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I hereby recommend that the thesis prepared under my supervision by Russell J. Dunkalter *entitled* On a Sequence of Factors Which Leaves ~~##~~ a Class of Double Fourier Series Unchanged

be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy

Approved by:

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A SEQUENCE OF FACTORS WHICH LEAVES A CLASS OF DOUBLE FOURIER SERIES UNCHANGED

A dissertation submitted to the

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1. Introduction. A Fourier series

$$(1) \quad \sum_{n=0}^{n=\infty} (a_n \cos nx + b_n \sin nx)$$

and its allied series

$$(2) \quad \sum_{n=1}^{n=\infty} (b_n \cos nx - a_n \sin nx)$$

may be classified according to the properties of the function f(x) to which they correspond. The following classification has been used by M.Fekete: integrable Lebesque, bounded, continuous, bounded and integrable Riemann, bounded variation, and an integral, that is, of the form $f(x) = \int_a^x g(x)\,dx$.

The problem of finding the necessary and sufficient condition on a sequence of factors λ_n which transform a Fourier series (1) of a certain class into a Fourier series

$$(3) \quad \sum_{n=0}^{n=\infty} \lambda_n (a_n \cos nx + b_n \sin nx)$$

of the same class; and which transform an allied series (2), whether a Fourier series or not, of a certain class into a Fourier series

$$(4) \quad \sum_{n=1}^{n=\infty} \lambda_n (b_n \cos nx - a_n \sin nx)$$

of the same class has been investigated by many mathematicians. In the first case it may be said that the class is invariant to the sequence λ_n and in the second case that the class is covariant to the sequence λ_n.

F.Riesz, W.H.Young, H.Steinhaus, and S.Szidon discovered theorms which M.Fekete (1) and later S.Bochner (2) have combined into general theorems applying to the six classes mentioned above.

(1) M.Fekete, Uber Faktorenfolgen welche die Klasse einer Fouriersche Reihe unverandert lossen, Acta Litterarum ac Scientiarum, Bd.1 (1923), S.148.

(2) S. Bochner, Uber Faktorenfolgen fur Fouriersche Reihen, Acta Litterarum ac Scientiarum, Bd.4 (1928-29), S.125.

These theorems are

INVARIANT THEOREM 1. The necessary and sufficient condition that the sequence λ_n transform the Fourier series

(1) $$\sum_{n=0}^{n=\infty} (a_n \cos nx + b_n \sin nx)$$

into a Fourier series

(2) $$\sum_{n=0}^{n=\infty} \lambda_n (a_n \cos nx + b_n \sin nx)$$

of the same class is that the series

(3) $$\sum_{n=1}^{n=\infty} \frac{\lambda_n \sin nx}{n}$$

be a Fourier series of an odd function of bounded variation.

COVARIANT THEOREM 1. The necessary and sufficient condition that the sequence λ_n transform the allied series

(1) $$\sum_{n=1}^{n=\infty} (b_n \cos nx - a_n \sin nx)$$

into a Fourier series

(2) $$\sum_{n=1}^{n=\infty} \lambda_n (b_n \cos nx - a_n \sin nx)$$

of the same class is that the series

(3) $$\sum_{n=1}^{n=\infty} \frac{\lambda_n \cos nx}{n}$$

be the Fourier series of an even function of bounded variation.

In this paper, following the methods of M.Fekete, A. Zygmund, W.H.Young, and S.Szidon the above two theorems are generalized to double Fourier series for the same six classes. The results are

INVARIANT THEOREM II. The necessary and sufficient condition that the sequence λ_{mn} transform the Fourier series

$$(1)\quad \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}(a_{mn}\cos mx\cos ny + b_{mn}\cos mx\sin ny + c_{mn}\sin mx\cos ny + d_{mn}\sin mx\sin ny)$$

Into a Fourier series

$$(2)\quad \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}\lambda_{mn}(a_{mn}\cos mx\cos ny + b_{mn}\cos mx\sin ny + c_{mn}\sin mx\cos ny + d_{mn}\sin mx\sin ny)$$

of the same class is that the series

$$(3)\quad \sum_{m=1}^{m=\infty}\frac{\lambda_{m0}\sin mx}{m} + \sum_{n=1}^{n=\infty}\frac{\lambda_{0n}\sin ny}{n} + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\lambda_{mn}\sin mx\sin ny}{mn}$$

be the Fourier series of an odd function of bounded variation according to the definition of Hardy-Krause.

COVARIANT THEOREM II. The necessary and sufficient condition that the sequence λ_{mn} transform the allied series

$$(1)\quad \sum_{m=1}^{m=\infty}(c_{m0}\cos mx - a_{m0}\sin mx) +$$

$$+\sum_{n=1}^{n=\infty}\left(b_{on}\cos ny - a_{on}\sin ny\right)+$$

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left(d_{mn}\cos mx\cos ny - c_{mn}\cos mx\sin ny -\right.$$

$$\left.-b_{mn}\sin mx\cos ny + a_{mn}\sin mx\sin ny\right)$$

into a Fourier Series

$$(2)\quad \sum_{m=1}^{m=\infty}\lambda_{mo}\left(c_{mo}\cos mx - a_{mo}\sin mx\right)+$$

$$+\sum_{n=1}^{n=\infty}\lambda_{on}\left(b_{on}\cos ny - a_{on}\sin ny\right)+$$

$$+\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\left(d_{mn}\cos mx\cos ny - c_{mn}\cos mx\sin ny\right.$$

$$\left.-b_{mn}\sin mx\cos ny + a_{mn}\sin mx\sin ny\right)$$

of the same class is that the series

$$(3)\quad \sum_{m=1}^{m=\infty}\frac{\lambda_{mo}\cos mx}{m} + \sum_{n=1}^{n=\infty}\frac{\lambda_{on}\cos ny}{n}$$

$$+\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\lambda_{mn}\cos mx\cos ny}{mn}$$

be a Fourier series of an even function of bounded variation according to the definition of Hardy-Krause.

2. Preliminary Remarks and Definitions. Let f(x,y) be a real function of the real variables x and y defined everywhere in the domain $\Delta \equiv (-\pi \leq x \leq \pi,\ -\pi \leq y \leq \pi)$. Also let f(x,y) be periodic of period 2 in x and y. We shall consider the following classes of functions:

Class C_1. Functions integrable Lebesque (L)

Class C_2. Bounded functions (B)

Class C_3. Continuous functions (C)

Class C_4. Functions bounded and integrable Riemann (R)

Class C_5. Functions of bounded variation according to the definition of Hardy-Krause (H)

Class C_6. Functions which are integrals, that is, of the form $f(x,y) = \int_a^x \int_b^y g(x,y)\,dx\,dy$ (I)

A function is said to be in class C_γ ($\gamma = 1,2,3,4,5,6$) if f(X,y) satisfies the class property everywhere in the domain Δ.

We shall use the following definitions:

Bounded and integrable Riemann. A function f(x,y) is said to be bounded and integrable Riemann in Δ if

$$|f(x,y)| \leqq G$$

and

$$\sum_{k=1}^{k=m} \sum_{l=1}^{l=n} |f(\alpha_k, \beta_l) - f(\alpha_k', \beta_l')| (x_k - x_{k-1})(y_l - y_{l-1}) \leqq \epsilon$$

where

$$-\pi \leqq x_0 < x_1 < \cdots < x_m \leqq \pi$$

$$-\pi \leqq y_0 < y_1 < \cdots < y_n \leqq \pi$$

and the maximum for

$$x_1 - x_0, \cdots, x_m - x_{m-1} \leqq \delta(\epsilon)$$

$$y_1 - y_0, \cdots, y_n - y_{n-1} \leqq \delta(\epsilon)$$

and

$$x_{k-1} \leqq \alpha_k < \alpha_k' \leqq x_k, \quad y_{l-1} \leqq \beta_l < \beta_l' \leqq y_l.$$

Continuous. A function f(x,y) is said to be continuous in Δ if

$$|f(\alpha,\beta) - f(\alpha',\beta')| \leqq \epsilon$$

where

$$|\alpha' - \alpha| \leqq \delta(\epsilon), \quad |\beta' - \beta| \leqq \delta(\epsilon)$$

or $|\alpha' - \alpha + 2\pi| \leqq \delta(\epsilon), \quad |\beta' - \beta + 2\pi| \leqq \delta(\epsilon)$

Bounded variation H. We define

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_i, y_{j+1}) - f(x_{i+1}, y_j) + f(x_i, y_j),$$

$$\Delta_{10} f(x_i, y) = f(x_{i+1}, y) - f(x_i, y),$$

$$\Delta_{01} f(x, y_j) = f(x, y_{j+1}) - f(x, y_j).$$

A function f(x,y) is said to be of bounded variation H (according to the definition of Hardy-Krause) in Δ if

$$\sum_{i=0}^{i=m-1} \sum_{j=0}^{j=n-1} |\Delta_{11} f(x_i, y_j)| < V$$

for all possible nets fitted Δ such that

$$-\pi = x_0 < x_1 < \text{-----------} < x_m = \pi,$$

$$-\pi = y_0 < y_1 < \text{-----------} < y_n = \pi;$$

and if in addition

$$\sum_{i=0}^{i=m-1} |\Delta_{10} f(x_i, y)| < V_1$$

for some fixed value of y, and

$$\sum_{j=0}^{j=n-1} |\Delta_{01} f(x, y_j)| < V_2$$

for some fixed value of x.

It has been shown by Clarkson and Adams (3) that

$$\sum_{i=0}^{i=m-1} |\Delta_{10} f(x_i, y)| < V_1$$

for all values of y, and

$$\sum_{j=0}^{j=n-1} |\Delta_{01} f(x, y_j)| < V_2$$

for all values of x; and that the necessary and sufficient condition that a function f(x,y) be of bounded variation H is that it be the difference of two bounded functions $\phi(x,y)$ and $\psi(x,y)$ such that

$$\phi(x_{i+1}, y_{j+1}) - \phi(x_i, y_j) \geqq 0,$$

$$\Delta_{11} \phi(x_i, y_j) \geqq 0,$$

$$\Delta_{10} \phi(x_i, y) \geqq 0,$$

$$\Delta_{01} \phi(x, y_j) \geqq 0,$$

with similar relations for $\psi(x,y)$. We say that such a function is monotone.

(3) C.R.Adams, J.A.Clarkson, Properties of Functions f(x,y) of Bounded Variation, Trans. Amer. Math. Soc., 1934, vol.36.

Double Fourier series. The double Fourier series corresponding to f(x,y) may be written

$$f(x,y) \sim \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} \phi_{mn}(x,y)$$

where

$$\phi_{mn}(x,y) = a_{mn}\cos mx\cos ny + b_{mn}\cos mx\sin ny$$
$$+ c_{mn}\sin mx\cos ny + d_{mn}\sin mx\sin ny, \quad m>0,\ n>0,$$
$$\phi_{mo}(x,y) = \phi_{mo}(x) = a_{mo}\cos mx + c_{mo}\sin mx, \quad m>0,$$
$$\phi_{on}(x,y) = \phi_{on}(y) = a_{on}\cos ny + b_{on}\sin ny, \quad n>0,$$
$$\phi_{oo}(x,y) = a_{oo}.$$

The Fourier constants are

$$a_{mn} = \frac{1}{\pi^2}\int_\Delta f(x,y)\cos mx\cos ny\,dx\,dy,$$
$$a_{mo} = \frac{1}{2\pi^2}\int_\Delta f(x,y)\cos mx\,dx\,dy,$$
$$a_{on} = \frac{1}{2\pi^2}\int_\Delta f(x,y)\sin mx\,dx\,dy,$$
$$a_{oo} = \frac{1}{4\pi^2}\int_\Delta f(x,y)\,dx\,dy,$$

with similar formulas for the other constants. The above integrals are Lebesque integrals. Thus every function integrable Lebesque has a Fourier series. It can be assumed without loss of generality that $a_{oo}=0$. For, if f(x,y) belongs to class C_γ so also does f(x,y)- a_{oo} which has no constant term in its Fourier series.

The functions

(1) $$f_1(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dy,$$

(2) $$f_2(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dx,$$

and

(3) $$f_3(x,y) = f(x,y) - f_1(x) - f_2(y)$$

have as their Fourier series, respectively, the series

(4) $\sum_{m=1}^{m=\infty} \phi_{mo}(x)$,

(5) $\sum_{n=1}^{n=\infty} \phi_{on}(y)$,

and

(6) $\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \phi_{mn}(x,y)$.

Allied series. The series

$$\sum_{m=1}^{m=\infty} \bar{\phi}_{mo}(x) + \sum_{n=1}^{n=\infty} \bar{\phi}_{on}(y) + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \bar{\phi}_{mn}(x,y),$$

where

$$\bar{\phi}_{mo}(x) = c_{mo}\cos mx - a_{mo}\sin mx,$$

$$\bar{\phi}_{on}(y) = b_{on}\cos ny - a_{on}\sin ny,$$

and

$$\bar{\phi}_{mn}(x,y) = d_{mn}\cos mx\cos ny - c_{mn}\cos mx\sin ny$$
$$- b_{mn}\sin mx\cos ny + a_{mn}\sin mx\sin ny$$

is defined as the series allied with the Fourier series

$$\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \phi_{mn}(x,y).$$

An allied series is said to be in class C_γ if the Fourier series with which it is allied is in class C_γ. A Fourier series is said to be in class C_γ if the function f(x,y) to which it corresponds is in class C_γ.

It is apparent from the above definitions that the series

(7) $\displaystyle\sum_{m=1}^{m=\infty} \overline{\Phi}_{m0}(x)$,

(8) $\displaystyle\sum_{n=1}^{n=\infty} \overline{\Phi}_{0n}(y)$,

and

(9) $\displaystyle\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \overline{\Phi}_{mn}(x,y)$

are, respectively, the series allied with the Fourier series (4), (5), and (6).

Method of Generalization. The two theorems of page 2 will be generalized in the following manner. If the function f(x,y) belongs to class C_γ (γ=1,2,3,4,5,6) so also do the functions $f_1(x)$, $f_2(y)$, and $f_3(x,y)$; and conversely, if $f_1(x)$, $f_2(y)$, and $f_3(x,y)$ belong to class C_γ so also does f(x,y). A proof is given for each class.

1° Integrable Lebesque

$$\int_{-\pi}^{\pi} f_1(x)\,dx = \int_{-\pi}^{\pi} dx \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dy\right\} = \frac{1}{2\pi}\int_{\Delta} f(x,y)\,dx\,dy.$$

Obviously $f_3(x,y) = f(x,y) - f_1(x) - f_2(y)$ is integrable L; and if $f_1(x)$, $f_2(y)$, and $f_3(x,y)$ are integrable L so is f(x,y).

2° Bounded

$$|f_1(x)| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x,y)|\,dy < V,$$

$$|f_3(x,y)| \leqq |f(x,y)| + |f_1(x)| + |f_2(y)| < 3V.$$

Conversely

$$|f(x,y)| \leqq |f_1(x)| + |f_2(y)| + |f_3(x,y)| < V_1 + V_2 + V_3$$

3° Continuous

$$|f(\alpha,y)-f(\alpha',y)| \leqq \epsilon$$ for all values of y,

when

$$|\alpha'-\alpha| \leqq \delta(\epsilon), \quad |\alpha'-\alpha+2\pi| \leqq \delta(\epsilon).$$

Then

$$|f_1(\alpha)-f_1(\alpha')| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\alpha,y)-f(\alpha',y)|\,dy \leqq \epsilon$$

Similarly for $f(x,y)$ and the converse.

4° Bounded and Integrable Riemann

That the function $f(x)$ is integrable R is proved in E.W. Hobson, The Theory of Functions of a Real Variable, vol. 1, p.511. Obviously $f(x,y)$ is then integrable R; and conversely, $f(x,y)$ is integrable R when $f_1(x)$, $f_2(y)$ and $f(x,y)$ are.

