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CONTENTS

CHAPTER I

INTRODUCTION

A large number of problems have arisen through the generalizations of ideas involving functions of a single variable to functions of two or more variables. Many of these extensions have proved to be direct analogues of the results obtained for functions of one variable, while others have lead to a wide variety of possible extensions. In this paper some of the consequences of generalizing the idea of the Fourier series of a function of one variable to a function of two variables are investigated. The Fourier series corresponding to a function of two variables is called the double Fourier series of the function.

A fundamental problem of double Fourier series is that of developing a given function of two variables in a Fourier's series and proving that, if the function is subjected to suitable restrictions, this development will actually converge to the value of the function. Considerable work has already been done on such convergence criteria for double Fourier series[1]. Many other important questions related to this subject have also been treated. In particular, as in single series, there is a large body of literature dealing with summability methods for double Fourier series

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[1] For a discussion of these questions and additional references, see Geiringer [25], Gergen [26] and Tonelli [66]. Numbers in brackets refer to the bibliography at the end of this paper.

in which methods are given for associating ''sums'' to nonconvergent series. It has also been found that there are many ways in which the double Fourier series of a given function may be said to represent that function. These questions and related ones have been investigated for extremely broad classes of functions. A phase of this theory which is concerned with the absolute convergence of double Fourier series has been discussed but slightly*. One purpose of this paper is to obtain analogues for some of the results on absolute convergence of single Fourier series and to secure other conditions which lead to the absolute convergence of double Fourier series.

There are two important situations wherein the treatment of the single Fourier series differs from that of the double Fourier series. One difference occurs in the examination of tests for convergence at a point and the other difference arises from a comparison of the corresponding Taylor's series of one and of two complex variables respectively. For the convergence of simple Fourier series at a point, the only conditions imposed on the function, other than integrability, are neighborhood conditions. This is possible since by means of the Riemann-Lebesgue theorem, it is shown that the behavior of an integrable function $f(x)$ in $(-\pi,-\delta)$ and (δ,π), for $\delta > 0$, has no effect on the convergence

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*The only results that seem to be known are those of Hilda Geiringer [25], C.N. Moore [49] and Bochner [8]. Geiringer states two theorems without proofs, Bochner proved the Theorem A.20 stated page 16 of this paper, and Moore published an abstract of a generalization of Zygmund's Theorem A.7 stated page 13.

of the single Fourier series at the origin. To obtain convergence of the double Fourier series at a point, however, it is essential to impose some condition on the function, other than integrability, in a ''cross-neighborhood'' of the point and usually the tests impose a condition on the function over the entire fundamental square of definition. This is due to the fact that, although by the analogue of the Riemann-Lebesgue theorem it follows that the convergence at the origin of the double Fourier series of an integrable function f(x,y) is independent of the behavior of f(x,y) in the regions $R_1(\delta,\delta;\pi,\pi)$, $R_2(-\pi,\delta;-\delta,\pi)$, $R_3(\delta,-\pi;\pi,-\delta)$, $R_4(-\pi,-\pi;-\delta,-\delta)$, $(\delta > 0)$, the convergence is not independent of the behavior of f(x,y) in the remainder of $R(-\pi,-\pi;\pi,\pi)$[3].

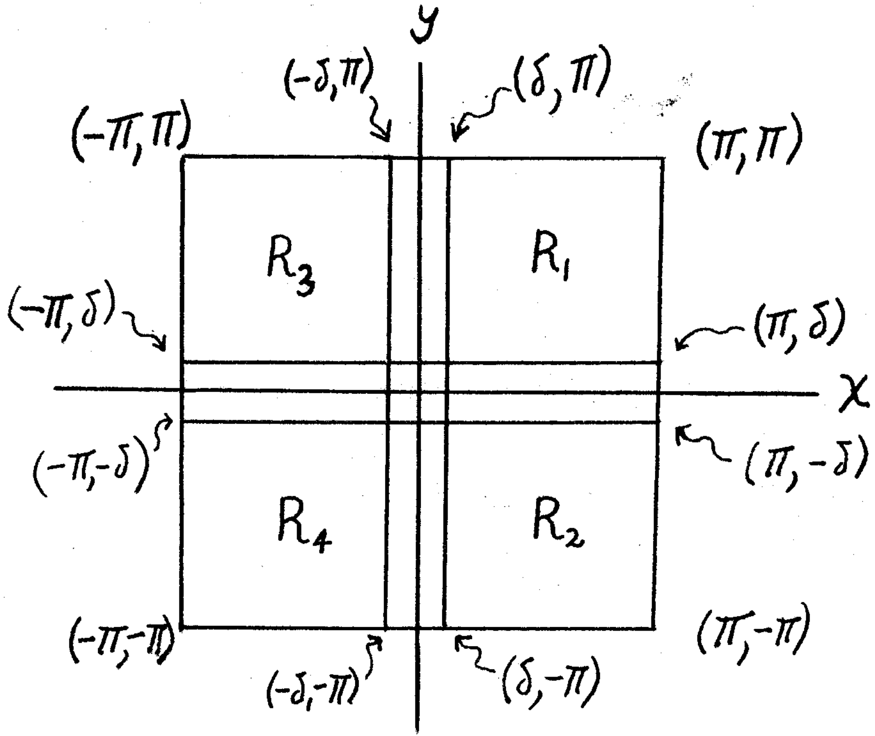

Fig.1

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[3] See Geiringer [25, p.106] and Gergen [26, p.33].

In the second situation concerning the power series it is well known that the discussion of the Fourier series of a function of one variable amounts to the discussion of the real part of the Taylor's series of a function of one complex variable. This is not the case with double series since the discussion of the real part of the Taylor's power series of a function of two complex variables does not correspond to the discussion of the double Fourier series of a function of two real variables. The distinction will be shown in detail later.

There are several ways of defining the sum of a double trigonometric series according to the manner of summing employed. Thus, one is immediately confronted with many possible extensions of the convergence of single trigonometric (or Fourier) series to that of double trigonometric (or double Fourier) series. Sums by rows and columns, over triangular regions, rectangular or square regions, and by the spherical process have been employed. Usually the sum is taken in the sense of Pringsheim[4] but the spherical sum of Fourier series employed by Bochner [8] removes the necessity of a ''cross-neighborhood'' condition for that type of sum. We shall assume the sum to be taken in the sense of Pringsheim, al-

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[4] Pringsheim [53]: The series $\sum_{m,n=0}^{\infty} u_{mn}$ converges, to sum S, or $\lim_{m,n=\infty} S_{mn}=S$ in the Pringsheim sense, if there corresponds to every number $0 < \varepsilon$ an integer N such that, if $N \leq m, N \leq n$, then $|S_{mn}-S| \leq \varepsilon$.

though if the series converges absolutely, the sum is independent of the manner of summing.

CHAPTER II

NOTATION, DEFINITIONS AND KNOWN RESULTS

The function $f(x,y)$ is assumed to be defined in the square $R(-\pi \leq x \leq \pi; -\pi \leq y \leq \pi)$ as a real valued function of the two real variables x and y, of period 2π in each variable, and integrable in the sense of Lebesgue. If, moreover, $f(x,y)$ is integrable to the p'th power in absolute value in R, it will be said to be in class L^p in R, or simply in $L^p(R)$.

We shall use the notation

(2.1) $\Delta f(x,y;h,k) = f(x+h,y+k)-f(x,y)$

(2.2) $\Delta_{10} f(x,y;h) = f(x+h,y)-f(x,y)$

(2.3) $\Delta_{01} f(x,y;k) = f(x,y+k)-f(x,y)$

(2.4) $\Delta_{11} f(x,y;h,k)=f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)$.

By a ''net covering R'' we shall mean a set of parallels to the axes:

$$(2.5)\quad \begin{aligned} &x=x_i (i=0,1,\dots,m), \quad -\pi=x_0 < x_1 < \dots < x_m=\pi \\ &y=y_j (j=0,1,\dots,n), \quad -\pi=y_0 < y_1 < \dots < y_n=\pi. \end{aligned}$$

If the points $(x_i,y_j),(x_{i+1},y_{j+1}),(x_{i+1},y_j),(x_i,y_{j+1})$ represent the corners of a cell of a net covering R, then we let

(2.6) $\Delta f(x_i,y_j) = f(x_{i+1},y_{j+1})-f(x_i,y_j)$

(2.7) $\Delta_{10} f(x_i,y_j) = f(x_{i+1},y_j)-f(x_i,y_j)$

(2.8) $\Delta_{01} f(x_i,y_j) = f(x_i,y_{j+1})-f(x_i,y_j)$

(2.9) $\Delta_{11} f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_{i+1},y_j)-f(x_i,y_{j+1})+f(x_i,y_j)$.

We also employ the following notation

$$(2.10)\begin{cases} \lambda_{oo}=\frac{1}{4},\ \lambda_{mo}=\lambda_{on}=\frac{1}{2},\ \lambda_{mn}=1, \quad (m>0,n>0), \\ A_{oo}=\frac{1}{4}a_{oo},\ A_{mo}(x)=\lambda_{mo}(a_{mo}\cos mx+b_{mo}\sin mx),\ m>0, \\ A_{on}(y)=\lambda_{on}(a_{on}\cos ny+c_{on}\sin ny),\quad n>0, \\ A_{mn}(x,y)=\lambda_{mn}(a_{mn}\cos mx\cos ny+b_{mn}\sin mx\cos ny \\ \qquad +c_{mn}\cos mx\sin ny+d_{mn}\sin mx\sin ny), \\ \qquad\qquad m\geq 0,\ n\geq 0. \end{cases}$$

With these expressions we write a double trigonometric series

$$(2.11)\quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{mn}(a_{mn}\cos mx\cos ny+b_{mn}\sin mx\cos ny + c_{mn}\cos mx\sin ny+d_{mn}\sin mx\sin ny),$$

in the form

$$(2.11)'\quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}(x,y)=A_{oo}+\sum_{m=1}^{\infty}A_{mo}(x)+\sum_{n=1}^{\infty}A_{on}(y)+\sum_{m,n=1}^{\infty}A_{mn}(x,y).$$

The double Fourier series of f(x,y) may also be written in the form (2.11)' if the coefficients are defined in the following manner

$$(2.12)\left.\begin{cases} a_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\cos nv\,du\,dv \\ b_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\cos nv\,du\,dv \\ c_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\sin nv\,du\,dv \\ d_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\sin nv\,du\,dv \end{cases}\right\} m,n\geq 0.$$

It is well known that the extension of the idea of the continuity of a function of one real variable leads to the so-called "modulus of continuity" and "Lipschitz" conditions [cf. Zygmund 71,p.17] . These Lipschitz conditions and integrated Lipschitz conditions give classes of functions which are important for the discussion of the convergence, uniform convergence, and absolute convergence of single Fourier series. Hardy and Littlewood [34] , [36] and [35] , have obtained relations between different Lipschitz classes and they have made many important applications. Particular results concerning the absolute convergence of the single Fourier series have been demonstrated by Bernstein 5 and [6] , Bochner [8] , Hille and Tamarkin [38] , Salem [56] , Szasz [59] , [60] and [61] , Tonelli [64] and [65] , Wiener [68] , and Zygmund [71] , [72] , [73] and [74] *.

Tonelli [65,pp.450-472] , Faedo [20] , Cesari [13] and others have shown that a function of two real variables contained in certain classes of functions defined by means of Lipschitz conditions possesses a convergent double Fourier series. In order to investigate the relations between such classes, as well as to consider the question of the absolute convergence of the corresponding double Fourier series, we wish to define some "Lipschitz" and " Integrated Lipschitz" conditions on f(x,y).

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*See page 12 of this paper for a statement of some of these results.

In the following definitions, α and β are positive constants (>0) and $K=K_f$ is a constant depending only on the function $f(x,y)$ in R, and independent of (x,y).

Def. 1. Lip(α,β) denotes the class of functions such that

$$|\Delta f(x,y;h,k)| \equiv |f(x+h,y+k)-f(x,y)| \leq K(|h|^{\alpha}+|k|^{\beta}),$$

for all points of R.

Def. 2. Lip$_{11}(\alpha,\beta)$ denotes the class of functions such that

$$|f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)| \leq K(|h|^{\alpha}+|k|^{\beta}),$$

for all points of R (i.e. $|\Delta_{11}f(x,y;h,k)| \leq K(|h|^{\alpha}+|k|^{\beta})$).

Def. 3. Lip α in x, denotes the class of functions such that

$$|\Delta_{10}f(x,y;h)| \equiv |f(x+h,y)-f(x,y)| \leq K\,|h|^{\alpha},$$

K independent of x and y, for all points of R.

Def. 4. Lipβ in y, denotes the class of functions such that

$$|\Delta_{01}f(x,y;k)| \equiv |f(x,y+k)-f(x,y)| \leq K\,|k|^{\beta},$$

K independent of x and y, for all points of R.

Def. 5. LipS$_{11}(\alpha,\beta)$ denotes the class of functions such that

$$|f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)| \leq K\,|h|^{\alpha}\,|k|^{\beta},$$

for all points of R (i.e. $|\Delta_{11}f(x,y;h,k)| \leq K\,|h|^{\alpha}\,|k|^{\beta}$).

Def. 6. Lip$(\alpha,\beta;p)$ denotes the class of functions in $L^p(R)$ such that

$$M(\Delta f;h,k;p)=\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x,y)|^p\,dxdy\right)^{\frac{1}{p}}\leq K(|h|^{\alpha}+|k|^{\beta}).$$

Def. 7. Lip$(\alpha,\beta;p)$ denotes the class of functions in $L^p(R)$ such that

$$M(\Delta f;h,k;p) \leq K(|h|^{\alpha} + |k|^{\beta})$$

where

$$M(\Delta_{11}f;h,k;p) = \left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\Delta_{11}f(x,y;h,k)|^p\,dx\,dy\right)^{\frac{1}{p}}$$

and $\Delta_{11}f(x,y;h,k;p) = f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)$.

Def. 8. Lip $S_{11}(\alpha,\beta;p)$ denotes the class of functions in $L^p(R)$ such that

$$M(\Delta_{11}f;h,k;p) \leq K\,|h|^{\alpha}\,|k|^{\beta},$$

where $M(\Delta_{11}f;h,k;p)$ is defined in Def. 7 above.

Def. 9. $\mathrm{Lip}_{10}(\alpha;p)$ in x, denotes the class of functions in $L^p(R)$ such that

$$M(\Delta_{10}f;h,y;p) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h,y)-f(x,y)|^p\,dx\right)^{\frac{1}{p}} \leq K\,|h|^{\alpha}$$

uniformly for y in $(-\pi,\pi)$.

Def. 10. $\mathrm{Lip}_{01}(\beta;p)$ in y, denotes the class of functions in $L^p(R)$ such that

$$M(\Delta_{01}f;x,k;p) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x,y+k)-f(x,y)|^p\,dy\right)^{\frac{1}{p}} \leq K\,|k|^{\beta}$$

uniformly for x in $(-\pi,\pi)$.

We also quote four Lipschitz classes for a function $g(x)$ of one variable.

Def. 11. A function $g(x)$ is in $\mathrm{Lip}\,\alpha$ $(\alpha > 0)$ if

$$|g(x+h)-g(x)| \leq K\,|h|^{\alpha} \text{ uniformly in } x \text{ in } (-\pi,\pi),$$

and $K=K_g$ is a constant depending only on $g(x)$.

Note: The class $\mathrm{Lip}\,\alpha$ in x, given in Def. 3, reduces to this class $\mathrm{Lip}\,\alpha$ when $f(x,y)\equiv g(x)$ since $\Delta_{10}f(x,y;h)$ reduces to $g(x+h)-g(x)$.

Def. 12. A function $g(x)$ in L^p is in $\mathrm{Lip}(\alpha;p)$ if

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x+h)-g(x)|^p dx\right)^{\frac{1}{p}} \leq K\,|h|^{\alpha}$$

Note: If $f(x,y)=g(x)$, then $\mathrm{Lip}_{10}(\alpha;p)$ in x reduces to $\mathrm{Lip}(\alpha;p)$.

Def. 13. A function $g(x)$ in L^p is in $\mathrm{Lip}_1(\alpha;p)$ if

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x+h)-g(x-h)|^p dx\right)^{\frac{1}{p}} \leq K\,|h|^{\alpha}$$

for $0<\alpha \leq 1$, $1< p\leq 2$.

Def. 14. A function $g(x)$ in L^p is in $\mathrm{Lip}_2(\alpha;p)$ if

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x+2h)+g(x-2h)-2g(x)|^p dx\right)^{\frac{1}{p}} \leq K\,|h|^{\alpha}$$

for $0<\alpha \leq 1$, $1<p\leq 2$.

For convenience we state here several known results on the absolute convergence of single series. Let f(x) be a real-valued function of the real variable x, of period 2π, and f(x) integrable in the sense of Lebesgue. For simplicity in writing we designate by FS the formal Fourier series of f(x)

$$(2.13)\qquad f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$(2.14)\qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int} dt = \tfrac{1}{2}(a_n - ib_n),\ c_{-n} = \overline{c_n},\ n = 0, \pm 1, \ldots$$

We shall also use the abbreviation ab.cv. for absolutely convergent everywhere. The known results may now be listed:

(A.1) If f(x) is in Lipα, $\alpha > \frac{1}{2}$, then the FS is ab.cv.; for $\alpha < \frac{1}{2}$ this is no longer true. Bernstein [5].

(A.2) If f(x) is in Lipα, $0 < \alpha \leq 1$, the series

$$(2.15)\qquad \sum_{n=1}^{\infty} \left(|a_n|^{\lambda} + |b_n|^{\lambda} \right)$$

converges for every $\lambda > \frac{2}{2\alpha + 1}$, but not necessarily for $\lambda < \frac{2}{2\alpha + 1}$. Moreover, for $\alpha > \frac{1}{2}$ it is shown by examples that even for $\lambda = \frac{2}{2\alpha + 1}$ the series (2.15) may diverge. Szasz [59]. Zygmund [71, p.138] quotes examples of Hardy and Littlewood where (2.15) diverges for $0 < \alpha \leq \frac{1}{2}$, $\lambda = \frac{2}{2\alpha + 1}$.

(A.3) If f(x) is in $\mathrm{Lip}_2(\alpha;2)$, $0<\alpha\leq 1$, then (2.15) converges for every $\lambda > \frac{2}{2\quad +1}$ and there exist functions in $\mathrm{Lip}_2(\alpha;2)$ such that the series (2.15) diverges for $\lambda = \frac{2}{2\alpha+1}$. Szasz [59].

(A.4) If f(x) is absolutely continuous and f' is in L^p, $p>1$, then the FS is ab.cv. Tonelli [64].

(A.5) The N. & S. condition that FS be ab.cv. is that f(x) be represented in the form

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(t)\, f_2(x-t)\, dt, \quad \text{where } f_1 \text{ and } f_2 \text{ are in } L^2.$$

Hardy and Littlewood [34].

(A.6) If f(x) is in $\mathrm{Lip}_1(\alpha;p)$, $\alpha p=1$, $p<2$ and also f(x) in $\mathrm{Lip}\,\beta$, $\beta>0$, then FS is ab.cv. Hardy and Littlewood [34].

(A.7) If f(x) is of bounded variation and if f(x) is in $\mathrm{Lip}\,\alpha$ for any $\alpha>0$, then the FS is ab.cv. Zygmund [74]. (This result includes that of A.4.)

(A.8) If f(x) is continuous and of bounded variation over $(-\pi,\pi)$ with the modulus of continuity

$$\omega(h) = \max_{|t|\leq h}\ \max_{x} |f(x+t)-f(x)| \tag{2.16}$$

being such that

$$\int_0^{\pi}\sqrt{\omega(h)}\,\frac{dh}{h}<\infty, \quad \text{then the FS is ab.cv.}$$

Hille and Tamarkin [38], and implicitly by Zygmund [74].

(A.9) If f(x) is of bounded variation over $(-\pi,\pi)$ and also f(x) is in $\mathrm{Lip}_1(\alpha;p)$, $\alpha p>1$, then the FS is ab.cv. Hardy and Littlewood [34].

(A.10) If FS and its conjugate

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \tag{2.17}$$

are both Fourier series of functions of bounded variation, then FS is ab.cv. Hardy and Littlewood [34]. (For a statement and proof of this theorem cf. Zygmund [71,p.139] and [pp.157-160]).

(A.11) If f(x) is in $\mathrm{Lip}_1(\alpha;p)$, $0<\alpha\leq 1$, $\alpha p>1$, $p\leq 2$, then the FS is ab.cv. In fact, (2.15) converges for $\lambda>\frac{p}{p+p\alpha-1}$ whenever $0<\alpha\leq 1$, $1<p\leq 2$. Hardy and Littlewood [35]. Moreover there are functions in $\mathrm{Lip}_1(\alpha;p)$ where (2.15) diverges for $\lambda=\frac{p}{p+p\alpha-1}$. (Theorem A.3 contains the case p=2.)

(A.12) If f(x) is in $\mathrm{Lip}_2(\alpha;p)$, $0<\alpha\leq 1$, $1<p\leq 2$, then (2.15) converges for $\lambda>\frac{p}{p+p\alpha-1}$, and there are functions in $\mathrm{Lip}_2(\alpha;p)$ where (2.15) diverges for $\lambda=\frac{p}{p+p\alpha-1}$. Szasz [60].

(A.13) If f(x) is absolutely continuous, F'=f, and $|f|\log^+|f|$ integrable, then the FS is ab.cv. Zygmund [73].

(A.14) If $f(x)$ is of bounded variation and in Lipα, $0<\alpha\leq 1$, then (2.15) converges for $\lambda>\frac{2}{2+\alpha}$. Waraszkiewicz [67] and Zygmund [72].

(A.15) If to every point $x°$ corresponds a neighborhood $I_{x°}$ of $x°$ and a function $g(x)=g_{x°}(x)$ such that (i) the FS of g is ab.cv., and (ii) $g(x)=f(x)$ in $I_{x°}$, then the FS of f is ab.cv. Wiener [68].

(A.16) Let the FS of $f(t)$ be ab.cv., and let the values of $f(t)$ belong to an interval (a,b). If $g(z)$ is a function of a complex variable, regular at every point of (a,b), the FS of $g\{f(t)\}$ is ab.cv. Wiener [68].

(A.17) If $f(x)$ is of bounded variation over $(-\pi,\pi)$ and in addition,

$$\sum_{n=1}^{\infty} n^{-1}\, W\left(\frac{\pi}{2n}\right) < \infty, \text{ where } W(h)=\frac{1}{2\pi}\int_{-\pi}^{\pi} |d_t[f(t+h)-f(t-h)]|,$$

then the FS is ab.cv. Hille and Tamarkin [38].

(A.18) If the modulus of continuity of $f(x)$, $\omega(h)$, is such that

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}}\, \omega\left(\frac{1}{n}\right) < \infty,$$

then the FS is ab.cv. Bernstein [6].

(A.19) Let $\sum_{n=1}^{\infty}\rho_n\cos(nx-\alpha_n)$ be the FS of a continuous function which has the modulus of continuity $\omega(h)$. Let $F(u)$ be a positive function, increasing and concave, for positive u, tending to zero with u. If the series

$$\sum_{n=1}^{\infty} F\left[\frac{1}{n}\,\omega^2\!\left(\frac{1}{n}\right)\right]$$

converges, then the series $\sum_{n=1}^{\infty} F(\rho_n^2)$

converges also. Salem [56].

(A.20) Let $x=(x_1,x_2,\dots x_k)$ be a point in k-dimensional Euclidean space and

$$f(x)\sim \sum a_{n_1\dots n_k}\, e^{i(n_1x_1+n_2x_2+\dots+n_kx_k)}$$

$$a_{n_1\dots n_k} = \frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi} f(x)\, e^{-i(n_1x_1+n_2x_2+\dots+n_kx_k)}dx$$

and consider the spherical sums

$$S_R(x) = \sum_{\gamma\le R} a_{n_1\dots n_k}\, e^{i(n_1x_1+\dots+n_kx_k)}, \quad \gamma^2=n_1^2+n_2^2+\dots+n_k^2.$$

Then if for a fixed point x, the function

$$f_x(t) = (2\pi)^{-\frac{k}{2}}\int_{\sigma} f(x_1+t\zeta_1,\dots,x_k+t\zeta_k)\,d\sigma_\zeta,$$

$\zeta_1^2+\dots+\zeta_k^2=1$ is the unit sphere, $d\sigma_\zeta$ the (k-1)-dim.vol. element of σ,

has k+4 derivatives which are all bounded in $0\le t<\infty$, then

$$\sum_{\gamma=0}^{\infty}\left|\sum_{n=\gamma^2} a_{n_1\dots n_k}\, e^{i(n_1x_1+\dots+n_kx_k)}\right|$$

converges toward $f_x(0)=f(x)$. Bochner [8].

(A.21) Let
$$\begin{aligned}\Delta_0 f(x;t)&=f(x)\\ \Delta_1 f(x;t)&=f(x+t)-f(x-t)\\ \Delta_2 f(x;t)&=\Delta_1\Delta_1 f(x;t)=f(x+2t)+f(x-2t)-2f(x)\\ \Delta_m f(x;t)&=\Delta_1\Delta_{m-1} f(x;t)\\ &=\sum_{v=0}^{m}(-1)^v C_{m,v} f(x+(m-2v)t),m=1,2,..\end{aligned}$$

and let $\omega_p^{(m)}(\delta)=\omega_p^{(m)}(f;\delta)=\sup_{0\leq t\leq\delta}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_m f(x;t)|^p dx\right)^{\frac{1}{p}}$

where for p=1, $\sup|g(x)|$ denotes the "effective upper bound of $|g(x)|$" as defined in [37,p.135].

(i) If f(x) in L^p, $k\ 0$, and if

$$\begin{cases}\sum_{n=1}^{\infty} n^{-\frac{k}{p'}}\left(\omega_p^{(m)}\left(\frac{\pi}{2n}\right)\right)^k<\infty & \text{for } 1<p\leq 2,\\ \sum_{n=1}^{\infty}\left(\omega_1^{(m)}\left(\frac{\pi}{2n}\right)\right)^k<\infty & \text{for } p=1,\end{cases}$$

then $\sum_{n=0}^{\infty}|c_n|^k<\infty$. Szasz [61,p.376].

(ii) If f(x) in L^{p_2}, $1\leq p\leq 2$, $0<p_1\leq p\leq p_2$, $\rho=\rho_1 p_1+\rho_2 p_2$, $\rho_1+\rho_2=1$, $\rho_1\geq 0$, $\rho_2\geq 0$, and if

$$\sum_{n=1}^{\infty} n^{-\frac{k}{p'}}\left(\omega_{p_1}^{(m)}\left(\frac{\gamma}{n}\right)^{\rho_1 p_1}\omega_{p_2}^{(m)}\left(\frac{\gamma}{n}\right)^{\rho_2 p_2}\right)^{\frac{k}{p}}<\infty,\quad 0<k\leq p',$$

where γ is an arbitrary positive constant, then

$\sum_{n=0}^{\infty}|c_n|^k<\infty$. Szasz [61,p.378].

These theorems contain the following theorems as special cases:(A.1),(A.2),(A.3),(A.6),(A.11),(A.12),(A.14) and (A.18).

(B.1) If the trigonometric series (2.13) converges for all values of x in $(-\pi,\pi)$, then

$$\lim_{n=\infty} a_n=0 \quad \text{and} \quad \lim_{n=\infty} b_n=0. \quad \text{Cantor [10]}.$$

More generally, the conclusion holds if merely

$$\lim_{n=\infty} (a_n \cos nx + b_n \sin nx) = 0$$

for all points of the interval $(-\pi,\pi)$. Osgood [51,p.337].

(B.2) If $\lim_{n=\infty} (a_n \cos U_n x + b_n \sin U_n x) = 0$ for all points of $(-\pi,\pi)$, where U_n is any function of n which merely becomes infinite with n (i.e. $\lim_{n=\infty} |U_n|=\infty$), then

$$\lim_{n=\infty} a_n = 0 \quad \text{and} \quad \lim_{n=\infty} b_n = 0.$$

Osgood [51,p.339].

(B.3) If the trigonometric series (2.13) is ab.cv. at every point of $(-\pi,\pi)$, then the two series of coefficients

(2.18) $$\sum_{n=1}^{\infty} a_n$$

(2.19) $$\sum_{n=1}^{\infty} b_n$$

are ab.cv. Fatou [21].

(B.4) If the trigonometric series (2.13) is ab.cv. in a set E of positive measure, then the two series of coefficients (2.18) and (2.19) are ab.cv. Denjoy [17], and Lusin [46].

(B.5) If the series

$$\sum_{n=1}^{\infty} a_n \cos nx\,, \qquad |a_1| \geq |a_2| \geq \ldots,$$

is ab.cv. at a point x°, then (2.18) is ab.cv. Likewise, if

$$\sum_{n=1}^{\infty} b_n \sin nx, \qquad |b_1| \geq |b_2| \geq \ldots,$$

is ab.cv. at a point $x^\circ \not\equiv 0 \pmod{\pi}$, then (2.19) is ab.cv. and

$$\sum_{n=1}^{\infty} |b_n| |\sin nx^\circ| < \infty\,. \qquad \text{Fatou [22].}$$

(B.6) A point x° is said to be a point of symmetry for a set of points if a property which holds at the two points x° and $x^\circ+h$ (or $x^\circ-h$) implies the same property at $x^\circ-h$ (or $x^\circ+h$). If B represents the set of points where (2.13) is convergent, $\overline{B}$ the set where the conjugate series of (2.13) is convergent, A the set of ab.cv. of (2.13) and $\overline{A}$ the set of ab.cv. of the conjugate series, then every point of A is a point of symmetry for the sets $A,\overline{A},B,\overline{B}$. Fatou [22].

(B.7) With the notation of (B.6), if A possesses an infinite number of points, then B, and similarly $\overline{B}$, is either of measure 0 or the measure of the whole interval. Lusin[47].

(B.8) If (2.13) is ab.cv. in a set of the second catagory(i.e. a set which cannot be expressed as the sum of a countable set of point sets each of which is in no part of a given interval everywhere dense), even if it is of measure zero, the series (2.18) and (2.19) are ab.cv. Lusin [47].

(B.9) A point set A is called a basis, if every real x can be represented in the form $d_1x_1+d_2x_2+\ldots+d_kx_k$, where $d_1, d_2, \ldots$ are integers, and $x_1, x_2, \ldots$ belong to A. If A is a basis, every trigonometric series ab.cv. in A is ab.cv. everywhere. Cf. Niemytzki [50] for the case of sine series and Zygmund [71,p.133] for the general case.

(B.10) If

$$\sum_{k=1}^{\infty}(a_k\cos n_kx + b_k\sin n_kx)$$

is a FS of a bounded f(x) with $\frac{n_{k+1}}{n_k} > \lambda > 1$, then the FS is ab.cv. Szidon [62].

We now quote two well known theorems which will be needed in the sequel.

(C.1) Theorem of Baire. The necessary and sufficient condition that a function f(x,y), defined in a plane set E, which is either perfect, or open or the set of points common to a perfect and an open set, is the limit of a sequence of functions, all of which are continuous in E, is that f(x,y) be, at most, pointwise discontinuous with respect to every perfect set contained in E. Hobson [40,p.273]. Baire [1].

(C.2) Theorem of Egoroff. If E be a measurable two dimensional set of finite measure $|E|$, the necessary and sufficient condition that a sequence of measurable finite functions should converge to a finite function f(x,y), almost everywhere in E, is that a set H_ε, contained in E, and of measure greater than $|E|-\varepsilon$, where ε is an arbitrarily chosen positive number, exists, such that the sequence converges uniformly in H_ε to $f(x,y)$. Egoroff [19]. Hobson [40,p.144].

The following inequalities will be frequently used.

(i) <u>Hoelder inequalities</u>. Assuming $p>1$, we have

$$(2.20)\quad \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)g(x,y)|\,dxdy \leq$$

$$\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^{p}dxdy\right)^{\frac{1}{p}}\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|g(x,y)|^{\frac{p-1}{p}}dxdy\right)^{\frac{p-1}{p}}$$

$$(2.21)\quad \sum_{m,n=1}^{\infty}|P_{mn}Q_{mn}| \leq \left(\sum_{m,n=1}^{\infty}|P_{mn}|^{p}\right)^{\frac{1}{p}}\left(\sum_{m,n=1}^{\infty}|Q_{mn}|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

By placing

$$P_{mn} = D_{mn}^{\frac{1}{p}}E_{mn},\quad Q_{mn} = D_{mn}^{\frac{p-1}{p}},\quad p\geq 1,\ D_{mn}\geq 0,\ E_{mn}\geq 0,$$

we obtain a special finite form of (2.21)

$$(2.22)\quad \sum_{m=m_1}^{m_2}\sum_{n=n_1}^{n_2}D_{mn}E_{mn} \leq \left(\sum_{m=m_1}^{m_2}\sum_{n=n_1}^{n_2}D_{mn}E_{mn}^{p}\right)^{\frac{1}{p}}\left(\sum_{m=m_1}^{m_2}\sum_{n=n_1}^{n_2}D_{mn}\right)^{1-\frac{1}{p}}$$

Let $D_{mn}=1$, then (2.22) becomes

$$(2.23)\quad \sum_{m=m_1}^{m_2}\sum_{n=n_1}^{n_2}E_{mn} \leq (m_0n_0)^{1-\frac{1}{p}}\left(\sum_{m=m_1}^{m_2}\sum_{n=n_1}^{n_2}E_{mn}^{p}\right)^{\frac{1}{p}},\ \begin{cases} m_0=m_2-m_1+1\\ n_0=n_2-n_1+1\end{cases}$$

(ii) Minkowski inequalities. If $p \geq 1$, then

$$(2.24)\quad \left[\int_{-\pi}^{\pi} dx\left(\int_{-\pi}^{\pi} |f(x,y)|\,dy\right)^{p}\right]^{\frac{1}{p}} \leq \int_{-\pi}^{\pi} dy\left(\int_{-\pi}^{\pi} |f(x,y)|^{p}\,dx\right)^{\frac{1}{p}}$$

$$(2.25)\quad \left(\sum_{m,n=1}^{\infty} |P_{mn}+Q_{mn}|^{p}\right)^{\frac{1}{p}} \leq \left(\sum_{m,n=1}^{\infty} |P_{mn}|^{p}\right)^{\frac{1}{p}} + \left(\sum_{m,n=1}^{\infty} |Q_{mn}|^{p}\right)^{\frac{1}{p}} .$$

We refer to Hardy, Littlewood and Polya [37] for the proofs of (2.20),(2.21),(2.24) and (2.25).

