geq n_\epsilon) \\ \log |L_n^a(z)| &> 2n^{\frac{1}{2}} \left\{ \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} - \frac{1}{2}\epsilon(z) \right\} & (n \geq n_\epsilon) \end{aligned}$$

It follows from the above that:

$$(2.22) \quad |L_n^a(z)| = \exp(\log |L_n^a(z)|) < \exp(2n^{\frac{1}{2}} \left\{ \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} + \frac{1}{2}\epsilon(z) \right\}) \quad (n \geq n_\epsilon)$$

$$(2.23) \quad |L_n^a(z)| > \exp(2n^{\frac{1}{2}} \left\{ \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} - \frac{1}{2}\epsilon(z) \right\}) \quad (n \geq n_\epsilon)$$

where z is excluded from the non-negative real axis.

For any point z exclusive of the positive real axis we obtain from (2.20) and (2.22) that given an $\epsilon(z) > 0$ there exists an n_ϵ such that:

$$(2.24) \quad |a_n^a L_n^a(z)| < \exp(-2n^{\frac{1}{2}} \{d_a - \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} - \epsilon(z)\}) \quad (n \geq n_\epsilon)$$

$$(2.25) \quad |a_n^a L_n^a(z)| > \exp(-2n^{\frac{1}{2}} \{d_a - \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} + \epsilon(z)\}) \quad (n = n_k; k = 1, 2, \dots)$$

Let \bar{b}_a, b_a be such that $0 < b_a < \bar{b}_a < d_a$. Let $z \in p(b_a)$. Consequently by lemma (2.3):

$$(2.26) \quad \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} < b_a < \bar{b}_a.$$

Select the positive number ϵ so that:

$$(2.27) \quad \epsilon < \frac{1}{2}(d_a - \bar{b}_a).$$

Then since $\bar{b}_a < \frac{1}{2}(\bar{b}_a - d_a) < d_a$ we have:

$$(2.28) \quad \begin{aligned} d_a - \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} - \epsilon &> \frac{1}{2}(\bar{b}_a - d_a) - \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} - \epsilon \\ &= \bar{b}_a - \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} - \frac{1}{2}(d_a - \bar{b}_a) - \epsilon > \epsilon > 0 \quad (z \in p(b_a)) \end{aligned}$$

by virtue of (2.26) and (2.27).

Now let $z \in \bar{S}^*(b_a)$. Then $z \in p^*(b_a)$ by definition (2.7) p.7, and hence $z \in p(b_a)$ by definition (2.3) p.6. Moreover we recall from lemma (2.2) that relations (2.22) and (2.23)

hold uniformly for all $z \in \overline{S^*}(b_a)$. Hence the number $\epsilon(z)$ in (2.22) and (2.23) will not depend on z provided only that $z \in \overline{S^*}(b_a)$. In addition since the number ϵ in (2.21) is independent of z , the number $\epsilon(z)$ of (2.24) and (2.25) will be also provided that $z \in \overline{S^*}(b_a)$.

Therefore we may apply (2.28) to inequality (2.24) so that for the choice of ϵ indicated by inequality (2.27) there exists an n_ϵ such that:

$$(2.29) \quad |a_n^a L_n^a(z)| < \exp(-cn^{\frac{1}{2}}) \quad (n \geq n_\epsilon; z \in \overline{S^*}(b_a))$$

The series:

$$(2.30) \quad \sum_{n=0}^{\infty} \exp(-cn^{\frac{1}{2}})$$

converges for $c > 0$, since

$$(2.31) \quad \exp(-cn^{\frac{1}{2}}) = 1/\exp(cn^{\frac{1}{2}}) < 4/c^{\frac{1}{2}} n^2$$

Hence by (2.29) the series (1.2) p.1 converges absolutely and uniformly for $z \in \overline{S^*}(b_a)$ by comparison with the series (2.30)

On the otherhand suppose $z \notin \overline{p}(d_a)$. Then by definition

(2.2) p. 6:

$$(2.32) \quad \left(\frac{1}{2}(r-x) \right)^{\frac{1}{2}} \geq d_a + \epsilon$$

and by inequality (2.25) for a given z such that $z \notin \overline{p}(d_a)$

we have $|a_n^a L_n^a(z)| > 1 \quad (n = n_k; k = 1, 2, \dots)$.

Hence series (1.2) p.1 diverges for $z \notin \overline{p}(d_a)$. This completes the proof of lemma (2.4).

LEMMA (2.5)

Let be $d_a > 0$, $D \geq 0$, where d_a is given by statement (1.5)

p.2. Then the series (1.2) p.1 converges absolutely and uniformly ($0 \leq x \leq d$; $y = 0$).

Proof: We have given by (1) p.219:

$$(2.33) \quad |L_n^\alpha(x)| \leq \exp(\frac{1}{2}x) \Gamma(n+\alpha+1) / \Gamma(\alpha+1) \Gamma(n+1) \quad (x \geq 0)$$

We use the following formula from the theory of gamma functions, see (7) p. 58 formula (2).

$$(2.34) \quad \Gamma(n+\alpha+1) / \Gamma(n+1) \sim (n+1)^\alpha = \left(\frac{n-1}{n} \right)^\alpha n^\alpha \sim n^\alpha \quad (n \rightarrow \infty)$$

Hence inequality (2.33) becomes:

$$(2.35) \quad |L_n^\alpha(x)| \leq K \exp(\frac{1}{2}x) n^\alpha \leq K \exp(\frac{1}{2}D) n^\alpha = K_1 n^\alpha \quad (0 \leq x \leq D; n)$$

From inequality (2.35) and inequality (2.20) p. 10 given an $\epsilon > 0$ there exists an n_ϵ such that:

$$(2.36) \quad |a_n^\alpha L_n^\alpha(x)| \leq K_1 n^\alpha \exp(-2n^{\frac{1}{2}}(\bar{d}_\alpha + \epsilon)) \\ \leq K_1 n^\alpha p! / (2n^{\frac{1}{2}}(\bar{d}_\alpha - \epsilon))^p \quad (n \geq n_\epsilon; 0 \leq x \leq D; p=0,1,\dots)$$

Select the number $p > 2(1+\alpha)$, then from inequality (2.36):

$$(2.37) \quad |a_n^\alpha L_n^\alpha(x)| \leq K_2 / \exp n(\frac{1}{2}p - \alpha) \quad (n \geq n_\epsilon; 0 \leq x \leq D)$$

Thus by comparison with a convergent series, the series (1.2) p.1 converges absolutely and uniformly for $0 \leq x \leq D$; $y = 0$.

LEMMA (2.6)

If $\bar{d}_\alpha > 0$, where \bar{d}_α is defined by statement (1.5) p.2, then series (1.2) p.1 converges absolutely for $z \in p(\bar{d}_\alpha)$ and diverges for $z \notin \bar{p}(\bar{d}_\alpha)$.

Proof: Let $z \in p(\bar{d}_\alpha)$. Then either z is on the non-

negative real axis, in which case series (1.2) p.1 converges absolutely at this point by lemma (2.5), or else we can take b_a . $0 < b_a < d_a$ so large that $z \in \overline{S^*}(b_a)$ in which case series (1.2) p.1 converges absolutely at this point also, by lemma (2.4). If $z \notin \overline{p}(d_a)$ the lemma is merely a restatement of lemma (2.4).

DEFINITION(2.8)

The parabola $p(d_a)$ (1.4) p.2, where d_a is defined by (1.5) p. 2 is called the parabola of convergence of the series (1.2) p.1, provided $d_a > 0$. If $d_a \leq 0$, we have no parabola.

LEMMA(2.7)

If the series (1.2) p.1 converges at a point $z_0 = x_0 + iy_0$ ($y_0 \neq 0$) then $d_a > 0$, where d_a is defined by (1.5) p.2 and the series (1.2) converges absolutely for $z \in p(d_a)$ and for $-1 < a$.

Proof: If $d_a < 0$ then $d_a - \left(\frac{r_0 - x_0}{2} \right)^{\frac{1}{2}} < 0$.

By choosing ϵ small enough $d_a - \left(\frac{1}{2}(r_0 - x_0) \right)^{\frac{1}{2}} + \epsilon < 0$.

So that by (2.25) p.11 there exists a sequence of integers $\{n_k\}$ depending on z_0 such that:

$$|a_n^a L_n(z)| > 1 \quad (y_0 \neq 0; n = n_k; k = 0, 1, \dots)$$

Thus if $d_a \leq 0$ series (1.2) diverges at $z = z_0$. Hence $d_a > 0$.

Therefore by lemma (2.6) the proposition follows.

LEMMA(2.8)

Let $d > 0$. If for every b such that $0 \leq b < d$, there

exists a positive number $B(\alpha, b)$ such that

$$(2.38) \quad |a_n^\alpha| \leq B(\alpha, b) \exp(-2n^{\frac{1}{2}}b)$$

then $d_\alpha \geq d > 0$, where d_α is defined by (1.5) p.2. That is, the parabola, $p(d_\alpha)$ (1.4) p.1, is the parabola of convergence of the series (1.2) p.1.

Proof: By (2.38) and (2.22) p. 11 (z excluded from the non-negative real axis) we have that given $\epsilon(z) > 0$ there is an n_ϵ such that:

$$(2.39) \quad |a_n^\alpha L_n^\alpha(z)| \leq B(\alpha, b) \exp(-2n^{\frac{1}{2}}\{b - (\frac{1}{2}(r-x))^{\frac{1}{2}} - \epsilon(z)\})$$

($n \geq n_\epsilon$)

If we have:

$$(2.40) \quad d > d_\alpha,$$

then select b and b_0 such that $0 < d_\alpha < b_0 < b < d$. Let us select z_0 so that z_0 lies on the $p(b_0)$, (1.4) p.2, i.e. x_0 and y_0 satisfy equation (1.4). Then by a proof similar to that of lemma (2.3) p.9:

$$(2.41) \quad d_\alpha < (\frac{1}{2}(r_0 - x_0))^{\frac{1}{2}} = b_0 < b.$$

Then select the number $\epsilon(z_0)$ so small that $b - b_0 - \epsilon(z_0) > 0$.

Consequently by inequalities (2.41) and (2.39) the series (1.2) p.1 converges at the point z_0 (which is not on the real axis) since we can compare it with the series (2.30) p.12. By lemma (2.7) the number $d_\alpha > 0$, where d_α is given by (1.5) p.2. On the otherhand if $d_\alpha < d$, then by inequality (2.41) and definition (2.2) p. 6, we get that $z_0 \notin \bar{p}(d_\alpha)$. But since the number $d_\alpha > 0$, by lemma (2.4) p. 10, series (1.2) diverges at

the point z_0 . Thus assumption (2.40) is false. This gives the theorem.

EXAMPLES

We are going to find the parabola of convergence given by definition (2.8) p. 14, of several special Laguerre series. As the parabola of convergence is given by (1.4) p. 2, it is uniquely determined by the number d_α as defined by (1.5) p.2.

Example(2.1)

Let us consider the Laguerre development of the function:

$$f(z) = \exp(qz) \quad (q \text{ real})$$

By (1.1) and (1.3) p. 1 we obtain:

$$a_n^\alpha = \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} (k!)^{-1} \int_0^\infty \exp(-(1-q)x) x^{\alpha+k} dx$$

$$a_n^\alpha = \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} \int_0^\infty e^{-x} x^\alpha \exp(qx) \left\{ \sum_{k=0}^n \binom{n+\alpha}{n-k} (-x)^k / k! \right\} dx$$

Let us put $(1-q)x=y$, then for $q < 1$:

$$a_n^\alpha = \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} \sum_{k=0}^n \binom{n+\alpha}{n-k} (-1)^k (k!)^{-1} (1-q)^{-\alpha-k} \int_0^\infty e^{-y} y^{\alpha+k} dy$$

By the definition of the gamma function, see (7) p. 55 formula (1):

$$a_n^\alpha = \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} \sum_{k=0}^n \binom{n+\alpha}{n-k} (-1)^k (k!)^{-1} (1-q)^{-\alpha-k-1} \Gamma(\alpha+k+1)$$

Since $\Gamma(\alpha+k+1) = (\alpha+k)(\alpha+k+1)\dots(\alpha+1)\Gamma(\alpha+1)$

$$(2.54) \quad a_n^\alpha = \left\{ \binom{n+\alpha}{n} \right\}^{-1} \sum_{k=0}^n \binom{n+\alpha}{n-k} (-1)^k (k!)^{-1} (1-q)^{-\alpha-k-1} (\alpha+1)\dots(\alpha+k)$$

$$= (1-q)^{-\alpha-1} n! \sum_{k=0}^n \frac{(-1)^k}{(n-k)! k! (1-q)^k}$$

$$= (1-q)^{-\alpha-1} \sum_{k=0}^n \binom{n}{k} (-(1-q))^{-k}$$

$$= (1-q)^{-\alpha-1} (1 - (1-q)^{-1})^n$$

$$= (1-q)^{-\alpha-1} (-1)^n (q/(1-q))^n$$

$$a_n^\alpha = (1-q)^{-\alpha-1} (q/(1-q))^n$$

By statement (1.5) p.2:

$$d_\alpha = -\lim_n \sup (2n^{\frac{1}{2}})^{-1} \{ (\alpha+1) \log(1/(1-q)) - n \log(q/(1-q)) \}$$

hence:

$$d_\alpha \rightarrow -\frac{1}{2} n^{\frac{1}{2}} \log(q/(1-q))$$

From this we get:

$$(2.55) \quad \begin{aligned} d_\alpha &= +\infty & (q < \frac{1}{2}) \\ d_\alpha &= 0 & (q = \frac{1}{2}) \\ d_\alpha &= -\infty & (1 > q > \frac{1}{2}) \end{aligned}$$

From the equation (2.54) and (1.2) p.1 the Laguerre development of $\exp(qz)$ is:

$$(1-q)^{-\alpha-1} \sum_{n=0}^{\infty} (-1)^n (q/(1-q))^n L_n^\alpha(z)$$

From equation (2.55) and lemma (2.6) p. 13, we see

that the Laguerre development of $\exp(qz)$ converges for all points z if $q < \frac{1}{2}$, diverges for all points z , except possibly $y=0, x \geq 0$, if $q = \frac{1}{2}$ and diverges for all points z if $q > \frac{1}{2}$. If $q = \frac{1}{2}$:

$$e^{\frac{1}{2}z} \sim \sum_{n=0}^{\infty} (-1)^n L_n^{\alpha}(z) z^{\alpha+1}$$

Example(2.2)

We are going to calculate the parabola of convergence given by definition (2.8) p. 14, of the Laguerre development of the following function:

$$(2.56) \quad (1/(1-t))^{d+1} \exp(tz/(t-1))$$

The generating function for Laguerre polynomials is found in Szego (6) p. 97 formula (5.1.9) and is as follows:

$$(2.57) \quad \sum_{n=0}^{\infty} t^n L_n^{\alpha}(z) = (1-t)^{-\alpha-1} \exp(tz/(t-1)) \quad (t < 1)$$

By equality (1.3) p.1 for function (2.56) we have:

$$\begin{aligned} a_n^{\alpha} &= \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} (1-t)^{-\alpha-1} \int_0^{\infty} e^{-z} z^{\alpha} L_n^{\alpha}(z) \exp(tz/(t-1)) dz \\ &= \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} (1-t)^{-1} \int_0^{\infty} \exp(z/(t-1)) z^{\alpha} L_n^{\alpha}(z) dz \end{aligned}$$

By calculations similar to those of example (2.1) we get:

$$a_n^{\alpha} = \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\}^{-1} \left\{ \binom{n+\alpha}{n} \Gamma(\alpha+1) \right\} (1-(1-t))^{-1} = t^n$$

Thus we see that equation (2.57) is the Laguerre

development of functions (2.56). From statement (1.5) p.

$$2: \quad \delta_a = -\lim_n \sup \left(\frac{1}{2n} \log |t| \right)$$

from which we obtain:

$$\delta_a = \infty \quad (t < 1), \quad \delta_a = 0 \quad (t = 1), \quad \delta_a = -\infty \quad (t > 1).$$

From the above and lemma (2.6) p. 13, we see that the parabola of convergence of series (2.57) is infinite for $t < 1$ i.e. series (2.57) converges through out the whole z -plane whereas it diverges everywhere for $t > 1$ and diverges everywhere except possibly for $y = 0; x \geq 0$ if $t = 1$, since the parabola of convergence is the non-negative real axis.

Example(2.3)

Wigert (6) p. 17 has calculated the Fourier coefficients (1.3) p. 1 for $\alpha = 0$ of the Laguerre development of the function:

$$f(x) = x^s \quad (s > 0)$$

He obtains:

$$a_n^0 = (-1)^n \Gamma(s-1) \binom{s}{n}$$

We can now calculate the parabola of convergence of the Laguerre development of the given function. By formula (1.2) p.1 the development is:

$$(2.58) \quad \sum_{n=0}^{\infty} (-1)^n \Gamma(s-1) \binom{s}{n} L_n^0(z)$$

$$a_n^0 = (-1)^n s(s-1)\dots(s-n+1)/n! = (-s)(-s+1)\dots(-s-1+n)/n!$$

$$\binom{n+(s-1)}{n} \sim K \expn(-s-1) \quad (n \rightarrow \infty; s \neq 1, 2, \dots)$$

by (10) p. 42 formula (2) and paragraph (3.12).

$$(2n^{\frac{1}{2}})^{-1} \log |a_n^0| \sim (2n^{\frac{1}{2}})^{-1} (\log k - (s+1) \log n) \quad (n \rightarrow \infty)$$

The parabola of convergence of the Laguerre development of the given function degenerates therefore to the non-negative real axis. That is, the series (2.58) diverges everywhere except possibly on the non-negative real axis. Wigert shows (6) p. 17-18 that the series converges to $f(x)$ there. Moreover, it is absolutely convergent.