5° Bounded Variation H

$$\sum_{i=0}^{i=m-1}|\Delta_{10}f_1(x_i)| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{i=0}^{i=m-1}|\Delta_{10}f(x_i,y)|\,dy < V_1 .$$

Thus $f_1(x)$ is of bounded variation. That $f_3(x,y)$ is of bounded variation H follows immediately from the fact that

$$\Delta_{11}f_3(x_i,y_j) = \Delta_{11}\{f(x_i,y_j)-f_1(x_i)-f_2(y_j)\} = \Delta_{11}f(x_i,y_j).$$

Similarly $f(x,y)$ is of bounded variation H if $f_1(x)$, $f_2(y)$, and $f_3(x,y)$ are, for,

$$\Delta_{11}f(x_i,y_j) = \Delta_{11}\{f_1(x_i)+f_2(y_j)+f_3(x_i,y_j)\} = \Delta_{11}f_3(x_i,y_j).$$

6° An Integral

If the function $f(x,y)$ is of the form

$$f(x,y) = \int_a^x\int_b^y g(t_1,t_2)\,dt_1\,dt_2$$

Then f(x,y) is also absolutely continuous and of bounded variation H. By absolutely continuous we mean that

$$\sum\sum |\Delta_{11} f(x_i, y_i)| < \epsilon$$

for every set of non-overlapping cells of measure less than η contained in Δ. The necessary and sufficient condition that a function be an integral is that it be absolutely continuous. From the above equation it is apparent that for any constant value of y, say y_1, $f(x,y_1)$ is an integral in x or is absolutely continuous in x. Thus

$$\sum |\Delta_{10} f(x_i, y_1)| < \epsilon_1$$

for every set of non-overlapping intervals of x of total measure less than η. In general, for the same η

$$\sum |\Delta_{10} f(x_i, y_m)| < \epsilon_n .$$

But since f(x,y) is of bounded variation H

$$\sum_{i=0}^{i=m-1} |\Delta_{10} f(x_i, y)| < V_1 \qquad \text{for all values of } y,$$

that is, the ϵ_m are bounded by V_1. Thus ~~for~~ we can choose an η so that

$$\sum |\Delta_{10} f(x_i, y)| < \epsilon$$

for all values of y. Then

$$\sum |\Delta_{10} f_1(x_i)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum |\Delta_{10} f(x_i, y)| \, dy < \epsilon$$

for every set of non-overlapping intervals of x of total measure less than η. Thus f(x) is an integral. As indicated above in 5°, $\Delta_{11} f_3(x_i, y_j) = \Delta_{11} f(x_i, y_j)$ so that $f_3(x,y)$ is also an integral when $f(x,y)$ is an integral. Similarly the converse is true, that is,

$$f(x,y) = f_1(x) + f_2(y) + f_3(x,y)$$

is an integral when $f_1(x)$, $f_2(y)$, and $f_3(x,y)$ are integrals.

Thus it is sufficient to generalize the theorems of page 2 for the function $f_3(x,y)$, that is, for a double Fourier series having no constant term and no terms in one variable. The results for such a series combined with the theorems of page 2 will give the theorems of page 3.

Fourier Stieltjes series. If $f(x,y)$ is of bounded variation H then there exists a Riemann Stieltjes integral of $f(x,y)$ with respect to a continuous function. The series

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \phi_{mn}(x,y) = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} (A_{mn} \cos mx \cos ny + B_{mn} \cos mx \sin ny + C_{mn} \sin mx \cos ny + D_{mn} \sin mx \sin ny),$$

Where

$$A_{mn} = \frac{1}{\pi^2}\int_\Delta \cos mx \cos ny \, df(x,y),$$

$$B_{mn} = \frac{1}{\pi^2}\int_\Delta \cos mx \sin ny \, df(x,y),$$

$$C_{mn} = \frac{1}{\pi^2}\int_\Delta \sin mx \cos ny \, df(x,y),$$

$$D_{mn} = \frac{1}{\pi^2}\int_\Delta \sin mx \sin ny \, df(x,y), \quad m>0,\ n>0,$$

is called the Fourier Stieltjes series of $f(x,y)$.

If f(x,y) is periodic of period 2π in x and y then the other constants, corresponding to those of a Fourier series, A_{oo}, A_{mo}, A_{on}, C_{mo}, B_{on}, are all zero.

$$\int_\Delta df(x,y) = f(\pi,\pi) - f(\pi,-\pi) - f(-\pi,\pi) + f(-\pi,-\pi) = 0$$

Consider

$$A_{mo} = \frac{1}{2\pi^2}\int_\Delta \cos mx \, df(x,y).$$

Let the domain Δ be divided into a network of cells by lines parallel to the axes and let the upper bound of cos(mx) in the cell $(x_i, y_j; x_{i+1}, y_{j+1})$ be represented by U_{ij}. Then the above integral equals

$$\frac{1}{2\pi^2}\lim_{m,n\to\infty}\sum_{i=0}^{i=m-1}\sum_{j=0}^{j=n-1} U_{ij}\,\Delta_{11} f(x_i, y_j)$$

where the dimensions of the largest cell diminish toward zero. Since U_{ij} is independent of j we can write

$$\frac{1}{2\pi^2}\sum_{i=0}^{i=m-1} U_i \sum_{j=0}^{j=n-1}\Delta_{11} f(x_i, y_j) =$$

$$= \frac{1}{2\pi^2}\sum_{i=0}^{i=m-1} U_i\left\{f(x_{i+1},\pi) - f(x_i,\pi) - f(x_{i+1},-\pi) + f(x_i,-\pi)\right\} =$$

$$= \frac{1}{2\pi^2}\sum_{i=0}^{i=m-1} U_i\left\{f(x_{i+1},\pi) - f(x_i,\pi)\right\} - \frac{1}{2\pi^2}\sum_{i=0}^{i=m-1} U_i\left\{f(x_{i+1},-\pi) - f(x_i,-\pi)\right\}.$$

Then passing to the limit we obtain

$$\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\cos mx \, df(x,\pi) - \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\cos mx \, df(x,-\pi)$$

$$= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\cos mx \, d\left[f(x,\pi) - f(x,-\pi)\right] = 0.$$

Integration

By a transformation (4) of the Riemann Stieltjes double integral we obtain

$$A_{mn} = \frac{1}{\pi^2}\int_\Delta \cos mx \cos ny \, df(x,y) = \frac{mn}{\pi^2}\int_\Delta f(x,y) \sin mx \sin ny \, dx\, dy = mn\, d_{mn}$$

$$B_{mn} = \frac{1}{\pi^2}\int_\Delta \cos mx \sin ny \, df(x,y) = \frac{-mn}{\pi^2}\int_\Delta f(x,y) \sin mx \cos ny \, dx\, dy = -mn\, c_{mn}$$

$$C_{mn} = \frac{1}{\pi^2}\int_\Delta \sin mx \cos ny \, df(x,y) = \frac{-mn}{\pi^2}\int_\Delta f(x,y) \cos mx \sin ny \, dx\, dy = -mn\, b_{mn}$$

$$D_{mn} = \frac{1}{\pi^2}\int_\Delta \sin mx \sin ny \, df(x,y) = \frac{mn}{\pi^2}\int_\Delta f(x,y) \cos mx \cos ny \, dx\, dy = mn\, a_{mn}$$

where a_{mn} , b_{mn} , c_{mn} , and d_{mn} are the Fourier constants of f(x,y).

(4) E.W.Hobson, The Theory of Functions of a Real Variable, vol.1, p.666.

Thus the differentiated double Fourier series of a function f(x,y) of bounded variation H is identical with the Fourier Stieltjes series of f(x,y). ?

3. Lemmas.

LEMMA 1. If a function f(x,y) satisfies the condition B, C, or R in a set M of measure $4\pi^2$ in Δ , then there exists an equivalent function f*(x,y) which satisfies the condition B, C, or R everywhere in Δ .

Let M_0 be the set of measure zero in which the condition is not satisfied. We define

$$f^*(x,y) = f(x,y) \text{ in } M,$$

$$f^*(x,y) = \overline{\lim_{\mu\to\infty}} f(\alpha_\mu, \beta_\mu) \quad \text{in } M_o,$$

where (α_μ, β_μ) is a sequence of points of M whose limit is a point of M_o.

1° Bounded Obviously $f^*(x,y)$ is bounded in Δ.

2° Continuous. If $f(x,y)$ is continuous in M, then by definition we can choose a δ such that

$$|f(\alpha,\beta) - f(\alpha',\beta')| \leqq \epsilon$$

where

$$|\alpha - \alpha'| \leqq \delta, \quad |\beta - \beta'| \leqq \delta.$$

Since the oscillation of $f(x,y)$ in M within a rectangle of dimensions less than δ is less than ϵ, we have

$$\lim_{\mu\to\infty} f(\alpha_\mu, \beta_\mu)$$

exists and is unique. Hence

$$|f^*(x_o, y_o) - f(\alpha_\mu, \beta_\mu)| \leqq \epsilon$$

where

$$|x_o - \alpha_\mu| \leqq \delta, \quad |y_o - \beta_\mu| \leqq \delta$$

and (x_o, y_o) is a point of M_o.

Then

$$|f^*(\alpha,\beta) - f^*(\alpha',\beta')| =$$

$$|f^*(\alpha,\beta) - f(\alpha_\mu,\beta_\mu) - f^*(\alpha',\beta')$$

$$+ f(\alpha'_\mu, \beta'_\mu) + f(\alpha_\mu,\beta_\mu) - f(\alpha'_\mu,\beta'_\mu)| \leqq$$

$$\leq |f^*(\alpha,\beta) - f(\alpha_\mu,\beta_\mu)| + |f^*(\alpha',\beta') - f(\alpha'_\mu,\beta'_\mu)| +$$

$$+ |f(\alpha_\mu,\beta_\mu) - f(\alpha'_\mu,\beta'_\mu)| < \epsilon + \epsilon + \epsilon = 3\epsilon,$$

where

$$|\alpha - \alpha'| \leq \delta, \quad |\beta - \beta'| \leq \delta$$

and all points involved lie within a rectangle of dimensions equal or less than δ. Thus f*(x,y) is continuous according to the definition.

3°. Bounded and Integrable Riemann. For the set M we can choose a number 2δ such that

$$(1) \sum_{k=1}^{k=m} \sum_{l=1}^{l=n} |f(\alpha_k,\beta_l) - f(\alpha'_k,\beta'_l)| (\bar{x}_k - \bar{x}_{k-1})(\bar{y}_l - \bar{y}_{l-1}) < \epsilon$$

where

$$\bar{x}_{k-1} \leq \alpha_k < \alpha'_k \leq \bar{x}_k, \quad \bar{y}_{l-1} \leq \beta_l < \beta'_l \leq \bar{y}_l$$

$$|\bar{x}_k - \bar{x}_{k-1}| \leq 2\delta, \quad |\bar{y}_l - \bar{y}_{l-1}| \leq 2\delta.$$

Let us choose a series of positive numbers ϵ_{mn} such that

$$(2) \qquad \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \epsilon_{mn} \leq \frac{\epsilon}{\delta^2}$$

Consider, for points in Δ and a division such that

$$|x_k - x_{k-1}| \leq \delta, \quad |y_l - y_{l-1}| \leq \delta,$$

the sum

$$(3) \sum_{k=1}^{k=m} \sum_{l=1}^{l=n} |f^*(\alpha_k,\beta_l) - f^*(\alpha'_k,\beta'_l)| (x_k - x_{k-1})(y_l - y_{l-1}),$$

where

$$x_{k-1} \leq \alpha_k < \alpha'_k \leq x_k, \quad y_{l-1} \leq \beta_l < \beta'_l \leq y_l.$$

This sum may always be divided into a sum of those terms containing only points of M and a sum of those terms containing one or two points of M_o. That is, (3) can be written as

$$(4)\quad \sum^{M}\sum |f(\alpha_k,\beta_\ell)-f(\alpha_k',\beta_\ell')|(x_k-x_{k-1})(y_\ell-y_{\ell-1}) + \sum^{M_o}\sum |f^*(\alpha_k,\beta_\ell)-f^*(\alpha_k',\beta_\ell')|(x_k-x_{k-1})(y_\ell-y_{\ell-1}).$$

Corresponding to any point, say α_k, β_ℓ, of M_o we can choose a point u_k, v_ℓ of M such that

$$|f^*(\alpha_k,\beta_\ell)-f(u_k,v_\ell)| < \frac{\epsilon_{k\ell}}{2},$$

and if necessary a point u_k', v_ℓ' such that

$$|f^*(\alpha_k',\beta_\ell')-f(u_k',v_\ell')| < \frac{\epsilon_{k\ell}}{2},$$

where

$$|u_k-u_k'| < 2\delta, \quad |v_\ell-v_\ell'| < 2\delta.$$

Then

$$|f^*(\alpha_k,\beta_\ell)-f^*(\alpha_k',\beta_\ell')|(x_k-x_{k-1})(y_\ell-y_{\ell-1}) < |f(u_k,v_\ell)-f(u_k',v_\ell')|(2\delta)(2\delta) + 4\delta^2\cdot\epsilon_{k\ell}.$$

Hence, using this inequality in the second sum of (4), we have

$$(5)\quad \sum_{k=1}^{k=\infty}\sum_{\ell=1}^{\ell=\infty} |f^*(\alpha_k,\beta_\ell)-f^*(\alpha_k',\beta_\ell')|(x_k-x_{k-1})(y_\ell-y_{\ell-1}) < \epsilon+\epsilon+4\epsilon = 6\epsilon,$$

$$|x_k-x_{k-1}| \leqq \delta, \quad |y_\ell-y_{\ell-1}| \leqq \delta$$

for all points in Δ such that

$$\gamma_{k-1} \leq \alpha_k < \alpha'_k \leq \gamma_k , \quad \gamma_{l-1} \leq \beta_l < \beta'_l \leq \gamma_l .$$

LEMMA 2. If $C_{mn}(x,y)$ is a sequence of trigonometric polynomials such that

$$\iint_\Delta |C_{mn}(x,y)|\,dx\,dy < K$$

and if $f(x,y)$ satisfies the condition B.C.R. or H everywhere in Δ, then the sequence

$$\sigma_{mn}(x,y) = \iint_\Delta f(x+t_1, y+t_2)\,C_{mn}(t_1,t_2)\,dt_1\,dt_2$$

satisfies the same condition uniformly in Δ.