CHAPTER III

CLASSES OF FUNCTIONS AND RELATIONS BETWEEN THEM

3.A. Properties of the Lipschitz classes.

Property 1. If $f(x,y)$ is in the class $\mathrm{Lip}(\alpha, \beta)$ in R, then $f(x,y)$ is continuous in the two variables in R.

Proof. By hypothesis we have

$$(3.1)\quad |\Delta f(x,y;h,k)| \equiv |f(x+h,y+k)-f(x,y)| \leq K(|h|^{\alpha}+|k|^{\beta}), \alpha>0, \beta>0,$$

for all points (x,y) in R. It must be shown that for an arbitrarily preassigned $\varepsilon > 0$ a $\delta > 0$ can be chosen such that

$$(3.2)\qquad |f(x+h,y+k)-f(x,y)| \leq \varepsilon \quad \text{for} \quad |h|<\delta, \quad |k|<\delta.$$

Now, for a given $\varepsilon > 0$, we choose δ as the smaller of the two numbers δ_1 and δ_2 given by the relations

$$\delta_1 = \left(\frac{\varepsilon}{2K}\right)^{\frac{1}{\alpha}} > 0, \quad \delta_2 = \left(\frac{\varepsilon}{2K}\right)^{\frac{1}{\beta}} > 0, \quad \alpha > 0, \quad \beta > 0.$$

Then for $|h| < \delta \leq \delta_1$, $|k| < \delta \leq \delta_2$, we have by (3.1) that

$$|f(x+h,y+k)-f(x,y)| \leq K(|h|^{\alpha}+|k|^{\beta}) < K\left(\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}\right) = \varepsilon,$$

which is the desired result (3.2).

Property 2. If $f(x,y)$ is independent of x or of y, then

$$\Delta_{11}f(x,y;h,k) \equiv f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y) \equiv 0.$$

Proof. Let $f(x,y) \equiv g(x)$, then

$$\begin{aligned} \Delta_{11}f(x,y;h,k) &\equiv g(x+h)-g(x+h)-g(x)+g(x) \\ &\equiv 0. \end{aligned}$$

Similarly for the other case.

<u>Property 3</u>. There are infinitely many functions $f(x,y)$ in $\mathrm{Lip}_{11}(\alpha,\beta)$ and in $\mathrm{Lip}\ S_{11}(\alpha,\beta)$ that are not continuous in (x,y).

<u>Proof</u>. Consider $f(x,y)=g(x)$, where $g(x)$ is a discontinuous function of x. Then $f(x,y)$ is discontinuous in (x,y), while by Property 2

$$|\Delta_{11}f(x,y;h,k)|=0\leq K(|h|^\alpha+|k|^\beta) \quad \text{or} \leq K|h|^\alpha|k|^\beta,$$

and hence $f(x,y)$ is in $\mathrm{Lip}_{11}(\alpha,\beta)$ and $\mathrm{Lip}\ S_{11}(\alpha,\beta)$ by the definitions given on page 9.

<u>Property 4</u>. There are functions in $\mathrm{Lip}\ S_{11}(\alpha,\beta)$ that are not in $\mathrm{Lip}(\alpha,\beta)$.

<u>Proof</u>. This is an immediate consequence of Property 1 and Property 3.

<u>Property 5</u>. If $f(x,y)=G(x)H(y)$ and if in addition $G(x)$ is in $\mathrm{Lip}(\alpha;p)$ and $H(y)$ is in $\mathrm{Lip}(\beta;p)$, then $f(x,y)$ is in $\mathrm{Lip}\ S_{11}(\alpha,\beta;p)$. If $G(x)$ in $\mathrm{Lip}\,\alpha$ and $H(y)$ in $\mathrm{Lip}\,\beta$, then $f(x,y)$ is in $\mathrm{Lip}\ S_{11}(\alpha,\beta)$.

<u>Proof</u>. Since

$$\Delta_{11}f(x,y;h,k)=f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y),$$

then

$$\begin{aligned}\Delta_{11}f(x,y;h,k)&=G(x+h)H(y+k)-G(x+h)H(y)-G(x)H(y+k)+G(x)H(y)\\ &=G(x+h)\left[H(y+k)-H(y)\right]-G(x)\left[H(y+k)-H(y)\right]\\ &=\left[G(x+h)-G(x)\right]\left[H(y+k)-H(y)\right]\\ &=\Delta_{10}G(x,h)\,\Delta_{01}H(y,k)\end{aligned}$$

With the assumption that G(x) is in Lip α and H(y) is in Lip β this relation together with Def.11 gives

$$|\Delta_{11}f(x,y;h,k)| \leq (K|h|^{\alpha})(K|k|^{\beta}) = K' \; |h|^{\alpha}\,|k|^{\beta}$$

and by Def. 5 we have f(x,y) is in Lip $S_{11}(\alpha,\beta)$. This proves the second part of Property 5.

Since $\Delta_{11}f(x,y;h,k) = \Delta_{10}G(x,h)\,\Delta_{01}H(y,k)$, then

$$|\Delta_{11}f(x,y;h,k)|^{p} = |\Delta_{10}G(x,h)|^{p}\,|\Delta_{01}H(y,k)|^{p}$$

$$\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta_{11}f(x,y;h,k)|^{p}dx\,dy\right)^{\frac{1}{p}}$$

$$=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_{10}G(x,h)|^{p}dx\right)^{\frac{1}{p}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_{01}H(y,k)|^{p}dy\right)^{\frac{1}{p}}$$

$$\leq (K_1|h|^{\alpha})(K_2|k|^{\beta})$$

$$\leq K\;|h|^{\alpha}\,|k|^{\beta},$$

whence f(x,y) is in Lip $S_{11}(\alpha,\beta;p)$ by Def. 8.

3.B. Relations between the Lipschitz classes. The following relations between the Lipschitz classes will be discussed and established in the sequel. In the writing of the relations we employ the symbol ($\geq$) between classes to mean that any function in the class on the right hand side is also in the class on the left hand side.

R1.
$$\begin{cases} \text{(i)} & \left.\begin{matrix}\text{Lip}\,\alpha \text{ in } x\\ \text{Lip}\,\beta \text{ in } y\end{matrix}\right\} \geq \text{Lip}(\alpha,\beta), \\ \\ \text{(ii)} & \text{Lip}_{11}(\alpha,\beta) \geq \text{Lip}(\alpha,\beta). \end{cases}$$

R2.
$$\begin{cases} \text{(i)} & \text{Lip}(\alpha,\beta) \geq \begin{cases}\text{Lip}\,\alpha \text{ in } x\\ \text{with}\\ \text{Lip}\,\beta \text{ in } y,\end{cases} \\ \\ \text{(ii)} & \text{Lip}_{11}(\alpha,\beta) \geq \begin{cases}\text{Lip}\,\alpha \text{ in } x\\ \text{with}\\ \text{Lip}\,\beta \text{ in } y.\end{cases} \end{cases}$$

R3.
$$\begin{cases} \text{(i)} & \text{Lip}_{10}(\alpha;p) \ngeq \text{Lip}(\alpha,\beta;p), \\ \text{(ii)} & \text{Lip}_{01}(\beta;p) \ngeq \text{Lip}(\alpha,\beta;p). \end{cases}$$

R4.
$$\begin{cases} \text{(i)} & \text{Lip}(\alpha,\beta;p) \geq \begin{cases}\text{Lip}_{10}(\alpha;p)\\ \text{with}\\ \text{Lip}_{01}(\beta;p),\end{cases} \\ \\ \text{(ii)} & \text{Lip}_{11}(\alpha,\beta;p) \geq \text{Lip}(\alpha,\beta;p). \end{cases}$$

R5. $\text{Lip}_{11}(\gamma,\gamma) \geq \text{Lip}\ S_{11}(\alpha,\beta)$ for $\gamma = \alpha+\beta$.

R6. $\text{Lip}(\alpha,\beta;p) \geq \text{Lip}(\alpha,\beta)$ for all $p > 0$.

R7. $\text{Lip}(\alpha,\beta;p) \downarrow$ as $p \uparrow$.

In the following relation, we assume the constant K given in the definitions of the Lipschitz conditions is independent of p.

R8. $\lim\limits_{p=\infty} \text{Lip}(\alpha,\beta;p) = \text{Lip}(\alpha,\beta)$, $\quad f(x,y)$ in $L^p(R)$

for each p.

<u>R1</u>. $$\text{(i)}\quad \left.\begin{array}{ll}\text{Lip}\,\alpha & \text{in } x\\ \text{or} & \\ \text{Lip}\,\beta & \text{in } y\end{array}\right\} \geqq \text{Lip}(\alpha,\beta),$$

$$\text{(ii)}\ \text{Lip}_{11}(\alpha,\beta) \geqq \text{Lip}(\alpha,\beta).$$

<u>Proof</u>. We first establish the relation (i). Since $f(x,y)$ is in $\text{Lip}(\alpha,\beta)$ we have by Def.1, page 9, that

$$(3.3)\qquad |\Delta f(x,y;h,k)| \equiv |f(x+h,y+k)-f(x,y)| \leq K(|h|^{\alpha}+|k|^{\beta}).$$

If $k=0$, we get

$$(3.4)\qquad |\Delta_{10}f(x,y;h)| \equiv |f(x+h,y)-f(x,y)| \leq K|h|^{\alpha},$$

which by Def.3, page 9, is $\text{Lip}\,\alpha$ in x; and if $h=0$, then

$$(3.5)\qquad |\Delta_{01}f(x,y;k)| \equiv |f(x,y+k)-f(x,y)| \leq K|k|^{\beta},$$

which by Def.4, page 9, is $\text{Lip}\,\beta$ in y.

We now show that the relation (ii) holds. We see that

$$(3.6)\qquad \begin{aligned} |\Delta_{11}f(x,y;h,k)| &= |f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)| \\ &= |f(x+h,y+k)-f(x,y)+f(x,y)-f(x+h,y)-f(x,y+k)+f(x,y)| \\ &\leq |f(x+h,y+k)-f(x,y)|+|f(x+h,y)-f(x,y)|+|f(x,y+k)-f(x,y)| \\ &\leq |\Delta f(x,y;h,k)| + |\Delta_{10}f(x,y;h)| + |\Delta_{01}f(x,y;k)| \end{aligned}$$

so that by (3.3),(3.4) and (3.5), we have

$$\begin{aligned} |\Delta_{11}f(x,y;h,k)| &\leq K(|h|^{\alpha}+|k|^{\beta}) + K|h|^{\alpha} + K|k|^{\beta} \\ &\leq 2K(|h|^{\alpha}+|k|^{\beta}), \end{aligned}$$

which is the condition $\text{Lip}_{11}(\alpha,\beta)$ given in Def.2, page 9.

<u>R2</u>. $$\text{(i)}\quad \text{Lip}(\alpha,\beta) \geqq \left\{\begin{array}{l}\text{Lip}\,\alpha \text{ in } x\\ \text{with}\\ \text{Lip}\,\beta \text{ in } y,\end{array}\right.$$

$$\text{(ii)}\ \text{Lip}_{11}(\alpha,\beta) \geqq \left\{\begin{array}{l}\text{Lip}\,\alpha \text{ in } x\\ \text{with}\\ \text{Lip}\,\beta \text{ in } y.\end{array}\right.$$

<u>Proof</u>. We first establish part (i). By hypothesis

$$|\Delta_{10}f(x,y;h)| \leq K|h|^{\alpha}, \qquad |\Delta_{01}f(x,y;k)| \leq K|k|^{\beta},$$

so that

$$
(3.7)\qquad \begin{aligned}
|\Delta f(x,y;h,k)| &= |f(x+h,y+k)-f(x,y)| \\
&= |f(x+h,y+k)-f(x+h,y)+f(x+h,y)-f(x,y)| \\
&\leq |f(x+h,y+k)-f(x+h,y)|+|f(x+h,y)-f(x,y)| \\
&\leq |\Delta_{01}f(x+h,y;k)| + |\Delta_{10}f(x,y;h)|.
\end{aligned}
$$

Applying the above hypothesis we have

$$|\Delta f(x,y;h,k)| \leq K|k|^{\beta} + K|h|^{\alpha} = K(|h|^{\alpha}+|k|^{\beta}),$$

which by Def.1, page 9, is the class Lip(α,β).

Part (ii) follows from part (ii) of R1.

<u>Remark</u>. It is obvious from (i) of R1 and R2 that Lip(α,β) is equivalent to the combination of Lip α in x and Lip β in y.

<u>R4</u>. (i) If f(x,y) is in each of the two classes $\mathrm{Lip}_{10}(\alpha;p)$ and $\mathrm{Lip}_{01}(\beta;p)$, then f(x,y) is in Lip$(\alpha,\beta;p)$;

(ii) If f(x,y) is in Lip$(\alpha,\beta;p)$, then f(x,y) is in the class $\mathrm{Lip}_{11}(\alpha,\beta;p)$. (The Lipschitz classes are defined on page 10.)

<u>Proof</u>. We first prove part (i). By assumption we have

$$
(3.8)\qquad \begin{aligned}
M(\Delta_{10}f;h,y;p) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h,y)-f(x,y)|^{p}dx\right)^{\frac{1}{p}} \leq K|h|^{\alpha} \quad \text{uniformly in } y, \\
M(\Delta_{01}f;x,k;p) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x,y+k)-f(x,y)|^{p}dy\right)^{\frac{1}{p}} \leq K|k|^{\beta} \quad \text{uniformly in } x.
\end{aligned}
$$

From (3.7) we have

$$|\Delta f(x,y;h,k)|\equiv|f(x+h,y+k)-f(x,y)|$$
$$\leq|\Delta_{10}f(x,y;h)| + |\Delta_{01}f(x+h,y;k)|,$$

whence

$$|\Delta f(x,y;h,k)|^p\leq(|\Delta_{10}f(x,y;h)|+|\Delta_{01}f(x+h,y;k)|)^p.$$

Upon integrating this inequality with respect to x and y between $-\pi$ and π and applying Minkowski's inequality, we have

$$M(\Delta f;h,k;p)\equiv\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x,y)|^p dxdy\right)^{\frac{1}{p}}$$

$$\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta_{10}f(x,y;h)|^p dxdy\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta_{01}f(x+h,y;k)|^p dxdy\right)^{\frac{1}{p}},$$

which by (3.8) gives

$$(3.9)\quad M(\Delta f;h,k;p)\leq\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(K|h|^{\alpha})^p dy\right)^{\frac{1}{p}}+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(K|k|^{\beta})^p dx\right)^{\frac{1}{p}}$$

$$\leq K(|h|^{\alpha}+|k|^{\beta}),$$

so that by Def.6, page 9, $f(x,y)$ is in $\mathrm{Lip}(\alpha,\beta;p)$.

We now prove part (ii), i.e. that if $f(x,y)$ is in the class $(\alpha,\beta;p)$ just stated above, then $f(x,y)$ is in $\mathrm{Lip}_{11}(\alpha,\beta;p)$. Thus by Def.7, page 10, we have to show that

$$M(\Delta_{11}f;h,k;p)\leq 2K(|h|^{\alpha}+|k|^{\beta}).$$

From (3.6) we have

$$|\Delta_{11}f(x,y;h,k)|=|f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)|$$
$$\leq|\Delta f(x,y;h,k)|+|\Delta_{10}f(x,y;h)|+|\Delta_{01}f(x,y;k)|.$$

Then for $p\geq 1$, we have

$$|\Delta_{11}f(x,y;h,k)|^p\leq(|\Delta f(x,y;h,k)|+|\Delta_{10}f(x,y;h)|+|\Delta_{01}f(x,y;k)|)^p,$$

and by Minkowski's inequality

$$M(\Delta_{11}f;h,k;p)\leq\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta f|^p dxdy\right)^{\frac{1}{p}}+\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta_{10}f|^p dxdy\right)^{\frac{1}{p}}$$
$$+\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta_{01}f|^p dxdy\right)^{\frac{1}{p}}$$
$$\leq M(\Delta f;h,k;p)+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[M(\Delta_{10}f;p)\right]^p dy\right)^{\frac{1}{p}}$$
$$+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[M(\Delta_{01}f;p)\right]^p dx\right)^{\frac{1}{p}}.$$

Applying (3.8) and (3.9), we have

$$M(\Delta_{11}f;h,k;p)\leq K(|h|^{\alpha}+|k|^{\beta})+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(K|h|^{\alpha})^p dy\right)^{\frac{1}{p}}$$
$$+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(K|k|^{\beta})^p dx\right)^{\frac{1}{p}}$$
$$\leq 2K(|h|^{\alpha}+|k|^{\beta}),$$

which is the desired result.

R5. If $f(x,y)$ is in Lip $S_{11}(\alpha,\beta)$, i.e.

$$|\Delta_{11}f(x,y;h,k)|=|f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)|$$
$$\leq K|h|^{\alpha}|k|^{\beta},$$

then $f(x,y)$ is in $\mathrm{Lip}_{11}(\gamma,\gamma)$ for $\gamma=\alpha+\beta$, i.e.

$$|\Delta_{11}f(x,y;h,k)|\leq K(|h|^{\gamma}+|k|^{\gamma}) \quad \text{for } \gamma=\alpha+\beta.$$

Proof. By hypothesis

$$|\Delta_{11}f(x,y;h,k)|\leq K|h|^{\alpha}|k|^{\beta},$$

and since $|h|^{\alpha}|k|^{\beta}\leq |h|^{\gamma}+|k|^{\gamma}$ for $\alpha+\beta=\gamma$,

we have $|\Delta_{11}f(x,y;h,k)|\leq K(|h|^{\gamma}+|k|^{\gamma})$, for $\alpha+\beta=\gamma$.

R6. If $f(x,y)$ is in Lip(α,β), then $f(x,y)$ is in Lip$(\alpha,\beta;p)$ for all $p>0$.

Proof. From Def.1, page 9, we have

$$|\Delta f(x,y;h,k)|\leq K(|h|^{\alpha}+|k|^{\beta}),$$

so that

$$|\Delta f(x,y;h,k)|^{p}\leq K^{p}(|h|^{\alpha}+|k|^{\beta})^{p} \quad \text{for all } p>0, \tag{3.10}$$

and since $f(x,y)$ is integrable, it follows that $\Delta f(x,y;h,k)$ is integrable. From Property 1, page 23, we have $f(x,y)$ is continuous in R, whence $\Delta f(x,y;h,k)$ and $|\Delta f(x,y;h,k)|^{p}$ are continuous in R, and thus $|\Delta f(x,y;h,k)|^{p}$ is integrable in the region R. Integrating (3.10), we obtain

$$M(\Delta f;h,k;p)^{p}=\frac{1}{4\pi^{2}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta f(x,y;h,k)|^{p}dxdy$$
$$\leq\frac{K^{p}(|h|^{\alpha}+|k|^{\beta})^{p}}{4\pi^{2}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}dx\,dy=K^{p}(|h|^{\alpha}+|k|^{\beta})^{p}$$

and hence $M(\Delta f;h,k;p)\leq K(|h|^{\alpha}+|k|^{\beta})$ for all $p>0$.

<u>R7</u>. Lip$(\alpha, \beta; p) \downarrow$ as $p \uparrow$.

<u>Discussion and proof of R7</u>. Lip$(\alpha, \beta; p)$ is the class of all functions $f(x,y)$ satisfying the inequality

$$M(\Delta f;p)=\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x,y)|^p dxdy\right)^{\frac{1}{p}} \leq K(|h|^{\alpha}+|k|^{\beta}),$$

where α, β, p are numerical constants independent of any parameter, and K is a constant depending only on the function $f(x,y)$. The relation R7, "Lip$(\alpha, \beta; p) \downarrow$ as $p\uparrow$", means that the classes Lip$(\alpha, \beta; p)$ decrease monotonically as p increases. To prove R7 we first show that the function $M(\Delta f;p)$ is a monotonic increasing function of increasing p for all $p > 0$, i.e.

$$M(\Delta f;q)\equiv\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x,y)|^q dxdy\right)^{\frac{1}{q}}$$

$$\leq\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x,y)|^p dxdy\right)^{\frac{1}{p}} \equiv M(\Delta f;p)$$

for $0 < q < p$.

By Hoelder's inequality we have

$$M(g,w;1)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|g(x,y)w(x,y)|dxdy \leq M(g;r)M(w;r), \quad r > 1.$$

On letting $w(x,y) \equiv 1$, this gives

$$M(g;1)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|g(x,y)|dxdy \leq \left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{+\pi}^{\pi}|g(x,y)|^r dxdy\right)^{\frac{1}{r}} \equiv M(g;r),$$

$r > 1$.

In this inequality we set $g(x,y) \equiv (\Delta f)^q$ and $rq = p > q$, for arbitrary q. This gives

$$M(\Delta f;q) \leq M(\Delta f;rq) = M(\Delta f;p).$$

Since now $0 < q < p$ were arbitrary numbers it follows that

$$M(\Delta f;q) \leq M(\Delta f;p), \quad 0 < q < p.$$

Thus $M(\Delta f;p)$ is a monotonic increasing function of p. The classes $\mathrm{Lip}(\alpha, \beta;p)$, on the other hand, therefore, decrease monotonically as p increases. This completes the proof of R7.

Before discussing R8 we give a definition of the "effective upper bound" of $|f(x,y)|$ analogous to that for a single variable given by Hardy, Littlewood and Polya [37, p.135], and denote it by $\mathrm{Sup}|f(x,y)| = \mathfrak{J}$.

Definition. We define $\mathrm{Sup}|f(x,y)|$ in the region R as the *largest* number $\mathfrak{J}$ which has the property:

(3.7) to any ε (>0), there is a two-dimensional set E of positive measure in which $|f(x,y)| > \mathfrak{J} - \varepsilon$.

It may now be shown that, as a consequence of the above definition, $\mathfrak{J}$ has the following property:

(3.8) given any $\varepsilon > 0$, then there exists a set A_ε such that $m(A_\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and $|f(x,y)| \leq \mathfrak{J} + \varepsilon$ in $C(A_\varepsilon)$.

To prove that $\mathfrak{J}$ has the property (3.8) we assume the contrary and arrive at a contradiction to the definition of $\mathfrak{J}$. We assume then that there is an $A_{\varepsilon'}$ corresponding to an $\varepsilon' > 0$, such that

$$|f(x,y)| > \mathfrak{J} + \varepsilon' \quad \text{in } C(A_{\varepsilon'}) \text{ with } m(A_{\varepsilon'}) = d > 0.$$

Hence we have a $\mathfrak{J}' = \mathfrak{J} + \varepsilon'$, greater than $\mathfrak{J}$, which has the property (3.7). This contradiction shows that (3.8) must hold.

In order to prove the relation R8 we shall need the following

Lemma 3.1. If $\zeta = \operatorname{Sup}|f(x,y)|$ as defined above, then

$$\lim_{p=\infty} M(f;p) = \zeta, \text{ where } M(f;p) = \left(\frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)|^p dxdy \right)^{\frac{1}{p}} .$$

Proof. Since ζ is finite and since from property (3.8) we have $|f(x,y)| \leq \zeta + \varepsilon$ in $C(A_\varepsilon)$ for a preassigned $\varepsilon > 0$ and $m(A_\varepsilon) \to 0$ as $\varepsilon \to 0$, then

$$M(f;p) = \left[\frac{1}{4\pi^2} \iint_{C(A_\varepsilon)} |f(x,y)|^p dxdy + \frac{1}{4\pi^2} \iint_{A_\varepsilon} |f(x,y)|^p dxdy \right]^{\frac{1}{p}}$$

$$\leq \left[\frac{1}{4\pi^2} \iint_{C(A_\varepsilon)} (\zeta + \varepsilon)^p dxdy + \frac{1}{4\pi^2} \iint_{A_\varepsilon} |f(x,y)|^p dxdy \right]^{\frac{1}{p}}$$

$$\leq \left(\frac{(\zeta + \varepsilon)^p}{4\pi^2} \iint_{C(A_\varepsilon)} dxdy \right)^{\frac{1}{p}} + \left(\frac{1}{4\pi^2} \iint_{A_\varepsilon} |f(x,y)|^p dxdy \right)^{\frac{1}{p}}$$

$$\leq (\zeta + \varepsilon) \left(\frac{m(CA_\varepsilon)}{4\pi^2} \right)^{\frac{1}{p}} + I, \quad I = \left(\frac{1}{4\pi^2} \iint_{A_\varepsilon} |f(x,y)|^p dxdy \right)^{\frac{1}{p}} .$$

Letting $\varepsilon \to 0$, we have $m(A_\varepsilon) \to 0$ so that $I \to 0$ and $m(CA_\varepsilon)$ tends to $4\pi^2$. Hence

(a) $$M(f;p) \leq \zeta .$$

Now by the definition of ζ we have for given $\varepsilon > 0$,

$|f(x,y)| > \delta - \varepsilon$ in a plane set E_ε with $m(E_\varepsilon)=N(\varepsilon)>0$,

and so

$$M(f;p)=\left[\frac{1}{4\pi^2}\iint_{CE_\varepsilon}|f(x,y)|^p\,dxdy+\frac{1}{4\pi^2}\iint_{E_\varepsilon}|f(x,y)|^p\,dxdy\right]^{\frac{1}{p}}$$

$$\geqq\left(\frac{1}{4\pi^2}\iint_{E_\varepsilon}|f(x,y)|^p\,dxdy\right)^{\frac{1}{p}}$$

$$\geqq\left(\frac{1}{4\pi^2}(\delta-\varepsilon)^p\iint_{E_\varepsilon}dxdy\right)^{\frac{1}{p}}$$

$$\geqq(\delta-\varepsilon)\left(\frac{N(\varepsilon)}{4\pi^2}\right)^{\frac{1}{p}}.$$

Therefore, $\underline{\lim}_{p=\infty} M(f;p) \geqq \delta-\varepsilon.$

Combining with (a) we have $\delta-\varepsilon \leqq \lim_{p=\infty} M(f;p) \leqq \delta$,

so that

$$\lim_{p=\infty} M(f;p) = \delta.$$

This completes the proof of the lemma.

Remarks. In view of R7 we may also write

$$\mathrm{Lip}(\alpha,\beta;p)\downarrow \text{ to } \mathrm{Lip}(\alpha,\beta),$$

which means that every member of $\mathrm{Lip}(\alpha,\beta)$ is in $\mathrm{Lip}(\alpha,\beta;p)$ for every p, and the cross-section of all classes $\mathrm{Lip}(\alpha,\beta;p)$ is in $\mathrm{Lip}(\alpha,\beta)$. We shall now prove the following relation R8.

R8. $\lim_{p=\infty} \mathrm{Lip}(\alpha,\beta;p) = \mathrm{Lip}(\alpha,\beta).$

Proof. The first part of the proof is obtained from

R6, since R6 states that if f(x,y) is in the class of functions Lip(α,β), then it is in the class Lip($\alpha,\beta;p$) for every p. We now prove the second part, i.e. the cross-section of all the classes Lip($\alpha,\beta;p$) is in Lip(α,β). Thus we suppose that f(x,y) is in the cross-section of all the classes Lip($\alpha,\beta;p$), and by Def.7, page 10, a function is in Lip($\alpha,\beta;p$) if it satisfies the inequality

$$(3.9)\quad M(\Delta f;p) \equiv \left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta f(x,y;h,k)|^p dxdy\right)^{\frac{1}{p}} \leq K(|h|^\alpha+|k|^\beta),$$

where K depends only on the function f. Since K is to be independent of p, we have by Lemma 3.1 that

$$\lim_{p=\infty} M(\Delta f;p) = \operatorname{Sup}|\Delta f(x,y;h,k)| \equiv \delta(h,k) \equiv \delta$$

and from (3.9) it follows that

$$(3.10)\qquad \delta \leq K(|h|^\alpha+|k|^\beta).$$

By (3.8), for given $\varepsilon > 0$ and a corresponding set A_ε with $m(A_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have

$$|\Delta f(x,y;h,k)| \leq \delta + \varepsilon \leq K(|h|^\alpha+|k|^\beta)+\varepsilon \quad \text{in } C(A_\varepsilon).$$

On denoting $\lim_{\varepsilon\to 0} A_\varepsilon = A$, we have $m(A)=0$ and letting $\varepsilon \to 0$, we obtain

$$|\Delta f(x,y;h,k)| \leq K(|h|^\alpha+|k|^\beta) \text{ in } C(A), \text{ where } m(A)=0.$$

To complete the proof of R8 we show that this inequality implies that f(x,y) is in Lip(α,β) in R. We note that by Lemma 3.1 f(x,y) is uniformly continuous in R. If now (x_0,y_0) is a point of A; let $(x_n,y_n) \longrightarrow (x_0,y_0)$, where (x_n,y_n) are points of C(A); thus

$$|\Delta f(x_n,y_n;h,k)| = |f(x_n+h,y_n+k)-f(x_n,y_n)| \leq K(|h|^\alpha+|k|^\beta),$$
$$n \geq 1.$$

Hence as $n \to \infty$,

$$|f(x_0+h, y_0+k)-f(x_0,y_0)| \quad K(|h|^{\alpha}+|k|^{\beta}).$$

Since (x_0,y_0) is any point of A, this gives $f(x,y)$ in Lip(α,β) for every point of A; and as it has already been shown that $f(x,y)$ is in Lip(α,β) for (x,y) in C(A), it follows that $f(x,y)$ is in Lip(α,β) everywhere in R. This completes the proof of R8.

Lemma 3.2. If $f(x,y)$ is in Lip $S_{11}(\alpha_1,\beta_1;p_1)$ and in Lip $S_{11}(\alpha_2,\beta_2;p_2)$, $p_1 < p_2$, then $f(x,y)$ is in Lip$S_{11}(\alpha,\beta;p)$ where for $p_1 \leq p \leq p_2$,

$$\alpha = \alpha_1 \frac{p_1(p_2-p)}{p(p_2-p_1)} + \alpha_2 \frac{p_2(p-p_1)}{p(p_2-p_1)} \qquad \beta = \beta_1 \frac{p_1(p_2-p)}{p(p_2-p_1)} + \beta_2 \frac{p_2(p-p_1)}{p(p_2-p_1)}.$$

The result is still true when $p_2 = \infty$, i.e. if $f(x,y)$ is in Lip $S_{11}(\alpha_1,\beta_1;p_1)$ and in Lip $S_{11}(\alpha_2,\beta_2)$, then $f(x,y)$ is in Lip $S_{11}(\alpha,\beta;p)$, where for $p > p_1$,

$$\alpha = \alpha_2 + (\alpha_1-\alpha_2)\frac{p_1}{p}, \qquad \beta = \beta_2 + (\beta_1-\beta_2)\frac{p_1}{p}.$$

Proof. By assumption

$$M(\Delta_{11}f;h,k;p_1) \leq K|h|^{\alpha_1}|k|^{\beta_1}, \quad M(\Delta_{11}f;h,k;p_2) \leq K|h|^{\alpha_2}|k|^{\beta_2},$$

where

$$M(\Delta_{11}f;h,k;p_1) = \left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta_{11}f|^{p_1}dxdy\right)^{\frac{1}{p_1}}$$

$$M(\Delta_{11}f;h,k;p_2) = \left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta_{11}f|^{p_2}dxdy\right)^{\frac{1}{p_2}}$$

and $|\Delta_{11}f| = |f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)|$.

By Hoelder's inequality for double integrals we obtain

$$M(\triangle_{11}f;h,k;p)^{p} \leq M(\triangle_{11}f;h,k;p_1)^{p_1\frac{p_2-p}{p_2-p_1}} \quad M(\triangle_{11}f;h,k;p_2)^{p_2\frac{p-p_1}{p_2-p_1}}$$

$$\leq K\left(|h|^{\alpha_1\frac{p_1(p_2-p)}{p_2-p_1}+\alpha_2\frac{p_2(p-p_1)}{p_2-p_1}}\right)\left(|k|^{\beta_1\frac{p_1(p_2-p)}{p_2-p_1}+\beta_2\frac{p_2(p-p_1)}{p_2-p_1}}\right)$$

$$\leq K\ |h|^{\alpha p}\ |k|^{\beta p}$$

Therefore, $M(\triangle_{11}f;h,k;p)\leq K_1|h|^{\alpha}|k|^{\beta}$, where α and β satisfy the relations given in the statement of the Lemma above, and where $p_1 \leq p \leq p_2$. If now $p_2 \rightarrow \infty$, then

$$p_1\frac{p_2-p}{p_2-p_1} \longrightarrow p_1, \qquad p_2\frac{p-p_1}{p_2-p_1} \longrightarrow p-p_1,$$

and

$$M(\triangle_{11}f;h,k;p)^{p} \leq M(\triangle_{11}f;h,k;p_1)^{p_1} M(\triangle_{11}f;h,k;p_2)^{p-p_1}$$

$$\leq K_2|h|^{\alpha_1 p_1+\alpha_2(p-p_1)} \quad |k|^{\beta_1 p_1+\beta_2(p-p_1)}$$

$$\leq K_2|h|^{\alpha p}\ |k|^{\beta p}.$$

Therefore,

$$M(\triangle_{11}f;h,k;p) \leq K_3|h|^{\alpha}\ |k|^{\beta},$$

where

$$\alpha = \alpha_2+(\alpha_1-\alpha_2)\frac{p_1}{p} \quad \text{and} \quad \beta = \beta_2+(\beta_1-\beta_2)\frac{p_1}{p} \quad \text{for } p > p_1.$$

<u>Lemma 3.3</u>. If $f(x,y)$ is in Lip(α,α) throughout R with the constant K and $0<\alpha\leq 1$, then $f(x,y)$ is in Lip $S_{11}(\beta,\gamma)$ in R with the constant 2K and $\beta+\gamma=\alpha$.

<u>Proof</u>. We have

$$|f(x+h,y+k)-f(x,y)|=|\triangle f(x,y;h,k)|\leq K(|h|^{\alpha}+|k|^{\alpha}).$$

Now $|\Delta_{11}f| = |f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)|$

and

$$|\Delta_{11}f| \le \begin{cases} 2K|k|^{\alpha}=2K|k|^{\beta}|k|^{\gamma} \le 2K|h|^{\beta}|k|^{\gamma}, & \text{for } |k|\le|h|, \\ 2K|h|^{\alpha}=2K|h|^{\beta}|h|^{\gamma} \le 2K|h|^{\beta}|k|^{\gamma}, & \text{for } |h|\le|k|, \end{cases}$$

Hence we have $|\Delta_{11}f(x,y;h,k)| \le 2K|h|^{\beta}|k|^{\gamma}$ for $0<\alpha=\beta+\gamma\le 1$.

