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GIBBS' PHENOMENON IN THE DOUBLE FOURIER'S SERIES

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by

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Introduction. In the convergence of the Fourier's series of a discontinuous function, the peculiar fact that bears his name was noted by J.W.Gibbs[1] in connection with a simple function of a single variable. Since then it has been studied by several men, M.Bôcher[2] and others, in connection with more general functions. There is now available complete information concerning very general functions of a single variable, (see Carslaw, "Introduction to the Theory of Fourier's Series and Integrals", chapter IX) but apparently the question has not been studied for functions of two variables and the double series. In this paper I propose to extend the work to include certain functions of two variables.

In the single variable case the discussion was based on the behavior of the Fourier's expansion of a particular function, f(x), where $f(x) = \frac{1}{2}\pi, (0 < x < \pi); f(x) = -\frac{1}{2}\pi, (-\pi < x < 0)$ and f(0) = 0. The first extension I will make will be to consider the product of this function and the function y = constant. That will be followed by the discussion of the product f(x)g(y), of two functions, each of which is like

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[1] Nature, Vol.59,p.606 (1899).
[2] Introduction to the Theory of Fourier's Series, Ann.Math.,Princeton, Ser.2,Vol.7,1906.

the function f(x). This product has discontinuities along the axes. The final extension to special functions will be the consideration of a function having two lines of discontinuities not both of which are parallel to the axes, the function being represented by sections of plane surfaces parallel to the xy-plane.

With the discussions of these special functions as a basis, I shall then proceed to the consideration of a more general function of two variables. The conditions on this function make it one of a large class of functions.

Part I

SPECIAL FUNCTIONS

1. Let us begin our study of the Gibbs' phenomena with a simple generalization of the problem in the single variable case to the double series case where the function has a finite jump along one axis, by taking the product of two simple functions of the single variables x and y. We define the functions $f(x)$ and $f_1(y)$ in the region R, $R = (-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ as follows:

$$
\text{(A)} \quad
\begin{aligned}
&f(x) = \tfrac{1}{2}\pi \quad (0 < x < \pi), \qquad f(x) = -\tfrac{1}{2}\pi \quad (-\pi < x < 0) \\
&f(0) = f(\pi) = f(-\pi) = 0 \\
&f_1(y) = 1 \quad (-\pi \leq y \leq \pi)
\end{aligned}
$$

The double Fourier's series development of the product $f(x)f_1(y)$ is given by the product of the expansions of each. Now $f_1(x)$ is an even function, hence must have a cosine development, and being constant, that development reduces to the constant term. Thus we find that the discussion reduces to the discussion of the single variable expansion of $f(x)$, which Carslaw, in his book "Introduction to the Theory of Fourier's Series and Integrals", has completely discussed.

To continue our study, we will now consider a case with the discontinuities along the two axes. We will take the product of the two functions $f(x)$ and $g(y)$ defined on

R as follows:

f(x) defined as in (A).

(B) $g(y) = \frac{1}{2}\pi \quad (0 < y < \pi), \qquad g(y) = -\frac{1}{2}\pi \quad (-\pi < y < 0)$

$g(-\pi) = g(0) = g(\pi) = 0$

As the Fourier's developments,

$$(1) \quad f(x) = \sum_{m=0}^{\infty} \frac{2}{2m+1} \sin(2m+1)x \qquad (-\pi \leq x \leq \pi)$$

$$(2) \quad g(y) = \sum_{n=0}^{\infty} \frac{2}{2n+1} \sin(2n+1)y \qquad (-\pi \leq y \leq \pi)$$

have been completely discussed at points of discontinuity by Carslaw, we shall use these as a basis upon which to proceed to the double series case by proving the following theorem. In the following discussion we shall use $\sigma_{mn}(x,y)$ as the sum from i=0 to m and j=0 to n of the term of the Fourier's expansion of the function under consideration. Also we shall use $\rho_1 = -\int_{\pi}^{\infty} \frac{\sin t}{t}\,dt = .2811\text{----}$, and $\rho_2 = -\int_{\pi}^{2\pi} \frac{\sin t}{t}\,dt = .4341\text{--}$ which were evaluated by Bôcher[1].

Theorem I.

Hypothesis:

1. The function f(x) and g(y) are defined on R as in (B) above and extended beyond R by the principle of periodicity.

Conclusions:

1. The surface $z = \sigma_{mn}(x,y)$ approaches as near as we please to the surface $z = f(x)\,g(y)$ throughout the interior of R, except in the neighborhood of the lines x=0 and y=0, as m and n become infinite in

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[1] Bôcher, Loc. Cit. p. 129.

any manner.

2. Along the boundary of R, except in the neighborhood of $(\pm\pi, 0)$, $(0, \pm\pi)$, and $(\pm\pi, \pm\pi)$, and along the axes, except in the neighborhood of $(0, 0)$, as m and n become infinite in any manner, the surface approaches the vertical plane section from the xy-plane to the height $\frac{1}{2}\pi(\frac{1}{2}\pi+\rho_1)$ if the approach is from a region where f(x)g(y) > 0, and to $-\frac{1}{2}\pi(\frac{1}{2}\pi+\rho_1)$ if from a region where f(x)g(y) < 0.
3. At $(0,0)$, $(0, \pm\pi)$, $(\pm\pi, 0)$, and $(\pm\pi, \pm\pi)$, as m and n become infinite in any manner, the surface approaches the vertical line segment from the xy-plane to the height $(\frac{1}{2}\pi+\rho_1)^2$, if the approach is from a region where f(x)g(y) > 0 and $-(\frac{1}{2}\pi+\rho_1)^2$ if from a region where f(x)g(y) < 0.

Proof: The double Fourier's series development of the product f(x)g(y) is given by the product of the expansions (1) and (2). When we form this product of the first m and the first n terms we have:

$$\sigma_{mn}(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\frac{2}{2i+1}\,\frac{2}{2j+1}\,\sin(2i+1)x\,\sin(2j+1)y$$

$$\sigma_{mn}(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\frac{2}{2i+1}\,\frac{2}{2j+1}\,\sin(2i+1)x\,\sin(2j+1)y$$

$$= 4\int_0^x\int_0^y\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\cos(2i+1)\alpha\,\cos(2j+1)\beta\,d\alpha\,d\beta$$

$$(3).\quad \sigma_{mn}(x,y) = \int_0^x\frac{\sin 2m\alpha}{\sin\alpha}\,d\alpha\int_0^y\frac{\sin 2n\beta}{\sin\beta}\,d\beta,$$

we find that sufficient conditions for the point (x,y) to be a maximum point for fixed values of m and n are that:

$$(4)\quad \frac{\partial\sigma_{mn}(x,y)}{\partial x} = \frac{\partial\sigma_{mn}(x,y)}{\partial y} = 0$$

$$(5)\quad \frac{\partial^2\sigma_{mn}(x,y)}{\partial x^2}\cdot\frac{\partial\sigma_{mn}(x,y)}{\partial y^2} > \left[\frac{\partial^2\sigma_{mn}(x,y)}{\partial x\,\partial y}\right]^2$$

$$(6)\quad \frac{\partial^2\sigma_{mn}(x,y)}{\partial x^2} < 0$$

From the expression for $\sigma_{mn}(x,y)$ we have,

$$(7)\quad \frac{\partial \sigma_{mn}}{\partial x} = \frac{\sin 2mx}{\sin x}\int_0^y \frac{\sin 2n\beta}{\sin\beta}\,d\beta,$$

$$(8)\quad \frac{\partial \sigma_{mn}}{\partial y} = \frac{\sin 2ny}{\sin y}\int_0^x \frac{\sin 2m\alpha}{\sin\alpha}\,d\alpha,$$

$$(9)\quad \frac{\partial^2 \sigma_{mn}}{\partial x^2} = \frac{2m\cos 2mx\sin x - \sin 2mx\cos x}{\sin^2 x}\int_0^y \frac{\sin 2n\beta}{\sin\beta}\,d\beta,$$

$$(10)\quad \frac{\partial^2 \sigma_{mn}}{\partial y^2} = \frac{2n\cos 2ny\sin y - \sin 2ny\cos y}{\sin^2 y}\int_0^x \frac{\sin 2m\alpha}{\sin\alpha}\,d\alpha,$$

$$(11)\quad \frac{\partial^2 \sigma_{mn}}{\partial x\,\partial y} = \frac{\sin 2mx}{\sin x}\cdot\frac{\sin 2ny}{\sin y}.$$

Setting (7) and (8) each equal to zero, we find, since

$$\int_0^{\xi} \frac{\sin kt}{\sin t}\,dt > 0 \qquad (0 < \xi < \pi)$$

that the values

$$(12)\quad \begin{aligned} x &= \frac{\pi}{2m}, \frac{2\pi}{2m}, \frac{3\pi}{2m}, \text{----}, \frac{i\pi}{2m}, \text{---} \\ y &= \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \text{----}, \frac{j\pi}{2n}, \text{---} \end{aligned}$$

are the positive roots, and that $\frac{\partial^2 \sigma_{mn}}{\partial x\,\partial y}$ is zero for any pair of these. The second and third conditions for maxima then combine to require $\frac{\partial^2 \sigma_{mn}}{\partial x^2} < 0$, and $\frac{\partial^2 \sigma_{mn}}{\partial y^2} < 0$. Substituting the values $x = \frac{i\pi}{2m}$ and $y = \frac{j\pi}{2n}$ in (9) and (10) we find that when i and j are each odd integers, $\sigma_{mn}(x,y)$ will be a maximum, and that the one nearest the origin is $(\frac{\pi}{2m}, \frac{\pi}{2n})$. We will fix our attention on this maximum, where we have

$$(13)\quad \sigma_{mn}\left(\frac{\pi}{2m},\frac{\pi}{2n}\right) = \int_0^{\pi/2m}\frac{\sin 2m\alpha}{\sin\alpha}d\alpha\int_0^{\pi/2n}\frac{\sin 2n\beta}{\sin\beta}d\beta$$

$$= \int_0^{\pi/2m}\frac{\alpha}{\sin\alpha}\,\frac{\sin 2m\alpha}{\alpha}d\alpha\int_0^{\pi/2n}\frac{\beta}{\sin\beta}\,\frac{\sin 2n\beta}{\beta}d\beta .$$

By the first law of the means for integrals, we have

$$\sigma_{mn}\left(\frac{\pi}{2m},\frac{\pi}{2n}\right) = \frac{\theta_1\frac{\pi}{2m}}{\sin\frac{\theta_1\pi}{2m}}\cdot\frac{\theta_2\frac{\pi}{2n}}{\sin\frac{\theta_2\pi}{2n}}\int_0^{\pi/2m}\frac{\sin 2m\alpha}{\alpha}d\alpha\int_0^{\pi/2n}\frac{\sin 2n\beta}{\beta}d\beta,$$

where $0<\theta_1<1$ and $0<\theta_2<1$.

If we let $\alpha=\frac{u}{2m}$ and $\beta=\frac{v}{2n}$, we have

$$(14)\quad \sigma_{mn}\left(\frac{\pi}{2m},\frac{\pi}{2n}\right) = \frac{\theta_1\frac{\pi}{2m}}{\sin\frac{\theta_1\pi}{2m}}\,\frac{\theta_2\frac{\pi}{2n}}{\sin\frac{\theta_2\pi}{2n}}\int_0^{\pi}\frac{\sin u}{u}du\int_0^{\pi}\frac{\sin v}{v}dv$$

$$= \frac{\theta_1\frac{\pi}{2m}}{\sin\frac{\theta_1\pi}{2m}}\,\frac{\theta_2\frac{\pi}{2n}}{\sin\frac{\theta_2\pi}{2n}}\left\{\int_0^{\pi}\frac{\sin t}{t}dt\right\}^2.$$

The value of $\left\{\int_0^{\pi}\frac{\sin t}{t}dt\right\}^2$ does not change with m and n. The value of $\int_0^{\pi}\frac{\sin t}{t}dt$ is

$$\int_0^{\pi}\frac{\sin t}{t}dt = \int_0^{\infty}\frac{\sin t}{t}dt - \int_{\pi}^{\infty}\frac{\sin t}{t}dt$$

$$= \tfrac{1}{2}\pi + \rho_1 .$$

The other two factors are each greater than unity, and approach unity as m and n increase indefinitely. Hence

$$(15)\quad \lim_{\substack{m\to\infty\\ n\to\infty}}\sigma_{mn}\left(\frac{\pi}{2m},\frac{\pi}{2n}\right) = \left(\tfrac{1}{2}\pi+\rho_1\right)^2.$$

By a little consideration, we can easily see that the surface is symmetrical in each of the three axes, and that our results in the first quadrant of the xy-plane can be carried to the other three quadrants.

If f(x)g(y) is extended beyond the region R by $f(x) = f(x \pm \pi)$ and $g(y) = g(y \pm \pi)$ we find that the product f(x)g(y) and the approximating surface are symmetrical to the planes $x = \pm\frac{1}{2}\pi$ and $y = \pm\frac{1}{2}\pi$. From symmetry to the plane $x = \frac{1}{2}\pi$, we find that the situation at $(\pi, 0)$ is the same as that at $(0,0)$ if we interpret the portion $(x<\pi, y>0)$ of the xy-plane as corresponding to the first quadrant, the region $(x<\pi, y<0)$ as corresponding to the second, etc. In an entirely similar manner we can establish the remainder of conclusion 3.

To determine the behavior of the surface along the x and y-axes between the various points we have already considered, let us take a section parallel to the x-axis through the maximum point $(\frac{\pi}{2m}, \frac{\pi}{2n})$, that is, the section cut by plane $y = \frac{\pi}{2n}$. The equation of the curve of intersection with the surface is

$$(16) \qquad z = \sigma_{mn}(x, \tfrac{\pi}{2n}) = \int_0^x \frac{\sin 2m\alpha}{\sin\alpha}\,d\alpha \int_0^{\pi/2n} \frac{\sin 2n\beta}{\sin\beta}\,d\beta .$$

If we make the transformation $2m\alpha = u$, and $2n\beta = v$, we have

$$(17) \qquad \sigma_{mn}(x, \tfrac{\pi}{2n}) = \int_0^{2mx} \frac{\sin u}{2m\sin u/2m}\,du \int_0^{\pi} \frac{\sin v}{2n\sin v/2n}\,dv$$

$$= \int_0^{2mx} \frac{u/2m}{\sin u/2m}\,\frac{\sin u}{u}\,du \int_0^{\pi} \frac{\frac{v}{2n}}{\sin v/2n}\,\frac{\sin v}{v}\,dv,$$

and applying the first law of the mean

$$(18) \qquad \sigma_{mn}(x, \tfrac{\pi}{2n}) = \frac{\theta_1\frac{\pi}{2m}}{\sin\theta_1\frac{\pi}{2m}}\,\frac{\theta_2\,\pi/2n}{\sin\theta_2\frac{\pi}{2n}} \int_0^{2mx}\frac{\sin u}{u}\,du \int_0^{\pi}\frac{\sin v}{v}\,dv,$$

where $(0<\theta_1<1)$ and $(0<\theta_2<1)$. As m and n increase, it is evident that for $(0<x<\pi)$

$$(19)\quad \lim_{\substack{m\to\infty \\ n\to\infty}} \sigma_{mn}\left(x,\tfrac{\pi}{2n}\right) = \sigma(x,+0) = \tfrac{1}{2}\pi\left(\tfrac{1}{2}\pi+\beta_1\right),$$

and that for $(0>x>-\pi)$

$$(20)\quad \lim_{\substack{m\to\infty \\ n\to\infty}} \sigma_{mn}\left(x,\tfrac{\pi}{2n}\right) = \sigma(x,+0) = -\tfrac{1}{2}\pi\left(\tfrac{1}{2}\pi+\beta_1\right).$$

In an entirely analogous manner we can complete the establishment of conclusion (2). Conclusion (1) has been established in any complete treatment of the convergence of the double Fourier's series at points of continuity of the functions represented.

2. In turning our attention to the consideration of another special function, where the discontinuity lies along a line or lines through the origin not all of which may be parallel to the axes, we find a much more complicated problem. We will begin our study of this case by considering the following lemmas.

Lemma A. If $\lambda(x)$ is a positive non-decreasing function of x, $A>0$, and $\delta>0$; the integral

$$I = \int_\delta^A \frac{\sin mx}{x}\int_0^{\lambda(x)}\frac{\sin ny}{y}\,dy\,dx$$

tends to zero when m and n tend to infinity in any manner.

This is lemma α in E.C.Titchmarsh's paper, "The Double Fourier Series of a Discontinuous Function."[1] and

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[1] Proc. Royal Soc. of London, Vol.106, (1924) pp. 299.

lemma B, following, is lemma δ of the same paper.

Lemma B. If R is the region between two curves $y = \lambda_1(x)$, and $y = \lambda_2(x)$, where $ax^p \leq \lambda_1(x) \leq \lambda_2(x) \leq bx^p, (p>0)$, which intersect at $(0,0)$ and at (A,B) and $g(x,y)$ is a positive non-increasing function of either variable for fixed values of the other satisfying

$$g(x,y) - g(x',y) - g(x,y') + g(x',y') \geq 0 \qquad (x>x',\ y>y'),$$

then

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \iint_R \frac{\sin m\alpha}{2\sin\frac{1}{2}\alpha}\,\frac{\sin n\beta}{2\sin\frac{1}{2}\beta}\, g(\alpha,\beta)\, d\alpha\, d\beta = g(0,0) \lim_{\substack{m\to\infty \\ n\to\infty}} \iint_R \frac{\sin m\alpha \sin n\beta}{\alpha\,\beta}\, d\alpha\, d\beta$$

if m and n become infinite in such a manner that the limit on the right exists.