1°. Condition B.

$$|f(x+t_1, y+t_2)| < G, \qquad \text{all } x, y, t_1, t_2$$

$$|\sigma_{mn}(x,y)| \leq \iint_\Delta |f(x+t_1, y+t_2)|\,|C_{mn}(t_1,t_2)|\,dt_1\,dt_2 < GK.$$

2°. Condition C.

$$|f(\alpha+t_1, \beta+t_2) - f(\alpha'+t_1, \beta'+t_2)| < \epsilon$$

for α, α', β, β', satisfying the conditions of the definition and for all values of t_1, t_2.

$$|\sigma_{mn}(\alpha,\beta) - \sigma_{mn}(\alpha',\beta')| \leq$$

$$\iint_{\Delta} \left| f(\alpha+t_1, \beta+t_2) - f(\alpha'+t_1, \beta'+t_2) \right| \left| C_{mn}(t_1,t_2) \right| dt_1 dt_2 < \epsilon K.$$

3°. Condition R.

$$\sum_{k=1}^{k=\mu} \sum_{\ell=1}^{\ell=\nu} \left| f(\alpha_k+t_1, \beta_\ell+t_2) - f(\alpha_k'+t_1, \beta_\ell'+t_2) \right| (x_k - x_{k-1})(y_\ell - y_{\ell-1}) \leq \epsilon$$

for α, β, α', β', x,y, satisfying the conditions of the definition and for all values of t_1, t_2.

$$\sum_{k=1}^{k=\mu} \sum_{\ell=1}^{\ell=\nu} \left| \sigma_{mn}(\alpha_k, \beta_\ell) - \sigma_{mn}(\alpha_k', \beta_\ell') \right| (x_k - x_{k-1})(y_\ell - y_{\ell-1}) \leq$$

$$\leq \iint_{\Delta} \sum_{k=1}^{k=\mu} \sum_{\ell=1}^{\ell=\nu} \left| f(\alpha_k+t_1, \beta_\ell+t_2) - f(\alpha_k'+t_1, \beta_\ell'+t_2) \right| (x_k - x_{k-1})(y_\ell - y_{\ell-1}) \cdot$$
$$\cdot \left| C_{mn}(t_1, t_2) \right| dt_1 dt_2 < \epsilon K.$$

4°. Condition H.

$$\sum_{i=1}^{i=\mu-1} \sum_{j=1}^{j=\nu-1} \left| \Delta_{11} f(x_i+t_1, y_j+t_2) \right| \leq G_1, \text{ all } t_1, t_2,$$

$$\sum_{i=1}^{i=\mu-1} \left| \Delta_{10} f(x_i+t_1, y+t_2) \right| \leq G_2, \text{ all } t_1, t_2, y,$$

$$\sum_{j=1}^{j=\nu-1} \left| \Delta_{01} f(x+t_1, y_j+t_2) \right| \leq G_3, \text{ all } t_1, t_2, x,$$

$$\sum_{i=1}^{i=\mu-1} \sum_{j=1}^{j=\nu-1} \left| \Delta_{11} \sigma_{mn}(x_i, y_j) \right| \leq$$

$$\leq \iint_{\Delta} \sum_{i=1}^{i=\mu-1} \sum_{j=1}^{j=\nu-1} \left| \Delta_{11} f(x_i+t_1, y_j+t_2) \right| \left| C_{mn}(t_1,t_2) \right| dt_1 dt_2 < G_1 K,$$

$$\sum_{i=1}^{i=\mu-1} \left| \Delta_{10} \sigma_{mn}(x_i, y) \right| \leq$$

$$\leq \iint_{\Delta} \sum_{i=1}^{i=\mu-1} \left| \Delta_{10} f(x_i+t_1, y+t_2) \right| \left| C_{mn}(t_1,t_2) \right| dt_1 dt_2 < G_2 K.$$

Similarly

$$\sum_{j=1}^{j=\nu-1} \left| \Delta_{10} \sigma_{mn}(x, y_j) \right| < G_3 K.$$

Lemma 3. If the trigonometric polynomials $\sigma_{mn}(x,y)$ with the common constant term

$$a_{00} = \frac{1}{4\pi^2}\int_\Delta \sigma_{mn}(x,y)\,dx\,dy$$

satisfy everywhere in Δ the condition C, R, or H uniformly, then they satisfy also the condition B uniformly.

If $\sigma_{mn}(x,y)$ satisfies the condition C or R uniformly then it follows from the definitions that

$$|\sigma_{mn}(x,y) - \sigma_{mn}(-\pi,-\pi)| < G, \quad \text{all } m, n, x, y.$$

$$\sigma_{mn}(-\pi,-\pi) = \frac{1}{4\pi^2}\int_\Delta \sigma_{mn}(x,y)\,dx\,dy -$$

$$- \frac{1}{4\pi^2}\int_\Delta \{\sigma_{mn}(x,y) - \sigma_{mn}(-\pi,-\pi)\}\,dx\,dy.$$

$$|\sigma_{mn}(-\pi,-\pi)| < |a_{00}| + G.$$

$$|\sigma_{mn}(x,y)| = |\sigma_{mn}(x,y) - \sigma_{mn}(-\pi,-\pi) + \sigma_{mn}(-\pi,-\pi)|$$

$$< |\sigma_{mn}(x,y) - \sigma_{mn}(-\pi,-\pi)| + |\sigma_{mn}(-\pi,-\pi)| < |a_{00}| + 2G.$$

If $\sigma_{mn}(x,y)$ is of uniformly bounded variation H then

$$|\sigma_{mn}(-\pi,-\pi) - \sigma_{mn}(-\pi,y) - \sigma_{mn}(x,-\pi) + \sigma_{mn}(x,y)| < G_1,$$

$$|\sigma_{mn}(-\pi,-\pi) - \sigma_{mn}(-\pi,y)| < G_2,$$

$$|\sigma_{mn}(-\pi,-\pi) - \sigma_{mn}(x,-\pi)| < G_3,$$

for all m,n,x, and y.

$$|\sigma_{mn}(x,y)-\sigma_{mn}(-\pi,-\pi)| =$$
$$|\sigma_{mn}(x,y)-\sigma_{mn}(x,-\pi)-\sigma_{mn}(-\pi,y)+\sigma_{mn}(-\pi,-\pi)+$$
$$+\sigma_{mn}(x,-\pi)-\sigma_{mn}(-\pi,-\pi)+\sigma_{mn}(-\pi,y)-\sigma_{mn}(-\pi,-\pi)|$$
$$< G_1 + G_2 + G_3.$$

From this it follows as before that $\sigma_{mn}(x,y)$ is uniformly bounded.

LEMMA 4. Given a sequence of functions $f_{mn}(x,y)$ defined in Δ and of uniformly bounded variation H, either there exists a uniformly bounded subsequence $f_{m_k n_\ell}(x,y)$ which converges everywhere to a function F(x,y) of bounded variation H, or $|f_{mn}(x,y)|$ diverges uniformly to ∞ as m,n, tend toward ∞.

This lemma for functions $f_m(x)$ is given in A.Zygmund, Trigonometrical Series, p. 80. First let us suppose that all the functions $f_{mn}(x,y)$ are non negative, bounded and monotone (p.7).

Let $R(P_\mu)$ be the enumerable sequence of all the rational points of the domain Δ together with the points $(-\pi,-\pi)$, $(-\pi,\pi)$, $(\pi,-\pi)$, and (π,π). Write the single sequence $f_\mu(x,y)$, where $\mu = m_i, n_j$ such that both m and $n \to \infty$. The sequence $f_\mu(x,y)$ being bounded we can find a subsequence (S_1) $p_1^{(1)}, p_2^{(1)}, -----, p_k^{(1)}, --$ of indices such that $f_{p_k^{(1)}}(P_1)$ converges. Rejecting the first term $p_1^{(1)}$ we can find from the remaining indices $p_2^{(1)}, p_3^{(1)}, ---- ----$ a subsequence (S_2) $p_1^{(2)}, p_2^{(2)}, ------ p_k^{(2)}, ---$ such that $f_{p_k^{(2)}}(P_2)$ converges. Rejecting $p_1^{(2)}$ we choose from among the rest a subsequence (S_3) $p_1^{(3)}, p_2^{(3)}, ----$ such that $f_{p_k^{(3)}}(P_3)$ converges and so on.

The sequence $p_1^{(1)}, p_1^{(2)}, p_1^{(3)}, ----$ being from some place onwards a subsequence of every S_i we see that $f_{p_1^{(k)}}(P)$ converges, at least for the set $R(P_\mu)$, to a limit $F(P)$ with the same properties as $f_{mn}(x,y)$ over the set where it exists. Consider, for example,

$$f_{mn}(x_{i+1}, y_{j+1}) - f_{mn}(x_{i+1}, y_j) - f_{mn}(x_i, y_{j+1}) + f_{mn}(x_i, y_j) \geqq 0 .$$

Using only points of R and passing to the limit over the sequence $p_1^{(k)}$ we have

$$F(x_{i+1}, y_{j+1}) - F(x_{i+1}, y_j) - F(x_i, y_{j+1}) + F(x_i, y_j) \geqq 0 .$$

The functions $f_{mn}(\pi, y)$, $f_{mn}(-\pi, y)$, $f_{mn}(x, \pi)$, and $f_{mn}(x, -\pi)$ satisfy the conditions of the lemma given by Zygmund. Hence there exists some sequence of indices which we shall again designate by $p_1^{(k)}$ such that

$$\lim_{k \to \infty} f_{p_1^{(k)}}(x,y) = F(x,y)$$

for the set R together with the points on the boundaries of Δ. Let this set be $\overline{R}$.

For any point interior to Δ let

$$d(x,y) = \lim_{\substack{P \to x+0 \\ y+0}} F(P) - \lim_{\substack{P \to x-0 \\ y-0}} F(P),$$

where $P(r,\rho)$ is a point of $\overline{R}$. It follows from the properties of $F(P)$ that $\overline{R}$ in

$$[F(r_2,\rho_2) - F(r_1,\rho_1)] + [F(r_3,\rho_3) - F(r_2,\rho_2)] + \cdots$$

$$\cdots + [F(r_n,\rho_n) - F(r_{n-1},\rho_{n-1})] = F(r_n,\rho_n) - F(r_1,\rho_1) < V.$$

Let rational points r_n, ρ_n be chosen so that

$$r_1 < x_1 < r_2 < x_2 < r_3 < \cdots < r_n < x_n < r_{n+1},$$

$$\rho_1 < y_1 < \rho_2 < y_2 < \rho_3 < \cdots < \rho_n < y_n < \rho_{n+1},$$

then

$$d(x_1,y_1) + d(x_2,y_2) + \cdots + d(x_n,y_n) =$$

$$\Big[\lim_{\substack{P \to x_1+0 \\ y_1+0}} F(P) - \lim_{\substack{P \to x_1-0 \\ y_1-0}} F(P)\Big] + \cdots + \Big[\lim_{\substack{P \to x_n+0 \\ y_n+0}} F(P) - \lim_{\substack{P \to x_n-0 \\ y_n-0}} F(P)\Big]$$

$$< [F(r_2,\rho_2) - F(r_1,\rho_1)] + \cdots + [F(r_{n+1},\rho_{n+1}) - F(r_n,\rho_n)] < V.$$

It follows from this that the number of points where $d(x,y) \geqq \epsilon > 0$ is finite. Let $\overline{\Pi}$ be the at most enumerable set of points for which $d(x,y) > 0$. For an $\eta > 0$ and a point (x,y) not contained in $\overline{\Pi}$ we can find two rational points (r_1, ρ_1) and (r_2, ρ_2) such that for

$$r_1 < x < r_2$$
$$\rho_1 < y < \rho_2$$

we have

$$0 \leqq F(r_2,\rho_2) - F(r_1,\rho_1) < \eta$$

and

$$f_{p_1^{(k)}}(r_1,\rho_1) \leqq f_{p_1^{(k)}}(x,y) \leqq f_{p_1^{(k)}}(r_2,\rho_2).$$

Then passing to the limit we have

$$F(r_1,\rho_1) \leqq \lim_{k\to\infty} f_{p_1^{(k)}}(x,y) \leqq F(r_2,\rho_2).$$

The oscillation of $f_{p_1^{(k)}}(x,y)$ does not exceed η and hence the sequence converges. We now consider the enumerable set Π of points at which $f_{p_1^{(k)}}(x,y)$ diverges. We repeat with Π the same argument as with R and find a subsequence μ_k of $p_1^{(k)}$ such that $f_{\mu_k}(x,y)$ converges in Π, hence everywhere in Δ. The limit $F(x,y)$ possesses the same properties as $f_{mn}(x,y)$, that is, it is bounded, non negative, and monotone. In the general case, since $f_{mn}(x,y)$ is of bounded variation H in Δ, we can write

$$f_{mn}(x,y) = f_{mn}(-\pi,-\pi) + P_{mn}(x,y) + p_{mn}^{(1)}(-\pi,y) + p_{mn}^{(2)}(x,-\pi) - N_{mn}(x,y) - n_{mn}^{(1)}(x,-\pi) - n_{mn}^{(2)}(-\pi,y),$$

where $P_{mn}(x,y)$, $p_{mn}^{(1)}(-\pi,y)$, and $p_{mn}^{(2)}(x,-\pi)$ represent positive variations of functions of uniformly bounded variation and are thus uniformly bounded.(5)

(5) E.W.Hobson, The Theory of Functions of a Real Variable, vol. 1, p.325-345.

Thus

$$f_{mn}(x,y) = f_{mn}(-\pi,-\pi) + \overline{P}_{mn}(x,y) - \overline{N}_{mn}(x,y)$$

where $\overline{P}_{mn}(x,y)$ and $\overline{N}_{mn}(x,y)$ are non negative, uniformly bounded, and monotone functions such as those first considered. Note that in the above equations the terms $N_{mn}(x,y)$, $n^{(1)}_{mn}(x,-\pi)$, and $n^{(2)}_{mn}(-\pi,y)$ are negative variations and thus uniformly bounded. Let us suppose that we can find a sequence μ_k such that $f_{\mu_k}(-\pi,-\pi)$ converges to a finite limit. From μ_k we choose a subsequence μ'_k such that $\overline{P}_{\mu'_k}(x,y)$ converges and from μ'_k a subsequence μ''_k such that $\overline{N}_{\mu''_k}(x,y)$ and therefore $f_{\mu''_k}(x,y)$ converges. That

$$F(x,y) = \lim_{k\to\infty} f_{\mu''_k}(x,y)$$

is of bounded variation H follows from the fact that

$$F(x,y) = \lim_{k\to\infty} f_{\mu''_k}(-\pi,-\pi) + \lim_{k\to\infty} \overline{P}_{\mu''_k}(x,y) - \lim_{k\to\infty} \overline{N}_{\mu''_k}(x,y),$$

that is, $F(x,y)$ is the difference of two bounded monotone functions. If our assumption concerning $f_{mn}(-\pi,-\pi)$ does not hold, then $|f_{mn}(-\pi,-\pi)| \to \infty$. Since the functions $\overline{P}_{mn}(x,y)$ and $\overline{N}_{mn}(x,y)$ are uniformly bounded, $|f_{mn}(x,y)|$ diverges uniformly to $+\infty$ as $m,n \to \infty$.

LEMMA 5. Let

(1) $\overline{T}_{mn}(x,y) = \int_\Delta f(x+t_1, y+t_2)\,\sigma_{mn}(t_1,t_2)\,dt_1\,dt_2$

be a sequence which corresponds to the function $f(x,y)$.

If the trigonometric sequence $\sigma_{mn}(t_1, t_2)$ is such that

$$\iint_\Delta |\sigma_{mn}(t_1, t_2)|\, dt_1\, dt_2$$

is not bounded, then there exists a continuous function $\phi(x,y)$ such that $\overline{\lim}_{m,n\to\infty} |T_{mn}(x,y)|$, for the function $\phi(x,y)$, equals $+\infty$ at $x = 0$, $y = 0$, that is,

$$\overline{\lim}_{m,n\to\infty} \left| \iint_\Delta \phi(t_1,t_2)\, \sigma_{mn}(t_1,t_2)\, dt_1\, dt_2 \right| = \infty.$$

Let

$$\iint_\Delta |\sigma_{mn}(t_1,t_2)|\, dt_1\, dt_2 = \omega_{mn}.$$

We can choose a subsequence ω_{r_p} of ω_{mn} such that

$$\lim_{p\to\infty} \iint_\Delta |\sigma_{r_p}(t_1,t_2)|\, dt_1\, dt_2 = \infty$$

Let

$$v_{r_p}(t_1,t_2) = +1, -1, 0$$

according as

$$\sigma_{r_p}(t_1,t_2) \text{ is } +, -, 0.$$

We then have

$$|\sigma_{r_p}(t_1,t_2)| = v_{r_p}(t_1,t_2)\, \sigma_{r_p}(t_1,t_2)$$

and

$$\omega_{r_p} = \int_\Delta v_{r_p}(t_1,t_2)\, \sigma_{r_p}(t_1,t_2)\, dt_1\, dt_2 .$$

By comparison with (1) we see that ω_{r_p} is the r_p th term of the sequence corresponding to the function $v_{r_p}(t_1, t_2)$ at the point (0,0).