<u>Lemma 3.4</u>. Let (1),(2),(3),(a),(b),(c) and (d) represent the following conditions, respectively:

(1) $f(x,y)$ in Lip $S_{11}(\alpha,\beta)$ in R, (Def.8,p.10)

(2) $f(x,y)$ in Lipα in x for some $y=\bar{y}$ in R,

(3) $f(x,y)$ in Lip β in y for some $x=\bar{x}$ in R,

(a) $f(x,y)$ in Lip α in x uniformly for y in R, (Def.3,p.9)

(b) $f(x,y)$ in Lip β in y uniformly for x in R, (Def.4,p.9)

(c) $f(x,y)$ in Lip(α,β) in R, (Def.1,p.9)

(d) $f(x,y)$ in Lip$_{11}(\alpha,\beta)$ in R. (Def.2,p.9)

Then we have the following relations:

(i) (1) and (2) together imply (a),

(ii) (1) and (3) together imply (b),

(iii) (1),(2) and (3) together imply both (c) and (d).

<u>Proof</u>. Part (i). For a fixed value of k, $0<k<2\pi$, we have the distinct points (x,y_1) and $(x,\bar{y})$ in R, where $y_1=\bar{y}+k$ and, therefore,

$$\begin{aligned} |f(x+h,y_1)-f(x,y_1)| &= |f(x+h,y_1)-f(x+h,\bar{y})-f(x,y_1)+f(x,\bar{y}) \\ &\qquad +f(x+h,\bar{y})-f(x,\bar{y})| \\ &\le |f(x+h,y_1)-f(x+h,\bar{y})-f(x,y_1)+f(x,\bar{y})| \\ &\qquad +|f(x+h,\bar{y})-f(x,\bar{y})| \end{aligned}$$

so that $|\Delta_{10}f(x,y_1;h)| \leq |\Delta_{11}f(x,\bar{y};h,k)| + |\Delta_{10}f(x,\bar{y};h)|$.

By Def.5, page 9,
$$|\Delta_{11}f(x,\bar{y};h,k)| \leq K_1|h|^{\alpha}|k|^{\beta}$$
and applying Def.3, page 9, on the line $y=\bar{y}$, we have
$$|\Delta_{10}f(x,\bar{y};h)| \leq K_2(\bar{y})\ |h|^{\alpha} \qquad (K_2 \text{ depends only on } \bar{y}).$$
Hence
$$|\Delta_{10}f(x,y_1;h)| \leq \left[K_1|k|^{\beta}+K_2(\bar{y})\right]|h|^{\alpha},$$
but for arbitrary y_1 in R, we have $k < 2\pi$ and thus
$$|\Delta_{10}f(x,y_1;h)| \leq \left[K_1(2\pi)^{\beta}+K_2(\bar{y})\right]|h|^{\alpha}$$
$$\leq K_3(y_1)\ |h|^{\alpha}.$$
Since K_1 is independent of x and y throughtout R and K_2 is independent of x and the arbitrary y_1, we have
$$|\Delta_{10}f(x,y;h)| \leq K_3|h|^{\alpha} \text{ uniformly in } y.$$
This gives the desired result (a) and concludes the proof of (i). An analogous argument gives the result (b) from (1) and (3), which gives paft (ii). Part (iii) follows from (a), (b) and R2.

3.C. <u>Bounded variation and absolute continuity in two variables</u>. From the list of known theorems quoted in Chapter 2 on pages 12 to 20, it is seen that bounded variation and absolute continuity of a function of a single variable play an important part in the investigation of absolute convergence of the corresponding Fourier series. It is also noticed in these theorems that with some slight assumption added to the condition of bounded variation or absolute continuity of the function, absolute convergence of the corresponding Fourier series is secured. As for functions of two variables, several concepts (which shall be listed below) of bounded variation and absolute continuity have been defined. These concepts have been used to determine convergence criteria of the corresponding double Fourier series, but as yet have not been used to obtain conditions on f(x,y) for the absolute convergence of the corresponding double Fourier series. It is to be expected, although this question is not to be discussed here, that again with an additional assumption in these classes of bounded variation and absolute continuity for two variables we should obtain absolute convergence of the double Fourier series. It is, therefore, of interest to list and examine the different types of bounded variation and absolute continuity for functions of two variables.

In the following definitions we use, unless otherwise specified, the definition of a net as given in (2.5) and the notation given in (2.6),(2.7),(2.8) and (2.9). For simplicity we shall also use the letters V,H,A,T,Va,Ha,Aa,Ta to represent the eight classes of functions satisfying the respective definitions. The class of bounded functions will be denoted by B and the class of continuous functions by C; a product, such as VTC, will stand for the common part of the two or more classes named. For a function of two variables f(x,y), the total variation with respect to the variable x in the interval $(-\pi,\pi)$ for a given value $y=\bar{y}$ shall be denoted by $\psi(\bar{y})$. Similarly, we define $\phi(\bar{x})$ for the total variation with respect to y alone in the interval $(-\pi,\pi)$. The value ∞ is admitted for unbounded variation. The total variation $V(f)$ of $f(x,y)$ in the two variables (x,y) over R is defined as

$$V(f) = \text{l.u.b.} \sum_{i,j=0}^{m-1,n-1} |\Delta_{11} f(x_i,y_j)| \quad \text{for all nets over R.}$$

Definition V. The function $f(x,y)$ is said to be of bounded variation V if $V(f)$ is finite.

Definition Va. The function $f(x,y)$ is said to be absolutely continuous Va if for any given $\eta\ (>0)$, there exists an $\varepsilon = \varepsilon(\eta) > 0$ such that if S is any finite or denumerably infinite set of cells the sum of whose areas is less than or equal ε, then

$$\sum_{i,j=0}^{m-1,n-1} |\Delta_{11} f(x_i,y_j)| < \eta.$$

(Note that ε depends only on η and not on S]

Definition H. The function $f(x,y)$ is said to be of bounded variation H if $f(x,y)$ is in class V and if in addition $f(x,\bar{y})$ is of bounded variation in x for at least one $\bar{y}$ and $f(\bar{x},y)$ is of bounded variation in y for at least one $\bar{x}$[6].

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[6] The three conditions in definition H are sufficient to imply that the two latter conditions hold for every x or y respectively.

__Definition Ha__. The function $f(x,y)$ is said to be absolutely continuous Ha if $f(x,y)$ is in class Va and if in addition $f(x,\bar{y})$ is absolutely continuous in x for at least one $\bar{y}$ and $f(\bar{x},y)$ is absolutely continuous in y for at least one $\bar{x}$.

__Definition T__. The function $f(x,y)$ is said to be of bounded variation T, if the total variation function $\phi(\bar{x})$ is finite almost everywhere in $(-\pi,\pi)$, and its Lebesgue integral over $(-\pi,\pi)$ exists (finite), while a symmetric condition is satisfied by $\psi(\bar{y})$.

__Definition Ta__. The function $f(x,y)$ is said to be absolutely continuous Ta if $f(x,y)$ is in class T and if in addition $f(x,\bar{y})$ is absolutely continuous in x for almost every $\bar{y}$ and $f(\bar{x},y)$ is absolutely continuous in y for almost every $\bar{x}$.

__Definition A__. Let (x_i,y_i) $(i=0,1,2,\dots,m)$ be any set of points in the plane satisfying the conditions

$$-\pi=x_0\le x_1\le x_2\le \dots \le x_m=\pi;\quad -\pi=y_0\le y_1\le y_2 \quad \dots\le y_m=\pi.$$

Then $f(x,y)$ is said to be of bounded variation A if the sum

$$\sum_{i=0}^{m-1}|\Delta f(x_i,y_i)| = \sum_{i=0}^{m-1}|f(x_{i+1},y_{i+1})-f(x_i,y_i)|$$

is bounded for all such sets of points.

__Definition Aa__. Let (x_i,y_i) be any set of points as defined in definition A. Then $f(x,y)$ is said to be absolutely continuous Aa if for all sets of points (x_i, y_i) and for any given $\eta\ (>0)$, there exists an $\varepsilon = \varepsilon(\eta) > 0$ such that if the sum of the broken lengths from (x_j, y_j) to (x_k, y_k) $(k \leq m, j \geq 0)$ is less than or equal to ε, then

$$\sum_{i=j}^{k-1} |\Delta f(x_i, y_i)| < \eta.$$

For a treatment of relations between the six distinct definitions of bounded variation in two variables found in the literature, we refer to Clarkson and Adams in [14] and [15]. We now quote a number of the known relations among the several definitions of bounded variation discussed by Clarkson and Adams[7]:

(1) $\left.\begin{matrix}V\\ T\\ A\\ AT\\ VB\\ TB\end{matrix}\right\} > H,$ (2) $\left.TC \begin{matrix}>AC\\ VC\end{matrix}\right\} > HC = VTC,$

(3) $AV = VT = H,$ (4) $A \ngeq V,\ V \ngtr A,$

(5) $V \ngeq T,\ T \ngtr V,$ (6) $A \ngeq T,\ T \ngtr A,$

(7) $V \ngeq AT,\ AT \ngtr V,$ (8) $AC \ngeq VC,\ VC \ngtr AC,$

(9) $TC \ngeq VC,\ VC \ngtr TC,$ (10) $A \ngeq VB,\ VB > A,$

(11) $VB \ngeq TB,\ TB \ngtr VB,$ (12) $A \ngeq TB,\ TB \ngtr A.$

We also know the following relations and theorems involving the several definitions of absolute continuity:

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[7] By $V \geq H$ we mean that any function in H is also in V. Accordingly for the other symbols.

(13) $H \geqq Ha$, Hahn [30,p.542] (14) $C > Ha$ Hahn [30,p.542].

(D.1) If $f(x,y)$ is in class Ha, it is the indefinite integral of some summable function; and conversely, the indefinite integral of a summable function is in class Ha. Caratheodory [12,p.659], Hobson [39,p.607].

(D.2) If a function $f(x,y)$ is such that $f(x,\bar{y})$ is absolutely continuous in x for every $\bar{y}$ and $f(\bar{x},y)$ is absolutely continuous in y for every $\bar{x}$, it does not follow that $f(x,y)$ is in class Ha. Caratheodory [12,p.655].

(D.3) The function $f(x,y)$ is in class Ha if it satisfies the three Lipschitz conditions

$$|f(x+h,y+k)-f(x,y+k)-f(x+h,y)+f(x,y)| \leq K|h||k|,$$

$$|f(x+h,y)-f(x,y)| \leq K|h|, \qquad |f(x,y+k)-f(x,y)| \leq K|k|.$$

Caratheodory [12,p.655].

(D.4) If $f(x,y)$ is in class TaC, the first partial derivatives exist(finite) almost everywhere and are Lebesgue integrable and

$$\left|\iint \frac{\partial f}{\partial x}\, dx\, dy\right| \leq \int \psi(y)\, dy, \quad \left|\iint \frac{\partial f}{\partial y}\, dx\, dy\right| \leq \int \phi(x)\, dx.$$

Tonelli [66].

(D.5) The necessary and sufficient condition that the area S of the surface $z=f(x,y)$ be given by the formula $S=\iint\sqrt{1+p^{*}+q^{*}}\, dxdy$, where $p(x,y)$ and $q(x,y)$ are the first order partials of $f(x,y)$, is that $f(x,y)$ be in TaC. Tonelli [66].

By using some of the above known results we shall establish the following:

(1a) $H > Ha$, (2a) $V > Va$, (3a) $T > Ta$,

(4a) $HC > Ha$, (5a) $Va > Ha$, (6a) $Ta > Ha$,

(7a) $VaC > HaC$, (8a) $TaC > HaC$, (9a) $Va \ngeq Ta$, $Ta \ngtr Va$,

(10a) $VaTa = Ha$, (11a) $Aa \ngtr Va$, $Va \ngtr Aa$, (12a) $C > Aa$,

(13a) $A > Aa$, (14a) $Aa > Ha$, (15a) $AaVa = Ha$.

<u>Proof of (1a)</u>. From (13) we have $H \geq Ha$. Consider a function $f(x,y) = g(x)$, where $g(x)$ is of bounded variation but not absolutely continuous. It follows immediately that $f(x,Y)$ is in H but not in Ha. This gives $H > Ha$.

<u>Proof of (2a)</u>. We first show $V \geq Va$. Choose a numerically fixed $\eta > 0$. Since by assumption $f(x,y)$ is in Va, we have by the definition Va that corresponding to the given η there exists an $\varepsilon (> 0)$ such that for any set of cells S with total area less than or equal to ε, then $\sum_S |\Delta_{11} f| < \eta$. We now decompose R into sets S each of which has the total area equal to ε. The total number of these sets is

$$\left[\frac{4\pi^2}{\varepsilon}\right] + L.$$

From the assumption the total variation in each of the sets is at most η and hence the total variation $V(f)$ over R is such that

$$V(f) \leq \eta + \frac{4\pi^2}{\varepsilon}\eta.$$

Thus, ε being > 0 and η a numerically fixed quantity we have $V(f)$ finite for every net over R and $f(x,y)$ is in V. To obtain the definite inequality $V > Va$ we now give a function in V which is not in Va. Let $f(x,y)=1$ for $x=\pi, y=\pi$ and otherwise $f(x,y)=0$, then obviously $f(x,y)$ is in V but not in Va. This completes the proof.

Proof of (3a). By the definitions we have $T \geq Ta$. The follwoing example gives a function in T not in Ta.

(D) $f(x,y) = \begin{cases} 0, & x < y \\ 1, & x \geq y \end{cases}$ in the unit square.

This example is given by Clarkson and Adams [14,p.836] as a function in T. That it is not in Ta is a consequence of the fact that it is not absolutely continuous in each variable separately.

Proof of (4a). From (14) we have $C > Ha$, and by (1a) we have $H > Ha$. Therefore, $HC \geq Ha$. Now let $f(x,y)=g(x)$, where $g(x)$ is taken as a continuous function of bounded variation which is not absolutely continuous. Then $f(x,y)$ is in HC but not in Ha. This gives $HC > Ha$.

Proof of (5a). From the definitions we have $Va \geq Ha$. The following example is in Va but not in Ha.

(E) $f(x,y) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$ in the unit square.

For this function the total variation is $V(f)=0$, and hence $f(x,y)$ is in Va. However, $f(x,y)$ is not in Ha since it is not absolutely continuous in x.

Proof of (6a). We assume the relation $Ta \geqq Ha$ which will be proved later in Theorem 3.1. We now give a function in Ta not in Ha.

$$\text{(F)} \qquad f(x,y) = \left\{\begin{array}{l} 1, \text{ x and y both rational} \\ 0, \text{ otherwise} \end{array}\right\} \text{ in the unit square.}$$

This function is in Ta since it is zero almost everywhere, but it is not in Ha since it is not absolutely continuous in each variable for every constant value of the other.

Proof of (7a). From (14) we have $C > Ha$, and by (5a) we have $Va > Ha$. Hence $VaC \geqq Ha$. The continuous example (E) has already been shown to be in Va but not in Ha in the proof of (5a). Since the function is continuous it is in VaC and we have VaC Ha.

Proof of (8a). From (14) we have $C > Ha$, and by (6a) we see that $TaC \geqq HaC$. The following example is in TaC but not in Ha.

(G) Let I, the unit square, be divided into quarter squares, and let S_1 be the upper left-hand quarter square. Next divide the lower right-hand square into quarter squares, and let S_2 be that quarter which has a common vertex with S_1, etc. (cf. Fig. 2). We obtain in this way an infinite sequence of square subdivisions of I converging toward the point (1,0). In each S_j let f(x,y) be defined by the surface of a regular square pyramid whose base is S_j and height $\frac{1}{j}$, and let f(x,y) vanish over the rest of I.

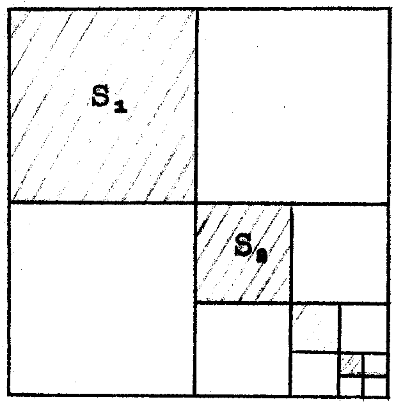

Fig.2

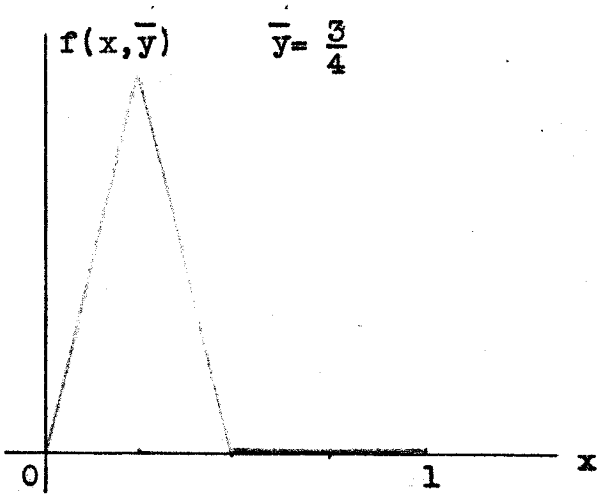

Fig.3

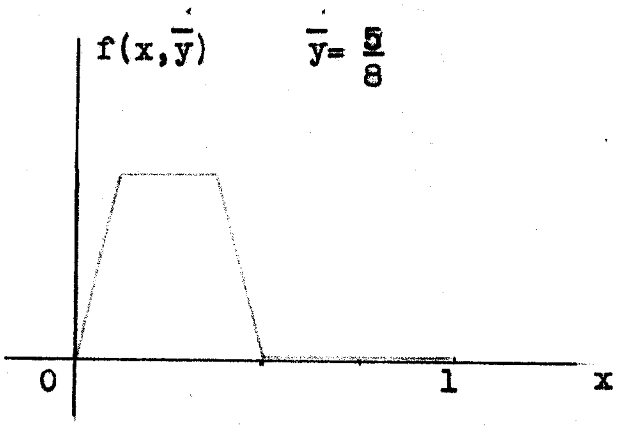

Fig.4

The function $f(x,y)$ given in (G) is continuous and has been shown to be in T and not in V by Clarkson and Adams [14,p.846]. It is obvious from the definition of $f(x,y)$ (cf. Figs.3 & 4) that $f(x,\bar{y})$ or $f(\bar{x},y)$ is absolutely continuous in x or y for every $\bar{y}$ or $\bar{x}$, and hence $f(x,y)$ is in Ta. However, since $f(x,y)$ is not in V it follows by (2a) that $f(x,y)$ is not in Va and by (5a) is not in Ha. This gives $TaC > HaC$.

Proof of (9a). We have seen that example (F) gives a function in Ta. This function is not in Va since upon choosing a net with each cell having only one corner with rational coordinates, the total variation is $V(f)=\infty$. Hence $Va \not\geq Ta$. The relation $Ta \not> Va$ follows from example (E). This function (E) being a function of one variable is in Va with $V(f)=0$, but $f(x,y)$ is not in Ha since it is not an absolutely continuous function of the one variable.

<u>Proof of (10a)</u>. From (5a) and (6a) we have $VaTa \geq Ha$. But if $f(x,y)$ is in VaTa it satisfies the first condition of definition Ha, and by definition Ta the functions $f(x,\bar{y})$ and $f(\bar{x},y)$ are absolutely continuous in x and y respectively for almost all values of $\bar{y}$ and $\bar{x}$, respectively. Thus we have $VaTa \leq Ha$ and hence $Ha=VaTa$.

<u>Proof of (11a)</u>. The relation $Aa \not\geq Va$ follows from example (E) which is in Va but not in Aa. That $f(x,y)$ is not in Aa follows by taking the sequence of points (x_i,y_i) along a line y=constant and remembering that $x \sin\frac{1}{x}$ is not absolutely continuous. Now the second relation $Va \not> Aa$ follows from example (G) which is in Aa and not in Va. We have already seen that this function is not in Va in the proof of (8a). To see that the function is in Aa we notice that for any set of points (x_i,y_i) as used in definition Aa, $f(x_i,y_i)$ vanishes except at such points as lie within one square S_j. For points in S_j the function has a graph similar to that in $(0,\frac{1}{2})$ in Fig.3 or Fig.4 and consequently

$$\sum_{i=0}^{k-1} |\Delta f(x_i,y_i)| < \eta .$$

when the sum of the broken lengths from (x_0,y_0) to (x_k,y_k) is less than ε. This gives the relation $Va \nrightarrow Aa$.

<u>Proof of (12a)</u>. To prove that $f(x,y)$ in Aa implies $f(x,y)$ is continuous we choose any point (x_1,y_1) and show that the assumption implies continuity at (x_1,y_1) and hence everywhere. Let (x,y) be a neighboring point of (x_1,y_1) such that

$$|x-x_1| < \delta, \qquad |y-y_1| < \delta, \qquad \delta(>0).$$

Select a sequence of monotone non-decreasing points (x_i,y_i) with

$$x_1=x_1,\ y_1=y_1, \ldots, x_n=x,\ y_n=y.$$

Since by hypothesis $f(x,y)$ is in Aa and since the broken distance d from (x_1,y_1) to (x,y) is obviously less than $\sqrt{2}\,\delta$, we can say that for $\eta(>0)$ we can take d such that

$$\sum_{i=1}^{n-1} |\Delta f(x_i,y_i)| < \eta .$$

However,

$$|f(x,y)-f(x_1,y_1)| \sum_{i=1}^{n-1} |\Delta f(x_i,y_i)| < \eta, \quad \text{for } d<\sqrt{2}\,\delta,$$

and hence

$$|f(x,y)-f(x_1,y_1)| < \eta \text{ for } |x-x_1| < \delta,\ |y-y_1| < \delta,$$

so that $f(x,y)$ is continuous in (x,y) at (x_1,y_1). Since (x_1,y_1) is any point this completes the proof.

Proof of (13a). From the definitions we have $A \geq Aa$. By (12a), we need only construct a discontinuous function which is in A in order to have a function in A which is not in Aa. The following is a discontinuous function in class A.

(J) Let E be a non-measurable set in $0 \leq x \leq 1$, and let E_1 be the set of points on the diagonal from (0,1) to (1,0) whose projection on the x-axis is the set E. f(x,y) is defined as follows:

$$f(x,y) = \begin{cases} 1, & x \text{ in } E_1 \\ 0, & \text{otherwise} \end{cases} \text{ in the unit square.}$$

Proof of (14a). Let f(x,y) be of class Ha in the unit square (0,0;1,1). We shall show that f(x,y) is in Aa. Let

$$x=x(t), \qquad y=y(t), \qquad 0 \leq t \leq 1,$$

be the arc of a monotone non-decreasing curve joining (0,0) and (1,1), and let any sequence of points on this curve be determined by

$$0=t_0 < t'_0 \leq t_1 < t'_1 \leq t_2 < t'_2 \leq \dots \leq t_n < t'_n = 1.$$

For brevity we write

$$x(t_i)=x_i, \ x(t'_i)=x'_i; \quad y(t_i)=y_i, \ y(t'_i)=y'_i \quad (i=0,1,\dots,n).$$

Now let (x_0, y_0) be a fixed point (any point), then

$$\begin{aligned} |f(x_i,y_i))-f(x_i',y'_i)| \leq & |f(x_0,y'_i)-f(x_0,y_i)-f(x_i,y'_i)+f(x_i,y_i)| \\ & +|f(x'_i,y_0)-f(x_i,y_0)-f(x'_i,y'_i)+f(x_i,y'_i)| \\ & +|f(x_0,y_i)-f(x_0,y'_i)|+|f(x_i,y_0)-f(x'_i,y_0)|. \end{aligned}$$

Since f(x,y) is in Ha we know that for a preassigned η we can select an ε such that

$$\sum_{i=0}^{n-1} |f(x_0,y_i')-f(x_0,y_i)-f(x_i,y_i')+f(x_i,y_i)| < \frac{\eta}{4}$$

$$\sum_{i=0}^{n-1} |f(x_i',y_0)-f(x_i,y_0)-f(x_i',y_i')+f(x_i,y_i')| < \frac{\eta}{4}$$

$$\sum_{i=0}^{n-1} |f(x_0,y_i)-f(x_0,y_i')| < \frac{\eta}{4}$$

$$\sum_{i=0}^{n-1} |f(x_i,y_0)-f(x_i',y_0)| < \frac{\eta}{4}$$

for the sum of the areas of the cells less than ε^2 and for

$$\sum_{i=0}^{n-1} |x_i'-x_i| < \varepsilon\ , \qquad \sum_{i=0}^{n-1} |y_i'-y_i| < \varepsilon .$$

If now we choose points of the sequence (x_i,y_i) beginning with the point (x_0,y_0) to the last point is less than ε, then

$$\sum_{i=0}^{n-1} |f(x_i,y_i)-f(x_i',y_i')| < \frac{\eta}{4}+\frac{\eta}{4}+\frac{\eta}{4}+\frac{\eta}{4} = \eta .$$

Thus $f(x,y)$ is in class Aa.

In order to obtain the relation (15a) we shall need the following Lemma.

Lemma 3.5. If $f(x,y)$ is in the class Aa, then $f(\bar{x},y)$ is absolutely continuous in y for every $\bar{x}$ and $f(x,\bar{y})$ is absolutely continuous in x for evefy $\bar{y}$.

Proof. Take a net over the unit square as follows,

$$\bar{x}=x_0=x_1=x_2=\dots=x_n; \quad y_0 < y_1 < y_2 \quad \dots < y_n .$$

Since $f(x,y)$ is in Aa we know that for preassigned η we can select a δ so that

$$\sum_{i=0}^{n-1} |\Delta f(x_i,y_i)| = \sum_{i=0}^{n-1} |f(\bar{x},y_{i+1})-f(\bar{x},y_i)| < \eta$$

for

$$\sum_{i=0}^{n-1} |y_{i+1}-y_i| < \delta.$$

But $\bar{x}$ is any value of x in $(0,1)$, and hence we have $f(\bar{x},y)$ is absolutely continuous in y for all x in $(0,1)$. A symmetric argument gives $f(x,\bar{y})$ is absolutely continuous in x for all y in $(0,1)$.

Proof of (15a). A function of class Va satisfies the first condition of definition Ha, and by the above lemma 3.5 we see that a function of class Aa satisfies the other conditions of definition Ha. Hence, Ha$\geq$AaVa. On the other hand we have by (5a) and (14a) that AaVa$\geq$Ha. This gives the desired result AaVa=Ha.

Theorem 3.1. If $f(x,y)$ is in class Ha, then $f(\bar{x},y)$ is absolutely continuous in y for every $\bar{x}$ and $f(x,\bar{y})$ is absolutely continuous in x for every $\bar{y}$. (Hence we obtain the relation Ta$\geq$Ha.)

Proof. Our assumption gives us $f(x,y)$ in Va and absolute continuity of $f(\bar{x},y)$ in y for some $\bar{x}$ and we shall show that the hypothesis of $f(x,y)$ in Ha will enable us to obtain absolute continuity of $f(x,y)$ in y for any other x.

Being given an η we can select an ε such that

$$\sum_{i,j=0}^{m-1,n-1} |\Delta_{11}f(x_i,y_j)| < \frac{\eta}{2} \qquad \sum_{j=0}^{n-1} |f(\bar{x},y_{j+1})-f(\bar{x},y_j)| < \frac{\eta}{2}.$$

Now, since f(x,y) is in Va and $f(\bar{x},y)$ absolutely continuous in y we have

$$|f(x,y_{j+1})-f(x,y_j)| \leq |f(x,y_{j+1})-f(x,y_j)-f(\bar{x},y_{j+1})+f(\bar{x},y_j)|$$
$$+|f(\bar{x},y_{j+1})-f(\bar{x},y_j)|$$

so that

$$\sum_{j=0}^{n-1} |f(x,y_{j+1})-f(x,y_j)| < \frac{\eta}{2}+\frac{\eta}{2}=\eta$$

for

$$\sum_{j=0}^{n-1} |y_{j+1}-y_j| < \varepsilon .$$

Since x is arbitrary this gives absolute continuity in y for every x. A symmetric argument gives absolute continuity in x for every y. Now by relations (1) and (1a) we have $T>H>Ha$, so that a function in Ha satisfies the first condition of definition Ta. Since we have just shown that a function in Ha satisfies the other conditions of absolute continuity in each variable (almost everywhere) it follows that $Ta \geq Ha$.

In order to prove Theorem 3.2 below, we shall need the three following known theorems.

(C.3) Assume X,Y,Z are metric spaces subject to no further restrictions and assume that f(x,y) is defined on the product space XxY and that the values of f(x,y) are in the space Z. Then if f(x,y) is continuous in x and of Baire's class α in y, it is of class $\alpha+1$ in (x,y). Montgomery [48,p.530].

(C.4) If f(x,y) be a measurable function, defined in a two dimensional set of points E, of finite measure, and finite almost everywhere in E, there exists in E a perfect set of points, of measure arbitrarily near to m(E), relative to which the function f(x,y) is continuous. Hobson [40,p.256].

(C.5) The necessary and sufficient condition that a finite function f(x), defined on a bounded measurable set E, be measurable is that for every $\varepsilon > 0$ there exist a closed sub-set F of E such that f(x) is continuous in F and $m(E-F) < \varepsilon$. Saks [54,p.44].

<u>Theorem 3.2</u>. If f(x,y) is in the class of functions Ta in the square $0 \leq x, y \leq 1$, then f(x,y) is a measurable function in this square.

<u>Proof</u>. By the definition of a function in class Ta we can say there are sets E_0 and E_1 contained in (0,1) with $m(E_0)=1$ and $m(E_1)=1$, such that

$$f(x,y) \begin{cases} \text{is continuous in } y \text{ for } x \text{ in the set } E_0 \\ \text{is continuous in } x \text{ for } y \text{ in the set } E_1 \end{cases}$$

and the two dimensional set $E_0 x E_1$ is of measure 1. Now from a property of measurable sets there exist, for every $\varepsilon > 0$, closed sub-sets $E_{0\varepsilon}$, $E_{1\varepsilon}$ of E_0 and E_1 respectively, with $m(E_{0\varepsilon}) > 1-\varepsilon$, and $m(E_{1\varepsilon}) > 1-\varepsilon$. For these sets we have

$$f(x,y)\begin{cases}\text{is continuous in } y \text{ for } x \text{ in the set } E_{0\varepsilon}\\ \text{is continuous in } x \text{ for } y \text{ in the set } E_{1\varepsilon}.\end{cases}$$

Let the two dimensional set $E_{0\varepsilon} \times E_{1\varepsilon}$, obtained by the intersection of vertical and horizontal lines erected on the two linear sets, be denoted by E. We can consider the sets $E_{0\varepsilon}$, E_{13} and E as metric spaces with the Euclidean definition of distance and hence apply the Theorem (C.3) of Deane Montgomery to see that for (x,y) in E the function $f(x,y)$ belongs to Baire's first class. Consequently $f(x,y)$ is Lebesgue measurable on E with $m(E)=(1-\varepsilon)^2$. Thus by the property of two dimensional measurable sets given in (C.4), there exists for every $\eta > 0$ a closed sub-set F of E with $m(F) > (1-\varepsilon)^2 - \eta$ such that $f(x,y)$ is continuous on the set F. Since ε and η are arbitrary we can use Lusin's Theorem (C.5)[*] to obtain the result that $f(x,y)$ is measurable on the square $0 \leq x, y \leq 1$.

<u>Remark</u>. Hardy and Littlewood [35], [36] have shown that bounded variation in one variable is equivalent to Lip(1;1). Does an analogous situation hold for a function of two variables? For example, is the class Lip $S_{11}(1,1;1)$ equivalent to the class V? This is an open question.

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[*] Lusin's Theorem is readily extensible to the case of a function of two variables. Cf. Nelson Dunford [18, p.475].

CHAPTER IV

GENERALIZATIONS OF FOURIER DEVELOPMENTS FROM ONE TO TWO VARIABLES

4.A. Harmonic Functions in the Plane. In order later to undertake some generalizations of the Fourier developments of a function of two variables from that of one variable, we shall first consider the single variable situation. It may be recalled that a function H(u,v), (u,v) rectangular coordinates, is said to be ''harmonic'' in a region of the plane when the function satisfies the Laplace differential equation

$$(4.1) \qquad \nabla^2 H = \frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2} = 0$$

at every point of the given region. In the polar coordinates (r,x) the equation (4.1) takes the form

$$(4.2) \qquad \frac{\partial^2 H}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{r}\frac{\partial H}{\partial r} = 0.$$

The problem of showing that there exists a function, harmonic in a plane region, and taking on preassigned continuous boundary values and of finding it when it exists, is known as the Dirichlet problem for the plane region.[Cf.Zygmund 71,p.51.]

We shall now point out the well known [Cf. Tonelli 65, pp.379-345] connection of the single Poisson integral, defined for a given function f(x) by

$$(4.3) \qquad \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)\,du}{1-2r\cos(u-x)+r^2}, \qquad (r,x) \text{ polar coordinates, } r<1,$$

with the single Fourier series (2.13) and with the solution of Dirichlet's problem for the unit circle. The Poisson integral (4.3) may readily be shown to satisfy (4.2), hence (4.3) defines a continuous harmonic function of the two variables (r,x) everywhere inside the unit circle. Upon taking f(x) a continuous periodic function, we know [Cf. Tonelli 65,pp.379-385] that the function defined by Poisson's integral tends uniformly to f(x) as $r \to 1$. Consequently Poisson's integral (4.3) furnishes the solution of the Dirichlet problem for the unit circle and for the boundary values f(x). In fact, in this special case of the unit circle, (4.3) gives the solution for the more general problem where the limit function f(x) is an arbitrary integrable function[*].

We give now the development of the Poisson integral in powers of r. Let f(x) be a continuous and periodic function of x and consider the function F(z) analytic inside the unit circle defined by the power series

$$(4.4)\qquad F(z) = c_0 + \sum_{n=1}^{\infty} 2c_n z^n, \quad z=re^{ix},\ 0 \leq r < 1,\ -\pi \leq x \leq \pi,$$

with the coefficients

$$(4.5)\qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\, e^{-inu}\, du, \quad (n=0,\pm1,\pm2,\ldots).$$

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[*] [Cf. Zygmund 71,p.54] for the following theorem: Let f_1 be an integral of f(x). For any x_1 where $f(x_1)$ is finite and equal to $f_1'(x_1)$, (which is the case almost everywhere), the Poisson integral tends to $f(x_1)$, as (r,x) tends to $(1,x_1)$ along any path not touching the circle.