Lemma C. If a_1, a_2, and δ are any positive constants, and the point (x,y) satisfies $(-\frac{1}{2}\delta \leq y \leq \frac{1}{2}\delta)$ and $|x|$ less than the smaller of the products $\frac{1}{2}a_1\delta$, and $\frac{1}{2}a_2\delta$, the integral

$$I = \frac{1}{4}\int_0^{\delta} \frac{\sin\frac{1}{2}n(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\, d\beta \int_{-a_2\delta}^{a_1\delta} \frac{\sin\frac{1}{2}m(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\, d\alpha + \frac{1}{4}\int_0^{-\delta} \frac{\sin\frac{1}{2}n(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\, d\beta \int_{a_2\delta}^{-a_1\delta} \frac{\sin\frac{1}{2}m(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\, d\alpha$$

tends to the value π^2 as m and n become infinite in any manner.

Proof: By use of lemma B, we have

$$(21)\quad I = \frac{1}{4}\int_0^{\delta} \frac{\sin\frac{1}{2}n(\beta-y)}{\frac{1}{2}(\beta-y)}\, d\beta \int_{-a_2\delta}^{a_1\delta} \frac{\sin\frac{1}{2}m(\alpha-x)}{\frac{1}{2}(\alpha-x)}\, d\alpha + \frac{1}{4}\int_0^{-\delta} \frac{\sin\frac{1}{2}n(\beta-y)}{\frac{1}{2}(\beta-y)}\, d\beta \int_{a_2\delta}^{-a_1\delta} \frac{\sin\frac{1}{2}m(\alpha-x)}{\frac{1}{2}(\alpha-x)}\, d\alpha.$$

In the first integral of I, let $\beta - y = 2v$, and $\alpha - x = 2u$, and in the second $\beta + y = -2v$, $\alpha + x = -2u$. Then

$$(22)\quad I = \int_0^{\frac{1}{2}(\delta-y)} \frac{\sin nv}{v}\, dv \int_{-\frac{1}{2}(a_2\delta+x)}^{\frac{1}{2}(a_1\delta-x)} \frac{\sin mu}{u}\, du + \int_0^{\frac{1}{2}(\delta+y)} \frac{\sin nv}{v}\, dv \int_{-\frac{1}{2}(a_2\delta-x)}^{\frac{1}{2}(a_1\delta+x)} \frac{\sin mu}{u}\, du$$

$$- \int_0^{-\frac{1}{2}y} \frac{\sin nv}{v}\, dv \int_{-\frac{1}{2}(a_2\delta+x)}^{\frac{1}{2}(a_1\delta+x)} \frac{\sin mu}{u}\, du - \int_0^{\frac{1}{2}y} \frac{\sin nv}{v}\, dv \int_{-\frac{1}{2}(a_2\delta-x)}^{\frac{1}{2}(a_2\delta+x)} \frac{\sin mu}{u}\, du$$

$$= I_1 + I_2 + I_3 + I_4 .$$

In I_1, I_2, and I_4, let $mu = \alpha$ and $nv = \beta$, then

$$(23)\quad I_1 = \int_0^{\frac{1}{2}n(\delta-y)} \frac{\sin\beta}{\beta}\, d\beta \int_{-\frac{1}{2}m(a_2\delta+x)}^{\frac{1}{2}m(a_1\delta-x)} \frac{\sin\alpha}{\alpha}\, d\alpha,$$

$$(24) \quad I_2 = \int_0^{\frac{1}{2}n(\delta+y)} \frac{\sin\beta}{\beta}\,d\beta \int_{-\frac{1}{2}m(a_2\delta-x)}^{\frac{1}{2}m(a_1\delta+x)} \frac{\sin\alpha}{\alpha}\,d\alpha,$$

$$(25) \quad I_4 = -\int_0^{\frac{1}{2}ny} \frac{\sin\beta}{\beta}\,d\beta \int_{-\frac{1}{2}m(a_2\delta-x)}^{\frac{1}{2}m(a_1\delta+x)} \frac{\sin\alpha}{\alpha}\,d\alpha,$$

and in I_3, let $mu = \alpha$, and $nv = -\beta$ to get,

$$(26) \quad I_3 = \int_0^{\frac{1}{2}ny} \frac{\sin\beta}{\beta}\,d\beta \int_{\frac{1}{2}m(a_2\delta+x)}^{\frac{1}{2}m(a_1\delta-x)} \frac{\sin\alpha}{\alpha}\,d\alpha.$$

Now $a_1\delta \pm x$, $a_2\delta \pm x$, and $\delta \pm y$, are all positive quantities for all values of x and y under consideration, hence in I_1, and I_2 each upper limit of integration is positive and increasing with m or n and the two non-zero lower limits are negative approaching negative infinity with -m or -n, then

$$(27) \quad \lim_{\substack{m\to\infty \\ n\to\infty}} (I_1+I_2) = 2\int_0^{\infty} \frac{\sin\beta}{\beta}\,d\beta \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\,d\alpha = \pi^2.$$

Also since $\left|\int_0^{\frac{1}{2}ny} \frac{\sin\beta}{\beta}\,d\beta\right| < M$, for all values of n and y

$$(28) \quad \lim_{\substack{m\to\infty \\ n\to\infty}} |I_3+I_4| \le M \lim_{m\to\infty}\left|\int_{-\frac{1}{2}m(a_2\delta+x)}^{\frac{1}{2}m(a_1\delta-x)} \frac{\sin\alpha}{\alpha}\,d\alpha - \int_{-\frac{1}{2}m(a_2\delta-x)}^{\frac{1}{2}m(a_1\delta+x)} \frac{\sin\alpha}{\alpha}\,d\alpha\right|$$

$$\le M\left|\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\,d\alpha - \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\,d\alpha\right| = 0.$$

Hence the integral I tends to the value π^2 as m and n increase indefinitely in any manner.

Lemma D. If a and δ are positive constants such that $(0 \le \delta \le \frac{1}{2}\pi)$ and $(-\frac{1}{2}\delta \le b \le \frac{1}{2}\delta)$ and $(0 \le a\delta \le \frac{1}{2}\pi)$, the integral

$$I = \int_0^{ah} \frac{\sin bt}{t} \int_{k\delta}^{\infty} \frac{\sin\beta \cos(t\beta/k)}{\beta}\,d\beta\,dt$$

tends to zero as h and k increase indefinitely in any manner.

Proof: We may write the integral I as

$$(29). \quad I = \int_{k\delta}^{\infty} \frac{\sin\beta}{\beta} \int_{0}^{\infty} \frac{\sin bt \, \cos(t\beta/k)}{t} \, dt \, d\beta$$

$$- \int_{k\delta}^{\infty} \frac{\sin\beta}{\beta} \int_{ah}^{\infty} \frac{\sin bt \, \cos(t\beta/k)}{t} \, dt \, d\beta$$

$$= I_1 - I_2 .$$

Now in the integral, I_1, let $bt = \alpha$. Then

$$(30) \quad I_1 = \int_{k\delta}^{\infty} \frac{\sin\beta}{\beta} \int_{0}^{\infty} \frac{\sin\alpha \, \cos(\alpha\beta/bk)}{\alpha} \, d\alpha \, d\beta,$$

and in the interval $k\delta \leq \beta < \infty$, we have

$$\left| \frac{\beta}{bk} \right| \geq \frac{k\delta}{|b|k} = \frac{\delta}{|b|} \geq 2,$$

and by a well known formula [1], we have

$$(31) \quad I_1 = \int_{k\delta}^{\infty} \frac{\sin\beta}{\beta} \cdot (0) \cdot d\beta = 0.$$

Applying the second law of the mean to I_2, we get

$$I_2 = \frac{1}{k\delta} \int_{ah}^{\infty} \frac{\sin bt}{t} \int_{k\delta}^{k\xi_1} \sin\beta \, \cos(t\beta/k) \, d\beta \, dt, \qquad (\delta < \xi_1 < \infty)$$

and performing the inner integration,

$$(32) \quad I_2 = \frac{1}{k\delta} \int_{ah}^{\infty} \frac{\sin bt}{t} \left[\frac{\cos(t-k)\beta/k}{2(t-k)/k} - \frac{\cos(t+k)\beta/k}{2(t+k)/k} \right]_{k\delta}^{k\xi_1} dt$$

$$= \frac{1}{2\delta} \int_{ah}^{\infty} \frac{dt}{t(t-k)} \left[\cos(t-k)\xi_1 - \cos(t-k)\delta \right] \sin bt$$

$$+ \frac{1}{2\delta} \int_{ah}^{\infty} \frac{dt}{t(t+k)} \left[\cos(t+k)\xi_1 - \cos(t+k)\delta \right] \sin bt$$

$$= I_3 + I_4$$

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[1] B.O.Peirce, "A short table of integrals." (Second Revised) No. 485. Also, Bieren de Haan, "Nouvelles Tables D'Integrals Definies". Table 151 No. 8

Applying the second law of the mean to I_4, we get

$$(33)\quad I_4 = \frac{1}{2\delta ah(ah+k)}\int_{ah}^{\xi_2}[\cos(t+k)\delta - \cos(t+k)\xi_1]\sin bt\,dt \quad (ah \le \xi_2 < \infty)$$

$$= \frac{1}{2\delta ah(ah+k)}\left\{-\frac{\cos\{(b+\delta)t+k\delta\}}{2(b+\delta)}\Big|_{ah}^{\xi_2} - \frac{\cos\{(b-\delta)t-k\delta\}}{2(b-\delta)}\Big|_{ah}^{\xi_2}\right.$$

$$\left. + \frac{\cos\{(b+\xi_1)t+k\xi_1\}}{2(b+\xi_1)}\Big|_{ah}^{\xi_2} + \frac{\cos\{(b-\xi_1)t-k\xi_1\}}{2(b-\xi_1)}\Big|_{ah}^{\xi_2}\right\}.$$

Since $|b\pm\delta| \geq \frac{1}{2}\delta$, and $|b\pm\xi_1| \geq \frac{1}{2}\delta$, given $\epsilon>0$, there exists an h_1 and a k_1 such that when $h>h_1$, and $k>k_1$,

$$(34)\quad |I_4| \leq \frac{1}{2\delta ah(ah+k)}\left|\frac{8}{\delta}\right| \leq \frac{4}{\delta^2 ah(ah+k)} < \frac{1}{2}\epsilon.$$

If $ah \leq k-\pi$, we will write

$$(35)\quad I_3 = \frac{1}{2\delta}\int_{ah}^{k-\pi}\frac{dt}{t(t-k)}\{\cos(t-k)\xi_1 - \cos(t-k)\delta\}\sin bt$$

$$+\frac{1}{2\delta}\int_{k-\pi}^{k}\frac{dt}{t(t-k)}\{\cos(t-k)\xi_1 - \cos(t-k)\delta\}\sin bt$$

$$+\frac{1}{2\delta}\int_{k}^{k+\pi}\frac{dt}{t(t-k)}\{\cos(t-k)\xi_1 - \cos(t-k)\delta\}\sin bt$$

$$+\frac{1}{2\delta}\int_{k+\pi}^{\infty}\frac{dt}{t(t-k)}\{\cos(t-k)\xi_1 - \cos(t-k)\delta\}\sin bt$$

$$\doteq I_5 + I_6 + I_7 + I_8.$$

Applying the second law of the mean to I_5, we have

$$(36)\quad I_5 = \frac{1}{2\delta\pi ah}\int_{ah}^{\xi_3}\{\cos(t-k)\xi_1 - \cos(t-k)\delta\}\sin bt\,dt, \quad (ah \le \xi_3 \le k-\pi)$$

$$= \frac{1}{2\delta\pi ah}\left[-\frac{\cos\{(b+\xi_1)t-k\xi_1\}}{2(b+\xi_1)} - \frac{\cos\{(b-\xi_1)t+k\xi_1\}}{2(b-\xi_1)}\right.$$

$$\left. + \frac{\cos\{(b+\delta)t-k\delta\}}{2(b+\delta)} + \frac{\cos\{(b-\delta)t+k\delta\}}{2(b-\delta)}\right]_{ah}^{\xi_3},$$

and there exists an h_2 and a k_2 such that when h > h_2 and k > k_2,

(37) $|I_5| \leq \frac{1}{2\delta\pi a h} \left|\frac{8}{\delta}\right| \leq \frac{4}{\pi\delta^2 a h} < \frac{1}{8}\epsilon$

And applying it to I_8, we have

(38) $I_8 = \frac{1}{2\delta(k+\pi)\pi}\int_{k+\pi}^{\xi_4}\{\cos(t-k)\xi_1 - \cos(t-k)\delta\}\sin bt\,dt, \quad (k+\pi \leq \xi_4 < \infty),$

with an h_3 and a k_3 such that, when $h > h_3$ and $k > k_3$, we have

(39) $|I_8| \leq \frac{1}{2\delta(k+\pi)\pi} \left|\frac{8}{\delta}\right| \leq \frac{4}{\pi\delta^2(k+\pi)} < \frac{1}{8}\epsilon.$

But,

(40) $$I_6 = \frac{1}{\delta}\int_{k-\pi}^{k}\frac{\sin\frac{1}{2}(\delta-\xi_1)(t-k)}{t(t-k)}\sin\tfrac{1}{2}(\delta+\xi_1)(t-k)\sin bt\,dt,$$

and since $\sin\frac{1}{2}(\delta-\xi_1)(t-k)/t(t-k)$ is a bounded monotonic function in $k-\pi \leq t \leq k$, we may apply the second law of the mean to get,

(41) $$I_6 = \frac{\frac{1}{2}(\delta-\xi_1)}{\delta k}\int_{k-\pi}^{\xi_5}\sin\tfrac{1}{2}(\delta+\xi_1)(t-k)\sin bt\,dt, \quad (k-\pi \leq \xi_5 \leq k),$$

$$= \frac{\delta-\xi_1}{2\delta k}\left[\frac{\sin\frac{1}{2}\{(2b-\xi_1-\delta)t+(\xi_1+\delta)k\}}{2b-\delta-\xi_1} - \frac{\sin\frac{1}{2}\{(2b+\xi_1+\delta)t-(\delta+\xi_1)k\}}{2b+\delta+\xi_1}\right]_{k-\pi}^{\xi_5}.$$

There exists an h_4 and a k_4 such that when $h > h_4$ and $k > k_4$

(42) $|I_6| = \frac{1}{\delta k}\left|\frac{\delta-\xi_1}{2b-\delta-\xi_1}\right| + \frac{1}{\delta k}\left|\frac{\delta-\xi_1}{2b+\delta+\xi_1}\right| < \frac{1}{8}\epsilon$

By arguments similar to those for I_6 we can show that

(43) $$I_7 = \frac{\delta-\xi_1}{2\delta k}\int_{k}^{\xi_6}\sin\tfrac{1}{2}(\delta+\xi_1)(t-k)\sin bt\,dt, \quad (k \leq \xi_6 < \pi+\pi)$$

and find an h_5 and a k_5 such that when $h > h_5$ and $k > k_5$,

$$(44) \quad |I_7| \leq \frac{1}{\delta k}\left|\frac{\delta-\xi_1}{2b-\delta-\xi_1}\right| + \frac{1}{\delta k}\left|\frac{\delta-\xi_1}{2b+\delta+\xi_1}\right| < \frac{1}{8}\epsilon.$$

Thus, if h_6 is the largest of $h_1 --- h_5$, and k_6 of $k_1 -- k_5$ and $h > h_6$ and $k > k_6$, we have

$$(45) \quad |I| \leq |I_4| + |I_5| + |I_6| + |I_7| + |I_8| \leq \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon + \tfrac{1}{8}\epsilon + \tfrac{1}{8}\epsilon + \tfrac{1}{8}\epsilon = \epsilon,$$

and we have established our propositions when $ah \leq k-\pi$. When $ah > k-\pi$, we can easily alter the above arguments to complete the proof, since one or more of the integrals I_5, I_6, I_7, I_8 will disappear from the consideration in part or completely.

3. With these lemmas at our disposal, we are now ready to define our special function and begin a reduction of the integrals obtained in the summing of the Fourier's series expansion of the function.

Let the region $R = (-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ be divided into two parts R_1 and R_2 by the line $x = ay$ $(a > 0)$. For lines cutting the other quadrants a simple transformation serves to reduce it to this case. Let R_1 signify the part of R containing the second quadrant and define $F(x,y)$ as follows:

$$F(x,y) = \tfrac{1}{4}\pi^2 \quad \text{on } R_1,$$

$$(C) \quad F(x,y) = -\tfrac{1}{4}\pi^2 \quad \text{on } R_2,$$

$F(x,y) = 0$ along $x = ay$ and on the boundary of R. Since the Fourier's expansion is not disturbed by a simple translation of the axes, we may imply the behavior of the series at all points of discontinuity of our function, from the behavior at the origin. In the remainder of this paper we will use $h=2m+1$ and $k=2n+1$ in order to simplify notation, and $S_{hk}(x,y)=\sigma_{mn}(x,y)$ shall have the obvious meaning. The behavior at the origin we will express in the form of a theorem.