LEMMA (2.9)

The following inequality holds in the complex z -plane for general Laguerre polynomials.

$$(2.42) \quad |L_n^\alpha(z)| \leq L_n^\alpha(-|z|) \leq L_n^\alpha(-b) \quad (\alpha > -1; |z| \leq b)$$

Proof: Recalling the definition of the general binomial coefficient, we have:

$$(2.43) \quad \binom{n+\alpha}{0} = 1 \quad (n=0, 1, 2, \dots)$$

$$(2.44) \quad \binom{n+\alpha}{n-k} = (n+\alpha)(n+\alpha-1)\dots(\alpha+k+1)/(n-k)!$$

From which we conclude that:

$$(2.45) \quad \binom{n+\alpha}{n-k} \geq 0 \quad (\alpha > -1; k=0,1,2,\dots,n; n=0,1,\dots)$$

From (1.1) p. 1 we conclude by virtue of inequality (2.45)

$$(2.46) \quad |L_n^\alpha(z)| = \left| \sum_{k=0}^n \binom{n+\alpha}{n-k} (-z)^k / k! \right| = \sum_{k=0}^n \binom{n+\alpha}{n-k} (|z|)^k / k!$$

$$= L_n^\alpha(-|z|) \leq L_n^\alpha(-b) \quad (\alpha > -1; |z| \leq b)$$

DEFINITION (2.9)

Through out this paper we will adhere to the following notational convention:

a) The symbol S shall indicate the set of points \bar{S} exclusive of the boundary points of \bar{S} , provided the set \bar{S} has been defined and conversely the symbol \bar{S} shall indicate the set of points S inclusive of the boundary points of S , provided the set S has been defined.

b) The symbol $z \in S$ shall mean $z \in \bar{S}$ exclusive of the boundary points of \bar{S} provided the symbol $z \in \bar{S}$ has been defined and conversely the symbol $z \in \bar{S}$ shall mean $z \in S$ inclusive of the boundary of S provided the symbol $z \in S$ has been defined.

c) If the symbol S indicates a set of points, then the symbol S^* shall indicate the set of points S exclusive of the non-negative real axis.

LEMMA (2.10)

Let $d_\alpha > 0$ where d_α is defined by (1.5) p.2, then for every b_α , $0 < b_\alpha < d_\alpha$ the general Laguerre series:

$$(2.47) \quad \sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(z)$$

converges absolutely and uniformly for all points z such that

$$(2.48) \quad z \in \overline{C}(b_\alpha) \equiv |z| \leq b$$

Proof: By the previous lemma we have:

$$(2.49) \quad |a_n^\alpha L_n^\alpha(z)| \leq a_n^\alpha L_n^\alpha(-b_\alpha) \quad (|z| \leq b_\alpha)$$

From equality (1.5) p.2 and the definition of limit superior we have: for any $\epsilon > 0$ there exists an n_ϵ , independent of z such that:

$$(2n^{\frac{1}{2}})^{-1} \log |a_n^\alpha| < -d_\alpha - \epsilon \quad (n \geq n_\epsilon)$$

$$\log |a_n^\alpha| < 2n^{\frac{1}{2}}(-d_\alpha + \epsilon) \quad (n \geq n_\epsilon)$$

so that

$$(2.50) \quad |a_n^\alpha| = \exp(\log |a_n^\alpha|) < \exp(-2n^{\frac{1}{2}}(d_\alpha - \epsilon)) \quad (n \geq n_\epsilon)$$

By relation (2.11) p.8, given an $\epsilon(b_\alpha) > 0$, there exists an n_ϵ such that:

$$n^{-\frac{1}{2}} \log |L_n^\alpha(-b_\alpha)| < 2 \left(\frac{1}{2}(b_\alpha + b_\alpha) \right)^{\frac{1}{2}} = 2b_\alpha^{\frac{1}{2}} + \epsilon(b_\alpha)$$

$$(b_\alpha > 0; n \geq n_\epsilon)$$

$$\log |L_n^\alpha(-b_\alpha)| < 2n^{\frac{1}{2}} \left(b_\alpha^{\frac{1}{2}} + \frac{1}{2} \epsilon(b_\alpha) \right)$$

$$(2.51) \quad |L_n^{\alpha}(-b_{\alpha})| = \exp(\log |L_n^{\alpha}(-b_{\alpha})|) < \exp(2n^{\frac{1}{2}}\{d_{\alpha} - \frac{1}{2}\epsilon(b_{\alpha})\}) \\ (n \geq n_{\epsilon})$$

From (2.50) and (2.51) we get that given an $\epsilon(b_{\alpha}) > 0$, there exists an n_{ϵ} such that:

$$(2.52) \quad |a_n^{\alpha} L_n^{\alpha}(-b_{\alpha})| < \exp(-2n^{\frac{1}{2}}\{d_{\alpha} - b_{\alpha}^{\frac{1}{2}} - \epsilon(b_{\alpha})\}) \quad (n \geq n_{\epsilon})$$

From (2.52) and (2.49), given $\epsilon(b_{\alpha}) > 0$ there exists an n_{ϵ} such that:

$$|a_n^{\alpha} L_n^{\alpha}(z)| \leq |a_n^{\alpha} L_n^{\alpha}(-b_{\alpha})| \leq \exp(-2n^{\frac{1}{2}}\{d_{\alpha} - b_{\alpha}^{\frac{1}{2}} - \epsilon(b_{\alpha})\}) \quad (|z| \leq b_{\alpha}; n \leq n_{\epsilon})$$

Let us select the number $\epsilon(b_{\alpha}) < d_{\alpha} - b_{\alpha}^{\frac{1}{2}}$, then

$$d_{\alpha} - b_{\alpha}^{\frac{1}{2}} - \epsilon(b_{\alpha}) = c > 0$$

so that:

$$a_n^{\alpha} L_n^{\alpha}(z) \leq \exp(-2n^{\frac{1}{2}}c) \quad (c > 0; \alpha > -1; |z| \leq b_{\alpha}; n \geq n_{\epsilon})$$

Hence series (2.47) and hence series (1.2) p.1 converges absolutely and uniformly in the circle $C(b_{\alpha})$ by comparison with the convergent series (2.30) p. 12.

LEMMA (2.11)

If the Laguerre coefficient, a_n^{α} is given by (1.3) p.1 then:

$$(2.59) \quad a_n^{\alpha+1} = a_n^{\alpha} - a_{n+1}^{\alpha}$$

Proof: From Szego (6) p. 98 formula (5.1.14):

$$(2.60) \quad L_n^{\alpha-1}(x) = x^{-1} \{(n+\alpha+1) L_n^{\alpha}(x) - (n+1) L_{n+1}^{\alpha}(x)\}$$

From (1.3) p. 1:

$$a_n^{\alpha+1} = \left\{ \binom{n+\alpha+1}{n} \Gamma(\alpha+2) \right\}^{-1} \int_0^{\infty} e^{-x} x^{\alpha+1} L_n^{\alpha+1}(x) f(x) dx$$

By equation (2.60) the above relation becomes:

$$a_n^{\alpha+1} = \left\{ \binom{n+\alpha+1}{n} \Gamma(\alpha+2) \right\}^{-1} \left\{ (n+\alpha+1) \int_0^{\infty} e^{-x} x^{\alpha} L_n^{\alpha}(x) f(x) dx - (n+1) \int_0^{\infty} e^{-x} x^{\alpha} L_{n+1}^{\alpha}(x) f(x) dx \right\}$$

Substituting for the above integrals as given by (1.3)p.1

$$(2.61) \quad a_n^{\alpha+1} = \left\{ \binom{n+\alpha+1}{n} \Gamma(\alpha+2) \right\}^{-1} \times \\ \left\{ (n+\alpha+1) \binom{n+\alpha}{n} \Gamma(\alpha+1) a_n^{\alpha} - (n+1) \binom{n+1+\alpha}{n+1} \Gamma(\alpha+1) a_{n+1}^{\alpha} \right\}$$

We obtain the following identities by direct calculation:

$$\left\{ \binom{n+\alpha+1}{n} \right\}^{-1} \binom{n+\alpha}{n} = (\alpha+1)/(\alpha+n+1)$$

$$\left\{ \binom{n+\alpha+1}{n} \right\}^{-1} \binom{n+1+\alpha}{n-1} = (\alpha+1)/(n+1)$$

Substituting in (2.61) by means of the last two identities:

$$a_n^{\alpha+1} = \left\{ \Gamma(\alpha+1)(\alpha+1)/\Gamma(\alpha+2) \right\} (a_n^{\alpha} - a_{n+1}^{\alpha})$$

Since by (7) p. 55 formula (2) we have $(\alpha+1)\Gamma(\alpha+1) = \Gamma(\alpha+2)$ the lemma follows.

LEMMA (2.12)

If the following series:

$$(2.62) \quad \sum_{n=0}^{\infty} a_n^{\alpha} L_n^{\alpha}(z)$$

converges for $z \in p(b_{\alpha})$, then

$$(2.63) \quad \sum_{n=0}^{\infty} a_n^{\alpha} L_n^{\alpha}(z) = \sum_{n=0}^{\infty} a_n^{\alpha+1} L_n^{\alpha+1}(z) \quad (z \in p(b_{\alpha}))$$

Proof: We define the sum of Laguerre polynomials:

$$(2.64) \quad s_n^{\alpha} = \sum_{k=0}^n L_k^{\alpha}(z) \quad (n=0,1,2,\dots)$$

$$\sum_{k=0}^n a_k^{\alpha} L_k^{\alpha}(z) = a_0^{\alpha} s_0^{\alpha} + \sum_{k=1}^n a_k^{\alpha} (s_k^{\alpha} - s_{k-1}^{\alpha}) = \sum_{k=0}^{n-1} (a_k^{\alpha} - a_{k+1}^{\alpha}) s_k^{\alpha} + a_n^{\alpha} s_n^{\alpha}$$

From lemma (2.11) p. 23: $a_{n-1}^{\alpha} - a_n^{\alpha} = a_{n-1}^{\alpha+1}$

From Szego (6) p. 98 formula (5.1.13):

$$s_n^{\alpha} = \sum_{k=0}^n L_k^{\alpha}(z) = L_n^{\alpha+1}(z) \quad \text{so that}$$

$$(2.65) \quad \sum_{k=0}^n a_k^{\alpha} L_k^{\alpha}(z) = \sum_{k=0}^{n-1} a_k^{\alpha+1} L_k^{\alpha+1}(z) + a_n^{\alpha} L_n^{\alpha+1}(z)$$

Employing inequalities (2.20) p. 10 and (2.22) p. 11 by the same method used in obtaining (2.24) p. 11, we have:

$$|a_n^{\alpha} L_n^{\alpha+1}(z)| < \exp\{-2n^{\frac{1}{2}}(d_{\alpha} - \{\frac{1}{2}(r-x)\}^{\frac{1}{2}} - \ell(z))\} \quad (z \in p^*(b_{\alpha}))$$

Since we assume that series (2.62) converges for $z \in p(b_{\alpha})$ we conclude by lemma (2.7) p. 14: $d_{\alpha} > 0$.

Hence for a given point z excluded from the non-negative real axis and which satisfies the inequality $(\frac{1}{2}(r-x))^{\frac{1}{2}} < b_\alpha$:

$$|a_n^\alpha L_n^{\alpha+1}(z)| \rightarrow 0 \quad (n \rightarrow \infty; b_\alpha < d_\alpha)$$

But by lemma (2.3) p. 9 $(\frac{1}{2}(r-x))^{\frac{1}{2}} < b_\alpha \Rightarrow z \in p(b_\alpha)$, so that by definition (2.3) p. 6 the above limit relation reads:

$$|a_n^\alpha L_n^{\alpha+1}(z)| \rightarrow 0 \quad (n \rightarrow \infty; z \in p^*(b_\alpha))$$

Applying this to identity (2.65) the lemma follows for z restricted to the region $z \in p^*(b_\alpha)$.

Now for z on the non-negative real axis, we get from relation (2.33) and (2.34) p. 13:

$$|L_n^{\alpha+1}(x)| < K e^{\frac{1}{2}x} n^{\alpha+1}$$

Using inequality (2.20) p. 10 with the above inequality we get that given an $\epsilon > 0$ independent of z there is an n_ϵ such that:

$$(2.66) \quad |a_n^\alpha L_n^{\alpha+1}(x)| < K e^{\frac{1}{2}x} n^{\alpha+1} \exp(-2n^{\frac{1}{2}}(d_\alpha - \epsilon)) \quad (n \geq n_\epsilon)$$

From considerations of the power series expansion of $\exp(2n^{\frac{1}{2}}(d_\alpha - \epsilon))$ we get the inequality:

$$\exp(2n^{\frac{1}{2}}(d_\alpha - \epsilon)) > (2n^{\frac{1}{2}}(d_\alpha - \epsilon))^p / p! \quad (p = 0, 1, 2, \dots)$$

so that from inequality (2.66) we have:

$$|a_n^{\alpha} L_n^{\alpha+1}(x)| < K e^{\frac{1}{2}x} n^{\alpha+1} p! / (2n^{\frac{1}{2}}(d_{\alpha} - \epsilon))^p$$

$$= K e^{\frac{1}{2}x} p! (2(d_{\alpha} - \epsilon))^{-p} \exp(-\frac{1}{2}p + \alpha + 1) \rightarrow 0 \quad (n \rightarrow \infty; p > 2(\alpha + 2))$$

Considering this fact in conjunction with (2.65) the lemma follows for a given z on the x -axis.

COROLLARY (2.12)

If d_{α} is given by equality (1.5) p. 2, the following inequality holds:

$$(2.67) \quad d_{\alpha} \leq d_{\alpha+1} \quad (d_{\alpha} \geq 0)$$

Proof: By relation (2.63) and lemma (2.6) p. 13:

$$z \in p(d_{\alpha}) \Rightarrow z \in p(d_{\alpha+1})$$

From lemma (2.3) p. 9 the corollary follows.

LEMMA (2.13)

Let the points z of the z -plane and the points w of the w -plane be made to correspond by the transformation:

$$(2.68) \quad z = w^2 = (u + iv)^2$$

then

$$(2.69) \quad z \in p(b) \Leftrightarrow w \in s(b)$$

Proof: From transformation (2.68) we have:

$$(2.70) \quad \begin{aligned} x &= u^2 - v^2 \\ y &= 2uv \end{aligned}$$

Now let $z \in p(b)$. Hence by lemma (2.3) p. 9 we have:

$$(2.71) \quad \frac{1}{2}(r - x) < b^2$$

and by means of the transformation (2.70) we have:

$$(2.72) \quad \frac{1}{2}(r-x) = \frac{1}{2}\left\{ (u^2 + v^2)^{\frac{1}{2}} + (u^2 - v^2) \right\} = v^2$$

so that by the transformation (2.70) inequality (2.71)

$$\text{becomes:} \quad v^2 < b^2 \quad -\infty < u < \infty$$

Hence by definition (1.2) p. 1 we conclude that $w \in s(b)$.

Conversely, let $w \in s(b)$. Then by definition (1.2) p. 1 we obtain:

$$v^2 < b^2 \quad -\infty < u < \infty$$

and by means of the transformation (2.70) we see from (2.72)

that the above inequality becomes:

$$\frac{1}{2}(r-x) < b^2$$

Hence by lemma (2.3) p.9, we get that $z \in p(b)$.

LEMMA (2.14)

If for $h(w^2)$, an "arbitrary" function of w , there exists a positive number $B(b)$ such that:

$$(2.73) \quad |h(w^2)| < B(b) \exp\left\{ (u^2 - v^2)q^{-1} - k|u^2 - v^2|^{\frac{1}{2}} (b^2 - v^2)^{\frac{1}{2}} \right\}$$

($w \in s(b)$; $q, k > 0$)

then there exists a positive number $B(b)$ such that:

$$(2.74) \quad |h(w^2)| < B(b) \exp\left\{ (u^2/q) - k(u^2)^{\frac{1}{2}} (b^2 - v^2)^{\frac{1}{2}} \right\}$$

($w \in s(b)$; $q, k > 0$)

(where the positive number $B(b)$ may differ from line to line) and conversely.

Proof: Let us assume that inequality (2.73) holds.