$F(z)$ thus has the form

$$(4.6)\quad F(z)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n-ib_n)z^n,\quad \text{for } 2c_n=a_n-ib_n,\ n=0,\pm1,\pm2,\ldots$$

where a_n, b_n are the Fourier coefficients of $f(x)$. We write

$$G(r,x)=\mathcal{R}F(z)=\mathcal{R}\Big(c_0+\sum_{n=1}^{\infty}2c_nz^n\Big)$$

$$=c_0+\sum_{n=1}^{\infty}c_nz^n+\sum_{n=1}^{\infty}c_{-n}\bar{z}^n$$

$$(4.7)\qquad =\sum_{n=-\infty}^{\infty}c_nr^{|n|}e^{inx}$$

$$=c_0+\sum_{n=1}^{\infty}c_nr^ne^{inx}+\sum_{n=1}^{\infty}c_{-n}r^ne^{-inx}$$

$$=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}r^nB_n(x),\quad \text{where } B_n(x)=a_n\cos nx+b_n\sin nx.$$

On substituting the values given in (4.5) for the coefficients, we have

$$G(r,x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)\,du+\sum_{n=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)r^ne^{in(x-u)}\,du$$

$$+\sum_{n=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)r^ne^{-in(x-u)}\,du$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)\Big[1+\sum_{n=1}^{\infty}r^ne^{in(x-u)}+\sum_{n=1}^{\infty}r^ne^{-in(x-u)}\Big]du.$$

$$G(r,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\left[1+ \frac{r\cos(x-u)-r^2+ir\sin(x-u)}{1-2r\cos(x-u)+r^2}\right.$$

$$\left. + \frac{r\cos(x-u)-r^2-ir\sin(x-u)}{1-2r\cos(x-u)+r^2}\right] du,$$

and finally

$$(4.8)\quad G(r,x) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(u)}{1-2r\cos(x-u)+r^2}\, du, \qquad r<1,$$

which is the Poisson integral (4.3). From (4.7) it is seen that the Poisson integral, $G(r,x)$, which is harmonic in the unit circle, corresponds to the real part of the analytic function $F(z)$ of one complex variable given in (4.6) and also that $\lim_{r=1} G(r,x)$ gives the Poisson or Abel sum of the single Fourier series (2.13) of $f(x)$.

4.B. Double Fourier series. It is known [cf. Zygmund 71, p.8] that the trigonometric system

$$(4.9)\quad 1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x, \ldots, \cos mx,\ \sin mx, \ldots \quad (m=0,1,2,\ldots),$$

is a complete orthogonal system of functions in $(-\pi,\pi)$[10].

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[10] A system of real or complex functions $g_m(x)$, $(m=0,1,2,\ldots)$, defined in an interval (a,b) is said to be orthogonal in this interval if

$$\int_a^b g_m(x)\,\overline{g_n(x)}\,dx = \begin{cases} 0, & m\neq n \\ k_n>0, & m=n \end{cases} \quad (m,n=0,1,2,\ldots),$$

and $\overline{g_n(x)}$ denotes the conjugate of $g_n(x)$. In particular, (from the last equation), no $g_m(x)\equiv 0$. If $k_0=k_1=k_2=\ldots=1$, the system is called normal. A sequence of real or complex

Zygmund also states [71,p.13] that if $\{g_m(x)\}$ and $\{h_n(y)\}$ are orthogonal, normal and complete in $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$ respectively, then the doubly infinite system $\{g_m(x)\ h_n(y)\}$ is orthogonal, normal and complete in R. Therefore, if we let both $\{g_m(x)\}$ and $\{h_n(y)\}$ be the above trigonometric system (4.9), it follows that the double trigonometric system

$$(4.10)\quad \{g_m(x)h_n(y)\}:\begin{cases}\cos mx\ \cos ny, & \sin mx\ \cos ny\\ \cos mx\ \sin ny, & \sin mx\ \sin ny\end{cases}\quad (m,n=0,1,2,\ldots)$$

is orthogonal and complete in R. Thus the formal development of f(x,y) into the double Fourier series (2.11)' with the coefficients in (2.12) is obtained by means of a complete orthogonal system of functions. It is well known that it is important in the development of functions for the orthogonal system to be complete since otherwise there would exist a

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functions $g_m(x)$ in (a,b) is called complete in the space L^* and interval (a,b) if the infinitely many equations for a function f(x) in L^*:

$$\int_a^b f(x)\ \overline{g_m(x)}\ dx = 0, \qquad m=0,1,2,\ldots,$$

imply f(x)=0 almost everywhere in (a,b).

Similarly a double system $g_{mn}(x,y)$ is orthogonal in (a,c;b,d) if

$$\int_a^b\int_c^d g_{mn}(x,y)g_{ij}(x,y)\ dxdy = \begin{cases}0, & (m\neq i,\ \substack{n=j\\ n\neq j})\text{or } (m=i,n\neq j)\\ k_{mn}>0, & (m=i,n=j)\end{cases}$$

for $m,n,i,j=0,1,2,\ldots$.

The definitions of ''normal'' and"complete'' double systems analogous to the above normal and complete single systems may also be formulated.

function in L^* whose Fourier series with respect to the system would consist entirely of zeros.

We have seen in (2.13) that the single Fourier series may be given a complex form by means of the single infinity of complex functions

$$(4.11) \qquad e^{inx} \qquad (n=0,\pm1,\pm2,,\dots).$$

The orthogonality of this system in $(-\pi,\pi)$ is obvious since

$$\int_{-\pi}^{\pi}(e^{imx})(e^{-inx})dx = \begin{cases} 0, & m\neq n \\ 2\pi, & m=n \end{cases} \quad (m,n=0,\pm1,\pm2,\dots).$$

By using the Euler relations

$$(4.12) \qquad \cos mx = \frac{1}{2}(e^{imx}+e^{-imx}), \quad \sin mx = \frac{1}{2i}(e^{imx}-e^{-imx})$$

we see that completeness of the system (4.11) in $(-\pi,\pi)$ follows from completeness of the trigonometric system (4.9). For in order to approximate any function $f(x)$ in L^* by the system (4.11), it is enough to approximate by (4.9) and use (4.12) which gives an approximation in the desired exponentials. We may again apply the method of obtaining a doubly infinite complete orthogonal system by multiplying elements of two singly infinite complete systems. Using the complete complex system (4.11), we obtain the doubly infinite complete system of complex functions

$$(4.13) \qquad e^{i(mx+ny)} \qquad (m,n=0,\pm1,\pm2,\dots)$$

in the region R.

Now we give the double series $(2.11)'$ a complex form. Applying Euler's formulae (4.12) to cosmx, cosny, sinmx, sinny we may write (in the notation of (2.10))

$$\begin{aligned}
\frac{1}{\lambda_{mn}}A_{mn}(x,y) &= a_{mn}\cos mx\,\cos ny + b_{mn}\sin mx\,\cos ny \\
&\quad + c_{mn}\cos mx\,\sin ny + d_{mn}\sin mx\,\sin ny, \quad m,n \geq 0, \\
&= \frac{1}{4}\Big[a_{mn}(e^{imx}+e^{-imx})(e^{iny}+e^{-iny}) \\
&\quad + ib_{mn}(e^{-imx}-e^{imx})(e^{iny}+e^{-iny}) \\
&\quad + ic_{mn}(e^{imx}+e^{-imx})(e^{-iny}-e^{iny}) \\
&\quad - d_{mn}(e^{-imx}-e^{imx})(e^{-iny}-e^{iny})\Big]
\end{aligned}$$

$$\begin{aligned}
(4.14) \qquad &= \frac{1}{4}\Big[(a_{mn}-ib_{mn}-ic_{mn}-d_{mn})e^{imx}e^{iny} \\
&\quad +(a_{mn}-ib_{mn}+ic_{mn}+d_{mn})e^{imx}e^{-iny} \\
&\quad +(a_{mn}+ib_{mn}-ic_{mn}+d_{mn})e^{-imx}e^{iny} \\
&\quad +(a_{mn}+ib_{mn}+ic_{mn}-d_{mn})e^{-imx}e^{-iny}\Big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\Big[\big\{(a_{mn}-d_{mn})-i(b_{mn}+c_{mn})\big\}e^{imx}e^{iny} \\
&\quad + \big\{(a_{mn}+d_{mn})-i(b_{mn}-c_{mn})\big\}e^{imx}e^{-iny} \\
&\quad + \big\{(a_{mn}+d_{mn})+i(b_{mn}-c_{mn})\big\}e^{-imx}e^{iny} \\
&\quad + \big\{(a_{mn}-d_{mn})+i(b_{mn}+c_{mn})\big\}e^{-imx}e^{-iny}\Big].
\end{aligned}$$

We now define the $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ for negative indices.

Let

$$
(4.15)\quad
\begin{aligned}
&a_{-m,-n} = a_{-m,n} = a_{m,-n} = a_{mn},\\
&b_{-m,-n} = b_{-m,n} = -b_{m,-n} = -b_{m,n},\; b_{oo}=0,\; b_{on}=0,\\
&c_{-m,-n} = -c_{-m,n} = c_{m,-n} = -c_{m,n},\; c_{oo}=0,\; c_{mo}=0, \qquad (m,n \geq 0)\\
&d_{-m,-n} = -d_{-m,n} = -d_{m,-n} = d_{m,n},\; d_{oo}=d_{mo}=d_{on}=0,
\end{aligned}
$$

and for $m,n=0,\pm 1,\pm 2,\ldots$ these relations may be written:

$$
(4.16)\quad
\begin{aligned}
&a_{m,n}=a_{|m|,|n|}, &\qquad& b_{m,n}=(\text{sign } m)b_{|m|,|n|},\\
&c_{m,n}=(\text{sign } n)c_{|m|,|n|}, && d_{m,n}=(\text{sign } mn)d_{|m|,|n|},
\end{aligned}
$$

where sign a = +1 for $a > 0$, sign a = −1 for $a < 0$ and sign0 = 0. From the last equation of (4.14) we define $C_{m,n}$ for positive and negative indices by the following relations:

$$
(4.17)\quad
\begin{aligned}
C_{m,n} &= \tfrac{1}{4}\left[a_{mn}-d_{mn}-i(b_{mn}+c_{mn})\right]\\
C_{m,-n} &= \tfrac{1}{4}\left[a_{mn}+d_{mn}-i(b_{mn}-c_{mn})\right]\\
C_{-m,n} &= \tfrac{1}{4}\left[a_{mn}+d_{mn}+i(b_{mn}-c_{mn})\right]\\
C_{-m,-n} &= \tfrac{1}{4}\left[a_{mn}-d_{mn}+i(b_{mn}+c_{mn})\right]
\end{aligned}
\qquad m \geq 0,\; n \geq 0,
$$

whence

$$
(4.18)\quad C_{m,n}=\overline{C}_{-m,-n}, \qquad C_{m,-n}=\overline{C}_{-m,n}, \qquad m \geq 0,\; n \geq 0.
$$

From (4.16) we may write the relations (4.17) in the form

$$
(4.19)\quad C_{m,n}= \tfrac{1}{4}\left[a_{m,n}-d_{m,n}-i(b_{m,n}+c_{m,n})\right] \qquad m,n=0,\pm 1,\pm 2,\ldots
$$

and (4.18) still holds with $m,n=0,\pm 1,\pm 2,\ldots$

With this notation we write (2.11)' in the complex form

$$
\sum_{m,n=0}^{\infty} A_{mn}(x,y) = \sum_{m,n=-\infty}^{\infty} C_{mn}e^{imx}e^{iny}
$$

which may be written

$$\sum_{m,n=0}^{\infty} A_{mn}(x,y) = C_{00} + \sum_{m=1}^{\infty}(C_{m0}e^{imx} + C_{-m,0}e^{-imx})$$

$$(4.20) \qquad + \sum_{n=1}^{\infty}(C_{0n}e^{iny} + C_{0,-n}e^{-iny})$$

$$+ \sum_{m,n=1}^{\infty}(C_{mn}e^{imx}e^{iny} + \bar{C}_{mn}e^{-imx}e^{-iny})$$

$$+ \sum_{m,n=1}^{\infty}(C_{m,-n}e^{imx}e^{-iny} + \bar{C}_{m,-n}e^{-imx}e^{iny}).$$

Upon introducing the two complex variables $z_1=e^{ix}$, $z_2=e^{iy}$, we have

$$(4.21) \qquad \sum_{m,n=0}^{\infty} A_{mn}(x,y) = C_{00} + 2\Re\sum_{m=1}^{\infty} C_{m0}z_1^m + 2\Re\sum_{n=1}^{\infty} C_{0n}z_2^n$$

$$+ 2\Re\sum_{m,n=1}^{\infty} C_{mn}z_1^m z_2^n + 2\Re\sum_{m,n=1}^{\infty} C_{m,-n}z_1^m \bar{z}_2^n$$

If $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ are the double Fourier coefficients in (2.12), then since the relations in (4.16) hold for $m,n=0,\pm1,\ldots$ in (2.12), we have from (4.19) that

$$4C_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos mx \cos ny \, dxdy - \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin mx \sin ny \, dxdy$$

$$-i\left[\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin mx \cos ny \, dxdy + \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos mx \sin ny \, dxdy\right]$$

$$4C_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\left[\cos(mx+ny)-i\,\sin(mx+ny)\right]dxdy,$$

whence

$$(4.22)\quad C_{mn} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\,e^{-i(mx+ny)}dxdy,\quad (m,n=0,\pm1,\pm2,\ldots).$$

Conversely, this process may be reversed and the coefficients in (2.12) may be uniquely determined from (4.22) and (4.19). With the coefficients (4.22) we may write the double Fourier series of f(x,y) in the complex form (4.21) or in the form

$$(4.23)\quad f(x,y)\sim\sum_{m,n=0}^{\infty} A_{mn}(x,y) = \sum_{m,n=-\infty}^{\infty} C_{mn}z_1^m z_2^n,\quad z_1=e^{ix},\ z_2=e^{iy}.$$

Two important differences between single and double Fourier series were mentioned in the introduction (page 2). Confusion results from a failure to understand the distinction between the correspondence of the single power series to a single Fourier series and the analogous situation in double series. We, therefore, wish to consider this distinction in detail.

On placing r=1 in the last equation of (4.7), it is seen that the single trigonometric series (2.13) may be treated as the real part of the power series (4.6) on the unit circle $z=e^{ix}$ [cf. 65,p.1]. We shall see, however, that the real part of a double power series in two complex variables

$$(4.24) \qquad \sum_{m,n=0}^{\infty} B_{m,n} z_1^m z_2^n, \quad B_{mn}=\alpha_{mn}-i\,\beta_{mn}, \quad (\alpha_{mn}, \beta_{mn} \text{ real}),$$

on the circles $z_1=e^{ix}$, $z_2=e^{iy}$ is not the most general double trigonometric series (2.11)' or the complex form given in (4.23). The series (4.24) may be formally written

$$\sum_{m,n=0}^{\infty} B_{mn} e^{i(mx+ny)} = \sum_{m,n=0}^{\infty} B_{mn}(\cos mx + i \sin mx)(\cos ny + i \sin ny)$$

$$= \sum_{m,n=0}^{\infty} B_{mn}(\cos mx \cos ny - \sin mx \sin ny + i \cos mx \sin ny + i \sin mx \cos ny)$$

so that

$$(4.26) \qquad \mathcal{R} \sum_{m,n=0}^{\infty} B_{mn} e^{i(mx+ny)} = \sum_{m,n=0}^{\infty} (\alpha_{mn} \cos mx \cos ny + \beta_{mn} \sin mx \cos ny + \beta_{mn} \cos mx \sin ny - \alpha_{mn} \sin mx \sin ny)$$

$$= \sum_{m,n=0}^{\infty} \left[\alpha_{mn} \cos(mx+ny) + \beta_{mn} \sin(mx+ny)\right]$$

and it is obvious that the coefficients do not give the most general double series since in the notation of (2.10) we have here that necessarily

$$(4.27) \qquad a_{mn} = -d_{mn}, \qquad b_{mn} = c_{mn},$$

and consequently two of the four coefficients are not arbitrary. Thus the real part of the double Taylor's series (4.24) on $z_1=e^{ix}$, $z_2=e^{iy}$ is not as general as the double Fourier series (4.23) of $f(x,y)$.

4.C. Harmonic, Double Harmonic, and Biharmonic Functions of several variables. Laplace Series. We now introduced the following concepts.

Harmonic function of several variables: A real function $U(x_1,x_2,\dots,x_n)$ of n real variables $(x_1,x_2,\dots,x_n)$ in rectangular coordinates is said to be harmonic in a region of n-space if it satisfies the Laplace differential equation

$$(4.28) \qquad \frac{\partial^2 U}{\partial x_1^2}+\frac{\partial^2 U}{\partial x_2^2}+\frac{\partial^2 U}{\partial x_3^2}+\cdots+\frac{\partial^2 U}{\partial x_n^2}=0$$

at every point of the given region.

Bicylinder: Let z_1 and z_2 be two complex variables with

$$(4.29) \qquad \begin{array}{l} z_1=re^{ix}=u_1+iv_1 \\ z_2=se^{iy}=u_2+iv_2 \end{array} \qquad (r,s,x,y,u_1,v_1,u_2,v_2 \text{ real}),$$

then the four-fold field of variation of z_1 and z_2 consisting of the ∞^4 places given by combining any point within or upon the circle $|z_1|=1$ with any point within or upon the circle $|z_2|=1$ is defined as the closed unit bicylinder. The interior given by the two-fold field

$$(4.30) \qquad |z_1|=r<1, \qquad |z_2|=s<1$$

is designated as the unit bicylinder B. The boundary of B is the three-dimensional manifold consisting of the two parts

$$(4.31) \qquad \begin{array}{lll} (a) & |z_1|=r=1, & |z_2|=s\leq 1 \\ (b) & |z_1|=r\leq 1, & |z_2|=s=1. \end{array}$$

The common part of (a) and (b) given by

(4.32) $|z_1|=r=1, \qquad |z_2|=s=1$

is called the ''distinguished boundary surface'' of B or more briefly the ''distinguished surface of B[11]. This distinguished surface constitutes the points of the boundary of the unit torus. The distinguished surface of the bicylinder B plays a role in the theory of functions of two complex variables similar to that played by the unit circle as the boundary curve of a region in functions of one complex variable.

<u>Double harmonic function</u>: A function $U(u_1,v_1,u_2,v_2)$, (u_1,v_1,u_2,v_2 rectangular coordinates), is said to be a double harmonic function in the bicylinder B if it satisfies simultaneously the two Laplace differential equations

$$(4.33) \qquad \frac{\partial^2 U}{\partial u_1^2} + \frac{\partial^2 U}{\partial v_1^2} = 0 \qquad \frac{\partial^2 U}{\partial u_2^2} + \frac{\partial^2 U}{\partial v_2^2} = 0$$

in B. A fortiori U satisfies (4.28).

<u>Biharmonic function</u>: A function $U(u_1,v_1,u_2,v_2)$, (u_1,v_1,u_2,v_2 rectangular coordinates), is said to be a biharmonic function in the bicylinder B if it is the real part (or imaginary part) of an analytic function of two complex

---

[11] The term ''frontier'' is used by A.R. Forsyth [24,p.20] to designate the surface (4.32). The terms used here are those used by S.Bergman [2]. This distinguished surface is a special case of the ''three-dimensional distinguished manifold'' which is the real part of the boundary of the four-dimensional domain of two complex variables such that for every function $F(z_1,z_2)$ regular in the domain, $|F(z_1,z_2)|$ takes its maximum not only on the boundary, but more specifically on the distinguished boundary manifold of the domain.

variables in B. U then satisfies (as can be seen easily) not only the Laplace differential equations (4.33) but in addition the two differential equations

$$(4.34)\qquad \frac{\partial^2 U}{\partial u_1 \partial u_2} + \frac{\partial^2 U}{\partial v_1 \partial v_2} = 0 \qquad \frac{\partial^2 U}{\partial u_1 \partial v_2} - \frac{\partial^2 U}{\partial v_1 \partial u_2} = 0$$

in B.

From these definitions it is seen that both double harmonic and biharmonic functions are harmonic functions of the four variables (u_1, v_1, u_2, v_2) and that also the class of biharmonic functions is included in the class of double harmonic functions.

In division 4.B we have outlined the double Fourier series as a development of a function of two variables analogous to the single Fourier series development of a function of one variable. In the case of one variable, it was seen that the Poisson integral G(r,x) in (4.8) furnishes the solution of the Dirichlet problem for the unit circle and that $\lim_{r=1} G(r,x)$ gives the Poisson sum of the single Fourier series (2.13) of f(x). What is the situation now when dealing with a function of two variables f(x,y)? What are the Dirichlet problems corresponding to analogues of the unit circle? Can Poisson-type integrals be shown to give solutions of the corresponding problems? What connection, if any, exists between such Dirichlet analogues and the double Fourier series (4.23) or with the real part of a double power series of the form (4.24) or other developments? We shall not attempt to give a complete answer to these questions but merely wish to make clear the part played by the double Fourier series.

We shall now consider another generalization of the single Fourier series by giving a series development corresponding to the surface of a three dimensional sphere. We shall thus consider the following Dirichlet problem corresponding to the unit spher: With a given periodic function $f(\theta,\phi)$ of two variables defined on the boundary of the unit sphere, when does there exist a function harmonic on the interior of the sphere taking on the given values of $f(\theta,\phi)$ on the boundary?

If we use spherical coordinates the function depends on two angles (x,y), the longitude and latitude, and $f(x,y)$ has a period of 2π in x and a period of π in y. It is known [39, p.241] that the surface Poisson integral defined by

$$(4.35) \quad G(r,x,y) = \frac{1-r^2}{4\pi} \int_{-\pi}^{\pi} \int_{0}^{\pi} \frac{\sin\theta \, f(\theta,\phi)}{(1-2r\cos\gamma+r^2)^{\frac{3}{2}}} \, d\theta \, d\phi, \quad r<1,$$

(r,x,y spherical coordinates),

where $\cos\gamma = \cos x \cos\theta + \sin x \sin\theta \cos(y-\phi)$, is the harmonic function which takes on the given boundary values $f(x,y)$ and gives the solution of the Dirichlet problem for the unit sphere. For continuous $f(x,y)$ the function $G(r,x,y)$ approaches $f(x,y)$ at every boundary point. But if $f(x,y)$

while remaining integrable, has discontinuities, $G(r,x,y)$ cannot always approach $f(x,y)$ at every boundary point. In this case Kellogg [44] states that $G(r,x,y)$ approaches $f(x,y)$ at every boundary point where the function $f(x,y)$ is continuous, and $G(r,x,y)$ lies between the least upper and greatest lower bound of $f(x,y)$ at a point of discontinuity. Tt is thus seen that the surface Poisson integral $G(r,x,y)$ in(4.35) is a harmonic function which plays a role for the unit sphere similar to that played by the single Poisson integral $G(r,x)$ in (4.8) in that it gives the solution of the corresponding Dirichlet problem. On the other hand if we use the spherical coordinates $r=1$, $-\pi\leq x\leq\pi$, $0\leq y\leq\pi$, $f(x,y)$ may be given a Laplace series representation

$$(4.36)\quad f(x,y)=\sum_{n=0}^{\infty}\left[A_nP_n(\cos x)+\sum_{m=1}^{n}(A_n^{(m)}\cos my+B_n^{(m)}\sin my)P_n^{m}(\cos x)\right]$$

with

$$(4.37)\begin{cases} A_n^{(m)}=(2n+1)\dfrac{(n-m)!}{(n+m)!\,2\pi}\displaystyle\int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta,\phi)P_n^{m}(\cos\theta)\cos m\phi\ \sin\theta\ d\theta d\phi \\ B_n^{(m)}=(2n+1)\dfrac{(n-m)!}{(n+m)!\,2\pi}\displaystyle\int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta,\phi)P_n^{m}(\cos\theta)\sin m\phi\ \sin\theta\ d\theta d\phi \\ A_n=(2n+1)\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta,\phi)P_n(\cos\theta)\ \sin\theta\ d\theta\ d\phi \end{cases}$$

where $P_n(\cos\theta)$ denotes the well known Legendre functions:

$$\left\{\begin{aligned}
P_n(\cos\theta) &= \frac{2}{\pi}\int_0^{\theta} \frac{\cos(n+\frac{1}{2})u}{\left[2(\cos u-\cos\theta)\right]^{\frac{1}{2}}}\,du \\
&= \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\left[2\cos n\theta + \frac{1\cdot(2n)}{2(2n-1)}\,2\cos(n-2)\theta\right. \\
&\qquad\left. + \frac{1\cdot 3\cdot(2n)(2n-2)}{2\cdot 4(2n-1)(2n-3)}\,2\cos(n-4)\theta+\cdots\right] \\
P_n^m(\cos\theta) &= (-1)^m\sin^m\theta\,\frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m}\,.
\end{aligned}\right. \tag{4.38}$$

It has been seen that the limiting value as r tends to 1 of the single Poisson integral G(r,x) in (4.8) gives the Poisson sum of the single Fourier series of f(x). Here the surface Poisson integral G(r,x,y) in (4.35) plays an analogous role for the Laplace series (4.36) of f(x,y) since $\lim_{r=1} G(r,x,y)$ gives the Poisson sum of the Laplace series (4.36).

Additional analogues of the Dirichlet problem for the unit circle lead us to a discussinn of the members of two other classes of functions: a biharmonic function as the real part of a double power series in two complex variables and a double harmonic function as the Poisson sum of the general double Fourier series.

We first consider the Dirichlet problem of determining a function defined in the bicylinder B which assumes the values of a given continuous function f(x,y) on the distinguished surface

(4.39) $\quad z_1=e^{ix}=u_1+iv_1, \quad z_2=e^{iy}=u_2+iv_2, \quad -\pi \leq x \leq \pi, -\pi \leq y \leq \pi,$

and is the real part of an analytic function of two complex variables in B (biharmonic in B).

Let $F_1(z_1,z_2)$ be an analytic function of the two complex variables z_1,z_2 in the bicylinder $B(|z_1|<1, |z_2|<1)$ whose real part assumes the values of $f(x,y)$ on (4.39). It is to be kept in mind that $f(x,y)$ being the real part of an analytic function of two complex variables is not an arbitrary function of the two variables. Now $f(x,y)$ can be developed into the special double Fourier series

$$(4.40) \qquad f(x,y) \sim \sum_{m,n=0}^{\infty} \left[\alpha_{mn}\cos(mx+ny) + \beta_{mn}\sin(mx+ny)\right]$$

with coefficients

$$\alpha_{00} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\,dudv = \frac{1}{4}a_{00},$$

(4.41)

$$\alpha_{mn} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\cos(mu+nv)dudv = \frac{1}{2}(a_{mn}-d_{mn})$$
$$= a_{mn} = -d_{mn},$$

$$\beta_{mn} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sin(mu+nv)dudv = \frac{1}{2}(b_{mn}+c_{mn})$$
$$= b_{mn} = c_{mn},$$

where $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ are as defined in (2.12). The analytic function $F_1(z_1,z_2)$ has the double power series development

(4.42) $$F_1(z_1,z_2)=\sum_{m,n=0}^{\infty} B_{mn} z_1^m z_2^n, \qquad z_1=re^{ix}, z_2=se^{iy}, \ r<1, s<1,$$

and since the series (4.40) is the real part of the double power series (4.42) for r=1, s=1, the coefficients B_{mn} are given by the relations

(4.43)
$$B_{oo}=\alpha_{oo}-i\,\beta_{oo}=\frac{1}{4}a_{oo}=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)dudv, \quad \beta_{oo}=0,$$

$$B_{mn}=\alpha_{mn}-i\,\beta_{mn}=\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\ e^{-i(mu+nv)}dudv.$$

We write
$$G_1(r,s,x,y)=\mathcal{R}F_1(z_1,z_2)=\mathcal{R}\sum_{m,n=0}^{\infty} B_{mn}\ z_1^m\ z_2^n$$

$$=\mathcal{R}\sum_{m,n=0}^{\infty} B_{mn}\ r^m\ s^n\ e^{i(mx+ny)}$$

(4.44)
$$=\mathcal{R}\left[B_{oo}+\sum_{m=1}^{\infty} r^m B_{mo}e^{imx}+\sum_{n=1}^{\infty} s^n B_{on}e^{iny}+\sum_{m,n=1}^{\infty} r^m s^n B_{mn}e^{i(mx+ny)}\right]$$

$$=\mathcal{R}\left[2B_{oo}+\sum_{m=1}^{\infty} r^m B_{mo}e^{imx}+\sum_{n=1}^{\infty} s^n B_{on}e^{iny}+\sum_{m,n=1}^{\infty} r^m s^n B_{mn}e^{i(mx+ny)}-B_{oo}\right]$$

$$=\sum_{m,n=0}^{\infty} r^m s^n \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} 2f(u,v)e^{i\ m(x-u)+n(y-v)}\ dudv$$
$$-\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)dudv$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\left[\sum_{m,n=0}^{\infty} 2r^m s^n e^{i(m\phi+n\theta)} - 1\right] du dv, \text{ where } \begin{matrix}\phi = x-u\\ \theta = y-v,\end{matrix}$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\left[\frac{2}{(1-re^{i\phi})(1-se^{i\theta})} - 1\right] du\, dv$$

$$= \frac{(1-r^2)\ (1-s^2)}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{f(u,v)\ du\ dv}{(1-2r\ \cos\phi+r^2)(1-2s\ \cos\theta+s^2)}$$

$$+ \frac{1}{2\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\left[\frac{1-r\ \cos\phi}{1-2r\ \cos\phi + r^2} + \frac{1-s\ \cos\theta}{1-2s\ \cos\theta+s^2}\right.$$

$$\left. - \frac{(1-r\ \cos\phi)(1-s\ \cos\theta)+rs\ \sin\phi\ \sin\theta}{(1-2r\ \cos\phi+r^2)(1-2s\ \cos\theta + s^2)} - 1\right] dudv.$$

Now we use the following notation for the double Poisson integral defined by

$$(4.45)\quad G_2(r,s,x,y) = \frac{(1-r^2)(1-s^2)}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{f(u,v)\ du\ dv}{[1-2r\cos(x-u)+r^2][1-2s\cos(y-v)+s^2]}$$

$$r<1,\ s<1,$$

and letting

$$(4.46)\quad \begin{cases} \Delta = \Delta_1\Delta_2, & \phi = x-u, & \theta = y-v, \\ \Delta_1 = 1-2r\ \cos\phi + r^2, & h_1 = 1-r\ \cos\phi, & g_1 = r\ \sin\phi \\ \Delta_2 = 1-2s\ \cos\theta + s^2, & h_2 = 1-s\ \cos\theta, & g_2 = s\ \sin\theta \end{cases}$$

and

$$(4.47)\quad G_3(r,s,x,y) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\left[\frac{h_1}{\Delta_1} + \frac{h_2}{\Delta_2} - \frac{h_1h_2+g_1g_2}{\Delta} - 1\right] dudv,$$

we may write $G_1(r,s,x,y)$ as the sum of the two Poisson-type integrals

$$G_1(r,s,x,y) = G_2(r,s,x,y) + G_3(r,s,x,y). \tag{4.48}$$

Jacob [39] states that for any integrable function $f(x,y)$ the Poisson-type integral $G_1(r,s,x,y)$ gives a biharmonic function in the bicylinder B but $G_1(r,s,x,y)$ does not in general tend to $f(x,y)$ when one approaches the distinguished surface (4.39). With the aid of Fatou's classic results on Poisson integrals, Jacob obtains the following necessary and sufficient condition that $G_1(r,s,x,y)$ furnish the solution of the above Dirichlet problem for biharmonic functions in a bicylinder:

$$\begin{aligned} f(\xi,\zeta) &+ \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[f(x,y) - f(\xi,y) - f(x,\zeta)\right]dxdy \\ &+ \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cot\frac{\xi-x}{2}\cot\frac{\zeta-y}{2}\,dxdy = 0, \end{aligned} \tag{4.49}$$

the last integral being taken with its principal value in the sense of Cauchy. This condition (4.49) includes the following one stated by Bergman [2]

$$\left.\begin{aligned} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(mx-ny)\,dxdy = 0 \\ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(mx-ny)\,dxdy = 0 \end{aligned}\right\} \quad (m,n=1,2,\ldots). \tag{4.50}$$

On the other hand Bergman [3] has pointed out that one can easily define a continuous function on the distinguished surface (4.39) such that there does not exist a biharmonic function which takes on these boundary values on the distinguished boundary surface (4.39). Consequently, a more general class of functions than biharmonic functions is needed for it to be possible to find a harmonic function which assumes preassigned values on the distinguished boundary for every continuous function given on the distinguished boundary surface.