Theorem II

Hypotheses:

1. F(x,y) is defined on R as in (C).
2. x = ey $(e \neq a)$ is any line.

Conclusion:

There is a manner for (x,y) to approach the origin from R_1 along $x=ey$ such that $S_{hk}(x,y)$ is a maximum, approaching the value $\frac{1}{2}\pi(\frac{1}{2}\pi+\beta)$, and from R_2 along $x=ey$ such that $S_{hk}(x,y)$ is a minimum approaching the value $-\frac{1}{2}\pi(\frac{1}{2}\pi+\beta)$, as h and k increase indefinitely in any manner.

Proof: Let δ be a constant such that $0<\delta\le\frac{1}{2}\pi$, and $0\le a\delta\le\frac{1}{2}\pi$, and let (x,y) be so situated that $-\frac{1}{2}\delta<y<\frac{1}{2}\delta$ and $-\frac{1}{2}a\delta<x<\frac{1}{2}a\delta$. By the usual method for summing the double series we have

$$(45)\quad S_{hk}(x,y)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\,\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,F(\alpha,\beta)\,d\alpha\,d\beta.$$

Now substitute the value of $F(\alpha,\beta)$ and then by use of lemmas A and B, and the Riemann-Lebesgue theorem, given $\epsilon>0$, there is an h_1 and a k_1 such that when $h>h_1$ and $k>k_1$, we have

$$(46)\quad S_{hk}(x,y) = \frac{1}{16}\int_{-(2\delta-y)}^{2\delta+y} \frac{\sin\frac{1}{2}k(\beta-y)}{\frac{1}{2}(\beta-y)} \int_{-(2a\delta-x)}^{a\beta} \frac{\sin\frac{1}{2}h(\alpha-x)}{\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta$$

$$-\frac{1}{16}\int_{-(2\delta-y)}^{2\delta+y} \frac{\sin\frac{1}{2}k(\beta-y)}{\frac{1}{2}(\beta-y)} \int_{a\beta}^{2a\delta+x} \frac{\sin\frac{1}{2}h(\alpha-x)}{\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta + E,$$

where $|E| < \frac{1}{2}\epsilon$. Let $\beta - y = 2v$, $\alpha - x = 2u$ and $2ab = ay - x$, then $-\frac{1}{2}\delta < b < \frac{1}{2}\delta$, and

$$(47)\quad S_{hk}(x,y) = \frac{1}{4}\int_{-\delta}^{\delta} \frac{\sin kv}{v} \int_{-a\delta}^{a(v+b)} \frac{\sin hu}{u}\,du\,dv + \frac{1}{4}\int_{-\delta}^{\delta} \frac{\sin kv}{v} \int_{a\delta}^{a(v+b)} \frac{\sin hu}{u}\,du\,dv$$

$$= \frac{1}{2}\int_{-\delta}^{\delta} \frac{\sin kv}{v} \int_{0}^{a(v+b)} \frac{\sin hu}{u}\,du\,dv$$

$$+\frac{1}{4}\int_{-\delta}^{\delta} \frac{\sin kv}{v} \left\{\int_{-a\delta}^{0} \frac{\sin hu}{u}\,du - \int_{0}^{a\delta} \frac{\sin hu}{u}\,du\right\} dv.$$

But if we let $-u = u'$,

$$(48)\quad \int_{-a\delta}^{0} \frac{\sin hu}{u}\,du = -\int_{a\delta}^{0} \frac{\sin hu'}{u'}\,du' = \int_{0}^{a\delta} \frac{\sin hu'}{u'}\,du'.$$

Thus, we have

$$(49)\quad S_{hk}(x,y) = \frac{1}{2}\int_{-\delta}^{\delta} \frac{\sin kv}{v} \int_{0}^{a(v+b)} \frac{\sin hu}{u}\,du\,dv$$

$$= \frac{1}{2}\int_{0}^{\delta} \frac{\sin kv}{v} \int_{0}^{a(v+b)} \frac{\sin hu}{u}\,du\,dv$$

$$-\frac{1}{2}\int_{0}^{-\delta} \frac{\sin kv}{v} \int_{0}^{a(v+b)} \frac{\sin hu}{u}\,du\,dv.$$

In the first integral let $hu = (v+b)t$, and in the second let $v = -v'$ and $hu = -(v'-b)t$. Then

$$S_{hk}(x,y) = \frac{1}{2}\int_{0}^{\delta} \frac{\sin kv}{v} \int_{0}^{ah} \frac{\sin(v+b)t}{t}\,dt\,dv$$

$$-\frac{1}{2}\int_{0}^{\delta} \frac{\sin kv'}{v'} \int_{0}^{ah} \frac{\sin(v'-b)t}{t}\,dt\,dv$$

$$= \frac{1}{2}\int_{0}^{\delta} \frac{\sin kv}{v} \int_{0}^{ah} \frac{\sin(v+b)t - \sin(v-b)t}{t}\,dt\,dv.$$

$$(50)\quad S_{hk}(x,y) = \int_0^{\delta} \frac{\sin kv}{v} \int_0^{ah} \frac{\sin bt \cos vt}{t}\, dt\, dv.$$

Let $kv = \beta$, and we get

$$(51)\quad S_{hk}(x,y) = \int_0^{k\delta} \frac{\sin\beta}{\beta} \int_0^{ah} \frac{\sin bt \cos(t\beta/k)}{t}\, dt\, d\beta$$

$$= \int_0^{ah} \frac{\sin bt}{t} \int_0^{\infty} \frac{\sin\beta \cos(t\beta/k)}{\beta}\, d\beta\, dt$$

$$- \int_0^{ah} \frac{\sin bt}{t} \int_{k\delta}^{\infty} \frac{\sin\beta \cos(t\beta/k)}{\beta}\, d\beta\, dt$$

$$= I_1 + I_2 .$$

By lemma D, there is an h_2 and a k_2 such that when h $> h_2$, and k $> k_2$, we have $|I_2| < \frac{1}{2}\epsilon$ and by the formula used in (30) Lemma D, if $ah \geq k$,

$$(52)\quad I_1 = \tfrac{1}{2}\pi \int_0^{k} \frac{\sin bt}{t}\, dt$$

$$= \tfrac{1}{2}\pi \int_0^{kb} \frac{\sin\alpha}{\alpha}\, d\alpha .$$

But $S_{hk}(x,y)$ will be a maximum when I_1 is a maximum and a minimum when I_1, is a minimum. The roots of

$$(53)\quad \frac{dI_1}{db} = \tfrac{1}{2} k\pi \frac{\sin kb}{kb} = \frac{\pi}{2b} \sin kb = 0,$$

are $b = \pm\, i\pi/k$, $(i = 1, 2 \cdots)$, and a little investigation shows that $b = \pi/k$ is the maximum nearest the origin and $b = -\pi/k$ the minimum nearest the origin. Since $x = ey$, we have $b = \frac{\pi}{k} = \frac{1}{2}(y - \frac{x}{a})$ in R_1, whence $y = 2a\pi/k(a-e)$, $x = 2ea\pi/k(a-e)$, and

$$(54)\quad I_1 = \tfrac{1}{2}\pi \int_0^{\pi} \frac{\sin\alpha}{\alpha}\, d\alpha = \tfrac{1}{2}\pi(\tfrac{1}{2}\pi + \rho_1),$$

or we have $b = -\pi/k = \frac{1}{2}(y - x/a)$ in R_2, whence $y = -2a\pi/k(a-e)$, $x = -2ea\pi/k(a-e)$, and

$$(55)\quad I_1 = \tfrac{1}{2}\pi\int_0^{-\pi}\frac{\sin\alpha}{\alpha}\,d\alpha = -\tfrac{1}{2}\pi(\tfrac{1}{2}\pi + \rho_1).$$

Or if $ah < k$, *the formula gives,*

$$(56)\quad I_1 = \tfrac{1}{2}\pi\int_0^{ah}\frac{\sin bt}{t}\,dt = \tfrac{1}{2}\pi\int_0^{abh}\frac{\sin\alpha}{\alpha}\,d\alpha.$$

The roots of

$$(57)\quad \frac{dI_1}{db} = \tfrac{1}{2}ah\pi\frac{\sin abh}{abh} = \frac{\pi}{2b}\sin abh = 0,$$

are $b = \pm i\pi/ah$, $i = (1,3,\ldots)$ and $b = \pi/ah$ is the maximum nearest the origin and $b = -\pi/ah$ the minimum. Since $x = ey$, we have $b = \pi/ah = \frac{1}{2}(y - x/a)$, in R_1, whence $x = 2e\pi/h(a-e)$, $y = 2\pi/h(a-e)$, and

$$(58)\quad I_1 = \tfrac{1}{2}\pi\int_0^{\pi}\frac{\sin\alpha}{\alpha}\,d\alpha = \tfrac{1}{2}\pi(\tfrac{1}{2}\pi + \rho_1),$$

or we have $b = -\pi/ah = \frac{1}{2}(y - x/a)$, in R_2, whence $x = -2e\pi/h(a-e)$, $y = 2\pi/h(a-e)$, and

$$(59)\quad I_1 = \tfrac{1}{2}\pi\int_0^{-\pi}\frac{\sin\alpha}{\alpha}\,d\alpha = -\tfrac{1}{2}\pi(\tfrac{1}{2}\pi + \rho_1).$$

Thus we see there is a manner of approach to the origin from R_1 for (x,y) such that,

$$(60)\quad S_{hk}(x,y) = \tfrac{1}{2}\pi(\tfrac{1}{2}\pi + \rho_1) + E_1.$$

And if h_3 is the larger of h_1 and h_2 and k_3 of k_1 and k_2 we have $|E_1| < \epsilon$ when $h > h_3$ and $k > k_3$. Similarly, there is a manner of approach from R_2, such that

$$(61)\quad S_{hk}(x,y) = -\tfrac{1}{2}\pi(\tfrac{1}{2}\pi + \rho_1) + E_2,$$

where $|E_2| < \epsilon$ when $h > h_3$ and $k > k_3$.

4. If our function has two lines of discontinuity, our further interest will be only at the intersections of these lines, since all other points of discontinuity have their neighboring points of discontinuity along single lines and thus fall under theorem II. We will make the origin the point of intersection of the lines of discontinuity.

Let the region $R = (-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ be divided into four parts R_1, R_2, R_3, R_4 by the lines $x = a_1 y$ and $x = a_2 y$ $(a_1 > a_2)$. The direction of increasing y will be called positive on these lines and R_1 used to designate the portion of R between these positive directions, R_2, R_3, and R_4 follow in order as we proceed around the origin in a counter-clockwise direction. We define F(x,y) on the region R as follows:

$$(D)\quad \begin{aligned} F(x,y) &= \tfrac{1}{4}\pi^2 && \text{on } R_1 \text{ and } R_3. \\ F(x,y) &= -\tfrac{1}{4}\pi^2 && \text{on } R_2 \text{ and } R_4 \\ F(x,y) &= 0 && \text{along } x = a_1 y,\ x = a_2 y, \text{ and the boundary of } R. \end{aligned}$$

Theorem III.

Hypotheses:

1. F(x,y) is defined on R as in (D) above.
2. δ is a positive constant in $0 < \delta < \pi$ such that $(-\pi \leq a_2\delta < a_1\delta < \pi)$.
3. The point (x,y) satisfies $-\frac{1}{2}\delta < y < \frac{1}{2}\delta$ and $|x|$ less than the smaller of $\frac{1}{2}|a_1|\delta$ and $\frac{1}{2}|a_2|\delta$.
4. $\lambda_1(v,x,y) = a_1 v + \frac{1}{2}(a_1 y - x)$ and $\lambda_2(v,x,y) = a_2 v + \frac{1}{2}(a_2 y - x)$.

Conclusion :

The sum $S_{hk}(x,y)$ tends to the same limit as

$$-\frac{1}{4}\pi^2+\frac{1}{2}\int_{-\frac{1}{2}y}^{\delta}\frac{\sin kv}{v}\int_0^{\lambda_1(v,x,y)}\frac{\sin hu}{u}\,du\,dv$$

$$+\frac{1}{2}\int_{\frac{1}{2}y}^{\delta}\frac{\sin kv}{v}\int_0^{\lambda_1(v,-x,-y)}\frac{\sin hu}{u}\,du\,dv$$

$$+\frac{1}{2}\int_{-\frac{1}{2}y}^{\delta}\frac{\sin kv}{v}\int_0^{\lambda_2(v,x,y)}\frac{\sin hu}{u}\,du\,dv$$

$$+\frac{1}{2}\int^{\delta}\frac{\sin kv}{v}\int^{\lambda_2(v,-x,-y)}\sin hu$$

(64) $$S_{hk}(x,y)=\frac{1}{4\pi^2}\iint_{R_\delta}F(\alpha,\beta)\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,d\alpha\,d\beta+\frac{1}{4\pi^2}\iint_{R-R_\delta}F(\alpha,\beta)\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,d\alpha\,d\beta.$$

Proof: The double Fourier's series expansion for the function F(x,y,) is given by

(62) $$F(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{a_{ij}\sin ix\sin jy+b_{ij}\sin ix\cos jy+c_{ij}\cos ix\sin jy+d_{ij}\cos ix\cos jy\},$$

and when the coefficients are evaluated, and the series summed from i = 0 to m and j = 0 to n by the usual method,

(63) $$S_{hk}(x,y)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}F(\alpha,\beta)\,\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\,\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,d\alpha\,d\beta.$$

Let R_δ signify that portion of R in the two rectangles $(0\le y\le\delta,\ -|a_1|\delta\le x\le|a_1|\delta)$ and $(-\delta\le y\le 0,\ -|a_1|\delta\le x\le|a_1|\delta)$. We see that R_δ lies entirely within R and we have,

(64) $$S_{hk}(x,y)=\frac{1}{4\pi^2}\iint_{R_\delta}F(\alpha,\beta)\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,d\alpha\,d\beta+\frac{1}{4\pi^2}\iint_{R-R_\delta}F(\alpha,\beta)\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,d\alpha\,d\beta.$$

By the use of lemma A, lemma B, and the Riemann-Lebesgue theorem, we can show that the second integral in (64) is as small as we please when h and k are made sufficiently large (compare with 6 in Titchmarsh's paper).[1]

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[1] Loc. Cit. p. 308

In the first integral, say, I_1 of (64), insert the values of F(x,y) and we have,

$$(65)\quad I_1 = \frac{1}{8}\int_0^{\delta}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\int_{a_2\beta}^{a_1\beta}\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta$$

$$+\frac{1}{8}\int_{-\delta}^{0}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\int_{a_1\beta}^{a_2\beta}\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta$$

$$-\frac{1}{8}\int_0^{\delta}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\int_{-|a_2|\delta}^{|a_1|\delta}\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta$$

$$+\frac{1}{8}\int_0^{-\delta}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\int_{-|a_1|\delta}^{|a_2|\delta}\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta.$$

The sum of the third and fourth integrals in (65) approaches $-\frac{1}{4}\pi^2$ by virtue of lemma C, and if in the second integral we transform $-\alpha=\alpha$ and $-\beta=\beta$, we may write

$$(66)\quad S_{hk}(xy) = -\frac{1}{4}\pi^2+\frac{1}{8}\int_0^{\delta}\frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\int_{a_2\beta}^{a_1\beta}\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta$$

$$+\frac{1}{8}\int_0^{\delta}\frac{\sin\frac{1}{2}k(\beta+y)}{\sin\frac{1}{2}(\beta+y)}\int_{a_2\beta}^{a_1\beta}\frac{\sin\frac{1}{2}h(\alpha+x)}{\sin\frac{1}{2}(\alpha+x)}\,d\alpha\,d\beta+E$$

where E is a function of x and y, but $|E|$ can be made uniformly as small as we please in R_δ by taking h and k sufficiently large. By applying lemma B again we have,

$$(67)\quad S_{hk}(x,y) = -\frac{1}{4}\pi^2+\frac{1}{8}\int_0^{\delta}\frac{\sin\frac{1}{2}k(\beta-y)}{\frac{1}{2}(\beta-y)}\int_{a_2\beta}^{a_1\beta}\frac{\sin\frac{1}{2}h(\alpha-x)}{\frac{1}{2}(\alpha-x)}\,d\alpha\,d\beta$$

$$+\frac{1}{8}\int_0^{\delta}\frac{\sin\frac{1}{2}k(\beta+y)}{\frac{1}{2}(\beta+y)}\int_{a_2\beta}^{a_1\beta}\frac{\sin\frac{1}{2}h(\alpha+x)}{\frac{1}{2}(\alpha+x)}\,d\alpha\,d\beta+E_1.$$

Now if in the first integral we let $\beta-y=2v$ and $\alpha-x=2u$, and in the second we let $\beta+y=-2v$ and $\alpha+x=-2u$, we have

$$(68)\quad S_{hk}(x,y) = -\frac{1}{4}\pi^2+\frac{1}{2}\int_{-\frac{1}{2}y}^{\frac{1}{2}(\delta-y)}\frac{\sin kv}{v}\int_{\lambda_2(v,x,y)}^{\lambda_1(v,x,y)}\frac{\sin hu}{u}\,du\,dv$$

$$+\frac{1}{2}\int_{\frac{1}{2}y}^{\frac{1}{2}(\delta+y)}\frac{\sin kv}{v}\int_{\lambda_2(v,-x,-y)}^{\lambda_1(v,-x,-y)}\frac{\sin hu}{u}\,du\,dv+E_1.$$

But the integrals,

$$(69)\quad \int_{\frac{1}{2}(\delta-y)}^{\delta} \frac{\sin kv}{v} \int_{\lambda_2(v,x,y)}^{\lambda_1(v,x,y)} \frac{\sin hu}{u}\, du\, dv,$$

$$(70)\quad \int_{\frac{1}{2}(\delta+y)}^{\delta} \frac{\sin kv}{v} \int_{\lambda_2(v,-x,-y)}^{\lambda_1(v,-x,-y)} \frac{\sin hu}{u}\, du\, dv$$

each tend to zero as (x,y) approaches (0,0) and h and k become infinite in any manner. This follows, since we can choose (x,y) near enough to (0,0) that $\lambda_1(v,x,y)$, $\lambda_2(v,x,y)$ $\lambda_1(v,-x,-y)$ and $\lambda_2(v,-x,-y)$ each has the sign of the corresponding a_1 or a_2 throughout the interval under consideration for v, thus the integrals may be transformed to satisfy the condition of lemma A. Thus

$$(71)\quad S_{hk}(x,y) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{2}\int_{\frac{1}{2}y}^{\delta} \frac{\sin kv}{v} \int_0^{\lambda_1(v,x,y)} \frac{\sin hu}{u}\, du\, dv$$
$$-\tfrac{1}{2}\int_{\frac{1}{2}y}^{\delta} \frac{\sin kv}{v} \int_0^{\lambda_2(v,x,y)} \frac{\sin hu}{u}\, du\, dv$$
$$+\tfrac{1}{2}\int_{\frac{1}{2}y}^{\delta} \frac{\sin kv}{v} \int_0^{\lambda_1(v,-x,-y)} \frac{\sin hu}{u}\, du\, dv$$
$$-\tfrac{1}{2}\int_{\frac{1}{2}y}^{\delta} \frac{\sin kv}{v} \int_0^{\lambda_2(v,-x,-y)} \frac{\sin hu}{u}\, du\, dv + E_2,$$

where $|E_2|$ can be made as small as we please, as h and k become infinite in any manner, and our theorem is established.