Since the following inequality holds:

$$(2.76) \quad \exp(-v^2) \leq 1$$

we conclude from inequality (2.73) that:

$$(2.77) \quad |h(w^2)| < B(b) \exp(u^2 q^{-1} - k |u^2 - v^2|^{\frac{1}{2}} (b^2 - v^2)^{\frac{1}{2}}) \\ (w \in s(b); q, k > 0)$$

To arrive at our first conclusion we need to prove:

$$(2.78) \quad |u^2 - v^2|^{\frac{1}{2}} + b \geq |u^2 - v^2|^{\frac{1}{2}} + (v^2)^{\frac{1}{2}} > (u^2)^{\frac{1}{2}} \quad (w \in s(b))$$

We proceed as follows:

$$(|u^2 - v^2|^{\frac{1}{2}} - (v^2)^{\frac{1}{2}})^2 = u^2 - v^2 - 2 |u^2 - v^2|^{\frac{1}{2}} (v^2)^{\frac{1}{2}} - v^2 \\ > u^2 - v^2 - 2 |u^2 - v^2|^{\frac{1}{2}} (v^2)^{\frac{1}{2}} + v^2 = u^2 + 2 |u^2 - v^2|^{\frac{1}{2}} (v^2)^{\frac{1}{2}} > u^2$$

That is: $(|u^2 - v^2|^{\frac{1}{2}} + (v^2)^{\frac{1}{2}}) > u^2$

Hence: $|u^2 - v^2|^{\frac{1}{2}} + (v^2)^{\frac{1}{2}} > (u^2)^{\frac{1}{2}}$

Remembering that $w \in s(b)$ implies $|v| < b$ the above inequality establishes relation (2.78). From (2.78)

$$(2.79) \quad -|u^2 - v^2|^{\frac{1}{2}} < -(u^2)^{\frac{1}{2}} + b$$

From inequality (2.77) and (2.79) it follows that:

$$(2.80) \quad |h(w^2)| < B(b) \exp(u^2 q^{-1} - k [-(u^2)^{\frac{1}{2}} - b] (b^2 - v^2)^{\frac{1}{2}}) \\ (w \in s(b); q, k > 0, b \geq 0)$$

$$= B(b) \exp\{kb(b^2 - v^2)^{\frac{1}{2}}\} \exp\{u^2 q^{-1} - k(u^2)^{\frac{1}{2}}(b^2 - v^2)^{\frac{1}{2}}\}$$

But since $v^2 < b^2$ if $w \in s(b)$ we can write:

$$(2.81) \quad \exp(kb(b^2 - v^2)^{\frac{1}{2}}) \leq \exp(kb(b^2)^{\frac{1}{2}}) = B(b) \quad (w \in s(b))$$

so that from relation (2.81) and (2.80) we conclude:

$$(2.82) \quad |h(w^2)| < B(b) \exp(u^2 q^{-1} - k(u^2)^{\frac{1}{2}} (b^2 - v^2)^{\frac{1}{2}}) \quad (w \in s(b); q, k > 0)$$

Conversely, we want to prove, that if inequality (2.74) is true then inequality (2.73) follows.

The following inequalities hold:

$$(u^2)^{\frac{1}{2}} + b \geq (u^2)^{\frac{1}{2}} + (v^2)^{\frac{1}{2}} \geq (u^2 + v^2)^{\frac{1}{2}} \geq |u^2 - v^2|^{\frac{1}{2}} \quad (w \in s(b))$$

Hence:

$$(2.82) \quad -(u^2)^{\frac{1}{2}} < b - |u^2 - v^2|^{\frac{1}{2}} \quad (w \in s(b))$$

Applying inequality (2.82) to (2.74) we have:

$$(2.83) \quad |h(w^2)| < B(b) \exp\{u^2 q^{-1} + k(b - |u^2 - v^2|^{\frac{1}{2}})(b^2 - v^2)^{\frac{1}{2}}\} \\ (w \in s(b); q, k > 0)$$

$$= B(b) \exp(kb(b^2 - v^2)^{\frac{1}{2}}) \exp(u^2 q^{-1} - k|u^2 - v^2|^{\frac{1}{2}}(b^2 - v^2)^{\frac{1}{2}}) \\ (w \in s(b); q, k > 0)$$

From inequalities (2.83) and (2.81) we conclude:

$$|h(w^2)| \leq B(b) \exp\{u^2 q^{-1} - k|u^2 - v^2|^{\frac{1}{2}}(b^2 - v^2)^{\frac{1}{2}}\} \quad (w \in s(b); q, k > 0) \\ = B(b) \exp(v^2 q^{-1}) \exp(q^{-1}(u^2 - v^2) - k|u^2 - v^2|^{\frac{1}{2}}(b^2 - v^2)^{\frac{1}{2}}) \\ (w \in s(b); q, k > 0)$$

$$\text{But: } \exp(v^2 q^{-1}) < \exp(b^2 q^{-1}) = B(b) \quad (w \in s(b); q > 0)$$

so that inequality (2.73) follows.

The next lemma gives an estimate of the following integral which will be essential in proving the necessity of theorem \bar{B} .

LEMMA (2.15)

Given a number b , $0 \leq b < d$ there exists a positive number $B(b)$ such that:

$$(2.84) \quad 0 \leq \bar{I}(z) \equiv$$

$$\int_0^{\infty} t^{-3/2} \exp(-4c^2/t) (1-e^{-t})^{-1} \exp\{-2x(e^t-1)^{-1} + 2|z|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})\} dt$$

$$\leq B(b) \exp\left(x - 2|x| \frac{1}{2} \left\{ b^2 - \frac{1}{2}(r-x) \right\}^{\frac{1}{2}} \right) \quad (z \in p(b); c \equiv \frac{1}{2}(b+d))$$

Proof: Pollard shows (5) p. 359-60 that given a number b $0 \leq b < d$ there exists a positive number $B(b)$ such that:

$$(2.85) \quad 0 \leq \bar{I}(w^2) =$$

$$\int_0^{\infty} t^{-3/2} (1-e^{-t})^{-1} \exp\left(\frac{-4c^2}{t}\right) \exp\{-2(u^2-v^2)(e^t-1)^{-1} + 2|w^2|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})\} dt$$

$$\leq B(b) \exp\left(u^2 - 2|u^2| \frac{1}{2} (b^2 - v^2)^{\frac{1}{2}} \right) \quad (w \in s(b))$$

By lemma (2.14) p. 28 with $q=1$, $k=2$ we see that inequality (2.85) implies:

$$I(w^2) \leq B(b) \exp\left(u^2 - v^2 - 2|u^2 - v^2| \frac{1}{2} (b^2 - v^2)^{\frac{1}{2}} \right) \quad (w \in s(b))$$

Now by lemma (2.13) and transformation (2.68), (2.70) p. 27 and identity (2.72) p. 28 the above inequality gives:

$$0 \leq I(z) \leq B(b) \exp\left(x - 2|x| \frac{1}{2} \left\{ b^2 - \frac{1}{2}(r-x) \right\}^{\frac{1}{2}} \right) \quad (z \in p(b))$$

This completes the proof of the lemma.

LEMMA (2.16)

(Riemann-Schwarz principle of reflection)

If two functions $f(z)$ and $f_1(z)$ are analytic in regions R and R_1 , separated by a contour C , and continuous on C , and $f(z) = f_1(z)$ along C , then the two functions are analytic continuations of each other.

For the proof see Titchmarsh (7) p. 155, section 4.51.

LEMMA (2.17)

Suppose the $L: \sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(z)$ converges for $z \in p(d)$

($d > 0$, $-1 < \alpha$) where the number d may depend on α , then the series L is analytic for $z \in p(d)$.

Proof: Let $z_0 \in p^*(d)$ (see definition (2.3) p. 6). Then, there exists a number $b < d$ and a set $S^*(b)$ such that $z_0 \in S^*(b)$ and hence by definition (2.8) part b) p. 21, we conclude that $z_0 \in \bar{S}^*(b)$. Since series L converges inside the parabola $p(d)$ there is a point z_1 not on the axis of reals at which series L converges. Consequently, we may use lemma (2.7) p. 14, so that we have $d_\alpha > 0$ where d_α is defined by statement (1.5) p.2. Therefore by lemma (2.4) p. 10 we conclude that $d \leq d_\alpha$ and the series L converges absolutely and uniformly for all $z \in \bar{S}^*(b)$.

As a result of this and by a theorem on a uniformly convergent series of analytic functions (see (3) p. 74

theorem 3) the series L is analytic in $\overline{S^*}(b)$ and therefore also analytic in $S^*(b)$. Thus the series L is analytic in a neighborhood of every point z such that $z \in p^*(d)$.

Let us define the region R of lemma (2.16) to be the set of all points z such that $z \in p(d)$ and $y > 0$, the region R_1 to be the set of all points z such that $z \in p(d)$ and $y < 0$, the contour C to be all points z on the axis of reals for which $x \geq -d$ the function $f(z)$ to be series L for $z \in R$ and the function $f_1(z)$ to be the series L for $z \in R_1$.

On the part of contour C consisting of the non-negative real axis series L is uniformly convergent by lemma (2.5) p. 12 and hence is continuous, while on the negative part series L is analytic by the first part of this proof and hence is continuous.

Hence by lemma (2.16) the series L in the upper half of the parabola $p(d)$ is an analytic continuation of the series L in the lower half. That is, the series L is analytic throughout the parabola $p(d)$. This completes the proof.

3. The Necessity: The Fundamental Inequality

The aim of this section is to show that:

Given $b_\alpha, 0 \leq b_\alpha < d_\alpha$ there is a positive number $B(\alpha, b)$ such that:

$$(3.1) \quad 0 \leq I_\alpha(z) \equiv$$

$$\int_0^\infty \frac{\exp(-4c_\alpha^2/t) \exp\{-2x/(e^t-1)\} \exp(\frac{1}{2}\alpha t) J_\alpha(i|z|\operatorname{csch}\frac{1}{2}t) dt}{\Gamma(\alpha+1) t^{3/2} (1-e^{-t}) |z|^\alpha i^\alpha}$$

$$\leq B(\alpha, b) \exp\left(x - 2|x|^{1/2} (b_\alpha^2 - \frac{1}{2}(r-x))^{1/2}\right) \quad (z \in p(b_\alpha); -1 < \alpha)$$

where the symbol c_α is defined $c_\alpha \equiv \frac{1}{2}(b_\alpha + d_\alpha)$ and the number $B(\alpha, b), C_\alpha$ may differ from line to line.

The first step in the long and tedious discourse which follows is to show that:

There exists positive numbers C_α such that

$$(3.2) \quad I_\alpha(z) \leq$$

$$t(z) \int_0^{c_\alpha} \frac{\exp(-4c_\alpha^2/t) \exp\{-2x/(e^t-1)\} \exp(\frac{1}{2}\alpha t) \exp\{2|z|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})\} dt}{t^{3/2} (1-e^{-t}) |z|^\alpha}$$

$$+C_{\alpha} \int_{t(z)}^{\infty} \frac{\exp(-4c_{\alpha}^2/t) \exp\{-2x/(e^t-1)\}}{t^{3/2}(1-e^{-t})} \left\{ \frac{2e^{\frac{1}{2}t}}{(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})} \right\}^{\alpha} dt \quad (z \in p(b_{\alpha}))$$

where for brevity we write $t(z)$ for the symbol $t(|z|)$ and the latter designates the solution of the equation:

$$(3.3) \quad |z| = e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \quad \text{or} \quad t(|z|) = \sinh^{-1} |z|/2$$

Let us define:

$$(3.4) \quad \overset{*}{I}_{\alpha}(z) \equiv \int_0^{t(z)} \frac{\exp(-4c_{\alpha}^2/t) \exp\{-2x/(e^t-1)\} \exp(\frac{1}{2}\alpha t) \exp\{2|z|/(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})\}}{t^{3/2}(1-e^{-t})|z|^{\alpha}} dt$$

$$(3.5) \quad \overset{0}{I}_{\alpha}(z) \equiv \int_{t(z)}^{\infty} t^{-3/2} \exp(-4c_{\alpha}^2/t) (1-e^{-t}) \exp\{-2x/(e^t-1)\} \left\{ \frac{2e^{\frac{1}{2}t}}{(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})} \right\}^{\alpha} dt$$

Relation (3.2) will then take the form:

$$(3.6) \quad I_{\alpha}(z) \leq C_{\alpha} \overset{*}{I}_{\alpha}(z) + C_{\alpha} \overset{0}{I}_{\alpha}(z) \quad (z \in p(b_{\alpha}))$$

By direct calculation it can be verified from the power series expansion of $J_{\alpha}(z)$ found in (6) p. 14 formula (1.71.1) that:

$$i^{-\alpha} J_{\alpha}(is) = |J_{\alpha}(is)|$$

Hence from formula (3.1) we see that the integral $I_{\alpha}(z) \geq 0$ since its integrand is non-negative. In what follows in this section if an integral is declared to be non-negative without proof it will be because the integrand is

non-negative.

From Szego (6) p. 15 formula (1.71.7) we can derive the following relationship:

$$(3.7) \quad |J_\alpha(is)| < C_\alpha \exp(|I(is)|) \quad (s \geq 2)$$

where in the above formula the symbol $I(is)$ is to mean the imaginary part of is . The number 2 is selected for convenience. Also from Szego (6) p. 16 formula (1.71.10) we obtain:

$$(3.8) \quad |J_\alpha(is)| < C_\alpha s^\alpha \quad (0 \leq s \leq 2)$$

If in relation (3.8) we set $s = |z| \operatorname{csch} \frac{1}{2}t$, the result is:

$$(3.9) \quad |J_\alpha(i|z| \operatorname{csch} \frac{1}{2}t)| < C_\alpha |z|^\alpha (\operatorname{csch} \frac{1}{2}t)^\alpha \\ (0 \leq |z| \operatorname{csch} \frac{1}{2}t < 2)$$

Since by definition:

$$(3.7) \quad \operatorname{csch} \frac{1}{2}t \equiv 2/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}) \geq 0 \quad (t > 0)$$

Relation (3.9) is valid if z and t are connected by the relation:

$$(3.11) \quad |z| < e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \quad (t > 0)$$

From the estimate (3.7) we conclude:

$$(3.12) \quad |J_\alpha(i|z| \operatorname{csch} \frac{1}{2}t)| < C_\alpha \exp(|z| \operatorname{csch} \frac{1}{2}t) \\ (|z| \operatorname{csch} \frac{1}{2}t \geq 2; t > 0)$$

That is, relation (3.12) holds if z and t are connected by the relation:

$$(3.13) \quad |z| \geq e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \quad (t > 0)$$

Since by differentiation, we can easily verify that the function $e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}$ is a monotonic increasing function of t we get from equation (3.3) p. 36:

$$(3.14) \quad |z| = \exp\left(\frac{1}{2}t(z)\right) - \exp\left(-\frac{1}{2}t(z)\right) < e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \quad (t(z) < t)$$

Thus relation (3.11) and hence relation (3.9) holds for $t > t(z)$. Similarly relation (3.12) holds for all $0 < t \leq t(z)$. Now let us write for $I_\alpha(z)$ defined by statement (3.1) p. 35.

$$(3.15) \quad I_\alpha(z) = \int_0^{t(z)} + \int_{t(z)}^\infty \frac{\exp\{-2x/(e^t-1)\} \exp(\frac{1}{2}\alpha t) J_\alpha(i|z| \operatorname{csch}\frac{1}{2}t) dt}{t^{3/2} \exp(4c_\alpha^2/t) (1-e^{-t}) |z|^\alpha i^\alpha \Gamma(\alpha+1)}$$

Hence we can apply the estimate (3.12) to the first integral above and inequality (3.9) to the second getting inequality (3.2) p. 35 and hence (3.6) p. 36 obtaining our first objective.

In order to prove inequality (3.1) p. 35 we must split the proof into several different cases.

CASE I

$$(-1 < \alpha \leq 0)$$

Given a b_α , $0 \leq b_\alpha < d_\alpha$, there exists a positive number $B(\alpha, b)$ such that:

$$(3.16) \quad 0 \leq I_\alpha(z) \leq B(\alpha, b) \exp\left\{x-2|x|^{1/2} \left(b_\alpha^2 - \frac{1}{2}(r-z)^{1/2}\right)\right\}$$

where the above inequality holds for $z \in p(b_\alpha)$ and $-1 < \alpha < 0$.