We shall, therefore, be concerned with a generalized Dirichlet problem for the bicylinder where it is desired to find a double harmonic function in the bicylinder which assumes the values of a given continuous function $f(x,y)$ on the distinguished boundary surface (4.39). We,first, give the development of the double Poisson integral $G_2(r,s,x,y)$ given in (4.45) from the point of view of the Poisson sum of the double Fourier series (4.23) of $f(x,y)$. We again employ the complex form with the variables $z_1=re^{ix}, z_2=se^{iy}$, $r<1$, $s<1$. It has been seen that the general double Fourier series (4.23) can not be considered as the real part of an analytic function of two complex variables and consequently can not be a biharmonic function. However, as in (4.21) we obtain the desired generality if we consider the real part of the non-analytic function

(4.51) $$F(z_1,z_2) = U+iV = C_{oo} + \sum_{m=1}^{\infty} 2C_{mo} z_1^m + \sum_{n=1}^{\infty} 2C_{on} z_2^n + \sum_{m,n=1}^{\infty} 2C_{mn} z_1^m z_2^n + \sum_{m,n=1}^{\infty} 2C_{m,-n} z_1^m \bar{z}_2^n$$

with C_{mn} as given in (4.22), and $z_1 = re^{ix}$, $z_2 = se^{iy}$, $r < 1$, $s < 1$, $-\pi \leq x \leq \pi$, $-\pi \leq y \leq \pi$. With the aid of (4.18) we write

$$G(r,s,x,y) = U = \mathcal{R}\, F(z_1,z_2) =$$

$$= C_{oo} + \sum_{m=1}^{\infty} C_{mo} z_1^m + \sum_{m=1}^{\infty} C_{-mo} \bar{z}_1^m + \sum_{n=1}^{\infty} C_{on} z_2^n + \sum_{n=1}^{\infty} C_{o,-n} \bar{z}_2^n + \sum_{m,n=1}^{\infty} C_{mn} z_1^m z_2^n + \sum_{m,n=1}^{\infty} C_{m,-n} z_1^m \bar{z}_2^n + \sum_{m,n=1}^{\infty} C_{-mn} \bar{z}_1^m z_2^n + \sum_{m,n=1}^{\infty} C_{-m,-n} \bar{z}_1^m \bar{z}_2^n$$

(4.52) $$= \sum_{m,n=-\infty}^{\infty} C_{mn} r^{|m|} s^{|n|} e^{i(mx+ny)}$$

$$= \sum_{m,n=0}^{\infty} r^m s^n A_{mn}(x,y),$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\left[1+2\frac{r\cos\phi-r^2}{\Delta_1}+2\frac{s\cos\theta-s^2}{\Delta_2}\right.$$

$$\left.+4\frac{(r\cos\phi-r^2)(s\cos\theta-s^2)}{\Delta}\right]du\,dv$$

whence

$$(4.53)\quad G(r,s,x,y)=\frac{(1-r^2)(1-s^2)}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)\,du\,dv}{[1-2r\cos(x-u)+r^2][1-2s\cos(y-v)+s^2]}$$

$$= G_2(r,s,x,y).$$

In a similar manner the imaginary part V of $F(z_1,z_2)$ may be found and we have

$$F(z_1,z_2)=\frac{(1-r^2)(1-s^2)}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)}{\Delta}\,du\,dv$$

$$+i\left[\frac{r(1-s^2)}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)\sin\phi}{\Delta}\,du\,dv\right.$$

$$\left.+\frac{s}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)\sin\theta}{\Delta_2}\,du\,dv\right]$$

Substituting the values of C_{mn} from (4.22) into the equation (4.51) with $z_1=re^{ix}$, $z_2=se^{iy}$ and taking the real part we have

$$G(r,s,x,y)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\,du\,dv+\sum_{m=1}^{\infty}\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)r^m e^{im\phi}$$

$$+\sum_{n=1}^{\infty}\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)s^n e^{in\theta}\,du\,dv$$

$$+\sum_{m,n=1}^{\infty}\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)r^m s^n e^{im\phi}e^{in\theta}\,du\,dv$$

$$+\sum_{m,n=1}^{\infty}\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)r^m s^n e^{im\phi}e^{-in\theta}\,du\,dv,$$

where $\phi=x-u, \theta=y-v$,

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\Bigg[1+2\sum_{m=1}^{\infty}r^m e^{im0}+2\sum_{n=1}^{\infty}s^n e^{in\theta}$$

$$+2\sum_{m,n=1}^{\infty}r^m s^n e^{im\phi}e^{in\theta}+2\sum_{m,n=1}^{\infty}r^m s^n e^{im\phi}e^{-in\theta}\Bigg]\,du\,dv$$

which in the notation of (4.46) is

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\Bigg[1+2\frac{r\cos\phi-r^2}{\Delta_1}+2\frac{s\cos\theta-s^2}{\Delta_2}$$

$$+2\frac{(r\cos\phi-r^2)(s\cos\theta-s^2)-rs\sin\phi\sin\theta}{\Delta}$$

$$+2\frac{(r\cos\phi-r^2)(s\cos\theta-s^2)+rs\sin\phi\sin\theta}{\Delta}\Bigg]\,du\,dv.$$

An examination shows that the double Poisson integral $G_2(r,s,x,y)$ satisfies the equations (4.33) in polar form and thus is a double harmonic function in the bicylinder B, but it is not a biharmonic function since it is not the real part of an analytic function of two complex variables. Now for an approach to the distinguished surface (4.39) corresponding to a radial approach to the boundary of the unit circle, Kuestermann [45] showed that $G_2(r,s,x,y)$ takes on the values of a continuous $f(x,y)$ on the distinguished surface (4.39). He showed that

$$(4.54)\quad \lim_{\substack{x=y=0\\ r\to 1, s\to 1}} G_2(r,s,x,y) = \frac{1}{4}\Big[f(x+0,y+0) + f(x+0,y-0) + f(x-0,y+0) + f(x-0,y-0)\Big],$$

provided $f(x,y)$ is bounded over the fundamental square about (0,0) and the limit on the right exists. The double Poisson integral $G_2(r,s,x,y)$ thus gives the solution forthe Dirichlet problem for double harmonic functions in a bicylinder B, and $\lim_{\substack{r\to 1,\\ s\to 1}} G_2(r,s,x,y)$ gives the Poisson sum of the double Fourier series (4.23).

The relations between the different double series may now be summarized.

(1) On the circles $|z_1|=r=1$, $|z_2|=s=1$ we have seen in (4.27) that the real part of the double power series (4.24) is less general than the double Fourier series (4.23).

(2) For $|z_1|=r<1$, $|z_2|=s<1$, the real part of the double power series (4.42) is a biharmonic function and less

general than the double Poisson integral (4.53) which is a double harmonic function and which gives the Poisson sum of the general double Fourier series (4.23) of $f(x,y)$.

CHAPTER V

ABSOLUTE CONVERGENCE OF DOUBLE TRIGONOMETRIC SERIES

5.A. Summary of Chapters V and VI. In Chapter V we shall be concerned with some questions relative to the effect of absolute convergence of double trigonometric series on the absolute convergence of the series of coefficients.

With the aid of Lemma 5.1 and Lemma 5.2, we are able to obtain the Theorem 5.1 which is an analogue to double trigonometric series of the single trigonometric series theorem obtained by Cantor [10] (cf. Theorem B.1, p.17). Theorem 5.2 gneeralizes to double series a theorem of Fatou [21] which states that if a trigonometric series is absolutely convergent everywhere, then the series of coefficients are absolutely convergent (cf. Theorem B.3, p.17). On the other hand, Theorem 5.3 shows that in the case of a double power series, we need only to assume absolute convergence in a set of positive measure in order to get absolute convergence of the series of coefficients. This theorem is thus an analogue of a result obtained by Denjoy [17] and Lusin [46] in the sense of being a generalization to a double power series which is a special case of the double trigonometric series (cf. Theorem B.4,p.17). With a certain type of monotonicity assumption on the coefficients, we obtain in Theorem 5.4 an analogue of the result on the problem of the absolute convergence for sine or cosine series proved by

Fatou [22] (cf. Theorem B.5, p.18).

In Chapter VI we shall consider the absolute convergence of the double Fourier coefficients corresponding to a function f(x,y). There we shall first prove seven preliminary results in a sequence of lemmas needed for the proofs of subsequent theorems.

Integrated Lipschitz conditions on f(x,y) are employed to secure the absolute convergence of the double Fourier coefficients in Theorem 6.1 and Theorem 6.3. The two theorems, Theorem 6.2 and Theorem 6.4, which follow from the more general Theorem 6.1, are analogues to double Fourier series of the Bernstein theorem on the absolute convergence of single Fourier series (cf. Theorem A.1, p.11). These analogues are shown to contain the Bernstein theorem in the special case where f(x,y) reduces to a function of a single variable.

A generalization of Theorem 6.1 is given by Theorem 6.5 which gives the absolute convergence of the double Fourier coefficients when raised to some power. This theorem is an analogue of a theorem concerning the exponent of convergence of single Fourier series given by Szasz [59] (cf.Theorem A.5, p.11) and contains the Theorem 6.6 which is an analogue of the first generalization of Bernstein's theorem given by Szasz [59] (cf. Theorem A.2, p.11). ''Gegenbeispiele'' are given to show that these various results in Chapter VI are the ''best possible'' in a certain sense.

Theorem 6.7 is an analogue of a theorem by Zygmund [74] in which a combination of Lipschitz conditions and bounded variation gives absolute convergence of the Fourier coefficients (cf. Theorem A.7, p.12). This is generalized in the Theorem 6.8 to obtain the absolute convergence of the Fourier coefficients when raised to some power. The latter theorem is an analogue of the theorem given by Waraszkiewicz [67] and Zygmund [72] (cf. Theorem A.14, p.13).

In the last part of Chapter VI, we use the concept of a double harmonic function corresponding to a function f(x,y) to give an alternative proof of Theorem 6.4 in a manner analogous to G.H.Hardy's use of a harmonic function in his proof of Bernstein's theorem [32,p.321]. In obtaining the other analogues mentioned above our proofs generally use extensions to double series of the methods employed by the respective authors in their proofs in single series.

5.B. <u>Absolute convergence of double trigonometric series</u>. A double trigonometric series

$$\sum_{m,n=0}^{\infty}(a_{mn}\cos mx\cos ny + b_{mn}\sin mx\cos ny + c_{mn}\cos mx\sin ny + d_{mn}\sin mx\sin ny) \tag{5.1}$$

is said to converge absolutely at a point (x,y) if the series formed by replacing each term by its absolute value converges, i.e. if

$$\sum_{m,n=0}^{\infty}|a_{mn}\cos mx\cos ny + b_{mn}\sin mx\cos ny + c_{mn}\cos mx\sin ny + d_{mn}\sin mx\sin ny| < \infty . \tag{5.2}$$

If the series convârges absolutely at every point of a point-set or of a region, then the series is said to converge absolutely in the set or in the region. The problem to determine conditions on f(x,y) so that the double Fourier series will converge absolutely, may be considered for one point of R, or for all points of R, or for almost all points of R, or for some other plane set of points of R.

It is obvious from the definition of absolute convergence that the convergence of the series

$$(5.3)\qquad \sum_{m,n=0}^{\infty}\left[|a_{mn}| + |b_{mn}| + |c_{mn}| + |d_{mn}|\right]$$

implies the absolute convergence of the double series (5.1) at every point of R. This fact, when applied to double Fourier series, together with other results such as the Parseval theorem fof f(x,y), calls attention to the behavior of the double Fourier coefficients of given classes of functions. Several interesting properties of the double Fourier coefficients were given by Hilda Geiringer including the Riesz-Fischer theorem [25,p.78] and the Parseval theorem [25,p.138].

Although the convergence of (5.3) gives absolute convergence of the double series (5.1) in R, the double series (5.1) may be absolutely convergent at an infinite plane set of points without (5.3) being convergent. This may be seen by considering the example

$$(5.4) \qquad \sum_{m,n=0}^{\infty} \sin m!x \sin n!y.$$

Here we have $a_{mn}=b_{mn}=c_{mn}=0$, $(m \geq 0, n \geq 0)$, while the coefficients d_{mn} are either zero or one and there are an infinity of them equal to one so that

$$\sum_{m,n=0}^{\infty} |d_{mn}| = \infty .$$

On the other hand, (5.4) converges absolutely on the infinite plane set

$$x = \frac{p}{q}\pi, \qquad y = \frac{r}{s}\pi \qquad (p,q,r,s, \text{ any integers}),$$

since for all m,n such that $m \geq |q|$, $n \geq |s|$ we have

$$\sin(m!\frac{p}{q}\pi) = 0, \qquad \sin(n!\frac{r}{s}\pi) = 0.$$

Thus the series (5.4) converges absolutely on the (everywhere dense) plane set of points for which x and y are commensurable with π while the series of coefficients is not absolutely convergent.

We now point out the following useful relations. On letting

$$(5.5) \qquad \rho_{mn} = (a^2_{mn} + b^2_{mn} + c^2_{mn} + d^2_{mn})^{\frac{1}{2}}$$

and noticing that

$$(5.6) \quad |a_{mn}| \leq \rho_{mn}, \ |b_{mn}| \leq \rho_{mn}, \ |c_{mn}| \leq \rho_{mn}, \ |d_{mn}| \leq \rho_{mn},$$

it is seen that

(5.7) $$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty$$

implies

(5.8) $$\sum_{m,n=0}^{\infty} |a_{mn}| < \infty , \quad \sum_{m,n=0}^{\infty} |b_{mn}| < \infty , \quad \sum_{m,n=0}^{\infty} |c_{mn}| < \infty , \quad \sum_{m,n=0}^{\infty} |d_{mn}| < \infty ,$$

whence (5.3) converges and (5.1) converges absolutely at every point of R. On the other hand, if (5.8) holds we know since

(5.9) $$\rho_{mn} \leq |a_{mn}| + |b_{mn}| + |c_{mn}| + |d_{mn}|$$

that (5.7) holds.

We now give two lemmas to be used in the proof of Theorem 5.1.

<u>Lemma 5.1</u>. Let T(n) be an arbitrary positive function having only integral values such that $\lim_{n=\infty} T(n) = \infty$.

(i) If
$$\lim_{m,n=\infty} \alpha_{mn} = 0,$$

then for any T(n)

(5.10) $$\lim_{n=\infty} \alpha_{T(n),n} = 0$$

and conversely,

(ii) if $$\lim_{n=\infty} \alpha_{T(n),n} = 0$$

for every T(n), then

(5.11) $$\lim_{m,n=\infty} \alpha_{m,n} = 0.$$

Proof of (i). We have (5.5) given and wish to show that for any T(n) we obtain (5.4). Suppose (5.4) is not true, i.e. there exists a T(n) such that

$$\overline{\lim_{n=\infty}}\ |\alpha_{T(n),n}| > 0.$$

Thus there exists a $\delta > 0$ and corresponding to this δ there exists a sequence N_j, ($j=1,2,\ldots$), and an integer n_1, such that $N_j > n_1$ for all j and

$$|\alpha_{T(N_j),N_j}| > \delta\ , \qquad j=1,2,\ldots$$

But by hypothesis, we must have, for any preassigned positive and sufficiently large m_1 and n_1

$$|\alpha_{MN}| < \varepsilon, \qquad \text{for all } M > m_1,\ N > n_1.$$

Choosing $\varepsilon = \delta$ and $M=T(N)$, (integer), we have a contradiction which proves part (i).

Proof of (ii). For any integer T(n) we have (5.4) given with $\lim_{n=\infty} T(n)=\infty$ and wish to obtain (5.5). Assume (5.5) is not true, i.e.

$$\overline{\lim_{m,n=\infty}}\ |\alpha_{mn}| > 0.$$

Thus there is a $\delta > 0$ and corresponding to this δ there exists a sequence M_i, N_i of values of m,n and integers m_1, n_1 such that $M_i > m_1$, $N_i > n_1$, for all i, and

$$|\alpha_{M_i,N_i}| > \delta, \qquad \text{and } M_i \longrightarrow \infty\ , \quad N_i \longrightarrow \infty\ .$$

But since T(n) can be choosen as an arbitrary integer, with $\lim_{n=\infty} T(n)=\infty$, we can define $T(N_i)=M_i$, $(i=1,2,\ldots)$, whence

$$|\alpha_{T(N_i),N_i}| = |\alpha_{M_iN_i}| > \delta, \qquad i=1,2,\ldots .$$

Choosing $\delta = \varepsilon$ we see that this contradicts the hypothesis that for any T(N) and preassigned $\varepsilon > 0$,

$$|\alpha_{T(N),N}| < \varepsilon, \qquad \text{for all } N > n_1,$$

and part (ii) is proved.

<u>Lemma 5.2</u>. If

$$\lim_{m,n=\infty} \alpha_{mn} \sin mx \sin ny = 0 \tag{5.12}$$

for every x and y in R, then

$$\lim_{m,n=\infty} \alpha_{mn} = 0. \tag{5.13}$$

<u>Proof</u>. Let T(n) be an arbitrary positive number theoretic function of integer value such that $\lim_{n=\infty} T(n)=\infty$ and let

$$u_n = \alpha_{T(n),n}, \qquad D_n(x) = u_n \sin T(n)x.$$

For $m=T(n)$, we have from (5.12) that $\lim_{n=\infty} D_n(x) \sin ny = 0$ for every x and y in R. For each fixed value of x in $(-\pi,\pi)$ we have

$$\lim_{n=\infty} D_n \sin ny = 0 \qquad \text{for every } y \text{ in } (-\pi,\pi)$$

and by Cantor's theorem B.1, page 18, it follows that fore each fixed x, $\lim_{n=\infty} D_n(x)=\lim_{n=\infty} u_n \sin T(n)x=0$, i.e. for every x in $(-\pi,\pi)$. Now applying Theorem (B.2), page 18, we see that

$$\lim_{n=\infty} u_n = \lim_{n=\infty} \alpha_{T(n),n} = 0.$$

Since $T(n)$ is an arbitrary number theoretic function such that $\lim_{n=\infty} T(n)=\infty$, we may now apply part (ii) of Lemma 5.1 and thus obtain

$$\lim_{m,n=\infty} \alpha_{mn} = 0.$$

We now give an analogue to double series of the Cantor theorem for single trigonometric series.

__Theorem 5.1__. __If__ $A_{mn}(x,y)$, (in the notation of (2.10), page 7), __tends to zero, as__ m __and__ n __tend to__ ∞, __for every__ x __and__ y __in__ R, __then__

$$\lim_{\substack{m=\infty\\ n=\infty}} a_{mn} = 0, \quad \lim_{\substack{m=\infty\\ n=\infty}} b_{mn} = 0, \quad \lim_{\substack{m=\infty\\ n=\infty}} c_{mn} = 0, \quad \lim_{\substack{m=\infty\\ n=\infty}} d_{mn} = 0. \tag{5.14}$$

__Proof__. By hypothesis, we have

$$\lim_{m,n=\infty} A_{mn}(x,y)=\lim_{m,n=\infty} \lambda_{mn}(a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny) = 0$$

for every (x,y) in R. For $x=0$, $y=0$, this yields

$$\lim_{m,n=\infty} a_{mn} = 0.$$

Hence only the remaining three sequences for b_{mn}, c_{mn}, d_{mn} need be considered. It now follows that

$$\lim_{m,n=\infty} B_{mn}(x,y)= \lim_{m,n=\infty} \lambda_{mn}(b_{mn}\sin mx \cos ny + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny) = 0$$

for every (x,y) in R. Hence

$$\lim_{m,n=\infty} B_{mn}(x,0)=\lim_{m,n=\infty} \lambda_{mn} b_{mn}\sin mx = 0$$

for every x in $(-\pi,\pi)$. The existende of this double limit means that in whatever way maand n tend to infinity the double sequence approaches zero, hence if $n=m$ we have the single sequence with $u_m=b_{mm}$ and $\lim_{m=\infty} u_m \sin mx = 0$. By Cantor's Theorem B.1, we have

$$\lim_{m=\infty} u_m=0, \quad \text{i.e.} \quad \lim_{m=\infty} b_{mm} = 0.$$

In fact, we can say that for $n=T(m)$, where $T(m)$ is an arbitrary positive function tending to ∞ with m, we have the single sequence with $u_m=b_{T(m),m}$ and

$$\lim_{m=\infty} u_m \sin mx = 0 \qquad \text{for every } x \text{ in } (-\pi,\pi),$$

and by Cantor's Theorem B.1,

$$\lim_{m=\infty} u_m = \lim_{m=\infty} b_{T(m),m} = 0.$$

It follows from part (ii) of Lemma 5.1 that whatever manner of simultaneous approach to ∞ we take for m and n, the sequence has the limit zero, i.e.

$$\lim_{m,n=\infty} b_{mn} = 0.$$

Now only the two sequences for c_{mn} and d_{mn} are left to consider and we have

$$\lim_{m,n=\infty}{}_{mn}(c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny) = 0$$

for every (x,y) in R. For $x=0$, this yields

$$\lim_{m,n=\infty} c_{mn} \sin ny =0 \qquad \text{for every } y \text{ in } (-\pi,\pi).$$

A repetition of the argument used above gives $\lim_{m,n=\infty} c_{mn}=0$.

This leaves the one final sequence for d_{mn} to consider. From the above results and the original hypothesis we know

$$\lim_{m,n=\infty} d_{mn} \sin mx \sin ny = 0$$

for every (x,y) in R. Applying Lemma 5.2, we have

$$\lim_{m,n=\infty} d_{mn} = 0.$$

This completes the proof of the theorem.

We now give a lemma to be used in the proof of Theorem 5.2.

Lemma 5.3. If

$$(5.15) \qquad \sum_{m,n=1}^{\infty} |d_{mn} \sin mx \sin ny| < \infty \quad \text{for all } (x,y) \text{ in } R,$$

then

$$(5.16) \qquad \sum_{m,n=1}^{\infty} |d_{mn}| < \infty .$$

Proof. We define Q(x,y) by

$$(5.17) \qquad Q(x,y) = \sum_{m,n=1}^{\infty} |d_{mn}||\sin mx \sin ny|, \quad (x,y) \text{ in } R.$$

By assumption (5.15), Q(x,y) has a finite value at every point of R. From (5.17) this function Q(x,y) is the limit of the sequence of continuous functions

$$(5.18) \qquad Q_{ij}(x,y) = \sum_{m,n=1}^{i,j} |d_{mn}||\sin mx \sin ny|,$$

so that by the theorem of Baire (cf. Theorem C.1, p.20) it follows that $Q(x,y)$ is at most point wise discontinuous with respect to every perfect set contained in R. An application of Egoroff's theorem (cf. Theorem C.2, p.20) gives a plane set E_1 with $m(E_1)=|E_1|>4\pi^2-\varepsilon$, such that the sequence Q_{ij} converges uniformly in E_1 to $Q(x,y)$. Since $Q(x,y)$ is the limit of the uniformly bounded continuous functions $Q_{ij}(x,y)$ on E_1, $Q(x,y)$ is bounded on E_1, say $Q(x,y)$ M on E_1. The partial sums $Q_{ij}(x,y)$ being uniformly bounded on E_1 we may integrate the series for $Q(x,y)$ term wise over E_1. We have

$$\sum_{m,n=1}^{\infty}|d_{mn}|\iint_{E_1}|\sin mx \sin ny|\,dx\,dy \leq M|E_1|.$$

To prove (5.16) it is sufficient now to show that all the integrals

$$I_{mn}=\iint_{E_1}|\sin mx \sin ny|\,dx\,dy,\quad (m,n=1,2,\ldots),$$

are bounded away from zero by a positive quantity L. Let

$$I'_{mn}=\iint_{E_1}\sin^2 mx \sin^2 ny\,dx\,dy,\qquad (m,n=1,2,\ldots).$$

Since $I_{mn}\geq I'_{mn}$ it is sufficient to prove that $I'_{mn}>L$. Applying the formula $2\sin^2 X = 1-\cos 2X$, we have

$$I'_{mn}=\frac{1}{4}\iint_{E_1}(1-\cos 2mx)(1-\cos 2ny)dxdy = \frac{1}{4}\iint_{E_1}dxdy + I''_{mn}$$

where
$$\lim_{m,n=\infty} I''_{mn} = 0,$$

since
$$\lim_{m,n=\infty} \iint_{E_1} \cos 2mx \, dxdy = 0, \quad \lim_{m,n=\infty} \iint_{E_1} \cos 2ny \, dxdy = 0,$$

$$\lim_{m,n=\infty} \iint_{E_1} \cos 2mx \cos 2ny \, dxdy = 0.$$

Hence
$$\lim_{m,n=\infty} I'_{mn} = \tfrac{1}{4}|E_1| > L > 0.$$

This completes the proof.

Theorem 5.2. If the trigonometric series

$$\sum_{m,n=0}^{\infty} (a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny) \tag{5.19}$$

is absolutely convergent for every point in R, *then*

$$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty, \quad \text{where} \quad \rho^2_{mn} = a^2_{mn}+b^2_{mn}+c^2_{mn}+d^2_{mn}. \tag{5.20}$$

Proof. From the assumption that (5.19) converges absolutely at every point in R it follows that the two single series
$$\sum_{m=1}^{\infty}(a_{mo}\cos mx + b_{mo}\sin mx)$$
$$\sum_{n=1}^{\infty}(a_{on}\cos ny + c_{on}\sin ny)$$
are absolutely convergent for every x and y, respectively.

Hence by Fatou's theorem (cf. Theorem B.3,p.17), we have

$$(5.21) \qquad \sum_{m=1}^{\infty} \rho_{mo} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \rho_{on} < \infty .$$

To show that

$$(5.22) \qquad \sum_{m,n=1}^{\infty} \rho_{mn} < \infty$$

we consider the double series

$$(5.23) \qquad \sum_{m,n=1}^{\infty} (a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny)$$

which by assumption converges absolutely for every (x,y) in R. Let y=0 in (5.23), then by assumption

$$(5.24) \qquad \sum_{m,n=1}^{\infty} |a_{mn} \cos mx + b_{mn} \sin mx| < \infty \text{ for every } x \text{ in } (-\pi,\pi).$$

For x=0, this gives

$$(5.25) \qquad \sum_{m,n=1}^{\infty} |a_{mn}| < \infty .$$

If now we show that

$$(5.26) \qquad \sum_{m,n=1}^{\infty} |b_{mn}| < \infty ,$$

it will follow in the first place that

$$(5.27) \qquad \sum_{m,n=1}^{\infty} (a_{mn}^2 + b_{mn}^2)^{\frac{1}{2}} < \sum_{m,n=1}^{\infty} \left[|a_{mn}| + |b_{mn}| \right] < \infty$$

Now

$$\sum_{m,n=1}^{\infty} |b_{mn}\sin mx| = \sum_{m,n=1}^{\infty} |a_{mn}\cos mx + b_{mn}\sin mx - a_{mn}\cos mx|$$

$$\leq \sum_{m,n=1}^{\infty} |a_{mn}\cos mx + b_{mn}\sin mx| + \sum_{m,n=1} |a_{mn}\cos mx|$$

so that by (5.24) and (5.25) we have

$$(5.28)\quad \sum_{m,n=1}^{\infty} |b_{mn}\sin mx| = \sum_{m=1}^{\infty}\Big[|\sin mx| \sum_{n=1}^{\infty} |b_{mn}|\Big] < \infty \quad \text{for all } x.$$

Now sin mx=0 for any given m and suitable x, and upon placing

$$\sigma_m = \sum_{n=1}^{\infty} |b_{mn}|,$$

we see from (5.28) that $\sum_{m=1}^{\infty} \sigma_m |\sin mx| < \infty$ for all x.

Hence by Fatou's theorem (B.3) again, we obtain

$$\sum_{m=1}^{\infty} \sigma_m < \infty, \quad \text{i.e.} \quad \sum_{m,n=1}^{\infty} |b_{mn}| < \infty,$$

from which (5.27) follows.

If now we let x=0 in (5.23) our assumption gives

$$(5.29)\quad \sum_{m,n=1}^{\infty} |a_{mn}\cos ny + c_{mn}\sin ny| < \infty \quad \text{for all } y.$$

Now

$$\sum_{m,n=1}^{\infty} |c_{mn}\sin ny| = \sum_{m,n=1}^{\infty} |a_{mn}\cos ny + c_{mn}\sin ny - a_{mn}\cos ny|$$

$$\leq \sum_{m,n=1}^{\infty} |a_{mn}\cos ny + c_{mn}\sin ny| + \sum_{m,n=1}^{\infty} |a_{mn}\cos ny|,$$

whence by (5.25) and (5.29),

$$(5.30)\qquad \sum_{m,n=1}^{\infty} |c_{mn}| < \infty .$$

Since
$$\sum_{m,n=1}^{\infty} (a_{mn}^2+b_{mn}^2+c_{mn}^2)^{\frac{1}{2}} \leq \sum_{m,n=1}^{\infty} (a_{mn}^2+b_{mn}^2)^{\frac{1}{2}} + \sum_{m,n=1}^{\infty} |c_{mn}|$$

we have by (5.27) and (5.30) that

$$(5.31)\qquad \sum_{m,n=1}^{\infty} (a_{mn}^2+b_{mn}^2+c_{mn}^2)^{\frac{1}{2}} < \infty .$$

We now write

$$\begin{aligned}\sum_{m,n=1}^{\infty} |d_{mn} \sin mx \sin ny| = \sum_{m,n=1}^{\infty} |a_{mn} &\cos mx \cos ny + b_{mn} \sin mx \cos ny \\ &+ c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny \\ &\quad -(a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny \\ &\qquad\qquad + c_{mn} \cos mx \sin ny)|\end{aligned}$$

which by (5.19) and (5.31) gives

$$\sum_{m,n=1}^{\infty} |d_{mn} \sin mx \sin ny| < \infty \quad \text{for every } (x,y) \text{ in } R.$$

Applying Lemma 5.3 we have $\displaystyle\sum_{m,n=1}^{\infty} |d_{mn}| < \infty$,

which combined with (5.31) gives (5.22). From (5.22) and (5.21) we get the desired result (5.20). This completes the proof of the theorem.

Theorem 5.3. *Let* E_1 *and* E_2 *be linear point sets of positive measure on the* x *and* y *axes, respectively, and let a plane set* E *of positive measure be determined by the points of intersection of lines parallel to the coordinate axes erected on the points of* E_1 *and* E_2. *If the double trigonometric series* (5.19) *is such that the coefficients satisfy the condition*

$$(5.32)\qquad a_{mn}d_{mn}=b_{mn}c_{mn}\qquad (m,n=0,1,2,\ldots),$$

and if (5.19) *is absolutely convergent in* E, *then*

$$(5.33)\qquad \sum_{m,n=0}^{\infty}\rho_{mn}<\infty,\qquad \rho_{mn}^2=a_{mn}^2+b_{mn}^2+c_{mn}^2+d_{mn}^2.$$

Proof. By assumption we have

$$\sum_{m=1}^{\infty}A_{mo}(x)=\frac{1}{2}\sum_{m=1}^{\infty}(a_{mo}\cos mx+b_{mo}\sin mx)$$

converges absolutely in the set E_1 of positive measure and

$$\sum_{n=1}^{\infty}A_{on}(y)=\frac{1}{2}\sum_{n=1}^{\infty}(a_{on}\cos ny+c_{on}\sin ny)$$

converges absolutely in the set E_2 of positive measure. An application of Theorem B.4, page 18, to each of these single series gives

$$(5.34)\qquad \sum_{m=1}^{\infty}\rho_{mo}<\infty,\quad \sum_{n=1}^{\infty}\rho_{on}<\infty,\quad \rho_{mo}^2=a_{mo}^2+b_{mo}^2,\ \rho_{on}^2=a_{on}^2+c_{on}^2.$$

In order to complete the proof of the theorem we now show that

$$\sum_{m,n=1}^{\infty} \rho_{mn} < \infty$$

follows from the hypothesis that

$$(5.35) \quad \sum_{m,n=1}^{\infty} |a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny| < \infty$$

in a plane set E with $|E| > 0$. We first show that (5.32) enables us to write (5.35) in the form

$$(5.36) \quad \sum_{m,n=1}^{\infty} \rho_{mn}|\cos(mx+X)\ \cos(ny+Y)| < \infty$$

for suitably defined X and Y. In order to write (5.35) in the form (5.36) it is seen upon equating coefficients of corresponding terms that we must have

$$(5.37) \quad \begin{aligned} a_{mn} &= \rho_{mn}\cos X \cos Y & c_{mn} &= -\rho_{mn}\cos X \sin Y \\ b_{mn} &= -\rho_{mn}\sin X \cos Y & d_{mn} &= \rho_{mn}\sin X \sin Y, \end{aligned}$$

and hence $a_{mn}d_{mn}=b_{mn}c_{mn}$, which is the hypothesis (5.32). Also if (5.37) is to hold, then

$$a_{mn}^2+c_{mn}^2 = \rho_{mn}^2\cos^2 X, \qquad a_{mn}^2+b_{mn}^2 = \rho_{mn}^2\cos^2 Y.$$

We now define $|X|, |Y|$ by the relations

$$\cos X=\cos X_{mn}= \frac{1}{\rho_{mn}}(a_{mn}^2+c_{mn}^2)^{\frac{1}{2}}, \quad \cos Y=\cos Y_{mn}= \frac{1}{\rho_{mn}}(a_{mn}^2+b_{mn}^2)^{\frac{1}{2}}$$

so that
$$\sin^2 X=1-\frac{a_{mn}^2+c_{mn}^2}{\rho_{mn}^2} = \frac{b_{mn}^2+d_{mn}^2}{\rho_{mn}^2} \qquad \sin^2 Y = \frac{c_{mn}^2+d_{mn}^2}{\rho_{mn}^2}$$

and we choose +X and +Y so that

$$\sin X = \frac{1}{\rho_{mn}}(b_{mn}^2+d_{mn}^2)^{\frac{1}{2}}, \qquad \sin Y = \frac{1}{\rho_{mn}}(c_{mn}^2+d_{mn}^2)^{\frac{1}{2}}.$$

Hence the relation $\rho_{mn}^2 = a_{mn}^2+b_{mn}^2+c_{mn}^2+d_{mn}^2$ is satisfied and the coefficients have the form (5.37), which is permitted by hypothesis (5.32). From (5.37) we write

$$\begin{aligned} A_{mn}(x,y) &= \rho_{mn}(\cos mx \cos ny \cos X \cos Y - \sin mx \cos ny \sin X \cos Y \\ &\qquad - \cos mx \sin ny \cos X \sin Y + \sin mx \sin ny \sin X \sin Y) \\ &= \rho_{mn}(\cos mx \cos X - \sin mx \sin X)(\cos ny \cos Y - \sin ny \sin Y) \\ &= \rho_{mn} \cos(mx+X) \cos(ny+Y), \end{aligned}$$

so that (5.35) may be given the form (5.36) under the condition (5.32). We now define $Q(x,y)$ by

$$Q(x,y) = \sum_{m,n=1}^{\infty} |A_{mn}(x,y)| = \sum_{m,n=1}^{\infty} \rho_{mn} |\cos(mx+X)\cos(ny+Y)| \quad \text{on } E.$$

From the absolute convergence of this series on E, $Q(x,y)$ has a finite value at every point of E. Now let

$$Q_{ij}(x,y) = \sum_{m,n=1}^{i,j} \rho_{mn} |\cos(mx+X) \cos(ny+y)|.$$

Q_{ij} are evidently continuous on the set E and since

$$Q(x,y) = \lim_{i,j=\infty} Q_{ij}(x,y)$$

it follows that $Q(x,y)$ is a function of Baire's class 1 on E. From the theorem of Baire (cf. Theorem C.1,p.20) we have $Q(x,y)$ is at most point wise discontinuous with respect to every perfect set contained in E. We may therefore apply Egoroff's theorem (cf. Theorem C.2,p.20) to the sequence $Q_{ij}(x,y)$ and the function $Q(x,y)$. This gives a plane set E_3 contained in E, with $|E_3|>0$, such that the sequence $Q_{ij}(x,y)$ converges uniformly in E_3 to $Q(x,y)$. $Q(x,y)$ being the limit of the uniformly bounded continuous functions Q_{ij} on E_3, is bounded on E_3, say $Q(x,y)\quad M$; and the partial sums Q_{ij} being uniformly bounded on E_3, we may integrate the series for $Q(x,y)$ term wise over the plane set E_3. We have

$$\sum_{m,n=1}^{\infty} \rho_{mn} \iint_{E_3} |\cos(mx+X)\cos(ny+Y)|dxdy \leq M|E_3|.$$

To prove the convergence of $\sum_{m,n=1}^{\infty} {}_{mn}$ it is sufficient now to show that all the integrals

$$I_{mn} = \iint_{E_3} |\cos(mx+X)\cos(ny+Y)|dxdy, \qquad (m,n=1,2,\ldots),$$

are bounded from below by a positive quantity L. Let

$$I'_{mn} = \iint_{E_3} |\cos^2(mx+X)\cos^2(ny+Y)|dxdy, \qquad (m,n=1,2,\ldots).$$

Since $I_{mn}\geq I'_{mn}$, it is sufficient to prove that $I'_{mn}>L$. Applying the formula $2\cos^2 x=1+\cos 2x$ we obtain

$$I'_{mn}= \frac{1}{4}\iint_{E_3} (1+\cos 2mx\ \cos 2X-\sin 2mx\ \sin 2X)(1+\cos 2ny\ \cos 2Y$$
$$-\sin 2ny\ \sin 2Y)dxdy$$

whence

$$I'_{mn} = \frac{1}{4}\iint_{E_s} dxdy + I''_{mn},$$

where I''_{mn} is the sum of eight integrals, each of which tends to zero as m,n tend to ∞, since

$$\lim_{m,n=\infty}\iint_{E_s}\cos 2mx\, dxdy = 0, \quad \lim_{m,n=\infty}\iint_{E_s}\sin 2mx\, dxdy = 0,$$

$$\lim_{m,n=\infty}\iint_{E_s}\cos 2mx \cos 2ny\, dxdy=0, \quad \lim_{m,n=\infty}\iint_{E_s}\sin 2mx \cos 2ny\, dxdy=0$$

$$\lim_{m,n=\infty}\iint_{E_s}\cos 2mx \sin 2ny\, dxdy=0, \quad \lim_{m,n=\infty}\iint_{E_s}\sin 2mx \sin 2ny\, dxdy = 0.$$

Consequently

$$\lim_{m,n=\infty} I'_{mn} = \frac{1}{4}\iint_{E_s} dxdy = \frac{1}{4}|E_s| > L > 0.$$

This completes the proof since all I'_{mn} are bounded away from zero.