We next construct a sequence of continuous functions

$$g_{r_1}(t_1,t_2),\ g_{r_2}(t_1,t_2),\ -\ -\ -\ -\ -$$

such that

$$|g_{r_n}(t_1,t_2)| \leqq 1$$

and

(2) $$\iint_\Delta \{v_{r_p}(t_1,t_2) - g_{r_p}(t_1,t_2)\}^2 dt_1\,dt_2 < \delta_p$$

where p = 1,2,3, --- and δ_p is a sequence of positive numbers to be determined later. The construction of a function such as $g_{r_p}(t_1, t_2)$ is described in E.W.Hobson, The Theory of Functions of a Real Variable, vol. 1, p.632-634.

Consider

$$\left|\iint_\Delta g_{r_p}(t_1,t_2)\,\sigma_{r_p}(t_1,t_2)\,dt_1\,dt_2\right| =$$

$$\left|\iint_\Delta v_{r_p}(t_1,t_2)\,\sigma_{r_p}(t_1,t_2)\,dt_1\,dt_2 - \iint_\Delta \{v_{r_p}(t_1,t_2) - g_{r_p}(t_1,t_2)\}\,\sigma_{r_p}(t_1,t_2)\,dt_1\,dt_2\right|$$

$$\geqq \omega_{r_p} - \left[\iint_\Delta \sigma_{r_p}^2(t_1,t_2)\,dt_1\,dt_2 \cdot \iint_\Delta \{v_{r_p}(t_1,t_2) - g_{r_p}(t_1,t_2)\}^2 dt_1\,dt_2\right]^{\frac{1}{2}}.$$

Let δ_p be so chosen that

$$\frac{1}{2}\,\omega_{r_p} > \left\{\delta_p \iint_\Delta \sigma_{r_p}^2(t_1,t_2)\,dt_1\,dt_2\right\}^{\frac{1}{2}}.$$

Then

(3) $\left| \iint_{\Delta} g_{r_p}(t_1,t_2)\, \sigma_{r_p}(t_1,t_2)\, dt_1\, dt_2 \right| > \frac{1}{2}\, \omega_{r_p}$.

The r_1 th term of the sequence corresponding to the function $\phi^{(1)}(t_1, t_2) \equiv g_{r_1}(t_1, t_2)$ at the point (0,0) is greater than $\frac{1}{2}\omega_{r_1}$.

If the sequence $T_{mm}(0,0)$ is convergent, choose the number $G^{(1)}$ so that

$$\left| T_{mm}(0,0) \right| < G^{(1)}$$

Let $r' = r_1$ and from the sequence r_2, r_3 - - ---------choose r^2 so that

$$\omega_{r^2} > 6 \cdot 4\,(G^{(1)} + 1).$$

Take $\phi^{(2)}(t_1,t_2) = g_{r^1}(t_1,t_2) + \frac{1}{4}\, g_{r^2}(t_1,t_2)$.

If $T_{mm}(0,0)$ converges for $\phi^{(2)}(t_1,t_2)$ choose $G^{(2)}$ so that

$$\left| T_{mm}(0,0) \right| < G^{(2)}.$$

Then choose

$$\omega_{r^3} > 6 \cdot 4^2\,(G^{(2)} + 2)$$

and form

$$\phi^{(3)}(t_1,t_2) = g_{r^1}(t_1,t_2) + \frac{1}{4}\, g_{r^2}(t_1,t_2) + \frac{1}{4^2}\, g_{r^3}(t_1,t_2).$$

Proceeding in this manner we form a function

$$\phi^{(q-1)}(t_1,t_2) = g_{r^1}(t_1,t_2) + \frac{1}{4}\, g_{r^2}(t_1,t_2) + - - - - + \frac{1}{4^{q-2}}\, g_{r^{q-1}}(t_1,t_2).$$

If $T_{mm}(0,0)$ corresponding to $\phi^{(q-1)}(t_1,t_2)$ is convergent choose a number $G^{(q-1)}$ so that

$$(4) \qquad |T_{mm}(0,0)| < G^{(q-1)}$$

Then choose an index r^q from r_p so that

$$\omega_{r^q} > 6 \cdot 4^{q-1}\left(G^{(q-1)} + q - 1\right).$$

If this process does not come to an end by discovery of a value of q for which $T_{mm}(0,0)$ corresponding to $\phi^{(q)}(t_1,t_2)$ does not converge we consider the function $\phi(t_1,t_2)$ given by the infinite series

$$\phi(t_1,t_2) = g_{r^1}(t_1,t_2) + \frac{1}{4} g_{r^2}(t_1,t_2) + \cdots + \frac{1}{4^{q-1}} g_{r^q}(t_1,t_2) + \cdots$$

This series converges in Δ uniformly because the functions $g_{r^q}(t_1,t_2)$ are all equal to or less than one in absolute value. Thus $\phi(t_1,t_2)$ exists and is continuous in Δ.

In order to calculate $T_{mm}(0,0)$ corresponding to $\phi(t_1,t_2)$ we write

$$\phi(t_1,t_2) = \left\{ g_{r^1}(t_1,t_2) + \cdots + \frac{1}{4^{q-2}} g_{r^{q-1}}(t_1,t_2)\right\} + \frac{1}{4^{q-1}} g_{r^q}(t_1,t_2)$$

$$+ \left\{ \frac{1}{4^q} g_{r^{q+1}}(t_1,t_2) + \cdots \right\}.$$

Thus

$$|T_{r^q}(0,0)| = \left|\iint_\Delta \phi(t_1,t_2)\,\sigma_{r^q}(t_1,t_2)\,dt_1\,dt_2\right| =$$

$$= \left|\iint_\Delta \left\{ g_{r^1}(t_1,t_2) + \cdots + \frac{1}{4^{q-2}} g_{r^{q-1}}(t_1,t_2)\right\} \sigma_{r^q}(t_1,t_2)\,dt_1\,dt_2 + \right.$$

$$+ \int_{\Delta} \frac{1}{4^{q-1}} g_{rq}(t_1,t_2)\, \sigma_{rq}(t_1,t_2)\, dt_1\, dt_2 +$$

$$+ \int_{\Delta} \Big\{ \frac{1}{4^q} g_{rq+1}(t_1,t_2) + \cdots \Big\} \sigma_{rq}(t_1,t_2)\, dt_1\, dt_2 \Big| .$$

According to (4) we have

$$\Big| \int_{\Delta} \Big\{ g_{r1}(t_1,t_2) + \cdots + \frac{1}{4^{q-2}} g_{rq-1}(t_1,t_2) \Big\} \sigma_{rq}(t_1,t_2)\, dt_1\, dt_2 \Big| < G^{(q-1)} .$$

According to (3) we have

$$\Big| \int_{\Delta} \frac{1}{4^{q-1}} g_{rq}(t_1,t_2)\, \sigma_{rq}(t_1,t_2)\, dt_1\, dt_2 \Big| > \frac{1}{4^{q-1}} \cdot \frac{1}{2} \omega_{rq} .$$

Also

$$\Big| \int_{\Delta} \Big\{ \frac{1}{4^q} g_{rq+1}(t_1,t_2) + \cdots \Big\} \sigma_{rq}(t_1,t_2)\, dt_1\, dt_2 \Big| <$$

$$< \Big| \frac{1}{4^q} g_{rq+1}(t_1,t_2) + \cdots \Big| \int_{\Delta} \big| \sigma_{rq}(t_1,t_2) \big|\, dt_1\, dt_2 < \frac{1}{3 \cdot 4^{q-1}} \omega_{rq} .$$

Hence

$$\big| T_{rq}(0,0) \big| > \frac{1}{2 \cdot 4^{q-1}} \omega_{rq} - G^{(q-1)} - \frac{1}{3 \cdot 4^{q-1}} \omega_{rq}$$

$$= \frac{1}{6 \cdot 4^{q-1}} \omega_{rq} - G^{(q-1)}$$

But

$$\omega_{rq} > 6 \cdot 4^{q-1} \big(G^{(q-1)} + q - 1 \big)$$

Hence

$$/T_{rq}(0,0)/ > q-1$$

It follows that for the indices $r', r'', ---$ the sequence T_{mn} (0,0) corresponding to ϕ(x,y) increases indefinitely, and hence

$$\overline{\lim_{m,n\to\infty}} \left/ \int_{\Delta} \phi(t_1,t_2)\,\sigma_{mn}(t_1,t_2)\,dt_1\,dt_2 \right/ = \infty$$

This lemma follows closely a theorem for functions f(x) given in E.W.Hobson, The Theory of Functions of a Real Variable, vol. 11, p.757.

4. Theorems

The first Cesàro mean of the Fourier series

$$\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \phi_{mn}(x,y)$$

is defined as

$$\sigma_{mn}(x,y) = \sum_{k=0}^{k=m-1}\sum_{l=0}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\phi_{kl}(x,y)$$

whether the constant and the terms in one variable exist or not. The Cesàro mean can also be written in the form

$$\sigma_{mn}(x,y) = \frac{1}{\pi^2}\int_\Delta f(x+t_1, y+t_2)\, C_{mn}(t_1,t_2)\, dt_1\, dt_2,$$

where

$$C_{mn}(t_1,t_2) = \left[\tfrac{1}{2} + \left(\tfrac{m-1}{m}\right)\cos t_1 + \left(\tfrac{m-2}{m}\right)\cos 2t_1 + \cdots + \tfrac{1}{m}\cos(m-1)t_1\right]\cdot$$

$$\cdot\left[\tfrac{1}{2} + \left(\tfrac{n-1}{n}\right)\cos t_2 + \left(\tfrac{n-2}{n}\right)\cos 2t_2 + \cdots + \tfrac{1}{n}\cos(n-1)t_2\right] =$$

$$= \frac{1}{2m}\left\{\frac{\sin \frac{m}{2}t_1}{\sin \frac{1}{2}t_1}\right\}^2 \cdot \frac{1}{2n}\left\{\frac{\sin \frac{n}{2}t_2}{\sin \frac{1}{2}t_2}\right\}^2.$$

Also

$$\int_\Delta C_{mn}(t_1,t_2)\,dt_1\,dt_2 = \int_\Delta |C_{mn}(t_1,t_2)|\,dt_1\,dt_2 = \pi^2$$

Theorem 1. The necessary and sufficient condition that the series

$$(1) \qquad \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \phi_{mn}(x,y)$$

be a Fourier series of class C_2 is that the Cesàro mean $\sigma_{mn}(x,y)$ be uniformly bounded.

If (1) is a Fourier series of class C_2 then f(x,y) is bounded. Since

$$\sigma_{mn}(x,y) = \frac{1}{\pi^2}\int_\Delta f(x+t_1, y+t_2)\, C_{mn}(t_1,t_2)\, dt_1\, dt_2$$

and

$$\int_\Delta |C_{mn}(t_1,t_2)|\, dt_1\, dt_2 = \pi^2$$

it follows immediately from Lemma 2 that $\sigma_{mn}(x,y)$ is uniformly bounded.

If

$$|\sigma_{mn}(x,y)| \leqq G$$

then

$$\frac{1}{\pi^2}\int_\Delta\int \{\sigma_{mn}(x,y)\}^2 dx\,dy \leqq 4G^2.$$

When $\sigma_{mn}(x,y)$ is squared and integrated term by term we obtain

$$\sum_{k=0}^{k=m-1}\sum_{l=0}^{l=n-1} \theta\left\{a_{kl}^2+b_{kl}^2+c_{kl}^2+d_{kl}^2\right\}\left(\frac{m-k}{m}\right)^2\left(\frac{n-l}{n}\right)^2 < 4G^2,$$

where

$\theta = 1$ when $k>0, l>0$; $\theta = 2$ when $k=0$ or $l=0$;

and $\theta = 4$ when $k=0$, $l=0$.

For,

$$\frac{1}{\pi^2}\int_\Delta \cos m_1x\cos n_1y\cos m_2x\cos n_2y\,dx\,dy = \begin{cases} 1 & m_1=m_2,\ n_1=n_2 \\ 0 & m_1\neq m_2,\ n_1\neq n_2 \end{cases}$$

$$\frac{1}{\pi^2}\int_\Delta \cos m_2x\cos m_1x\cos n_1y\,dx\,dy = 0,$$

$$\frac{1}{\pi^2}\int_\Delta \cos m_2x\cos m_1x\,dx\,dy = \begin{cases} 2 & m_1=m_2 \\ 0 & m_1\neq m_2 \end{cases}$$

$$\frac{1}{\pi^2}\int_\Delta a_{00}^2\,dx\,dy = 4a_{00}^2,$$

with similar formulas for the other terms.

Then we can write

$$\sum_{k=0}^{k=m-1}\sum_{l=0}^{l=n-1}(a_{kl}^2+b_{kl}^2+c_{kl}^2+d_{kl}^2)\left(\frac{m-k}{m}\right)^2\left(\frac{n-l}{n}\right)^2 < 4G^2.$$

Also, let us write

$$\left(\frac{m-k}{m}\right)^2\left(\frac{n-l}{n}\right)^2 = \left(1-\frac{k}{m}-\frac{l}{n}+\frac{kl}{mn}\right)^2.$$

For any value of k, l say k_0, l_0 we can choose an m and n so large that

$$\left(1-\frac{k}{m}-\frac{l}{n}+\frac{kl}{mn}\right)^2 > \frac{1}{2}$$

Then

$$\frac{1}{2}\sum_{k=0}^{k=k_0}\sum_{l=0}^{l=l_0}(a_{kl}^2+b_{kl}^2+c_{kl}^2+d_{kl}^2) <$$

$$< \sum_{k=0}^{k=k_0}\sum_{l=0}^{l=l_0}(a_{kl}^2+b_{kl}^2+c_{kl}^2+d_{kl}^2)\left(1-\frac{k}{m}-\frac{l}{n}+\frac{kl}{mn}\right)^2 <$$

$$< \sum_{k=0}^{k=m-1}\sum_{l=0}^{l=n-1}(a_{kl}^2+b_{kl}^2+c_{kl}^2+d_{kl}^2)\left(1-\frac{k}{m}-\frac{l}{n}+\frac{kl}{mn}\right)^2 < 4G^2.$$

Thus

$$\sum_{k=0}^{k=m}\sum_{l=0}^{l=n}(a_{kl}^2+b_{kl}^2+c_{kl}^2+d_{kl}^2) < 8G^2.$$

Since the series is non decreasing and bounded it is convergent. (6)

(6) E.W.Hobson, The Theory of Functions of a Real Variable, vol. 1, p.414.

Consequently there exists according to the Riesz-Fischer theorem a function g(x,y) whose square is integrable L and whose Fourier constants are those of the series (7).

(7) Ibid., vol. 11, p. 719.

Since the function g(x,y) is integrable L^2 the first

Cesàro mean $\sigma_{mn}(x,y)$ converges almost everywhere to the value $g(x,y)$. (8)

(8) A. Zygmund, Fundamenta Mathematicae, vol. 23, p.148.

Thus $g(x,y)$ is bounded almost everywhere in Δ . But according to Lemma 1 there is a function $g^*(x,y)$, equivalent to $g(x,y)$, which is bounded everywhere in Δ . Thus the series (1) is a Fourier series, of class C_2.

THEOREM 2. The necessary and sufficient condition that the series

$$(1) \qquad \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \phi_{mn}(x,y)$$

be a Fourier series of class C_3 or C_4 is that the Cesàro mean $\sigma_{mn}(x,y)$ possess the class property uniformly.

The necessity follows immediately from Lemma 2.

If $\sigma_{mn}(x,y)$ possesses the class property C or R uniformly then according to Lemma 3 $\sigma_{mn}(x,y)$ is uniformly bounded. As before we can infer the existence of a function $g(x,y)$, integrable L, whose Fourier series coincides with (1). Then $\sigma_{mn}(x,y)$ converges almost everywhere to $g(x,y)$ which fulfills almost everywhere in Δ the condition C or R since $\sigma_{mn}(x,y)$ fulfills the same condition uniformly. Finally, according to Lemma 1, there is a function $g^*(x,y)$, equivalent to $g(x,y)$, which fulfills the condition C or R everywhere in Δ. Thus the series (1) is a Fourier series of class C_3 or C_4.