Proof: It follows immediately that:

$$(3.17) \quad 0 \leq \left(\frac{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}{2e^{\frac{1}{2}t}} \right)^{-\alpha} \leq 1 = e^0 \leq \exp\left(2|z| / \left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \right) \right) \quad (0 \leq t; -1 < \alpha < 0)$$

Going back to relation (2.60) p. 23 define:

$$(3.18) \quad \begin{aligned} \bar{I}_\alpha(z) &\equiv \bar{I}(z) && (b = b_\alpha; c = c_\alpha; d = d_\alpha) \\ \bar{I}_\alpha(z) &= \\ &\int_0^\infty \frac{\exp(-2x/(e^t-1)) \exp(2|z|/e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})}{t^{3/2} \exp(4c_\alpha^2/t)(1-e^{-t})} dt \end{aligned}$$

Then by lemma (2.15) p.31, given a positive number b_α ,

$0 \leq b_\alpha < d_\alpha$ there exists a positive number $B(\alpha, b)$ such that:

$$(3.19) \quad 0 \leq \bar{I}_\alpha(z) \leq B(\alpha, b) \exp\left(x-2|x| \frac{1}{2} \left\{ b_\alpha^2 - \frac{1}{2}(r-x) \right\}^{\frac{1}{2}} \right) \\ (z \in p(b_\alpha); c_\alpha \equiv \frac{1}{2}(b_\alpha - d_\alpha))$$

By virtue of relation (3.5) p. 35, (3.17) and (3.19) we get that there is a $B(\alpha, b)$ such that:

$$(3.20) \quad 0 \leq \overset{0}{I}_\alpha(z) \leq \bar{I}_\alpha(z) \leq B(\alpha, b) \exp\left(x-2|x| \frac{1}{2} \left\{ b_\alpha^2 - \frac{1}{2}(r-x) \right\}^{\frac{1}{2}} \right) \\ (z \in p(b_\alpha); -1 < \alpha \leq 0)$$

We now hope to show (see (3.4) p.35) that there is a $B(\alpha, b)$ such that:

$$(3.21) \quad 0 \leq \overset{*}{I}_\alpha(z) \leq B(\alpha, b) \exp\left(x-2|x| \frac{1}{2} \left\{ b_\alpha^2 - \frac{1}{2}(r-x) \right\}^{\frac{1}{2}} \right)$$

where the above inequality is valid for $z \in p(b_\alpha)$ and $-1 < \alpha \leq 0$.

Relations (3.20) and (3.21) applied to (3.6) p.35 will establish inequality (3.16).

Given a b_α such that $0 \leq b_\alpha < d_\alpha$, pick \bar{b}_α such that $0 \leq b_\alpha < \bar{b}_\alpha < d_\alpha$. Then by inequality (3.19) there exists a positive number $B(\bar{b}_\alpha)$ such that:

$$(3.22) \quad \bar{I}_\alpha(z) \leq B(\alpha, \bar{b}) \exp(x-2|x|^{\frac{1}{2}} \{ \bar{b}_\alpha^2 - \frac{1}{2}(r-x) \}^{\frac{1}{2}})$$

($z \in p(\bar{b}_\alpha)$)

Since $0 < \exp(\frac{1}{2}\alpha t) \leq 1$ ($0 \leq t; -1 < \alpha \leq 0$)

We see from the estimate (3.22) and (3.4) p. 35 if we recall the form of $\bar{I}_\alpha(z)$ as defined by (3.18) p. 38, that

$$(3.23) \quad 0 \leq \bar{I}_\alpha^*(z) \leq |z|^{-\alpha} \bar{I}_\alpha(z) \leq |z|^{-\alpha} B(\alpha, \bar{b}) \exp(x-2|x|^{\frac{1}{2}} \{ \bar{b}_\alpha^2 - \frac{1}{2}(r-x) \}^{\frac{1}{2}})$$

($z \in p(\bar{b}_\alpha); -1 < \alpha \leq 0$)

$$= B(\alpha, \bar{b}) \exp(2|x|^{\frac{1}{2}} \{ \bar{b}_\alpha^2 - \frac{1}{2}(r-x) \}^{\frac{1}{2}} - \alpha \log |z|) \times \times$$

$$\exp(x-2|x|^{\frac{1}{2}} \{ \bar{b}_\alpha^2 - \frac{1}{2}(r-x) \}^{\frac{1}{2}})$$

($z \in p(b_\alpha)$)

The function $(\bar{b}_\alpha^2 - \frac{1}{2}(r-x))^{\frac{1}{2}} - (\bar{b}_\alpha^2 - \frac{1}{2}(r-x))^{\frac{1}{2}}$
 The function $(\bar{b}_\alpha^2 - \frac{1}{2}(r-x))^{\frac{1}{2}} - (\bar{b}_\alpha^2 - \frac{1}{2}(r-x))^{\frac{1}{2}}$

is a decreasing function of $\frac{1}{2}(r-x)$ as the first derivative will show. Since the function $\frac{1}{2}(r-x)$ is non-negative the above expression therefore has its largest value for $\frac{1}{2}(r-x) = 0$. Hence from (3.23) :

$$(3.24) \quad 0 \leq \overset{*}{I}_\alpha(z) \leq |z|^{-\alpha} \bar{I}_\alpha(z) \leq$$

$$B(\alpha, \bar{b}) \exp\left(x - 2|x| \frac{1}{2} \left\{ b_\alpha^2 - \frac{1}{2}(r-x) \right\} \frac{1}{2} \right) \exp\left\{ 2|x| \frac{1}{2} (b_\alpha - \bar{b}_\alpha) - \alpha \log |z| \right\}$$

Since $z \in p(b_\alpha)$ we have by definition (1.1) p. 2

$$y^2 < 4b_\alpha(x + b_\alpha^2) \quad \text{so that:}$$

$$\log |z| \leq 4!(|z|)^{\frac{1}{4}} \leq 4!(x^2 + 4b_\alpha^2(x + b_\alpha^2))^{\frac{1}{2}})^{\frac{1}{4}} = 4!(x + 2b_\alpha^2)^{\frac{1}{4}}$$

Hence for $-1 < \alpha \leq 0$:

$$(3.25) \quad \exp\left(2|x| \frac{1}{2} (b_\alpha - \bar{b}_\alpha) - \alpha \log |z| \right) \leq$$

$$\exp\left(2|x| \frac{1}{2} (b_\alpha - \bar{b}_\alpha) - \alpha 4!(x + 2b_\alpha^2)^{\frac{1}{4}} \right) \leq B(\alpha, b)$$

As the above exponent has a maximum with respect to x for $z \in p(b_\alpha)$, $b_\alpha < \bar{b}_\alpha$ and $-1 < \alpha \leq 0$.

From inequalities (3.24) and (3.25) since \bar{b}_α is a function of b_α , we get that there exists a positive number $B(\alpha, b)$ such that:

$$(3.26) \quad 0 \leq \overset{*}{I}_\alpha(z) \leq |z|^{-\alpha} \bar{I}_\alpha(z) \leq$$

$$B(\alpha, b) \exp\left(x - 2|x| \frac{1}{2} \left\{ b_\alpha^2 - \frac{1}{2}(r-x) \right\} \frac{1}{2} \right) \quad (z \in p(b_\alpha); -1 < \alpha \leq 0)$$

Hence inequality (3.21) p. 38 is established, and recalling that relations (3.20) and (3.21) applied to (3.6) establish (3.16), we see that obtaining inequality (3.26) completes the proof of case I.

CASE II

$$(z \in p(b_\alpha); |z| \leq r_1(\alpha) < 1; \alpha > 0)$$

Given b_α , $0 \leq b_\alpha < d_\alpha$, there exists a positive number $B(\alpha, b)$ such that:

$$(3.27) \quad 0 \leq I_\alpha(z) \leq B(\alpha, b) \exp\left(x - 2|x| \frac{1}{2} \{b_\alpha^2 - \frac{1}{2}(r-x)\}^{\frac{1}{2}}\right) \\ (z \in p(b_\alpha); |z| \leq r_1(\alpha) < 1; \alpha > 0)$$

Proof: First, we want to show that for the integral $I_\alpha(z)$ defined by (3.5) p. 35, there exists a positive number $B(\alpha, b)$ such that:

$$(3.28) \quad 0 \leq I_\alpha(z) \leq B(\alpha, b) \quad (z \in p(b_\alpha); \alpha \geq 0)$$

From (3.5) p. 35 we have:

$$(3.29) \quad 0 \leq I_\alpha(z) = \\ \int_0^\infty \frac{\exp(-2(x-b^2)/(e^t-1)) \exp(2b^2/(e^t-1))}{\exp(4c_\alpha^2/t) t^{3/2} (1-e^{-t})} \left[\frac{2e^{\frac{1}{2}t}}{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}} \right]^\alpha dt \\ \leq \int_0^1 \frac{\exp(-2(x-b^2)/(e^t-1)) \exp(2b^2/(e^t-1))}{\exp(4c_\alpha^2/t) t^{3/2} (1-e^{-t})} \left[\frac{2e^{\frac{1}{2}t}}{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}} \right]^\alpha dt \\ + \int_1^\infty \frac{\exp(-4c_\alpha^2/t) \exp(-2x/(e^t-1))}{t^{3/2} (1-e^{-t})} \left[\frac{2e^{\frac{1}{2}t}}{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}} \right]^\alpha dt$$

We are going to use the following inequalities on the integrals of relationship (3.29). By a consideration of the power series expansion of e^{-t} we get:

$$(3.30) \quad (1 - e^{-t})^{-1} < 1/(t - t^2/2!) = t^{-1}(1 - \frac{1}{2}t)^{-1} < t^{-1}(\frac{1}{2})^{-1} = 2/t$$

$$(0 \leq t \leq 1)$$

By a consideration of the power series expansion of e^t we get:

$$(3.31) \quad 1/(e^t - 1) < 1/t \quad (t \geq 0)$$

By definition (1.1) and inequality (1.7) p.2 the proposition $z \in p(b_\alpha)$ implies the proposition $-b_\alpha^2 < x$.

From the form of the following expressions it follows that:

$$(3.32) \quad 0 \leq (2e^{\frac{1}{2}t} / (e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}))^\alpha = (2/(1 - e^{-t}))^\alpha < C_\alpha$$

$$(1 \leq t < \infty; \alpha > 0)$$

$$(3.33) \quad 0 \leq \exp(-2x/(e^t - 1)) < C_\alpha \quad (1 \leq t < \infty; \alpha > 0; -b_\alpha^2 \leq x)$$

$$(3.34) \quad 0 \leq 1/(1 - e^{-t}) < K \quad (1 \leq t < \infty)$$

$$(3.35) \quad 0 \leq \exp(-4c_\alpha^2/t) < 1 \quad (0 \leq t < \infty)$$

By using inequalities (3.32) (3.33) (3.34) and (3.35) on the second integral of (3.29) we find that it is less than:

$$(3.36) \quad B(\alpha, b) \int_1^\infty t^{-3/2} dt = B(\alpha, b) \quad (z \in p(b_\alpha); \alpha > 0)$$

We now want to show that the first integral of relation (3.29) is less than a positive number $B(\alpha, b)$. By inequality (3.30) we have:

$$(3.37) \quad 2e^{\frac{1}{2}t} / (e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}) = 2 / (1 - e^{-t}) < 4/t \quad (0 \leq t \leq 1)$$

If $z \in p(b_\alpha)$ then by inequality (1.7) p.2 we get that $x + b_\alpha^2 > 0$ so that:

$$(3.38) \quad \exp(-2(x + b_\alpha^2) / (e^t - 1)) \leq 1 \quad (z \in p(b_\alpha); \alpha > 0; t \geq 0)$$

By use of inequalities (3.30) (3.38) (3.31) and (3.37) the first integral of (3.29) p. 41 is less than:

$$(3.39) \quad B(\alpha, b) \int_0^1 t^{-3/2} \exp(-4c_\alpha^2/t) (2/t) \exp(2b_\alpha^2/t) (4/t)^\alpha dt \\ = 2^{2\alpha-1} B(\alpha, b) \int_0^1 \exp t(-3/2 - \alpha - 1) \exp\{-4c_\alpha^2 + 2b_\alpha^2\}/t\} dt$$

Since $c_\alpha \equiv \frac{1}{2}(b_\alpha + d_\alpha)$, we have $b_\alpha < c_\alpha < d_\alpha$.

Let $a \equiv (4c_\alpha^2 - 2b_\alpha^2) > 0$, and $s = a/t$.

Then the above integral becomes:

$$0 \leq B(\alpha, b) \exp a(\frac{1}{2} - (\alpha + 1)) \int_a^\infty \exp s(-\frac{1}{2} + (\alpha + 1)) e^{-s} ds \\ \leq B(\alpha, b) \Gamma(\alpha + 3/2)$$

So that the first integral of (3.29) p. 41 is less than $B(\alpha, b) \Gamma(\alpha + 3/2)$. This together with relations (3.56) p.

42 and (3.29) p. 41 establishes (3.28) p. 41.

If the integral $I_{\alpha}^*(z)$ is given by (3.4) p. 35 we now hope to show that there exists a positive number $B(\alpha, b)$ such that:

$$(3.39) \quad 0 \leq I_{\alpha}^*(z) \leq B(\alpha, b) \quad (z \in p(b_{\alpha}); |z| \leq r_1(\alpha) \leq 1; \alpha > 0)$$

$$\max\{\exp(\alpha t/2) \mid 0 \leq t \leq t(z)\} = \exp(\alpha t(z)/2)$$

Hence from (3.4) p. 35:

$$(3.40) \quad 0 \leq I_{\alpha}^*(z) \leq$$

$$\exp(\alpha t(z)/2) |z|^{-\alpha} \int_0^{t(z)} \frac{\exp(-2x/(e^t-1)) \exp\{2|z|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})\}}{t^{3/2} \exp(4c_{\alpha}^2/t)(1-e^{-t})} dt$$

Consider the function:

$$(3.41) \quad |z| = e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}$$

By subtracting the power series expansion of $e^{-\frac{1}{2}t}$ from the power series expansion of $e^{\frac{1}{2}t}$ we verify that:

$$(3.42) \quad |z| = e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} > t \quad (t \geq 0)$$

But by relationship (3.5) p. 35 we can write equation (3.42)

$$(3.43) \quad |z| = \exp\{\frac{1}{2}t(z)\} - \exp\{-\frac{1}{2}t(z)\} > t(z)$$

From relations (3.43) and (3.40) we get:

$$(3.44) \quad 0 \leq I_{\alpha}^*(z) \leq$$

$$\int_0^{|z|} \frac{\exp\{-2(x+b_\alpha^2)/e^t-1\} \exp\{2b_\alpha^2/(e^t-1)\} \exp\{2|z|e^{-\frac{1}{2}t}/(1-e^{-t})\} dt}{\exp(-\alpha t(z)/2) |z|^\alpha t^{-\alpha} \exp t(3/2+\alpha) \exp(4c_\alpha^2/t)(1-e^{-t})}$$

Moreover:

$$(3.45) \quad e^t = 1+t+t(t/2! + t^2/3! + \dots) \quad 1+t+t(e^t-1) \\ < 1+t+t(3-1) = 1+3t \quad (0 \leq t \leq 1)$$

From which we conclude that:

$$(3.46) \quad -1/(e^t-1) < -1/3t \quad (0 \leq t \leq 1)$$

In addition we have:

$$(3.47) \quad 0 \leq \exp(\alpha \frac{1}{2}t(z)) \leq C_\alpha \quad (|z| < 1)$$

$$(3.48) \quad 0 \leq e^{-\frac{1}{2}t} \leq 1 \quad (t \geq 0)$$

$$(3.49) \quad \max(t^\alpha | 0 \leq t \leq |z|) = |z|^\alpha$$

By using relations (3.49) (3.47) (3.30) p. 42 (3.46) (3.31) p. 42 and (3.48) on (3.44) we have:

$$(3.50) \quad 0 \leq I_\alpha^*(z)$$

$$C_\alpha \int_0^{|z|} \frac{2 \exp\{-2(x+b_\alpha^2)/3t\} \exp(2b_\alpha^2/t) \exp(4|z|/t) dt}{|z|^\alpha |z|^{-\alpha} \exp t(3/2+\alpha) \exp(4c_\alpha^2/t) t}$$

$$B(\alpha, b) = \int_0^{|\alpha|} \frac{\exp\{-2c_\alpha^2 - (2/3)(x + b_\alpha^2) - 2(b_\alpha^2 + 2|\alpha|z)\}/t}{\exp t(5/2 + \alpha)} dt$$

Define:

$$\begin{aligned} a &\equiv 4c_\alpha^2 + (2/3)(x + b_\alpha^2) - 2(b_\alpha^2 + 2|\alpha|z) \\ &= 4c_\alpha^2 - 2b_\alpha^2 + (2/3)(x + b_\alpha^2) - 4|\alpha|z \end{aligned}$$

If $|\alpha| < \frac{1}{2}b_\alpha^2$; then we have:

$$a > 4(c_\alpha^2 - b_\alpha^2) + (2/3)(x + b_\alpha^2)$$

Select the positive number $r_1(\alpha)$ so that $r_1(\alpha) < \min(\frac{1}{2}b_\alpha^2, 1)$. Then remembering that the proposition $z \in p(b_\alpha)$ implies $x + b_\alpha^2 > 0$ by definition (1.1) p. 1 and in addition $c_\alpha^2 > b_\alpha^2$ by relation (2.60) p.23 we get

$$a > 4(c_\alpha^2 - b_\alpha^2) > 0 \quad (|\alpha| < r_1(\alpha))$$

So that by transformation $s = a/t$ the above integral becomes for $b_\alpha > 0$.

$$B(\alpha, b) = \int_{a/|\alpha|}^{\infty} \exp a(-3/2 - \alpha) \exp s(\frac{1}{2} + \alpha) e^{-s} ds$$

$$B(\alpha, b) \Gamma(\alpha + 3/2) \quad (z \in p(b_\alpha); 0 < |\alpha| < r_1(\alpha) < 1)$$

and alpha is greater than zero.

By inequality (3.50) for $b_\alpha = 0$, $z = 0$ we obtain $\hat{I}_\alpha(0) = 0$, so that:

$$(3.51) \quad 0 \leq \hat{I}_\alpha^*(z) \leq B(\alpha, b) \quad (z \in p(b_\alpha); |z| < r_1(\alpha) < 1; \alpha > 0)$$

This establishes proposition (3.39) p. 44.