<u>Theorem 5.4</u>. <u>If the double cosine series</u>

$$\sum_{m,n=0}^{\infty} a_{mn} \cos mx \cos ny \tag{5.38}$$

<u>is absolutely convergent at a point</u> (x_1,y_1), $x_1,y_1 \neq 0 \pmod{\pi}$, <u>and if the coefficients are such that</u>

$$\lim_{m=\infty} |a_{mj}|=0, \quad \lim_{n=\infty} |a_{in}|=0, \text{ for every } i \text{ and } j,$$

$$|a_{mn}|-|a_{m,n+1}| \geq 0, \quad |a_{mn}|-|a_{m+1,n}| \geq 0, \tag{5.39}$$

$$|a_{mn}|-|a_{m,n+1}|-|a_{m+1,n}|+|a_{m+1,n+1}| \geq 0,$$

<u>then</u>

$$(5.40) \qquad \sum_{m,n=0}^{\infty} |a_{mn}| < \infty .$$

<u>Proof</u>. Since $\cos(-x)=\cos x$ we may assume $0 < x_1 < \pi$, $0 < y_1 < \pi$ without loss of generality. By hypothesis

$$(5.41) \qquad \sum_{m,n=0}^{\infty} |a_{mn}| |\cos mx_1| |\cos ny_1| < \infty .$$

For $n=0$, we have $\displaystyle\sum_{m=0}^{\infty} |a_{mo}| |\cos mx_1| < \infty$,

and by (5.39) we know $|a_{mo}| - |a_{m+1,o}| \geqq 0$, so that we may apply Fatou's theorem (cf. Theorem B.5, p.19) for the single cosine series. This yields $\displaystyle\sum_{m=1}^{\infty} |a_{mo}| < \infty$.

Similarly, for $m=0$ in (5.38) we have by Fatou's theorem that

$$\sum_{n=1}^{\infty} |a_{on}| < \infty .$$

Since $\cos^2 mx_1 \cos^2 ny_1 < |\cos mx_1| |\cos ny_1|$, it follows from (5.41) that

$$\sum_{m,n=1}^{\infty} |a_{mn}| \cos^2 mx_1 \cos^2 ny_1 < \infty .$$

Substituting $\cos^2 mx_1 = \frac{1}{2}(1+\cos 2mx_1)$, $\cos^2 ny_1 = \frac{1}{2}(1+\cos 2ny_1)$ gives

$$\frac{1}{4} \sum_{m,n=1}^{\infty} |a_{mn}| (1+\cos 2mx_1)(1+\cos 2ny_1) < \infty$$

or

$$(5.42) \qquad \frac{1}{4} \sum_{m,n=1}^{\infty} |a_{mn}| (1+\cos 2mx_1+\cos 2ny_1+\cos 2mx_1 \cos 2ny_1) < \infty .$$

From a theorem given by H. Geiringer[1*] we have

$$\sum_{m=1}^{\infty} |a_{mn}| \cos 2mx_1 \quad \text{converges,} \qquad \sum_{n=1}^{\infty} |a_{mn}| \cos 2ny_1 \quad \text{converges,}$$

$$\sum_{m,n=1}^{\infty} |a_{mn}| \cos 2mx_1 \cos 2ny_1 \quad \text{converges.}$$

Combining this with (5.42) we get $\displaystyle\sum_{m,n=1}^{\infty} |a_{mn}| < \infty .$

Collecting results yields

$$\sum_{m,n=0}^{\infty} |a_{mn}| < \infty ,$$

and the theorem is proved.

[1*] Geiringer [25,p.69] gives the following theorem: If $a_{mn} > 0$, and if $\lim_{m=\infty} a_{mj}=0$, $\lim_{n=\infty} a_{in}=0$,

$$a_{ij} - a_{i,j+1} - a_{i+1,j} + a_{i+1,j+1} \geq 0,$$

then

$\sum a_{mn} \cos mx \cos ny$ converges for every $x \neq 2m\pi$, $y \neq 2n\pi$,

$\sum a_{mn} \sin mx \cos ny$ converges for every x and for $y \neq 2n\pi$,

$\sum a_{mn} \cos mx \sin ny$ converges for every y and for $x \neq 2m\pi$,

$\sum a_{mn} \sin mx \sin ny$ converges for every x and every y.

CHAPTER VI

ABSOLUTE CONVERGENCE OF DOUBLE FOURIER SERIES

6.A. __Lipschitz conditions and absolute convergence__. The substance of the following Lemma 6.1 is found in Hobson [37,p.714].

__Lemma 6.1__. Let f(x,y) be periodic, of period 2π in each variable, and integrable in the sense of Lebesgue. We employ the notation of (2.10) and suppose the double Fourier series of f(x,y) is given in (2.11)' with the coefficients defined in (2.12). Let G(x) and H(y) be defined by the expressions

$$(6.1)\qquad \begin{cases} G(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x,y)\,dy - \frac{1}{4}a_{oo} \\[2ex] H(y) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x,y)\,dx - \frac{1}{4}a_{oo} \end{cases}$$

and let

$$(6.1)'\qquad F(x,y) = f(x,y) - \frac{1}{4}a_{oo} - G(x) - H(y),$$

then G(x), H(y) and F(x,y) are integrable and the Fourier series corresponding to G(x), H(y) and F(x,y) are

$$(6.2)\qquad \begin{cases} G(x) \sim \displaystyle\sum_{m=1}^{\infty} A_{mo}(x) = \sum_{m=1}^{\infty} \frac{1}{2}(a_{mo}\cos mx + b_{mo}\sin mx) \\[2ex] H(y) \sim \displaystyle\sum_{n=1}^{\infty} A_{on}(y) = \sum_{n=1}^{\infty} \frac{1}{2}(a_{on}\cos ny + c_{on}\sin ny) \\[2ex] F(x,y) \sim \displaystyle\sum_{m,n=1}^{\infty} A_{mn}(x,y) = \sum_{m,n=1}^{\infty} (a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny \\ \qquad\qquad + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny) \end{cases}$$

Proof. By definition, the single Fourier series of $G(x)$ is

$$(6.3)\quad G(x) \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} G(u)du + \sum_{m=1}^{\infty}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} G(u)\cos mu \cos mx\, du + \frac{1}{\pi}\int_{-\pi}^{\pi} G(u)\sin mu \sin mx\, du\right].$$

Since

$$(6.4)\quad \begin{cases} \displaystyle\int_{-\pi}^{\pi}\cos ku\, du = \frac{1}{k}\sin ku\Big|_{-\pi}^{\pi} = \frac{2}{k}\sin k\pi = 0, \quad (k=0,1,2,..) \\ \displaystyle\int_{-\pi}^{\pi}\sin ku\, du = -\frac{1}{k}\cos ku\Big|_{-\pi}^{\pi} = -\frac{1}{k}(\cos k\pi - \cos k\pi) = 0, \end{cases}$$

the determination of the Fourier coefficients of (6.3) for the $G(x)$ given in (6.1) is as follows

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(u)du = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u,v)dv - \frac{1}{4}a_{00}\right]du$$

$$= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)dudv - \frac{1}{4\pi}a_{00}\int_{-\pi}^{\pi} du,$$

and by (2.12)

$$= \frac{1}{2}a_{00} - \frac{1}{2}a_{00} = 0,$$

also

$$\frac{1}{\pi}\int_{-\pi}^{\pi} G(u)\cos mu \, du = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u,v)dv - \frac{1}{4}a_{oo}\right]\cos mu \, du$$

$$= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\cos mu \, dudv - \frac{1}{4\pi}a_{oo}\int_{-\pi}^{\pi}\cos mu \, du$$

and by (2.12) and (6.4)

$$= \frac{1}{2}a_{mo}, \qquad (m > 0).$$

Likewise

$$\frac{1}{\pi}\int_{-\pi}^{\pi} G(u)\sin mu \, du = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u,v)dv - \frac{1}{4}a_{oo}\right]\sin mu \, du, \; m > 0,$$

$$= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sin mu \, dudv - \frac{1}{4\pi}a_{oo}\int_{-\pi}^{\pi}\sin mu \, du$$

and by (2.12) and (6.4),

$$\frac{1}{\pi}\int_{-\pi}^{\pi} G(u)\sin mu \, du = \frac{1}{2}b_{mo}, \; m > 0,$$

so that upon substituting these values for the coefficients, the Fourier series (6.3) of G(x) becomes

$$G(x) \quad \sum_{m=1}^{\infty}\frac{1}{2}(a_{mo}\cos mx + b_{mo}\sin mx) = \sum_{m=1}^{\infty}A_{mo}(x).$$

A similar calculation of the coefficients of the Fourier series of H(y),

$$(6.5)\quad H(y) \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} H(v)\,dv + \sum_{n=1}^{\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} H(v)\cos nv \cos ny\, dv$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} H(v)\sin nv \sin ny\, dv\ ,$$

yields

$$H(y) \sim \sum_{n=1}^{\infty} \tfrac{1}{2}(a_{on}\cos ny + c_{on}\sin ny) = \sum_{n=1}^{\infty} A_{on}(y).$$

Finally, with the use of (6.1)', a similar calculation for the Fourier coefficients of F(x,y) yields

$$F(x,y) \sim \sum_{m,n=1}^{\infty} A_{mn}(x,y).$$

<u>Lemma 6.2</u>. Let u_{mn} be any double sequence of constants, $u_{mn} \geq 0$, then

$$(6.6)\quad \sum_{m,n=1}^{\infty} u_{mn}^2 \sin^2 mt \sin^2 nw \leq K_1 |t|^{2\alpha} |w|^{2\beta},\ 0 < \alpha \leq 1,\ 0 < \beta \leq 1,$$

implies the following three inequalities: with any given integers M,N

$$(6.7)\quad M^{-2}N^{-2} \sum_{m,n=1}^{M,N} m^2 n^2 u_{mn}^2 + \sum_{M+1,N+1}^{\infty} u_{mn}^2 \leq K_2\, M^{-2\alpha} N^{-2\beta}$$

$$(6.8)\quad \sum_{m,n=1}^{M,N} u_{mn}^2\, M^2 n^2 \leq K_3\, M^{2(1-\alpha)}\, N^{2(1-\beta)}$$

$$(6.9)\quad \sum_{m=2}^{2^{\,+1}-1}\ \sum_{,n=2^{v}}^{2^{v+1}-1} u_{mn}^{2} \leq K_4\, 2^{-2(\alpha\mu+\beta v)},\quad (\mu, v=0,1,2,\ldots),$$

where K_1, K_2, K_3 and K_4 are constants depending only on α and β.

Proof. In order not to interrupt the subsequent argument we first interpose the following :

for any integers M,N with $m<M, n<N$ and $t,w,\delta,\varepsilon,$

$$(6.10)\quad \begin{cases} \text{such that} \quad t\leq\delta\leq\frac{\pi}{2M}<\frac{\pi}{2m},\quad w\leq\varepsilon\leq\frac{\pi}{2N}<\frac{\pi}{2n} \\ \text{it is known that} \quad \sin mt>\frac{2mt}{\pi} \text{ for } 0<mt<\frac{\pi}{2} \\ \qquad\qquad\qquad\quad \sin nw>\frac{2nw}{\pi} \text{ for } 0<nw<\frac{\pi}{2}. \end{cases}$$

We now show that (6.6) implies (6.7). Let (6.6) be satisfied and let M,N be any integers. Next allow $\delta>0$, $\varepsilon>0$ to be such that

$$(6.11)\quad \begin{array}{lll} M\leq\frac{\pi}{2\delta}<M+1, & \text{i.e.} & \frac{\pi}{2(M+1)}<\delta\leq\frac{\pi}{2M} \\ N\leq\frac{\pi}{2\varepsilon}<N+1, & \text{i.e.} & \frac{\pi}{2(N+1)}<\varepsilon\leq\frac{\pi}{2N}. \end{array}$$

We have

$$\sum_{m,n=1}^{j,k} u_{mn}^2\sin^2 mt\,\sin^2 nw\leq\sum_{m,n=1}^{\infty} u_{mn}^2\sin^2 mt\,\sin^2 nw\leq K_1|t|^{2\alpha}|w|^{2\beta}.$$

Hence

$$\sum_{m,n=1}^{j,k} u_{mn}^2\sin^2 mt\,\sin^2 nw\leq K_1\,|t|^{2\alpha}\,|w|^{2\beta},$$

and upon integrating this inequality over $(0,0;\delta,\varepsilon)$ we get

$$\sum_{m,n=1}^{j,k} \int_0^{\delta}\int_0^{\varepsilon} u_{mn}^2 \sin^2 mt \, \sin^2 nw \, dtdw \leq K_4 \frac{\delta^{2\alpha+1}\,\varepsilon^{2\beta+1}}{(2\alpha+1)(2\beta+1)}$$

Letting $j \longrightarrow \infty$, $k \longrightarrow \infty$ we see that

$$(6.12) \quad \sum_{m,n=1}^{\infty} \int_0^{\delta}\int_0^{\varepsilon} u_{mn}^2 \sin^2 mt \, \sin^2 nw \, dtdw \leq K_5 \, \delta^{2\alpha+1}\,\varepsilon^{2\beta+1} .$$

Now consider a single term of the left side of (6.12). An application of (6.10) gives

$$\int_0^{\delta}\int_0^{\varepsilon} \sin^2 mt \, \sin^2 nw \, dtdw > \left(\frac{2m}{\pi}\right)^2 \left(\frac{2n}{\pi}\right)^2 \int_0^{\delta}\int_0^{\varepsilon} t^2 w^2 dtdw > \frac{1}{144} m^2 n^2 \delta^3 \varepsilon^3$$

$m=1,2,\ldots,M$
$n=1,2,\ldots,N,$

and by using (6.12) it follows that

$$(6.13) \quad \frac{1}{144} \sum_{m,n=1}^{M,N} u_{mn}^2 m^2 n^2 \, \delta^3 \varepsilon^3 < \sum_{m,n=1}^{M,N} u_{mn}^2 \int_0^{\delta}\int_0^{\varepsilon} \sin^2 mt \, \sin^2 nw dtdw$$

$$< K_5 \, \delta^{2\alpha+1}\,\varepsilon^{2\beta+1}$$

or

$$(6.14) \quad \delta^2 \varepsilon^2 \sum_{m,n=1}^{M,N} u_{mn}^2 m^2 n^2 < K_6 \, \delta^{2\alpha}\,\varepsilon^{2\beta} .$$

We now estimate the second part on the left side of (6.7). We thus consider $m > M$, $n > N$ (i.e. $m=M+1,\ldots;$ $n=N+1,\ldots$), which combined with (6.11) gives

$$m\delta \geq \frac{m\pi}{2(M+1)} \geq \frac{\pi}{2}, \quad n\varepsilon > \frac{n\pi}{2(N+1)} \geq \frac{\pi}{2}.$$

In fact, upon placing

$$p = \left[\frac{2m\delta}{\pi}\right] , \quad q = \left[\frac{2n\varepsilon}{\pi}\right]$$

we have

$$p \le \frac{2m}{\pi} < p+1, \qquad m\delta \ge \frac{\pi}{2}p, \qquad p > \frac{m\delta}{\pi}$$

(6.15)

$$q \le \frac{2n\varepsilon}{\pi} < q+1, \qquad n\varepsilon \ge \frac{\pi}{2}q, \qquad q > \frac{n\varepsilon}{\pi}.$$

Then making the change of variables $t = \frac{x}{m}$, $w = \frac{y}{n}$, $dt = \frac{1}{m}dx$, $dw = \frac{1}{n}dy$ we see that

$$(6.16) \quad \int_0^{\delta}\int_0^{\varepsilon} \sin^2 mt \sin^2 nw \, dtdw = \frac{1}{mn}\int_0^{m\delta}\int_0^{n\varepsilon} \sin^2 x \sin^2 y \, dxdy$$

$$= \frac{1}{mn}\left(\int_0^{m\delta} \sin^2 x \, dx\right)\left(\int_0^{n\varepsilon} \sin^2 y \, dy\right).$$

Since, by (6.15), $m\delta > \frac{\pi}{2}p$, $n\varepsilon > \frac{\pi}{2}q$, it follows that

$$\int_0^{m\delta} \sin^2 x \, dx > p\int_0^{\frac{\pi}{2}} \sin^2 x \, dx, \qquad \int_0^{n\varepsilon} \sin^2 y \, dy > q\int_0^{\frac{\pi}{2}} \sin^2 y \, dy$$

so that (6.16) gives, on using (6.15) and evaluating the integral,

$$(6.17) \quad \int_0^{\delta}\int_0^{\varepsilon} \sin^2 mt \sin^2 nw \, dtdw \ge \frac{pq}{mn}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2 x \sin^2 y \, dxdy = \frac{pq\pi^2}{16mn}$$

$$\ge \frac{m\delta \, n\varepsilon \, \pi^2}{16mn\pi^2} = \frac{\delta\varepsilon}{16}.$$

This gives, after multiplying by u_{mn}^2 and summing,

$$(6.18)\quad \frac{1}{16}\sum_{m=M+1}^{\infty}\ \sum_{n=N+1}^{\infty} u_{mn}^2\,\delta\varepsilon \le \sum_{M+1,N+1}^{\infty} u_{mn}^2 \int_0^{\delta}\!\!\int_0^{\varepsilon} \sin^2 mt\ \sin^2 nw\ dt\,dw$$

$$\le K_5\,\delta^{2\alpha+1}\,\varepsilon^{2\beta+1}.$$

The combination of (6.13) and (6.18) yields

$$\sum_{m,n=1}^{M,N} u_{mn}^2 m^2 n^2 \delta^2 \varepsilon^2 + \sum_{M+1,N+1}^{\infty} u_{mn}^2\ \delta\,\varepsilon \le K_7\,\delta^{2\alpha+1}\,\varepsilon^{2\beta+1}$$

whence, by (6.11),

$$M^{-2}N^{-2}\sum_{m,n=1}^{M,N} u_{mn}^2 m^2 n^2 + \sum_{M+1,N+1}^{\infty} u_{mn}^2 \le K_8\, M^{-2\alpha}N^{-2\beta},$$

and the inequality (6.7) is proved.

Now to show that (6.6) implies (6.8) we need only to show that (6.7) implies (6.8) since (6.6) implies (6.7). However, (6.8) follows immediately from (6.7) by using only the first part of the left side of (6.7) and consequently (6.6) implies (6.8). Likewise, we now show that (6.9) is a consequence of (6.7) by using only the second part of (6.7) so that (6.6) also implies (6.9). This may be seen by placing $M=2^{\mu}-1$, $N=2^{v}-1$, and noticing that

$$(2^{\mu}-1)^{-2\alpha} < (\tfrac{1}{2})^{2\alpha}\,2^{-2\mu\alpha},\quad (2^{v}-1)^{-2\beta} < (\tfrac{1}{2})^{2\beta}\,2^{-2v\beta}$$

whence

$$\sum_{M+1,N+1}^{2M-1,\,2N-1} u_{mn}^2 = \sum_{2^{\mu},\,2^{v}}^{2^{+1}-1,\,2^{v+1}-1} u_{mn}^2 \le K_8\,(2^{\mu}-1)^{-2\alpha}\,(2^{v}-1)^{-2\beta}$$

$$\le K_4\, 2^{-2(\alpha\mu+\beta v)}, \qquad (\mu, v=0,1,2,\ldots),$$

which is the desired inequality (6.9).

We now give a second proof that (6.6) implies (6.8). Let M,N be given integers. Then on taking $t=\frac{\pi}{2M}$, $w=\frac{\pi}{2N}$ we have from (6.6) that

$$\sum_{m,n=1}^{M,N} u_{mn}^2 \sin^2\frac{m\pi}{2M}\, \sin^2\frac{n\pi}{2N} \le K_1\left(\frac{\pi}{2M}\right)^{2\alpha}\left(\frac{\pi}{2N}\right)^{2\beta} \le K_2 M^{-2\alpha} N^{-2\beta}$$

and since

$$\sin^2\frac{m\pi}{2M} > \frac{m^2}{M^2}, \qquad \sin^2\frac{n\pi}{2N} > \frac{n^2}{N^2} \qquad \text{for } m<M,\ n<N,$$

it follows that

$$\sum_{m,n=1}^{M,N} u_{mn}^2 \frac{m^2 n^2}{M^2 N^2} \le \sum_{m,n=1}^{M,N} u_{mn}^2 \sin^2\frac{m\pi}{2M}\, \sin^2\frac{n\pi}{2N} \le K_2\, M^{-2\alpha} N^{-2\beta}$$

whence

$$\sum_{m,n=1}^{M,N} u_{mn}^2\, m^2\, n^2 \le K_2\, M^{2(1-\alpha)}\, N^{2(1-\beta)}.$$

Lemma 6.3. If f(x,y) is in the Lipschitz class Lip $S_{11}(\alpha,\beta;2)$, i.e.

$$(6.19)\quad \left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+h,y+k)-f(x-h,y+k)-f(x,y-k)+f(x-h,y-k)|^2 dxdy\right)^{\frac{1}{2}} \le K_0 |h|^{\alpha} |k|^{\beta} \text{ for } 0<\alpha\le 1, 0<\beta\le 1,$$

then

$$(6.20)\qquad \sum_{m,n=1}^{\infty} \rho_{mn}^2 \sin^2 mh\, \sin^2 nk \le K_1 |h|^{2\alpha} |k|^{2\beta},$$

where the K's are constants depending only on the function

$f(x,y)$ and where $\rho^*_{mn} = a^*_{mn}+b^*_{mn}+c^*_{mn}+d^*_{mn}$, with $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ the Fourier coefficients defined in (2.12).

<u>Proof</u>. We use the following notation

$$\Delta'_{11}f(x,y) = f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+f(x-h,y-k)$$

with $f(x,y) \equiv \frac{1}{4}a_{oo} + G(x) + H(y) + F(x,y)$, as in Lemma 6.1. From Property 2, page 23, it follows that

$$\Delta'_{11}G(x) = 0, \qquad \Delta'_{11}H(y) = 0,$$

so that

$$\Delta'_{11}f(x,y) = F(x+h,y+k)-F(x-h,y+k)-F(x+h,y-k)+F(x-h,y-k).$$

From the double Fourier series expansion for $f(x,y)$ wehave

$$\begin{aligned}|\Delta'_{11}f| = \Bigg|\sum_{m,n=1}^{\infty}\Big\{&a_{mn}\left[\cos m(x+h)-\cos m(x-h)\right]\left[\cos n(y+k)-\cos n(y-k)\right]\\ &+b_{mn}\left[\cos m(x+h)-\cos m(x-h)\right]\left[\sin n(y+k)-\sin n(y-k)\right]\\ &+c_{mn}\left[\sin m(x+h)-\sin m(x-h)\right]\left[\cos n(y+k)-\cos n(y-k)\right]\\ &+d_{mn}\left[\sin m(x+h)-\sin m(x-h)\right]\left[\sin n(y+k)-\sin n(y-k)\right]\Big\}\Bigg|\\ =4\Bigg|\sum_{m,n=1}^{\infty}\Big[\Big(&a_{mn}\sin mx \sin ny \sin mh \sin nk\\ &-b_{mn}\sin mx \cos ny \sin mh \sin nk\\ &-c_{mn}\cos mx \sin ny \sin mh \sin nk\\ &+d_{mn}\cos mx \cos ny \sin mh \sin nk\Big)\Big]\Bigg|\end{aligned}$$

$$(6.21) \qquad = 4\left|\sum_{m,n=1}^{\infty}\sin mh \sin nk \, L_{mn}(x,y)\right|,$$

where $L_{mn}(x,y) = a_{mn}\sin mx \sin ny - b_{mn}\sin mx \cos ny$
$- c_{mn}\cos mx \sin ny + d_{mn}\cos mx \cos ny.$

Now let

$$I_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} L^2_{mn}(x,y)\,dx\,dy$$

then

$$\begin{aligned}I_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big[(&a^2_{mn}\sin^2 mx\,\sin^2 ny + b^2_{mn}\sin^2 mx\,\cos^2 ny\\ &+c^2_{mn}\cos^2 mx\,\sin^2 ny + d^2_{mn}\cos^2 mx\,\cos^2 ny\\ &-2a_{mn}b_{mn}\sin^2 mx\,\sin ny\,\cos ny\\ &+2a_{mn}d_{mn}\sin mx\,\cos mx\,\sin ny\,\cos ny\\ &+2b_{mn}c_{mn}\sin mx\,\cos mx\,\sin ny\,\cos ny\\ &-2b_{mn}d_{mn}\sin mx\,\cos mx\,\cos^2 ny\\ &-2\,c_{mn}\,d_{mn}\,\cos^2 mx\,\sin ny)\Big]\,dx\,dy\end{aligned}$$

$$= I_1+I_2+I_3+I_4-I_5-I_6+I_7+I_8-I_9-I_{10}\,,$$

and

$$I_1 = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} a^2_{mn}\sin^2 mx\,\sin^2 ny\,dxdy$$

$$= \frac{a^2_{mn}}{\pi^2}\int_{-\pi}^{\pi}\sin^2 mx\,dx\int_{-\pi}^{\pi}\sin^2 ny\,dy$$

$$= \frac{a^2_{mn}}{16\pi^2}\left(\frac{2mx - \sin 2mx}{m}\right)\Bigg|_{-\pi}^{\pi}\left(\frac{2ny - \sin 2ny}{n}\right)\Bigg|_{-\pi}^{\pi} = a^2_{mn},$$

and similarly since

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 ku\,du = 1, \qquad \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 ku\,du = 1, \quad (k=0,1,2,\ldots),$$

we have $I_2=b^2_{mn}$, $I_3=c^2_{mn}$ and $I_4=d^2_{mn}$. The remaining integrals

I_5, I_6, I_7, I_8, I_9, and I_{10} contain an integral of the form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin ku \cos ku \, du = \frac{1}{2k\pi}\sin^2 ku \Big|_{-\pi}^{\pi} = 0,$$

hence are all equal to zero, so that

$$I_{mn} = a_{mn}^2 + b_{mn}^2 + c_{mn}^2 + d_{mn}^2 = \rho_{mn}^2 \tag{6.22}$$

Squaring (6.21) and integrating over R we have in view of (6.22) that

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\Delta_{11} f(x,y;h,k)|^2 dxdy = \frac{16}{\pi^2}\sum_{m,n=1}^{\infty} \sin^2 mh \sin^2 nk \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} L_{mn}^2 dxdy$$

$$= 16 \sum_{m,n=1}^{\infty} \sin^2 mh \sin^2 nk \cdot I_{mn}$$

$$= 16 \sum_{m,n=1}^{\infty} \rho_{mn}^2 \sin^2 mh \sin^2 nk. \tag{6.23}$$

By the hypothesis (6.19) the left hand side of (6.23) is less than or equal to $K_0^2 |h|^{2\alpha} |k|^{2\beta}$ so that we have

$$\sum_{m,n=1}^{\infty} \rho_{mn}^2 \sin^2 mh \sin^2 nk \leq K_1 |h|^{2\alpha} |k|^{2\beta}$$

and the inequality (6.20) is proved.

<u>Lemma 6.4</u>. Let $f(x,y) \equiv Q(x)W(y)$, where $Q(x) \neq 0, W(y) \neq 0$. If the Fourier series corresponding to $Q(x)$ and $W(y)$ are

$$(6.24)\begin{cases} Q(x) \sim \frac{1}{2}\alpha_o + \sum_{m=1}^{\infty}(\alpha_m \cos mx + \beta_m \sin mx) \\ W(y) \sim \frac{1}{2}\alpha_o' + \sum_{n=1}^{\infty}(\alpha_n' \cos ny + \beta_n' \sin ny), \end{cases}$$

then the double Fourier series of f(x,y) is

$$f(x,y) \sim \sum_{m,n=0}^{\infty} A_{mn}(x,y) \equiv \sum_{m,n=0}^{\infty} \lambda_{mn}(\alpha_m \cos mx + \beta_m \sin mx)(\alpha_n' \cos ny + \beta_n' \sin ny)$$

$$\lambda_{oo} = \frac{1}{4},\ \lambda_{mo} = \lambda_{on} = \frac{1}{2},\ \lambda_{mn} = 1, m > 0, n > 0,$$

$$(6.25)\qquad \equiv \frac{1}{4}\alpha_o \alpha_o' + \frac{\alpha_o'}{2}\sum_{m=1}^{\infty}(\alpha_m \cos mx + \beta_m \sin mx)$$

$$+ \frac{\alpha_o}{2}\sum_{n=1}^{\infty}(\alpha_n' \cos ny + \beta_n' \sin ny)$$

$$+ \sum_{m,n=1}^{\infty}(\alpha_m \cos mx + \beta_m \sin mx)(\alpha_n' \cos ny + \beta_n' \sin ny)$$

and
(6.26) $\rho_{mn}^{*} = \rho_m^{*}\rho_n'^{*}$, where $\rho_{mn}^{*} = a_{mn}^{*} + b_{mn}^{*} + c_{mn}^{*} + d_{mn}^{*}$, $\rho_m^{*} = \alpha_m^{*} + \beta_m^{*}$, $\rho_n'^{*} = \alpha_n'^{*} + \beta_n'^{*}$.

Proof. By the definition of single Fourier series coefficients, we have

$$\alpha_m = \frac{1}{\pi}\int_{-\pi}^{\pi} Q(u)\cos mu\, du, \qquad \beta_m = \frac{1}{\pi}\int_{-\pi}^{\pi} Q(u)\sin mu\, du$$

$$\alpha'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} W(v)\cos nv\, dv, \qquad \beta'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} W(v)\sin nv\, dv,$$

and by the definition (2.12) for the double Fourier coefficients

$$a_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q(u)W(v)\cos mu \cos nv\, du\, dv, \quad m \geq 0, n \geq 0$$

$$= \left[\frac{1}{\pi}\int_{-\pi}^{\pi} Q(u)\cos mu\, du\right]\left[\frac{1}{\pi}\int_{-\pi}^{\pi} W(v)\cos nv\, dv\right] = \alpha_m \alpha'_n.$$

$$b_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q(u)W(v)\ \sin mu \cos nv\, du dv = \beta_m \alpha'_n, \quad m \geq 0, n \geq 0,$$

$$c_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q(u)W(v)\ \cos mu \sin nv\, du dv = \alpha_m \beta'_n, \quad m \geq 0, n \geq 0,$$

$$d_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q(u)W(v)\ \sin mu \sin nv\, du dv = \beta_m \beta'_n, \quad m \geq 0, n \geq 0.$$

From these relations we see that for $m \geq 0$, $n \geq 0$,

$$\frac{1}{\lambda_{mn}} A_{mn}(x,y) = a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny$$

$$= \alpha_m \alpha'_n \cos mx \cos ny + \beta_m \alpha'_n \sin mx \cos ny + \alpha_m \beta'_n \cos mx \sin ny + \beta_m \beta'_n \sin mx \sin ny$$

$$=(\alpha_m\cos mx+\beta_m\sin mx)(\alpha'_n\cos ny+\beta'_n\sin ny),$$

hence

$$\sum_{m,n=0}^{\infty} A_{mn}(x,y)=\sum_{m,n=0}^{\infty}\lambda_{mn}(\alpha_m\cos mx+\beta_m\sin mx)(\alpha'_n\cos ny+\beta'_n\sin ny).$$

Also

$$\rho^*_{mn}=a^*_{mn}+b^*_{mn}+c^*_{mn}+d^*_{mn}=\alpha^*_m\alpha'^*_n+\beta^*_m\alpha'^*_n+\alpha^*_m\beta'^*_n+\beta^*_m\beta'^*_n$$

$$=(\alpha^*_m+\beta^*_m)(\alpha'^*_n+\beta'^*_n)=\rho^*_m\rho'^*_n.$$

<u>Lemma 6.5</u>. If $f(x,y)$ is in the Lipschitz class $\mathrm{Lip}_{10}(\alpha;p)$, i.e.

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|^p dx\right)^{\frac{1}{p}}\leq K|h|^{\alpha}\text{ uniformly in } y,$$

and again letting $f(x,y)=\frac{1}{4}a_{00}+G(x)+H(y)+F(x,y)$ as defined in Lemma 6.1, then $G(x)$ is in the Lipschitz class $\mathrm{Lip}(\alpha;p)$ (cf.Def.12,p.10). Likewise if $f(x,y)$ is in $\mathrm{Lip}_{01}(\alpha;p)$, then $H(y)$ is in $\mathrm{Lip}(\alpha;p)$.