THEOREM 3. The necessary and sufficient condition that the series

(1) $$\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \phi_{mn}(x,y)$$

be a Fourier series of class C_5 is that the first Cesàro mean $\sigma_{mn}(x,y)$ be of uniformly bounded variation H.

The necessity follows immediately from Lemma 2.

If $\sigma_{mn}(x,y)$ is of uniformly bounded variation H it follows from Lemma 3 that $\sigma_{mn}(x,y)$ is uniformly bounded. Then there exists some subsequence m_α, n_β of m,n such that

$$\lim_{\alpha,\beta\to\infty} \sigma_{m_\alpha n_\beta}(-\pi,-\pi)$$

exists. It then follows from Lemma 4 that there is a subsequence m_i, n_j of m,n such that

$$\lim_{i,j\to\infty} \sigma_{m_i n_j}(x,y) = f(x,y)$$

where f(x,y) is of bounded variation H everywhere in Δ.

$$a_{kl}\left(1-\frac{k}{m_i}\right)\left(1-\frac{l}{n_j}\right) = \frac{1}{\pi^2}\int_\Delta \sigma_{m_i n_j}(x,y)\cos kx \cos ly \, dx\, dy,$$

and letting $i,j\to\infty$ we obtain

$$a_{kl} = \frac{1}{\pi^2}\int_\Delta f(x,y)\cos kx\cos ly\,dx\,dy.$$

Similarly

$$a_{k0}\left(1-\frac{k}{m_i}\right) = \frac{1}{2\pi^2}\int_\Delta \sigma_{m_i n_j}(x,y)\cos kx\,dx\,dy.$$

$$a_{k0} = \frac{1}{2\pi^2}\int_\Delta f(x,y)\cos kx\,dx\,dy$$

and so on.

Thus the series (1) is a Fourier series of class C_5.

We can now combine the above three theorems into

THEOREM 4. The necessary and sufficient condition that the series

(1) $$\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}\phi_{mn}(x,y)$$

be a Fourier series of class C_γ ($\gamma = 2,3,4,5$) is that the Cesàro mean $\sigma_{mn}(x,y)$ possess the class property uniformly.

THEOREM 5. The necessary and sufficient condition that the series

(1) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\phi_{mn}(x,y)$$

without a constant and without terms in one variable, be the Fourier series of a function f(x,y) of bounded variation H is that the integral of the absolute value of the derivative with respect to x and y, of the Cesaro mean be uniformly bounded, that is,

$$\iint_\Delta \left|\sigma''_{mn}(x,y)\right| dx\,dy < G.$$

If the series (1) is the Fourier series of a function f(x,y) of bounded variation H, then $\sigma''_{mn}(x,y)$ is the Cesàro mean of the Fourier Stieltjes series of f(x,y) (p.15).

$$\sigma''_{mn}(x,y) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\phi''_{mn}(x,y)$$

$$= \frac{1}{\pi^2}\int_\Delta C_{mn}(x-t_1,\, y-t_2)\, df(t_1,t_2).$$

Since f(x,y) is of bounded variation H it is the difference of two bounded monotone functions P(x,y) and Q(x,y). Thus

$$\sigma''_{mn}(x,y) = \frac{1}{\pi^2}\int_\Delta C_{mn}(x-t_1, y-t_2)\,dP(t_1,t_2) - \frac{1}{\pi^2}\int_\Delta C_{mn}(x-t_1, y-t_2)\,dQ(t_1,t_2).$$

Since

$$\Delta_{11}P(x_i, y_j) \geqq 0, \quad \Delta_{11}Q(x_i, y_j) \geqq 0, \quad C_{mn}(x,y) \geqq 0,$$

we have

$$|\sigma''_{mn}(x,y)| < \frac{1}{\pi^2}\int_\Delta C_{mn}(x-t_1, y-t_2)\,dP(t_1,t_2) + \frac{1}{\pi^2}\int_\Delta C_{mn}(x-t_1, y-t_2)\,dQ(t_1,t_2).$$

Let us integrate each side of the inequality and interchange the order of the integration on the right.

$$\iint_\Delta |\sigma''_{mn}(x,y)|\,dx\,dy < \frac{1}{\pi^2}\int_\Delta dP(t_1,t_2)\int_\Delta C_{mn}(x-t_1, y-t_2)\,dx\,dy +$$

$$+ \frac{1}{\pi^2}\int_\Delta dQ(t_1,t_2)\int_\Delta C_{mn}(x-t_1, y-t_2)\,dx\,dy$$

$$= \int_\Delta dP(t_1,t_2) + \int_\Delta dQ(t_1,t_2)$$

$$= P(\pi,\pi) - P(-\pi,\pi) - P(\pi,-\pi) + P(-\pi,-\pi) + Q(\pi,\pi)$$

$$- Q(-\pi,\pi) - Q(\pi,-\pi) + Q(-\pi,-\pi) < 8|f(x,y)| < 8K.$$

The above operation is permissible since $C_{mn}(x,y)$ is continuous. Consider

$$\phi(x,y) = \int_\Delta g(x-u, y-v)\,df(u,v)$$

where g(x,y) is continuous and f(u,v) is bounded and monotone; and both are periodic.

Since g(x,y) is continuous we have

$$\left| g(\alpha-u,\beta-v)-g(\alpha'-u,\beta'-v)\right| \leq \epsilon, \quad \text{all } u, v.$$

when

$$|\alpha-\alpha'| \leq \delta, \quad |\beta-\beta'| \leq \delta.$$

Then

$$\left|\phi(\alpha,\beta)-\phi(\alpha',\beta')\right| \leq \int_\Delta \left| g(\alpha-u,\beta-v)-g(\alpha'-u,\beta'-v)\right| d f(u,v)$$

$$< \epsilon \int_\Delta d f(u,v)$$

$$= \epsilon\left[f(\pi,\pi)-f(\pi,-\pi)-f(-\pi,\pi)+f(-\pi,-\pi)\right] < \epsilon G.$$

Thus ϕ (x,y) is continuous and has a Riemann integral. Let the intervals of integration be divided by points x_μ and y_ν, where $x_0 = -\pi$, $x_m = \pi$ and $y_0 = -\pi$, $y_n = \pi$, so that

$$\sum_{\mu=0}^{\mu=m-1}\sum_{\nu=0}^{\nu=n-1} g(\alpha_\mu-u,\beta_\nu-v)(x_{\mu+1}-x_\mu)(y_{\nu+1}-y_\nu) =$$

$$= \int_\Delta g(x-u,y-v)\,dx\,dy \pm \epsilon,$$

and

$$\sum_{\mu=0}^{\mu=m-1}\sum_{\nu=0}^{\nu=n-1} \int_\Delta g(\alpha_\mu-u,\beta_\nu-v)\,d f(u,v)\cdot(x_{\mu+1}-x_\mu)(y_{\nu+1}-y_\nu) =$$

$$= \int_\Delta \sum_{\mu=0}^{\mu=m-1}\sum_{\nu=0}^{\nu=n-1} g(\alpha_\mu-u,\beta_\nu-v)(x_{\mu+1}-x_\mu)(y_{\nu+1}-y_\nu)\,d f(u,v) =$$

$$= \int_\Delta \phi(x,y)\,dx\,dy \pm \epsilon,$$

where

$$x_{\mu+1} - x_\mu \leqq \delta\,, \quad y_{\nu+1} - y_\nu \leqq \delta\,,$$

$$x_\mu \leqq \alpha_{\mu+1} \leqq x_{\mu+1}\,, \quad y_\nu \leqq \beta_{\nu+1} \leqq y_{\nu+1}\,.$$

Then by substitution we obtain

$$\int_\Delta \left\{ \int_\Delta g(x-u, y-v)\,dx\,dy \mp \epsilon \right\} df(u,v) = \int_\Delta \phi(x,y)\,dx\,dy \mp \epsilon,$$

$$\int_\Delta g(x-u, y-v)\,dx\,dy \int_\Delta df(u,v) = \int_\Delta \phi(x,y)\,dx\,dy \mp \epsilon \pm \epsilon \int_\Delta df(u,v).$$

Then since ϵ is arbitarily small and $\int_\Delta df(u,v)$ is bounded we have

$$\int_\Delta \phi(x,y)\,dx\,dy = \int_\Delta g(x-u, y-v)\,dx\,dy \int_\Delta df(u,v).$$

Now we assume that

$$\int_\Delta \left| \sigma''_{mn}(x,y) \right| dx\,dy < G.$$

Let

$$F_{mn}(x,y) = \int_{-\pi}^{x}\int_{-\pi}^{y} \sigma''_{mn}(x,y)\,dx\,dy.$$

$$\Delta_{11} F_{mn}(x_i, y_j) = F_{mn}(x_{i+1}, y_{j+1}) - F_{mn}(x_{i+1}, y_j) - F_{mn}(x_i, y_{j+1}) + F_{mn}(x_i, y_j)$$

$$= \int_{-\pi}^{x_{i+1}}\int_{-\pi}^{y_{j+1}} \sigma''_{mn}(x,y)\,dx\,dy - \int_{-\pi}^{x_{i+1}}\int_{-\pi}^{y_j} \sigma''_{mn}(x,y)\,dx\,dy -$$

$$- \int_{-\pi}^{x_i}\int_{-\pi}^{y_{j+1}} \sigma''_{mn}(x,y)\,dx\,dy + \int_{-\pi}^{x_i}\int_{-\pi}^{y_j} \sigma''_{mn}(x,y)\,dx\,dy$$

$$= \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \sigma''_{mn}(x,y)\,dx\,dy.$$

Then for any network on Δ we have

$$\sum_{i=0}^{i=\mu-1}\sum_{j=0}^{j=\nu-1} |\Delta_{11} F_{mn}(x_i, y_j)| \leqq \sum_{i=0}^{i=\mu-1}\sum_{j=0}^{j=\nu-1} \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} |\sigma''_{mn}(x,y)|\,dx\,dy$$

$$\leqq \iint_{\Delta} |\sigma''_{mn}(x,y)|\,dx\,dy < G.$$

Also

$$\Delta_{10} F_{mn}(x_i, y) = \int_{x_i}^{x_{i+1}}\int_{-\pi}^{y} \sigma''_{mn}(x,y)\,dx\,dy$$

$$\sum_{i=0}^{i=\mu-1} |\Delta_{10} F_{mn}(x_i, y)| \leqq \sum_{i=0}^{i=\mu-1} \int_{x_i}^{x_{i+1}}\int_{-\pi}^{y} |\sigma''_{mn}(x,y)|\,dx\,dy$$

$$\leqq \int_{-\pi}^{\pi}\int_{-\pi}^{y} |\sigma''_{mn}(x,y)|\,dx\,dy < \iint_{\Delta} |\sigma''_{mn}(x,y)|\,dx\,dy < G,$$

for all $y \leqq \pi$.

Similarly

$$\sum_{j=0}^{j=\nu-1} |\Delta_{01} F_{mn}(x, y_j)| < G.$$

Thus the sequence $F_{mn}(x,y)$ is of uniformly bounded variation H. Since $F_{mn}(-\pi,-\pi) = 0$ it follows from Lemma 4 that there exists a subsequence m_i, n_j of m,n such that

$$\lim_{i,j\to\infty} F_{m_i n_j}(x,y) = F(x,y)$$

where F(x,y) is a function of bounded variation H. Note that since $\sigma''_{mn}(x,y)$ contains no constant and no terms in one variable we have

$$F_{mn}(\pi,\pi) = F_{mn}(\pi,y) = F_{mn}(x,\pi) = 0.$$

The functions $F_{mn}(x,y)$ are periodic of period 2π in x and y, for

$$F_{mn}(x+2\pi,y)-F_{mn}(x,y)=\int_{-\pi}^{x+2\pi}\int_{-\pi}^{y}\sigma''_{mn}(x,y)\,dx\,dy-\int_{-\pi}^{x}\int_{-\pi}^{y}\sigma''_{mn}(x,y)\,dx\,dy=$$
$$=\int_{x}^{x+2\pi}\int_{-\pi}^{y}\sigma''_{mn}(x,y)\,dx\,dy=\int_{-\pi}^{\pi}\int_{-\pi}^{y}\sigma''_{mn}(x,y)\,dx\,dy=0.$$

Similarly $F_{mn}(x,y+2\pi)-F_{mn}(x,y)=0$. It follows that $F(x,y)$ is also periodic.

By a consideration of the terms of $\sigma''_{mn}(x,y)$ it is apparent that

$$kl\,d_{kl}\left(1-\frac{k}{m}\right)\left(1-\frac{l}{n}\right)=\frac{1}{\pi^2}\int_{\Delta}\sigma''_{mn}(x,y)\cos kx\cos ly\,dx\,dy,\quad k>0,\ l>0.$$

Let

$$F^{(1)}_{mn}(x,y)=\int_{-\pi}^{x}\sigma''_{mn}(x,y)\,dx\,\cancel{dy},\qquad F^{(2)}_{mn}(x,y)=\int_{-\pi}^{y}\sigma''_{mn}(x,y)\,\cancel{dx}\,dy.$$

Then

$$F_{mn}(x,y)=\int_{-\pi}^{y}F^{(1)}_{mn}(x,y)\,dy=\int_{-\pi}^{x}F^{(2)}_{mn}(x,y)\,dx.$$

$$(2)\quad kl\,d_{kl}\left(1-\frac{k}{m}\right)\left(1-\frac{l}{n}\right)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\cos kx\left\{\int_{-\pi}^{\pi}\sigma''_{mn}(x,y)\cos ly\,dy\right\}dx.$$

Integrating by parts the integral in braces we have

$$\int_{-\pi}^{\pi}\sigma''_{mn}(x,y)\cos ly\,dy=\cos ly\,F^{(2)}_{mn}(x,y)\Big|_{-\pi}^{\pi}+$$
$$+\,l\int_{-\pi}^{\pi}F^{(2)}_{mn}(x,y)\sin ly\,dy=$$
$$=\,l\int_{-\pi}^{\pi}F^{(2)}_{mn}(x,y)\sin ly\,dy.$$

Substituting this in the right side of (2) we have

$$\frac{l}{\pi^2}\int_{\Delta}F^{(2)}_{mn}(x,y)\cos kx\sin ly\,dx\,dy=$$
$$\frac{l}{\pi^2}\int_{-\pi}^{\pi}\sin ly\left\{\int_{-\pi}^{\pi}F^{(2)}_{mn}(x,y)\cos kx\,dx\right\}dy.$$

Integrating by parts the integral in braces we have

$$\int_{-\pi}^{\pi} F_{mn}^{(2)}(x,y)\cos kx\,dx = \cos kx\, F_{mn}(x,y)\Big|_{-\pi}^{\pi} +$$

$$+k\int_{-\pi}^{\pi} F_{mn}(x,y)\sin kx\,dx = k\int_{-\pi}^{\pi} F_{mn}(x,y)\sin kx\,dx.$$

Thus

$$(3)\quad kl\,d_{kl}\left(1-\frac{k}{m}\right)\left(1-\frac{l}{n}\right) = \frac{kl}{\pi^2}\int_{\Delta} F_{mn}(x,y)\sin kx \sin ly\,dx\,dy.$$

Let us choose the subsequence m_i, n_j and pass to the limit.

$$(4)\quad kl\,d_{kl} = \frac{kl}{\pi^2}\int_{\Delta} F(x,y)\sin kx \sin ly\,dx\,dy$$

or

$$d_{kl} = \frac{1}{\pi^2}\int_{\Delta} F(x,y)\sin kx \sin ly\,dx\,dy.$$

Similar formulas exist for the other constants for $k>0, l>0$.

Again, it is apparent from the terms of $\sigma''_{mn}(x,y)$ that

$$\frac{1}{2\pi^2}\int_{\Delta} \sigma''_{mn}(x,y)\cos kx\,dx\,dy = 0.$$

And from this it follows by partial integration as before that

$$c_{k0} = \frac{1}{2\pi^2}\int_{\Delta} F(x,y)\sin kx\,dx\,dy = 0$$

Similarly the constants $a_{k0}, a_{0l}, b_{0l}, a_{00}$ for the function $F(x,y)$ are all zero. Thus the series (1) is the Fourier series of the function $F(x,y)$ of bounded variation H.