5. To study the convergence of the integrals in (71) we are interested in the maximum and minimum points on the surface $z = S_{hk}(x,y)$ near the origin, and the behavior of these points and of $S_{hk}(x,y)$ as h and k in-

crease indefinitely. We find it more convenient to study a related function, defined from equation (66) as $S^*_{hk}(x,y) = S_{hk}(x,y) - E$. The maxima and minima of $S_{hk}(x,y)$ and $S^*_{hk}(x,y)$ may not occur for the same values of (x,y). To each point on the surface $Z = S^*_{hk}(x,y)$ there corresponds a point on the surface $Z = S_{hk}(x,y)$ and these points can be made to remain as close together as we please by taking h and k sufficiently large, since E converges uniformly to zero as h and k increase. Thus any points on $z = S^*_{hk}(x,y)$ showing a Gibbs phenomenon will also have a corresponding point showing the same for $z = S_{hk}(x,y)$. A set of sufficient conditions that a point be a maximum or a minimum point are given in (4) and (5). Deriving $S^*_{hk}(x,y)$ with respect to x, holding y fixed, we have

$$(72)\quad \frac{\partial S^*_{hk}}{\partial x} = \frac{1}{\delta}\int_0^{\delta} \frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)} \int_{a_2\beta}^{a_1\beta} \frac{\partial}{\partial x}\left\{\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\right\} d\alpha\, d\beta$$

$$+ \frac{1}{\delta}\int_0^{\delta} \frac{\sin\frac{1}{2}k(\beta+y)}{\sin\frac{1}{2}(\beta+y)} \int_{a_2\beta}^{a_1\beta} \frac{\partial}{\partial x}\left\{\frac{\sin\frac{1}{2}h(\alpha+x)}{\sin\frac{1}{2}(\alpha+x)}\right\} d\alpha\, d\beta .$$

But observing that,

$$(73)\quad \frac{\partial}{\partial x}\left\{\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\right\} = -\frac{\partial}{\partial \alpha}\left\{\frac{\sin\frac{1}{2}h(\alpha-x)}{\sin\frac{1}{2}(\alpha-x)}\right\},$$

and that,

$$(74)\quad \frac{\partial}{\partial x}\left\{\frac{\sin\frac{1}{2}h(\alpha+x)}{\sin\frac{1}{2}(\alpha+x)}\right\} = \frac{\partial}{\partial \alpha}\left\{\frac{\sin\frac{1}{2}h(\alpha+x)}{\sin\frac{1}{2}(\alpha+x)}\right\},$$

we may perform the inner integration to get,

$$(75)\quad \frac{\partial S^*_{hk}}{\partial x} = \frac{1}{8}\int_0^\delta \frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,\frac{\sin\frac{1}{2}h(a_2\beta-x)}{\sin\frac{1}{2}(a_2\beta-x)}\,d\beta$$

$$-\frac{1}{8}\int_0^\delta \frac{\sin\frac{1}{2}k(\beta-y)}{\sin\frac{1}{2}(\beta-y)}\,\frac{\sin\frac{1}{2}h(a_1\beta-x)}{\sin\frac{1}{2}(a_1\beta-x)}\,d\beta$$

$$-\frac{1}{8}\int_0^\delta \frac{\sin\frac{1}{2}k(\beta+y)}{\sin\frac{1}{2}(\beta+y)}\,\frac{\sin\frac{1}{2}h(a_2\beta+x)}{\sin\frac{1}{2}(a_2\beta+x)}\,d\beta$$

$$+\frac{1}{8}\int_0^\delta \frac{\sin\frac{1}{2}k(\beta+y)}{\sin\frac{1}{2}(\beta+y)}\,\frac{\sin\frac{1}{2}h(a_1\beta+x)}{\sin\frac{1}{2}(a_1\beta+x)}\,d\beta\,,$$

which, if we remember that $h=2m+1$ and $k=2n+1$, reduces to

$$(76)\quad \frac{\partial S^*_{hk}}{\partial x} = \frac{1}{8}\int_0^\delta\Big\{\frac{[\cos(m\beta-\overline{m+1}y)-\cos(\overline{m+1}\beta-my)][\cos(na_2\beta-\overline{n+1}x)-\cos(\overline{n+1}a_2\beta-nx)]}{(\cos\beta-\cos y)(\cos a_2\beta-\cos x)}$$

$$-\frac{[\cos(m\beta+\overline{m+1}y)-\cos(\overline{m+1}\beta+my)][\cos(na_2\beta+\overline{n+1}x)-\cos(\overline{n+1}a_2\beta+nx)]}{(\cos\beta-\cos y)(\cos a_2\beta-\cos x)}\Big\}d\beta$$

$$-\frac{1}{8}\int_0^\delta\Big\{\frac{[\cos(m\beta-\overline{m+1}y)-\cos(\overline{m+1}\beta-my)][\cos(na_1\beta-\overline{n+1}x)-\cos(\overline{n+1}a_1\beta-nx)]}{(\cos\beta-\cos y)(\cos a_1\beta-\cos x)}$$

$$-\frac{[\cos(m\beta+\overline{m+1}y)-\cos(\overline{m+1}\beta+my)][\cos(na_1\beta+\overline{n+1}x)-\cos(\overline{n+1}a_1\beta+nx)]}{(\cos\beta-\cos y)(\cos a_1\beta-\cos x)}\Big\}d\beta.$$

By a similar process we find that the expression for $\frac{\partial S^*_{hk}(xy)}{\partial y}$ is equally as complex, and that the second derivatives are much more complicated. The determination of the zeros of $\frac{\partial S^*_{hk}(x,y)}{\partial x}$ and $\frac{\partial S^*_{hk}(xy)}{\partial y}$ for any finite value of h and k, involves the evaluation of certain integrals by some satisfactory method of approximation. Because of that fact we will not attempt to definitely locate the maximum or minimum points.

6. The forms of these derivatives, and the values

(12), in the former problem suggest that probably the maximum and minimum point vary with $1/h$ and $1/k$. Now if the ratio h/k varies, to take $x = \frac{e}{h}$ and $y = \frac{d}{k}$ may lead us to be considering a point that approaches (0,0) in such a way as to cross one or more of the lines of discontinuity, and in order to prevent this, we will require (x,y) to move along a line $x = ey$ by letting $y = \frac{2d}{k}$. We will need the following lemmas in discussing the behavior of $S_{hk}(x,y)$ as (x,y) approaches the origin in the manner specified above.

Lemma E. If a and δ are any positive constants, is positive, either fixed or approaching zero, and as h and k become infinite, the integral

$$I = \int_{d/k}^{\delta} \frac{\sin kv}{v} \int_0^{a(v+b/k)} \frac{\sin hu}{u}\,du\,dv,$$

(a) approaches the value $\frac{1}{2}\pi \int_d^{\infty} \frac{\sin \beta}{\beta}\,d\beta$ when $b \geq -d$.

(b) approaches the value $\frac{1}{2}\pi \int_{|b|}^{\infty} \frac{\sin \beta}{\beta}\,d\beta - \frac{1}{2}\pi \int_d^{|b|} \frac{\sin \beta}{\beta}\,d\beta$ when $b < -d$

Proof: Part (a). Choose δ_1 such that $\frac{1}{2}\pi > \delta_1 > 0$, and let $kv = \beta$ and $hu = (\beta + b)\alpha$. Then

$$(77) \quad I = \int_d^{k\delta} \frac{\sin \beta}{\beta} \int_0^{ah/k} \frac{\sin(\beta+b)\alpha}{\alpha}\,d\alpha\,d\beta$$

$$= \int_d^{k\delta} \frac{\sin \beta}{\beta} \int_0^{\infty} \frac{\sin(\beta+b)\alpha}{\alpha}\,d\alpha\,d\beta - \int_{d+\delta_1}^{k\delta} \frac{\sin \beta}{\beta} \int_{ah/k}^{\infty} \frac{\sin(\beta+b)\alpha}{\alpha}\,d\alpha\,d\beta$$

$$- \int_d^{d+\delta_1} \frac{\sin \beta}{\beta} \int_{ah/k}^{\infty} \frac{\sin(\beta+b)\alpha}{\alpha}\,d\alpha\,d\beta = I_1 + I_2 + I_3.$$

By use of the second law of the the mean for integrals, we have

$$(78)\qquad I_2 = -\frac{k}{ah}\int_{d+\delta_1}^{k\delta}\frac{\sin\beta}{\beta}\int_{ah/k}^{\xi_1}\sin(\beta+b)\alpha\, d\alpha\, d\beta\,, \quad (ah/k \le \xi_1 < \infty),$$

$$= -\frac{k}{ah}\int_{d+\delta_1}^{k\delta}\frac{\sin\beta}{\beta(\beta+b)}\left\{\cos[(\beta+b)ah/k] - \cos(\beta+b)\xi_1\right\}d\beta\,,$$

and applying the second law of the mean again,

$$(79)\qquad I_2 = \frac{-k}{ah(d+\delta_1)(d+\delta_1+b)}\int_{d+\delta_1}^{\xi_2}\left\{\sin\beta\cos(\beta+b)ah/k - \sin\beta\cos(\beta+b)\xi_1\right\}d\beta,$$

where $d+\delta_1 \le \xi_2 < k\delta_1$ and By integrating,

$$(80)\qquad I_2 = \frac{-k}{ah(d+\delta_1)(d+\delta_1+b)}\left\{\frac{\cos(\beta+\overline{\beta+b}\,ah/k)}{2(1+ah/k)} - \frac{\cos(\beta-\overline{\beta+b}\,ah/k)}{2(1-ah/k)}\right.$$

$$\left. + \frac{\cos(\beta+\overline{\beta+b}\,\xi_1)}{2(1+\xi_1)} + \frac{\cos(\beta-\overline{\beta+b}\,\xi_1)}{2(1-\xi_1)}\right\}\Bigg]_{d+\delta_1}^{\xi_2}.$$

Given $\epsilon > 0$, choose an h_1 and a k_1 such that, when $h > h_1$ and $k > k_1$,

$$(81)\qquad |I_2| \le \frac{k}{ah(d+\delta_1)(d+\delta_1+b)}\left|\frac{2}{ah/k+1} + \frac{2}{ah/k-1}\right|$$

$$\le \frac{k}{h}\left|\frac{4}{a(d+\delta_1)(d+\delta_1+b)(ah/k-1)}\right| \le \frac{\epsilon}{3}.$$

We may rewrite I_3 in the form

$$(82)\qquad I_3 = -\int_{ah/k}^{\infty}\frac{d\alpha}{\alpha}\int_{d}^{d+\delta_1}\frac{\sin\beta\sin(\beta+b)\alpha}{\beta}\,d\beta.$$

If $d \ge \pi - \delta$, by the use of the second law of the mean,

$$(83)\qquad I_3 = -\frac{1}{d}\int_{ah/k}^{\infty}\frac{d\alpha}{\alpha}\int_{d}^{\xi_3}\sin\beta\,\sin(\beta+b)\alpha\, d\beta \qquad (d \le \xi_3 \le d+\delta_1)$$

$$= -\frac{1}{d}\int_{ah/k}^{\infty}\frac{1}{\alpha}\left\{\frac{\sin(\overline{\alpha-1}\,\beta+b\alpha)}{2(\alpha-1)} - \frac{\sin(\overline{\alpha+1}\,\beta+b\alpha)}{2(\alpha+1)}\right\}\Bigg]_{d}^{\xi_3}d\alpha.$$

There exists an $\epsilon_1 > 0$ such that $|\log(1\pm\epsilon_1)| < \epsilon/6d$, and if we choose h_2 and k_2 such that $h/k > 1/a\epsilon_1$, when $h>h_2$ and $k>k_2$, we have

$$(84)\quad |I_3| \le \frac{1}{d}\Big|\int_{ah/k}^{\infty}\frac{d\alpha}{\alpha(\alpha+1)}\Big| + \frac{1}{d}\Big|\int_{ah/k}^{\infty}\frac{d\alpha}{\alpha(\alpha-1)}\Big|$$

$$\le \frac{1}{d}\Big\{\Big|\log\frac{\alpha-1}{\alpha}\Big]_{ah/k}^{\infty}\Big| + \Big|\log\frac{\alpha+1}{\alpha}\Big]_{ah/k}^{\infty}\Big|\Big\}$$

$$\le \frac{1}{d}\Big\{\big|\log 1-\log(1-\tfrac{k}{ah})\big| + \big|\log 1-\log(1+\tfrac{k}{ah})\big|\Big\} < \frac{\epsilon}{6}+\frac{\epsilon}{6} = \frac{1}{3}\epsilon.$$

If $d<\pi-\delta$, we have that $\sin\beta/\beta$ is a positive, monotonic decreasing function in the interval $0\le\beta\le d+\delta_1$ and if we apply the second law of the mean,

$$(85)\quad I_3 = \int_{ah/k}^{\infty}\frac{d\alpha}{\alpha}\;\frac{\sin d}{d}\int_{d}^{\xi_4}\sin(\beta+b)\alpha\,d\beta \qquad (d\le\xi_4\le\delta_1+d)$$

$$= \frac{\sin d}{d}\int_{ah/k}^{\infty}\frac{1}{\alpha^2}\Big[\cos(\beta+b)\alpha\Big]_{d}^{\xi_4}d\alpha.$$

If h_3 and k_3 are chosen so that $h/k > \frac{6}{a\epsilon}$ when $h>h_3$ and $k>k_3$, we have

$$(86)\quad |I_3| \le 2\Big|\int_{ah/k}^{\infty}\frac{d\alpha}{\alpha^2}\Big| \le 2\Big|-\frac{1}{\alpha}\Big]_{ah/k}^{\infty}\Big|$$

$$\le \frac{2k}{ah} < \frac{1}{3}\epsilon.$$

Now the integral $\int_0^{\infty}\frac{\sin mx}{x}dx$ has the value $\frac{1}{2}\pi$, when m is positive and the value zero when m=0. In I_1 we have $\beta+b\ge d+b>0$, and

$$(87)\quad I_1 = \frac{1}{2}\pi\int_{d}^{k\delta}\frac{\sin\beta}{\beta}d\beta.$$

But there is a k_4 such that for $k>k_4$,

$$(88)\quad \Big|I_1 - \frac{1}{2}\pi\int_{d}^{\infty}\frac{\sin\beta}{\beta}d\beta\Big| < \frac{1}{3}\epsilon.$$

Thus if h_5 is the largest of the numbers h_1, h_2, and h_3 and k_5 is the largest of the numbers k_1, k_2, k_3 and k_4 we have, when $h > h_5$ and $k > k_5$,

(89) $$\left| I - \tfrac{1}{2}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta \right| \leq \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon,$$

whence

(90 $$\lim I = \tfrac{1}{2}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta.$$

Part (b). When $b < -d$, we have

(91) $$I = \int_{d/k}^{\delta} \frac{\sin kv}{v}\int_0^{a(v+b/k)} \frac{\sin hu}{u}\,du\,dv$$
$$= \int_{d/k}^{|b|/k} \frac{\sin kv}{v}\int_0^{a(v+b/k)} \frac{\sin hu}{u}\,du\,dv$$
$$+ \int_{|b|/k}^{\infty} \frac{\sin kv}{v}\int_0^{a(v+b/k)} \frac{\sin hu}{u}\,du\,dv = I_1 + I_2,$$

and from the results of part (a) we have

(92) $$\lim I_2 = \tfrac{1}{2}\pi\int_{|b|}^\infty \frac{\sin\beta}{\beta}\,d\beta.$$

In I_1 let $kv = \beta$ and $hu = (\beta+b)\alpha$. Then

(93) $$I_1 = \int_d^{|b|} \frac{\sin\beta}{\beta}\int_0^\infty \frac{\sin(\beta+b)\alpha}{\alpha}\,d\alpha\,d\beta$$
$$- \int_d^{|b|} \frac{\sin\beta}{\beta}\int_{\ldots}^{\infty} \frac{\sin(\beta+b)\alpha}{\alpha}\,d\alpha\,d\beta = I_1' + I_1''$$

By an argument $\int_d^{|b|} \frac{\sin\beta}{\beta}\int_{ah/k}^{\infty} \frac{\sin(\beta+b)\alpha}{\alpha}\,d\alpha\,d\beta = I_1' + I_1''$ we see that $\lim I_1'' = 0$.