From inequalities (3.28) and (3.39) p. 43, we conclude from (3.6) p. 35 that there exists a positive number $B(\alpha, b)$ such that:

$$(3.52) \quad 0 \leq I_\alpha(z) \leq B(\alpha, b) \quad (z \in p(b_\alpha); |z| < r_1(\alpha) < 1; \alpha > 0)$$

Lemma (2.3) p.9, the fact that $z \in p(b_\alpha)$ and other considerations implies:

$$d_\alpha^2 > b_\alpha^2 > b_\alpha^2 - \frac{1}{2}(r-x) > 0 \quad \text{so that}$$

$$d_\alpha > (b_\alpha^2 - \frac{1}{2}(r-x))^{\frac{1}{2}} > 0$$

By means of the derivative, it can be shown that the expression $x - 2|x|^{\frac{1}{2}}d_\alpha$ has a minimum for some value of x in the range $x > -b_\alpha^2$. Hence there exists a positive number $B(\alpha, b)$ such that:

$$(3.53) \quad B(\alpha, b) \leq \exp(x - 2|x|^{\frac{1}{2}}d_\alpha)$$

$$\exp(x - 2|x|^{\frac{1}{2}}\{b_\alpha^2 - \frac{1}{2}(r-x)\}^{\frac{1}{2}}) \quad (z \in p(b_\alpha))$$

By an adjustment of $B(\alpha, b)$ inequalities (3.52) and (3.53) give (3.27) p. 41 thus completing the proof of case II.

CASE III

$$(|z| \geq r_0 > 1; \alpha > 0)$$

Given b_α , $0 \leq b_\alpha < d_\alpha$, there exists a positive number $B(\alpha, b)$ such that:

$$(3.54) \quad 0 \leq I_\alpha(z) \leq B(\alpha, b) \exp(x - 2|x|^{1/2} (b_\alpha^2 - \frac{1}{2} [x-x])^{1/2})$$

$$(z \in p(b_\alpha); |z| \geq r_0 > 1; \alpha > 0)$$

Proof: From the definition of $t(z)$ given by (3.3) p.

35:

$$|z| = \exp(\frac{1}{2}t(z)) - \exp(-\frac{1}{2}t(z)) < \exp(\frac{1}{2}t(z)) \quad (t(z) \geq 0)$$

From which we conclude that:

$$0 < 2 \log |z| \leq t(z) \quad (|z| \geq r_0 > 1)$$

$$(3.55) \quad 0 \leq I_\alpha^*(z) =$$

$$\int_0^{2 \log |z|} + \int_{2 \log |z|}^{t(z)} \frac{\exp\{-2x/(e^t - 1)\} \exp\{2|z|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})\} \exp(\frac{1}{2}\alpha t)}{t^{3/2} \exp(4c_\alpha^2/t)(1 - e^{-t}) |z|^{-\alpha}} dt$$

In the first integral of relationship (3.55) we use the fact that:

$$(3.56) \quad \exp(\alpha \frac{1}{2}t) \leq \exp(\alpha \log |z|) = (\exp(\log |z|))^\alpha = |z|^\alpha$$

$$(0 \leq t \leq 2 \log |z|)$$

As we have already shown on p. 37 that the function $e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}$ is a monotonic increasing function of t we have:

$$|z| = \exp(\frac{1}{2}t(z)) - \exp(-\frac{1}{2}t(z)) \geq e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \quad (t(z) \geq t)$$

Hence:

$$|z|^{-\alpha} \leq (e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})^{-\alpha} \quad (t \leq t(z))$$

Using (3.56) on the first integral of (3.55) and the above inequality on the second gives us:

$$(3.57) \quad 0 \leq \bar{I}_\alpha^*(z) \leq$$

$$|z|^\alpha |z|^{-\alpha} \int_0^{2 \log |z|} \frac{\exp(-2x/(e^t-1)) \exp(2|z|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}))}{t^{3/2} \exp(4c_\alpha^2/t)(1-e^{-t})} dt$$

$$+ \int_{2 \log |z|}^{t(z)} \frac{\exp(-2x/(e^t-1)) \exp(2|z|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}))}{t^{3/2} \exp(4c_\alpha^2/t)(1-e^{-t})} \left[\frac{+ e^{\frac{1}{2}t}}{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}} \right]^\alpha dt$$

It follows immediately that:

$$(e^{\frac{1}{2}t}/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}))^\alpha = (1/(1-e^{-t}))^\alpha < C_\alpha$$

$$(\alpha > 0; t \geq 2 \log |z| \geq 2 \log r_0)$$

Recalling the form of $\bar{I}_\alpha(z)$ from (3.18) p.38 and applying (3.19) p. 38 to (3.57) we have:

$$(3.58) \quad 0 \leq \bar{I}_\alpha^*(z) \leq B(\alpha, b) \bar{I}_\alpha(z) \leq$$

$$B(\alpha, b) \exp(x-2|x|)^{\frac{1}{2}} \{b_\alpha^2 - \frac{1}{2}(r-x)\}^{\frac{1}{2}}$$

$$(z \in p(b_\alpha); |z| > r_0 > 1; \alpha > 0)$$

Since inequality (3.28) p. 41 was established without

any restrictions on z we can use it in this case also. By use of relation (3.53) p. 47 we conclude that there exists a positive number $B(\alpha, b)$ such that:

$$(3.59) \quad 0 \leq \overset{0}{I}_\alpha(z) \leq B(\alpha, b) \exp(x-2|x| \frac{1}{2} \{ b_\alpha^2 - \frac{1}{2}(r-x) \} \frac{1}{2}) \\ (z \in p(b_\alpha))$$

Inequalities (3.58) and (3.59) used on (3.6) p. 35 gives inequality (3.54) p. 47 thus completing the proof of case III.

CASE IV

$$(\alpha > 0; r_1(\alpha) < |z| < r_0)$$

Proof: In this case:

$$|z|^{-\alpha} < (r_1(\alpha))^{-\alpha} = C_\alpha \quad (0 < r_1(\alpha) < |z| < r_0)$$

$$\exp(\alpha \frac{1}{2}t) < C_\alpha \quad (0 < t < t(z) < t(r_0))$$

So that by relation (3.4) p. 35 and (3.18) p. 38 we get:

$$(3.60) \quad 0 \leq \overset{*}{I}_\alpha(z) \leq \bar{I}_\alpha(z)$$

Using relation (3.19) p. 38 on (3.60) and applying this result as well as inequality (3.59) on (3.5) p. 35 we get the proof of case IV. Q. E. D.

Cases I, II, III and IV establishes the fundamental inequality (3.1) p. 34. This completes the task of section 3.

4. The Necessity of Theorem \bar{B} : Conclusion

In section 3 we established the fundamental inequality for the necessity of Theorem \bar{B} given by (3.1) p. 34. In this section we will use inequality (3.1) to prove the necessity of Theorem \bar{B} .

We have given that $d_\alpha > 0$ and that the series (1.2) p. 1 converges for $z \in p(d_\alpha)$ to the function $f(z)$, that is:

$$(4.1) \quad f(z) = \sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(z) \quad (z \in p(d_\alpha); d_\alpha > 0)$$

By lemma (2.17) p. 32 the series on the right side of equality (4.1) is analytic for $z \in p(d_\alpha)$ and hence $f(z)$ being represented by the series is also analytic for $z \in p(d_\alpha)$.

Thus we already have established part of the necessity of Theorem \bar{B} p. 4, namely that $f(z)$ be analytic for $z \in p(d_\alpha)$.

We must now show that to every b_α , $0 \leq b_\alpha < d_\alpha$, there corresponds a positive number $B(\alpha, b)$ such that:

$$(4.2) \quad |f(z)| < B(\alpha, b) \exp\left(\frac{1}{2}x - |x|^{\frac{1}{2}} \left\{ b_\alpha^2 - \frac{1}{2}(r-x) \right\}^{\frac{1}{2}}\right)$$

$$(z \in p(b_\alpha); -1 < \alpha)$$

where we define the symbol $r \equiv (x^2 + y^2)^{\frac{1}{2}}$.

We recall that from the definition of d_α given by (1.5) p. 2 we obtained inequality (2.20) p. 10 and since inequality (2.20) holds for all but a finite number of n , it follows from this inequality that we can find a positive number $m(\alpha, \epsilon)$ such that:

$$(4.3) \quad |a_n^\alpha| < m(\alpha, \epsilon) \exp(2n^{\frac{1}{2}}(-d_\alpha + \epsilon)) \quad (n)$$

Let $b_\alpha < d_\alpha$ be given. As in relation (3.1) p. 34 we define:

$$(4.4) \quad c_\alpha \equiv \frac{1}{2}(b_\alpha + d_\alpha) > b_\alpha$$

Then the series:

$$(4.5) \quad A^2(\alpha, b) = \sum_{n=0}^{\infty} |a_n^\alpha|^2 \exp(4c_\alpha n^{\frac{1}{2}}) \binom{n+\alpha}{n}$$

converges. We will prove this statement as follows: From (10) p. 42 formula (2) and section (3.12) we obtain the formula:

$$(4.6) \quad \binom{n+\alpha}{n} \simeq n^\alpha / \Gamma(\alpha+1) \quad (n \rightarrow \infty; \alpha \neq -1, -2, \dots)$$

Applying relations (4.3), (4.4) and (4.6) to (4.5) we obtain:

$$(4.7) \quad |A^2(\alpha, b)| \leq c_\alpha m^2(\alpha, \epsilon) \sum_{n=0}^{\infty} n^\alpha \exp(4n^{\frac{1}{2}}(-d_\alpha + c_\alpha + \epsilon)) = \\ c_\alpha m^2(\alpha, \epsilon) \sum_{n=0}^{\infty} n^\alpha \exp(2n^{\frac{1}{2}}(b_\alpha - d_\alpha + 2\epsilon))$$

By hypothesis $b_\alpha - d_\alpha < 0$, so that we can pick ϵ so small that $b_\alpha - d_\alpha + 2\epsilon < 0$. From the power series expansion of $n = \exp(\log n)$ we conclude that:

$$\log n < (4!n)^{\frac{1}{4}}$$

Hence for $\alpha \geq 0$ there exists a $c > 0$ such that:

$$2n^{\frac{1}{2}}(d_\alpha - b_\alpha - 2\epsilon) - \alpha \log n > 2n^{\frac{1}{2}}(d_\alpha - b_\alpha - 2\epsilon) - (4!n)^{\frac{1}{4}}$$

$$n^{\frac{1}{4}}(2n^{\frac{1}{2}}(d_\alpha - b_\alpha - 2\epsilon) - (4!)^{\frac{1}{4}}) > n^{\frac{1}{4}}c \quad (n \geq n_{\alpha, \epsilon})$$

Hence:

$$n^{-\alpha} \exp(2n^{\frac{1}{2}}\{d_\alpha - b_\alpha - 2\epsilon\}) = \exp(2n^{\frac{1}{2}}(d_\alpha - b_\alpha - 2\epsilon) - \alpha \log n)$$

$$> \exp(cn^{\frac{1}{4}}) > (1/8!)(cn^{\frac{1}{4}})^8 = (1/8!)c^8 n^2$$

Hence the series (4.5) converges by comparison with the

series: $\sum_{n=0}^{\infty} 1/n^2$ for $\alpha \geq 0$. The series in the case $\alpha < 0$

converges by comparison with the series:

$$\sum_{n=0}^{\infty} n^{|\alpha|} \exp(2n^{\frac{1}{2}}(b_\alpha - d_\alpha + 2\epsilon))$$

By direct calculation we can verify that series (4.1) can be put in the form:

$$f(z) = \sum_{n=0}^{\infty} a_n^\alpha \exp(2c_\alpha^{\frac{1}{2}} n^{\frac{1}{2}}) \left\{ \binom{n+\alpha}{n} \right\}^{\frac{1}{2}} L_n^\alpha(z) \exp(-2c_\alpha^{\frac{1}{2}} n^{\frac{1}{2}}) \left\{ \binom{n+\alpha}{n} \right\}^{-\frac{1}{2}}$$

By the use of Cauchy's inequality on the above:

$$(4.8) \quad |f(z)|^2 \leq$$

$$\left[\sum_{n=0}^{\infty} |a_n^\alpha|^2 \exp(4c_\alpha n^{\frac{1}{2}}) \binom{n+\alpha}{n} \right] \left[\sum_{n=0}^{\infty} |L_n^\alpha(z)|^2 \exp(-4c_\alpha n^{\frac{1}{2}}) \left\{ \binom{n+\alpha}{n} \right\}^{-1} \right]$$

For proof of Cauchy's inequality see Hardy-Littlewood-Polya, Inequalities, p. 16 theorem 7.

By use of the integral:

$$\exp(-s^{\frac{1}{2}}) = (1/2 \pi^{\frac{1}{2}}) \int_0^{\infty} e^{-st} t^{-3/2} e^{-\frac{1}{4}t} dt$$

in which we let $s = 16c_\alpha^2 n$ we obtain:

$$\exp(-s^{\frac{1}{2}}) = \exp(-4c_\alpha n^{\frac{1}{2}}) \quad \text{and}$$

$$\exp(-4c_\alpha n^{\frac{1}{2}}) = (1/2 \pi^{\frac{1}{2}}) \int_0^{\infty} \exp(-16c_\alpha^2 nt) t^{-3/2} e^{-\frac{1}{4}t} dt$$

where $c_\alpha = \frac{1}{2}(b_\alpha + d_\alpha) < d_\alpha$.

Let $16c_\alpha^2 t = q$, then:

$$\exp(-4c_\alpha n^{\frac{1}{2}}) = (4c_\alpha/2 \pi^{\frac{1}{2}}) \int_0^{\infty} e^{-nq} q^{-3/2} \exp(-4c_\alpha^2/q) dq$$

Recalling from complex algebra that for the polynomial $p_n(z)$ we have $|p_n(z)|^2 = p_n(\bar{z})p_n(z)$ we get by equality (4.5) applied to (4.8) and finally making use of the formula given by (6) p. 98 formula (5.1.15):

$$(4.9) \quad |f(z)|^2 \leq A^2(\alpha, b) \sum_{n=0}^{\infty} |L_n^\alpha(z)|^2 \left\{ \binom{n+\alpha}{n} \right\}^{-1} (4c_\alpha/2 \pi^{\frac{1}{2}}) \int_0^{\infty} e^{-nt} t^{-3/2} \exp(-4c_\alpha^2/t) dt$$

$$\begin{aligned}
& (4c_\alpha/2\pi^{3/2})A^2(\alpha, b) \int_0^\infty t^{-3/2} \exp(-4c_\alpha^2/t) \left[\sum_{n=0}^\infty |L_n^\alpha(z)|^2 \left\{ \binom{n+\alpha}{n} \right\}^{-1} e^{-nt} \right] dt \\
& = B(\alpha, b) \int_0^\infty t^{-3/2} \exp(-4c_\alpha^2/t) \left[\sum_{n=0}^\infty L_n^\alpha(z) L_n^\alpha(\bar{z}) \left\{ \binom{n+\alpha}{n} \right\}^{-1} e^{-nt} \right] dt \\
& = B(\alpha, b) \Gamma(\alpha+1) \int_0^\infty \frac{\exp(-2x/(e^t-1)) J_\alpha(1/z \operatorname{csch} \frac{1}{2}t) e^{\frac{1}{2}\alpha t}}{t^{3/2} \exp(4c_\alpha^2/t) (1-e^{-t}) |z|^\alpha} dt
\end{aligned}$$

Before we proceed with the proof we will justify the term by term integration over the infinite interval $0 \leq t < \infty$ in (4.9)

We want to show that for a fixed $z \in p(b_\alpha)$ the series:

$$\begin{aligned}
(4.10) \quad & t^{-3/2} \exp(-4c_\alpha^2/t) \sum_{n=0}^\infty |L_n^\alpha(z)|^2 \left\{ \binom{n+\alpha}{n} \right\}^{-1} e^{-nt} \\
& = \sum_{n=0}^\infty t^{-3/2} \exp(-4c_\alpha^2/t) |L_n^\alpha(z)|^2 \left\{ \binom{n+\alpha}{n} \right\}^{-1} e^{-nt}
\end{aligned}$$

is uniformly convergent in the finite interval:

$$D_1 \leq t \leq D_2 \quad (0 < D_1 < D_2)$$

For $z \in p^*(b_\alpha)$ we get by using inequality (2.22) p. 11 and (4.6) p. 52 on series (4.10) and recalling (3.35) p. 42 we get that this series is less than:

$$(4.11) \quad C_\alpha \sum_{n=0}^\infty t^{-3/2} \exp(-4c_\alpha^2/t) \exp(4n \frac{1}{2} \left\{ \frac{1}{2}(r-x) \right\}^{\frac{1}{2}} + \epsilon(z)) n^{-\alpha} e^{-nt}$$

$$C_\alpha \sum_{n=0}^{\infty} D_1^{-3/2} \exp\left(4n^{\frac{1}{2}} \left[\left\{ \frac{1}{2}(r-x) \right\}^{\frac{1}{2}} - n^{\frac{1}{2}} D_1/4 + \epsilon(z) \right] \right) n^{-\alpha}$$

($D_1 \leq t \leq D_2$)

Then there are positive numbers C_α and k such the above series is less than:

$$C_\alpha \sum_{n=0}^{\infty} n^{-\alpha} e^{-4n^{\frac{1}{2}}k} \quad (n)$$

The above series converges by comparison with (4.7) p. 52 This establishes the uniform convergence of (4.10) for a given z such that $z \in p^*(b_\alpha)$ and for all t such that:

$$D_1 \leq t \leq D_2$$

For $x \geq 0$; $y = 0$, from (2.25) p. 11 we get that series (4.10) is less than:

$$C_\alpha \sum_{n=0}^{\infty} D_1^{-3/2} n^{-\alpha} 2^\alpha n^\alpha e^{-nt}$$

As before we than prove that series (4.10) is uniformly convergent for $x \geq 0$, $y = 0$; $D_1 \leq t \leq D_2$.