INVARIANT THEOREM 6. The necessary and sufficient condition that the series

$$(1)\qquad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\,\phi_{mn}(x,y),$$

without a constant and without terms in one variable, be a Fourier series of class C_{γ} ($\gamma = 2,3,4,5$) when the series

$$(2)\qquad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \phi_{mn}(x,y),$$

without a constant and without terms in one variable, is a Fourier series of the same class is that the series

$$(3)\qquad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{\lambda_{mn}\sin mx \sin ny}{mn};$$

without a constant and without terms in one variable, be the Fourier series of an odd function of bounded variation H.

The Cesàro mean of the series (1) is

$$(4)\qquad T_{mn}(x,y) = \frac{1}{\pi^2}\int_{\Delta} f(x+t_1, y+t_2)\, C^{(\lambda)}_{mn}(t_1,t_2)\,dt_1\,dt_2$$

where f(x,y) is the function whose Fourier series is (2) and

$$C^{(\lambda)}_{mn}(t_1,t_2) = \frac{1}{4} + \frac{1}{2}\sum_{k=1}^{k=m-1}\left(\frac{m-k}{m}\right)\cos kt_1 +$$

$$+\frac{1}{2}\sum_{l=1}^{l=n-1}\left(\frac{n-l}{n}\right)\cos lt_2 + \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\lambda_{kl}\cos kt_1\cos lt_2 .$$

But since f(x,y) has no constant and no terms in one variable in its Fourier series we can write

$$(5)\qquad T_{mn}(x,y) = \frac{1}{\pi^2}\int_{\Delta} f(x+t_1, y+t_2)\, \sigma''_{mn}(t_1,t_2)\,dt_1\,dt_2$$

where

$$\sigma''_{mn}(t_1,t_2) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\lambda_{kl}\cos kt_1\cos lt_2$$

which corresponds to

$$\sigma_{mn}(t_1,t_2) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\frac{d_{kl}\sin kt_1 \sin lt_2}{kl}.$$

Assuming that the series (3) is the Fourier series of a function of bounded variation H it follows from Theorem 5 that

$$\iint_{\Delta} |\sigma''_{mn}(x,y)|\,dx\,dy < G$$

Then if f(x,y) belongs to class C_r ($r=2,3,4,5$) it follows from Lemma 2 that the Cesàro mean $T_{mn}(x,y)$ of the series (1) fulfills the corresponding class property uniformly. Hence according to Theorem 4 the series (1) is a Fourier series of Class C_r ($r=2,3,4,5$).

Since the series

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\sin mx \sin ny}{mn}$$

is itself of class C_5, which may be proved by applying Theorem 5, no proof is required for class C_5. We assume now that the series (1) and (2) are Fourier series of the same class. If the series (3) is not the Fourier series of a function of bounded variation H then

$$\iint_{\Delta} |\sigma''_{mn}(x,y)|\,dx\,dy$$

is not uniformly bounded. Then according to Lemma 5 there exists a continuous function $\phi(x,y)$ (which belongs to classes C_2, C_3, and C_4) such that

$$\limsup_{m,n\to\infty} |T_{mn}(0,0)| = \infty,$$

that is, the Cesàro mean is not uniformly bounded at the point $x=0, y=0$. But if the series (1) is of class C_ν ($\nu = 2,3,4$) then according to Lemma 3 and Theorem 4 the Cesàro mean should be uniformly bounded everywhere. Thus we have a contradiction and the theorem is proved.

INVARIANT THEOREM 7. A sufficient condition that the series

$$(1) \qquad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\phi_{mn}(x,y),$$

without a constant and without terms in one variable, be a Fourier series (class C_1) when the series

$$(2) \qquad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \phi_{mn}(x,y),$$

without a constant and without terms in one variable, is a Fourier series (class C_1) is that the series

$$(3) \qquad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{\lambda_{mn}\sin mx \sin ny}{mn},$$

without a constant and without terms in one variable, be a Fourier series of an odd function of bounded variation H.

Let $f(x,y)$ and $g(x,y)$ be respectively the functions corresponding to the series (2) and (3). Let

$$F(x,y) = \int_{-\pi}^{x}\int_{-\pi}^{y} f(x,y)\,dx\,dy.$$

Then

$$F(x,y) = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left\{\frac{d_{mn}}{mn}\cos mx\cos ny - \frac{c_{mn}}{mn}\cos mx\sin ny - \frac{b_{mn}}{mn}\sin mx\cos ny + \frac{a_{mn}}{mn}\sin mx \sin ny\right\}. \qquad (9)$$

(9) E.W.Hobson, The Theory of Functions of a Real Variable, vol. 11, p.712.

The statement of the theorem referred to is

"The necessary and sufficient condition that a double trigonometrical series, without a constant term, and without terms which contain one variable only, shall be a Fourier's double series is that the series obtained by integrating each term with respect to both variables shall converge uniformly to an indefinite integral."

That is if the series

$$\text{(a)} \quad \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} (a_{mn} \cos mx \cos ny + - - - + d_{mn} \sin mx \sin ny)$$

is the Fourier series of f(x,y) then the series

$$\text{(b)} \quad \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \left\{ \frac{d_{mn}}{mn} \cos mx \cos ny - \frac{c_{mn}}{mn} \cos mx \sin ny - \frac{b_{mn}}{mn} \sin mx \cos ny + \frac{a_{mn}}{mn} \sin mx \sin ny \right\}$$

is a Fourier series which converges uniformly to the function

$$F(x,y) = \int_{-\pi}^{x} \int_{-\pi}^{y} f(x,y)\,dx\,dy,$$

and conversely, if the series

$$\text{(c)} \quad \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} (a_{mn} \cos mx \cos ny + - - - - - + d_{mn} \sin mx \sin ny)$$

is a Fourier series which converges uniformly to the indefinite integral $F(x,y) = \int_{-\pi}^{x} \int_{-\pi}^{y} f(x,y)\,dx\,dy$, then the series

$$\text{(d)} \quad \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \left\{ mn\, d_{mn} \cos mx \cos ny - mn\, c_{mn} \cos mx \sin ny - mn\, b_{mn} \sin mx \cos ny + mn\, a_{mn} \sin mx \sin ny \right\}$$

is a Fourier series of an integrable function f(x,y).

Also

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn} \cos mx \cos ny$$

is the Fourier Stieltjes series of g(x,y) (p.15).

$$\frac{1}{\pi^2}\int_\Delta F(x+t_1, y+t_2)\, dg(t_1,t_2) =$$

$$= \frac{1}{\pi^2}\iint_\Delta \Big\{\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\Big[\frac{d_{mn}}{mn}\cos m(x+t_1)\cos n(y+t_2) -$$

$$- \frac{c_{mn}}{mn}\cos m(x+t_1)\sin n(y+t_2) - \frac{b_{mn}}{mn}\sin m(x+t_1)\cos n(y+t_2)$$

$$+ \frac{a_{mn}}{mn}\sin m(x+t_1)\sin n(y+t_2)\Big]\Big\}\, dg(t_1,t_2) =$$

$$= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\Big\{\frac{d_{mn}}{mn}\lambda_{mn}\cos mx\cos ny - \frac{c_{mn}}{mn}\lambda_{mn}\cos mx \sin ny$$

$$- \frac{b_{mn}}{mn}\lambda_{mn}\sin mx\cos ny + \frac{a_{mn}}{mn}\lambda_{mn}\sin mx\sin ny\Big\}.$$

we shall show that the function

$$\frac{1}{\pi^2}\int_\Delta F(x+t_1, y+t_2)\, dg(t_1,t_2)$$

is an integral, that is, absolutely continuous. It follows (see reference above) that the series

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\{a_{mn}\cos mx\cos ny + b_{mn}\cos mx\sin ny$$

$$+ c_{mn}\sin mx\cos ny + d_{mn}\sin mx \sin ny\}$$

is a Fourier series (class C_1).

Since F(x,y) is absolutely continuous we have

$$\sum_R |\Delta_{11} F(x_i, y_j)| < \epsilon$$

for every set of non overlapping cells R in Δ of plane measure less than η.

This is not only true for $F(x+t_1, y+t_2)$ for any fixed value of t_1, t_2 but it is also true when t_1 and t_2 vary. For,

$\Delta_{11}F(x_i+t_1, y_j+t_2)$ for a cell $(x_i, y_j;\ x_{i+1}, y_{j+1})$

is equal to

$\Delta_{11}F(x_i, y_j)$ for a cell $(x_i+t_1, y_j+t_2;\ x_{i+1}+t_1, y_{j+1}+t_2)$.

Thus the variation of $F(x+t_1, y+t_2)$ over a set of non overlapping cells of plane measure less than η is equal to that of $F(x,y)$ over another set of cells of measure less than η, and is therefore less than ϵ, whatever value t_1 and t_2 have. Thus the total variation of

$$\int_{\Delta} F(x+t_1, y+t_2)\, dg(t_1, t_2)$$

over a set of non overlapping cells of measure less than η is less than

$$\epsilon \int_{\Delta} |dg(t_1, t_2)|.$$

But $\int_{\Delta} |dg(t_1, t_2)|$ is bounded since $g(t_1, t_2)$ is of bounded variation H. Therefore

$$\int_{\Delta} F(x+t_1, y+t_2)\, dg(t_1, t_2)$$

is an integral.

THEOREM 8. If the Cesàro mean of the series

$$\sum_{k=1}^{k=\infty} \sum_{\ell=1}^{\ell=\infty} \lambda_{k\ell}\, a_{k\ell}$$

where $\lambda_{mn} = \alpha_m \beta_n$ and α_n, β_n are convex null sequences, is bounded, then so is the Cesàro mean of the series

$$\sum_{k=1}^{k=\infty} \sum_{\ell=1}^{\ell=\infty} a_{k\ell}.$$

That α_m is a convex null sequence means

$$\lim_{m \to \infty} \alpha_m = 0$$

and

$$\Delta^2 \alpha_m = \alpha_{m+2} - 2\alpha_{m+1} + \alpha_m \geqq 0 .$$

It follows that α_m is also monotone decreasing. This theorem is partly a generalization to double series of a theorem of S.Szidon in Mathematische Zeitschrift, vol. 10, p.122. In his proof he employs the following two lemmas:

(a) To any sequence $c_1, c_2, \dots, c_m, \dots$ for which $\lim_{n \to \infty} c_n = \infty$ there is always a concave sequence $b_1, b_2, \dots, b_m, \dots$ for which $\lim_{m \to \infty} b_m = \infty$ and $b_m < c_m$,

(b) If the sequence $b_1, b_2, \dots, b_m, \dots$ is concave and $\lim_{n \to \infty} b_m = \infty$, then the sequence $\frac{1}{b_1}, \frac{1}{b_2}, \dots, \frac{1}{b_m}, \dots$ is convex.

Also, Szidon uses the following identity

$$(1) \qquad S_m = \sum_{k=1}^{k=m-1} C_{mk} S'_k + \mu_m S'_m ,$$

where

$$C_{mk} = \frac{(m-k)k\,\Delta^2 \mu_k - k(\Delta\mu_k + \Delta\mu_{k+1})}{m},$$

$$S_m = \sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) a_k ,$$

$$S'_m = \sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) \lambda_k a_k ,$$

$$\mu_k = \frac{1}{\lambda_k} , \qquad \Delta\mu_k = \mu_{k+1} - \mu_k .$$

Since μ_k is a concave sequence, that is,

$\Delta^2\mu_k \leqq 0,\ \Delta\mu_k > 0$, the expression C_{mk} is negative.

The identity (1) may be written in the form

$$(2)\quad S_m = \sum_{k=1}^{k=m-1} C_{mk} \sum_{i=1}^{i=k} \left(\frac{k-i+1}{k}\right) \lambda_k a_k + \mu_m \sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) \lambda_k a_k$$

Let

$$(3)\quad S_{mn} = \sum_{k=1}^{k=m} \sum_{\ell=1}^{\ell=n} \left(\frac{m-k+1}{m}\right)\left(\frac{n-\ell+1}{n}\right) a_{k\ell}$$

$$= \sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) \sum_{\ell=1}^{\ell=n} \left(\frac{n-\ell+1}{n}\right) a_{k\ell},$$

$$(4)\quad S'_{mn} = \sum_{k=1}^{k=m} \sum_{\ell=1}^{\ell=n} \left(\frac{m-k+1}{m}\right)\left(\frac{n-\ell+1}{n}\right) \alpha_k \beta_\ell a_{k\ell}$$

$$= \sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) \alpha_k \sum_{\ell=1}^{\ell=n} \left(\frac{n-\ell+1}{n}\right) \beta_\ell a_{k\ell},$$

and $\mu_m = \dfrac{1}{\alpha_m}, \quad \gamma_n = \dfrac{1}{\beta_n}$.

By using the identity (2) we obtain a relationship between the summations over n in (3) and (4). Thus

$$(5)\quad S_{mn} = \sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) \left\{ \sum_{\ell=1}^{\ell=n-1} C_{n\ell} \left[\sum_{j=1}^{j=\ell} \left(\frac{\ell-j+1}{\ell}\right) \beta_j a_{kj} \right] + \right.$$

$$\left. + \gamma_n \sum_{\ell=1}^{\ell=n} \left(\frac{n-\ell+1}{n}\right) \beta_\ell a_{k\ell} \right\}$$

$$= \sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) \left\{ \sum_{\ell=1}^{\ell=n-1} C_{n\ell} \sum_{j=1}^{j=\ell} \left(\frac{\ell-j+1}{\ell}\right) \beta_j a_{kj} \right\} +$$

$$+ \gamma_n \sum_{k=1}^{k=m} \sum_{\ell=1}^{\ell=n} \left(\frac{m-k+1}{m}\right)\left(\frac{n-\ell+1}{n}\right) \beta_\ell a_{k\ell}\ .$$

Employing the identity (2) again we can obtain a relationship between the two series

$$\sum_{k=1}^{k=m} \left(\frac{m-k+1}{m}\right) \left\{ \sum_{\ell=1}^{\ell=n-1} C_{n\ell} \sum_{j=1}^{j=\ell} \left(\frac{\ell-j+1}{\ell}\right) \beta_j a_{kj} \right\}$$

and

$$\sum_{k=1}^{k=m}\left(\frac{m-k+1}{m}\right)\alpha_k\left\{\sum_{l=1}^{l=n-1} C_{nl}\sum_{j=1}^{j=l}\left(\frac{l-j+1}{l}\right)\beta_j\, a_{kj}\right\}.$$

Thus

$$(6)\quad S_{mn} = \sum_{k=1}^{k=m-1} C_{mk}\left[\sum_{i=1}^{i=k}\left(\frac{k-i+1}{k}\right)\alpha_i\left\{\sum_{l=1}^{l=n-1} C_{nl}\sum_{j=1}^{j=l}\left(\frac{l-j+1}{l}\right)\beta_j\, a_{ij}\right\}\right]$$

$$+\mu_m\sum_{k=1}^{k=m}\left(\frac{m-k+1}{m}\right)\alpha_k\left\{\sum_{l=1}^{l=n-1} C_{nl}\sum_{j=1}^{j=l}\left(\frac{l-j+1}{j}\right)\beta_j\, a_{kj}\right\}+$$

$$+\nu_n\sum_{k=1}^{k=m}\sum_{l=1}^{l=n}\left(\frac{m-k+1}{m}\right)\left(\frac{n-l+1}{n}\right)\beta_l\, a_{kl}.$$