And since $\beta + b \leq 0$ in $d \leq \beta \leq -b$ we have, as h and k become infinite so that h/k becomes infinite,

(94) $$\lim I_1' = -\tfrac{1}{2}\pi\int_d^{|b|} \frac{\sin\beta}{\beta}\,d\beta.$$

Thus

$$(95)\quad \lim I = \tfrac{1}{2}\pi\int_{|b|}^{\infty}\frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{2}\pi\int_{d}^{|b|}\frac{\sin\beta}{\beta}\,d\beta,$$

Lemma F. If a and δ are any positive constants, d is positive, either constant or approaching zero and $h/k \to \infty$ as h and k increase indefinitely, then the integral

$$I = \int_{d/k}^{\infty}\frac{\sin kv}{v}\int_{0}^{a(v+b/k)}\frac{\sin hu}{u}\,du\,dv$$

approaches the value

(a) $\tfrac{1}{4}\pi^2+\tfrac{1}{2}\pi\int_0^d \frac{\sin t}{t}\,dt$ when $b \geq d$,

(b) $\tfrac{1}{2}\pi\int_d^\infty \frac{\sin t}{t}\,dt + \pi\int_0^{|b|}\frac{\sin t}{t}\,dt$ when $d > b > -d$,

(c) $\tfrac{1}{4}\pi^2 - \pi\int_d^{|b|}\frac{\sin t}{t}\,dt + \tfrac{1}{2}\pi\int_0^d\frac{\sin t}{t}\,dt$ when $-d \geq b$.

Proof:

$$(96)\quad I = \int_{-d/k}^{\delta}\frac{\sin kv}{v}\int_0^{a(v+b/k)}\frac{\sin hu}{u}\,du\,dv$$

$$= \int_{d/k}^{\delta}\frac{\sin kv}{v}\int_0^{a(v+b/k)}\frac{\sin hu}{u}\,du\,dv + \int_0^{d/k}\frac{\sin kv}{v}\int_0^{a(v+b/k)}\frac{\sin hu}{u}\,du\,dv$$

$$-\int_0^{-d/k}\frac{\sin kv}{v}\int_0^{a(v+b/k)}\frac{\sin hu}{u}\,du\,dv = I_1+I_2+I_3.$$

And now let $kv=\beta$ and $hu=(\beta+b)\alpha$ in I_2 and $kv=-\beta$ and $hu=-(\beta+b)\alpha$ in I_3. Then

$$(97)\quad I_2+I_3 = \int_0^d\frac{\sin\beta}{\beta}\int_0^{ah/k}\left\{\frac{\sin(\beta+b)\alpha}{\alpha} - \frac{\sin(\beta-b)\alpha}{\alpha}\right\}d\alpha\,d\beta$$

$$= 2\int_0^d\frac{\sin\beta}{\beta}\int_0^{ah/k}\frac{\sin b\alpha\cos\beta\alpha}{\alpha}\,d\alpha\,d\beta$$

$$= 2\int_0^d\frac{\sin\beta}{\beta}\int_0^{\infty}\frac{\sin b\alpha\cos\beta\alpha}{\alpha}\,d\alpha\,d\beta + E,$$

where

$$(98)\qquad E = -2\int_0^d \frac{\sin\beta}{\beta}\int_{ah/k}^{\infty} \frac{\sin b\alpha \cos\beta\alpha}{\alpha}\, d\alpha\, d\beta$$

$$= -2\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha}\int_0^d \frac{\sin\beta\cos\alpha\beta}{\beta}\, d\beta\, d\alpha$$

Now $\sin\beta/\beta$ is a positive, monotonic decreasing function of β in $0 \le \beta \le \pi$, and if $d > \pi$, by writing

$$(99)\qquad E = -2\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha}\int_0^{\pi} \frac{\sin\beta\cos\alpha\beta}{\beta}\, d\beta\, d\alpha$$

$$-2\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha}\int_{\pi}^{d} \frac{\sin\beta\cos\alpha\beta}{\beta}\, d\beta\, d\alpha = E_1 + E_2,$$

We may apply the second law of the mean to E_1 to get

$$(100)\qquad E_1 = -2\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha}\int_0^{\xi_1} \cos\alpha\beta\, d\beta\, d\alpha \qquad (0 \le \xi_1 \le \pi)$$

$$= -2\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha^2}\Big[\sin\alpha\beta\Big]_0^{\xi_1} d\alpha = -2\int_{ah/k}^{\infty} \frac{\sin b\alpha \sin\xi_1\alpha}{\alpha^2}\, d\alpha.$$

Given $\epsilon > 0$, there exist an h_1 and a k_1 such that when $h > h_1$ and $k > k_1$, we have

$$(101)\qquad |E_1| \le 2\left|\int_{ah/k}^{\infty} \frac{d\alpha}{\alpha^2}\right| = 2\left|-\frac{1}{\alpha}\right]_{ah/k}^{\infty}\Big| = \frac{2k}{ah} < \frac{1}{4}\epsilon,$$

And applying the second law of the mean to E_2, we get

$$(102)\qquad E_2 = -\frac{2}{\pi}\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha}\int_{\pi}^{\xi_2} \sin\beta\cos\alpha\beta\, d\beta\, d\alpha \qquad (\pi \le \xi_2 \le d)$$

$$= \frac{2}{\pi}\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha}\left[\frac{\cos(1-\alpha)\beta}{2(1-\alpha)}\right]_{\pi}^{\xi_2} d\alpha + \frac{2}{\pi}\int_{ah/k}^{\infty} \frac{\sin b\alpha}{\alpha}\left[\frac{\cos(1+\alpha)\beta}{2(1+\alpha)}\right]_{\pi}^{\xi_2} d\alpha,$$

Then,

$$|E_2| \le \frac{2}{\pi}\left|\int_{ah/k}^{\infty} \frac{d\alpha}{\alpha(\alpha-1)}\right| + \frac{2}{\pi}\left|\int_{ah/k}^{\infty} \frac{d\alpha}{\alpha(\alpha+1)}\right|.$$

$$(103)\quad |E_2| \le \frac{2}{\pi}\left|\log\frac{\alpha}{\alpha-1}\Big]_{ah/k}^{\infty}\right| + \frac{2}{\pi}\left|\log\frac{\alpha}{\alpha+1}\Big]_{ah/k}^{\infty}\right|$$

$$\le \frac{2}{\pi}|\log(1-k/ah)| + \frac{2}{\pi}|\log(1+k/ah)|,$$

and there exist an h_2 and a k_2 such that when $h>h_2$ and $k>k_2$

$$(104)\quad |E_2| < \tfrac{1}{4}\epsilon.$$

If $d \le \pi$ we need, for E_1 to use only the arguments for E_1 above, then in either case, if h_3 is the larger of the numbers h_1 and h_2, and k is the larger of k_1 and k_2

$$(105)\quad |E| < \tfrac{1}{2}\epsilon,$$

when $h>h_3$ and $k>k_3$. Then

$$Lim(I_2+I_3) = 2\int_0^d \frac{\sin\beta}{\beta}\int_0^\infty \frac{\sin b\alpha \cos\beta\alpha}{\alpha}\,d\alpha\, d\beta$$

$$(106)\qquad = \pi\int_0^d \frac{\sin\beta}{\beta}d\beta \quad \text{when } b \ge d$$

$$(107)\qquad = \pi\int_0^{|b|} \frac{\sin\beta}{\beta}d\beta \quad \text{when } d > b > -d$$

$$(108)\qquad = \pi\int_0^{d} \frac{\sin\beta}{\beta}d\beta \quad \text{when} -d \ge b$$

For the values of I_1 we go back to Lemma E and find

$$(109)\quad Lim I_1 = \tfrac{1}{2}\pi\int_d^\infty \frac{\sin\beta}{\beta}d\beta \quad \text{when} \quad b \ge -d$$

$$(110)\qquad = \tfrac{1}{2}\pi\int_{|b|}^\infty \frac{\sin\beta}{\beta}d\beta - \tfrac{1}{2}\pi\int_d^{|b|}\frac{\sin\beta}{\beta}d\beta, \text{ when } \quad b < -d.$$

Then by adding the values of I_1, I_2 and I_3 we get, when $\infty > b \ge d$,

$$(111)\quad Lim I = \tfrac{1}{2}\pi\int_d^\infty \frac{\sin\beta}{\beta}d\beta + \pi\int_0^d \frac{\sin\beta}{\beta}d\beta$$

$$= \tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\int_0^d \frac{\sin\beta}{\beta}d\beta,$$

and when $d > b \geq -d$,

$$(112)\quad \operatorname{Lim} I = \tfrac{1}{2}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta + \pi\int_0^{|b|}\frac{\sin\beta}{\beta}\,d\beta,$$

and finally when $-d > b > -\infty$,

$$(113)\quad \operatorname{Lim} I = \tfrac{1}{2}\pi\int_{|b|}^\infty \frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{2}\pi\int_d^{|b|}\frac{\sin\beta}{\beta}\,d\beta + \pi\int_0^d \frac{\sin\beta}{\beta}\,d\beta$$

$$= \tfrac{1}{4}\pi^2 - \pi\int_d^{|b|}\frac{\sin\beta}{\beta}\,d\beta + \tfrac{1}{2}\pi\int_0^d \frac{\sin\beta}{\beta}\,d\beta.$$

Lemma G. If δ is a positive constant, if c and b are bounded, if $c_1 > 0$ is any fixed number, and $|c| > c_1$ and if b is independent of k, then

$$I = \int_{k\delta}^\infty \frac{\sin\beta}{\beta}\int_0^{c(\beta+b)}\frac{\sin\alpha}{\alpha}\,d\alpha\,d\beta$$

tends to zero as k increases indefinitely.

Proof:

$$(114).\quad I = \int_{k\delta}^\infty \frac{\sin\beta}{\beta}\int_0^{c(\beta+b)}\frac{\sin\alpha}{\alpha}\,d\alpha\,d\beta$$

$$= \int_{k\delta}^\infty \frac{\sin\beta}{\beta}\int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\,d\beta - \int_{k\delta}^\infty \frac{\sin\beta}{\beta}\int_{c(\beta+b)}^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\,d\beta$$

$$= I_1 + I_2.$$

But,

$$(115)\quad I_1 = \tfrac{1}{2}\pi\int_{k\delta}^\infty \frac{\sin\beta}{\beta}\,d\beta,$$

and if we apply the second law of the mean, we have

$$(116)\quad I_1 = \frac{\pi}{2k\delta}\int_{k\delta}^{\xi_1}\sin\beta\,d\beta = \frac{\pi}{2k\delta}\left[-\cos\beta\right]_{k\delta}^{\xi_1},\quad (k\delta \leq \xi_1 < \infty).$$

Whence, given $\epsilon > 0$, there is a k_1 such that for $k > k_1$

$$(117)\quad |I_1| \leq \frac{\pi}{2k\delta}|2| = \frac{\pi}{k\delta} < \tfrac{1}{2}\epsilon,$$

~~and~~ Applying the second law of the mean to I_2, we have

$$(118)\quad I_2 = -\frac{1}{c}\int_{k\delta}^{\infty}\frac{\sin\beta}{\beta(\beta+b)}\int_{c(\beta+b}^{f_2}\sin\alpha\, d\alpha\, d\beta\,, \qquad (c(\beta+b< f_2<\infty),$$

$$= \frac{1}{c}\int_{k\delta}^{\infty}\frac{\sin\beta}{\beta(\beta+b)}\left[\cos\alpha\right]_{c(\beta+b)}^{f_2} d\beta.$$

If $b \neq 0$, we have

$$(119)\quad |I_2| \leq \left|\frac{2}{c}\int_{k\delta}^{\infty}\frac{d\beta}{\beta(\beta+b)}\right| \leq \frac{2}{|bc|}\left|\log\left(1+\frac{b}{\beta}\right)\right]_{k\delta}^{\infty}\Big|$$

$$\leq \frac{2}{|bc|}\left|\log\left(1+\frac{b}{k\delta}\right)\right|,$$

and there exists a k_2 such that when $k>k_2$, we have

$$|I_2| < \tfrac{1}{2}\epsilon.$$

If $b=0$, we have

$$(120)\quad |I_2| \leq \left|\frac{2}{c}\int_{k\delta}^{\infty}\frac{d\beta}{\beta^2}\right| \leq \left|\frac{2}{c}\left[-\frac{1}{\beta}\right]_{k\delta}^{\infty}\right|$$

$$\leq \frac{2}{k\delta|c|},$$

and there exists a k_3 such that when $k > k_3$, we have

$$|I_2| < \tfrac{1}{2}\epsilon.$$

Whence, if k is greater than the largest of k_1, k_2, and k_3, we have

$$|I| \leq |I_1| + |I_2| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

7. We are now ready to evaluate the sum $S_{hk}(x,y)$ in some particular cases, and will use the symbol $S(0,0)$ in the four following theorems to indicate the limit of $S_{hk}(x,y)$ as (x,y) approaches the origin in the prescribed manner, as h and k become infinite.

Theorem IV.

Hypotheses:

1. The hypotheses of theorem III are satisfied.
2. $a_1 > a_2 > 0$
3. The ratio h/k becomes infinite as h and k increase indefinitely.

Conclusion:

There is at least one manner for (x,y) to approach from each of the regions R_1, R_2, R_3 and R_4 to the origin such that the corresponding point on the approximating surface $z = S_{hk}(x,y)$ approaches a point at least $\frac{1}{2}\pi\rho_2$ units below the value $-\frac{1}{4}\pi^2$ approached by F(x,y) as (x,y) approaches the origin along or near the y-axis.

Proof: Let the point (x,y) move toward the origin along the line $x = ey$, so that $y = 2d/k$, that is, let (x,y) be the point $(2ed/k, 2d/k)$. By substituting this point for (x,y) in $S_{hk}(x,y)$ as given by theorem III, we have

$$(121)\quad S_{hk}\left(\frac{2ed}{k}, \frac{2d}{k}\right) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{2}\int_{d/k}^{\delta}\frac{\sin kv}{v}\int_0^{a_1v+d(a_1-e)/k}\frac{\sin hu}{u}\,du\,dv$$

$$+\tfrac{1}{2}\int_{d/k}^{\delta}\frac{\sin kv}{v}\int_0^{a_1v-d(a_1-e)/k}\frac{\sin hu}{u}\,du\,dv$$

$$-\tfrac{1}{2}\int_{d/k}^{\delta}\frac{\sin kv}{v}\int_0^{a_2v+d(a_2-e)/k}\frac{\sin hu}{u}\,du\,dv$$

$$-\tfrac{1}{2}\int_{d/k}^{\delta}\frac{\sin kv}{v}\int_0^{a_2v-d(a_2-e)/k}\frac{\sin hu}{u}\,du\,dv$$

$$= -\tfrac{1}{4}\pi^2 + I_1 + I_2 + I_3 + I_4$$

Now to make (x,y) approach the origin from R_1, if $a_1 < 2a_2$, let $a_1 > e > a_2$, or if $a_1 > 2a_2$, let $2a_2 > e > a_2$. In either case to apply Lemmas E and F, we let $b = d(a_1-e)/a_1$ in I_1 to find that $0 < b < d$, let $b = -d(a_1-e)/a_1$, in I_2 to find that $-d < b < 0$, let $b = d(a_2-e)/a_2$ in I_3, whence $-d < b < 0$, and let $b = -d(a_2 - e)/a_2$ in I_4 where $0 < b < d$. We then have

$$(122)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta + \tfrac{1}{2}\pi\int_0^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}\,d\beta$$

$$+\tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta$$

$$-\tfrac{1}{2}\pi\int_0^{d(e-a_2)/a_2} \frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta$$

$$= -\tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\int_{d(e-a_2)/a_2}^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}\,d\beta.$$

Let d $= \pi(2a_1+a_2)/(a_1-a_2)$, and $e = 3a_1a_2/(2a_1+a_2)$, and we find

$$(123)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\int_\pi^{2\pi} \frac{\sin\beta}{\beta}\,d\beta$$

$$= -\tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\rho_2.$$

The approach from R_3 can be inferred from that in R_1 by the principle of symmetry to the z-axis, and similarly that from R_4 from the approach in R_2, which we will now establish. If $a_1 > e > 0$ the point remains in R_2 and we have $0 < b < d$, $b = d(a_1-e)/a_1$ in I_1, $-d < b = -d(a_1-e)/a_1 < 0$ in I_2, $0 < b = d(a_2-e)/a_2 < d$ in I_3, and $-d < b = -d(a_2-e)/a_2 < 0$, in I_4, and may apply lemma E and F to get,

$$(124)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta + \tfrac{1}{2}\pi\int_0^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}\,d\beta$$

$$+\tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta$$

$$-\tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{2}\pi\int_0^{d(a_2-e)/a_2} \frac{\sin\beta}{\beta}\,d\beta$$

$$(125)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\int_{d(a_2-e)/a_2}^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}\,d\beta.$$