Hence series (4.10) is uniformly convergent for:

$$0 < D_1 \leq t \leq D_2 \quad \text{and} \quad z \in p(b_\alpha).$$

Consequently we can integrate this series term by term in this interval. Since the first series on the right side of of (4.9) is convergent being less than:

$$\sum_{n=0}^{\infty} n^{-\alpha} \exp(2n^{\frac{1}{2}}(-c_{\alpha} + (\frac{1}{2}(r-x))^{\frac{1}{2}} + \epsilon(z))$$

and all the terms are positive and since we can integrate it term by term over any finite interval $0 < D_1 \leq t \leq D_2$ then we are justified in integrating series (4.9) term by term over $(0, \infty)$ by a theorem in Titchmarsh (8) p. 44.

From series (4.9) then, we have:

Given b_{α} , $0 \leq b_{\alpha} < d_{\alpha}$ there exists a positive number $B(\alpha, b)$ such that

$$(4.12) \quad |f(z)|^2 \leq B(\alpha, b) \int_0^{\infty} \frac{\exp(-2x/(e^t-1)) e^{\alpha \frac{1}{2}t} J_{\alpha}(i|z| \operatorname{csch} \frac{1}{2}t) dt}{t^{\frac{3}{2}} \exp(4c_{\alpha}^2/t)(1-e^{-t}) |z|^{\alpha} i^{\alpha}}$$

($z \in p(b_{\alpha}); -1 < \alpha$)

Now applying the fundamental inequality (3.1) p. 34 to (4.12) we get:

$$(4.13) \quad |f(z)|^2 \leq B(\alpha, b) \exp(x-2|x|^{\frac{1}{2}} \{ b_{\alpha}^2 - \frac{1}{2}(r-x) \}^{\frac{1}{2}})$$

($z \in p(b_{\alpha}); -1 < \alpha$)

From which it follows that given b_{α} , $0 \leq b_{\alpha} < d_{\alpha}$ there exists a positive number $B(\alpha, b)$ such that:

$$(4.14) \quad |f(z)| < B(\alpha, b) \exp(\frac{1}{2}x - |x|^{\frac{1}{2}} \{ b_{\alpha}^2 - \frac{1}{2}(r-x) \}^{\frac{1}{2}})$$

($z \in p(b_{\alpha}); -1 < \alpha$)

Hence the necessity of theorem \bar{B} has been proved.

5. The Sufficiency of Theorem B: Lemmas

In this section we propose to prove some lemmas which will be needed to prove the sufficiency of theorem B:

LEMMA (5.1)

We define:

$$(5.1) \quad g_{\alpha}(w) \equiv w \int_0^1 t^{\alpha} f(tw^2)(1-t)^{-\alpha-\frac{1}{2}} dt$$

$$(w = u + iv; -1 < \alpha < \frac{1}{2})$$

Given the number $d_{\alpha} > 0$, let $f(z)$ be analytic for $z \in p(d_{\alpha})$ and for every b_{α} , $0 \leq b_{\alpha} < d_{\alpha}$ let there exist a positive number $B(\alpha, b)$ such that:

$$(5.2) \quad |f(z)| \leq B(\alpha, b) \exp\left(\frac{1}{2}x - |x|^{\frac{1}{2}} \{b_{\alpha}^2 - \frac{1}{2}(r-x)\}^{\frac{1}{2}}\right)$$

$$(5.2) \quad |f(z)| \leq B(\alpha, b) \exp\left(\frac{1}{2}x - |x|^{\frac{1}{2}} \{b_{\alpha}^2 - \frac{1}{2}(r-x)\}^{\frac{1}{2}}\right) \quad (-1 < \alpha < \frac{1}{2})$$

Then for every b_{α} , $0 \leq b_{\alpha} < d_{\alpha}$ there exists a positive number $B(\alpha, b)$ such that:

$$(5.3) \quad |g_{\alpha}(w)| < B(\alpha, b) \exp\left(\frac{1}{2}u^2 - |u| (b_{\alpha}^2 - v^2)^{\frac{1}{2}}\right)$$

$$(w \in s(b_{\alpha}); -1 < \alpha < \frac{1}{2})$$

and $g_{\alpha}(w)$ is analytic for $w \in s(d_{\alpha})$.

Proof: In the integral (5.1) let us make the transformation $t = h^2$. Then formula (5.1) becomes:

$$(5.4) \quad g_{\alpha}(w) = 2w \int_0^1 h^{2\alpha+1} f((hw)^2) (1-h^2)^{-\alpha-\frac{1}{2}} dh$$

We make the transformation $z = w^2$ on inequality (5.2). Then by lemma (2.12) p. 24 the proposition $z \in p(b\alpha)$ implies $w \in s(b\alpha)$. Hence by (2.46) p. 21 and (2.48) p. 22 inequality (5.2) becomes:

$$(5.5) \quad |f(w^2)| \leq B(\alpha, b) \exp\left(\frac{1}{2}(u^2 - v^2) - |u^2 - v^2|^{\frac{1}{2}} (b_{\alpha}^2 - v^2)^{\frac{1}{2}}\right) \\ (w \in s(b\alpha); -1 < \alpha)$$

Then by lemma (2.13) p. 27 with $q=2$ and $k=1$, we get from inequality (5.5):

$$(5.6) \quad |f(w^2)| \leq B(\alpha, b) \exp\left(u^2 - (u^2)^{\frac{1}{2}} (b_{\alpha}^2 - v^2)^{\frac{1}{2}}\right) \\ (w \in s(b\alpha); -1 < \alpha)$$

Now it follows immediately from formula (5.4) that:

$$(5.7) \quad |g_{\alpha}(w)| \leq \\ 2|w| \left\{ \max_{0 \leq h \leq 1} |f((hw)^2)| \right\} \left| \int_0^1 (h^2/(1-h^2))^{\alpha + \frac{1}{2}} dh \right|$$

We want to show that the above integral is bounded by a positive number which depends only on α . Let T be such that $0 < T < 1$. Then we have the following relation for the integral in inequality (5.7):

$$0 \leq \int_0^1 \left(\frac{h^2}{1-h^2} \right)^{\alpha + \frac{1}{2}} dh = \left[\int_0^T + \int_T^1 \right] \left(\frac{h^2}{1-h^2} \right)^{\alpha + \frac{1}{2}} dh$$

By making the transformation $1-h^2 = H^2$ on the second integral the above two integrals become:

$$\int_0^T \left(\frac{h^2}{1-h^2} \right)^{\alpha + \frac{1}{2}} dh + \int_0^{\frac{(1-T)^{\frac{1}{2}}}{2}} \left\{ \frac{(1-H^2)}{H^2} \right\}^{\alpha} dH$$

$$\leq \max \left[\left(\frac{1}{1-h^2} \right)^{\alpha + \frac{1}{2}} \right] \int_0^T \left\{ h^2 \right\}^{\alpha + \frac{1}{2}} dh$$

$$+ \max_{0 \leq H \leq \frac{(1-T)^{\frac{1}{2}}}{2}} \left(\frac{(1-H^2)^{\alpha}}{H^{-2\alpha}} \right) \int_0^{\frac{(1-T)^{\frac{1}{2}}}{2}} H^{-2\alpha} dH$$

We can verify by integration that the first integral exists for $\alpha > -1$ and the second for $\alpha < \frac{1}{2}$. Hence the integral of inequality (5.7) is bounded by a positive function of alpha for $-1 < \alpha < \frac{1}{2}$.

Now Pollard shows that if the analytic function $G(w)$ (see Pollard (5) p. 362-363, in particular inequality (3.3)) satisfies order condition (5.3) then the function:

$$(5.8) \quad |w| \max_{0 \leq h \leq 1} |G(hw)|$$

also satisfies order condition (5.3).

Now since $f(z)$ is an analytic function of z and $z = w^2$ is an analytic function of w we have that $f(w^2)$ is an analytic function of w , since the function of an analytic

function is also analytic (see Titchmarsh (7) p. 65 section (2.12)). Hence let us select for our analytic function $G(w)$ the function $f(w^2)$ so that in our case $G(w) = f(w^2)$.

Then
$$G(hw) = f((hw)^2)$$

Hence by expression (5.8) the function:

$$(5.9) \quad |w| \left\{ \max_{0 \leq h \leq 1} f((hw)^2) \right\}$$

satisfies order condition (5.3).

We have proved that the integral in inequality (5.7) is bounded by a positive number which is a function of α only. Hence from (5.7) the function $g(w)$ satisfies order condition (5.3) because function (5.9) satisfies this order condition. This proves inequality (5.3).

We now hope to prove that the function $g(w)$ is analytic for $w \in s(d_\alpha)$.

Let the symbol $\bar{p}(b, D)$ represent the set of all points $z = x + iy$ such that x and y satisfy the inequality:

$$A: \quad y^2 \leq 4b^2(x + b^2); \quad x \leq D$$

Let the symbol $\bar{s}(b, D)$ represent the set of all points $w = u + iv$ such that u and v satisfy the condition:

$$B: \quad v^2 \leq b^2; \quad u^2 \leq b^2 \alpha + D$$

Given a number $d_\alpha > 0$, take the number b_α such that $0 \leq b_\alpha < d_\alpha$. Then by the maximum modulus theorem, (see

Titchmarsh (7) p. 165 section 5.1) since by hypothesis $f(z)$ is analytic for $z \in \bar{p}(b_\alpha, D)$, there exists a point z_0 on the boundary of $\bar{p}(b_\alpha, D)$ such that:

$$C: \quad |f(z)| \leq |f(z_0)| \quad (z \in \bar{p}(b_\alpha, D))$$

By means of inequalities A it follows that if $z \in \bar{p}(b_\alpha, D)$ then $tz \in \bar{p}(b_\alpha, D)$ for $0 \leq t \leq 1$. Hence from inequality C we conclude:

$$D: \quad |f(tz)| \leq |f(z_0)| \quad (z \in \bar{p}(b_\alpha, D); 0 \leq t \leq 1)$$

If we make the transformation $z = w^2$, then by a proof analogous to that of the necessity of lemma (2.3) p. 9, together with identities 2.70 and (2.72) p. 27 and p. 28, and inequalities B we conclude that the proposition $z \in \bar{p}(b_\alpha, D)$ implies $w \in \bar{s}(b_\alpha, D)$. Hence from inequality D:

$$E: \quad |f(tw^2)| \leq |f(tz)| \leq |f(z_0)| \quad (w \in \bar{s}(b_\alpha, D))$$

and for $0 \leq t \leq 1$.

From formula (5.1):

$$\begin{aligned} g_\alpha(w) &= w \int_0^1 t^\alpha f(tw^2)(1-t)^{-\alpha-\frac{1}{2}} dt \\ &= w \int_0^1 t^\alpha f(tw^2)(1-t)^{-\alpha+\frac{1}{2}-1} dt \\ &= w \int_0^1 t^\alpha f(tw^2) \left[\sum_{n=0}^{\infty} \binom{n + (\alpha - \frac{1}{2})}{n} t^n \right] dt \quad (t < 1) \end{aligned}$$

We may integrate the above series term by term provided the resulting series converges, see (7) p. 45. The resulting

series is:

$$w \sum_{n=0}^{\infty} \binom{n + (\alpha - \frac{1}{2})}{n} \int_0^1 t^{\alpha+n} f(tw^2) dt$$

By (10) p. 42 formula (2) and section (3.12) we have:

$$\binom{n + (\alpha - \frac{1}{2})}{n} = O(n^{\alpha - \frac{1}{2}})$$

so that the above estimate together with inequality E gives that the above series is in absolute value less than:

$$Kw \sum_{n=0}^{\infty} n^{\alpha - \frac{1}{2}} \left| \int_0^1 t^{\alpha+n} f(tw^2) dt \right| \leq Kw \sum_{n=0}^{\infty} n^{\alpha - \frac{1}{2}} |f(z_0)| \int_0^1 t^{\alpha+n} dt$$

($w \in \bar{s}(d_{\alpha}, D)$)

The series constituting the right side of the above inequality is then equal to:

$$Kw |f(z_0)| \sum_{n=0}^{\infty} n^{\alpha - \frac{1}{2}} (\alpha + n - 1)^{-1} \leq K_1 w |f(z_0)| \sum_{n=0}^{\infty} n^{\alpha - 3/2}$$

which converges for $\alpha < \frac{1}{2}$. This not only justifies the above term by term integration but shows that the series representing $g_{\alpha}(w)$ is absolutely and uniformly convergent in the closed region $\bar{s}(d_{\alpha}, D)$ and hence by Knopp (3) p. 73-74, the function $g_{\alpha}(w)$ is analytic for w in the simply connected region $s(d_{\alpha})$.

This completes the proof of the lemma.

LEMMA (5.2)

The following integral relationship between Laguerre polynomials of different order is valid:

$$(5.10) \quad \int_x^{\infty} e^{-u} (u-x)^{-t} L_n^{\alpha}(u) du = e^{-x} L_n^{\alpha+t-1}(x) \Gamma(1-t)$$

$$(\alpha > -1; t < 1; x \geq 0)$$

Proof: From Szego (6) p. 97 formula (5.1.9) we have:

$$(5.11) \quad \sum_{n=0}^{\infty} L_n^{\alpha}(u) s^n = (1-s)^{-\alpha-1} \exp(-us/(1-s)) \quad (|s| < 1)$$

From which we get:

$$(5.12) \quad \int_x^{\infty} e^{-u} (u-x)^{-t} \left[\sum_{n=0}^{\infty} L_n^{\alpha}(u) s^n \right] du =$$

$$\int_x^{\infty} e^{-u} (u-x)^{-t} (1-s)^{-\alpha-1} \exp(-us/(1-s)) du$$

By the transformation $u-x=v$ the above integral is:

$$(5.13) \quad \int_0^{\infty} e^{-x-v} v^{-t} (1-s)^{-\alpha-1} \exp(-(xs+sv)/(1-s)) dv =$$

$$(1-s)^{-\alpha-1} e^{-x} \exp(-xs/(1-s)) \int_0^{\infty} e^{-v} v^{-t} \exp(-sv/(1-s)) dv$$

$$= (1-s)^{-\alpha-1} e^{-x} \exp(-xs/(1-s)) \int_0^{\infty} v^{-t} \exp(-v/(1-s)) dv$$

By the transformation $T = v/(1-s)$ the above integral becomes.

$$\begin{aligned}
 (5.14) \quad & (1-s)^{-\alpha-1} e^{-x} \exp(-xs/(1-s)) \int_0^{\infty} e^{-T} (1-s)^{-t} T^{-t} (1-s) dT \\
 & = (1-s)^{(-\alpha-t+1)-1} \exp(-xs/(1-s)) e^{-x} \int_0^{\infty} e^{-T} T^{(-t+1)-1} dT \\
 & = e^{-x} \sum_{n=0}^{\infty} L_n^{(\alpha+t-1)}(x) s^n \Gamma(1-t) \quad (t < 1; \alpha > -1; x \geq 0)
 \end{aligned}$$

by use of equation (5.11).

We deduce from equations (5.12) (5.13) and (5.14) that:

$$(5.15) \quad \int_x^{\infty} e^{-u} (u-x)^{-t} \left[\sum_{n=0}^{\infty} L_n^{\alpha}(u) s^n \right] du = \Gamma(1-t) e^{-x} \sum_{n=0}^{\infty} L_n^{(\alpha+t-1)}(x) s^n$$

($\alpha > -1; t < 1; x \geq 0; |s| < 1$)

Before we can conclude the proof we need to justify the statement:

$$(5.16) \quad \int_x^{\infty} e^{-u} (u-x)^{-t} \sum_{n=0}^{\infty} L_n^{\alpha}(u) s^n du = \sum_{n=0}^{\infty} \int_x^{\infty} e^{-u} (u-x)^{-t} L_n^{\alpha}(u) s^n du$$

That is, we must show that the series:

$$(5.17) \quad e^{-u} (u-x)^{-t} \sum_{n=0}^{\infty} L_n^{\alpha}(u) s^n = \sum_{n=0}^{\infty} e^{-u} (u-x)^{-t} L_n^{\alpha}(u) s^n$$

can be integrated with respect to u term by term over the interval: $x \leq u < \infty$.