Finally, applying the identity (2) to the last term of equation (6) we obtain

$$(7)\quad \nu_n\sum_{k=1}^{k=m}\sum_{l=1}^{l=n}\left(\frac{m-k+1}{m}\right)\left(\frac{n-l+1}{n}\right)\beta_l\, a_{kl} =$$

$$=\nu_n\sum_{l=1}^{l=n}\left(\frac{n-l+1}{n}\right)\beta_l\left\{\sum_{k=1}^{k=m-1} C_{mk}\sum_{i=1}^{i=k}\left(\frac{k-i+1}{k}\right)\alpha_i\, a_{il}+\right.$$

$$\left.+\mu_m\sum_{k=1}^{k=m}\left(\frac{m-k+1}{m}\right)\alpha_k\, a_{kl}\right\}=$$

$$=\nu_n\sum_{l=1}^{l=n}\left(\frac{n-l+1}{n}\right)\beta_l\sum_{k=1}^{k=m-1} C_{mk}\sum_{i=1}^{i=k}\left(\frac{k-i+1}{k}\right)\alpha_i\, a_{il}+$$

$$+\mu_m\nu_n\sum_{k=1}^{k=m}\sum_{l=1}^{l=n}\left(\frac{m-k+1}{m}\right)\left(\frac{n-l+1}{n}\right)\alpha_k\beta_l\, a_{kl}.$$

Combining (6) and (7) we have

$$(8)\quad S_{mn} = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1} C_{mk}C_{nl}S'_{kl}+$$

$$+\mu_m \sum_{k=1}^{k=m}\left(\frac{m-k+1}{m}\right)\alpha_k\left\{\sum_{l=1}^{l=n-1} C_{nl}\sum_{j=1}^{j=l}\left(\frac{l-j+1}{l}\right)\beta_j a_{kj}\right\}+$$

$$+\gamma_n \sum_{l=1}^{l=n}\left(\frac{n-l+1}{n}\right)\beta_l\left\{\sum_{k=1}^{k=m-1} C_{mk}\sum_{i=1}^{i=k}\left(\frac{k-i+1}{k}\right)\alpha_i a_{il}\right\}+$$

$$+\mu_m\gamma_n S'_{mn}.$$

Since (8) is an identity in S_{mn} and S'_{mn} we can take

$$S_{11}=S_{12}=S_{21}=\cdots=S_{mn}=1,\quad S'_{mn}=\alpha_1\beta_1=\frac{1}{\mu_1\gamma_1}$$

Then from (8) we have

$$1=\sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\frac{C_{mk}C_{nl}}{\mu_1\gamma_1}+\mu_m\sum_{l=1}^{l=n-1}C_{nl}\cdot\frac{1}{\mu_1\gamma_1}+$$

$$+\gamma_n\sum_{k=1}^{k=m-1}C_{mk}\cdot\frac{1}{\mu_1\gamma_1}+\mu_m\gamma_n\cdot\frac{1}{\mu_1\gamma_1},$$

or

$$\mu_1\gamma_1=\sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}C_{mk}C_{nl}+\mu_m\sum_{l=1}^{l=n-1}C_{nl}+\gamma_n\sum_{k=1}^{k=m-1}C_{mk}+\mu_m\gamma_n.$$

Similarly from the identity (2)

$$\mu_1=\sum_{k=1}^{k=m-1}C_{mk}+\mu_m,$$

$$\gamma_1=\sum_{l=1}^{l=n-1}C_{nl}+\gamma_n.$$

Since

$$\sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}C_{mk}C_{nl}\geqq 0,\quad \sum_{k=1}^{k=m-1}C_{mk}\leqq 0;\quad \sum_{l=1}^{l=n-1}C_{nl}\leqq 0,$$

we have

$$(9)\quad \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}C_{mk}C_{nl}+\mu_m\sum_{l=1}^{l=n-1}|C_{nl}|+\gamma_n\sum_{k=1}^{k=m-1}|C_{mk}|+\mu_m\gamma_n=$$

$$=(\mu_m-\mu_1)(\gamma_n-\gamma_1)+(\gamma_n-\gamma_1)\mu_m+(\mu_m-\mu_1)\gamma_n+\mu_m\gamma_n=$$

$$=4\mu_m\gamma_n-2\mu_1\gamma_n-2\mu_m\gamma_1+\mu_1\gamma_1.$$

Now if

$$|S'_{mn}| \leq M$$

then from (8) and (9)

$$(10)\ |S_{mn}| \leq M \sum_{k=1}^{k=m-1} \sum_{l=1}^{l=n-1} C_{mk} C_{nl} + \mu_m M \sum_{l=1}^{l=n-1} |C_{nl}| +$$

$$+ \nu_n M \sum_{k=1}^{k=m-1} |C_{mk}| + M \mu_m \nu_n =$$

$$= M (4\mu_m \nu_n - 2\mu_1 \nu_n - 2\mu_m \nu_1 + \mu_1 \nu_1)$$

$$< M(4\mu_m \nu_n + \mu_1 \nu_1) < 5M\mu_m \nu_n .$$

If we assume that S_{mn} is not uniformly bounded then there is a subsequence $m_i n_j \equiv p_k$ of m,n such that

$$\lim_{k \to \infty} |S_{p_k}| = \infty .$$

Let us choose

$$(11) \qquad \mu_{m_i} < |S_{p_k}|^{1/4}$$

$$\nu_{n_j} < |S_{p_k}|^{\frac{1}{4}}$$

where $m_i n_j \equiv p_k$. This is permissible for, according to the two lemmas mentioned at the beginning of this theorem, there exist two convex null sequences corresponding to μ_{m_i} and ν_{n_j}. It follows from (10) and (11) that for the subsequence $m_i n_j \equiv p_k$

$$|S_{p_k}| < 5M |S_{p_k}|^{1/2}$$

Thus we have a contradiction and the assumption that S_{mn} is not uniformly bounded is false.

By analogy with the above theorem we obtain

THEOREM 8A. Let $\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} f_{mn}(x,y)$ be a series whose terms are integrable L in Δ. Then the sequence

$$\int_\Delta\int |S_{mn}(x,y)|\,dx\,dy$$

is uniformly bounded, where $S_{mn}(x,y)$ is the Cesàro mean of the given series, when the sequence

$$\int_\Delta\int |S'_{mn}(x,y)|\,dx\,dy$$

is uniformly bounded, where $S'_{mn}(x,y)$ is the Cesàro mean of the series

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \alpha_m \beta_n f_{mn}(x,y)$$

and α_m, β_n are two convex null sequences.

THEOREM 9. The integral of the absolute value of the Cesàro mean

(1) $$\sigma_{mn}(x,y) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\phi_{kl}(x,y)$$

of a Fourier series

(2) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\phi_{mn}(x,y)$$

without a constant and without terms in one variable, is uniformly bounded, that is,

$$\int_\Delta\int |\sigma_{mn}(x,y)|\,dx\,dy < G.$$

Let the function corresponding to the Fourier series (2) be f(x,y). Then the Fourier series corresponding to the integral

$$F(x,y) = \int_{-\pi}^{x}\int_{-\pi}^{y} f(x,y)\,dx\,dy$$

is

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left\{\frac{d_{mn}}{mn}\cos mx\cos ny-\frac{c_{mn}}{mn}\cos mx\sin ny\right.$$

$$\left.-\frac{b_{mn}}{mn}\sin mx\cos ny+\frac{a_{mn}}{mn}\sin mx\sin ny\right\}.$$

The function $F(x,y)$ is of bounded variation H. (10)

(10) E.W.Hobson, The Theory of Functions of a Real Variable, vol. 1, p.608.

It follows from Theorem 5 that

$$\iint_\Delta |T''_{mn}(x,y)|\,dx\,dy < G$$

where $T_{mn}(x,y)$ is the Cesàro mean of the Fourier series of $F(x,y)$. But

$$T''_{mn}(x,y) = \sigma_{mn}(x,y).$$

Hence

$$\iint_\Delta |\sigma_{mn}(x,y)|\,dx\,dy < G$$

INVARIANT THEOREM 10. The necessary condition that the series

(1) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\phi_{mn}(x,y),$$

without a constant and without terms in one variable, be a Fourier series (class C_1) when the series

(2) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\phi_{mn}(x,y),$$

without a constant and without terms in one variable, is a Fourier series (class C_1) is that the series

(3) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\lambda_{mn}\sin mx\sin ny}{mn},$$

without a constant and without terms in one variable, be the Fourier series of an odd function of bounded variation H.

W.H.Young (11) has shown that the series

$$(4) \qquad \sum_{m=1}^{m=\infty} \alpha_m \cos mx$$

is a Fourier series when α_m is a convex null sequence.

(11) W.H.Young, on the Fourier Series of Bounded Functions, Proc. Lond. Math. Soc., vol. 12, (1912) Series 2, p.45.

Thus

$$(5) \qquad \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \alpha_m \beta_n \cos mx \cos ny$$

Is a Fourier series when α_m, β_n are both convex null sequences. If the sequence λ_{mn} always transforms a Fourier series into a Fourier series then

$$(6) \qquad \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \lambda_{mn} \alpha_m \beta_n \cos mx \cos ny$$

is a Fourier series. If $\sigma'_{mn}(x,y)$ is the Cesàro mean of the Fourier series (6) it follows from Theorem 9 that

$$\iint_{\Delta} |\sigma'_{mn}(x,y)|\, dx\, dy$$

is uniformly bounded. It follows from Theorem 8A that

$$\iint_{\Delta} |\sigma_{mn}(x,y)|\, dx\, dy$$

where $\sigma_{mn}(x,y)$ is the Cesàro mean of the series

$$\sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \lambda_{mn} \cos mx \cos ny$$

is uniformly bounded. But $\sigma_{mn}(x,y)$ is the derivative with respect to x and y of the Cesàro mean of the series (3).

Hence from Theorem 5 the series (3) must be a Fourier series of a function of bounded variation H.

INVARIANT THEOREM 11. The necessary and sufficient condition that the sequence λ_{mn} transform the Fourier series

(1) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\phi_{mn}(x,y),$$

without a constant and without terms in one variable, of class C_b into a Fourier series

(2) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\phi_{mn}(x,y)$$

of class C_b is that the series

(3) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\lambda_{mn}\sin mx\sin ny}{mn},$$

without a constant and without terms in one variable, be the Fourier series of an odd function of bounded variation H.

Necessity. It is assumed that

(4) $$F(x,y)=\int_{-\pi}^{x}\int_{-\pi}^{y}f(x,y)\,dx\,dy=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\phi_{mn}(x,y)$$

and

(5) $$G(x,y)=\int_{-\pi}^{x}\int_{-\pi}^{y}g(x,y)\,dx\,dy=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\phi_{mn}(x,y).$$

Then the Fourier series of the function f(x,y) is

(6) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\phi''_{mn}(x,y),$$

and the Fourier series of the function g(x,y) is

(7) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\phi''_{mn}(x,y),$$

where " indicates the derivative with respect to x and y. (12).

(12) Ibid. p.48.

But the condition that the sequence λ_{mn} transform the Fourier series (6) into the Fourier series (7) is precisely the condition stated. Hence the necessity is proved.

Sufficiency. If the sequence λ_{mn} satisfy the condition stated then it transforms the Fourier series (6) into a Fourier series (7). This follows from Invariant Theorem 7. But if the series (7) is a Fourier series (class C_1) then the series (2) is the Fourier series of an integral (p.48). Thus the sufficiency is proved.

If we combine Invariant Theorems 6,7,10, and 11 with Invariant Theorem 1 (p.2) we obtain

INVARIANT THEOREM II. The necessary and sufficient condition that the sequence λ_{mn} transform the Fourier series

$$(1) \quad \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \phi_{mn}(x,y)$$

of class C_γ (γ=1,2,3,4,5,6) into a Fourier series

$$(2) \quad \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \lambda_{mn}\,\phi_{mn}(x,y)$$

of the same class is that the series

$$(3) \quad \sum_{m=1}^{m=\infty} \frac{\lambda_{m0}\sin mx}{m} + \sum_{n=1}^{n=\infty} \frac{\lambda_{0n}\sin ny}{n} + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{\lambda_{mn}\sin mx \sin ny}{mn}$$

be a Fourier series of an odd function of bounded variation H.

The validity of this theorem is apparent from the discussion under the topic Method of Generalization, p.10. For convenience we shall speak only of class C_1.

Sufficiency. Each of the above series may be resolved into three parts.

(1a) $\sum_{m=1}^{m=\infty} \phi_{mo}(x)$

(1b) $\sum_{n=1}^{n=\infty} \phi_{on}(y)$

(1c) $\sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \phi_{mn}(x, y)$

(2a) $\sum_{m=1}^{m=\infty} \lambda_{mo} \phi_{mo}(x)$

(2b) $\sum_{n=1}^{n=\infty} \lambda_{on} \phi_{on}(y)$

(2c) $\sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \lambda_{mn} \phi_{mn}(x, y)$

(3a) $\sum_{m=1}^{m=\infty} \lambda_{mo} \sin mx \cdot \frac{1}{m}$

(3b) $\sum_{n=1}^{n=\infty} \lambda_{on} \sin ny \cdot \frac{1}{n}$

(3c) $\sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \lambda_{mn} \sin mx \sin ny \cdot \frac{1}{mn}$.

The two series (3a) and (3b) satisfy the condition of Invariant Theorem 1. Hence if the series (1) is of class C_1 so are both the series (1a) and (1b) and it follows that the series (2a) and (2b) are Fourier series of class C_1. Also, if series (1) is of class C_1 so is the series (1c). Similarly the series (3c) satisfies the condition of Invariant Theorem 7 andit follows that the series (2c) is a Fourier series of class C_1. If the series (2a), (2b), and (2c) are all of class C_1 so is the series (2) (p.10).

Necessity. If the series (3) is not a Fourier series of a function of bounded variation H then not all the series (3a), (3b), (3c) are Fourier series of functions of bounded variation. Hence not all the series (2a), (2b), and (2c) are Fourier series of Class C_1. Then the series (2) cannot be a Fourier series of class C_1 for if it were then the series (2), (2b), and (2c) would all be of class C_1.

5. Covariant Theorems. We shall now obtain covariant theorems corresponding to the Invariant Theorems 6,7,10,11.

COVARIANT THEOREM 6. The necessary and sufficient condition that the sequence λ_{mn} transform the allied series

(1) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\overline{\Phi}_{mn}(x,y),$$

without terms in one variable, or class C_γ (γ=2,3,4,5) into a Fourier series

(2) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\overline{\Phi}_{mn}(x,y),$$

without a constant and without terms in one variable, of the same class is that the series

(3) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\lambda_{mn}\cos mx\cos ny}{mn},$$

without a constant and without terms in one variable, be the Fourier series of an even function of bounded variation H.

Sufficiency. Let the function corresponding to the allied series (1), without terms in one variable, be f(x,y). The Cesàro mean of the series (2) is

$$\overline{T}_{mn}(x,y)=\frac{1}{\pi^2}\int_\Delta f(x+t_1,y+t_2)\,C_{mn}^{(\lambda)}(t_1,t_2)\,dt_1\,dt_2,$$

where

$$C_{mn}^{(\lambda)}(t_1,t_2)=\frac{1}{2}\sum_{k=1}^{k=m-1}\left(\frac{m-k}{m}\right)\sin kt_1+\frac{1}{2}\sum_{l=1}^{l=n-1}\left(\frac{n-l}{n}\right)\sin lt_2+$$

$$+\sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\lambda_{kl}\sin kt_1\sin lt_2.$$

But since f(x,y) has an allied series without terms in one variable we can write,

$$(4) \quad T_{mn}(x,y) = \frac{1}{\pi^2}\int_{\Delta} f(x+t_1, y+t_2)\,\sigma''_{mn}(t_1,t_2)\,dt_1\,dt_2$$

where

$$\sigma''_{mn}(t_1,t_2) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\lambda_{kl}\sin kt_1 \sin lt_2 .$$

Corresponding to $\sigma''_{mn}(t_1,t_2)$ we have

$$\sigma_{mn}(t_1,t_2) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\frac{\lambda_{kl}\cos kt_1 \cos lt_2}{kl}$$

which is the Cesàro mean of the series (3). If the series (3) satisfies the condition stated then according to Theorem 5

$$\int_{\Delta} |\sigma''_{mn}(x,y)|\,dx\,dy$$

is uniformly bounded. The remainder of the proof goes through precisely as in Invariant Theorem 6.