If we let $d = \frac{\pi(2a_1-a_1)}{a_1-a_2}$, and $e = \frac{a_1a_2}{2a_1-a_2}$, we find

$$(126)\quad S(0,0) = -\tfrac{1}{4}\pi^2+\tfrac{1}{2}\pi\int_{\pi}^{2\pi}\frac{\sin\beta}{\beta}d\beta = -\tfrac{1}{4}\pi^2-\tfrac{1}{2}\pi\rho_2 .$$

Or if $-\infty < e < 0$, the point remains in R_2 and we have

$d < b = d(a_1-e)/a_1 < \infty$ in I_1, $-\infty < b = -d(a_1-e)/a_1 < -d$ in I_2

$d < b = d(a_2-e)/a_2 < \infty$ in I_3, and $-\infty < b = -d(a_2-e)a_2 < -d$ in I_4,

giving, when we apply lemma E and F,

$$(127)\quad S(0,0) = -\tfrac{1}{4}\pi^2+\tfrac{1}{8}\pi^2+\tfrac{1}{4}\pi\int_0^d\frac{\sin\beta}{\beta}d\beta+\tfrac{1}{4}\pi\int_{d(a_1-e)/a_1}^{\infty}\frac{\sin\beta}{\beta}d\beta$$
$$-\tfrac{1}{4}\pi\int_d^{d(a_1-e)/a_1}\frac{\sin\beta}{\beta}d\beta-\tfrac{1}{8}\pi^2-\tfrac{1}{4}\pi\int_0^d\frac{\sin\beta}{\beta}d\beta$$
$$-\tfrac{1}{4}\pi\int_{d(a_2-e)/a_2}^{\infty}\frac{\sin\beta}{\beta}d\beta+\tfrac{1}{4}\pi\int_d^{d(a_2-e)/a_2}\frac{\sin\beta}{\beta}d\beta$$
$$= -\tfrac{1}{4}\pi^2+\tfrac{1}{2}\pi\int_{d(a_1-e)/a_1}^{d(a_2-e)a_2}\frac{\sin\beta}{\beta}d\beta,$$

And if $a_1 > 2a_2$, we may let $d = \frac{\pi(a_1-2a_2)}{a_1-a_2}$, and $e = \frac{a_1a_2}{a_1-2a_2}$, to get

$$(128)\quad S(0,0) = -\tfrac{1}{4}\pi^2+\tfrac{1}{2}\pi\int_{\pi}^{2\pi}\frac{\sin\beta}{\beta}d\beta = -\tfrac{1}{4}\pi^2-\tfrac{1}{2}\pi\rho_2 .$$

We can show an effect of greater magnitude by letting the point (x,y) move along $x=a_1y$ as follows. If $a_1 < 2a_2$, let $2a_2 > e \geq a_1$, to find $-d < b = d(a_1-e)/a_1 \leq 0$ in I_1, $0 \leq b = -d(a_1-e)/a_1 < d$ in I_2, $-d < b = d(a_2-e)/a_2 \leq 0$ in I_3, and $0 \leq b = -d(a_2-e)/a_2 < d$ in I_4, and by lemmas E and F,

$$(129)\quad S(0,0) = -\tfrac{1}{4}\pi^2+\tfrac{1}{4}\pi\int_d^{\infty}\frac{\sin\beta}{\beta}d\beta+\tfrac{1}{2}\pi\int_0^{d(e-a_1)/a_1}\frac{\sin\beta}{\beta}d\beta$$
$$+\tfrac{1}{4}\pi\int_d^{\infty}\frac{\sin\beta}{\beta}d\beta-\tfrac{1}{2}\pi\int_d^{\infty}\frac{\sin\beta}{\beta}d\beta$$
$$-\tfrac{1}{2}\pi\int_0^{d(e-a_2)/a_2}\frac{\sin\beta}{\beta}-\tfrac{1}{4}\pi\int_d^{\infty}\frac{\sin\beta}{\beta}d\beta,$$

$$(130)\quad S(0,0) = -\tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_{d(e-a_1)/a_1}^{d(e-a_2)/a_2} \frac{\sin\beta}{\beta}\,d\beta.$$

Let $a_1 = e$ and $d = \frac{a_2\pi}{a_1-a_2}$ and we find

$$(131)\quad S(0,0) = -\tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_0^\infty \frac{\sin\beta}{\beta}\,d\beta + \tfrac{1}{2}\pi\int_\pi^\infty \frac{\sin\beta}{\beta}\,d\beta$$
$$= -\tfrac{1}{2}\pi^2 - \tfrac{1}{2}\pi\rho_1.$$

Or if $a_1 > 2a_2$, let $2a_1 > e > 2a_2$ to find $-d < b = d(a_1-e)/a_1 < d$ in I_1; $-d < b = -d(a_1-e)/a_1 < d$, in I_2; $-\infty < b = d(a_2-e)/a_2 < -d$, in I_3; and $d < b = -d(a_2-e)/a_2 < \infty$ in I_4, and by lemmas E and F, we have,

$$(132)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta + \tfrac{1}{2}\pi\int_0^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}\,d\beta$$
$$+ \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{8}\pi^2 + \tfrac{1}{2}\pi\int_d^{d(e-a_2)/a_2} \frac{\sin\beta}{\beta}\,d\beta$$
$$-\tfrac{1}{4}\pi\int_0^d \frac{\sin\beta}{\beta}\,d\beta - \tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}\,d\beta$$
$$= -\tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_{d(a_1-e)/a_1}^{d} \frac{\sin\beta}{\beta}\,d\beta + \tfrac{1}{2}\pi\int_d^{d(e-a_2)/a_2} \frac{\sin\beta}{\beta}\,d\beta$$

Let $d=\pi$ and $e=a_1$, and we find

$$(133)\quad S(0,0) = -\tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_0^\pi \frac{\sin\beta}{\beta}\,d\beta + \tfrac{1}{2}\pi\int_\pi^{\pi(a_1-a_2)/a_2} \frac{\sin\beta}{\beta}\,d\beta$$
$$= -\tfrac{1}{2}\pi^2 - \tfrac{1}{2}\pi\rho_1 + \tfrac{1}{2}\pi\int_\pi^{\pi(a_1-a_2)/a_2} \frac{\sin\beta}{\beta}\,d\beta,$$

and we know, since $(a_1-a_2)/a_2 > 1$, that

$$0 < \int_\pi^{\pi(a_1-a_2)/a_2} \frac{\sin\beta}{\beta}\,d\beta \leq \rho_2.$$

Since the value approached by F(x,y) is $-\frac{1}{4}\pi^2$ we have demonstrated our theorem.

Theorem V

Hypotheses:

1. F(x,y) is defined on R as in (D).
2. $1/a_2 > 1/a_1 > 0$
3. The ratio $h/k \to 0$ as h and k become infinite.

Conclusion:

There is at least one manner for (x,y) to approach from each of the regions R_1, R_2, R_3, and R_4 to the origin such that the corresponding point on the approximating surface $z = S_{hk}(x,y)$ approaches a point at least $\frac{1}{2}\pi\rho_2$ units below the value $-\frac{1}{4}\pi^2$ approached by F(x,y) as (x,y) approaches the origin along or near the x-axis.

The proof of this theorem follows along the lines of the proofs of Theorems III and IV by an interchange of x and y throughout.

Theorem VI.

Hypotheses:

1. The hypotheses of Theorem III are satisfied.
2. $a_1 > 0 > a_2 = -a_3$
3. The ratio h/k becomes infinite as h and k become infinite.

Conclusions:

1. There are two manners for (x,y) to approach the origin from R_1 and two from R_2 such that the corresponding point on the surface $z = S_{hk}(x,y)$ approaches $\frac{1}{4}\pi^2 + \frac{1}{2}\pi\rho_2$, thus demonstrating a Gibbs' phenomenon of at least $\frac{1}{2}\pi\rho_2$ magnitude.
2. There are two manners for (x,y) to approach the origin from R_2 and two from R_4 such that the corresponding point on the surface $z = S_{hk}(x,y)$ approaches a point above $\frac{1}{4}\pi^2 + \frac{1}{2}\pi\rho_2$ showing a Gibbs' phenomenon of more than $\frac{1}{2}\pi\rho_2$ magnitude.

Proof: Let the point (x,y) move toward the origin along the line $x = ey$, so that $y = 2d/k$. By substituting in the expression for $S_{nk}(x,y)$ as given by theorem III when a_2 is replaced by $-a_3$, we find

$$(134) \quad S_{nk}\left(\frac{2ed}{k}, \frac{2d}{k}\right) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{2}\int_{-d/k}^{\delta} \frac{\sin kv}{v} \int_0^{a_1 v + d(a_1 - e)/k} \frac{\sin hu}{u}\, du\, dv$$

$$+ \tfrac{1}{2}\int_{d/k}^{\delta} \frac{\sin kv}{v} \int_0^{a_1 v - d(a_1 - e)/k} \frac{\sin hu}{u}\, du\, dv$$

$$+ \tfrac{1}{2}\int_{-d/k}^{\delta} \frac{\sin kv}{v} \int_0^{a_3 v + d(a_3 + e)/k} \frac{\sin hu}{u}\, du\, dv$$

$$+ \tfrac{1}{2}\int_{d/k}^{\delta} \frac{\sin kv}{v} \int_0^{a_3 v - d(a_3 + e)/k} \frac{\sin hu}{u}\, du\, dv$$

$$= -\tfrac{1}{4}\pi^2 + I_1 + I_2 + I_3 + I_4$$

We may infer from symmetry the z-axis that the behavior in R_3 is the same as that in R_1, thus to make the point (x,y) approach from R_1 is sufficient. We must require that $a_1 \geq e \geq -a_3$, and to use lemmas E and F, when $a_1 \geq e > 0$, we have in I_1, $0 < b = d(a_1-e)/a_1 < d$; in I_2, $-d < b = -d(a_1-e)/a_1 < 0$; and in I_3, $d < b = d(a_3+e)/a_3 < \infty$; and in I_4, $-\infty < b = -d(a_3+e)/a_3 < -d$. Then

$$(135) \quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{4}\pi\int_d^{\infty} \frac{\sin\beta}{\beta}\, d\beta + \tfrac{1}{2}\pi\int_0^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}\, d\beta$$

$$+ \tfrac{1}{4}\pi\int_d^{\infty} \frac{\sin\beta}{\beta}\, d\beta + \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\int_0^{d} \frac{\sin\beta}{\beta}\, d\beta$$

$$+ \tfrac{1}{4}\pi\int_{d(a_3+e)/a_3}^{\infty} \frac{\sin\beta}{\beta}\, d\beta - \tfrac{1}{4}\pi\int_0^{d(a_3+e)/a_3} \frac{\sin\beta}{\beta}\, d\beta$$

$$= +\tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_{d(a_1-e)/a_1}^{d(a_3+e)/a_3} \frac{\sin\beta}{\beta}\, d\beta.$$

Let $d = \frac{\pi(2a_3+a_1)}{a_3+a_1}$, and $e = \frac{a_1a_3}{2a_3+a_1}$, and we find

$$(136)\quad S(0,0) = \tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_{\pi}^{2\pi}\frac{\sin\beta}{\beta}d\beta$$
$$= \tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\rho_2.$$

Or when $0 < e \leq -a_3$, we have in I_1, $d < b = d(a_1-e)/a_1 < \infty$; in I_2, $-\infty < b = -d(a_1-e)/a_1 < -d$; in I_3, $0 < b = d(a_3+e)/a_3 < d$; and in I_4, $-d < b = -d(a_3+e)/a_3 < 0$. Then

$$(137)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\int_0^d\frac{\sin\beta}{\beta}d\beta + \tfrac{1}{4}\pi\int_{d(a_1-e)/a_1}^{\infty}\frac{\sin\beta}{\beta}d\beta$$
$$-\tfrac{1}{4}\pi\int_d^{d(a_1-e)/a_1}\frac{\sin\beta}{\beta}d\beta + \tfrac{1}{4}\pi\int_d^{\infty}\frac{\sin\beta}{\beta}d\beta$$
$$+\tfrac{1}{2}\pi\int_0^{d(a_3+e)/a_3}\frac{\sin\beta}{\beta}d\beta + \tfrac{1}{4}\pi\int_d^{\infty}\frac{\sin\beta}{\beta}d\beta$$
$$= \tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_{d(a_3+e)/a_3}^{d(a_1-e)/a_1}\frac{\sin\beta}{\beta}d\beta.$$

Let $d = \pi(2a_1+a_3)/(a_1+a_3)$ and $e = a_1a_3/(2a_1+a_3)$, and we find

$$(138)\quad S(0,0) = \tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_{\pi}^{2\pi}\frac{\sin\beta}{\beta}d\beta = \tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\rho_2.$$

If $e > 2a_1$, we find that (x,y) is in R_4 if $d>0$, and R_2 if $d<0$. Since we have symmetry to the z-axis the behavior will be the same, and as it is a little simpler, we will use $d>0$. In I_1 (134), we have $-\infty < b = d(a_1-e)/a_1 < -d$; in I_2, $d < b = -d(a_1-e)/a_1 < \infty$; in I_3, $d < b = d(a_3+e)/a_3 < \infty$; and in I_4, $-\infty < b = -d(a_3+e)/a_3 < -d$; and by substituting from lemma E and F, we get

$$(139)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{8}\pi^2 - \tfrac{1}{2}\pi\int_d^{d(e-a_1)/a_1} \frac{\sin\beta}{\beta}d\beta + \tfrac{1}{4}\pi\int_0^d \frac{\sin\beta}{\beta}d\beta$$
$$+\tfrac{1}{4}\pi\int_d^\infty \frac{\sin\beta}{\beta}d\beta + \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\int_0^d \frac{\sin\beta}{\beta}d\beta$$
$$+\tfrac{1}{4}\pi\int_{d(a_3+e)/a_3}^\infty \frac{\sin\beta}{\beta}d\beta - \tfrac{1}{4}\pi\int_d^{d(a_3+e)/a_3} \frac{\sin\beta}{\beta}d\beta$$
$$= \tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_d^{d(e-a_1)/a_1} \frac{\sin\beta}{\beta}d\beta - \tfrac{1}{2}\pi\int_d^{d(a_3+e)/a_3} \frac{\sin\beta}{\beta}d\beta.$$

To make $S(0,0)$ a maximum for this approach, we must make d and e the solution near $d=\pi$ and $e=3a_1$ of the equations,

$$(140)\quad (e+a_3)\sin d(e/a_1 - 1) + (e-a_1)\sin d(e/a_3 + 1) = 0,$$
$$(141)\quad 2\sin d - \sin d(e/a_1 - 1) - \sin d(e/a_3 + 1) = 0,$$

obtained from $\frac{\partial S(00)}{\partial e} = \frac{\partial S(00)}{\partial d} = 0$. Since this solution in general, will require some method of approximate solution we will use $d=\pi$ and $e=3a_1$ to obtain some measure of the value. Then

$$(142)\quad S(0,0) = \tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\rho_2 - \tfrac{1}{2}\pi\int_\pi^{\pi(1+3a_1/a_3)} \frac{\sin\beta}{\beta}d\beta,$$

and since $1+3a_1/a_3 > 1$, we have

$$0 < -\int_\pi^{\pi(1+3a_1/a_3)} \frac{\sin\beta}{\beta}d\beta \le \rho_2.$$

Thus one part of our conclusion is established.

If $e \le -2a_3$, (x,y) is in R_2 when $d>0$ and we have in I_1, (134), $d < b = d(a_1-e)/a_1 < \infty$; in I_2, $-\infty < b = -d(a_1-e)/a_1 < -d$; in I_3, $-\infty < b = d(a_3+e)/a_3 < -d$; and in I_4, $d < b = -d(a_3+e)/a_3 < \infty$. Then substituting from lemmas E and F,

$$(143)\quad S(0,0) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\int_0^d \frac{\sin\beta}{\beta}d\beta + \tfrac{1}{4}\pi\int_{d(a_1-e)/a_1}^{\infty} \frac{\sin\beta}{\beta}d\beta$$

$$-\tfrac{1}{4}\pi\int_d^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}d\beta + \tfrac{1}{8}\pi^2 - \tfrac{1}{2}\pi\int_d^{-d(a_3+e)/a_3} \frac{\sin\beta}{\beta}d\beta$$

$$+\tfrac{1}{4}\pi\int_0^d \frac{\sin\beta}{\beta}d\beta + \tfrac{1}{4}\pi\int_d^{\infty} \frac{\sin\beta}{\beta}d\beta$$

$$= \tfrac{1}{4}\pi^2 - \tfrac{1}{2}\pi\int_d^{d(a_1-e)/a_1} \frac{\sin\beta}{\beta}d\beta - \tfrac{1}{2}\pi\int_d^{-d(a_3-e)/a_3} \frac{\sin\beta}{\beta}d\beta.$$

From $\frac{\partial S(0,0)}{\partial e} = \frac{\partial S(0,0)}{\partial d} = 0$ we get the same equations as (140) and (141) above, but for a maximum require the solution near $d=\pi$ and $e=-3a_3$. At these values we have

$$(144)\quad S(0,0) = \tfrac{1}{4}\pi^2 + \tfrac{1}{2}\pi\rho_1 - \tfrac{1}{2}\pi\int_{\pi}^{\pi(1+3a_3/a_1)} \frac{\sin\beta}{\beta}d\beta,$$

and we have our two manners of approach.