We prove this as follows;

We first hope to show that the series:

$$(5.18) \quad e^{-u}(u-x)^{-t} \sum_{n=0}^{\infty} |L_n^\alpha(u)| s^n = \sum_{n=0}^{\infty} e^{-u}(u-x)^{-t} |L_n^\alpha(u)| s^n$$

$$(\alpha > -1; x \geq 0; t < 1)$$

is uniformly convergent with respect to u in the interval $x < D_1 \leq u \leq D_2$. By estimates (2.33) and (2.34) p. 13 we obtain from equation (5.18):

$$(5.19) \quad 0 \leq \sum_{n=0}^{\infty} e^{-u}(u-x)^{-t} |L_n^\alpha(u)| s^n$$

$$\leq \sum_{n=0}^{\infty} e^{-u} s^n e^{\frac{1}{2}u} (u-x)^{-t} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$$

$$\leq C_\alpha M \sum_{n=0}^{\infty} n^\alpha s^n \quad (D_1 \leq u \leq D_2)$$

Where $u-x < M$ for $D_1 \leq u \leq D_2$. Since we have $(1/n^\alpha)^{1/n} \rightarrow 1$ ($n \rightarrow \infty$) then by the n th root test it follows that the last series converges for $|s| < 1$.

Thus for a given s such that $|s| < 1$, $t < 1$, $x \geq 0$,

$\alpha > -1$ the series (5.18) converges uniformly for $D_1 \leq u \leq D_2$ and hence the series (5.17) does also. Therefore series (5.17) can be integrated with respect to u term by term over the interval $D_1 \leq u \leq D_2$.

As in inequality (5.19) we get by (2.33) and (2.34) p. 13 that:

$$\begin{aligned}
 (5.20) \quad 0 &\leq \sum_{n=0}^{\infty} \int_x^{\infty} e^{-u} (u-x)^{-t} |L_n^{\alpha}(u)| s^n du \\
 &\leq c_{\alpha} \sum_{n=0}^{\infty} n^{\alpha} s^n \int_x^{\infty} e^{-\frac{1}{2}u} (u-x)^{-t} du
 \end{aligned}$$

which by the transformation $u-x=v$ becomes:

$$c_{\alpha} \sum_{n=0}^{\infty} n^{\alpha} s^n e^{-\frac{1}{2}x} \int_0^{\infty} e^{-\frac{1}{2}t} v^{-t} dv$$

The last integral becomes under the transformation $p = \frac{1}{2}v$:

$$\begin{aligned}
 &c_{\alpha} \sum_{n=0}^{\infty} n^{\alpha} s^n e^{-\frac{1}{2}x} 2^{1-t} \int_0^{\infty} e^{-p} \exp((1-t)-1) dp \\
 &= c_{\alpha} e^{-\frac{1}{2}x} 2^{1-t} \Gamma(1-t) \sum_{n=0}^{\infty} n^{\alpha} s^n \quad (t < 1; |s| < 1; \alpha > -1; x \geq 0)
 \end{aligned}$$

As in relation (5.19) the latter series converges. Therefore series (5.20) converges.

Now since the first series of (5.20) converges and series (5.17) can be integrated with respect to u , term by term over the interval (for u) $0 \leq x < D_1 \leq u \leq D_2$ by a theorem in Titchmarsh (7) p. 45 we may integrate the series (5.17) term by term over the interval $x \leq u < \infty$.

Having this matter out of the way we continue with the derivation of formula (5.10).

From equalities (5.15) and (5.16) we get:

$$(5.21) \quad \sum_{n=0}^{\infty} s^n \int_x^{\infty} e^{-u} (u-x)^{-t} L_n^{\alpha}(u) du =$$

$$\Gamma(1-t) e^{-x} \sum_{n=0}^{\infty} L_n^{\alpha+t-1}(x) s^n$$

Equating the coefficients of the two power series in s , we get formula (5.10).

LEMMA (5.3)

The following integral formula connecting $L_n^{\alpha}(x)$ with $L_n^{\frac{1}{2}}(x)$ holds for $\alpha < \frac{1}{2}$.

$$(5.22) \quad \int_x^{\infty} e^{-u} (u-x)^{-\alpha-\frac{1}{2}} L_n^{\frac{1}{2}}(u) du = \Gamma(-\alpha+\frac{1}{2}) e^{-x} L_n^{\alpha}(x)$$

$$(\alpha > \frac{1}{2}; x \geq 0)$$

Proof: If in formula (5.10) p. 65 we set $\alpha = \frac{1}{2}$ we get:

$$\int_x^{\infty} e^{-u} (u-x)^{-t} L_n^{\frac{1}{2}}(u) du = \Gamma(1-t) e^{-x} L_n^{t-\frac{1}{2}}(x)$$

$$(t < 1; x \geq 0)$$

In the above formula let $\alpha = t - \frac{1}{2} < \frac{1}{2}$ then $\alpha + \frac{1}{2}$, and formula (5.22) follows.

This proves the lemma and completes the work of this chapter.

6. The Sufficiency of Theorem \bar{B} : The Fundamental Inequality

LEMMA (6.1)

Let $d > 0$ and suppose $f(z)$ is analytic for $z \in p(d)$. Moreover, suppose for every b_α , $0 \leq b_\alpha < d$ there exists a positive number $B(\alpha, b)$ such that:

$$(6.1) \quad |f(z)| < B(\alpha, b) \exp\left(\frac{1}{2}x - |x|^{\frac{1}{2}} \left\{b_\alpha^2 - \frac{1}{2}(r-x)\right\}^{\frac{1}{2}}\right) \\ (z \in p(b_\alpha))$$

then:

$$(6.2) \quad |a_n| \leq B(\alpha, b) \exp(-2n^{\frac{1}{2}} b_\alpha) \quad (-1 < \alpha < \frac{1}{2})$$

Proof: By estimate (2.33) p. 13 and inequality (6.1) the following integral converges. Hence we may write:

$$(6.3) \quad a_n^\alpha = \left[\binom{n+\alpha}{n} \Gamma(\alpha+1) \right]^{-1} \int_0^\infty t^\alpha e^{-t} f(t) L_n^\alpha(t) dt$$

From formula (5.22) p. 69 we have:

$$\int_t^\infty e^{-s} L_n^{\frac{1}{2}}(s) (s-t)^{-\alpha-\frac{1}{2}} ds = \Gamma(-\alpha+\frac{1}{2}) e^{-t} L_n^\alpha(t)$$

It is this formula along with lemma (5.1) which restricts the proof to the range $-1 < \alpha < \frac{1}{2}$.

Substituting by means of equality (6.4) in (6.3) we have:

$$(6.4a) \quad a_n^\alpha = \frac{\int_0^\infty t^\alpha f(t) dt \int_t^\infty e^{-t} L_n^{1/2}(s)(s-t)^{-\alpha-1/2} ds}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+1/2)}$$

And by a reversal of the order of integration which we will justify later we obtain:

$$a_n^\alpha = \left[\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+1/2) \right]^{-1} \int_0^\infty e^{-s} L_n^{1/2}(s) ds \int_0^\infty t^\alpha f(t)(s-t)^{-\alpha-1/2} dt$$

which by the transformation $t = ps$ becomes:

$$\frac{\int_0^\infty e^{-s} L_n^{1/2}(s) s^{1/2} ds \int_0^1 p^\alpha f(ps)(1-p)^{-\alpha-1/2} dp}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+1/2)} = a_n^\alpha$$

and by the transformation: $s = q^2$

$$a_n^\alpha = \frac{\int_0^\infty 2e^{-q^2} L_n^{1/2}(q^2) q^2 dq \int_0^1 p^\alpha f(pq^2)(1-p)^{-\alpha-1/2} dp}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+1/2)}$$

From Szego (6) p. 102 formula (5.6.1) we have the formula:

$$(6.5) \quad q L_n^{\frac{1}{2}}(q^2) = \left((-1)^n 2^{2n+1} n! \right)^{-1} H_{2n+1}(q) \quad (q \text{ real})$$

Moreover we define:

$$(6.6) \quad g_\alpha(q) = q \int_0^1 p^\alpha f(pq^2)(1-p)^{-\alpha-\frac{1}{2}} dp \quad (q \text{ real})$$

so that

$$(6.7) \quad a_n = \frac{2 \int_0^\infty \exp(-q^2) H_{2n+1}(q) g_\alpha(q) dq}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+\frac{1}{2}) \left((-1)^n 2^{2n+1} n! \right)}$$

From formula (6.6) we see that $g_\alpha(q)$ is an odd function of q . Solving for $H_{2n+1}(q)$ in equality (6.5) we see that the Hermite polynomial $H_{2n+1}(q)$ is also an odd function of q . Hence the function $\exp(-q^2) H_{2n+1}(q) g_\alpha(q)$ is an even function of q so that:

$$(6.8) \quad 2 \int_0^\infty \exp(-q^2) H_{2n+1}(q) g_\alpha(q) dq = \int_{-\infty}^\infty \exp(-q^2) H_{2n+1} g_\alpha(q) dq$$

According to lemma (5.1) p. 59 the function of a complex variable $g_\alpha(w)$ where $w = u + iv$ satisfies Hille's condition for an analytic function to possess a convergent Hermetian series i.e. condition (5.3) p. 59 (see (2) p. 81 theorem 1). Hence we may write:

$$(6.9) \quad g_\alpha(w) = \sum_{n=0}^{\infty} \left(2^n n! \Gamma^{\frac{1}{2}} \right)^{-\frac{1}{2}} f_n H_n(w)$$

$$(6.10) \quad (2^{2n+1} (2n+1)! \pi^{\frac{1}{2}})^{\frac{1}{2}} f_{2n+1}^{\alpha} = \int_{-\infty}^{\infty} \exp(-q^2) H_{2n+1}(q) g_{\alpha}(q) dq$$

Substituting by means of equality (6.10) in (6.8) and then substituting this result in (6.7) we have:

$$(6.11) \quad a_n^{\alpha} = \frac{(2^{2n+1} (2n+1)! \pi^{\frac{1}{2}})^{\frac{1}{2}}}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+\frac{1}{2}) (-1)^n 2^{2n+1} n!}$$

Pick any $b_{\alpha} < d$. Then select \bar{b}_{α} such that $b_{\alpha} < \bar{b}_{\alpha} < d$. Then by Hille's article (2) p. 90 we have that there exists a positive number $B(\alpha, \bar{b})$ such that:

$$(6.12) \quad |f_{2n+1}| \leq B(\alpha, \bar{b}) \exp(-\bar{b}_{\alpha}(4n+3)^{\frac{1}{2}})$$

Apply inequality (6.12) to (6.11) we have:

$$(6.13) \quad a_n^{\alpha} = \frac{(2^{2n+1} (2n+1)! \pi^{\frac{1}{2}})^{\frac{1}{2}} B(\alpha, \bar{b}) \exp(-\bar{b}_{\alpha}(4n+3)^{\frac{1}{2}})}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+\frac{1}{2}) (-1)^n 2^{2n+1} n!}$$

We now want to show that there exists a positive number $B(\alpha, b)$ such that:

$$(6.14) \quad \frac{B(\alpha, \bar{b}) (2^{2n+1} (2n+1)! \pi^{\frac{1}{2}})^{\frac{1}{2}} \exp(-\bar{b}_{\alpha}(4n+3)^{\frac{1}{2}})}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+\frac{1}{2}) (-1)^n 2^{2n+1} n!} \leq B(\alpha, b) \exp(-2b_{\alpha} n^{\frac{1}{2}})$$

We now want to show that there exists a positive number $B(\alpha, b)$ such that:

$$(6.14) \quad \frac{B(\alpha, b) (2^{2n+1} (2n+1)! \pi^{\frac{1}{2}})^{\frac{1}{2}} \exp(-b_\alpha (4n+3)^{\frac{1}{2}})}{\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+\frac{1}{2}) (-1)^n 2^{2n+1} n!} \leq B(\alpha, b) \exp(-2n^{\frac{1}{2}} b_\alpha)$$

The following formulas are well known:

$$(6.15) \quad \binom{n+\alpha}{n} \sim n^\alpha / \Gamma(\alpha+1) \quad (n \rightarrow \infty)$$

For proof see Zygmund (10) p. 42 formula (2) and section (3.12).

$$(6.16) \quad n! \sim (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} / e^n \quad (n \rightarrow \infty)$$

This is a special case of Stirlings formula. For proof of the general real case see Titchmarsh (7) formula (4) p. 58.

$$(6.17) \quad \Gamma(n) = (n-1)!$$

$$(6.18) \quad \Gamma(n) = (n-1)\Gamma(n-1)$$

Proofs of these formulas are given by Titchmarsh (7) p. 55 formulas (3) and (2) respectively. From formula (6.16):

$$(6.19) \quad (2n+1)! = (2n+1)(2n)! = \{(2n+1)/2n\} 2n(2n)! < K(2\pi)^{\frac{1}{2}} (2n)^{2n-3/2} e^{-2n} \quad (n \rightarrow \infty)$$

For future use on inequality (6.14) we calculate:

$$(6.20) \quad \left[\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+\frac{1}{2}) (-1)^n 2^{2n+1} n! \right]^{-1} 2^{2n+1} (2^{2n+1} (2n+1)! \pi^{\frac{1}{2}})^{\frac{1}{2}}$$

$$= \left[\binom{n+\alpha}{n} \Gamma(\alpha+1) \Gamma(-\alpha+\frac{1}{2}) 2^{n+\frac{1}{2}} n! \right]^{-1} \pi^{\frac{1}{4}} (2n+1)!^{\frac{1}{2}}$$

which by use of relationships (6.15) (6.16) and (6.19) is less than:

$$(6.21) \quad \frac{K_1 \pi^{\frac{1}{4}} (2\pi)^{\frac{1}{2}} (2n)^{2n+3/2} e^{-2n})^{\frac{1}{2}}}{n^{\alpha} \Gamma(-\alpha+\frac{1}{2}) 2^{n-\frac{1}{2}} (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n}} =$$

$$K_1 n^{-\alpha+\frac{1}{4}} 2\pi \Gamma(-\alpha+\frac{1}{2})$$

Applying equality (6.21) to (6.20) we see that the left side of (6.14) in absolute value is less than:

$$(6.22) \quad B(\alpha, \bar{b}) n^{-\alpha+\frac{1}{4}} \exp(-\bar{b}_{\alpha}(4n+3)^{\frac{1}{2}})$$

$$= B(\alpha, \bar{b}) n^{-\alpha+\frac{1}{4}} \exp\left\{-\bar{b}_{\alpha} \left(\frac{4n+3}{4n}\right)^{\frac{1}{2}} (4n)^{\frac{1}{2}}\right\}$$

$$\leq B(\alpha, \bar{b}) n^{-\alpha+\frac{1}{4}} \exp(-2\bar{b}_{\alpha} n^{\frac{1}{2}})$$

$$= B(\alpha, \bar{b}) \exp\left\{(-\alpha+\frac{1}{4}) \log n + 2n^{\frac{1}{2}}(b_{\alpha}-\bar{b}_{\alpha})\right\} \exp(-2b_{\alpha} n^{\frac{1}{2}})$$

But remembering that $\bar{b}_{\alpha} > b_{\alpha}$

$$(6.23) \quad \exp\left[2n^{\frac{1}{2}}(b_{\alpha}-\bar{b}_{\alpha}) + (-\alpha+\frac{1}{4}) \log n\right] \leq K(\alpha, b) \quad (n)$$

$$(6.24) \quad B(\alpha, b) \equiv B(\alpha, \bar{b}) K(\alpha, b)$$

Apply (6.24) to inequality (6.22) gives (6.14). The inequality (6.14) applied to (6.13) gives:

$$(6.25) \quad |a_n^{\alpha}| \leq B(\alpha, b) \exp(-2b_{\alpha} n^{\frac{1}{2}}) \quad (-1 < \alpha < \frac{1}{2})$$

Hence we have proved the lemma.