Necessity. The series

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\cos mx \cos ny}{mn}$$

is an allied series of class C_5 for it is the series allied with

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\sin mx \sin ny}{mn}.$$

Thus no proof of necessity is required for class C5. If the series (2) is a Fourier series of class C_γ (γ=2,3,4) when the series (1) is an allied series of class C_γ (γ=2,3,4) then it follows from Theorem 4 that the Cesàro mean $T_{mn}(x,y)$ fulfills the corresponding class property uniformly. Then by Lemma 3 $T_{mn}(x,y)$ is uniformly bounded everywhere. Suppose

The series (3) does not satisfy the condition stated, then

$$\int_\Delta\int |\sigma''_{mn}(x,y)|\,dx\,dy$$

is not uniformly bounded. Then by Lemma 5 there exists a continuous function $\phi(x,y)$, with an allied series in class $C_\gamma(\gamma = 2,3,4)$, for which $T_{mn}(x,y)$ is not uniformly bounded at the point $x = 0, y = 0$. Thus we have a contradiction and the theorem is proved.

COVARIANT THEOREM 7. A sufficient condition that the sequence λ_{mn} transform an allied series

$$(1) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \overline{\Phi}_{mn}(x,y),$$

without terms in one variable, of class C_1 into a Fourier series

$$(2) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\overline{\Phi}_{mn}(x,y)$$

of class C_1 is that the series

$$(3) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{\lambda_{mn}\cos mx\cos ny}{mn}$$

without a constant and without terms in one variable, be a Fourier series of an even function of bounded variation H.

Let $f(x,y)$ and $g(x,y)$ be respectively the functions corresponding to the series (1) and (3). Let

$$F(x,y) = \int_{-\pi}^{x}\int_{-\pi}^{y} f(x,y)\,dx\,dy.$$

Then

$$F(x,y) = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\Big\{\frac{d_{mn}}{mn}\cos mx\cos ny - \frac{c_{mn}}{mn}\cos mx\sin ny$$

$$- \frac{b_{mn}}{mn}\sin mx\cos ny + \frac{a_{mn}}{mn}\sin mx\sin ny\Big\}.$$

Also,
$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn} \sin mx \sin ny$$

is the Fourier Stieltjes series of g(x,y). Consider

$$\frac{1}{\pi^2}\int_\Delta F(x+t_1, y+t_2)\, dg(t_1,t_2) =$$

$$= \frac{1}{\pi^2}\iint_\Delta \Big\{ \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \Big[\frac{d_{mn}}{mn}\cos m(x+t_1)\cos n(y+t_2) -$$

$$- \frac{c_{mn}}{mn}\cos m(x+t_1)\sin n(y+t_2) - \frac{b_{mn}}{mn}\sin m(x+t_1)\cos n(y+t_2)$$

$$+ \frac{a_{mn}}{mn}\sin m(x+t_1)\sin n(y+t_2)\Big]\Big\}\, dg(t_1,t_2)$$

$$= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\Big\{ \frac{a_{mn}}{mn}\cos mx \cos ny\, \lambda_{mn} + \frac{b_{mn}}{mn}\cos mx \sin ny\, \lambda_{mn}$$

$$+ \frac{c_{mn}}{mn}\sin mx \cos ny\, \lambda_{mn} +$$

$$+ \frac{d_{mn}}{mn}\sin mx \sin ny\, \lambda_{mn}\Big\}.$$

It can be proved as before that the function

$$\frac{1}{\pi^2}\int_\Delta F(x+t_1, y+t_2)\, dg(t_1,t_2)$$

is an integral. Therefore its differentiated series is a Fourier series (class C_1), that is,

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\Big\{ d_{mn}\cos mx \cos ny - c_{mn}\cos mx \sin ny -$$

$$- b_{mn}\sin mx \cos ny + a_{mn}\sin mx \sin ny\Big\}$$

$$= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\, \overline{\Phi}_{mn}(x,y)$$

is a Fourier series (class C_1).

COVARIANT THEOREM 10. The necessary condition that the sequence λ_{mn} transform the allied series

$$(1) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \overline{\Phi}_{mn}(x,y),$$

without terms in one variable, of class C_1 into a Fourier series

$$(2) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\overline{\Phi}_{mn}(x,y)$$

of the same class is that the series

$$(3) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{\lambda_{mn}\cos mx \cos ny}{mn},$$

without a constant and without terms in one variable, be the Fourier series of an even function of bounded variation H.

The series

$$(4) \quad \sum_{m=1}^{m=\infty}(-\alpha_m \sin mx)$$

is an allied series, allied with the Fourier series

$$(5) \quad \sum_{m=1}^{m=\infty}\alpha_m \cos mx$$

of class C_1. See page 58. Thus

$$(6) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \alpha_m\beta_n \sin mx \sin ny$$

is an allied series of class C_1. Then if the sequence λ_{mn} fulfills the property of transforming every allied series of class C_1 into a Fourier series of class C_1, the series

$$(7) \quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\alpha_m\beta_n \sin mx \sin ny$$

is a Fourier series of class C_1. It follows from Theorems 9 and 8A that

$$\int_\Delta |\sigma''_{mn}(x,y)|\,dx\,dy < G$$

where

$$\sigma''_{mn}(x,y) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\lambda_{mn}\sin mx \sin ny.$$

Corresponding to the derived sequence $\sigma''_{mn}(x,y)$ we have

$$\sigma_{mn}(x,y) = \sum_{k=1}^{k=m-1}\sum_{l=1}^{l=n-1}\left(\frac{m-k}{m}\right)\left(\frac{n-l}{n}\right)\frac{\lambda_{kl}}{kl}\cos kx \cos ly.$$

Then according to Theorem 5 the series

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\lambda_{mn}\cos mx \cos ny}{mn}$$

is a Fourier series of a function of bounded variation H.

COVARIANT THEOREM 11. The necessary andsufficient condition that the sequence λ_{mn} transform the allied series

(1) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\overline{\Phi}_{mn}(x,y),$$

without terms in one variable, of class C_6 into a Fourier series

(2) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\lambda_{mn}\overline{\Phi}_{mn}(x,y)$$

of the same class C_6 is that the series

(3) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{\lambda_{mn}\cos mx \cos ny}{mn},$$

without a constant and without terms in one variable, be the Fourier series of an even function of bounded variation H.

We employ the theorem of Hobson mentioned on p.48. If the series (1) is an allied series of class C_6 then the series

(4) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\{mn\,a_{mn}\cos mx \cos ny + mn\,b_{mn}\cos mx \sin ny + mn\,c_{mn}\sin mx \cos ny + mn\,d_{mn}\sin mx \sin ny\}$$

is an allied series of class C_1. If the sequence λ_{mn} satisfies the condition stated then by Covariant Theorem 7 the series

$$(5)\quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\Big\{mn\, a_{mn}\cos mx\cos ny + mn\, b_{mn}\cos mx\sin ny + mn\, c_{mn}\sin mx\cos ny + mn\, d_{mn}\sin mx\sin ny\Big\}$$

is a Fourier series of the class C_1. But if the series 5 is a Fourier series then the series

$$(6)\quad \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\Big\{d_{mn}\cos mx\cos ny - c_{mn}\cos mx\sin ny - b_{mn}\sin mx\cos ny + d_{mn}\sin mx\sin ny\Big\}$$

$$= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\overline{\Phi_{mn}}(x,y)$$

is a Fourier series of an integral, that is, of class C_6. Thus the sufficiency of the condition is proved.

If the series (1) and (2) are both series of class C_6 then the series (4) is an allied series of class C_1 and the series (5) is a Fourier series of class C_1. But in order that this be so it is necessary, according to Covariant Theorem 10, that the sequence λ_{mn} satisfy the condition given. Thus the necessity of the condition is proved.

Now by combining Covariant Theorems 6,7,10,11 with the covariant Theorem 1 of p.2 we obtain

COVARIANT THEOREM 11. The necessary and sufficient condition that the sequence λ_{mn} transform the allied series

$$(1) \sum_{m=1}^{m=\infty} \bar{\phi}_{m0}(x) + \sum_{n=1}^{n=\infty} \bar{\phi}_{0n}(y) + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \bar{\phi}_{mn}(x,y)$$

of class $C_\gamma(\gamma=1,2,3,4,5,6)$ into a Fourier series

$$(2) \sum_{m=1}^{m=\infty} \lambda_{m0}\bar{\phi}_{m0}(x) + \sum_{n=1}^{n=\infty} \lambda_{0n}\bar{\phi}_{0n}(y) + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\bar{\phi}_{mn}(x,y)$$

of the same class is that the series

$$(3) \sum_{m=1}^{m=\infty} \frac{\lambda_{m0}\cos mx}{m} + \sum_{n=1}^{n=\infty} \frac{\lambda_{0n}\cos ny}{n} + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{\lambda_{mn}\cos mx \cos ny}{mn}$$

be the Fourier series of an even function of bounded variation H.

The detailed proof is similar to that of Invariant Theorem II.

6. Application. We can apply the theorems of Fekete and Bochner and of the present paper to the problem of convergence factors for Fourier series.

THEOREM 1. The necessary and sufficient condition that the sequence λ_n transform any Fourier series

$$(1) \quad \sum_{n=1}^{n=\infty} (a_n \cos nx + b_n \sin nx),$$

without a constant term, whether convergent or not, into a Fourier series

$$(2) \quad \sum_{n=1}^{n=\infty} \lambda_n (a_n \cos nx + b_n \sin nx)$$

which converges uniformly to an indefinite integral, is that the series

$$(3) \quad \sum_{n=1}^{n=\infty} \lambda_n \cos nx$$

be a Fourier series of an even function of bounded variation.

Necessity. If the series (2) is a Fourier series of class C_6 then the series

$$(4)\quad \sum_{n=1}^{n=\infty} \lambda_n(-n a_n \sin nx + n b_n \cos nx)$$
$$= \sum_{n=1}^{n=\infty} n\lambda_n (b_n \cos nx - a_n \sin nx)$$

is a Fourier series (class C_1) (13).

(13) E.W.Hobson, The Theory of Functions of a Real Variable, vol. II, p. 553.

The series

$$(5)\quad \sum_{n=1}^{n=\infty} (b_n \cos nx - a_n \sin nx)$$

is the series allied with the Fourier series (1). According to Fekete's Covariant Theorem I the necessary condition that the series (4) be a Fourier series (class C_1 at least) when the series (5) is an allied series of class C_γ (γ=1,2,3, 4,5,6) is that the series

$$(6)\quad \sum_{n=1}^{n=\infty} \frac{n\lambda_n \cos nx}{n} = \sum_{n=1}^{n=\infty} \lambda_n \cos nx$$

be a Fourier series of a function of bounded variation.

Sufficiency. If the series

$$\sum_{n=1}^{n=\infty} \lambda_n \cos nx = \sum_{n=1}^{n=\infty} \frac{n\lambda_n \cos nx}{n}$$

is the Fourier series of a function of bounded variation then according to Fekete's Covariant Theorem I the allied series (5) of any class C_γ is transformed into a Fourier series (4) of the same class. But if the series (4) is a Fourier series then the series (2) is a Fourier series which converges uniformly to an indefinite integral.

THEOREM 2. The necessary and sufficient conditon that the sequence λ_{mn} transform any Fourier series

(1) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \phi_{mn}(x,y),$$

without a constant term and without terms in one variable, whether convergent or not, into a Fourier series

(2) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\,\phi_{mn}(x,y)$$

which converges uniformly to an indefinite integral is that the series

(3) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn} \cos mx \cos ny$$

be a Fourier series of an even function of bounded variation H.

The proof goes through as in the preceeding theorem using Hobson's theorem mentioned on page 48 and Covariant Theorems 6,7,10, and all.

By combining Theorems 1 and 2 have

THEOREM 3. The necessary and sufficient condition that the sequence λ_{mn} transform the Fourier series

(1) $$\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \phi_{mn}(x,y),$$

whether convergent or not, into a Fourier series

(2) $$\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \lambda_{mn}\,\phi_{mn}(x,y)$$

which converges uniformly to an indefinite integral is that the series

(3) $\sum_{m=1}^{m=\infty} \lambda_{m0} \cos mx + \sum_{n=1}^{n=\infty} \lambda_{0n} \cos ny + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn} \cos mx \cos ny$

be a Fourier series of an even function of bounded variation H.

THEOREM 4. The necessary and sufficient condition that the sequence λ_n transform the allied series

(1) $\sum_{n=1}^{n=\infty} (b_n \cos nx - a_n \sin nx)$

of any class into a Fourier series

(2) $\sum_{n=1}^{n=\infty} \lambda_n (b_n \cos nx - a_n \sin nx)$

which converges uniformly to an indefinite integral is that the series

(3) $\sum_{n=1}^{n=\infty} (-\lambda_n \sin nx)$

be a Fourier series of an odd function of bounded variation.

Necessity. The Fourier series corresponding to the allied series (1) is

(4) $\sum_{n=1}^{n=\infty} (a_n \cos nx + b_n \sin nx)$

If the series (2) is the Fourier series of an indefinite integral, that is, class C_b, then

(5) $\sum_{n=1}^{n=\infty} -n \lambda_n (a_n \cos nx + b_n \sin nx)$

is a Fourier series (class C_1). (14).

(14) Ibid. p.48.

But according to Fekete's Invariant Theorem I the necessary condition that the series (5) be a Fourier series is that the series

$$\sum_{m=1}^{m=\infty} \frac{-m\lambda_m \sin mx}{m} = \sum_{m=1}^{m=\infty} -\lambda_m \sin mx$$

be a Fourier series of an odd function of bounded variation.

Sufficiency. If the series

$$\sum_{m=1}^{m=\infty} -\lambda_m \sin mx = \sum_{m=1}^{m=\infty} -\frac{m\lambda_m \sin mx}{m}$$

is a Fourier series of a function of bounded variation then by Fekete's Invariant Theorem I the series (5) is a Fourier series when the series (4) is a Fourier series. But if the series (5) is a Fourier series then the series

$$\sum_{m=1}^{m=\infty} \lambda_m (b_m \cos mx - a_m \sin mx)$$

which is the series (2) is the Fourier series of an indefinite integral. (15)

(15) Ibid. p.70.

THEOREM 5. The necessary and sufficient condition that the sequence λ_{mn} transform an allied series

(1) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \overline{\Phi}_{mn}(x,y),$$

without terms in one variable, of any class, into a Fourier series

(2) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\overline{\Phi}_{mn}(x,y)$$

which converges uniformly to an indefinite integral is that the series

(3) $$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn} \sin mx \sin ny$$

be the Fourier series of an odd function of bounded variation H.

The proof goes through as in the previous theorem using Hobson's theorem, p. 48, and Invariant Theorems 6,7,10, and 11.

By combining Theorems 4 and 5 we have

THEOREM 6. The necessary and sufficient condition that the sequence λ_{mn} transform the allied series

$$(1)\quad \sum_{m=1}^{m=\infty} \overline{\Phi}_{m0}(x) + \sum_{n=1}^{n=\infty} \overline{\Phi}_{0n}(y) + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \overline{\Phi}_{mn}(x,y)$$

of any class into a Fourier series

$$(2)\quad \sum_{m=1}^{m=\infty} \lambda_{m0}\overline{\Phi}_{m0}(x) + \sum_{n=1}^{n=\infty} \lambda_{0n}\overline{\Phi}_{0n}(y) + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\overline{\Phi}_{mn}(x,y)$$

which converges uniformly to an indefinite integral is that the series

$$(3)\quad \sum_{m=1}^{m=\infty} -\lambda_{m0}\sin mx + \sum_{n=1}^{n=\infty} -\lambda_{0n}\sin ny + \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \lambda_{mn}\sin mx \sin ny$$

be the Fourier series of an odd function of bounded variation H.