Theorem VII.

Hypotheses:

1. $F(x,y)$ is defined on R as in (D).
2. $1/a_1 > 0$ and $1/a_2 = -1/a_3 < 0$
3. The ratio $h/k \to 0$ as h and k increase indefinitely.

Conclusions:

1. There are two manners for (x,y) to approach the origin from R_2 and two from R_4 such that the corresponding point on the surface $z=S_{hk}(x,y)$ approaches $-\tfrac{1}{4}\pi^2-\tfrac{1}{2}\pi\rho_2$.
2. There are two manners for (x,y) to approach the origin from R_1 and two from R_3 such that the corresponding point on the surface $z=S_{hk}(x,y)$ approaches a point below $-\tfrac{1}{4}\pi^2-\tfrac{1}{2}\pi\rho_2$.

The proof of this theorem follows that of theorem III and theorem VI by an interchange of x and y throughout.

Theorem VIII.

Hypotheses:

1. The hypotheses of theorem III are satisfied.
2. The point (x,y) approaches the origin along the line $x = ey$ so that $y = 2d/k$.
3. The ratio $h/k = c$ approaches a finite positive constant c, as h and k increase indefinitely.
4. $\theta_1(\beta) = \frac{1}{2} dc(a_1 - a_2)(1-\beta)$ and $\theta_2(\beta) = \frac{1}{2} cd\{(a_1+a_2)(\beta-1)+2e\}$.

Conclusions:

As h and k increase indefinitely, the sum $S_{hk}\left(\frac{2ed}{k}, \frac{2d}{k}\right)$ approaches the limit

$$-\tfrac{1}{4}\pi^2 + \int_0^1 \frac{\cos(a_1-e)cdt}{4t}\log\left(\frac{1+a_1ct}{1-a_1ct}\right)^2 dt - \int_0^1 \frac{\cos(a_2-e)cdt}{4t}\log\left(\frac{1+a_2ct}{1-a_2ct}\right)^2 dt$$

$$+2\int_0^1 \frac{\sin\beta d}{\beta}\int_0^1 \frac{\sin\theta_1(\beta)t\,\cos\theta_2(\beta)t}{t}\,dt\,d\beta,$$

when that limit exists.

Proof: Substituting the values of x and y in the expression (71) from theorem III, we have

$$(145)\quad S_{hk}\left(\frac{2ed}{k}, \frac{2d}{k}\right) = -\tfrac{1}{4}\pi^2 + \tfrac{1}{2}\int_{-d/k}^{\delta} \frac{\sin kv}{v}\int_0^{a_1v+d(a_1-e)/k} \frac{\sin hu}{u}\,du\,dv$$

$$+\tfrac{1}{2}\int_{d/k}^{\delta} \frac{\sin kv}{v}\int_0^{a_1v-d(a_1-e)/k} \frac{\sin hu}{u}\,du\,dv$$

$$-\tfrac{1}{2}\int_{-d/k}^{\delta} \frac{\sin kv}{v}\int_0^{a_2v+d(a_2-e)/k} \frac{\sin hu}{u}\,du\,dv$$

$$-\tfrac{1}{2}\int_{d/k}^{\delta} \frac{\sin kv}{v}\int_0^{a_2v-d(a_2-e)/k} \frac{\sin hu}{u}\,du\,dv + E_2$$

$$= -\tfrac{1}{4}\pi^2 + I_1 + I_2 + I_3 + I_4 + E_2.$$

In I_1, let $b_1 = d(a_1-e)/a_1$, and make the transformation $\beta = kv$ and $\alpha = hu$. Then

$$(146)\quad I_1 = \tfrac{1}{2}\int_{-d}^{k\delta} \frac{\sin\beta}{\beta}\int_0^{a_1c(\beta+b_1)} \frac{\sin\alpha}{\alpha}\,d\alpha\,d\beta$$

$$= \tfrac{1}{2}\int_{-d}^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1c(\beta+b_1)} \frac{\sin\alpha}{\alpha}\,d\alpha\,d\beta - \tfrac{1}{2}\int_{k\delta}^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1c(\beta+b_1)} \frac{\sin\alpha}{\alpha}\,d\alpha\,d\beta,$$

and by lemma G the second integral of I_1 tends to zero. In the other integral, let $\alpha = ct(\beta + b_1)$, to get,

$$(1147)\quad I_1 = \frac{1}{2}\int_{-d}^{\infty} \frac{\sin\beta}{\beta} \int_0^{a_1} \frac{\sin tc(\beta+b_1)}{t}\, dt\, d\beta + E_3$$

$$= -\frac{1}{2}\int_0^{-d} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta+b_1)}{t}\, dt\, d\beta$$

$$+ \frac{1}{2}\int_0^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta+b_1)}{t}\, dt\, d\beta + E_3$$

$$= -\frac{1}{2}\int_0^{d} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta-b_1)}{t}\, dt\, d\beta$$

$$+ \frac{1}{2}\int_0^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta+b_1)}{t}\, dt\, d\beta + E_3.$$

In I_2 let $\beta = kv$ and $\alpha = hu$. Then

$$(1148)\quad I_2 = \frac{1}{2}\int_d^{k\delta} \frac{\sin\beta}{\beta}\int_0^{a_1 c(\beta-b_1)} \frac{\sin\alpha}{\alpha}\, d\alpha\, d\beta$$

$$= \frac{1}{2}\int_d^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1 c(\beta-b_1)} \frac{\sin\alpha}{\alpha}\, d\alpha\, d\beta - \frac{1}{2}\int_{k\delta}^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1 c(\beta-b_1)} \frac{\sin\alpha}{\alpha}\, d\alpha\, d\beta.$$

By lemma G the second integral of I_2 tends to zero. In the first let $\alpha = ct(\beta - b_1)$, to get,

$$(1149)\quad I_2 = -\frac{1}{2}\int_0^{d} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta-b_1)}{t}\, dt\, d\beta$$

$$+ \frac{1}{2}\int_0^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta-b_1)}{t}\, dt\, d\beta + E_4.$$

Then by adding, we have

$$I_1 + I_2 = \frac{1}{2}\int_0^{\infty} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta+b_1) + \sin tc(\beta-b_1)}{t}\, dt\, d\beta$$

$$- \int_0^{d} \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc(\beta-b_1)}{t}\, dt\, d\beta \quad + E_3 + E_4$$

$$(150)\quad I_1+I_2 = \frac{1}{2}\int_0^\infty \frac{\sin\beta}{\beta}\int_0^{a_1} \frac{\sin tc\beta \cos b_1ct}{t}\,dt\,d\beta$$

$$-\int_0^d \frac{\sin\beta}{\beta}\int_0^{a_1}\frac{\sin tc(\beta-b_1)}{t}\,dt\,d\beta + E_3 + E_4$$

$$= \frac{1}{2}\int_0^{a_1}\frac{\cos b_1ct}{t}\int_0^\infty \frac{\sin\beta \sin tc\beta}{\beta}\,d\beta\,dt$$

$$-\int_0^d \frac{\sin\beta}{\beta}\int_0^{a_1}\frac{\sin tc(\beta-b_1)}{t}\,dt\,d\beta + E_3 + E_4$$

But $\int_0^\infty \frac{\sin\beta \sin tc\beta}{\beta}\,d\beta = \frac{1}{2}\log\left(\frac{1+tc}{1-tc}\right)^2$ (Bierens de Haan)[1].

Thus we have,

$$(151)\quad I_1+I_2 = \int_0^{a_1}\frac{\cos b_1ct}{4t}\log\left(\frac{1+tc}{1-tc}\right)^2 dt$$

$$-\int_0^d \frac{\sin\beta}{\beta}\int_0^{1}\frac{\sin ta_1c(\beta-b_1)}{t}\,dt\,d\beta + E_3 + E_4.$$

Similarly if $b_2 = d(a_2-e)/a_2$, we have

$$(152)\quad I_3+I_4 = -\int_0^{a_2}\frac{\cos b_2ct}{4t}\log\left(\frac{1+tc}{1-tc}\right)^2 dt$$

$$+\int_0^d \frac{\sin\beta}{\beta}\int_0^1 \frac{\sin ta_2c(\beta-b_2)}{t}\,dt\,d\beta + E_5 + E_6.$$

If we let $E = E_2+E_3+E_4+E_5+E_6$, we have

$$(153)\quad S_{11k}\left(\frac{2cd}{k},\frac{2d}{k}\right) = -\frac{1}{4}\pi^2 + \int_0^d \frac{\sin\beta}{\beta}\int_0^1 \frac{\sin ta_2c(\beta-b_2) - \sin ta_1c(\beta-b_1)}{t}\,dt\,d\beta$$

$$+\int_0^{a_1}\frac{\cos b_1ct}{4t}\log\left(\frac{1+tc}{1-tc}\right)^2 dt - \int_0^{a_2}\frac{\cos b_2ct}{4t}\log\left(\frac{1+tc}{1-tc}\right)^2 dt + E.$$

By replacing the values of b_1 and b_2 and transforming with $t=a_1\alpha$ in the first integral, $t=a_2\alpha$ in the second

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[1] Loc. Cit. Table 151, No. 7.

and $\beta = vd$, in the third, we get

$$(15A) \quad S_{hk}\left(\frac{2ed}{k}, \frac{2d}{k}\right) = -\frac{1}{4}\pi^2 + \int_0^1 \frac{\cos(a_1-e)dct}{4t} \log\left(\frac{1+tca_1}{1-tca_1}\right)^2 dt$$

$$+2\int_0^1 \frac{\sin vd}{v} \int_0^1 \frac{\sin(\theta_1(v)t) \cos\theta_1(v)t}{t} dt\, dv$$

$$-\int_0^1 \frac{\cos(a_2-e)dct}{4t} \log\left(\frac{1+tca_2}{1-tca_2}\right)^2 dt + E.$$

Given $\epsilon > 0$, there exist an h_1 and a k_1 such that when $h > h_1$, and $k > k_1$, $|E| < \epsilon$, (theorem III and lemma G) and our theorem is established.

It is interesting to note that when we fix the point (x,y) at the origin by making $d=0$, the results of theorem VIII are the same as those obtained by E.C. Titchmarsh [1]. However the evaluation of the limit when $d \neq 0$ requires some method of approximate integration for each particular set of values for a_1, a_2, d, e, and c_1 as it has been shown by Titchmarsh, C.M.Moore [2], and others, that the series does not converge but depends upon the value of c_1, we will not attempt the evaluation in this paper.

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[1] Loc. Cit. Theorem III

[2] "On the Summability of the Double Fourier's Series of a Discontinuous Function," Math. Annalen, Vol. 74 (1913) pp. 555-572. Here it is shown that for summability C_1 the result depends on the ratio h/k, hence we may deduce the fact for convergence.

Part II GENERAL FUNCTIONS

1. We have studied the behavior of the double Fourier's series development of several functions relative to the existence of a Gibbs' phenomenon at various points of discontinuity. By use of these results we are now able to proceed to the discussion of any general function having a convergent double Fourier's series expansion.

Definition E. Let the region $R = (-\pi \le x \le \pi, \ -\pi \le y \le \pi)$ contain a network, W, of curves defined as follows : (a) each curve has a definite tangent everywhere; (b) any line parallel to either axis cuts the network in only a finite number of points; (c) there are only a finite number of points where two curves intersect, and none where more than two intersect. Let $\Phi(x,y)$ be a summable function on R such that in the interior of each cell formed by the network, W, $\Phi(x,y)$ is continuous and of bounded variation in the sense used by Titchmarsh[1] and has a finite jump as (x,y) crosses any curve of the network.

The conditions on $\Phi(x,y)$ are sufficient to give a Fourier's series for $\Phi(x,y)$ which is uniformly convergent in the interior of any of the cells, hence at any point of continuity of $\Phi(x,y)$ the surface $z = S_{h,k}(x,y)$ approaches the

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[1] Loc. Cit. p. 301

surface $z=\Phi(x,y)$ as near as we please as h and k increase in any manner. At any point (x,y) on the network, W, at which all the neighboring points of continuity lie along a single curve of the network, we take a section parallel to either axis and consider the resulting function of a single variable, say $\Phi(x,y_1)$. We may take the more convenient section, since by virtue of theorem II, and the work of Bôcher[1] and others, we know that in the Gibbs' phenomena the magnitude is not dependent on the function but on the magnitude of the jump. Thus the curve cut from the approximating surface $z=S_{hk}(xy)$ by the plane $y=y_1$ will approach the curve $z=\Phi(x,y)$, $y=y_1$ as near as we please at all points in the neighborhood of (x_1,y_1), except at (x_1,y_1). When (x,y) approaches (x_1,y_1) as x increases, the approximating curve rises to the height

$$(155) \qquad z=\Phi(x_1-0,y_1)-\{\Phi(x_1+0,y_1)-\Phi(x_1-0,y_1)\}\rho_1/\pi,$$

and then drops to

$$(156 \qquad z=\Phi(x_1+0,y_1)+\{\Phi(x_1+0,y_1)-\Phi(x_1-0,y_1)\}\rho_1/\pi,$$

as (x,y) passes through (x_1,y_1) and then back to the original curve.

2. For the consideration of points of our network

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[1] Loc. Cit. § 9, p. 123

where the neighboring points of discontinuity of $\Phi(x,y)$ lie along two curves intersecting at the point, we need first, a lemma.

Lemma H. Of x and y are coordinates in a Cartesian system, rectangular or oblique, if R is the region $(-\pi \le x \le \pi, -\pi \le y \le \pi)$, if f(x) and g(y) are the functions defined in (B), if B(x,y) is any function continuous in x and y on R except along x = 0 and y = 0, if B(x,y) has a finite jump along the axes, and if

$$B(x,0) = \tfrac{1}{2}\{B(x,+0)+B(x,-0)\}, \quad B(0,y) = \tfrac{1}{2}\{B(+0,y)+B(-0,y)\}$$
$$B(0,0) = \tfrac{1}{4}\{B(+0,+0)+B(+0,-0)+B(-0,+0)+B(-0,-0)\} \quad \text{and}$$
$$A = B(+0,+0) - B(-0,+0) - B(+0,-0) + B(-0,-0)$$

there exist three continuous functions $\Psi(x,y)$, $H(x)$ and $G(y)$ such that

$$B(x,y) = \Psi(x,y) + G(y)\,f(x)/\pi + H(x)\,g(y)/\pi + A\,f(x)g(y)/\pi^2.$$

Proof: Let

$$(157) \quad \theta(x) = B(x,+0) - B(x,-0) \qquad (-\pi \le x < 0) \text{ and } (0 < x \le \pi),$$
$$\theta(0) = \tfrac{1}{2}\{\theta(+0) + \theta(-0)\},$$

and

$$(158) \quad \varphi(y) = B(+0,y) - B(-0,y) \qquad (-\pi \le y < 0) \text{ and } (0 < x \le \pi),$$
$$\varphi(0) = \tfrac{1}{2}\{\varphi(+0) + \varphi(-0)\}.$$

Then

$$(159) \quad \theta(+0) - \theta(-0) = B(+0,+0) - B(+0,-0) - B(-0,+0) + B(-0,-0)$$
$$= \phi(+0) - \phi(-0) = A.$$

Define

$$(160) \quad \Psi(x,y) = B(x,y) - \theta(x)\,\phi(y)/A.$$

Then

$$(161) \quad \Psi(x,+0) - \Psi(x,-0) = B(x,+0) - B(x,-0) - \theta(x)\{\phi(+0)-\varphi(-0)\}/A$$
$$= 0$$

$$(162)\quad \Psi(x,+0) - \Psi(x,0) = \tfrac{1}{2}\{B(x,+0)-B(x,-0)\} - \tfrac{1}{2}\theta(x)\{\varphi(+0)-\varphi(-0)\}/A$$
$$= 0,$$

$$(163)\quad \Psi(+0,y) - \Psi(-0,y) = B(+0,y) - B(-0,y) - \{\theta(+0)-\theta(-0)\}\varphi(y)/A$$
$$= 0,$$

$$(164)\quad \Psi(+0,y) - \Psi(0,y) = \tfrac{1}{2}\{B(+0,y)-B(-0,y)\} - \tfrac{1}{2}\{\theta(+0)-\theta(-0)\}\varphi(y)/A$$
$$= 0.$$

From these, we have,

$$(165)\quad \Psi(0,\pm 0) = \Psi(\pm 0,0) = \Psi(\pm 0,\pm 0),$$

and

$$(166)\quad \Psi(0,+0) - \Psi(0,0) = \tfrac{1}{4}\{B(+0,+0)-B(+0,-0)+B(-0,+0)-B(-0,-0)\}$$
$$-\tfrac{1}{4}\{\theta(+0)+\theta(-0)\}\{\varphi(+0)-\varphi(-0)\}/A$$
$$\doteq 0.$$

Hence $\Psi(x,y)$ is continuous in x and y on R.