<u>Proof</u>. By the definitions of $G(x)$ and $H(y)$ given in Lemma 6.1, we have

$$G(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x,y)dy-\frac{1}{4}a_{00},\quad H(y)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x,y)dx-\frac{1}{4}a_{00}$$

and thus

$$G(x+h)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+h,y)dy-\frac{1}{4}a_{00},\quad H(y+k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x,y+k)dx-\frac{1}{4}a_{00},$$

so that

$$|G(x+h)-G(x)|^p = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x+h,y)-f(x,y)\right]dy\right|^p$$

$$\leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|dy\right)^p ,$$

$$|H(y+k)-H(y)|^p = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x,y+k)-f(x,y)\right]dx\right|^p$$

$$\leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x,y+k)-f(x,y)|dx\right)^p ,$$

and

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(x+h)-G(x)|^p dx\right.^{\frac{1}{p}} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|dy\right]^p\right)^{\frac{1}{p}}$$

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(y+k)-H(y)|^p dy\right)^{\frac{1}{p}} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}dy\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x,y+k)-f(x,y)|dx\right]^p\right)^{\frac{1}{p}} .$$

By the Minkowski inequality (2.24), we have

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(x+h)-G(x)|^p dx\right)^{\frac{1}{p}} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}dy\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|^p dx\right]^{\frac{1}{p}}$$

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(y+k)-H(y)|^p dy\right)^{\frac{1}{p}} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}dx\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x,y+k)-f(x,y)|^p dy\right]^{\frac{1}{p}} .$$

Now since $f(x,y)$ is in $\mathrm{Lip}_{10}(\alpha;p)$ the first inequality becomes

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(x+h)-B(x)|^p dx\right)^{\frac{1}{p}} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(K|h|^{\alpha})dy = K|h|^{\alpha},$$

which by Def.12, page 11, means $G(x)$ is in $\mathrm{Lip}(\alpha;p)$. Likewise $f(x,y)$ in $\mathrm{Lip}_{01}(\alpha;p)$ gives

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(y+k)-H(y)|^p dy\right)^{\frac{1}{p}} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(K|k|^{\alpha})dx = K|k|^{\alpha}$$

and $H(y)$ is also in $\mathrm{Lip}(\alpha;p)$.

<u>Lemma 6.6</u>. Let $F(t,w)$ be a periodic function of period 2π in each variable and integrable in the sense of Lebesgue, then for finite constants a,b we have

$$I = \int_{-\pi-a}^{\pi-a}\int_{-\pi-b}^{\pi-b} F(t,w)\ dtdw = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(t,w)\ dtdw.$$

<u>Proof</u>. We have from the periodicity that $F(t,w)=F(t+2\pi,w)=F(t,w+2\pi)=F(t+2\pi,w+2\pi)$. We write

$$I = \int_{-\pi}^{\pi-a}\int_{-\pi}^{\pi-b} F(t,w)dtdw + \int_{-\pi}^{\pi-a}\int_{-\pi-b}^{-\pi} F(t,w)dtdw$$

$$+ \int_{-\pi-a}^{-\pi}\int_{-\pi}^{\pi-b} F(t,w)dtdw + \int_{-\pi-a}^{-\pi}\int_{-\pi-b}^{-\pi} F(t,w)dtdw = I_1+I_2+I_3+I_4.$$

Now

$$I_2 = \int_{-\pi}^{\pi-a}\int_{-\pi-b}^{-\pi} F(t,w)\,dtdw = \int_{-\pi}^{\pi-a}\int_{-\pi-b}^{-\pi} F(t,w+2\pi)\,dtdw$$

and on letting $w+2\pi=y$, $t=x$, $dw=dy$, $dt=dx$, then

$$I_2 = \int_{-\pi}^{\pi-a}\int_{\pi-b}^{\pi} F(x,y)\ dxdy.$$

Likewise

$$I_3 = \int_{-\pi-a}^{-\pi}\int_{-\pi}^{\pi-b} F(t,w)\,dtdw = \int_{-\pi-a}^{-\pi}\int_{-\pi}^{\pi-b} F(t+2\pi,w)\,dtdw$$

and letting $t+2\pi=x, w=y$, $dt=dx$, $dw=dy$, then

$$I_3 = \int_{\pi-a}^{\pi}\int_{-\pi}^{\pi-b} F(x,y)\ dxdy.$$

Also

$$I_4 = \int_{-\pi-a}^{-\pi}\int_{-\pi-b}^{-\pi} F(t,w)\,dtdw = \int_{-\pi-a}^{-\pi}\int_{-\pi-b}^{-\pi} F(t+2\pi,w+2\pi)\,dtdw$$

and letting $t+2\pi=x$, $w+2\pi=y$, $dt=dx$, $dw=dy$ we obtain

$$I_4 = \int_{\pi-a}^{\pi}\int_{\pi-b}^{\pi} F(x,y)\ dxdy.$$

Collecting we have

$$I = \int_{-\pi}^{\pi-a}\int_{-\pi}^{\pi-b} F(x,y)\,dxdy + \int_{-\pi}^{\pi-a}\int_{\pi-b}^{\pi} F(x,y)\,dxdy$$

$$+ \int_{\pi-a}^{\pi}\int_{-\pi}^{\pi-b} F(x,y)\,dxdy + \int_{\pi-a}^{\pi}\int_{\pi-b}^{\pi} F(x,y)\,dxdy$$

so that

$$I = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y)\,dxdy.$$

This completes the proof.

Lemma 6.7. If $f(x,y)$ is in the Lipschitz classes

(6.27) Lipα in x: $|f(x+h,y)-f(x,y)| \leq K|h|^{\alpha}$ uniformly in y,

(6.28) Lipβ in y: $|f(x,y+k)-f(x,y)| \leq K|k|^{\beta}$ uniformly in x,

and if $G(x)$ and $H(y)$ are as defined in (6.1), the $G(x)$ is in Lip α and $H(y)$ is in Lipβ.

Proof. We shall first show that $G(x)$ is in Lipα, i.e.

$$(6.29) \qquad |G(x+h)-G(x)| \leq K|h|^{\alpha} \text{ uniformly in } (-\pi,\pi).$$

On integrating the inequality in (6.27) with respect to y over the interval $(-\pi,\pi)$, we have

$$(6.30) \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h,y)-f(x,y)|dy \leq \frac{K|h|^{\alpha}}{2\pi}\int_{-\pi}^{\pi} dy = K|h|^{\alpha}.$$

By the definition of $G(x)$ in (6.1) we see that

$$G(x+h)-G(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+h,y)dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x+h,y)-f(x,y)\right]dy,$$

and

$$|G(x+h)-G(x)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\left[f(x+h,y)-f(x,y)\right]dy\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|dy$$

which by (6.30) gives the desired result (6.29). A similar argument shows that (6.28) implies H(y) is in Lipβ.

Theorem 6.1. *If* f(x,y) *is in each of the three Lipschitz classes*

Lip $S_{11}(\alpha,\alpha;2)$:

$$\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+f(x-h,y-k)|^2dxdy\right)^{\frac{1}{2}}$$

(6.31) $$\leq K|h|^{\alpha}|k|^{\alpha},$$

Lip$_{10}(\alpha;2)$:

(6.32) $$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|^2dx\right)^{\frac{1}{2}}\leq K|h|^{\alpha}$$ *uniformly in* y,

Lip$_{01}(\alpha;2)$:

(6.33) $$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x,y+k)-f(x,y)|^2dy\right)^{\frac{1}{2}}\leq K|k|^{\alpha}$$ *uniformly in* x,

with $\frac{1}{2}<\alpha\leq 1$, *then the double Fourier series* (2.11)' *of* f(x,y) *with the coefficients defined in* (2.12) *converges absolutely in* R. *For* $\alpha=\frac{1}{2}$ *the double series may not converge absolutely*.

Proof. In view of the hypothesis (6.31) we may have by (6.20) of Lemma 6.3 and (6.8) of Lemma 6.2,

$$\sum_{m=2^{\mu}}^{2^{+1}-1}\sum_{n=2^{v}}^{2^{v+1}-1}\rho_{mn}^{2}\leq K_2\,2^{-2\alpha(\mu+v)},\quad(\mu,v=0,1,2,..).$$

Then by Schwarz's inequality

$$\sum_{m=2^{\mu}}^{2^{\mu+1}-1}\ \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn} \le \left[\sum_{m=2^{\mu}}^{2^{\mu+1}-1}\ \sum_{n=2^{v}}^{2^{v+1}-1} 1^2\right]^{\frac{1}{2}} \left[\sum_{m=2^{\mu}}^{2^{\mu+1}-1}\ \sum_{n=2^{v}}^{2^{v+1}-1} \rho^2_{mn}\right]^{\frac{1}{2}}$$

$$\le \left[2^{\frac{1}{2}(\mu+v)}\right]\left[K_2 2^{-\alpha(\mu+v)}\right]$$

$$\le K_2\ 2^{\mu(\frac{1}{2}-\alpha)+v(\frac{1}{2}-\alpha)}.$$

Therefore,

$$\sum_{m,n=1}^{\infty} \rho_{mn} = \sum_{\mu=0}^{\infty}\ \sum_{v=0}^{\infty}\ \sum_{m=2^{\mu}}^{2^{\mu+1}-1}\ \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn} \le K_2 \sum_{\mu=0}^{\infty}\ \sum_{v=0}^{\infty} 2^{\mu(\frac{1}{2}-\alpha)}\, 2^{v(\frac{1}{2}-\alpha)}$$

and the double series converges since $\alpha > \frac{1}{2}$. To show that the complete double series of coefficients is absolutely convergent, it must now be shown that

$$\sum_{m=1}^{\infty} \rho_{mo} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \rho_{on}$$

are convergent. Applying the hypothesis (6.32) we have in view of Lemma 6.1 and Lemma 6.5 that $G(x)$ is in $\mathrm{Lip}(\alpha;2)$. Also by (6.33), Lemma 6.1 and Lemma 6.5 $H(y)$ is in $\mathrm{Lip}(\alpha;2)$. It is also seen by Lemma 6.1 that the single Fourier series corresponding to $G(x)$ and $H(y)$ are

$$G(x) \sim \sum_{m=1}^{\infty} \tfrac{1}{2}(a_{mo}\cos mx + b_{mo}\sin mx), \quad \rho^2_{mo} = a^2_{mo} + b^2_{mo},$$

$$H(y) \sim \sum_{n=1}^{\infty} \tfrac{1}{2}(a_{on}\cos ny + c_{on}\sin ny), \quad \rho^2_{on} = a^2_{on} + c^2_{on}.$$

Consequently, since G(x) and H(y) are in Lip(α;2) which class is included in the class Lip$_*$(α;2), (cf.Def.14,p.11), we may apply Theorem A.3(cf. page 13) for $\alpha > \frac{1}{2}$. This gives

$$\sum_{m=1}^{\infty} \rho_{mo} < \infty, \qquad \sum_{n=1}^{\infty} \rho_{on} < \infty.$$

Collecting results we have

$$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty,$$

which completes the proof of the theorem except for a ''Gegenbeispiel'' for the case $\alpha = \frac{1}{2}$.

Gegenbeispiel to Theorem 6.1. Let f(x,y)=Q(x)W(y), where

$$(6.34) \qquad Q(x) \sim \sum_{m=2}^{\infty} m^{-(\frac{1}{2}+\alpha)} \cos(m \log m + mx)$$

$$= \sum_{m=2}^{\infty} \left[\frac{\cos(m \log m)}{m^{\frac{1}{2}+\alpha}} \cos mx - \frac{\sin(m \log m)}{m^{\frac{1}{2}+\alpha}} \sin mx \right]$$

and where similarly

$$(6.35) \qquad W(y) \sim \sum_{n=2}^{\infty} n^{-(\frac{1}{2}+\alpha)} \cos(n \log n + ny).$$

These series (6.34) and (6.35) are Fourier series in the sense that the Fourier coefficients are as follows

$$a_0 = a_1 = b_0 = b_1 = 0,$$

$$a_k = \frac{\cos(k \log k)}{k^{\frac{1}{2}+\alpha}}, \qquad b_k = \frac{\sin(k \log k)}{k^{\frac{1}{2}+\alpha}}, \quad k \geq 2.$$

By Lemma 6.4 we have the double Fourier series of $f(x,y)$ is

$$f(x,y) \sim \sum_{m,n=2}^{\infty} (mn)^{-(\frac{1}{2}+\alpha)} \cos(m\log m + mx)\cos(n\log n + ny)$$

and
$$\rho^*_{mn} = a^*_{mn}+b^*_{mn}+c^*_{mn}+d^*_{mn} = (a^*_m+b^*_m)(a^*_n+b^*_n) = \frac{1}{(mn)^{1+2\alpha}}$$

so that
$$\rho_{mn} = (mn)^{-(\frac{1}{2}+\alpha)}.$$

It has been shown by Hardy and Littlewood [35] that the functionss$Q(x)$ and $W(y)$ are in Lipα, ($\alpha > 0$), and consequently are in the more general class Lip$(\alpha;2)$. From the Property 5, page 24, it follows that $f(x,y)$ is in the class Lip $S_{11}(\alpha,\alpha;2)$ and thus $f(x,y)$ satisfies the hypothesis (6.31) of the theorem. From the above definition of $f(x,y)$ we also have

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h,y)-f(x,y)|^2 dx\right)^{\frac{1}{2}} = W(y)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |Q(x+h)-Q(x)|^2 dx\right)^{\frac{1}{2}}$$

and since
$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |Q(x+h)-Q(x)|^2 dx\right)^{\frac{1}{2}} \leq K|h|^{\alpha},$$

it follows that

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h,y)-f(x,y)|^2 dx\right)^{\frac{1}{2}} \leq W(y)\,K\,|h|^{\alpha} \leq K_1|h|^{\alpha},$$

so that $f(x,y)$ satisfies the hypothesis (6.32). Likewise $f(x,y)$ satisfies the hypothesis (6.33). However, for $\alpha = \frac{1}{2}$ it is immediately obvious that

$$\sum_{m,n=2}^{\infty} \rho_{mn} = \sum_{m,n=2}^{\infty} \frac{1}{mn} = \infty .$$

We, therefore, have a function $f(x,y)$ satisfying the hypothesis of the Theorem 6.1 with the series of Fourier coefficients of $f(x,y)$ not absolutely convergent when $\alpha = \frac{1}{2}$.

Remark. It will be seen later that this ''gegenbeispiel'' enables us to give a stronger result than is obtained here since the function $f(x,y)$ as defined here actually belongs to a narrower class of functions than is considered in this Theorem 6.1.

Theorem 6.2. *If* $f(x,y)$ *is in each of the three Lipschitz classes*

(6.36) Lip $S_{11}(\alpha,\alpha)$:

$|f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+f(x-h,y-k)| \leq K|h|^{\alpha}|k|^{\alpha}$,

(6.37) Lip α in x: $|f(x+h,y)-f(x,y)| \leq K|h|^{\alpha}$ *uniformly in* y,

(6.38) Lip α in y: $|f(x,y+k)-f(x,y)| \leq K|k|^{\alpha}$ *uniformly in* x,

with K *a constant depending only on* $f(x,y)$ *and with* $\frac{1}{2} < \alpha \leq 1$, *then*

$$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty , \quad \overset{*}{\rho}_{mn} = a^*_{mn}+b^*_{mn}+c^*_{mn}+d^*_{mn} .$$

For $\alpha = \frac{1}{2}$, *the series* $\sum_{m,n=0}^{\infty} \rho_{mn}$ *may not converge*.

Remark. From (i) of R2, page 26, and from the assumptions (6.37) and (6.38) it follows that $f(x,y)$ is in Lip(α,α), and by Property 1, page 23, $f(x,y)$ is a continuous function in the two variables in R.

Proof of Theorem 6.2. The classes (6.36), (6.37) and (6.38) are included in the more general classes of Theorem 6.1 and

$$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty$$

follows as a special case of the Theorem 6.1.

Gegenbeispiel for Theorem 6.2. The function used as a gegenbeispiel for Theorem 6.1 may also be used here. Since $Q(x)$ and $W(y)$ are in Lip α we have by Property 5, page 24, that $f(x,y)$ is in class Lip $S_{11}(\alpha,\alpha)$ and thus (6.36) holds. It may also be shown by an argument similar to that employed for Theorem 6.1 that $f(x,y)$ satisfies (6.37) and (6.38) and thus is a function satisfying the hypotheses of Theorem 6.2 whose series of Fourier coefficients does not converge agsolutely if $\alpha = \frac{1}{2}$.

Remark. Theorem 6.2 reduces to the Bernstein theorem (cf. Theorem A.1, p.12) if $f(x,y)$ is simply a function of one variable only. To see this let

$$f(x,y) \equiv g(x) \sim \tfrac{1}{4}a_{00} + \tfrac{1}{2}\sum_{m=1}^{\infty}(a_{mo}\cos mx + b_{mo}\sin mx),$$

which is what the double Fourier series (2.11)' reduces to when we place $n=0$. Now from (2.12) and the definition of the Fourier coefficients of a function of one variable we have

$$a_{mo} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(u)\cos mu\, dudv = \frac{1}{\pi^2}\int_{-\pi}^{\pi} g(u)\cos mu\,(v)\Big|_{-\pi}^{\pi} du$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} g(u)\cos mu\, du = 2a_m, \quad \text{where } a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos mu\, du$$

and similarly

$$b_{mo} = 2b_m, \quad \text{where } b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin mu\, du, \quad m=0,1,2,\ldots$$

Thus the double Fourier series is seen to reduce to the single Fourier series of $g(x)$. Now for $f(x,y)=g(x)$, the hypothesis (6.37) in Theorem 6.2 becomes simply

$$|g(x+h)-g(x)| \leq K|h|^{\alpha}, \qquad (\tfrac{1}{2} < \alpha \leq 1)$$

which is the definition of $g(x)$ in the class Lip α and gives the hypothesis of the Bernstein theorem with $\alpha > \frac{1}{2}$.

Theorem 6.3. *If* $f(x,y)$ *is in each of the three Lipschitz classes*

(6.39) Lip $S_{11}(\alpha,\beta;2)$:

$$\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+f(x-h,y-k)|^2 dxdy\right)^{\frac{1}{2}} \leq K|h|^{\alpha}|k|^{\beta},$$

(6.40) $\mathrm{Lip}_{10}(\alpha;2)$ in x:

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|^2 dx\right)^{\frac{1}{2}} \leq K|h|^{\alpha} \text{ uniformly in } y,$$

(6.41) $\mathrm{Lip}_{01}(\beta;2)$ in y:

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x,y+k)-f(x,y)|^2 dy\right)^{\frac{1}{2}} \leq K|k|^{\beta} \text{ uniformly in } x,$$

with $\frac{1}{2}<\alpha\leq 1$, $\frac{1}{2}<\beta\leq 1$, then the double Fourier series (2.11)' of f(x,y), with coefficients given in (2.12), converges absolutely in R. If $\alpha=\frac{1}{2}$ and $\beta>\frac{1}{2}$, the double series of Fourier coefficients may not converge absolutely.

<u>Proof</u>. Let

$$\min(\alpha,\beta)=\gamma>\tfrac{1}{2},$$

then (6.39) implies Lip $S_{11}(\gamma,\gamma;2)$ in R, i.e. (6.31); similarly (6.40) implies $\mathrm{Lip}_{10}(\gamma;2)$ in x in R, i.e. (6.32); and finally (6.41) implies $\mathrm{Lip}_{01}(\gamma;2)$ in y in R, i.e.(6.33). We thus have the hypotheses of Theorem 6.1, whence the double Fourier series of f(x,y) converges absolutely in R.

<u>Gegenbeispiel for Theorem 6.3</u>. Let $f(x,y)\equiv Q(x)W(y)$, where Q(x) is as defined in the ''gegenbeispiel'' for The Theorem 6.1, and where

$$W(y)\sim\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos ny, \qquad W(y)=y^2 \text{ for } -\pi\leq y\leq\pi.$$

It may be shown as in Theorem 6.1 that this function f(x,y) satisfies (6.39), (6.40) and (6.41), the hypotheses of Theorem 6.3. By Lemma 6.4, f(x,y) has the double Fourier series

$$f(x,y) \sim \frac{\pi^2}{3}Q(x) + \sum_{m=2}^{\infty}\sum_{n=1}^{\infty}(-1)^n \frac{4}{m^{\frac{1}{2}+\alpha}n^2}\cos(m\log m+mx)\cos ny,$$

with

$$\rho^*_{mn} = \left(\frac{1}{m^{1+2\alpha}}\right)\left(\frac{16}{n^4}\right), \quad \rho_{mn} = \frac{4}{m^{\frac{1}{2}+\alpha}n^2} \quad (m,n=2,3,\ldots).$$

Hence

$$\sum_{m,n=2}^{\infty}\rho_{mn} = \sum_{m,n=2}^{\infty}\frac{4}{m^{\frac{1}{2}+\alpha}n^2}$$

and for $\alpha = \frac{1}{2}$, this becomes

$$\sum_{m,n=2}^{\infty}\rho_{mn} = \sum_{m=2}^{\infty}\frac{1}{m}\sum_{n=2}^{\infty}\frac{4}{n^2} = \infty .$$

Theorem 6.4. If $f(x,y)$ *is in each of the three Lipschitz classes*

(6.42) Lip $S_{11}(\alpha,\beta)$:

$|f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+f(x-h,y-k)| \leq K|h|^{\alpha}|k|^{\beta}$,

(6.43) Lip α in x: $|f(x+h,y)-f(x,y)| \leq K|h|^{\alpha}$ *uniformly in* y,

(6.44) Lip β in y: $|f(x,y+k)-f(x,y)| \leq K|k|^{\beta}$ *uniformly in* x,

with $\frac{1}{2}<\alpha\leq 1$, $\frac{1}{2}<\beta\leq 1$, *then the double Fourier series* (2.11)' *of* $f(x,y)$, *with coefficients in* (2.12), *converges absolutely in* R. *If* $\alpha = \frac{1}{2}$ *and* $\beta > \frac{1}{2}$, *the double series of Fourier coefficients may not converge absolutely.* (As in Theorem 6.2, the $f(x,y)$ here must be continuous in (x,y).)

Proof. Since the Lipschitz classes in (6.42), (6.43) and (6.44) are included in the more general classes of

Theorem 6.3 the result of this theorem follows as a special case of Theorem 6.3. The gegenbeispiel used for Theorem 6.3 may be employed here to obtain a stronger result since the $f(x,y)$ defined there belongs to the narrower class of functions in Theorem 6.4. The value of β is $\beta = 1$ and as has been seen the series of Fourier coefficients does not converge absolutely if $\alpha = \frac{1}{2}$.

Remarks. From R1 and R2, page 26, we see that the conditions (6.43) and (6.44) in Theorem 6.4 when taken together are equivalent to the single class of functions $\mathrm{Lip}(\alpha,\beta)$, i.e.

$$|f(x+h,y+k)-f(x,y)| \leq K(|h|^{\alpha} + |k|^{\beta}) \text{ for all points of } R.$$

An equivalent theorem could, therefore, be stated with $f(x,y)$ in the two Lipschitz classes $\mathrm{Lip}\ S_{11}(\alpha,\beta)$ and $\mathrm{Lip}(\alpha,\beta)$. It is of interest here to point out the known relations between these two classes. Property 4, page 24, shows that the class $\mathrm{Lip}\ S_{11}(\alpha,\beta)$ contains functions not in the class $\mathrm{Lip}(\alpha,\beta)$ and Lemma 3.3 shows that a function in $\mathrm{Lip}(\alpha,\alpha)$ for $0<\alpha\leq 1$ is in $\mathrm{Lip}\ S_{11}(\gamma,\delta)$ for $\alpha = \gamma+\delta$. If $\gamma = \delta$, $\alpha=2\gamma$, $0<\alpha\leq 1$, then a function in $\mathrm{Lip}(\alpha,\alpha)$ is in $\mathrm{Lip}\ S_{11}(\frac{\alpha}{2},\frac{\alpha}{2})$.

Theorem 6.5. *If* $f(x,y)$ *is in each of the three Lipschitz classes*

(6.45) $\mathrm{Lip}\ S_{11}(\alpha,\alpha;2)$:

$$\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+f(x-h,y-k)|^2dxdy\right)^{\frac{1}{2}}$$
$$\leq K|h|^{\alpha}|k|^{\alpha},$$

(6.46) $\mathrm{Lip}_{10}(\alpha;2)$:

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h,y)-f(x,y)|^2dx\right)^{\frac{1}{2}} \leq K|h|^{\alpha} \text{ \underline{uniformly in} } y,$$

(6.47) $\mathrm{Lip}_{01}(\alpha;2)$:

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x,y+k)-f(x,y)|^2dy\right)^{\frac{1}{2}} \leq K|k|^{\alpha} \text{ \underline{uniformly in} } x,$$

with $0<\alpha\leq 1$, _and if_ $\lambda > \frac{2}{2\alpha+1}$, _then_ $\sum_{m,n=0}^{\infty}\rho_{mn}^{\lambda}<\infty$,

where $\rho_{mn}^{*}=a_{mn}^{*}+b_{mn}^{*}+c_{mn}^{*}+d_{mn}^{*}$. _For_ $\lambda=\frac{2}{2\alpha+1}$, _the series_

$$\sum_{m,n=0}^{\infty}\rho_{mn}^{\lambda} \text{ \underline{may diverge}.}$$

(Note: $\lambda > \frac{2}{2\ \ +1}$ and $\lambda=1$ implies $\alpha>\frac{1}{2}$, so that Theorem 6.1 is included in Theorem 6.5)

Proof of Theorem 6.5. From the hypothesis (6.45) we are able to apply Lemma 6.2 and Lemma 6.3, which yields

$$\sum_{m=2^{\mu}}^{2^{\mu+1}-1}\ \sum_{n=2^{v}}^{2^{v+1}-1}\rho_{mn}^{*} \leq K_3\ 2^{-2\alpha(\mu+v)}, \quad (\mu, v=0,1,2,\ldots).$$

Then writing
$$p = \frac{2}{\lambda} \geq 1, \quad Q_{mn} = \rho_{mn}^{\lambda},$$

$$m_0 = 2^{\mu+1}-1-2^{\mu}+1 = 2^{\mu}, \quad n_0 = 2^{v+1}-1-2^{v}+1 = 2^{v}$$

in the inequality (2.23), page 21, we obtain

$$\sum_{m=2^{\mu}}^{2^{\mu+1}-1} \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn}^{\lambda} \leq 2^{\mu(1-\frac{\lambda}{2})} 2^{v(1-\frac{\lambda}{2})} \left[\sum_{m=2^{\mu}}^{2^{\mu+1}-1} \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn}^{2}\right]^{\frac{\lambda}{2}}$$

$$\leq \left[2^{(\mu+v)(1-\frac{\lambda}{2})}\right]\left[K_2^{\frac{\lambda}{2}} 2^{-\alpha\lambda(\mu+v)}\right], \quad (\mu, v=0,1,..)$$

$$\leq K' \, 2^{(\mu+v)[1-(\alpha+\frac{1}{2})]}.$$

Hence
$$\sum_{m,n=1}^{\infty} \rho_{mn}^{\lambda} = \sum_{\mu=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=2^{\mu}}^{2^{\mu+1}-1} \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn}^{\lambda}$$

$$\leq K' \sum_{m=2^{\mu}}^{2^{\mu+1}-1} \sum_{n=2^{v}}^{2^{v+1}-1} 2^{(\mu+v)[1-(\alpha+\frac{1}{2})\lambda]}$$

and the series on the right converges since $\lambda > \frac{2}{2\alpha+1}$, i.e. $1-(\alpha+\frac{1}{2})\lambda < 0$, so that

$$\sum_{m,n=1}^{\infty} \rho_{mn}^{\lambda} < \infty .$$

Now from the hypotheses (6.46) and (6.47) together with Lemma 6.1 and Lemma 6.5 it follows that G(x) and H(y) as defined in Lemma 6.1 are in Lip(α;2) and consequently in the more general class $\mathrm{Lip}_2(\alpha;2)$. Thus an application of Theorem A.3, page 13, yields

$$\sum_{m=1}^{\infty} \rho_{mo}^{\lambda} < \infty, \qquad \sum_{n=1}^{\infty} \rho_{on}^{\lambda} < \infty \quad \text{for } \lambda > \frac{2}{2\alpha+1}.$$

Collecting results we have

$$\sum_{m,n=0}^{\infty} \rho_{mn}^{\lambda} < \infty .$$

The ''gegenbeispiel'' for Theorem 6.1 with $\alpha = \frac{1}{2}$ shows that

$$\sum_{m,n=0}^{\infty} \rho_{mn}^{\lambda} \quad \text{may diverge if } \lambda = \frac{2}{2\alpha+1}, \text{ for in}$$

that case $\lambda = 1$, which is the condition treated there. This completes the proof of the theorem.

<u>Theorem 6.6</u>. <u>If</u> $f(x,y)$ <u>is in each of the three Lipschitz classes</u>

(6.48) Lip $S_{11}(\alpha,\alpha)$:

$$|f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+f(x-h,y-k)| \leq K|h|^{\alpha} \, |k|^{\alpha},$$

(6.49) Lip α in x: $|f(x+h,y)-f(x,y)| \leq K|h|^{\alpha}$ <u>uniformly in</u> y,

(6.50) Lip α in y: $|f(x,y+k)-f(x,y)| \leq K|k|^{\alpha}$ <u>uniformly in</u> x,

<u>with</u> $0 < \alpha \leq 1$, <u>and if</u> $\lambda > \frac{2}{2\alpha+1}$, <u>then</u>

$$\sum_{m,n=0}^{\infty} \rho_{mn}^{\lambda} < \infty .$$

<u>For</u> $\lambda = \frac{2}{2\alpha+1}$, <u>the series</u> $\sum_{m,n=0}^{\infty} \rho_{mn}^{\lambda}$ <u>may diverge</u>.

Proof. This theorem is a special case of Theorem 6.5, since the result was obtained there for a more general calss of functions. The ''gegenbeispiel'' for Theorem 6.2 may be used here to show that for $\lambda = \frac{2}{2\alpha+1}$ the series

$$\sum_{m,n=0}^{\infty} \rho_{mn}^{\lambda}$$

may diverge.

We now introduce the definition of ''modulus of continuity of f(x,y)'' which is to be used in the sequel.

Definition. The modulus of continuity $\omega_{11}(h,k)$ of f(x,y) in R is defined as

$$\omega_{11}(h,k) = \mathrm{Max}|\Delta_{11}f(x_i,y_j)| \text{ for all } (x_i,y_j) \text{ in } R$$

with $|x_{i+1}-x_i| \leq h$, $|y_{j+1}-y_j| \leq k$, where

$$|\Delta_{11}f(x_i,y_j)| = |f(x_{i+1},y_{j+1})-f(x_{i+1},y_j)-f(x_i,y_{j+1})+f(x_i,y_j)|.$$

Theorem 6.7. *If* f(x,y) *is in each of the three Lipschitz classes*

(6.51) Lip $S_{11}(\alpha,\beta)$:

$$|\Delta_{11}f(x,y;h,k)| = |f(x+h,y+k)-f(x,y+k)-f(x+h,y)+f(x,y)| \leq K|h|^{\alpha}|k|^{\beta}$$

(6.52) Lip α in x: $|f(x+h,y)-f(x,y)| \leq K|h|^{\alpha}$ *uniformly in* y,

(6.53) Lip β in y: $|f(x,y+k)+f(x,y)| \leq K|k|^{\beta}$ *uniformly in* x,

with $\alpha > 0$, $\beta > 0$, *and if* f(x,y) *is of bounded variation* H (cf. definition page 42), *then the double Fourier series* (2.11)' *of* f(x,y) *with coefficients in* (2.12), *converges absolutely in* R.

Proof. We note first that, as in Theorem 6.4, the conditions (6.52) and (6.53) together with R2 (page 26) and Property 1 (page 23) imply that f(x,y) is continuous in (x,y) in R. The proof of the Theorem 6.7 will be carried out by applying the method of quadratic variation of N.Wiener [68] extended to the two variable case.