As an after thought we justify the change of order of integration carried out in equality (6.4a) p. 71 as follows:

Let us define:

$$F(s,t) \equiv t^{\alpha} f(t) e^{-s} L_n^{\alpha}(s)(s-t)^{-\alpha-\frac{1}{2}}$$

$$(t \leq s < \infty; 0 \leq t < \infty)$$

$$F(s,t) \equiv 0 \quad (0 \leq s < t; 0 \leq t < \infty)$$

Hence we may write integral (6.4a) as follows:

$$(6.26) \quad \int_0^{\infty} dt \int_t^{\infty} t^{\alpha} f(t) e^{-s} L_n^{\alpha}(s)(s-t)^{-\alpha-\frac{1}{2}} ds$$

$$= \int_0^{\infty} dt \int_0^{\infty} F(s,t) ds$$

Since the limits of integration are now constant we may call upon a theorem in Titchmarsh (7) section 1.85 p. 63 and especially p. 55. Therefore by the above mentioned theorem in (7) p. 55 we can justify the change of order of integration in relation (6.4a) by proving that the integral on the left side of equality (6.26) is absolutely convergent. We proceed to prove this:

Since $L_n^{\frac{1}{2}}(s)$ is a polynomial of the n th degree it is dominated by Ks^n as s becomes infinite and n remains fixed, so that for a fixed n :

$$(6.27) \quad \left| e^{-\frac{1}{2}s} L_n(s) \right| < c \quad (t \leq s < \infty)$$

Moreover, by relation (6.1) p. 70, we can take b_a , $0 < b_a < a$ and find a positive number $B(a, b)$ such that:

$$(6.28) \quad |f(t)| < B(a, b) \exp\left(\frac{1}{2}t - t^{\frac{1}{2}} b_a\right) \quad (0 \leq t < \infty)$$

From inequalities (6.27) and (6.28)

$$(6.29) \quad 0 \leq \int_0^{\infty} |t^a f(t)| dt \int_t^{\infty} e^{-s} L_n^{\frac{1}{2}}(s) (s-t)^{-a-\frac{1}{2}} ds \leq$$

$$B(a, b) \int_0^{\infty} t^a \exp\left(\frac{1}{2}t - t^{\frac{1}{2}} b_a\right) dt \int_t^{\infty} e^{-\frac{1}{2}s} (s-t)^{-a-\frac{1}{2}} ds$$

$$(a < \frac{1}{2})$$

By the transformation $s-t = 2p$ the above integral becomes:

$$B(a, b) 2^{-a+\frac{1}{2}} \int_0^{\infty} t^a e^{-\frac{1}{2}t} \exp\left(\frac{1}{2}t - t^{\frac{1}{2}} b_a\right) dt \int_0^{\infty} e^{-pp}^{-a+\frac{1}{2}-1} dp$$

$$= B(a, b) 2^{-a+\frac{1}{2}} \int_0^{\infty} t^a \exp(-t^{\frac{1}{2}} b_a) \Gamma(-a+\frac{1}{2}) dt \quad (a < \frac{1}{2})$$

by the definition of the gamma function (see (7) p. 55 formula (1)). The above gives us by the transformation $q = t^{\frac{1}{2}} b_a$:

$$B(a, b) 2^{-a+3/2} \Gamma(-a+\frac{1}{2}) b^a \int_0^{\infty} q^{2a+2-1} e^{-q} dq$$

$$= B(a, b) \quad (a < \frac{1}{2})$$

since:

$$\int_0^{\infty} q^{2a+2-1} e^{-q} dq = \Gamma(2a+2)$$

This establishes the absolute convergence of the integral (6.26), and completes the proof of lemma (6.1).

7. The Sufficiency of Theorem \bar{B} : Conclusion

In this section we hope to complete:

THE SUFFICIENCY OF THEOREM \bar{B} :

If for $-1 < \alpha$ and for every b_α , $0 \leq b_\alpha < d_\alpha$, there exists a positive number $B(\alpha, b)$ such that:

$$(7.1) \quad |f(z)| < B(\alpha, b) \exp\left\{\frac{1}{2}x - |x|^{\frac{1}{2}} \left(b^2 - \frac{1}{2}(r-x) \right)^{\frac{1}{2}}\right\} \\ (z \in p(b_\alpha))$$

and

$$(7.2) \quad f(z) \text{ is analytic} \quad (z \in p(d_\alpha))$$

then

$$(7.3) \quad f(z) = \sum_{n=0}^{\infty} a_n L_n(z) \quad (z \in p(d_\alpha); -1 < \alpha)$$

To complete the proof of the above theorem we need two more lemmas. The lemmas and their proofs follow.

LEMMA (7.1)

Let there exist a number $d > 0$ (depending on alpha) such that $f(z)$ is analytic for $z \in p(d)$, and such that for $-1 < \alpha$ and for every b_α , $0 \leq b_\alpha < d$ there exists a positive number $B(\alpha, b)$ for which:

$$(7.4) \quad |f(z)| < B(\alpha, b) \exp\left\{\frac{1}{2}x - |x|^{\frac{1}{2}} \left(b_\alpha^2 - \frac{1}{2}(r-x) \right)^{\frac{1}{2}}\right\}$$

where the above inequality holds for $z \in p(b_\alpha)$ and for $\alpha > -1$. Then for $-1 < \alpha$, we have that $d_\alpha > 0$ where d_α is defined by (1.5) p. 2 and the series:

$$(7.5) \quad \sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(z)$$

has the parabola of convergence $p(d_\alpha)$ as given by definition (2.8) p. 14.

Proof: In case $-1 < \alpha < \frac{1}{2}$ we use lemma (6.1) p. 70 and get that for every b_α , $0 \leq b_\alpha < d$, there is a positive number $B(\alpha, b)$ such that:

$$(7.6) \quad |a_n^\alpha| < B(\alpha, b) \exp(-2n^{\frac{1}{2}} b_\alpha) \quad (-1 < \alpha < \frac{1}{2})$$

Then by lemma (2.8) p. 14 the parabola is the parabola of convergence of series (7.5) for $-1 < \alpha < \frac{1}{2}$.

We will prove the theorem for the case $0 < \alpha$ by mathematical induction.

First, the theorem is true for $-1 < \alpha \leq 0$ as we have just proved.

Now assuming that the theorem is true for a series of a given order α we hope to show that it is also true for a series of order $\alpha + 1$. This will complete the induction proof.

Now let $d_1 > 0$ be the given number of the hypothesis of the lemma for the series of order $\alpha + 1$ and let $d > 0$

be the number for the series of order alpha. Define:

$$\bar{d} \equiv \min(d, d_1)$$

Now if d or d_1 satisfies the conditions imposed upon it by lemma (7.1) for a Laguerre series of a given order, then so does \bar{d} for the Laguerre series of the same order. Hence the hypothesis of lemma (7.1) are satisfied for \bar{d} for a series of order alpha. Since we assume that the lemma is true for a Laguerre series of order alpha we conclude that series (7.5) converges in a parabola $p(d_\alpha)$.

Hence there exists a point $z_0 = x_0 + iy_0$ ($y_0 \neq 0$) such that series (7.5) converges at z_0 . But by lemma (2.12) p. 24 the Laguerre series

$$\sum_{n=0}^{\infty} \frac{a+1}{a_n} L_n^{a+1}(z)$$

must also converge at z_0 and moreover by lemma (2.7) p. 14 must converge for all points z such that $z \in p(d_{\alpha+1})$. This establishes the lemma.

LEMMA (7.1)

(The identity theorem for analytic functions)

If two functions are analytic in a region R and if they coincide in a neighborhood of a point z_0 of R or along any path segment however small terminating in z_0 , or also for an infinite number of distinct points with the limit point z_0 , then the two functions are equal everywhere in R .

For the proof of the above theorem see Knopp (3) p.87.

We now complete the proof of the sufficiency of theorem B, stated at the beginning of this chapter.

Proof: Now by (7.1) p. 79 we conclude that for every b_α , $0 \leq b_\alpha < d_\alpha$ there exists a positive number $B(\alpha, b)$ such that:

$$(7.7) \quad |f(x)| < B(\alpha, b) \exp\left(\frac{1}{2}x - |x|^{\frac{1}{2}} b_\alpha\right) \quad (0 \leq x)$$

$$< B(\alpha, b) e^{\frac{1}{2}x} \quad (0 \leq x)$$

Now if $f(x)$ is integrable over $(0, D)$ for all $D > 0$ and satisfies the inequality

$$(7.8) \quad |f(x)| < A e^{\frac{1}{2}x} \quad (x \geq 0)$$

then by a theorem of Caton and Hille (1) p. 227 the series (7.5) p. 80 i. e. series (1.2) p. 1 is Abel summable to $f(x)$ for almost all x , including the points of continuity.

Since by hypothesis $f(z)$ is analytic for $z \in p(b_\alpha)$ $f(x)$ is continuous and bounded and hence integrable on $(0, D)$. Moreover by inequality (7.7) the function $f(x)$ satisfies inequality (7.8) and consequently $f(x)$ satisfies the conditions of Caton and Hille for Abel summability of its Laguerre series of order alpha to $f(x)$ at the points of continuity of $f(x)$, namely everywhere, on the non-negative real axis at least.

We conclude then that series (7.5) p. 80 is Abel summable to $f(z)$ at all points on the positive real axis. Since the number d_α by the sufficiency hypothesis (7.1) and (7.2) p. 79 satisfies the conditions imposed on d in lemma (7.1) we conclude by lemma (7.1) that the series (7.5) has the parabola of convergence $p(d_\alpha)$ for $-1 < \alpha$. Since the Cauchy sum always equals the Abel sum (Abel summability is regular) equation (7.3) p. 79 holds on the non-negative real axis.

We now hope to show that equation (7.3) holds for $z \in p^*(d_\alpha)$. This will prove the sufficiency of theorem \bar{B} .

Let $z_0 \in p^*(d_\alpha)$. Take $b_\alpha > 0$ so large that $z_0 \in p^*(b_\alpha)$. Let us select the finite closed region $\bar{S}^*(b_\alpha)$ of lemma (2.4) p. 10 so as to contain the point z_0 and an interior point of the set $\bar{C}(b_\alpha)$ mentioned in lemma (2.10) p. 21 and given by statement (2.48) p. 22.

Let us designate by the symbol $R(b_\alpha)$, the set of all points consisting of the points of $\bar{S}(b_\alpha)$ and $\bar{C}(b_\alpha)$ exclusive of the boundary points. The set $R(b_\alpha)$ is a region.

By lemma (2.4) p. 10 the series (7.5) converges absolutely and uniformly in $\bar{S}^*(b_\alpha)$. Moreover all the polynomials $L_n^\alpha(z)$ are analytic in $\bar{S}^*(b_\alpha)$. Then by Knopp (3) p. 74 theorem 3 the series (7.5) p. 80 represents a function analytic in $S^*(b_\alpha)$.

Now $d_\alpha > 0$, where in this case we mean the number defined by statement (1.5) p. 2 by virtue of lemma (7.1) p. 79. Then .

by lemma (2.10) p. 21, the series (7.5) converges absolutely and uniformly in $\bar{C}(b_a)$ and consequently by virtue of Knopp (3) p. 74 theorem 3, series (7.5) represents an analytic function in $C(b_a)$.

The series (7.5) is analytic at each point of $S^*(b_a)$ and $C(b_a)$ hence it is analytic at each point of $R(b_a)$. Therefore series (7.5) p. 80 represents an analytic function in $R(b_a)$.

Now we just proved above that equation (7.3) p. 79 is true along the non-negative real axis and hence it is true along an arc in region $R(b_a)$, namely along the interval of the non-negative real axis for which $x < b_a$.

Therefore by the identity theorem, lemma (7.2) p. 79 the function, $f(z)$ and the series

$$\sum_{n=0}^{\infty} a_n L_n(z)$$

must be identical at each point of $R(b_a)$ and in particular at z_0 . But z_0 is an arbitrary point for the set of all points z such that $z \in p^*(d_a)$.

This completes the proof of the sufficiency of theorem \bar{B} and hence the proof of theorem \bar{B} is finished.

8. The Equivalence of Theorems \bar{A} and \bar{B}

The purpose of this final chapter is to show that theorems \bar{A} and \bar{B} are equivalent. That is, theorem \bar{A} implies theorem \bar{B} and theorem \bar{B} implies theorem \bar{A} . These theorems are stated on page 3, but for the convenience of the reader we will restate them here.

THEOREM \bar{A}

In order that $g(w)$ possess a Laguerre series of order α for $\alpha > -1$ such that:

$$(8.1) \quad f(z) = f(w^2) = g(w) = \sum_{n=0}^{\infty} a_n^{\alpha} L_n^{\alpha}(w^2)$$

$$(w \in s(\bar{d}_{\alpha}); \bar{d}_{\alpha} > 0)$$

where

$$(8.2) \quad a_n^{\alpha} = \left[\binom{n+\alpha}{n} \Gamma(\alpha+1) \right]^{-1} \int_0^1 e^{-x} x^{\alpha} L_n^{\alpha}(x) dx$$

it is necessary and sufficient that

$$(8.3) \quad g(w) \text{ be analytic and even} \quad (w \in s(\bar{d}_{\alpha}))$$

and that to every b_{α} , $0 \leq b_{\alpha} < \bar{d}_{\alpha}$ there correspond a positive number $B(\alpha, b)$ such that:

$$(8.4) \quad |g(w)| \leq B(\alpha, b) \exp\left(\frac{1}{2}u^2 - |v|(b_{\alpha}^2 - v^2)^{\frac{1}{2}}\right)$$

where the above inequality holds for $w \in s(b_a)$.

THEOREM \bar{B}

In order that $f(z)$ possess a Laguerre series of order α for $\alpha > -1$ such that:

$$(8.5) \quad f(z) = \sum_{n=0}^{\infty} a_n L_n^{\alpha}(z)$$

where

$$(8.6) \quad a_n = \left[\binom{n+\alpha}{n} \Gamma(\alpha+1) \right]^{-1} \int_0^{\infty} e^{-x} x^{\alpha} L_n^{\alpha}(x) f(x) dx$$

($z \in p(d_a)$; $d_a > 0$)

it is necessary and sufficient that:

$$(8.7) \quad f(z) \text{ be analytic} \quad (z \in p(d_a))$$

and that to every b_a , $0 \leq b_a < d_a$, there correspond a positive number $B(\alpha, b)$ such that:

$$(8.8) \quad |f(z)| \leq B(\alpha, b) \exp\left(\frac{1}{2}x - |x|^{\frac{1}{2}} (b_a^2 - \frac{1}{2}(r-x))^{\frac{1}{2}} \right)$$

($z \in p(b_a)$)

We now hope to prove that theorems \bar{A} and \bar{B} are equivalent.

Proof: From (8.1) the equation $f(z) = f(w^2)$ implies that the transformation between the z and w -planes is:

$$(8.9) \quad z = w^2$$

from which we obtain:

$$(8.10) \quad x = u^2 - v^2; \quad y = 2uv$$

The equations (8.9) and (8.10) are equivalent. Hence by lemma (2.13) p. 27 we have :

$$(8.11) \quad \begin{cases} z \in p(d_a) \Leftrightarrow w \in s(d_a) \\ z \in p(b_a) \Leftrightarrow w \in s(b_a) \end{cases}$$

By identity (2.72) p. 28 we have:

$$(8.12) \quad \frac{1}{2}(r-x) = v^2$$

A: We first want to show that conditions (8.3) and (8.7) are equivalent. If $g(w)$ is an analytic function of w for $w \in s(d_a)$ then since $f(w^2) = g(w)$ the function $f(w^2)$ is an analytic function of w for $w \in s(d_a)$. And since $w = z^{\frac{1}{2}}$ is an analytic function of z except at the origin, (for instance $z^{\frac{1}{2}} = ((z-k) + k)^{\frac{1}{2}}$ can be expanded into a power series convergent for $|z-k| < |k|$, i. e. convergent in a circle with radius $|k|$ about the point k) and since an analytic function of an analytic function is analytic, the function $f(w^2) = f((z^{\frac{1}{2}})^2)$ is analytic function of z for $z \in p(d_a)$ by proposition (8.11) except possibly at $z=0$. Now since $g(w)$ is an even function and analytic at the origin i. e. for $w=0$ it can be expanded into a power series and hence we have:

$$f(w^2) = g(w) = \sum_{n=0}^{\infty} a_{2n} w^{2n}$$

and by the transformation $w^2 = z$ the above becomes:

$$f(z) = \sum_{n=0}^{\infty} a_{2n} z^n$$

Therefore $f(z)$ is analytic at the origin in the z -plane i. e. at the point $z=0$.

On the other hand if the function $f(z)$ is analytic for $z \in p(d_\alpha)$ since the transformation (8.9) is an analytic function of w the function, $g(w)$ where $g(w) = f(w^2) = f(z)$ is also analytic at the corresponding points in the w -plane, since the analytic function of an analytic function is likewise analytic. Therefore, by proposition (8.11) the function $g(w) = f(w^2)$ is analytic and even for $w \in s(d_\alpha)$.

B: We now want to show that conditions (8.4) and (8.8) are equivalent.

By lemma (2.14) p. 28 we have from inequality (8.4) that for every b_α , $0 \leq b_\alpha < d_\alpha$, there exists a positive number $B(\alpha, b)$ such that:

$$(8.13) \quad |f(w^2)| = |g(w)| <$$

$$B(\alpha, b) \exp\left(\frac{1}{2}(u^2 - v^2) - |u^2 - v^2|^{\frac{1}{2}}(b^2 - v^2)^{\frac{1}{2}}\right) \quad (w \in s(b)_\alpha)$$

By relations (8.10) (8.11) and (8.12) the above inequality becomes:

$$(8.14) \quad |f(z)| < B(a, b) \exp\left(\frac{1}{2}x - |x| \frac{1}{2} \left\{ b^2 - \frac{1}{2}(r-x) \right\}^{\frac{1}{2}}\right)$$

$$(z \in p(d_a))$$

Since the above steps are reversible we can go backward from relation (8.8) to (8.4). This completes the proof of part B.

C: We now want to show that equations (8.1) and (8.5) are equivalent.

By equality (8.2) and (8.6) and proposition (8.11) we see that by the transformation $z = w^2$ equation (8.1) becomes equation (8.5) and vice versa.

By propositions A, B, and C of this chapter we conclude that that theorems \bar{A} and \bar{B} are equivalent. This terminates the work of this chapter.

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