But $\theta(x) = H(x) + A f(x)/\pi$ where H(x) is continuous on $(-\pi \le x \le \pi)$ and $H(0) = \tfrac{1}{2}\{\theta(+0)+\theta(-0)\}$ since $H_1(x) = \theta(x) - \{\theta(+0)-\theta(-0)\}f(x)/\pi - \tfrac{1}{2}\{\theta(+0)+\theta(-0)\}$ is easily seen to be continuous at x = 0 where $H_1(0)=0$, and we can make $H(x) = H_1(x) + \tfrac{1}{2}\{\theta(+0) + \theta(-0)\}$. Similarly we have $\varphi(y) = G(y) + Ag(y)/\pi$, where G(y) is continuous on $(-\pi \le y \le \pi)$ and $G(0) = \tfrac{1}{2}\{\varphi(+0) + \varphi(-0)\}$

Thus

$$(167)\quad B(x,y) = \Psi(x,y) + \{H(x) + Af(x)/\pi\}\{G(y) + Ag(y)/\pi\}/A$$
$$= \psi(x,y) + G(y)f(x)/\pi + H(x)g(y)/\pi + A f(x)g(y)/\pi^2,$$

where

where $\psi(x,y) = \Psi(x,y) + H(x)G(y)/A$ is continuous.

3. There are only a finite number of points on the network, W, of Definition E at which two curves of the network intersect, hence only a finite number where the tangents to the intersecting curves are respectively parallel to the x and the y-axis. To study the behavior of the Fourier's development at these points we make repeated application of the following theorem.

Theorem IX

Hypotheses:

1. (a,b) is any point of discontinuity of the function $\Phi(x,y)$ defined in (E) near which all other points of discontinuity of $\Phi(x,y)$ lie along two curves intersecting at (a,b) and having tangents parallel to the x and y-axes respectively.
2. When the axes are translated so that (a,b) becomes the origin, $\Phi(x,y)$ becomes $\phi(x',y')$.
3. $G(y')$, $H(x')$ $f(x')$ and $g(y')$ are defined in lemma H.
4. $S_{nk}(x,y)$ is the sum from i = 0 to m, j = 0 to n of the terms of the double Fourier's expansion of $\phi(x',y')$.

Conclusion:

The approximating surface $z = S_{nn}(x',y')$ as m and n increase indefinitely in any manner, has a maximum (minimum) point approaching the point

(a) $\phi(+0,+0) + A\{\pi\rho_1 + \rho_1^2\}/\pi^2 + \{G(0)+H(0)\}\rho_1/\pi$
as $x' \to +0$ and $y' \to +0$.

(b) $\phi(+0,-0) - A\{\pi\rho_1+\rho_1^2\}/\pi^2 + \{G(0)-H(0)\}\rho_1/\pi$
as $x' \to +0$ and $y' \to -0$.

(c) $\phi(-0,-0) + A\{\pi\rho_1+\rho_1^2\}/\pi^2 - \{G(0)+H(0)\}\rho_1/\pi$
as $x' \to -0$ and $y' \to -0$.

(d) $\phi(-0,+0) - A\{\pi\rho_1+\rho_1^2\}/\pi^2 + \{H(0)-G(0)\}\rho_1/\pi$
as $x' \to -0$ and $y' \to +0$.

Proof: If we confine our attention to a small neighborhood of the origin, we can apply lemma H to get,

$$(167)\quad \phi(x',y') = \Psi(x',y') + G(y')f(x')/\pi + H(x')g(y')/\pi + A f(x') g(y')/\pi^2.$$

Hence the sum $S_{mn}(x',y')$ is made up of contributions from four different functions. Since $\Psi(x',y')$ is continuous in the neighborhood of the origin, its contribution to the sum converges as closely as we please to its value at (0,0), regardless of the manner of approach of (x,y) to (0,0) as m and n increase indefinitely in any manner. f(x) and g(y) are bounded, and H(x) and G(y) are continuous, hence the variation in the contributions of the products $G(y')f(x')/\pi$ and $H(x')g(y')/\pi$ to the sum caused by replacing $H(x')$ by H(0) and $G(y')$ by G(0) can be made as small as we please by keeping (x',y') near enough to the origin. The contribution of $G(0)f(x')/\pi$ to the sum will be a maximum when $x' = \frac{\pi}{2m}$, and converges to a value $G(0)\rho_1/\pi$ in excess of the value $G(0)f(+0)/\pi$, and of $H(0)g(y')/\pi$ a maximum when $y' = \frac{\pi}{2n}$, and converges to a value $H(0)\rho_1/\pi$ in excess of the value $H(0)g(+0)/\pi$ as m and n increase indefinitely in any manner. By theorem I, the contribution of $A f(x')g(y')/\pi^2$ to the sum will be a maximum when $x' = \frac{\pi}{2m}$ and $y' = \pi/2n$, (same values as above) and converges to a value $A(\pi\rho_1 + \rho_1^2)/\pi^2$ in excess of the value $A f(+0)g(+0)/\pi^2$ as m and n increase indefinitely in any manner. Adding all these contributions together we find that $S_{mn}(\frac{\pi}{2m}, \frac{\pi}{2n})$ tends to

(168) $\phi(+0,+0) + A(\pi\rho_1 + \rho_1^2)/\pi^2 + \{G(0)+H(0)\}\rho_1/\pi .$

The other parts of the conclusion follow from similar considerations.

It is interesting to note that under certain conditions on the values approached by $\phi(x,y)$ as (x,y) approaches a point of discontinuity, the Gibbs' phenomenon may disappear for the approach from one region, but not from all at once. This will be the case in the first quadrant when the function has the origin as a point of discontinuity if,

(169) $A\pi\rho_1 + A\rho_1^2 + \pi G(0)\rho_1 + \pi H(0)\rho_1 = 0,$

or substituting values and collecting,

(170) $(3\pi+\rho_1)\,\phi(+0,+0) - (\pi+\rho_1)\,\phi(+0,-0) - (\pi+\rho_1)\,\phi(-0,+0) - (\pi-\rho_1)\,\phi(-0,-0) = 0.$

From this we find that in choosing the function we may take arbitrarily the values to approach from three quadrants and find the value in the other to make the phenomenon disappear from the first quadrant.

4. There now remains at most a finite set of points in R to be considered. All these points are points of intersection on the network, W, at which at least one of the two tangents is oblique to the axes. Since we have

information concerning the corresponding special function only in the case where the ratio h/k approaches zero or infinity as h and k increase, and that not conclusive as to magnitudes, we will not attempt to state our results as a theorem. If (α, β) is any one of these points, translate the origin to that to get,

$$(171)\quad \Phi(x\,y) = \Phi(x'-\alpha, y'-\beta) = \varphi(x', y').$$

Now let $x' = a_1 y'$ and $x' = a_2 y'$ $(a_1 > a_2)$, be the equations of the tangents to the curves at the origin. If we consider a sufficiently small neighborhood of the origin, N, when we make the transformation,

$$x'' = x' - a_2 y' \quad \text{and} \quad y'' = a_1 y' - x'$$

to get

$$(172)\quad \phi(x', y') = \phi\left(\frac{a_1 x'' + a_2 y''}{a_1 - a_2}, \frac{x'' + y''}{a_1 - a_2}\right) = \varphi_1(x'', y'')$$

We can apply lemma H to $\phi_1(x'', y'')$ and we have,

$$(173)\quad \phi_1(x'', y'') = \Psi(x'', y'') + G(y'')f(x'')/\pi + H(x'')g(y'')/\pi + A f(x'')g(y'')/\pi^2;$$

However this is not in a form to expand as a Fourier's series, but by reversing the last transformation, noting that the product $f(x'-a_2 y')g(a_1 y'-x')$ is the function $F(x', y')$ defined in (D), and replacing $\Psi(x'-a_2 y', a_1 y'-x')$ by $\psi(x', y')$ we get,

$$(174)\quad \phi(x', y') = \psi(x', y') + G(a_1 y'-x')f(x'-a_2 y')/\pi +$$
$$+ H(x'-a_2 y')g(a_1 y'-x')/\pi + A F(x', y')/\pi^2,$$

in terms of the original axes. In this, $\psi(x',y')$ is continuous in x' and y' in the neighborhood N, and $G(a_1 y'-x')f(x'-a_2 y')$ and $H(x'-a_2 y')g(a_1 y'-x')$ each have a single line of discontinuity.

Since the function $\psi(x',y')$ is continuous, its contribution to the sum, $S_{h,k}(x',y')$, from i = 0 to m, j = 0 to n, ($k=2m+1$, $h=2n+1$) of the terms of the expansion of the function $\phi(x',y')$ converges to the value $\psi(0,0)$ regardless of the manner in which (x,y) approaches the origin and h and k become infinite. Also since the functions $f(x'-a_2 y')$ and $g(a_1 y'-x')$ are bounded and $H(x'-a_2 y')$ and $G(a_1 y'-x')$ are continuous, we can make N small enough that the variation caused by replacing $H(x'-a_2 y')$ by H(0) and $G(a_1 y'-x')$ by G(0) is as small as we please. The neighborhood N is divided into four parts by the tangents, and we will use N_1 to designate that portion that lies entirely in the first quadrant, or that includes the positive y-axis if the tangents are in different quadrants, and by N_2, N_3, N_4 the portions as they appear in order on a counter clockwise circuit of the origin.

To complete the discussion, we must consider several distinct cases. We will begin by requiring $a_1 > a_2 > 0$ for the first case, and let $h/k \to \infty$ as h and k become infinite. For maximum effect (greatest difference between function and limit of sum) from the contributions of $H(0)g(a,y'-x')/\pi$ and $G(0)f(x'-a_2y')$ to the sum as (x',y') approaches the origin from N_1, the line $x'= ey'$ (theorem II) must be the same for both functions, and such that the point (x',y') makes both a maximum. This happens when,

$$(175) \quad x' = 2ea_1\pi/k(a_1-e) = -2ea_2\pi/k(a_2-e)$$

from which

$$(176) \quad e = \frac{2a_1a_2}{a_1+a_2}, \quad x' = \frac{4a_1a_2\pi}{k(a_1-a_2)}, \quad y' = \frac{2(a_1+a_2)\pi}{k(a_1-a_2)}.$$

With these values, we find that the contribution of $G(0)f(x'-ay)/\pi$ to the sum is $G(0)\rho/\pi$ in excess of the value approached by the function, and of $H(0)\,g(a,y'-x')/\pi$ is $H(0)\rho/\pi$ in excess. Substituting these values in (122) theorem IV, we find that the contribution due to $A\,F(x',y')/\pi^2$ approaches the value $-\frac{1}{4}\,A/\pi^2\cdot\pi^2 = -\frac{1}{4}A$, whereas the function approaches $\frac{1}{4}A$, hence the excess is $-\frac{1}{2}A$. If we use the symbol ϕ_1 for the value approached by $\phi(x',y')$ as (x',y') approaches (0,0) from region N_1, ϕ_2 from N_2, ϕ_3 from N_3, and ϕ_4 from N_4, we have, when we add these excesses together, that the sum $S_{1,k}(x',y')$ tends to the value

$$(177) \quad \phi_1 + \rho/\pi\{G(0)+H(0)\} - \tfrac{1}{2}A$$

$$= \phi_1 + \{\phi_1-\phi_3\}\rho/\pi - \tfrac{1}{2}\{\phi_1-\phi_2+\phi_3-\phi_4\}$$

Similarly for approach from N_2, we find $e = 0 = x'$ and $y' = 2\pi/k$ giving,

$$(178) \quad \phi_2 + \{H(0)-G(0)\}\rho/\pi + 0 = \phi_2 + (\phi_2-\phi_4)\rho/\pi.$$

And from N_3, $e = \frac{2a_1a_2}{a_1+a_2}$, $x' = -\frac{4a_1a_2\pi}{k(a_1-a_2)}$ and $y' = -\frac{2(a_1+a_2)\pi}{k(a_1-a_2)}$, giving,

$$(179) \quad \phi_3 + \{-H(0) - G(0)\}A/\pi - \tfrac{1}{2}A$$
$$= \phi_3 + \{\phi_3 - \phi_1\}A/\pi - \tfrac{1}{2}\{\phi_1 - \phi_2 + \phi_3 - \phi_4\}$$

And from N_4, $e = 0 = x'$ and $y' = -2\pi/k$, giving,

$$(180) \quad \phi_4 + \{G(0) - H(0)\}A/\pi + 0 = \phi_4 + \{\phi_4 - \phi_2\}A/\pi.$$

For a second case when $a_1 > a_2 > 0$, let $h/k \to 0$ as h and k increase indefinitely, and from theorems II and V, we find the same results as when $h/k \to \infty$ if we make $e = \pm(a_1+a_2)$ with $x' = \pm\frac{2(a_1+a_2)\pi}{h(a_1-a_2)}$ and $y' = \pm\frac{4\pi}{h(a_1-a_2)}$ in N_1 and N_3, and $e = \infty, y' = 0$, $x' = \frac{2\pi}{h}$, in N_2 and N_4. The verification of this is simplified if we observe that in our special function F(x,y) the value approached by the sum of the series depended on the value of the function near the y-axis as $h/k \to \infty$ and near the x-axis as $h/k \to 0$. These are both $-\frac{1}{4}\pi^2$ in these cases.

If $a_1 > 0 > a_2 = -a_3$, we have a third case when $h/k \to \infty$. For maximum effect from $H(0)g(a_1y'-x')/\pi$ and $G(0)f(x'-a_1y')/\pi$ as (x',y') approaches the origin from N_1, we find $e = 0 = x'$ and $y' = \frac{2\pi}{k}$ and the excess is,

$$(181) \quad \{G(0) + H(0)\}A/\pi = (\phi_1 - \phi_3)A/\pi.$$

Also $AF(x',y')$ gives a contribution to the sum $S_{hk}(x',y')$ of $\frac{1}{4}A$ (see (135) theorem VI) which is the value of the function approaches from N_1, that is, as (x',y') approaches the origin from N_1, the sum $S_{hk}(x',y')$ tends to the value

$$(182) \quad \phi_1 + (\phi_1 - \phi_3)\,P_1/\pi$$

By similar methods, we find as (x',y') approaches from N_2, the sum $S_{hk}(x',y')$ tends to

$$(183) \quad \phi_2 + \{H(0) - G(0)\}P_1/\pi + \tfrac{1}{2}A$$
$$= \phi_2 + \{\phi_2 - \phi_4\}P_1/\pi + \tfrac{1}{2}\{\phi_1 - \phi_2 + \phi_3 - \phi_4\},$$

from N_3, tends to

$$(184) \quad \phi_3 + \{-G(0) - H(0)\}P_1/\pi = \phi_3 + \{\phi_3 - \phi_1\}P_1/\pi,$$

and from N_4, tends to the value

$$(185) \quad \phi_4 + \{G(0) - H(0)\}P_1/\pi + \tfrac{1}{2}A$$
$$= \phi_4 + (\phi_4 - \phi_2)P_1/\pi + \tfrac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4).$$

By the use of theorems II and VII, when $h/k \to 0$ as h and k become infinite, if we let $c = \tfrac{1}{2}\{a_1 - a_3\}$, $x' = \pm \frac{2(a_1 - a_3)\pi}{h(2a_3 - a_1)}$, and $y' = \pm \frac{4\pi}{h(2a_3 - a_1)}$ in N_1 and N_3, or $c = \infty$, $y' = 0$, and $x' = \pm 2\pi/h$ in N_2 and N_4, we get by approach from N_1,

$$(186) \quad \phi_1 + \{H(0) + G(0)\} - \tfrac{1}{2}A$$
$$= \phi_1 + \{\phi_1 - \phi_3\}P_1/\pi - \tfrac{1}{2}\{\phi_1 - \phi_2 + \phi_3 - \phi_4\};$$

from N_2,

$$(187) \quad \phi_2 + \{H(0) - G(0)\}P_1/\pi = \phi_2 + \{\phi_2 - \phi_4\}P_1/\pi;$$

from N_3,

$$(188) \quad \phi_3 + \{-H(0) - G(0)\}P_1/\pi - \tfrac{1}{2}A$$
$$= \phi_3 + \{\phi_3 - \phi_1\}P_1/\pi - \tfrac{1}{2}\{\phi_1 - \phi_2 + \phi_3 - \phi_4\};$$

and from N_4,

$$(189) \quad \Phi_4 + \{\Phi(0) - H(0)\} P/\pi = \Phi_4 + \{\Phi_1 - \Phi_2\} P_1/\pi,$$

as the values approached by the sum.

Conclusion. With these results we will close our present investigation. We have shown the existence of a Gibbs' phenomenon in the convergence of the double Fourier's series as a general rule. In the case where the point under consideration is a point of discontinuity on a single curve of discontinuities of the function, we have shown that the phenomenon always exists, and that its magnitude is the same as for the plane curve cut from the surface by any convenient section. In the case where the point under consideration is on two curves of discontinuities having tangents at the point which are parallel to the x- and y-axes, we have shown the existence of the phenomenon and that its magnitude differs from that at other points, exceeding it for some approaches.

At points on two curves of discontinuities where the tangents are not both parallel to an axis, it has been found that the series does not converge, but depends on the value approached by h/k, and we have found the same to be true concerning the Gibbs' phenomena. We have found that there is an effect but have not found an accurate measure of its magnitude.