For every pair of integers M,N form the net (x_m, y_n) defined by

$$x_m = x + \frac{m\pi}{M}, \quad x_{m-1} = x+(m-1)\frac{\pi}{M}, \quad h = \frac{\pi}{M}, \quad (m=1,2,\dots,2M)$$

$$y_n = y + \frac{n\pi}{N}, \quad y_{n-1} = y+(n-1)\frac{\pi}{N}, \quad k = \frac{\pi}{N}, \quad (n=1,2,\dots,2N).$$

Let

$$|\Delta_{11} f_{MN}(x_m, y_n)| = |f\left[x+\frac{m\pi}{M}, y+\frac{n\pi}{N}\right] - f\left[x+(m-1)\frac{\pi}{M}, y+\frac{n\pi}{N}\right]$$
$$-f\left[x+\frac{m\pi}{M}, y+(n-1)\frac{\pi}{N}\right] + f\left[x+(m-1)\frac{\pi}{M}, y+(n-1)\frac{\pi}{N}\right]$$
$$m=1,2,\dots,2M; \; n=1,2,\dots,2N,$$

and
$$\omega_{11}(h,k) = \omega_{11}\left(\frac{\pi}{M},\frac{\pi}{N}\right) = \operatorname{Max}|\Delta_{11} f_{MN}(x_m, y_n)|$$

For fixed M,N we have

$$|\Delta_{11} f_{MN}(x_m, y_n)|^2 \leq \omega_{11}\left(\frac{\pi}{M},\frac{\pi}{N}\right) |\Delta_{11} f_{MN}(x_m, y_n)|, \quad \begin{cases} m=1,2,\dots,2M \\ n=1,2,\dots,2N, \end{cases}$$

whence

$$\sum_{m,n=1}^{2M,2N} |\Delta_{11} f_{MN}(x_m, y_n)|^2 \leq \omega_{11}\left(\frac{\pi}{M},\frac{\pi}{N}\right) \sum_{m,n=1}^{2M,2N} |\Delta_{11} f_{MN}(x_m, y_n)|.$$

Since f(x,y) is of bounded variation H, the sum on the right hand side is not greater than a finite constant L for all possible nets over R. Thus,

$$(6.54)\qquad \sum_{m,n=1}^{2M,2N} |\Delta_{11} f_{MN}(x_m,y_n)|^2 \leq L\,\omega_{11}\left(\tfrac{\pi}{M},\tfrac{\pi}{N}\right).$$

From (6.51) we also have

$$\omega_{11}\left(\tfrac{\pi}{M},\tfrac{\pi}{N}\right) \leq K\left(\tfrac{\pi}{M}\right)^{\alpha}\left(\tfrac{\pi}{N}\right)^{\beta} \leq K_1\, M^{-\alpha} N^{-\beta}, \quad \text{where } K_1 = K(\pi)^{\alpha+\beta}.$$

The inequality (6.54) now yields

$$\sum_{m,n=1}^{2M,2N} |\Delta_{11} f_{MN}(x_m,y_n)|^2 \leq K_1\, L\, M^{-\alpha} N^{-\beta} \leq K_2\, M^{-\alpha} N^{-\beta}.$$

Integrating this inequality with respect to x,y over $(0,0;\frac{\pi}{M},\frac{\pi}{N})$ gives

$$(6.55)\qquad \sum_{m,n=1}^{2M,2N} \int_0^{\frac{\pi}{M}}\int_0^{\frac{\pi}{N}} |\Delta_{11} f_{MN}(x_m,y_n)|^2 dxdy \leq \pi^2 K_2 M^{-(\alpha+1)} N^{-(\beta+1)},$$

or

$$(6.56)\qquad \sum_{m,n=1}^{2M,2N} I_{mn} \leq K_3\, M^{-(\alpha+1)}\, N^{-(\beta+1)}.$$

Now consider the integrals

$$I_{mn} = \int_0^{\frac{\pi}{M}}\int_0^{\frac{\pi}{N}} |\Delta_{11} f_{MN}(x_m,y_n)|^2 dxdy \qquad \begin{cases} m=1,2,\ldots,2M \\ n=1,2,\ldots,2N, \end{cases}$$

and make the change of variables

$$x = u-(m-1)\tfrac{\pi}{M}, \quad y = v-(n-1)\tfrac{\pi}{N}, \quad dx=du, \quad dy=dv,$$

then

$$I_{mn} = \int_{(m-1)\frac{\pi}{M}}^{\frac{m\pi}{M}} \int_{(n-1)\frac{\pi}{N}}^{\frac{n\pi}{N}} \left[f(u+\tfrac{\pi}{M}, v+\tfrac{\pi}{N}) - f(u, v+\tfrac{\pi}{N}) - f(u+\tfrac{\pi}{M}, v) + f(u,v)\right]^2 dudv \qquad \begin{cases} m=1,2,\ldots,2M \\ n=1,2,\ldots,2N \end{cases}$$

$$= \int_{(m-1)\frac{\pi}{M}}^{\frac{m\pi}{M}} \int_{(n-1)\frac{\pi}{N}}^{\frac{n\pi}{N}} |\Delta_{11} f(u,v;\tfrac{\pi}{M},\tfrac{\pi}{N})|^2 \, du \, dv.$$

Summing these integrals we get

$$\sum_{m,n=1}^{2M,2N} I_{mn} = \sum_{m,n=1}^{2M,2N} \int_{(m-1)\frac{\pi}{M}}^{\frac{m\pi}{M}} \int_{(n-1)\frac{\pi}{N}}^{\frac{n\pi}{N}} |\Delta_{11} f(u,v;\tfrac{\pi}{M},\tfrac{\pi}{N})|^2 \, dudv$$

$$= \int_0^{2\pi}\int_0^{2\pi} |\Delta_{11} f(u,v;\tfrac{\pi}{M},\tfrac{\pi}{N})|^2 \, dudv,$$

so that by (6.56) we obtain

$$(6.57) \quad I \equiv \int_0^{2\pi}\int_0^{2\pi} |\Delta_{11} f(u,v;\tfrac{\pi}{M},\tfrac{\pi}{N})|^2 dudv \leq K_2 M^{-(\alpha+1)} N^{-(\beta+1)}.$$

Now making the change of variables

$$u = x - \frac{\pi}{2M}, \quad v = y - \frac{\pi}{2N}, \quad du=dx, \quad dv=dy,$$

we have

$$I = \int_{\frac{\pi}{2M}}^{2\pi+\frac{\pi}{2M}} \int_{\frac{\pi}{2N}}^{2\pi+\frac{\pi}{2N}} |f(x+\tfrac{\pi}{2M}, y+\tfrac{\pi}{2N}) - f(x-\tfrac{\pi}{2M}, y+\tfrac{\pi}{2N}) - f(x+\tfrac{\pi}{2M}, y-\tfrac{\pi}{2N}) + f(x-\tfrac{\pi}{2M}, y-\tfrac{\pi}{2N})|^2 dxdy$$

$$I = \int_{\frac{\pi}{2M}}^{2\pi+\frac{\pi}{2M}} \int_{\frac{\pi}{2N}}^{2\pi+\frac{\pi}{2N}} |\Delta'_{11} f(x,y;\tfrac{\pi}{2M},\tfrac{\pi}{2N})|^2 \, dxdy.$$

We now apply Lemma 6.6 with $a = -\frac{\pi}{2M}$, $b = -\frac{\pi}{2N}$, which together with (6.57) gives

$$I = \int_0^{2\pi}\int_0^{2\pi} |\Delta'_{11} f(x,y;\tfrac{\pi}{2M},\tfrac{\pi}{2N})|^2 dxdy \leq K_3 \, M^{-(\alpha+1)} N^{-(\beta+1)}.$$

Lemma 6.3 yields

$$\sum_{m,n=1}^{\infty} \rho_{mn}^2 \sin^2\frac{m\pi}{2M} \sin^2\frac{n\pi}{2N} \leq K_4 \, M^{-(\alpha+1)} \, N^{-(\beta+1)}.$$

With the aid of Lemma 6.2 we get

$$(6.58) \qquad \sum_{m=2^\mu}^{2^{\mu+1}-1} \; \sum_{n=2^v}^{2^{v+1}-1} \rho_{mn}^2 \leq K_5 \, 2^{-\mu(\alpha+1)} \, 2^{-v(\beta+1)},$$

$$\mu, v = 0,1,2,\ldots,$$

and by Schwarz's inequality

$$\sum_{m=2^\mu}^{2^{\mu+1}-1} \; \sum_{n=2^v}^{2^{v+1}-1} \rho_{mn} \leq K_6 \, 2^{-\frac{1}{2}\mu(\alpha+1)-\frac{1}{2}v(\beta+1)} \, 2^{\frac{1}{2}\mu+\frac{1}{2}v}$$

$$\leq K_6 \, 2^{-\frac{1}{2}(\mu\alpha+v\beta)} \qquad (\mu, v = 0,1,2,\ldots).$$

Hence

$$\sum_{m,n=1}^{\infty} \rho_{mn} = \sum_{\mu,v=0}^{\infty} \; \sum_{m=2^\mu}^{2^{\mu+1}-1} \; \sum_{n=2^v}^{2^{v+1}-1} \rho_{mn} \leq K_6 \sum_{\mu,v=1}^{\infty} 2^{-\frac{1}{2}(\mu\alpha+v\beta)},$$

and the double series on the right converges since $\alpha > 0$, $\beta > 0$, so that

$$\sum_{m,n=1}^{\infty} \rho_{mn} < \infty .$$

From bounded variation H we have bounded variation in each variable separately. This fact together with (6.52),(6.53), Lemma 6.1 and Lemma 6.7)gives the hypothesis of Zygmund's theorem (cf. Theorem A.7,p.13) for the functions G(x) and H(y) defined in Lemma 6.1. Hence by Zygmund's theorem the series

$$\sum_{m=1}^{\infty} \rho_{mo} \quad \text{and} \quad \sum_{n=1}^{\infty} \rho_{on} \quad \text{are convergent.}$$

A collection of results gives

$$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty .$$

This completes the proof of the theorem.

<u>Theorem 6.8</u>. <u>If</u> f(x,y) <u>is in each of the three Lipschitz classes</u>

(6.59) Lip $S_{11}(\alpha,\beta)$:

$$|\Delta_{11}f(x,y;h,k)| = |f(x+h,y+k)-f(x,y+k)-f(x+h,y)+f(x,y)|$$

$$\leq K\,|h|^{\alpha}|k|^{\beta},$$

(6.60) Lip α in x: $|f(x+h,y)-f(x,y)| \leq K|h|^{\alpha}$ <u>uniformly in</u> y,

(6.61) Lip β in y: $|f(x,y+k)-f(x,y)| \leq K|k|^{\beta}$ <u>uniformly in</u> x,

<u>with</u> $\alpha > 0$, $\beta > 0$, <u>and if</u> f(x,y) <u>is of bounded variation</u> H,

<u>then</u>

$$\sum_{m,n=0}^{\infty} \rho_{mn}^{\lambda} < \infty$$

for every λ greater than the larger of the two quantities

$\frac{2}{2+\alpha}$ and $\frac{2}{2+\beta}$.

<u>Proof</u>. Since the hypotheses (6.59), (6.60) and (6.61) are the same as for Theorem 6.7, we may begin the proof with the inequality (6.58)

$$\sum_{m=2^{\mu}}^{2^{\mu+1}-1} \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn}^{s} \leq K_s \, 2^{-\mu(\alpha+1)} \, 2^{-v(\beta+1)},$$

$$(\mu, v=0,1,2,\ldots).$$

Then by inequality (2.23), page 21, with

$$p = \frac{2}{\lambda} \geq 1, \qquad Q_{mn} = \rho_{mn}^{\lambda}$$

$$m_o = 2^{\mu+1}-1-2^{\mu}+1 = 2^{\mu}, \quad n_o = 2^{v+1}-1-2^{v}+1 = 2^{v}$$

we obtain

$$\sum_{m=2}^{2^{\mu+1}-1} \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn}^{\lambda} \leq 2^{\mu(1-\frac{\lambda}{s})+v(1-\frac{\lambda}{s})} \, K_s^{\frac{\lambda}{s}} \, 2^{-\frac{1}{s}\lambda\mu(\alpha+1)} \, 2^{-\frac{1}{s}\lambda v(\beta+1)}$$

$$\leq K_s \, 2^{\mu\left[1-\frac{2+\alpha}{2}\lambda\right]} \, 2^{v\left[1-\frac{2+\beta}{2}\lambda\right]}.$$

Thus

$$\sum_{m,n=1}^{\infty} \rho_{mn}^{\lambda} = \sum_{\mu,v=0}^{\infty} \sum_{m=2}^{2^{\mu+1}-1} \sum_{n=2^{v}}^{2^{v+1}-1} \rho_{mn} \leq \sum_{,v=0}^{\infty} K_s 2^{\mu\left[1-\frac{2+\alpha}{2}\lambda\right]+v\left[1-\frac{2+\beta}{2}\lambda\right]}$$

and the series on the right converges since

$\lambda > \frac{2}{2+\alpha}$, $\lambda > \frac{2}{2+\beta}$ means $1-\frac{2+\alpha}{2}\lambda < 0$, $1-\frac{2+\beta}{2}\lambda < 0$.

Hence

$$\sum_{m,n=1}^{\infty} \rho_{mn}^{\lambda} < \infty .$$

Now for $\lambda > \frac{2}{2+\alpha}$, $\lambda > \frac{2}{2+\beta}$ the hypotheses (6.60) and (6.61), together with Lemma 6.1, Lemma 6.7 and the bounded variation in each variable from definition H, enable us to apply Theorem A.14, page 15, to G(x) and H(y). This gives

$$\sum_{m=1}^{\infty} \rho_{mo}^{\lambda} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \rho_{on}^{\lambda} < \infty ,$$

so that from the above result it follows that

$$\sum_{m,n=0}^{\infty} \rho_{mn}^{\lambda} < \infty ,$$

and the theorem is proved.

6.B. <u>Double harmonic functions and absolute convergence</u>.

<u>Definition</u>. Let f(x,y) be a continuous periodic function of period 2π in each variable whose double Fourier series (2.11)' has the coefficients $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ defined in (2.12). The double harmonic function G(r,s,x,y) associated with f(x,y) is defined as follows:

(6.62) $$G(r,s,x,y) = \frac{1}{4}a_{oo} + \frac{1}{2}\sum_{m=1}^{\infty} r^m(a_{mo}\cos mx + b_{mo}\sin mx)$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} s^n(a_{on}\cos ny + c_{on}\sin ny)$$

$$+\sum_{m,n=1}^{\infty} r^m s^n (a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny)$$

with $0<r<1$, $0<s<1$, $-\pi\leq x\leq\pi$, $-\pi\leq y\leq\pi$, and

(6.63) $G(1,1,x,y)=f(x,y)$, $G(r,s,x,y)$ continuous for $r\leq 1, s\leq 1$.

Lemma 6.8. If $G(r,s,x,y)$ is the double harmonic function associated with $f(x,y)$, and if $f(x,y)$ is inthen Lipschitz class Lip $S_{11}(\alpha,\beta)$:

$$|\gamma(x,y;t,w)|\equiv|f(x+t,y+w)-f(x,y+w)-f(x+t,y)+f(x,y)|\leq K|t|^{\alpha}|w|^{\beta}$$

with $0<\alpha\leq 1$, $0<\beta\leq 1$, and K a constant depending only on the function $f(x,y)$, then

$$\left|\frac{\partial^2 G}{\partial x\partial y}\right|< M(K,\alpha,\beta,r,s)\cdot(1-r)^{\alpha-1}(1-s)^{\beta-1} \text{ uniformly in}(x,y),$$

with

$$M<\frac{1024K}{r^{\frac{\alpha}{2}}s^{\frac{\beta}{2}}}.$$

Proof. With $0<r<1$, $0<s<1$ it has been shown in (4.52) and (4.53) that $G(r,s,x,y)$ has the double Poisson integral form

$$G(r,s,x,y)=\frac{(1-r^2)(1-s^2)}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)\,dudv}{[1-2r\cos(u-x)+r^2][1-2s\cos(v-y)+s^2]}$$

Now in the notation of (4.46), page 79, we have

$$(6.64)\quad \frac{\partial G}{\partial x}=-\frac{(1-r^2)(1-s^2)2r}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)\sin(u-x)}{\Delta_1^2\Delta_2}\,dudv$$

and

$$(6.65)\quad \frac{\partial^2 G}{\partial x \partial y} = \frac{rs(1-r^2)(1-s^2)}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)\ \sin(u-x)\ \sin(v-y)}{\Delta^2}\,dudv.$$

We use the notation

$$(6.66)\qquad H = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(u,v)\ \sin(u-x)\ \sin(v-y)}{\Delta^2}\ dudv.$$

Introducing the change of variables $u=t+x, v=w+y$, $du=dt$, $dv=dw$, and writing $D_1=1-2r\cos t + r^2$, $D_2=1-2s\cos w+s^2$, $D=D_1D_2$, the integral H takes the form

$$H = \int_{-\pi-x}^{\pi-x}\int_{-\pi-y}^{x-y}\frac{\sin t\ \sin w\ f(x+t,y+w)}{D^2}\ dtdw,$$

which by Lemma 6.6 may be written

$$H = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin t\ \sin w\ f(x+t,y+w)}{D^2}\ dtdw.$$

We now introduce

$$(6.67)\ J = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin t\ \sin w}{D^2}\ \gamma(x,y;t,w)\ dtdw = H-I_1-I_2+I_3,$$

where

$$I_1=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin t\ \sin w}{D^2}f(x,y+w)dtdw=\int_{-\pi}^{\pi}\frac{\sin t\ dt}{[1-2r\ \cos t+r^2]^2}\int_{-\pi}^{\pi}\frac{\sin w f(x,y+w)}{[1-2s\ \cos w+s^2]^2}dw$$

$$I_2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sin t \sin w}{D^2} f(x+t,y)\,dt\,dw$$

$$= \int_{-\pi}^{\pi} \frac{\sin t\, f(x+t,y)}{[1-2r\cos t+r^2]^2}\,dt \int_{-\pi}^{\pi} \frac{\sin w\, dw}{[1-2s\cos w+s^2]^2}$$

$$I_3 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sin t \sin w}{D^2} f(x,y)\,dt\,dw$$

$$= f(x,y)m \int_{-\pi}^{\pi} \frac{\sin t\, dt}{[1-2r\cos t+t^2]^2} \int_{-\pi}^{\pi} \frac{\sin w\, dw}{[1-2r\cos t+r^2]^2}$$

Since

$$(6.68)\quad \int_{-\pi}^{\pi} \frac{\sin u\, du}{[1-2r\cos u+r^2]^2} = -\frac{1}{2r(1-2r\cos u+r^2)}\Bigg|_{-\pi}^{\pi} = 0,$$

it follows immediately that $I_1=I_2=I_3=0$ and thus $J=H-I_1-I_2+I_3=H$, and by (6.65) and (6.66) we may write

$$(6.69)\quad \frac{\partial^2 G}{\partial x\partial y} = \frac{rs(1-r^2)(1-s^2)}{\pi^2} H = \frac{rs(1-r^2)(1-s^2)}{\pi^2} J$$

$$= \frac{rs(1-r^2)(1-s^2)}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sin t \sin w}{D^2} \gamma(x,y;t,w)\,dt\,dw.$$

Applying the hypothesis

$$|\gamma(x,y;t,w)| < K|t|^{\alpha}|w|^{\beta},$$

we obtain

$$|J| \leq K \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|t|^{\alpha+1}\,|w|^{\beta+1}\,dt\,dw}{[1-2r\cos t+r^2]^2\,[1-2s\cos w+s^2]^2}$$

$$\leq 4K \int_{0}^{\pi}\int_{0}^{\pi} \frac{t^{\alpha+1}\,w^{\beta+1}\,dt\,dw}{[1-2r\cos t+r^2]^2\,[1-2s\cos w+s^2]^2}.$$

Since $1-r > \frac{1}{2}(1-r)$, $1-s > \frac{1}{2}(1-s)$, $\sin\frac{1}{2}t \geq \frac{1}{4}t$, $\sin\frac{1}{2}w \geq \frac{1}{4}w$,

then

$$\begin{cases} 1-2r\cos t + r^2 = (1-r)^2 + 4r\sin^2\frac{1}{2}t > \frac{1}{4}\left[(1-r)^2+rt^2\right] \\ 1-2s\cos w + s^2 = (1-s)^2 + 4s\sin^2\frac{1}{2}w > \frac{1}{4}\left[(1-s)^2+sw^2\right], \end{cases}$$

and

$$|J| \leq 1024\,K \int_{0}^{\pi}\int_{0}^{\pi} \frac{t^{\alpha+1}\,w^{\beta+1}\,dt\,dw}{[(1-r)^2+rt^2]^2\,[(1-s)^2+sw^2]^2}$$

$$\leq 1024\,K \int_{0}^{\infty}\int_{0}^{\infty} \frac{t^{\alpha+1}\,w^{\beta+1}\,dt\,dw}{[(1-r)^2+rt^2]^2\,[(1-s)^2+sw^2]^2}.$$

We now make the substitution

$$u^2 = \frac{rt^2}{(1-r)^2}, \quad v^2 = \frac{sw^2}{(1-s)^2}, \quad dt = \frac{1-r}{r^{\frac{1}{2}}}\,du, \quad dw = \frac{1-s}{s^{\frac{1}{2}}}\,dv,$$

so that $(1-r)^2+rt^2=(1-r)^2(1-u^2)$, $(1-s)^2+sw^2=(1-s)^2(1-v^2)$,

$$t^{\alpha+1} = \frac{(1-r)^{\alpha+1}\,u^{\alpha+1}}{r^{\frac{1}{2}(\alpha+1)}}, \quad w^{\beta+1} = \frac{(1-s)^{\beta+1}\,v^{\beta+1}}{s^{\frac{1}{2}(\beta+1)}},$$

and

$$|J| \leq \frac{1024\,K(1-r)^{\alpha-2}\,(1-s)^{\beta-2}}{r^{\frac{1}{2}(\alpha)+1}\,s^{\frac{1}{2}(\beta)+1}} \int_{0}^{\infty}\int_{0}^{\infty} \frac{u^{\alpha+1}\,v^{\beta+1}}{(1+u^2)^2(1+v^2)^2}\,du\,dv.$$

Therefore, by (6.69)

$$\left|\frac{\partial^2 G}{\partial x \partial y}\right| = \frac{rs(1-r^2)(1-s^2)}{\pi^2}|J| \leq M(K,\alpha,\beta,r,s)\cdot(1-r)^{\alpha-1}(1-s)^{\beta-1}$$

uniformly in x and y,

where

$$M = \frac{1024K(1+r)(1+s)}{\pi^2\, r^{\frac{\alpha}{2}}\, s^{\frac{\beta}{2}}}\int_0^\infty\int_0^\infty \frac{u^{\alpha+1}v^{\beta+1}\,dudv}{[(1+u^2)^2][(1+v^2)^2]} .$$

Now consider the integral in M and use the inequalities

$$u^{\alpha+1} < (1+u^2)^{\frac{\alpha+1}{2}}, \quad 2-\frac{\alpha+1}{2} = \frac{3-\alpha}{2} \geq 1, \quad 1+u^2 \leq (1+u^2)^{\frac{3-\alpha}{2}}$$

and symmetric inequalities for v and β, then

$$\int_0^\infty\int_0^\infty \frac{u^{\alpha+1}v^{\beta+1}\,dudv}{(1+u^2)^2(1+v^2)^2} \leq \int_0^\infty\int_0^\infty \frac{(1+u^2)^{\frac{\alpha+1}{2}}(1+v^2)^{\frac{\beta+1}{2}}}{(1+u^2)^2(1+v^2)^2}\,dudv$$

$$\leq \int_0^\infty\int_0^\infty \frac{dudv}{(1+u^2)^{\frac{3-\alpha}{2}}(1+v^2)^{\frac{3-\beta}{2}}}$$

$$\leq \int_0^\infty\int_0^\infty \frac{du\,dv}{(1+u^2)(1+v^2)} = \frac{\pi^2}{4},$$

and thus

$$M \leq \frac{1024\,K(1+r)(1+s)}{\pi^2 r^{\frac{\alpha}{2}}\, s^{\frac{\beta}{2}}}\left(\frac{\pi^2}{4}\right) = \frac{256\,K(1+r)(1+s)}{r^{\frac{\alpha}{2}}\, s^{\frac{\beta}{2}}} \leq \frac{1024\,K}{r^{\frac{\alpha}{2}}\, s^{\frac{\beta}{2}}}.$$

With the aid of Lemma 6.8 we now give a second proof of Theorem 6.4.

<u>Proof of Theorem 6.4</u>. Let $G(r,s,x,y)$ be the double harmonic funtion associated with $f(x,y)$ in (6.62). By Lemma 6.8 we have

$$\left|\frac{\partial^2 G}{\partial x\partial y}\right| = \left|\sum_{m,n=1}^{\infty} mnr^m s^n (a_{mn}\sin mx \sin ny - b_{mn}\cos mx \sin ny - c_{mn}\sin mx \cos ny + d_{mn}\cos mx \cos ny)\right|$$

$$\leq M(1-r)^{\alpha-1}(1-s)^{\beta-1} \text{ uniformly in } (x,y) \text{ in } R.$$

Squaring and integrating this inequality from 0 to 2π we get

$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial^2 G}{\partial x\partial y}\right|^2 dxdy \leq M^2(1-r)^{2(\alpha-1)}(1-s)^{2(\beta-1)},$$

whence

$$\sum_{m,n=1}^{\infty} m^2n^2r^{2m}s^{2n}(a_{mn}^2+b_{mn}^2+c_{mn}^2+d_{mn}^2) \leq M^2(1-r)^{2(\alpha-1)}(1-s)^{2(\beta-1)}.$$

Hence

$$\sum_{m,n=1}^{\infty} \rho_{mn}^2 m^2n^2r^{2m}s^{2n} \leq M^2(1-r)^{2(\alpha-1)}(1-s)^{2(\beta-1)}.$$

On putting $r = 1 - \frac{1}{\mu}$, $s = 1 - \frac{1}{v}$, we obtain

$$\sum_{m,n=1}^{\mu,\nu} m^2n^2\rho_{mn}^2 \leq M^2\mu^{2(1-\alpha)}v^{2(1-\beta)},$$

and by Schwarz's inequality we get

$$\sum_{m,n=1}^{\mu,\nu} mn\,\rho_{mn} \leq M(\mu^{1-\alpha}v^{1-\beta})(\mu^{\frac{1}{2}}v^{\frac{1}{2}}) = M(\mu^{\frac{3}{2}-\alpha}v^{\frac{3}{2}-\beta}).$$

Consider

$$\sum_{m,n=1}^{2^i-1,2^k-1} m^a n^b \rho_{mn} = \sum_{\mu,v=1}^{i,k} \sum_{m=2^{\mu-1}}^{2^\mu-1} \sum_{n=2^{v-1}}^{2^v-1} mn\, m^{a-1} n^{b-1} \rho_{mn}$$

$$(a \geq 0, b \geq 0),$$

$$\leq M_1 \sum_{\mu,v=1}^{i,k} 2^{(\mu-1)(a-1)+(v-1)(b-1)+\mu(\frac{3}{2}-\alpha)+v(\frac{3}{2}-\beta)-\frac{3}{2}+\alpha-\frac{3}{2}+\beta}$$

$$\leq M_2 \sum_{\mu,v=1}^{i,k} 2^{\mu(\frac{1}{2}-\alpha+a)}\, 2^{v(\frac{1}{2}-\beta+b)},$$

whence upon taking a=b=0,

$$\sum_{m,n=1}^{2^i-1,2^k-1} \rho_{mn} \leq M_2 \sum_{\mu=1}^{i} 2^{\mu(\frac{1}{2}-\alpha)} \sum_{v=1}^{k} 2^{v(\frac{1}{2}-\beta)}.$$

The series on the right converge sinne $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, and thus

$$\sum_{m,n=1}^{\infty} \rho_{mn} < \infty.$$

With G(x) and H(y) as given in (6.1) it follows from (6.43), (6.44) and Lemma 6.7 that the Bernstein theorem A.1, page 12, may be applied to G(x) and H(y). This gives

$$\sum_{m=1}^{\infty} \rho_{mo} < \infty, \qquad \sum_{n=1}^{\infty} \rho_{on} < \infty,$$

which combined with the above result gives

$$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty.$$

<u>Lemma 6.9</u>. If $G(r,s,x,y)$ is the double harmonic function associated with $f(x,y)$, and if $|\gamma'(x,y;t,w)|=|f(x+h,y+k)-f(x,y)|\leq K(|h|^{\alpha}+|k|^{\alpha})$ uniformly in (x,y) in R, with $\alpha > 1$, then

$$\left|\frac{\partial^2 G}{\partial x\partial y}\right|\leq M(K,r,s,\alpha)\cdot(1-r)^{-1}(1-s)^{-1}\left[(1-r)^{\alpha}+(1-s)^{\alpha}\right]$$ uniformly

in R, where $M\leq \frac{256\ K\ (1+r)(1+s)}{rs}\ L$, with L the larger of the two quantities $r^{-\frac{\alpha}{2}}$ and $s^{-\frac{\alpha}{2}}$.

<u>Proof</u>. We write

$$\begin{array}{lll} t = u-x & D_1 = 1-2r\cos t+r^2 & \\ & & D = D_1\cdot D_2. \\ w = v-y & D_2 = 1-2s\cos w+s^2 & \end{array}$$

As in the proof of Lemma 6.8 we get the expression (6.69)

$$\frac{\partial^2 G}{\partial x\partial y} = \frac{rs(1-r^2)(1-s^2)}{\pi^2}\ H,$$

where

$$H = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin t\ \sin w}{D^2}\ f(x+t,y+w)\ dtdw.$$

We introduce

$$J = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin t\ \sin w}{D^2}\ \gamma'(x,y;t,w)\ dtdw = H - I,$$

where

$$I = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin t\ \sin w}{D^2}\ f(x,y)\ dtdw.$$

By (6.68) we see that $I=0$, and hence $J=H$.

Using the inequalities

$$1-r > \frac{1}{2}(1-r),\ 1-s > \frac{1}{2}(1-s),\ \sin\frac{1}{2}t \geq \frac{1}{4}t,\ \sin\frac{1}{2}w \geq \frac{1}{4}w$$

and the hypothesis $|\gamma'(x,y;t,w)| \leq K(|t|^{\alpha}+|w|^{\alpha})$, we get

$$|J| \leq 1024\ K \int_0^{2\pi}\int_0^{2\pi} \frac{t\ w\ (t^{\alpha}+w^{\alpha})\ dtdw}{[(1-r)^2+rt^2]^2[(1-s)^2+sw^2]^2}$$

$$\leq 1024\ K \int_0^{\infty}\int_0^{\infty} \frac{t\ w\ (t^{\alpha}+w^{\alpha})\ dtdw}{[(1-r)^2+rt^2]^2[(1-s)^2+sw^2]^2}.$$

We now make the substitution

$$u^2 = \frac{rt^2}{(1-r)^2}, \qquad v^2 = \frac{sw^2}{(1-s)^2}.$$

Hence

$$|J| \leq \frac{1024\ K\ (1-r)^{\alpha-2}}{r^{\frac{\alpha}{2}+1}\ s(1-s)^2}\int_0^{\infty}\int_0^{\infty}\frac{u^{\alpha+1}v\ dudv}{(1+u^2)^2(1+v^2)^2}$$

$$+\frac{1024\ K\ (1-s)^{\alpha-2}}{rs^{\frac{\alpha}{2}+1}\ (1-r)^2}\int_0^{\infty}\int_0^{\infty}\frac{u\ v^{\alpha+1}dudv}{(1+u^2)^2(1+v^2)^2}$$

$$\leq L_1(1-r)^{\alpha-2}(1-s)^{-2} + L_2(1-r)^{-2}(1-s)^{\alpha-2},$$

where

$$L_1 = \frac{1024\ K}{r^{\frac{\alpha}{2}+1}s}\int_0^{\infty}\int_0^{\infty}\frac{u^{\alpha+1}v\ dudv}{(1+u^2)^2(1+v^2)^2} \leq \frac{256K\pi^2}{rs}.r^{-\frac{\alpha}{2}}$$

and

$$L_2 = \frac{1024K}{rs^{\frac{\alpha}{2}+1}}\int_0^{\infty}\int_0^{\infty}\frac{u\,v^{\alpha+1}\,du\,dv}{(1+u^2)^2(1+v^2)^2} \leq \frac{256K\pi^2}{rs}\cdot s^{-\frac{\alpha}{2}}.$$

If now we call L the larger of the two quantities $r^{-\frac{\alpha}{2}}$, $s^{-\frac{\alpha}{2}}$ we have

$$|J| \leq \frac{256\pi^2 K\,L\,(1-r)^{-2}(1-s)^{-2}}{rs}\left[(1-r)^{\alpha}+(1-s)^{\alpha}\right].$$

Therefore,

$$\left|\frac{\partial^2 G}{\partial x\partial y}\right| \leq 256\ K\ L\ (1+r)(1+s)(1-r)^{-1}(1-s)^{-1}\left[(1-r)^{\alpha}+(1-s)^{\alpha}\right]$$

$$\leq M(K,r,s,\ \ )(1-r)^{-1}(1-s)^{-1}\left[(1-r)^{\alpha}+(1-s)^{\alpha}\right],$$

where $M = 256\ K\ L\ (1+r)(1+s)$, and L is the larger of the two quantities $r^{-\frac{\alpha}{2}}$, $s^{-\frac{\alpha}{2}}$.

Theorem 6.9. If f(x,y) *is in the Lipschitz class* Lip(α,α): $|f(x+h,y+k)-f(x,y)| \leq K(|h|^{\alpha}+|k|^{\alpha})$ *uniformly in* R, *with* $\alpha > 1$, *then the double Fourier series of* f(x,y) *converges absolutely in* R.

Proof. For $r<1$, $s<1$, Lemma 6.9 gives

$$\frac{\partial^2 G}{\partial x\partial y} = \sum_{m,n=1}^{\infty} mn\ r^m s^n |a_{mn}\sin mx \sin ny - b_{mn}\cos mx \sin ny$$
$$-c_{mn}\sin mx \cos ny + d_{mn}\cos mx \cos ny|$$

$$\leq M(1-r)^{-1}(1-s)^{-1}\left[(1-r)^{\alpha}+(1-s)^{\alpha}\right], \text{(M independent of } x,y).$$

Squaring and integrating this inequality from 0 to 2π, we get

$$\sum_{m,n=1}^{\infty} m^2 n^2 r^{2m} s^{2n} \rho_{mn}^2 \leq M_1(1-r)^{-2}(1-s)^{-2}\left[(1-r)^{\alpha}+(1-s)^{\alpha}\right]^2,$$

and placing

$$r = 1 - \frac{1}{\mu}, \quad s = 1 - \frac{1}{v}$$

we get

$$\sum_{m,n=1}^{\mu,v} m^2 n^2 \rho_{mn}^2 \leq M_1 \mu^2 v^2 \left[\mu^{-\alpha} + v^{-\alpha}\right]^2 .$$

Applying Schwarz's inequality we have

$$\sum_{m,n=1}^{\mu,v} |mn\, \rho_{mn}| \leq M_2 \mu^{\frac{3}{2}} v^{\frac{3}{2}} \left[\mu^{-\alpha} + v^{-\alpha}\right],$$

or

$$\sum_{m,n=1}^{2^\mu, 2^v} mn\, \rho_{mn} \leq M_2\, 2^{\frac{3}{2}(\mu+v)} \left[2^{-\alpha\mu} + 2^{-\alpha v}\right].$$

Now we consider

$$\sum_{m,n=1}^{2^i, 2^k} m^a n^b \rho_{mn} = \sum_{\mu,v=1}^{i,k} \sum_{m=2}^{2^\mu} {}_{-1}\sum_{n=2^{v-1}}^{2^v} m^{a-1} n^{b-1} mn\, \rho_{mn}$$

$$\leq M_2 \sum_{\mu,v=1}^{i,k} 2^{(\mu-1)(a-1)+(v-1)(b-1)+\frac{3}{2}(\mu+v)} \left[2^{-\alpha\mu} + 2^{-\alpha v}\right]$$

$$\leq M_2 \sum_{\mu,v=1}^{i,k} 2^{\mu(a+\frac{1}{2})+v(b+\frac{1}{2})-a-b} \left[2^{-\alpha\mu} + 2^{-\alpha v}\right].$$

Whence if a=b=0,

$$\sum_{m,n=1}^{2^i, 2^k} \rho_{mn} \leq M_2 \sum_{\mu,v=1}^{i,k} 2^{\frac{1}{2}(\mu+v)} \left[2^{-\alpha\mu} + 2^{-\alpha v}\right].$$

Hence

$$\sum_{m,n=1}^{\infty} \rho_{mn} \leq M_2 \sum_{\mu,v=1}^{\infty} 2^{\frac{1}{2}(\mu+v)} \left[2^{-\alpha\mu} + 2^{-\alpha v}\right]$$

and thus

$$\sum_{m,n=1}^{\infty} \rho_{mn} < \infty ,$$

since the above series on the right converges for $\alpha > 1$. Now from R1, page 26, f(x,y) is in Lip α in x and f(x,y) is in Lip α in y. By Lemma 6.7 it follows that G(x) and H(y) as defined in Lemma 6.1 are in Lip α. We, therefore, may apply Bernstein's Theorem A.1, page 12, which yields

$$\sum_{m=1}^{\infty} \rho_{mo} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \rho_{on} < \infty .$$

Collecting results we have

$$\sum_{m,n=0}^{\infty} \rho_{mn} < \infty ,$$

and the theorem is proved.

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